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FACHBEREICH MATHEMATIK

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Introduction

It is well–known (see [M1], [M2]) that there is a coarse moduli scheme $M(c_1, c_2, c_3)$ of semistable coherent sheaves of rank 2 on $\mathbb{P}_3$ with Chern classes c_1, c_2, c_3; although so far very few of these schemes have been studied in detail and not much is known about their structure in general. This work is devoted to the study of $M(0, 2, 4)$, which is among the first non–trivial cases with extremal third Chern class, but still allowing explicit considerations. The open subset M_r of $M(0, 2, 4)$ of μ–stable reflexive sheaves was described by Chang in [C] and Okonek in [O]. They proved that M_r is irreducible, rational and smooth of dimension 13. Moreover, they showed that any $\mathcal{F} \in M_r$ can be represented as a cokernel of a 2×4–matrix of linear forms, i.e. has a short resolution

$$0 \longrightarrow k^2 \otimes \mathcal{O}(-2) \longrightarrow k^4 \otimes \mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

It turned out that M_r is dense in the Maruyama scheme $M(0, 2, 4)$ and that this compactification is the G.I.T.–quotient of the space of all semistable 2×4–matrices of linear forms under the natural action of $GL(2) \times GL(4)$, see theorem I, 1.3.

Thus the study of $M(0, 2, 4)$ is the study of 2×4–matrices of linear forms on a 4–dimensional vector space. There are some remarkable and astonishing phenomena related to these matrices. In section 3 we succeeded in giving normal forms to these matrices in the different geometric situations related to their Fitting ideals of quadrics. This helps us to prove results on the subvarieties of the sheaves with specified geometrical data, see 3.10 and theorem II, 4.5.

Our purpose is to describe the most important subvarieties of $M(0, 2, 4)$ and their sheaves. This classification is still far from revealing the structure of $M(0, 2, 4)$ completely. In sections 1 and 2 we determine the subvarieties S_0, S_1, S_2 of non–reflexive sheaves of $M(0, 2, 4)$ and prove theorem I. We show that $M(0, 2, 4)$ is singular along the irreducible subvariety S_0 of properly semistable sheaves. The variety S_2 of sheaves, which are singular along a conic, corresponds to the subvariety D_4' of reflexive sheaves, which are centred pullbacks of bundles on a plane, under a remarkable duality between the moduli spaces $M(0, 2, 4)$ over $\mathbb{P}_3$ and $\mathbb{P}_3^*$ respectively, see proposition 2.6.

The reflexive sheaves $\mathcal{F}$ in $M(0, 2, 4)$ are singular at a 0–dimensional subscheme $Z(\mathcal{F}) \subset \mathbb{P}_3$ of length 4. They can be classified by the multiplicities of the points of $Z(\mathcal{F})$. This leads to the closures D_ν of the subvarieties of sheaves with a ν–fold point in $Z(\mathcal{F})$. These are studied in section 4, together with the rational morphism (of generic fibre dimension 1) from $M(0, 2, 4)$ to the Hilbert scheme $\text{Hilb}^4(\mathbb{P}_3)$ of four points, defined by $\mathcal{F} \mapsto Z(\mathcal{F})$. In proposition 3.5.4 we improve a result of Chang, [C], about the fibre over a scheme with four simple points. The comparison of $M(0, 2, 4)$ with $\text{Hilb}^4(\mathbb{P}_3)$ also involves normal forms for the ideals of the schemes in $\text{Hilb}^4(\mathbb{P}_3)$, which we did not find in the literature, see remarks 3.8.2 and 3.9.2. As a final result we show in proposition 4.7 that the subvarieties S_1 and S_2 of non–reflexive sheaves are both contained in D_4, whereas the singular locus S_0 is not contained in D_3, but is in the closure $D_{2,2}$ of the subvariety of sheaves with two double points.

Notation

- Throughout the paper k will be an algebraically closed field of characteristic zero.

- $G_m V$ denotes the Grassmannian of m-dimensional subspaces of a vector space V, $\mathbb{P}_n = \mathbb{P}V = G_1 V$ the projective space, dim $V = n + 1$.

- The invertible sheaf of degree d on $\mathbb{P}V$ is $\mathcal{O}(d)$, so that $V^* = H^0(\mathbb{P}V, \mathcal{O}(1))$.

- For an $\mathcal{O}_{\mathbf{P}V}$-module $\mathcal{F}$ we use the abbreviations $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}(d)$, $H^i\mathcal{F} = H^i(\mathbb{P}V, \mathcal{F})$, and $h^i\mathcal{F}(d) = \dim_k H^i\mathcal{F}(d)$.

- The sheaf of the trivial vector bundle with fibre E is denoted by $E \otimes \mathcal{O}$ and $E \otimes \mathcal{F}$ is written for $(E \otimes \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{F}$. Moreover, we write $m\mathcal{F} = k^m \otimes \mathcal{F}$ for any $\mathcal{O}_{\mathbf{P}V}$-module and an integer $m \geq 1$.

- The dimension of a coherent sheaf is the dimension of its support.

- The Chern classes $c_i(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_n$ are considered as integers and we call the n-tuple $(c_1(\mathcal{F}), \ldots, c_n(\mathcal{F}))$ the Chern classes of $\mathcal{F}$.

- (Semi-)stability is always meant in the sense of Gieseker-Maruyama and μ-(semi-)stability in the sense of Mumford-Takemoto, see [OSS].

- $M(c_1, \ldots, c_n)$ denotes the Maruyama scheme of semistable coherent sheaves on $\mathbb{P}_n$ of rank 2 and Chern classes $(c_1, \ldots, c_n)$.

1 Description of M(0, 2, 4)

1.1. Sheaves $\mathcal{F}(A)$

The dual of a linear map $k^4 \xrightarrow{A} k^2 \otimes V^*$ induces a sheaf homomorphism $k^2 \otimes \mathcal{O}(-2) \xrightarrow{A^*} k^4 \otimes \mathcal{O}(-1)$, whose cokernel is denoted by $\mathcal{F}(A)$. We shall always assume that A^* is injective and hence are given an exact sequence

$$0 \longrightarrow k^2 \otimes \mathcal{O}(-2) \xrightarrow{A^*} k^4 \otimes \mathcal{O}(-1) \longrightarrow \mathcal{F}(A) \longrightarrow 0.$$

Then $\mathcal{F}(A)$ has rank 2, Chern classes $(0, 2, 4)$ and normalized Hilbert polynomial

$$\frac{1}{2} \mathcal{X} \mathcal{F}(A)(m) = \frac{m^3}{6} + m^2 + \frac{5}{6}m.$$

Moreover, it is easily verified that the above presentation of $\mathcal{F}(A)$ is at the same time the Beilinson presentation, [OSS], theorem 3.1.3. It follows that $\mathcal{F}(A) \cong \mathcal{F}(B)$ if and only if $A \sim B$, i.e. $B = S \circ A \circ R$ for some matrix $S \in GL(2, k)$, $R \in GL(4, k)$.

It is also easy to see that $\mathcal{F}(A)$ has $\mathcal{O}(-1)$ as a quotient (and thus is not μ–semistable) if and only if A is not injective. Therefore, and in order to eliminate the action of $GL(4, k)$, we consider the subspaces

$$\Lambda = Im A \in G_4(k^2 \otimes V^*).$$

Clearly the isomorphism class of $\mathcal{F}(A)$ only depends on $Im\, A$. Given a subspace Λ we can define a sheaf $\mathcal{F}(\Lambda)$ by the exact sequence

$$0 \longrightarrow k^2 \otimes \mathcal{O}(-2) \longrightarrow \Lambda^* \otimes \mathcal{O}(-1) \longrightarrow \mathcal{F}(\Lambda) \longrightarrow 0,$$

where the left–hand side is induced from the surjection $k^2 \otimes V \to \Lambda^*$. If $\Lambda = Im\, A$ then $\mathcal{F}(\Lambda) \cong \mathcal{F}(A)$. If $X \subset G_4(k^2 \otimes V^*)$ is the open set of Λ for which $\mathcal{F}(\Lambda)$ has rank 2, and if Λ is the tautological subbundle on the Grassmannian, we obtain the universal family $\mathbf{F}$ over $X \times \mathbb{P}_3$ with the presentation

$$0 \longrightarrow k^2 \otimes \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbf{P}_3}(-2) \longrightarrow \Lambda^* \boxtimes \mathcal{O}_{\mathbf{P}_3}(-1) \longrightarrow \mathbf{F} \longrightarrow 0 \qquad\qquad (UF)$$

