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On the Variance of the Number of Pivot Steps Required by the Simplex Algorithm

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Summary. The article provides an asymptotic probabilistic analysis of the variance of the number of pivot steps required by phase II of the "shadow vertex algorithm" — a parametric variant of the simplex algorithm, which has been proposed by Borgwardt [1]. The analysis is done for data which satisfy a rotationally invariant distribution law in the n-dimensional unit ball.

1. Introduction

Despite their very good empirical performance the simplex algorithm's variants require exponentially many pivot steps in terms of the problem dimensions of the given linear programming problem (LPP) in worst-case situtation. The first to explain the large gap between practical experience and the disappointing worst-case was Borgwardt [1], who could prove polynomiality on the average for a certain variant—the "Schatteneckenalgorithmus (shadow vertex algorithm)"—using a stochastic problem simulation.

Borgwardt studied LPPs of type

$$\max_{x \in X} v^T x\,,\ X_A := \{x \in \mathbb{R}^n \,|\, a^T x \leq 1,\ a \in A\}, \tag{1}$$

with $A := \{a_1, \ldots, a_m\}$, a_i, $v \in \mathbb{R}^n$, $m \geq n \geq 2$. The vectors a_i are supposed to be i.i.d. on $\mathbb{R}^n \setminus \{0\}$, whose common distribution is invariant under rotations around the origin. That means the polar representations of the a_i, $a_i = r_i w_i$, consist of stochastically independent radial parts $r_i \in \mathbb{R}^+$ and directional parts $w_i \in \omega_n$, ω_n being the unit sphere in $\mathbb{R}^n$. r_i is distributed by a radial distribution function (RDF) F, which we assume continuous from the right, and w_i is uniformly distributed on ω_n. Introducing the notation

$$Y_A := \{y \in \mathbb{R}^n \,|\, x^T y \leq 1;\ x \in X_A\} = \text{convhull}(\{0\} \cup A) \tag{2}$$

for the polar polyhedron Y_A of X_A we define corresponding to Borgwardt the random variable $s_{u,v}(X_A)$ for linearly independent vectors $u, v \in \mathbb{R}^n$ by

(3)
$$s_{u,v}(X_A) := \begin{cases} \text{numb. of bound. simpls. of } Y_A \text{ intersected by cone}(u,v) - 1 \\ \quad \text{if } \mathbb{R}^+ u \text{ intersects a bound. simpl. of } Y_A \\ \text{numb. of bound. simpls. of } Y_A \text{ intersected by cone}(u,v) \\ \quad \text{if } \mathbb{R}^+ u \text{ does not intersect a bound. simpl. of } Y_A \end{cases}$$

The number $s_{u,v}(X_A)$ equals the number of pivot steps, which phase II of the shadow vertex algorithm requires in order to maximize the functional $v^T x$ over X_A, when the iteration is started with a vertex of X_A, which maximizes the functional $u^T x$ over X_A.

Borgwardt analyzed the expectation of $s_{u,v}$ averaging on the choice of the polyhedron X_A and on the choice of the vectors u and v. The vectors u and v he assumed independently, identically and uniformly distributed on the n-dimensional unit sphere. Here, the restriction to the uniform distribution on the sphere has been done without loss of generality within the rotationally invariant distributions, as (3) depends on the directions of u and v only. The main result of Borgwardt's analysis is an estimation of the expectation value $\mathrm{E}(s_{u,v})$ from above, which is independent from the underlying rotationally invariant distribution. It holds, c.f. Borgwardt [1],

(4)
$$\mathrm{E}(s_{u,v}) \leq C n^3 m^{1/(n-1)},$$

where C is a small constant not depending on n and m.

But knowledge about the expected number of pivots alone does not completely reveal the algorithm's effort. Of great importance is the question for the probability of large deviations of the number of pivots required from its expectation value. So, many researchers, e.g. Shamir [5] or Borgwardt [1], raised the question for higher moments or even for the distribution of the random variable $s_{u,v}$. We are going to answer this question partially. As a first step to the analysis of $s_{u,v}$'s variance we investigate the variance of the functional $s_{u,v}$ asymptotically for $m \to \infty$ in case of distributions with compact domain.

2. Main results

Let $\mathcal{F}_{\Omega_n}$ be the set of RDFs of rotationally symmetric distributions in $\mathbb{R}^n$, whose mass is concentrated in the n-dimensional unit ball Ω_n, that means $F(1) = 1$, $F(t) < 1$ for $t < 1$. For each RDF F we denote with $\overline{F} := 1 - F$ the tail of the distribution function. Especially, we deal with two subsets of $\mathcal{F}_{\Omega_n}$, the set of distributions with algebraic tail $\mathcal{F}_{\mathrm{alg},\Omega_n}$,

(5)
$$\mathcal{F}_{\mathrm{alg},\Omega_n} := \{F \in \mathcal{F}_{\Omega_n} \mid \overline{F}(1-t) \sim L t^\alpha,\ t \to 0+,\ L, \alpha > 0\},$$

and the more general set of distributions with regular varying tail $\mathcal{F}_{\mathrm{reg},\Omega_n}$, cf. Feller [2],

(6)
$$\mathcal{F}_{\mathrm{reg},\Omega_n} := \{F \in \mathcal{F}_{\Omega_n} \mid \overline{F}(1-t) \sim L(1/t) t^\alpha,\ t \to 0+,\ L \in SV_\infty, \alpha > 0\},$$

where SV_∞ is the set of positive, slowly varying functions at infinity. A function L is slowly varying at infinity if $L \in \mathcal{L}^1[1,\infty)$ and $\lim_{t\to\infty} \frac{L(\rho t)}{L(t)} = 1$, $\rho \in \mathbb{R}^+$. $f_1(t) \sim f_2(t)$, $t \to t_0$, denotes asymptotic equivalence, which means for $t \to t_0$: $f_1(t) = f_2(t)(1+o(1))$. The uniform distribution in Ω_n is a special case in $\mathcal{F}_{\text{alg},\Omega_n}$ taking $L = n$, $\alpha = 1$. The uniform distribution on the sphere ω_n is a limiting case of $\mathcal{F}_{\text{reg},\Omega_n}$ for $\alpha \to 0+$ and appropriately chosen functions L.

First, we analyze the variance of the functional $s_{u,v}$ for fixed linearly independent vectors u and v, which means that we average on the choice of the polyhedron X_A only. For reasons of rotational symmetry moments of $s_{u,v}$ depend only on the angle $\gamma := \gamma(u,v) := \angle(u,v)$, which the vectors u and v enclose, and not on their individual directions in $\mathbb{R}^n$. If u and v are fixed, we denote moments of $s_{u,v}$ with index γ in order to indicate that u, v are assumed to be fixed with $\angle(u,v) = \gamma$, e.g. $\mathrm{E}_\gamma(s_{u,v})$, $\mathrm{Var}_\gamma(s_{u,v})$. Results on $s_{u,v}$, where we average on the choice of u, v as well, are supplied as corollaries. Here, moments are denoted without index.

Theorem 1
i) For $n \geq 2$, $\gamma \in (0,\pi)$ and $F \in \mathcal{F}_{\text{alg},\Omega_n}$:

$$\frac{\mathrm{Var}_\gamma(s_{u,v})}{\mathrm{E}_\gamma^2(s_{u,v})} = \mathcal{O}(\mathrm{E}_\gamma^{-1}(s_{u,v})) = \frac{1}{\gamma}\mathcal{O}(m^{-\frac{1}{n-1+2\alpha}}),\ m \to \infty. \tag{7}$$

ii) For $n \geq 2$ and $F \in \mathcal{F}_{\text{reg},\Omega_n}$ let G,

$$G(t) := \Pr(a^{(n)} \geq t),\ t \in [-1,1], \tag{8}$$

be the distribution function of a vector a's n-th component and $\tilde{G}$ be the inverse function of G. Then for $\gamma \in (0,\pi)$:

$$\frac{\mathrm{Var}_\gamma(s_{u,v})}{\mathrm{E}_\gamma^2(s_{u,v})} = \mathcal{O}(\mathrm{E}_\gamma^{-1}(s_{u,v})) = \frac{1}{\gamma}\mathcal{O}((1-\tilde{G}(\tfrac{1}{m}))^{\frac{1}{2}}),\ m \to \infty. \tag{9}$$

The asymptotic orders of the quotients we stated in line (7) and (9) are sharp for many distributions, whose RDF belongs to $\mathcal{F}_{\text{reg},\Omega_n}$. A typical example is the uniform distribution in Ω_n. But one can construct pathological cases of RDFs in $\mathcal{F}_{\text{reg},\Omega_n}$, where the investigated quotient tends faster to zero as $\mathrm{E}_\gamma^{-1}(s_{u,v})$. As a consequence of theorem 1 we obtain:

Corollary *For $n \geq 2$ and $F \in \mathcal{F}_{\text{reg},\Omega_n}$:*

$$\frac{\mathrm{Var}(s_{u,v})}{\mathrm{E}^2(s_{u,v})} = C_n + \mathcal{O}(\mathrm{E}^{-1}(s_{u,v})),\ m \to \infty, \tag{10}$$

where $C_2 = 1/3$ and

$$\frac{8}{5}\frac{n-2}{(2n-1)(2n-3)} \leq C_n \leq 4\frac{n-2}{(2n-1)(2n-3)},\ n \geq 3. \tag{11}$$

The constant C_n in line (11) equals the quotient $\mathrm{Var}(\gamma)/\mathrm{E}^2(\gamma)$, which means the variance of the angle enclosed by u and v dominates the variance of the number of pivot steps in case of regularly varying distributions. It is possible to generalize the statements (9) and (10) for all distributions supported by the unit ball Ω_n in weakened form.

Remark : For any RDF $F \in \mathcal{F}_{\Omega_n}$, $n \geq 2$ and $\gamma \in (0, \pi)$:

$$\frac{\mathrm{Var}_\gamma(s_{u,v})}{\mathrm{E}_\gamma^2(s_{u,v})} = \frac{1}{\gamma} o(1), \; m \to \infty. \tag{12}$$

Moreover, we have analogously to (10):

$$\frac{\mathrm{Var}(s_{u,v})}{\mathrm{E}^2(s_{u,v})} = C_n + o(1), \; m \to \infty, \tag{13}$$

where the constants C_n are the same as in (11).

The convergence rate on the right hand side of (12) and (13) depends on the special choice of the distribution F and there is no possibility to give decreasing asymptotic bounds in m, which are independent from the underlying distribution, as theorem 1 ii) suggests. For RDFs F with $F(t) < 1$, $t \in \mathbb{R}^+$, (12) is not true in general. Here, one can only show that the quotient investigated is always bounded in m . Due to Chebychev's inequality

$$\Pr\left(\left|\frac{s_{u,v}}{\mathrm{E}(s_{u,v})} - 1\right| > \eta\right) \leq \eta^{-2} \frac{\mathrm{Var}(s_{u,v})}{\mathrm{E}^2(s_{u,v})}, \tag{14}$$

it is a consequence of the statements (9) and (12) that even small relative deviations of $s_{u,v}$ from the mean $\mathrm{E}_\gamma(s_{u,v})$ become rare if n is fixed and m is large. This is not true if we average additionally on the vectors u and v, cf. (10) and (13). Here, the variance is asymptotically proportional to the squared expectation with a small factor.

3. Proof of the main theorem—additional results

In the present section we are going to prove theorem 1 and the stated corollary. The complicated and lengthy proof of the generalization given in the remark is omitted here for lack of space and will be published elsewhere.

The main trick in treating the random variable $s_{u,v}$ is to consider it a random variable of the bounded polar polyhedron Y_A, cf. (2) and (3). It is an important structural property of $s_{u,v}$ that it can be additively decomposed relative to the facets of Y_A. We call a random event $A := \{a_1, \ldots, a_m\}$, cf. (1), nondegenerate if every subset of $A_{u,v} := A \cup \{u, v\}$ with cardinality $\leq n$ is linearly independent and every $A_{u,v}$-subset with cardinality $\leq n+1$ is affinely independent. If A is nondegenerate, Y_A is simplicial, which means especially that every facet of Y_A is a simplex. In our stochastic model random events A are nondegenerate with probability one. For nondegenerate events A the functional $s_{u,v}(X_A)$ can be written as

$$s_{u,v}(X_A) = s_{u,v}(Y_A) = \sum_{I \subset \{1,\ldots,m\}} \chi_1(A, A_I)\chi_{u,v}(A_I), \tag{15}$$

where I is an arbitrary subset of indices $1, \dots, m$ with cardinality n corresponding to the $n-1$-simplex $S_I := \text{convhull}(A_I)$. $\chi_1(A, A_I)$ is the indicator functional, which indicates whether the simplex S_I is a facet of the polyhedron Y_A. $\chi_{u,v}$ is an indicator functional defined by

$$(16) \qquad \chi_{u,v}(A_I) := \chi(\text{cone}(u, v) \cap S_I \neq \emptyset, \ \mathbb{R}^+ u \cap S_I = \emptyset)$$

deciding, whether S_I is intersected by the convex cone generated by the vectors u and v and not by the ray generated by the vector u. (15) obviously meets definition (3).

As first step in our analysis we are going to look at the first moment of the random variable $s_{u,v}$ and derive asymptotic equivalents for $m \to \infty$ for the expectation $E_\gamma(s_{u,v})$ in case of classes (5) and (6) of distributions. For the vectors $a_i \in A$ are i.i.d., we have for any I

$$(17) \qquad E_\gamma(s_{u,v}) = q_0 \, E_\gamma(\chi_{u,v}(A_I)\chi_1(A, A_I)),$$

where $q_0 = \binom{m}{n}$ is the number of A-subsets of cardinality n. Let $H(A_I)$ denote the hyperplane generated by the affine hull of A_I, $h(A_I)$ its Euclidian distance from the origin and let finally $H^{(1)}(A_I)$ denote the closed halfspace generated by $H(A_I)$ containing the origin. Then, $\chi_1(A, A_I) = \prod_{a \in A \setminus A_I} \chi(a \in H^{(1)}(A_I))$. By the aid of rotationally symmetry of the underlying distribution we obtain

$$(18) \qquad \Pr(a \in H^{(1)}(A_I) \,|\, h(A_I) = h') = 1 - G(h'),$$

where G is defined as in (8). Particularly, (18) means that the probability of S_I being a boundary simplex of Y_A only depends on the distance $h(A_I)$. Thus, it is convenient to establish $h(A_I)$ as independent variable, which we do by the aid of Fubini's theorem. We get

$$(19) \qquad E_\gamma(s_{u,v}) = \binom{m}{n} \int_0^\infty (1 - G(h'))^{m-n} A_\gamma(h') dh',$$

where the function A_γ can be interpreted as a conditioned expectation value by

$$(20) \qquad A_\gamma(h') = E_\gamma(\chi_{u,v}(A_I) \,|\, h(A_I) = h') \, p(h'),$$

$p(h')$ being the density of the distribution function $P(h') := \Pr(h(A_I) \leq h')$. For reasons of symmetries in our stochastic model the conditioned expectation $E_\gamma(\chi_{u,v}(A_I) \,|\, h(A_I) = h')$ only depends on the angle γ, which u and v enclose and not on the special directions of u and v. Therefore, averaging on the choice of the directions of u and v under the additional assumption that u and v enclose the angle γ does not change the conditioned expectation's value:

$$(21) \qquad \begin{aligned} &E_\gamma(\chi_{u,v}(A_I) \,|\, h(A_I) = h') = E(\tilde{W}_\gamma(A_I) \,|\, h(A_I) = h'), \\ &\tilde{W}_\gamma(A_I) := E(\chi_{u,v}(A_I) \,|\, \angle(u, v) = \gamma). \end{aligned}$$

Elementary stochastics delivers a factorization of $\tilde{W}_\gamma(A_I)$. We obtain