On $G_4(k^2 \otimes V^*)$ we are left with the natural action of $SL(2, k)$ by $\Lambda \mapsto (g \otimes id)(\Lambda)$. This can be linearized via the Plücker embedding. By [N-T], prop. 5.1.1, the stable and semistable points of this action in the sense of [M-F] can be characterized by

1.1.1 Lemma: *A point $\Lambda \in G_4(k^2 \otimes V^*)$ is stable (semistable) if and only if for any nonzero $\xi \in k^2$*

$$\dim \Lambda \cap (\xi \otimes V^*) \leq 1 \qquad (\leq 2).$$

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In concrete terms this means that Im A is stable (semistable) if and only if

$$A \nsim \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \qquad \left(A \nsim \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right).$$

In particular, Im A is semistable and not stable if and only if

$$A \sim \begin{pmatrix} x & y & * & * \\ 0 & 0 & x' & y' \end{pmatrix}$$

with x, y and x', y' pairwise independent as vectors in V^*.

1.2 Stability properties of the sheaves $\mathcal{F}(A)$

If *Im A* is not semistable (but with A^* injective) then A can be given the form

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & z \end{pmatrix}$$

with $z \neq 0$ and it follows that $\mathcal{F}(A)$ has z–torsion. Therefore, if $\mathcal{F}(A)$ is μ–semistable then *Im A* is already semistable. It follows from lemma 1.2.1 that $\mathcal{F}(A)$ is μ–semistable if and only if it is semistable.

1.2.1 Lemma: *For a torsionfree coherent rank-2 sheaf $\mathcal{F}$ on $\mathbb{P}_3$ with Chern classes $(0, 2, 4)$ the following are equivalent:*

(a) *$\mathcal{F}$ is properly semistable*

(b) *$\mathcal{F}$ is an extension $0 \to \mathcal{I} \to \mathcal{F} \to \mathcal{I}' \to 0$, where $\mathcal{I}$ and $\mathcal{I}'$ are ideal sheaves of lines in $\mathbb{P}_3$*

(c) *$\mathcal{F} \cong \mathcal{F}(A)$ for some $\Lambda = Im\, A \in G_4(k^2 \otimes V^*)$ which is properly semistable.*

<u>Proof</u>: If $A = \begin{pmatrix} x & y & 0 & 0 \\ * & * & x' & y' \end{pmatrix}$ is properly semistable, there is an exact diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-2) & \xrightarrow{(x\ y)} & 2\mathcal{O}(-1) & \longrightarrow & \mathcal{I}_\ell & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\mathcal{O}(-2) & \xrightarrow{A^*} & 4\mathcal{O}(-1) & \longrightarrow & \mathcal{F}(A) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-2) & \xrightarrow{(x'y')} & 2\mathcal{O}(-1) & \longrightarrow & \mathcal{I}_{\ell'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

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where ℓ resp. ℓ' are the lines with equations x, y resp. x', y'. Conversely any such extension has a resolution of the above type. This proves the equivalence between (b) and (c). Clearly (b) implies (a), because for any line ℓ we have $\mathcal{X} \mathcal{I}_\ell(m) = \frac{1}{2} \mathcal{X} \mathcal{F}(m)$, [M2], lemma 1.4. If conversely $\mathcal{F}$ is properly semistable, we can find a torsionfree quotient $\mathcal{F} \to \mathcal{F}'' \to 0$ with $rk\mathcal{F}'' = 1$, $c_1 \mathcal{F}'' = 0$, $\mathcal{X} \mathcal{F}''(m) = \frac{1}{2} \mathcal{X} \mathcal{F}(m)$. By the second property there is an embedding $\mathcal{F}'' \subset \mathcal{O}$, and by the third $\mathcal{C} = \mathcal{O}/\mathcal{F}''$ satisfies $\mathcal{X}\mathcal{C}(m) = m + 1$. Using the reducing method as in 1.7 it is easy to see that $\mathcal{C}$ is the structure sheaf of a line ℓ' and hence $\mathcal{F}'' \cong \mathcal{I}_{\ell'}$. Analogically $Ker(\mathcal{F} \to \mathcal{F}'') \cong \mathcal{I}_\ell$ for a second line ℓ. This proves (a) $\Rightarrow$ (b).

1.2.2 Lemma: *Let $k^4 \overset{A}{\hookrightarrow} k^2 \otimes V^*$ be given and A^* injective. Then*

1) $\mathcal{F}(A)$ is semistable if and only if $Im\ A$ is semistable.

2) $\mathcal{F}(A)$ is stable if and only if $Im\ A$ is stable.

<u>Proof:</u> If $\mathcal{F}(A)$ is semistable then by the remark preceding lemma 1.2 $Im\ A$ must be semistable. Conversely, let $Im\ A$ be semistable. Then it is easy to prove that $\mathcal{F}(A)$ is torsionfree. Assume that $\mathcal{F}(A)$ were not semistable. Then there is a torsionfree quotient $\mathcal{F}(A) \to \mathcal{F}'' \to 0$ of $rank\ 1$ with $\mathcal{X} \mathcal{F}''(m) < \frac{1}{2} \mathcal{X} \mathcal{F}(A)(m)$ for $m >> 0$. Let $c = c_1 \mathcal{F}''$, such that $\mathcal{F}'' \subset \mathcal{O}(c)$ with Supp $\mathcal{O}(c)/\mathcal{F}''$ at most 1-dimensional. By the presentation of $\mathcal{F}(A)$ we must have $c \geq -1$, and since A is injective, it follows that $c \geq 0$. If $c > 0$ the Riemann-Roch formula would imply $\mathcal{X} \mathcal{F}''(m) > \frac{1}{2} \mathcal{X} \mathcal{F}(A)(m)$. Hence $c = 0$. Then we are given an exact sequence

$$4\mathcal{O}(-1) \overset{b}{\longrightarrow} \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{F}'' \longrightarrow 0$$

with $b \circ A^* = 0$. Hence $\mathcal{O}/\mathcal{F}''$ is the structure sheaf of a linear subspace $Z \subset \mathbb{P}_3$ with $\dim Z \leq 1$. But $\mathcal{X}\mathcal{O}_Z(m) = \mathcal{X}\mathcal{O}(m) - \mathcal{X}\mathcal{F}''(m) > \mathcal{X}\mathcal{O}(m) - \frac{1}{2}\mathcal{X}\mathcal{F}(A)(m) = m + 1$. Therefore, Z cannot be a point or a line, contradicting $\dim Z \leq 1$. This proves (1). (2) follows from (1) and lemma 1.2.1.

1.3.   The G.I.T. description of $M(0, 2, 4)$

Let $G_4(k^2 \otimes V^*)^{ss}$ be the open subscheme of semistable points. By lemma 1.3 the restriction to this open subset of the universal family, which is also flat over $G_4(k^2 \otimes V^*)^{ss}$ gives us a modular morphism to $M(0, 2, 4)$, which factorizes through the good quotient:

$$G_4(k^2 \otimes V^{**})^{ss}//SL(2) \overset{\varphi}{\longrightarrow} M(0, 2, 4).$$

Theorem I: *φ is an isomorphism.*

In particular, $M(0, 2, 4)$ is irreducible, reduced, normal, 13-dimensional. By the result of Chang [C], theorem 2.12, the open part $M_r(0, 2, 4)$ of all reflexive sheaves

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in $M(0,2,4)$ (any reflexive sheaf in $M(0,2,4)$ is already μ–stable) is rational, hence $M(0,2,4)$ is rational. Before proving the theorem in 1.7 we consider the subvarieties S_0, S_1, S_2 of $M(0,2,4)$, all irreducible and of dimension 8.

1.4. The semistable locus S_0 in $M(0,2,4)$

By lemma 1.2.1 the properly semistable classes of $M(0,2,4)$ are given by two lines as $[\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}]$. If $G = G_2 V$ denotes the Grassmannian of lines in $\mathbb{P}_3$, we get a bijective morphism

$$Sym^2(G) \longrightarrow S_0 \subset M(0,2,4)$$

from the symmetric product of G onto the subvariety of properly semistable points of $M(0,2,4)$. By lemma 1.2.1 S_0 is in the image of φ, and clearly irreducible of dimension 8. It will be shown in 2.2 that S_0 is exactly the singular locus of $M(0,2,4)$.