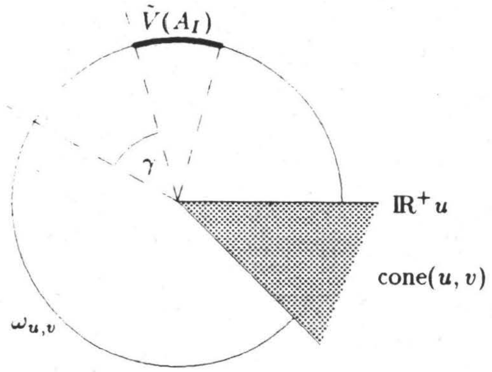


Fig. 1. Illustration of $\mathrm{E}(\chi_{u,v}(A_I))\,|\,\mathrm{span}(u,v)\cap S_I \neq \emptyset, \angle(u,v)=\gamma)$

$$(22)\qquad \begin{aligned}\tilde{W}_\gamma(A_I) &= \mathrm{E}(\chi_{u,v}(A_I))\,|\,\mathrm{span}(u,v)\cap S_I \neq \emptyset, \angle(u,v)=\gamma)\,\tilde{W}(A_I),\\ \tilde{W}(A_I) &= \Pr(\mathrm{span}(u,v)\cap S_I \neq \emptyset).\end{aligned}$$

The random event "span$(u,v)\cap S_I \neq \emptyset$" does not depend on the choice of γ. Hence, we are allowed to consider $\tilde{W}(A_I)$ as the probability of S_I being intersected by a plane generated by independently and uniformly on the unit sphere distributed vectors u and v without any conditions on the enclosed angle.

We evaluate the conditioned expectation in (22) by geometrical considerations, cf. figure 1. Let $\omega_{u,v} := \{x \in \mathbb{R}^n \cap \mathrm{span}(u,v)\,|\,\|x\|_2 = \frac{1}{2\pi}\}$ be the circle around the origin in the (u,v)-plane with unit circumference. The set of intersection points of the convex cone cone(u,v) with $\omega_{u,v}$ is uniformly distributed on $\omega_{u,v}$ due to the symmetry assumptions on the distribution of u and v. That means any fixed point of the set of intersection points is uniformly distributed on $\omega_{u,v}$. We take as fixed point the intersection point of the ray $\mathbb{R}^+u$ and $\omega_{u,v}$. As we can look at the plane span(u,v) from both of its sides we may assume u in counterclockwise direction from v in the (u,v)-plane drawn in figure 1 without loss of generality. Let $\tilde{V}(A_I)$ be the arc of intersection of the convex cone cone(S_I) generated by S_I and $\omega_{u,v}$. Then, $\chi_{u,v}(A_I)=1$, if and only if $\frac{1}{2\pi}u$ lies on the arc of length $\frac{\gamma}{2\pi}$ counterclockwise beside $\tilde{V}(A_I)$ on $\omega_{u,v}$. Therefore,

$$(23)\qquad \mathrm{E}(\chi_{u,v}(A_I))\,|\,\mathrm{span}(u,v)\cap S_I \neq \emptyset, \angle(u,v)=\gamma) = \frac{\gamma}{2\pi}.$$

If we abbreviate $\Lambda_{\tilde{W}}(h') := \mathrm{E}(\tilde{W}(A_I)\,|\,h(A_I)=h')$, (20-23) yield

$$(24)\qquad \Lambda_\gamma(h') = \frac{\gamma}{2\pi}\Lambda_{\tilde{W}}(h').$$

Integral representations for $\Lambda_{\tilde{W}}$ and p have been given by Borgwardt [1] and Raynaud [4] respectively exploiting symmetries and performing simultaneous rotations of the vectors a_i. An asymptotic analysis for $m \to \infty$ of the integral (19) in case of distributions supported by Ω_n is based on the following observation: if n is fixed and m is large only the values of Λ_γ for h' near one are relevant for the integral as $1-G(h')$ tends rapidly to zero for $h' \neq 1$. Thus, the asymptotic behaviour of the integral (19) is related to the asymptotic behaviour of $\Lambda_{\tilde{W}}$ near the boundary of the distribution's support. Hence, in order to investigate (19) asymptotically we have to analyze the asymptotic behaviour of $G(h)$, $\Lambda_{\tilde{W}}(h)$ and $p(h)$ for $h \to 1-$. Let g be the derivative of $-G$, then:

Lemma 2 *For an RDF $F \in \mathcal{F}_{\mathrm{reg},\Omega_n}$ and $n \geq 2$:*

$$G(h) \sim C_{n,\alpha}(1-h)^{(n-1)/2}\overline{F}(h),\ h \to 1-, \tag{25}$$

$$p(h) \sim \tilde{C}_{n,\alpha}G^{n-2}(h)g(h)\overline{F}(h),\ h \to 1-, \tag{26}$$

$$\Lambda_{\tilde{W}}(h) \sim \hat{C}_{n,\alpha}(1-h)^{-1/2}G^{n-1}(h)g(h),\ h \to 1-, \tag{27}$$

where $C_{n,\alpha}$, $\hat{C}_{n,\alpha}$ and $\tilde{C}_{n,\alpha}$ are positive constants depending on n and α.

In the special case of the uniform distribution in the unit ball (25) and (26) have been achieved by Raynaud [4]. Borgwardt [1] estimated $\Lambda_{\tilde{W}}(h)$ asymptotically from above and from below for the same special case. (25) can be shown by elementary analysis applied to Borgwardt's integral representation of G in [1]. The proof of (27) is very technical and therefore delayed to the appendix. (26) can be shown using similar methods as in the given proof of (27). The interested reader may take the proof of (27) as a model for proving (26) or is referred to Küfer [3]. If we substitute $G(h') = \psi$ in line (19) we receive

$$\mathrm{E}_\gamma(s_{u,v})) = \frac{\gamma}{2\pi}\binom{m}{n}\int_0^{1/2}(1-\psi)^{m-n}z_{\tilde{W}}(\psi)d\psi, \tag{28}$$

where $z_{\tilde{W}}$ is defined by

$$z_{\tilde{W}}(\psi) := \frac{\Lambda_{\tilde{W}}(\tilde{G}(\psi))}{g(\tilde{G}(\psi))}. \tag{29}$$

From (27) we derive that $z_{\tilde{W}}(\psi) \sim \hat{C}_{n,\alpha}\psi^{n-1}(1-\tilde{G}(\psi))^{-1/2}$ for $\psi \to 0+$ in case of $F \in \mathcal{F}_{\mathrm{reg},\Omega_n}$. Furthermore, using (25) it is elementary to show that $z_{\tilde{W}}(\psi)$ is regularly varying near zero if $F \in \mathcal{F}_{\mathrm{reg},\Omega_n}$, cf. Feller [2]. The integral in (28) can be asymptotically treated like an integral of Laplacian type. As a direct consequence of lemma 3 we obtain an explicit asymptotic equivalent for $\mathrm{E}_\gamma(s_{u,v})$, if we apply a generalization of Watson's lemma suitable for regularly varying functions, cf. Feller [2]:

Theorem 3 *For any RDF $F \in \mathcal{F}_{\mathrm{reg},\Omega_n}$, $\gamma \in (0,\pi)$ and $n \geq 2$ there is a positive constant $K_{n,\alpha}$ such that:*

$$\mathrm{E}_\gamma(s_{u,v}) \sim \gamma\, K_{n,\alpha}(1-\tilde{G}(\tfrac{1}{m}))^{-1/2},\ m \to \infty. \tag{30}$$

$\tilde{G}$ is the inverse function of G defined in (8) and $K_{n,\alpha}$ is a positive constant.

In the more special situation, when the radial distribution function F belongs to $\mathcal{F}_{\mathrm{alg},\Omega_n}$, the right hand side of (30) can be simplified:

Remark : For any RDF $F \in \mathcal{F}_{\text{alg},\Omega_n}$, $\gamma \in (0, \pi)$ and $n \geq 2$ there is a positive constant $\tilde{K}_{n,\alpha}$ such that:

$$\text{(31)} \qquad \mathrm{E}_\gamma(s_{u,v}) \sim \gamma\, \tilde{K}_{n,\alpha}\, m^{1/(n-1+2\alpha)},\ m \to \infty.$$

We exploit (30) in order to derive an asymptotic result on $\mathrm{E}(s_{u,v})$, where we average on the choice of u and v as well. For any two independently and uniformly on the unit sphere in $\mathbb{R}^n$ distributed random vectors the density $\hat{p}(\gamma)$ of the distribution of the enclosed angle γ is given by

$$\text{(32)} \qquad \hat{p}(\gamma) = \frac{|\omega_{n-1}|}{|\omega_n|} \sin^{n-2}\gamma,\ n \geq 2,$$

which is shown using elementary stochastic geometry. A proof of (32) can be found in Borgwardt [1]. An easy calculation delivers $\mathrm{E}(\frac{\gamma}{2\pi}) = 1/4$, which yields as a consequence of theorem 3:

Corollary *For $n \geq 2$ and $F \in \mathcal{F}_{\text{reg},\Omega_n}$:*

$$\text{(33)} \qquad \mathrm{E}(s_{u,v}) \sim \pi/2K_{n,\alpha}(1 - \tilde{G}(\tfrac{1}{m}))^{-1/2} \sim \frac{\pi}{2\gamma}\mathrm{E}_\gamma(s_{u,v}),\ m \to \infty$$

with $K_{n,\alpha}$ as in (30).

Theorem 3 and its corollary show that for all RDFs $F \in \mathcal{F}_{\text{reg},\Omega_n}$ the expectations $\mathrm{E}_\gamma(s_{u,v})$ and $\mathrm{E}(s_{u,v})$ respectively tend to infinity as m tends to infinity. This is true for all distributions with compact support, cf. Borgwardt [1], but wrong in the generality of all rotationally invariant distributions. Theorem 3 generalizes and sharpens results of Borgwardt, who achieved asymptotic lower and upper bounds in cases of uniform distribution in the unit ball and on the unit sphere respectively. For these special cases the constants can be calculated exactly, cf. Küfer [3]. In contrast to our approach Borgwardt did not analyze the functional $s_{u,v}$ itself. He related the number of pivots required by the shadow vertex to a purely polyhedral quantity, the number of shadow vertices of the polyhedron X_A in the (u, v)-plane, whose expectation is just four times as big as $s_{u,v}$'s in case of rotationally invariant distributions. For an analysis of the variance this approach fails, because the variance of $s_{u,v}$ and the variance of the number of shadow vertices is not simply related. In contrast to $s_{u,v}$'s variance the variance of the number of shadow vertices divided by the squared expectation tends to zero for fixed n and $m \to \infty$ for all RDFs $F \in \mathcal{F}_{\Omega_n}$ with the same asymptotic order as the quotient in (9).

Having studied the expectation value of $s_{u,v}$ so far, we are now going to analyse the second moment $\mathrm{E}_\gamma(s_{u,v}^2)$ of the random variable $s_{u,v}$. Making use of the identical, independent distribution of random vectors a_i we receive

$$\text{(34)} \qquad \mathrm{E}_\gamma(s_{u,v}^2) = \sum_{k=0}^{n} e_\gamma(k)$$

where for any fixed sets of indices I and J, $|I| = |J| = n$, $|I \cap J| = n - k$,

$$\text{(35)} \qquad e_\gamma(k) := q_k \mathrm{E}_\gamma(\chi_{u,v}(A_I)\tilde{\chi}_{u,v}(A_J)\chi_1(A, A_I)\chi_1(A, A_J)).$$

The quantity $e_\gamma(k)$ is the contribution of the pairs (A_I, A_J) consisting of susbsets A_I and A_J, which have exactly $n-k$ elements in common, to the expectation $E_\gamma(s^2_{u,v})$. The number of those pairs is abbreviated by q_k. Elementary combinatorics delivers

$$q_k := \binom{m}{n}\binom{m-n}{k}\binom{n}{k}. \tag{36}$$

It is our next objective to derive an integral representation of $e_\gamma(k)$, which allows an asymptotic analysis similar to (19)'s. If we slightly generalize the definition of the indicator functional χ_1, cf. (15), by

$$\chi_1(B,A) := \prod_{a\in B\setminus A} \chi(a \in H^{(1)}(A)) \tag{37}$$

for all A-subsets B and A with $|A| = n$ we have
(38)

$$\chi_1(A, A_I)\chi_2(A, A_J) = \prod_{a\notin A_{I\cup J}} \chi(a \in H^{(1)}(A_I)\cap H^{(1)}(A_J))\chi_1(A_I, A_J)\chi_1(A_J, A_I).$$

The probability $\Pr(a \in H^{(1)}(A_I) \cap H^{(1)}(A_J))$ depends only on the distance $h(A_I)$ of the hyperplane $H(A_I)$ from the origin, on the distance $h(A_J)$ of the hyperplane $H(A_J)$ from the origin and on the angle $\angle(A_I, A_J)$, which the normal vectors of both hyperplanes enclose, cf. figure 2. Thus, we define $G_{1,1}(h_1, h_2, \varphi)$:

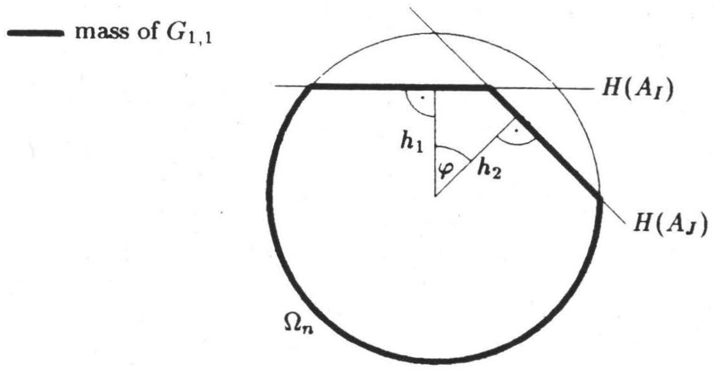


Fig. 2. Illustration of $G_{1,1}(h_1, h_2, \varphi)$, if $h_1 = h(A_I)$, $h_2 = h(A_J)$, $\varphi = \angle(A_I, A_J)$

$$G_{1,1}(h_1, h_2, \varphi) := \Pr(a \in H^{(1)}(A_I) \cap H^{(1)}(A_J) \mid h_1, h_2, \varphi), \tag{39}$$

where the condition (h_1, h_2, φ) in (39) abbreviates joint conditions $h(A_I) = h_1$, $h(A_J) = h_2$, $\angle(A_I, A_J) = \varphi$ on A_I and A_J. We remark that $G_{1,1}$ obviously satisfies

$$1 - G(h_1) - G(h_2) \le G_{1,1}(h_1, h_2, \varphi) \le 1 - G(\min(h_1, h_2)), \tag{40}$$

cf. figure 2, which we will need later. If we establish $h(A_I)$, $h(A_J)$ and $\angle(A_I, A_J)$ as independent variables, by the aid of Fubini's theorem we obtain integral representations for $e_{\bar{W}}(k)$, which correspond to formula (19):

(41)
$$e_\gamma(k) = \begin{cases} q_k \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\pi G_{1,1}^{m-n-k}(h_1, h_2, \varphi) A_{k,\gamma}(h_1, h_2, \varphi) d\varphi dh_1 dh_2, & 1 \leq k \leq n \\ q_0 \int\limits_0^\infty (1 - G(h))^{m-n} A_\gamma(h) dh = \mathrm{E}_\gamma(s_{u,v}), & k = 0 \end{cases};$$

The functions $A_{k,\gamma}$, $k = 1, \ldots, n$, are interpretable as conditioned expectations:

$$(42) \quad \frac{A_{k,\gamma}(h_1, h_2, \varphi)}{p_k(h_1, h_2, \varphi)} = \mathrm{E}_\gamma(\chi_{u,v}(A_I)\chi_{u,v}(A_J)\chi_1(A_I, A_J)\chi_1(A_J, A_I) | h_1, h_2, \varphi),$$