1.5   The subvariety $S_1 \subset M(0,2,4)$

Let $\mathcal{E}$ be a null-correlation bundle (see [B], §7) and ℓ a jumping line of $\mathcal{E}$ such that $\mathcal{E} \otimes \mathcal{O}_\ell = \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$. Then there is a unique exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow 0.$$

Since $\mathcal{E}$ is μ–stable, $\mathcal{F}$ is μ–stable, too, and by this representation $\mathcal{F}$ has rank 2 and Chern classes $(0,2,4)$. We thus can construct an 8-dimensional irreducible family F_1 of such sheaves and an injective morphism $F_1 \to M(0,2,4)$. We denote by S_1 its closure.

S_1 is contained in the image of φ.

Proof: Instead of using Beilinson's spectral sequence, we give a direct construction. A null-correlation bundle $\mathcal{E}$ is the cohomology of a monad $\mathcal{O}(-1) \to 4\mathcal{O} \to \mathcal{O}(1)$, [OSS]. Since neither arrow is degenerate anywhere, we easily get a resolution (R) of $\mathcal{E}$.

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{O}(-3) & \xrightarrow{C} & 4\mathcal{O}(-2) & \xrightarrow{B} & 5\mathcal{O}(-1) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \quad (R) \\
& & \downarrow b_3 & & \downarrow b_2 & & \downarrow b_1 & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-3) & \longrightarrow & 2\mathcal{O}(-2) & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \mathcal{O}_\ell(-1) & \longrightarrow & 0 \quad (r)
\end{array}
$$

If there is a surjection $\mathcal{E} \to \mathcal{O}_\ell(-1)$, ℓ a line, there is a projection b_1 compatible with the surjection. Moreover, there exists then a homomorphism of the resolutions. The homomorphism b_2 must be surjective, for otherwise B could be given the form

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$$B = \left(\begin{array}{c|c} B' & \begin{array}{c} z \\ 0 \\ 0 \\ 0 \end{array} \end{array} \right)$$

with z vanishing on ℓ. This, however, would imply that $\mathcal{E}$ is not locally free along $\{z = 0, \det B' = 0\}$. Also b_3 cannot be 0, since C is not degenerate anywhere and hence is an isomorphism. The kernel of the homomorphism $(R) \to (r)$ is now an exact sequence

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{A^\bullet} 4\mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

1.5.1 Normal form: A null correlation bundle is defined by an indecomposable vector $\xi \in \Lambda^2 V$ via $0 \to \Omega^3(3) \xrightarrow{\xi} \Omega^1(1) \to \mathcal{E} \to 0$, and $\ell = <x \wedge y> \in G_2 V$ is a jumping line of $\mathcal{E}$ if $\xi \wedge x \wedge y = 0$. Given ξ and ℓ it is possible to find a basis $e_0, \ldots, e_3 \in V$ such that $\xi = e_0 \wedge e_3 + e_1 \wedge e_2$ and $\ell = <e_0 \wedge e_1>$. Then the kernel sheaf of $\mathcal{E} \to \mathcal{O}_\ell(-1)$ can be presented in normal form by the matrix

$$\begin{pmatrix} z_0 & z_2 & z_3 & 0 \\ z_1 & 0 & z_2 & z_3 \end{pmatrix},$$

where $z_0, \ldots, z_3$ is dual to the above basis.

1.6 The subvariety $S_2 \subset M(0, 2, 4)$

Let C be a smooth conic in $\mathbb{P}_3$ and $\mathcal{O}_C(1)$ a line bundle of degree 1 on C. For each pair of generating sections of $\mathcal{O}_C(1)$ we get an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{O}_C(1) \longrightarrow 0$$

on $\mathbb{P}_3$. The kernel sheaf $\mathcal{F}$ again has rank 2 and Chern classes $(0, 2, 4)$. If $z_0, \ldots, z_3$ are homogeneous coordinates such that $C = \{z_3 = 0, \; z_0 z_2 - z_1^2 = 0\}$, then up to equivalence $\mathcal{O}_C(1)$ has the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{A} 4\mathcal{O}(-1) \xrightarrow{B} 2\mathcal{O} \longrightarrow \mathcal{O}_C(1) \longrightarrow 0$$

with

$$A = \begin{pmatrix} -z_0 & z_1 & z_3 & 0 \\ z_1 & -z_2 & 0 & z_3 \end{pmatrix} \qquad B = \begin{pmatrix} z_3 & 0 \\ 0 & z_3 \\ z_0 & -z_1 \\ -z_1 & z_2 \end{pmatrix}.$$

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Therefore, $\mathcal{F}$ is stable by lemma 1.2.2 and belongs to the image of φ. The isomorphism class of $\mathcal{F}$ depends only on C. Therefore, the Hilbert scheme of smooth conics in $\mathbb{P}_3$ defines an 8-dimensional irreducible family F_2 of sheaves in $M(0,2,4)$. We let S_2 be the closure of its modular image in $M(0,2,4)$, which again is 8-dimensional and irreducible. By the above, S_2 is contained in the image of φ.

1.7 Proof of theorem I:

(i) The morphism φ is injective. This is clear on the open set of stable points by the remark in 1.1. If

$$A = \begin{pmatrix} x & y & 0 & 0 \\ * & * & x' & y' \end{pmatrix}$$

is properly semistable, let

$$A_0 = \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x' & y' \end{pmatrix}.$$

Then $Im\ A_0$ is in the orbit closure of $Im\ A$, since $Im\ A_0$ is the limit of

$$Im \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} A$$

for $t \to 0$. Moreover, $Im\ A_0$ generates the closed minimal orbit in the fibre of the morphism $G_4(k^2 \otimes V^*)^{ss} \to M(0,2,4)$ over $[\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}]$, where $\ell = \{x = y = 0\}$, $\ell' = \{x' = y' = 0\}$. This proves injectivity of φ also over S_0.

(ii) The morphism φ is surjective: since S_0, S_1, S_2 are contained in the image of φ by 1.4, 1.5, 1.6, it is sufficient to investigate the stable sheaves $\mathcal{F}$ in $M(0,2,4) \backslash S_0 \cup S_1 \cup S_2$. We do this by classifying the quotients $\mathcal{F}^{**}/\mathcal{F}$. Since Supp $\mathcal{F}^{**}/\mathcal{F}$ is at least 2-codimensional $c_2(\mathcal{F}^{**}/\mathcal{F}) \leq 0$ and thus $c_2\mathcal{F}^{**} \leq c_2\mathcal{F} = 2$. On the other hand $0 \leq c_2\mathcal{F}^{**}$ because $\mathcal{F}^{**}$ is always μ–semistable, [OSS], lemma 1.2.4.

a) Let $\mathcal{F}$ be μ–stable. Then also $\mathcal{F}^{**}$ is μ–stable, [OSS], loc. cit. By [H], remark 4.2.0, $c_2\mathcal{F}^{**} > 0$.

a.1) If $c_2\mathcal{F}^{**} = 2$, then $\mathcal{C} = \mathcal{F}^{**}/\mathcal{F}$ must be 0–dimensional with Chern polynomial $c(\mathcal{C}) = 1 + 2\ell(\mathcal{C})h^3$, where $\ell(\mathcal{C})$ is the length of $\mathcal{C}$. But then from the short exact sequence of $\mathcal{C}$ we get

$$c_3\mathcal{F}^{**} = c_3\mathcal{F} + 2\ell(\mathcal{C}) = 4 + 2\ell(\mathcal{C}).$$

By [H], theorem 8.2, $c_3\mathcal{F}^{**} \leq c_2(\mathcal{F}^{**})^2 - c_2(\mathcal{F}^{**}) + 2 = 4$. This implies $\mathcal{C} = 0$ and $\mathcal{F} = \mathcal{F}^{**}$. By [C], theorem 2.12, and [O], lemma 2.4, $\mathcal{F}$ is in the image of φ.