$p_k(h_1, h_2, \varphi)$ being the density function of the joint distribution function

$$(43) \qquad P_k(h_1, h_2, \varphi) := \Pr(h(A_I) \leq h_1, h(A_J) \leq h_2, \angle(A_I, A_J) \leq \varphi)$$

of the quantities $h(A_I)$, $h(A_J)$ and $\angle(A_I, A_J)$ under the assumption that I and J have exactly $n - k$ elements in common. For the functions $G_{1,1}$, $A_{k,\gamma}$ and p_k explicit integral representations are given in Küfer [3]. Looking at (36) we observe that for large m and fixed n the number of pairs of index sets (A_I, A_J), which consist of non-disjoint sets A_I and A_J, is very small relative to the total number of pairs. Thus, it seems to be plausible that for large m the expectation $\mathrm{E}_\gamma(s_{u,v}^2)$ is dominated by $e_\gamma(n)$. In case of distributions with compact domain even more is true. We will show that for distributions supported by the unit ball only the contribution of those pairs of index sets (A_I, A_J) is relevant for the asymptotic behaviour of $\mathrm{E}_\gamma(s_{u,v}^2)$, where the corresponding hyperplanes $H(A_I)$ and $H(A_J)$ intersect outside the unit ball. Of course, sets A_I and A_J in those pairs are disjoint, otherwise at least the common points would lie in Ω_n. Let

$$(44) \qquad r(A_I, A_J) := \min\{\|a\|_2 \mid a \in H(A_I) \cap H(A_J)\}$$

be the distance $H(A_I)$ and $H(A_J)$'s intersection from the origin and define for disjoint sets A_I and A_J

$$(45) \quad \bar{e}_\gamma := q_n \mathrm{E}_\gamma(\chi_{u,v}(A_I)\chi_{u,v}(A_J)\chi_1(A, A_I)\chi_1(A, A_J)\chi(r(A_I, A_J) > 1)).$$

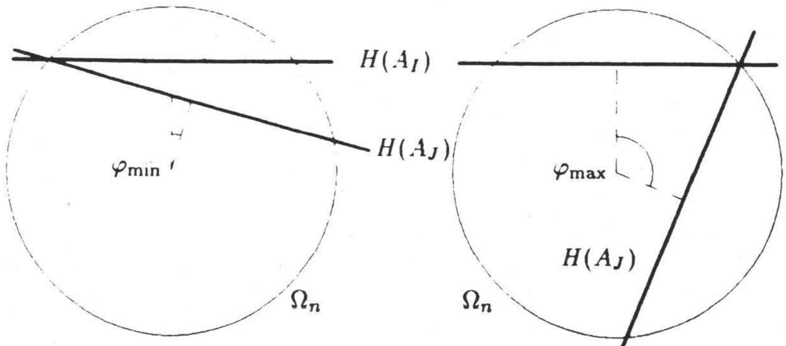
Then, $\bar{e}_\gamma$ represents the contribution of pairs of index sets, whose corresponding hyperplanes intersect outside the ball.

Lemma 4 *For any RDF $F \in \mathcal{F}_{\mathrm{reg},\Omega_n}$, $\gamma \in (0, \pi)$, $n \geq 2$ and m big enough, there is a constant $c_{n,\alpha}$ such that*

$$(46) \qquad \frac{\bar{e}_\gamma}{\mathrm{E}_\gamma(s_{u,v})} - 1 \leq \frac{c_{n,\alpha}}{m}.$$

Proof: Given $h_1 = h(A_I)$ and $h_2 = h(A_J)$ it is geometrically obvious, cf. figure 3, that $r(A_I, A_J) \leq 1$ is equivalent to

$$(47) \quad \angle(A_I, A_J) \in V(h_1, h_2) := [|\arccos h_1 - \arccos h_2|, \arccos h_1 - \arccos h_2].$$



$\varphi_{\min} = |\arccos h(A_I) - \arccos h(A_J)|$ $\quad \varphi_{\max} = \arccos h(A_I) + \arccos h(A_J)$

Fig. 3. Min. and max. possible angle, if $h(A_I) = h_1$, $h(A_J) = h_2$ and $r(A_I, a_J) \leq 1$

As $G_{1,1,}(h_1, h_2, \varphi) = 1 - G(h_1) - G(h_2)$ for $\varphi \in V^c(h_1, h_2) := [0, \pi] \setminus V(h_1, h_2)$, we receive

(48) $$\bar{e}_\gamma = q_n \int_0^1 \int_0^1 (1 - G(h_1) - G(h_2))^{m-2n} S_\gamma(h_1, h_2) dh_1 dh_2,$$

$$S_\gamma(h_1, h_2) := \int_{V^c(h_1, h_2)} \Lambda_{n,\gamma}(h_1, h_2, \varphi) d\varphi.$$

As $V(h_1, h_2) \subset [0, \pi]$ we estimate S_γ from above by

(49) $$S_\gamma(h_1, h_2) \leq \int_0^\pi \Lambda_{n,\gamma}(h_1, h_2, \varphi) d\varphi.$$

If A_I and A_J are disjoint sets, $h(A_I)$, $h(A_J)$ and $\angle(A_I, A_J)$ are independent random variables. Hence, the joint density p_n, cf. (42-43), can be factorized by

(50) $$p_n(h_1, h_2, \varphi) = \hat{p}(\varphi) p(h_1) p(h_2),$$

where p is defined by (20) and $\hat{p}$ by (32). Thus, by the aid of (42) we receive by integration over φ

(51) $$\int_0^\pi \Lambda_{n,\gamma}(h_1, h_2, \varphi) d\varphi = \mathrm{E}_\gamma(\chi_{u,v}(A_I)\chi_{u,v}(A_J) \,|\, h_1, h_2) p(h_1) p(h_2).$$

The conditioned expectation in (51) can be factorized exploiting independence of A_I and A_J by

(52) $$\mathrm{E}_\gamma(\chi_{u,v}(A_I)\chi_{u,v}(A_J) \,|\, h_1, h_2) = \mathrm{E}_\gamma(\chi_{u,v}(A_I) \,|\, h_1) \mathrm{E}_\gamma(\chi_{u,v}(A_J) \,|\, h_2).$$

Using (20) we obtain for S_γ

(53) $$S_\gamma(h_1,h_2) \le \Lambda_\gamma(h_1)\Lambda_\gamma(h_2).$$

By use of $q_n \le q_0^2$, line (24) and by substitution of $G(h_i) = \psi_i$, $i = 1,2$, in (48) we get the bilinear form

(54) $$\frac{\overline{e}_\gamma}{\mathrm{E}_\gamma^2(s_{u,v})} - 1 \le \int_0^{1/2}\int_0^{1/2} \mathcal{K}(\psi_1,\psi_2)\tilde{z}_{\bar{W}}(\psi_1)\tilde{z}_{\bar{W}}(\psi_2)d\psi_1 d\psi_2,$$

where the kernel $\mathcal{K}$ and the argument functions $\tilde{z}_{\bar{W}}$ are defined by

(55) $$\tilde{K}(\psi_1,\psi_2) := \frac{(1-\psi_1-\psi_2)^{m-2n}}{(1-\psi_1)^{m-n}(1-\psi_2)^{m-n}} - 1,$$
$$\tilde{z}_{\bar{W}}(\psi) := \frac{(1-\psi)^{m-n} z_{\bar{W}}(\psi)}{\int_0^{1/2}(1-\psi')^{m-n} z_{\bar{W}}(\psi')d\psi'}.$$

The function $z_{\bar{W}}$ has been defined in (29). A careful discussion of the bilinear form (54) yields the desired estimate provided in line (46). As the analysis of (54) is of more technical character it is delayed to the appendix.

We continue the proof of theorem 1 by showing that

(56) $$\mathrm{E}_\gamma(s_{u,v}^2) - \overline{e}_\gamma = \Theta((1-\tilde{G}(\tfrac{1}{m}))^{-1/2}) = \Theta(\mathrm{E}_\gamma(s_{u,v})),\ m\to\infty.$$

In particular, (56) implies that $\overline{e}_\gamma$ is asymptotically equivalent with the second moment $\mathrm{E}_\gamma(s_{u,v}^2)$. Hence, the second moment of $s_{u,v}$ is asymptotically equivalent with the square of the first moment. But, in case of the variance the asymptotic behaviour for $m\to\infty$ of $\overline{e}_\gamma$'s complement dominates the asymptotic behaviour.

Before we are going to estimate $e_\gamma(k)$, $k = 1,\dots,n-1$, we drop the cumbersome indicator functionals in $\Lambda_{k,\bar{W}}$, cf. (42), and define

(57) $$\tilde{\Lambda}_{k,\gamma}(h_1,h_2,\varphi) := \mathrm{E}_\gamma(\chi_{u,v}(A_I)\chi_{u,v}(A_J)\,|\,h_1,h_2,\varphi)p_k(h_1,h_2,\varphi).$$

We replace $\Lambda_{k,\gamma}$ in (41) by $\tilde{\Lambda}_{k,\gamma}$ and introduce

(58)
$$\tilde{e}_\gamma(k) := q_k \int_0^\infty\int_0^\infty\int_0^\pi G_{1,1}^{m-n-k}(h_1,h_2,\varphi)\tilde{\Lambda}_{k,\gamma}(h_1,h_2,\varphi)d\varphi dh_1 dh_2,\ 1\le k\le n-1,$$

which is an upper bound for $e_\gamma(k)$, $1\le k\le n-1$. Finally, we use $\tilde{e}_\gamma(n) := e_\gamma(n) - \overline{e}_\gamma$ as an abbreviation for the complement of $\overline{e}_\gamma$. $\tilde{e}_\gamma$ has the integral representation

(59) $$\tilde{e}_\gamma(n) := q_n \int_0^\infty\int_0^\infty\int_0^\pi G_{1,1}^{m-2n}(h_1,h_2,\varphi)\Lambda_{n,\gamma}(h_1,h_2,\varphi)\chi(r\le 1)d\varphi dh_1 dh_2,$$

where $r = r(A_I,A_J)|_{h_1,h_2,\varphi}$.

Lemma 5 *For any RDF $F \in \mathcal{F}_{\mathrm{reg},\Omega_n}$, $\gamma \in (0,\pi)$, $n \geq 2$ and $1 \leq k \leq n$:*

$$\tilde{e}_\gamma(k) = \mathcal{O}(\mathrm{E}_\gamma(s_{u,v})),\ m \to \infty. \tag{60}$$

Proof : The corresponding hyperplanes $H(A_I)$ and $H(A_J)$ for all pairs of sets A_I and A_J under consideration in $\tilde{e}_\gamma(k)$ intersect inside the unit ball. For that reason the angle $\angle(A_I, A_J)$, which the normal vectors of the hyperplanes enclose, must lie in the restricted interval $V(h_1, h_2)$, cf. (47) and figures 3 and 4. Outside the interval $V(h_1, h_2)$ the densities $p_k(\cdot,\cdot,\varphi)$ vanish for $k = 1, \ldots, n-1$. The same is trivially true for the indicator functional $\chi(r(A_I, A_J) \leq 1)$ in the integral representation (59) of $\tilde{e}_\gamma(n)$. If we estimate $G_{1,1}$ by the upper bound provided in (40), we receive for $k = 1, \ldots, n$

$$\tilde{e}_\gamma(k) \leq 2q_k \int_0^1 (1 - G(h_1))^{m-n-k} R_{k,\gamma}(h_1) dh_1 \tag{61}$$

where the function $R_{k,\gamma}$ is defined by

$$R_{k,\gamma}(h_1) := \int_{h_1}^1 \int_{V(h_1,h_2)} \tilde{\Lambda}_{k,\gamma}(h_1, h_2, \varphi) d\varphi dh_2. \tag{62}$$

In order to estimate $R_{k,\gamma}(h_1)$ asymptotically for $h_1 \to 1-$, we are going to estimate $\tilde{\Lambda}_{k,\gamma}$, cf. (57), from above. By Cauchy's inequality

$$\mathrm{E}_\gamma(\chi_{u,v}(A_I)\chi_{u,v}(A_J) \,|\, \cdot) \leq \mathrm{E}_\gamma^{1/2}(\chi_{u,v}(A_I) \,|\, \cdot)\mathrm{E}_\gamma^{1/2}(\chi_{u,v}(A_J) \,|\, \cdot). \tag{63}$$

Using the same arguments as in the proof of (21) and (22) we receive

$$\mathrm{E}_\gamma(\chi_{u,v}(A_K) \,|\, h_1, h_2, \varphi) = \frac{\gamma}{2\pi}\mathrm{E}(\tilde{W}(A_K) \,|\, h_1, h_2, \varphi) \tag{64}$$

for $K \in \{I, J\}$. We estimate $\tilde{W}(A_K)$ from above. If $A_K \subset \Omega_n$, there is an upper bound for $\tilde{W}(A_K)$, which we will prove in the appendix. It holds,

$$\tilde{W}(A_K) \leq \frac{n\sigma_{n-2}}{h^{n-1}(A_K)|\omega_{n-1}|}(1 - h^2(A_K))^{(n-2)/2}, \tag{65}$$

where σ_{n-2} is the maximal possible Lebesgue-volume of a simplex in Ω_{n-2}. Thus, (65) yields an upper bound for the expectations $\mathrm{E}(\tilde{W}(A_K) \,|\, h_1, h_2, \varphi)$ and therefore for $\tilde{\Lambda}_{k,\gamma}$ and for $R_{k,\gamma}$ as well. If we replace the right hand side of (63) by the upper bound resulting from (64-65), an upper bound for $\tilde{\Lambda}_{k,\gamma}$ is at hand, cf. (57). We obtain from (62)

$$R_{k,\gamma}(h_1) = \gamma\mathcal{O}((1-h_1)^{(n-2)/2} \int_{h_1}^1 \int_{V(h_1,h_2)} p_k(h_1, h_2, \varphi) d\varphi dh_2),\ h_1 \to 1-. \tag{66}$$

The remaining double integral in (66) can be interpreted stochastically. If A_J and A_I have $n-k$ vectors in common, the integral is equal to

(67) $\Pr(h(A_J) \geq h(A_I), \angle(A_I, A_J) \in V(h(A_I), h(A_J)) \,|\, h(A_I) = h_1)p(h_1)$.

Now, we estimate the conditioned probability in (67) making use of geometrical arguments. If $h(A_J)$ is bigger than the given distance $h(A_I) = h_1$, each of the k vectors in $A_J \setminus A_I$ must lie outside a ball of radius h_1, which occurs with probability $\overline{F}(h_1)$. On the other hand, the angle, which a vector in $A_J \setminus A_I$ and the normal vector $w(A_I)$ of the hyperplane $H(A_I)$ enclose, has to be in the interval

(68) $$\overline{V}(h_1) := [0, 3\arccos h_1] \cap [0, \pi],$$

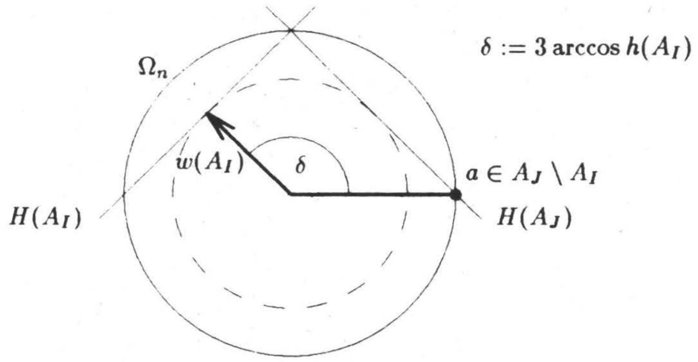


Fig. 4. Maximal possible angle between $w(A_I)$ and a vector $a \in A_J \setminus A_I$

as $H(A_I)$ and $H(A_J)$ intersect inside the unit ball, cf. figure 4. Hence, as any vector a in $A_J \setminus A_I$ and $w(A_I)$ are independent random variables, we receive:

(69) $$\Pr(\angle(a, w(A_I)) \in \overline{V}(h_1)) = \int_{\overline{V}(h_1)} \hat{p}(\varphi)d\varphi = \mathcal{O}((1-h_1)^{(n-1)/2}),\ h_1 \to 1-.$$

Thus, making use of (25) we get for all RDF $F \in \mathcal{F}_{\mathrm{reg},\Omega_n}$ the asymptotic estimation