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a.2) If, however, $c_2 \mathcal{F}^{**} = 1$ then by [H], example 4.2.1, $\mathcal{F}^{**} = \mathcal{E}$ is a null-correlation bundle. For $\mathcal{C} = \mathcal{E}/\mathcal{F}$ we get

$$\mathcal{X}\, \mathcal{C}(m) = m \quad \text{and} \quad c(\mathcal{C}) = 1 - h^2 - 4h^3.$$

Hence, $\mathcal{C}$ is supported by a line ℓ. By Nakayama's Lemma $\mathcal{C} \otimes \mathcal{O}_\ell$ has the same support and its Hilbert polynomial must be $m + c$. Hence $\mathcal{C} \otimes \mathcal{O}_\ell \cong \mathcal{T} \oplus \mathcal{O}_\ell(a)$ where $\mathcal{T}$ is the 0–dimensional torsion. We get an exact "reducing" sequence

$$0 \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{T} \oplus \mathcal{O}_\ell(a) \longrightarrow 0$$

with $\mathcal{C}_0$ at most 0–dimensional. Since $\mathcal{E} \otimes \mathcal{O}_\ell$ splits at most into $\mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$ we must have $a \geq -1$. But the Euler characteristics of the sequence give us

$$m = \mathcal{X}\, \mathcal{C}(m) = \ell(\mathcal{C}_0) + \ell(\mathcal{T}) + m + a + 1.$$

Therefore, $a = -1$ and $\mathcal{C}_0 = \mathcal{T} = 0$, i.e. $\mathcal{C} = \mathcal{O}_\ell(-1)$. Therefore, $\mathcal{F}$ occurs in S_1, which is in the image of φ.

b) Now let $\mathcal{F}$ be stable, but not μ–stable. Then $\mathcal{F}^{**}$ is not stable, [OSS], lemma 1.2.4, [H], remark 3.1.1, but is still μ–semistable.

b.1) If $c_2 \mathcal{F}^{**} = 2$ we again have a 0–dimensional quotient $\mathcal{C} = \mathcal{F}^{**}/\mathcal{F}$. By [H], thm 8.2 $c_3(\mathcal{F}^{**}) \leq c_2(\mathcal{F}^{**})^2 + c_2(\mathcal{F}^{**}) + 2 = 6$. As in a.1) we obtain conditions for c_3 and $\ell(\mathcal{C})$: there are only the possibilities $\ell(\mathcal{C}) = 0$ and $c_3 \mathcal{F}^{**} = 4$ or $\ell(\mathcal{C}) = 1$ and $c_3 \mathcal{F}^{**} = 6$. In the first case $\mathcal{F}^{**} = \mathcal{F}$ would be stable, and in the second we would have an exact sequence $0 \to \mathcal{F} \to \mathcal{F}^{**} \to k_p \to 0$, where k_p is the structure sheaf of a simple point. Now $\mathcal{F}^{**}$ has a non-zero section $\mathcal{O} \to \mathcal{F}^{**}$ which cannot factorize through $\mathcal{F}$. Hence, $\mathcal{F}$ contains the subsheaf $Ker(\mathcal{O} \to kp) = \mathcal{I}$ with

$$\mathcal{X}\, \mathcal{I}(m) = \mathcal{X}\, \mathcal{O}(m) - 1 > \frac{1}{2} \mathcal{X}\, \mathcal{F}(m),$$

contradicting stability.

b.2) If $c_2 \mathcal{F}^{**} = 1$ then by [C], lemma 2.1 $c_3 \mathcal{F}^{**} = 2$ and there is an exact sequence $0 \to \mathcal{O} \to \mathcal{F}^{**} \to \mathcal{I}_\ell \to 0$ for some line ℓ. If $\mathcal{F}''$ denotes the image of $\mathcal{F}$ in $\mathcal{I}_\ell$, we get $\mathcal{X}\, \mathcal{F}''(m) \leq \mathcal{X}\, \mathcal{I}_\ell(m) = \frac{1}{2}\mathcal{X}\, \mathcal{F}(m)$, contradicting again stability of $\mathcal{F}$. So we are left with the case

b.3) $c_3 \mathcal{F}^{**} = 0$ or $\mathcal{F}^{**} = 2\mathcal{O}$ by [H], lemma 9.7. Now $\mathcal{C} = \mathcal{F}^{**}/\mathcal{F}$ has

$$\mathcal{X}\, \mathcal{C}(m) = 2m + 2 \quad \text{and} \quad c(\mathcal{C}) = 1 - 2h^2 - 4h^3.$$

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Therefore, $C = Supp(\mathcal{C})$ is a curve of *degree* ≤ 2. If C is a **smooth conic**, we get a reducing diagram

$$0 \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{T} \oplus \mathcal{O}_C(a) \longrightarrow 0$$

where $\mathcal{T}$ is the 0–dimensional torsion of the rank–1 sheaf $\mathcal{O} \otimes \mathcal{O}_C$ on C, $a \geq 0$ is the degree of quotient $mod\ \mathcal{T}$ and $\mathcal{C}_0$ is 0–dimensional. We get

$$2m + 2 = \ell(\mathcal{C}_0) + \ell(\mathcal{T}) + a + 2m + 1.$$

If $a = 0$ there is an exact sequence $0 \to k_p \to \mathcal{C} \to \mathcal{O}_C \to 0$ which yields a contradiction to the stability of $\mathcal{F}$. Therefore, $\mathcal{C} = \mathcal{O}_C(1)$ and $\mathcal{F}$ belongs to S_2 in the image of φ.

If, however, $C = supp(\mathcal{C})$ is not a smooth conic, it must contain a line. Then by the following lemma 1.7.1 there is a line ℓ and a surjection $\mathcal{C} \to \mathcal{O}_\ell$. The composed homomorphism $2\mathcal{O} \to \mathcal{O}_\ell$ factorizes through $\mathcal{O}$ and we get the exact diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & 2\mathcal{O} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_\ell & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}_\ell & \longrightarrow & 0 \ . \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

The image of $\mathcal{F}$ in $\mathcal{I}_\ell$ then contradicts stability, since $\mathcal{X}\,\mathcal{I}_\ell(m) = \frac{1}{2}\mathcal{X}\,\mathcal{F}(m)$.

1.7.1 Lemma: *Let $\mathcal{C}$ be a coherent sheaf on $\mathbb{P}_3$ with Hilbert polynomial $2m + 2$, and let $\mathcal{C}$ be generated by two sections. If $Supp(\mathcal{C})$ contains a line, then there is a line ℓ and a surjection $\mathcal{C} \to \mathcal{O}_\ell \to 0$.*

<u>Proof:</u> Let $\ell \subset Supp(\mathcal{C})$ and consider the reducing sequence

$$0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{O}_\ell \longrightarrow 0.$$

It follows from the Koszul resolution of $\mathcal{O}_\ell$, that there is a surjection $2\mathcal{C}(-1) \to \mathcal{C}' \to 0$, and hence there is a surjection $4\mathcal{O}(-1) \to \mathcal{C}' \to 0$. We distinguish two cases.

<u>Case 1:</u> $\mathcal{C} \otimes \mathcal{O}_\ell$ has *rank* 1 on ℓ. Then we get the decompositions

$$\mathcal{C}' = \mathcal{T}' \oplus \mathcal{O}_{\ell'}(a') \quad \text{and} \quad \mathcal{C} \otimes \mathcal{O}_\ell = \mathcal{T} \oplus \mathcal{O}_\ell(a)$$

where $\mathcal{T}$ has finite length, $a \geq 0$, and where ℓ' is another line, with $\mathcal{T}'$ of finite length and $a' \geq -1$, since $\mathcal{X}\,\mathcal{C}'(m) = m + c'$. It follows that

$$a + a' + \ell(\mathcal{T}) + \ell(\mathcal{T}') = 0.$$

If $a = 0$, there is nothing to prove. If $a > 0$, then the equation implies $a = 1$, $a' = -1$, $\mathcal{T} = \mathcal{T}' = 0$. Hence we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\ell'}(-1) \longrightarrow \mathcal{C} \longrightarrow \mathcal{O}_\ell(1) \longrightarrow 0.$$

In this case $\ell \cap \ell' \neq \emptyset$ since by our assumption $\mathcal{C}$ is generated by two sections also on ℓ'. If now $\ell \cap \ell' = \{p\}$, and if we tensorise by $\mathcal{O}_{\ell'}$, we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\ell'}(-1) \longrightarrow \mathcal{C} \otimes \mathcal{O}_{\ell'} \longrightarrow k_p \longrightarrow 0,$$

since $\mathcal{O}_{\ell'}$ has no 0–dimensional subsheaf. Since $\mathcal{C} \otimes \mathcal{O}_{\ell'}$ is globally generated, too, we must have $\mathcal{C} \otimes \mathcal{O}_{\ell'} = \mathcal{O}_{\ell'}$, which proves the lemma in this case.