(70)
$$\Pr(h(A_J) \geq h(A_I), \angle(A_I, A_J) \in V(h(A_I), h(A_J)) \,|\, h(A_I) = h_1) = \mathcal{O}(G^k(h_1))$$

for $h_1 \to 1-$. If we finally invoke $p(h_1)$'s asymptotic equivalent (26) and make use of the asymptotic representation (25) for G, an asymptotic upper bound for $R_{k,\gamma}$ is at hand:

(71) $$R_{k,\gamma}(h_1) = \gamma\, \mathcal{O}(G^{n+k-1}(h_1)g_0(h_1)(1-h_1)^{-1/2}),\ h_1 \to 1-.$$

As $q_k = \mathcal{O}m^{n+k}$, $m \to \infty$, the substitution $G(h_1) = \psi$ in line (61) leads to an asymptotic estimation for $\tilde{e}_\gamma(k)$ for $k = 1, \ldots, n$:

$$(72)\quad \tilde{e}_\gamma(k) = \gamma\, \mathcal{O}(m^{n+k} \int_0^1 (1-\psi)^{m-n-k}\psi^{n+k-1}(1-\tilde{G}(\psi))^{-1/2}d\psi),\ m \to \infty.$$

Like in the proof of theorem 3 an application of a generalization of Watson's lemma, cf. Feller [2], yields the claim of lemma 5.

As $e_\gamma(0) = \mathrm{E}_\gamma(s_{u,v})$ for all rotationally distributions the proof of theorem 1 is complete. To the end we derive claim (10) of the corollary of theorem 1 from the proven lemmata. Obviously holds

$$(73)\qquad \frac{\mathrm{Var}(s_{u,v})}{\mathrm{E}^2(s_{u,v})} = \int_0^\pi \left(\frac{\mathrm{E}_\gamma(s_{u,v}^2)}{\mathrm{E}^2(s_{u,v})} - 1\right)\hat{p}(\gamma)d\gamma.$$

By the aid of lemmata 4 and 5, $\mathrm{E}_\gamma(s_{u,v}^2) = \mathrm{E}_\gamma^2(s_{u,v}) + \mathcal{O}(\mathrm{E}_\gamma(s_{u,v}))$ for $m \to \infty$. Hence, using $\mathrm{E}_\gamma(s_{u,v}) = \frac{2\gamma}{\pi}\mathrm{E}(s_{u,v})$ we get

$$(74)\qquad \frac{\mathrm{Var}(s_{u,v})}{\mathrm{E}^2(s_{u,v})} = C_n + \mathcal{O}(\mathrm{E}^{-1}(s_{u,v})),\ m \to \infty,$$

with $C_n := \int_0^\pi [(\frac{2\gamma}{\pi})^2 - 1]\hat{p}(\gamma)d\gamma = \frac{\mathrm{Var}(\gamma)}{\mathrm{E}^2(\gamma)}$. The proof of the the inclusion (11) for the constants C_n is easy analysis and is left to the reader.

We conclude the section with the remark that the proof we have given for theorem 1 is not transferable to the more general situation we mentioned in the remark. Mainly, the estimations we have used in the proof of lemma 5 are too rough. The general claim is proven by a very careful analysis of the explicit integral representations of the functions $\Lambda_{k,\bar{W}}$, we omitted here.

4. Appendix—proofs of auxiliary lemmata

4.1 Proof of lemma 2, (27):

We prove (27) for $n \geq 4$, the cases $n = 2$ and $n = 3$ being easier special cases. Due to Borgwardt [1], for a distribution with density function f and domain Ω_n, $\Lambda_{\bar{W}}$ has an integral representation of the form

$$(75)\quad \Lambda_{\bar{W}}(h) = n|\omega_n||\omega_{n-1}| \int_0^{\sqrt{1-h^2}} \lambda_W(\rho,h) \int_{-\sqrt{1-h^2}}^{\sqrt{1-h^2}} |\rho-\sigma|\tilde{g}_2((\sigma^2+h^2)^{1/2})d\sigma d\rho,$$

where

$$(76)\quad \lambda_W(\rho,h) := \int\limits_{\sqrt{1-\rho^2-h^2}\,\Omega_{n-2}}^{(n-1)} \det{}^2(C)W(\tilde{c}_1,\ldots,\tilde{c}_{n-1})\prod_{i=1}^{n-1} f(\tilde{c}_i)^T)dc_i$$

with $C := \begin{pmatrix} c_1 & \dots & c_{n-1} \\ 1 & \dots & 1 \end{pmatrix}$, $W(\tilde{c}_1, \dots \tilde{c}_{n-1}) := \frac{|\omega_{n-1} \cap \text{cone}(\tilde{c}_1, \dots, \tilde{c}_{n-1})|}{|\omega_{n-1}|}$, cf. (94), and $\tilde{c}_\ell = (c_\ell, \rho, h)^T$. $|\cdot|$ denotes Lebesgue-volume of appropriate dimension. The connection between the density function f and the RDF F is the following: If $\hat{f}(\rho) := f(a)$ for $\|a\|_2 = \rho$, then $dF(\rho) = |\omega_n|\rho^{n-1}\hat{f}(\rho)d\rho$. We formulate the integral representation (75) for RDFs with density function for ease of notation. The restriction to RDFs with density functions is done without loss of generality, as this subclass is pointwise dense in the class of arbitrary radial distribution functions. Furthermore, $\tilde{g}_2$ in (75) is a special case of

$$\tilde{g}_i(\sigma) := \frac{|\omega_{n-i}|}{|\omega_n|} \int_\sigma^1 \frac{(\rho^2 - \sigma^2)^{(n-2-i)/2}}{\rho^{n-2}} dF(\rho), \tag{77}$$

which we define for $i \leq n-1$. For $h \to 1$ and $0 \leq \rho \leq (1-h^2)^{1/2}$ holds, cf. Borgwardt [1]:

$$W(\tilde{c}_1, \dots, \tilde{c}_{n-1}) \sim \frac{|\det(\mathrm{C})|}{|\omega_{n-1}|(n-2)!}. \tag{78}$$

Inserting the asymptotic relation (78) into (76) we receive for $h \to 1-$ and $0 \leq \rho \leq (1-h^2)^{1/2}$

$$\lambda_W(\rho, h) \sim \frac{1}{|\omega_{n-1}|(n-2)!} \int_{\sqrt{1-\rho^2-h^2}\,\Omega_{n-2}}^{(n-1)} |\det{}^3(C)| \prod_{i=1}^{n-1} f(\tilde{c}_i)^T) dc_i. \tag{78}$$

As $\lambda_W(\rho, h)$ depends on $(\rho^2 + h^2)^{1/2}$ only, we define for $x \in [0,1]$:

(80)

$$\overline{\lambda}_W(x) := \frac{1}{|\omega_{n-1}|(n-2)!} \int_{\sqrt{1-x^2}\,\Omega_{n-2}}^{(n-1)} |\det{}^3(C)| \prod_{i=1}^{n-1} \hat{f}(\sqrt{\|c_i\|^2 + x^2}) dc_i.$$

Simultaneous rotations of the vectors c_ℓ in $\mathbb{R}^{n-2}$ lead to

(81)

$$\overline{\lambda}_W(x) = \frac{|\omega_{n-3}|}{|\omega_{n-1}|(n-2)!} \int_0^{\sqrt{1-x^2}} \tilde{\lambda}_W(\sqrt{\rho^2+x^2}) \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} |\rho - \sigma|^3 \tilde{g}_3(\sqrt{\sigma^2+x^2}) d\sigma d\rho$$

with $\tilde{g}_3$ being defined by (77) and

$$\tilde{\lambda}_W(y) := \int_{\sqrt{1-y^2}\,\Omega_{n-3}}^{(n-2)} \det{}^4(\overline{C}) \prod_{i=1}^{n-2} \hat{f}(\sqrt{\|\overline{c}_i\|_2^2 + y^2}) d\overline{c}_i, \tag{82}$$

where $\overline{C} := \begin{pmatrix} \overline{c}_1 & \dots & \overline{c}_{n-2} \\ 1 & \dots & 1 \end{pmatrix}$. We rescribe formula (82) in the general setting of RDFs F, which have not necessarily a density function:

$$(83)\quad \tilde\lambda_W(y) = \int\limits_y^1 {}^{(n-2)} \tilde D(\mu_1,\dots,\mu_{n-2},y) \prod_{i=1}^{n-2} \frac{|\omega_{n-3}|}{|\omega_n|} \frac{(\mu_i^2-y^2)^{(n-5)/2}}{\mu_i^{n-2}} dF(\mu_i),$$

where $\tilde D(\mu_1,\dots,\mu_{n-2},y)$ equals the integral

$$(84)\quad \int\limits_{\omega_{n-3}}^{(n-2)} \det{}^4 \begin{pmatrix} \sqrt{\mu_1^2-y^2}\bar c_1 & \dots & \sqrt{\mu_{n-2}^2-y^2}\bar c_{n-2} \\ 1 & \dots & 1 \end{pmatrix} \prod_{i=1}^{n-2} d^0_{\omega_{n-3}}(\bar c_i).$$

The differential $d^0_{\omega_{n-3}}(\bar c_i)$ in (84) is the normed spherical differential in direction of $\bar c_i$, i.e. $\int_{\omega_{n-3}} d^0_{\omega_{n-3}}(\bar c_i) = 1$. Now, we evaluate the determinant function $\tilde D$ along its last row and receive

$$(85)\quad \tilde D(\mu_1,\dots,\mu_{n-2},y) = \sum_{j=0}^{[(n-2)!]^4} \alpha_j(n) \prod_{i=1}^{n-2} (\mu_i^2-y^2)^{p_{i,j}/2},$$

the $\alpha_j(n)$ being constants depending on n, which result from integration over the spheres ω_{n-3}. The numbers $p_{i,j}\in\{0,\dots,4\}$ fulfill the equation

$$(86)\quad \sum_{i=1}^{n-2} p_{i,j} = 4(n-3).$$

If we replace $\tilde D$ in (83) by (85), after some calculations we obtain with the aid of (86) and the asymptotic formula

(87)

$$\int\limits_h^1 (\rho^2-h^2)^\beta dF(\rho) \sim \overline{F}(h)(1-h^2)^\beta(\alpha+\beta+1)B(\alpha+1,\beta+1),\ h\to 1,\ \beta>-1,$$

an asymptotic equivalent for $\tilde\lambda_W$. It holds

$$(88)\quad \tilde\lambda_W(y) \sim C^{(1)}_{n,\alpha}\overline{F}^{n-2}(y)(1-y^2)^{(n^2-3n-2)/2},\ y\to 1-.$$

The constant $C^{(1)}_{n,\alpha}$ depends on n and α only. Inserting (88) into (81) and inserting the resulting asymptotic equivalent for (79) into (75) we obtain finally

$$(89)\quad \Lambda_{\tilde W}(h) \sim C^{(2)}_{n,\alpha}\overline{F}^n(h)(1-h^2)^{(n^2-n-3)/2},\ h\to 1-.$$

Claim (27) is now an easy consequence of (25). The only matter left in order to complete the proof is to show the positivity of the coefficients $C_{n,\alpha}$. This is done with the aid of Borgwardt's theorem 9 in [1], where asymptotic estimates of $E(s_{u,v})$ are given, which confirm the established asymptotic order of (89).

4.2 Estimation of the bilinear form (54) in lemma 4:

We dissect the domain of integration in line (54) into four parts $Q_1,\dots,Q_4$, cf. figure 5.2. $\mathcal{K}(\frac{4n}{m+2n},\frac{4n}{m+2n})\le 0$, as is shown by elemtary analysis. Thus, by the monotonicity of $\mathcal{K}$, cf. figure 5.1, $\mathcal{K}(\psi_1, \psi_2) \leq 0$ for $(\psi_1, \psi_2) \in Q_4$. In the region Q_2 we have

$$\mathcal{K}(\psi_1, \psi_2) \leq (1-\psi_1)^{-n} - 1 \leq n 2^{n+1} \psi_1. \tag{90}$$

Replacing ψ_1 by ψ_2 the same estimation holds in Q_3 for reasons of symmetry. As figure 5.1. shows, $\mathcal{K}$ has a local maximum in $(\frac{n}{m}, \frac{n}{m})$. Hence, in region Q_1

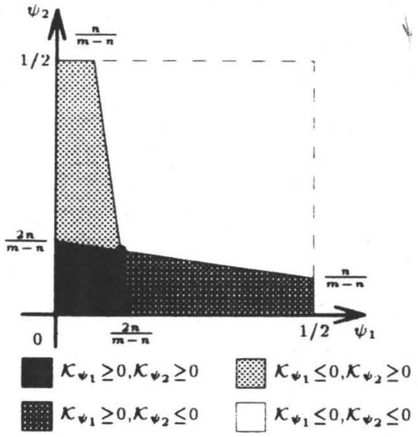


Fig. 5.1. Monotonicity of $\mathcal{K}$

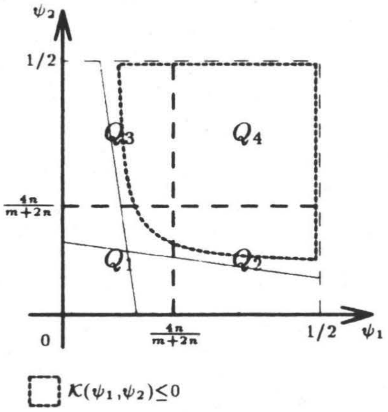


Fig. 5.2. Partition of the domain

$$\mathcal{K}(\psi_1, \psi_2) \leq \max(\mathcal{K}(0, \frac{4n}{m+2n}), \mathcal{K}(\frac{n}{m}, \frac{n}{m})) \leq \frac{n^2}{m+2n} 2^{n+3}. \tag{91}$$

Summarizing the discussion of the bilinear form we obtain from (54)

$$\frac{\overline{e}_\gamma}{\mathrm{E}_\gamma^2(s_{u,v})} - 1 \leq n 2^{n+2} \int_0^{1/2} \psi \tilde{z}_{\tilde{W}}(\psi) d\psi + \frac{n^2}{m+2n} 2^{n+3}. \tag{92}$$

If we use definitions (55) and (29) and replace the function $z_{\tilde{W}}$ by its asymptotic equivalent exploiting lemma 2, an application of a Watson-type lemma, cf. Feller [2], delivers

$$\int_0^{1/2} \psi \tilde{z}_{\tilde{W}}(\psi) d\psi \sim \frac{1+\alpha}{m}, \; m \to \infty, \tag{93}$$

and the claim of lemma 4 is proven.

4.3 Proof of lemma 5, line (65):

It is due to Borgwardt that with $I_k := I \setminus \{i_k\}$ and $S_{I_k} := \text{convhull}(A_{I_k})$,

$$\tilde{W}(A_I) = \sum_{k=1}^{n} W(A_{I_k}) \leq n \max_{1 \leq k \leq n} W(A_{I_k}) \tag{94}$$

$$W(A_{I_k}) := \Pr(\text{cone}(u, -v, v) \cap S_{I_k} \neq \emptyset) = \frac{|\omega_{n-1} \cap \text{cone}(S_{I_k})|}{|\omega_{n-1}|}$$

We estimate W from above. For any A_I-subset A_{I_k} with cardinality $n-1$, geometric insight delivers

$$W(A_{I_k}) \leq \frac{1}{|\omega_{n-1}| h^{n-1}(A_I)} |S_{I_k}|. \tag{95}$$

The set A_{I_k} lies in the intersection of the unit ball Ω_n and an affine subspace of dimension $n-2$ with distance $(h(A_I)^2 + \sigma^2)^{1/2}$, $0 \leq \sigma \leq \sqrt{1 - h^2(A_I)}$, from the origin. Therefore,

$$|S_{I_k}| \leq (1 - h^2(A_I) - \sigma^2)^{(n-2)/2} \sigma_{n-2} \leq (1 - h^2(A_I))^{(n-2)/2} \sigma_{n-2}, \tag{96}$$

σ_{n-2} being the maximal possible Lebesgue-volume of a simplex in Ω_{n-2}, which proves claim (65).

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