If $\ell = \ell'$ we again obtain, after tensorising by $\mathcal{O}_\ell$, an exact sequence

$$0 \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow \mathcal{C} \otimes \mathcal{O}_\ell \longrightarrow \mathcal{O}_\ell(1) \longrightarrow 0,$$

since $\mathcal{T}or_1^{\mathcal{O}}(\mathcal{O}_\ell(1), \mathcal{O}_\ell) \cong 2\mathcal{O}_\ell$. Again it follows that $\mathcal{C} \otimes \mathcal{O}_\ell = 2\mathcal{O}_\ell$ proving the statement of the lemma, or contradicting $rk\, \mathcal{C} \otimes \mathcal{O}_\ell = 1$.

Case 2: $\mathcal{X}(\mathcal{C} \otimes \mathcal{O}_\ell)(m) = 2m + c$. Here we get the reducing sequence

$$0 \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{O}_\ell \longrightarrow 0$$
$$\mathcal{C} \otimes \mathcal{O}_\ell = \mathcal{T} \oplus \mathcal{O}_\ell(a) \oplus \mathcal{O}_\ell(b)$$

where $\mathcal{C}_0, \mathcal{T}$ have finite length, $a, b \geq 0$. We obtain from this

$$a + b + \ell(\mathcal{C}_0) + \ell(\mathcal{T}) = 0,$$

and from this again $a = b = 0$, $\mathcal{C}_0 = \mathcal{T} = 0$, and $\mathcal{C} = 2\mathcal{O}_\ell$. This proves lemma 1.7.1.

(iii) To end the proof of the theorem it remains to show that φ is indeed an isomorphism.

To this end, using the relative Beilinson spectral sequence, we shall see that given a family $\mathbf{G} \to \mathbb{P}_3 \times S$ of *rank* 2 semistable sheaves on $\mathbb{P}_3$ with Chern classes $(0, 2, 4)$ parametrized by S then the natural morphism $S \to M(0, 2, 4)$ factorizes locally through $G_4(k^2 \otimes V^*)^{ss}$. We denote by $\pi : \mathbb{P}_3 \times S \to S$ the second projection. Now, for $s \in S(k)$, $H^2(\mathbf{G}_s(-3))$ and $H^2(\mathbf{G}_s(-2))$ are the only non-vanishing cohomology groups $H^q(\mathbb{P}_3, \mathbf{G}_s(\rho))$ in the range $-3 \leq \rho \leq 0$ and they have constant dimension 4 and 2, respectively. From the base change theorem it follows that $R^q\pi_*\mathbf{G}(\rho)$ are locally free in the range $-3 \leq \rho \leq 0$. The relative Beilinson spectral sequence [OSS], theorem 4.1.11, gives us $\mathbf{G}$ as the cokernel

$$0 \longrightarrow R^2\pi_*\mathbf{G}(-3)\boxtimes\Omega^3_{\mathbf{P}_3}(3) \longrightarrow R^2\pi_*\mathbf{G}(-2)\boxtimes\Omega^2_{\mathbf{P}_3}(2) \longrightarrow \mathbf{G} \longrightarrow 0$$

on $\mathbf{P}_3 \times S$.

Also $\mathbf{G}$ is the cokernel in

$$0 \longrightarrow R^1\pi_*(\mathbf{G}\otimes\Omega^2_{\mathbf{P}_3\times S/S}(2))\boxtimes\mathcal{O}_{\mathbf{P}_3}(-2) \longrightarrow R^1\pi_*(\mathbf{G}\otimes\Omega^1_{\mathbf{P}_3\times S/S}(1))\boxtimes\mathcal{O}_{\mathbf{P}_3}(-1) \longrightarrow$$
$$\mathbf{G} \longrightarrow 0,$$

which follows from the second relative Beilinson sequence. Choosing suitable trivialisations of the locally free sheaves $R^1\pi_*(\mathbf{G}\otimes\Omega^2_{\mathbf{P}_3\times S/S}(2))$ and $R^1\pi_*(\mathbf{G}\otimes\Omega^1_{\mathbf{P}_3\times S/S}(1))$ we obtain that the natural morphism $S \to M(0,2,4)$ locally factorizes through $G_4(k^2\otimes V^*)^{ss}//SL(2)$. The construction of the morphism $S \to G_4(k^2\otimes V^*)^{ss}//SL(2)$ is functorial and thus we have a transformation

$$\alpha : I \longrightarrow Hom(-, G_4(k^2\otimes V^*)^{ss}//SL(2))$$

where $I : Sch/k \to Sets$ is the functor $I(S) := \{\mathbf{G}|\mathbf{G}$ being an $\mathcal{O}_{\mathbf{P}_3\times S}$–coherent sheaf s.t. $\mathbf{G}_s$ is a $rank$ 2 semistable sheaf on $\mathbf{P}_3$ with Chern classes $(0,2,4)$ for any geometric point $s \in S(k)\}/\sim$, where $\mathbf{G}_1 \sim \mathbf{G}_2$ if there exists an invertible sheaf $\mathcal{L}$ on S such that $\mathbf{G}_1 \cong \mathbf{G}_2 \otimes \pi^*\mathcal{L}$.

Furthermore, $\alpha(k) : I(k) \to Hom(k, G_4(k^2\otimes V^*)^{ss}//SL(2))$ is bijective and by using the same arguments as in [OSS], p. 309, we prove that α is an isomorphism.

1.8 Remark: If $\mathcal{F}$ is semistable and a non–trivial extension $0 \to \mathcal{I}_\ell \to \mathcal{F} \to \mathcal{I}_{\ell'} \to 0$, then $\mathcal{F}^{**}$ is only μ–semistable but not $2\mathcal{O}$. In such a case $c_2\mathcal{F}^{**} = 1$, see 1.7, b.2) and there is an extension $0 \to \mathcal{O} \to \mathcal{F}^{**} \to \mathcal{I}_{\ell'} \to 0$ as well as a diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{I}_\ell & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I}_{\ell'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{F}^{**} & \longrightarrow & \mathcal{I}_{\ell'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & \mathcal{O}_\ell & = & \mathcal{O}_\ell & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0. & & & &
\end{array}
$$

If, in addition, ℓ and ℓ' are different lines, then the singular locus of $\mathcal{F}$, i.e. the support of $\mathcal{E}xt^1(\mathcal{F},\mathcal{O})$ is the union of ℓ and two points or a double point on ℓ'. The two points distinguish the isomophism classes of the sheaves $\mathcal{F}$ in the extension. If $\mathcal{F}$ is a direct sum, then its singular locus is the union of the two lines.

2 Subvarieties of non–reflexive sheaves and duality

After proving that the semistable locus S_0 is the singular locus, we show that $S_0 \cap S_1 = S_0 \cap S_2 = S_1 \cap S_2$ and equals the subvariety $S_0' \subset S_0$ of pairs of meeting lines. Moreover, we describe a remarkable duality between the moduli spaces $M(0, 2, 4)$ over $\mathbb{P}_3$ and $\mathbb{P}_3^*$.

2.1 As a corollary of the proof 1.7 we obtain

$$
\begin{aligned}
M_r(0, 2, 4) &= M(0, 2, 4) \backslash S_0 \cup S_1 \cup S_2 \\
M_\mu(0, 2, 4) &= M(0, 2, 4) \backslash S_0 \cup S_2,
\end{aligned}
$$

where M_r resp. M_μ denote the open subschemes of reflexive resp. μ–stable sheaves.

2.2 Proposition: *The 8–dimensional subvariety S_0 of semistable points is the singular locus of $M(0, 2, 4)$.*

Proof: We use the

Lemma: *Let $\mathcal{F}$ be a rank–2 semistable sheaf, $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ with $\mathcal{X} \, \mathcal{F}'(m) = \mathcal{X} \, \mathcal{F}''(m) = \frac{1}{2} \mathcal{X} \, \mathcal{F}(m)$ and $\mathcal{F}' \not\cong \mathcal{F}''$. If $Ext^2(\mathcal{F}, \mathcal{F}) = 0$, then the tangent space at $[\mathcal{F}]$ of the corresponding Maruyama scheme is isomorphic to*

$$
Ext^1(\mathcal{F}', \mathcal{F}') \oplus Ext^1(\mathcal{F}', \mathcal{F}'') \otimes Ext^1(\mathcal{F}'', \mathcal{F}') \oplus Ext^1(\mathcal{F}'', \mathcal{F}'').
$$

This lemma and a proof of it were communicated to the second author by J. Le Potier. Since under the assumption $Ext^2(\mathcal{F}, \mathcal{F}) = 0$ the germ of the moduli space at $[\mathcal{F}]$ is (analytically) isomorphic to $Ext^1(\mathcal{F}, \mathcal{F})/Aut(\mathcal{F})$, see also [LP], proof of proposition 6, and since $Aut(\mathcal{F}) = k^* \times k^*$ in this case, the proof consists in determining the affine quotient and its tangent space explicitly.

We apply this lemma to $\mathcal{F} = \mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}$, where ℓ, ℓ' are lines. First, from the simple representation of any sheaf $\mathcal{F}$ in $M(0, 2, 4)$ we immediately get $Ext^2(\mathcal{F}, \mathcal{F}) = 0$.

Hence, $M(0, 2, 4)$ is smooth away from S_0. Now

$$
\dim Ext^1(\mathcal{I}_\ell, \mathcal{I}_{\ell'}) = \left\{ \begin{array}{ll} 3 & \ell \neq \ell' \\ 4 & \ell = \ell' \end{array} \right.
$$

by direct computation. It follows that the tangent space at $[\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}] \in M(0, 2, 4)$ in the presentation of the lemma has dimension $4 + 3 \cdot 3 + 4 = 17$, if $\ell \neq \ell'$. Since $M(0, 2, 4)$ is 13–dimensional, this proves that this scheme is singular along $S_0 \backslash S_0''$ where S_0'' is the subvariety of double lines. Then $M(0, 2, 4)$ is also singular at S_0''.

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Remark: If we consider the same type of $SL(2)$-action on $G_4(k^2 \otimes W^*)$ where W is 3-dimensional, the quotient $G_4(k^2 \otimes W^*)^{ss}//SL(2)$ is the space $M(0,2)$ on $\mathbb{P}_2$ which is smooth. Indeed this space is isomorphic to the $\mathbb{P}_5 = |\mathcal{O}_{\mathbf{P}_2^*}(2)|$ by assigning to each sheaf its conic of jumping lines. The semistable locus S_0 here is the divisor of singular conics, see also [N-T], 5.2.1, and section 2.4.

2.3 **Proposition:** *Let $S_0' \subset S_0$ be the 7-dimensional irreducible subvariety of pairs (ℓ, ℓ') with $\ell \cap \ell' \neq \emptyset$. Then*

$$S_0' = S_0 \cap S_1 = S_0 \cap S_2 = S_1 \cap S_2.$$

Proof:

1) We prove first that $S_0' \subset S_1 \cap S_2$: Let $[\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}] \in S_0'$ and let $z_0, z_1, z_2, z_3 \in V^*$ homogeneous coordinates such that $\ell = \{z_2 = z_3 = 0\}$, $\ell' = \{z_1 = z_2 = 0\}$. Then $\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}$ can be presented by the matrix

$$A = \begin{pmatrix} z_3 & z_2 & 0 & 0 \\ 0 & 0 & z_2 & z_1 \end{pmatrix}.$$

We define two 1-parameter flat families $\mathcal{F}_t^i = \mathcal{F}(A_t^i)$ of deformations of $\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}$ by matrices

$$A_t^1 = \begin{pmatrix} z_3 & z_2 & 0 & tz_0 \\ 0 & tz_3 & z_2 & z_1 \end{pmatrix} \quad A_t^2 = \begin{pmatrix} z_3 & z_2 & 0 & tz_0 \\ tz_0 & 0 & z_2 & z_1 \end{pmatrix}.$$

For $t \neq 0$ the isomorphism classes of $\mathcal{F}_t^1$ resp. $\mathcal{F}_t^2$ belong to S_1 resp. S_2, as is easily verified. This proves $S_0' \subset S_1 \cap S_2$.

2) Next we show that $S_0 \cap S_1 \subset S_0'$. Then by 1) $S_0 \cap S_1 = S_0'$.

Let $\mathcal{F}_0 = \mathcal{I}_{\ell_1} \oplus \mathcal{I}_{\ell_2}$ belong to S_1 and let $\mathcal{F}_0 = \mathcal{F}(A_0)$. Then there is a 1-parameter family A_t such that $Im\, A_t$ converges to $Im\, A_0$ in $G_4(k^2 \otimes V^*)^{ss}$ and the biduality sequence of $\mathcal{F}_t = \mathcal{F}(A_t)$ for $t \neq 0$ is

$$0 \longrightarrow \mathcal{F}_t \longrightarrow \mathcal{E}_t \longrightarrow \mathcal{O}_{\ell_t}(-1) \longrightarrow 0 \qquad (\#)$$

where $\mathcal{E}_t$ is a null correlation bundle and ℓ_t is a jumping line of $\mathcal{E}_t$.

2.a) In order to show that $\ell_1 \cap \ell_2 \neq \emptyset$ we use the equations of the divisors of jumping lines of the sheaves involved.

Let $\mathbb{P}_3 \xleftarrow{p} \mathbf{F} \xrightarrow{q} G = G_2V$ be the incidence projections of points and lines. If we apply $R^* = R^* q_* \circ p^*$ to the sequence

$$0 \to 2\mathcal{O}(-3) \xrightarrow{A} 4\mathcal{O}(-2) \to \mathcal{F}(A)(-1) \to 0$$

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we get the short exact sequence

$$0 \to 2S \xrightarrow{\tilde{A}} 4\mathcal{O}_G \to R^1\mathcal{F}(A)(-1) \to 0,$$

see 3.10, such that $\det \tilde{A}$ is the equation of $Supp\ R^1\mathcal{F}(A)(-1)$ which is the divisor of jumping lines of $\mathcal{F}(A)$. It follows that

$$< \det \tilde{A}_t > \longrightarrow < \det \tilde{A}_0 >$$

in $|\mathcal{O}_G(2)|$.

2.b) Let $0 \neq \xi \in \Lambda^2 V$. Using an isomorphism $\sigma : \Lambda^4 V \cong k$, $\xi(\ell) := \sigma(\xi \wedge \ell) = 0$ is the equation of the hyperplane section of G defined by the polar hyperplane to $< \xi > \in \mathbb{P}\Lambda^2 V$. This is at the same time the divisor of jumping lines of the null–correlation sheaf $\mathcal{E}_\xi$ defined by $0 \to \Omega^3(3) \xrightarrow{\xi} \Omega^1(1) \to \mathcal{E}_\xi \to 0$. If ℓ_0 is a line then $R^0\mathcal{O}_{\ell_0}(-2) = R^1\mathcal{I}_{\ell_0}(-2)$ and the support of this sheaf is the cone of all lines ℓ meeting ℓ_0 with the equation $\ell_0(\ell) = 0$ or $\ell_0 \wedge \ell = 0$. (ℓ_0, ℓ considered as elements of $\Lambda^2 V$). Hence, $Supp\ R^1\mathcal{F}_0(-1)$ has equation $\ell_1(\ell) \cdot \ell_2(\ell) = 0$.

2.c) From (#) we get the exact sequence

$$0 \longrightarrow R^0\mathcal{O}_{\ell_t}(-2) \longrightarrow R^1\mathcal{F}_t(-1) \longrightarrow R^1\mathcal{E}_t(-1) \longrightarrow 0$$

for $t \neq 0$. Let $\mathcal{E}_t$ be defined by $\xi_t \in \Lambda^2 V$. Then up to a constant

$$\det \tilde{A}_t = \ell_t(\ell)\xi_t(\ell).$$

Because ℓ_t is a jumping line of $\mathcal{E}_t$, by 2.b) $\xi_t(\ell_t) = 0$ or $\xi_t \wedge \ell_t = 0$. Since, however, $< \det \tilde{A}_t > \longrightarrow < \det A_0 >$ and the latter is $\ell_1(\ell)\ell_2(\ell)$ up to a constant, ℓ_1 and ℓ_2 must satisfy $\ell_1 \wedge \ell_2 = 0$, too. This proves $\ell_1 \cap \ell_2 \neq \emptyset$ and $S_0 \cap S_1 \subset S'_0$.

3) It follows from the proof in 1.7 that the stable sheaves $\mathcal{F}$ belonging to S_1 resp. S_2 satisfy $c_2\mathcal{F}^{**} = 1$ resp. $c_2\mathcal{F}^{**} = 0$. It follows that $S_1 \cap S_2 \subset S_0$. But since $S_0 \cap S_1 \subset S'_0$ we even get $S_1 \cap S_2 \subset S'_0$. By 1) $S_1 \cap S_2 = S'_0$.

4) In order to show that also $S_0 \cap S_2 = S'_0$ it is sufficient to show that $S_0 \cap S_2 \subset S'_0$, since by 1) $S'_0 \subset S_0 \cap S_2$. Let $[\mathcal{F}_0]$ belong to $S_0 \cap S_2$. There is a flat 1–parameter deformation $\mathcal{F}_t$ of $\mathcal{F}_0$ with $\mathcal{F}_t$ stable for $t \neq 0$ and biduality sequence

$$0 \to \mathcal{F}_t \to 2\mathcal{O} \to \mathcal{O}_{C_t}(1) \to 0$$

where C_t is a smooth conic, which is also the singular locus of $\mathcal{F}_t$. It follows that the singular locus of $\mathcal{F}_0$ is also a conic, which consists of two meeting lines. This

terminates the proof of the proposition.

2.4 The subvariety D'_4 of centred pullbacks.

If $p = <x>$ is a point in $\mathbb{P}_3 = \mathbb{P}V$ we consider the natural inclusion and projection

$$\mathbb{P}V \xhookrightarrow{i} \mathbb{P}V \backslash \{p\} \xrightarrow{\pi} \mathbb{P}V/<x> \cong \mathbb{P}_2.$$

Given a rank–2 bundle $\mathcal{B}$ on $\mathbb{P}_2$ with Chern classes $(0,2)$, then

$$\mathcal{F} = i_*\pi^*\mathcal{B}$$

is a rank–2 reflexive sheaf with Chern classes $(0,2,4)$ having p as its only singular point. We call it a centered pullback. The following lemma classifies these sheaves.

2.4.1 Lemma: *Let $\mathcal{F}$ be a stable rank–2 reflexive sheaf on $\mathbb{P}_3$. The following are equivalent:*

  (i) $\mathcal{F}$ has Chern classes $(0,2,4)$ and there is a point p with $\dim \mathcal{F}_p/m_p\mathcal{F}_p = 4$;

  (ii) $\mathcal{F}$ is a centred pullback of a stable rank–2 bundle on $\mathbb{P}_2$ with Chern classes $(0,2)$.

<u>Proof:</u> If $\mathcal{B}$ is a stable rank–2 bundle with Chern classes $(0,2)$ on $P = \mathbb{P}V/<x>$, then it has Beilinson presentation

$$0 \longrightarrow 2\mathcal{O}_p(-2) \xrightarrow{A^*} 4\mathcal{O}_p(-1) \longrightarrow \mathcal{B} \longrightarrow 0$$

as in the case of $M(0,2,4)$. (The other Beilinson presentation would be $0 \to 2\Omega_p^3(3) \xrightarrow{B} 2\Omega_p^1(1) \to \mathcal{B} \to 0$ where $0 \to k^2 \xrightarrow{B} k^2 \otimes W \xrightarrow{A^*} k^4 \to 0$ is exact. Then $\det B$ is the equation of the conic of jumping lines in $\mathbb{P}_2^*$.) Now $\mathcal{F} = i_*\pi^*\mathcal{B}$ has the same presentation A on $\mathbb{P}_3$ and therefore Chern classes $(0,2,4)$. As in lemma 1.2.2 A is stable with $\mathcal{B}$ and hence $\mathcal{F}$ is stable. The entries of A are elements of $(V/<x>)^* \subset V^*$ and hence vanish at $p = <x>$. It follows that $\mathcal{F}_p/m_p\mathcal{F}_p \cong k^4$. If conversely $\mathcal{F}$ satisfies (i) then the entries of A vanish at p and therefore A defines a stable rank–2 bundle $\mathcal{B}$ on $\mathbb{P}V/<x>$ whose centered pullback is $\mathcal{F}$.

2.4.2 Remarks:

  1) By [N-T], lemma 5.3.1, the matrix A of $\mathcal{B}$ can be given the normal form $\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}$, where z_1, z_2, z_3 are independent.

  2) It follows that for such a sheaf $\mathcal{F}(A)$ the sheaf $\mathcal{E}xt^1(\mathcal{F}(A),\mathcal{O}) \cong k^2$.

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Since the space of stable bundles with Chern classes $(0,2)$ on $\mathbb{P}_2$ is 5–dimensional and irreducible, the centered pullbacks form an irreducible family of stable reflexive sheaves of dimension 8. We denote by D'_4 its closure in $M(0,2,4)$. The matrices of the stable sheaves in D'_4 can be given the normal form as mentioned in the remark. It follows from 2.5 that $S_0 \cap D'_4 = S'_0$.

2.5 A remarkable duality

Since $k^2 \otimes V^*$ is 8–dimensional there is the natural isomorphism $G_4(k^2 \otimes V^*) \cong G_4(k^2 \otimes V)$ given by $\Lambda \longleftrightarrow K$ for an exact sequence of $\Lambda \in G_4(k^2 \otimes V^*)$, $0 \to K \to k^2 \otimes V \to \Lambda^* \to 0$. We write $K(\Lambda) = K$. This leads to an isomorphism of the moduli spaces

$$M_{\mathbb{P}_3}(0,2,4) \quad \cong \quad M_{\mathbb{P}_3^*}(0,2,4)$$

as follows. The above isomorphism of the Grassmannians is not strictly equivariant, but we get

$$g \cdot \Lambda \longleftrightarrow g^{t,-1} \cdot K(\Lambda).$$

Using the above exact sequences, their duals and the criterion of lemma 1.1.1, one easily checks that $K(\Lambda)$ is stable (semistable) if and only if Λ is stable (semistable). Therefore, we obtain an induced isomorphism of the quotients

$$
\begin{array}{ccc}
G_4(k^2 \otimes V^*)^{ss}//SL(2) & \cong & G_4(k^2 \otimes V)^{ss}//SL(2) \\
\| \wr & & \| \wr \\
M_{\mathbb{P}_3}(0,2,4) & \overset{\phi}{\longrightarrow} & M_{\mathbb{P}_3^*}(0,2,4)
\end{array}
$$

If $\mathcal{G}(K)$ denotes the sheaf defined on $\mathbb{P}_3^*$ by

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}_3^*}(-2) \longrightarrow K^* \otimes \mathcal{O}_{\mathbb{P}_3^*}(-1) \longrightarrow \mathcal{G}(K) \longrightarrow 0$$

as $\mathcal{F}(\Lambda)$ on $\mathbb{P}_3$, then

$$[\mathcal{F}(\Lambda)] \overset{\phi}{\longleftrightarrow} [\mathcal{G}(K(\Lambda))].$$

2.5.1 Remark: The subspace $K(\Lambda) \subset k^2 \otimes V$ also defines a homomorphism $K(\Lambda) \otimes \Omega^3(3) \to k^2 \otimes V \otimes \Omega^3(3) \to k^2 \otimes \Omega^2(2)$ on $\mathbb{P}_3$, whose cokernel is $\mathcal{F}(\Lambda)$. This follows from the exact diagram

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & k^2 \otimes \Omega^3(2) & \cong & k^2 \otimes \mathcal{O}(-2) & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \to & K(\Lambda) \otimes \mathcal{O}(-1) & \to & k^2 \otimes V \otimes \mathcal{O}(-1) & \to & \Lambda^* \otimes \mathcal{O}(-1) & \to & 0 \\
& & \| \wr & & \downarrow & & \downarrow & & \\
0 & \to & K(\Lambda) \otimes \Omega^3(3) & \to & k^2 \otimes \Omega^2(2) & \to & \mathcal{F}(\Lambda) & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

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relating directly to the two Beilinson complexes of $\mathcal{F}(\Lambda)$.

2.5.2 Remark: The sheaf $\mathcal{G} = \mathcal{G}(K(\Lambda))$ on $\mathbb{P}_3^*$ cannot be constructed directly from $\mathcal{F} = \mathcal{F}(\Lambda)$ via the incidence transformation $\mathbb{P}_3 \xleftarrow{p} \mathbf{F} \xrightarrow{q} \mathbb{P}_3^*$. If $R^* = R^*q_* \circ p^*$ then for the general sheaf $\mathcal{F}$ we only get an exact sequence on $\mathbb{P}_3^*$

$$0 \longrightarrow \mathcal{G}(2) \longrightarrow R^1\mathcal{F}(-2)^* \longrightarrow \mathcal{E}xt^2(R^2\mathcal{F}(-2), \mathcal{O}) \longrightarrow 0$$

where $R^1\mathcal{F}(-2)$ is reflexive and equal to $\mathcal{G}^*(-2)$.

2.6 Proposition

1) If $\mathcal{F}(\Lambda)$ on $\mathbb{P}_3$ corresponds to $\mathcal{G}(K)$ on $\mathbb{P}_3^*$ under the above isomorphism ϕ, then the singular locus of $\mathcal{G}(K)$ is the set of unstable planes of $\mathcal{F}(\Lambda)$ and vice versa.

2) Let $\Sigma_0, \Sigma_1, \Sigma_2, \Delta_4'$ denote the subvarieties of $M_{\mathbb{P}_3^*}(0,2,4)$ defined as S_0, S_1, S_2, D_4' in $M_{\mathbb{P}_3}(0,2,4)$ respectively. Then $\phi(S_0) = \Sigma_0$, $\phi(S_1) = \Sigma_2$, $\phi(S_2) = \Delta_4'$ and $\phi(D_4') = \Sigma_2$.

Proof:

1) If P is a plane, then the restriction $\mathcal{F}_P$ of $\mathcal{F}(\Lambda)$ has the resolution $0 \to 2\mathcal{O}_P(-2) \xrightarrow{A^*|P} 4\mathcal{O}_P(-1) \to \mathcal{F}_P \to 0$ with $\frac{1}{2}\mathcal{X}\,\mathcal{F}_P(m) = \frac{m(m+3)}{2}$, since $A^*|P$ is again injective by the characterization of semistability in lemma 1.1.1. If $\mathcal{F}_P$ is not semistable it admits a torsion–free quotient $\mathcal{C}$ of rank 1 satisfying $\mathcal{X}\,\mathcal{C}(m) \not\geq \frac{1}{2}\mathcal{X}\,\mathcal{F}_P(m)$ for $m \gg 0$. Then $\mathcal{C} \subset \mathcal{O}_P(-1)$ and we can even replace $\mathcal{C}$ by $\mathcal{O}_P(-1)$. Hence, $\mathcal{F}_P$ is not semistable if and only if it has $\mathcal{O}_P(-1)$ as quotient. By the above representation of $\mathcal{F}_P$ this means that $Im A = \Lambda \subset k^2 \otimes V^*$ contains a vector $(\alpha, \beta) \otimes z$, where z is the equation of P. Finally, if $k^4 \xrightarrow{B} k^2 \otimes V$ represents the kernel of $k^2 \otimes V \xrightarrow{A^*} k^4$ such that $\mathcal{G}(K)$ is represented by B^*, then $< z >$ is a singular point of B^* or $\mathcal{G}(K)$ if and only if Λ contains a vector $(\alpha, \beta) \otimes z$. This proves 1).

2) can be easily verified by calculating a matrix B from A in the different cases described in lemma 1.2.1 for S_0, 1.5.1 for S_1, 1.6 for S_2, and 2.4.2 for D_4'. The remarkable fact is that S_2 corresponds to Δ_4' and D_4' to Σ_2: if $A = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}$ represents a stable sheaf $\mathcal{F}(A)$ in D_4', then $B = \begin{pmatrix} e_0 & 0 & e_3 & -e_2 \\ 0 & e_0 & -e_2 & e_1 \end{pmatrix}$ defines a typical sheaf in Σ_2 where $e_0, e_1, e_2, e_3 \in V$ is a basis and $z_0, \ldots, z_3 \in V^*$ the dual basis.

2.6.1 Corollary: $S_0 \cap D_4' = S_0'$.

Proof: $S_0 \cap D_4' = \phi^{-1}(\Sigma_0 \cap \Sigma_2) = \phi^{-1}(\Sigma_0') = S_0'$.

The position of the subvarieties S_0, S_1, S_2, D_4' (all irreducible of dimension 8) can now be illustrated by the following picture:
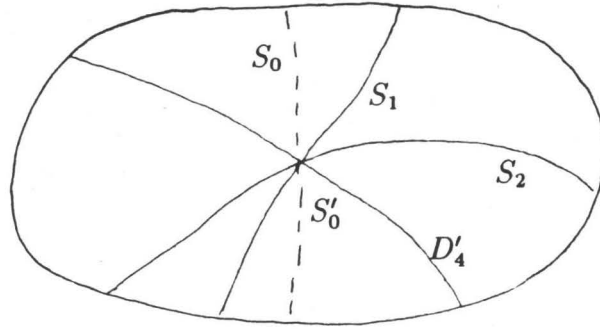
2.6.2 Remark: Each isomorphism $V \xrightarrow{b} V^*$ induces an automorphism of $M(0, 2, 4)$ by $\Lambda \mapsto (id \otimes b)(K(\Lambda))$, leaving S_0 and S_1 fixed but interchanging S_2 and D_4'.

3 Singular points and normal forms

In this section we investigate the structure of the reflexive sheaves in $M(0,2,4)$ w.r.t their singular points and give normal forms for the representing r..atrices in the different cases, thereby improving the results of Chang in [C]. This gives us new insight into the structure of the divisors of jumping lines, too. The results on normal forms will be used later.

3.1 Zero schemes of sections and singular points.

Let $\mathcal{F}$ be reflexive in $M(0,2,4)$ and let s be a non-zero section. By [H], prop 4.2., we get an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{F} \to \mathcal{I}_Y(1) \to 0$, where Y is the zero locus of s, and Y is non-degenerate since $\mathcal{F}$ is μ–stable. In our case Y is a rational cubic curve. This can be seen also from the following diagram, which is induced by the section s and the presentation of $\mathcal{F}$.

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}(-1) & = & \mathcal{O}(-1) & & \\
 & & \downarrow & & \downarrow s & & \\
0 \to & 2\mathcal{O}(-2) & \overset{A}{\to} & 4\mathcal{O}(-1) & \to & \mathcal{F} & \to 0 \\
 & \| & & \downarrow & & \downarrow & \\
0 \to & 2\mathcal{O}(-2) & \to & 3\mathcal{O}(-1) & \to & \mathcal{I}_Y(1) & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0. &
\end{array}
$$

Taking duals in the right-hand column we get the exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \to \mathcal{F}^*(-1) \longrightarrow \mathcal{O} \longrightarrow \omega_Y(2) \longrightarrow \mathcal{E}xt^1(\mathcal{F},\mathcal{O}) \longrightarrow 0,$$

where $\mathcal{E}xt^1(\mathcal{I}_Y(2),\mathcal{O}) = \omega_Y(2)$ for the Cohen–Macaulay curve Y. Hence, $\mathcal{E}xt^1(\mathcal{F},\mathcal{O})$ is an $\mathcal{O}_Y$–module and $Supp\,\mathcal{E}xt^1(\mathcal{F},\mathcal{O}) \subset Y$ as a subscheme. Note that this support is the singular locus of $\mathcal{F}$ and that the length $\ell(\mathcal{E}xt^1(\mathcal{F},\mathcal{O})) = 4$ in our case.

3.2 Proposition: *Let $\mathcal{F} \in M(0,2,4)$ be reflexive. Then $\mathcal{E}xt^1(\mathcal{F},\mathcal{O})$ is the structure sheaf $\mathcal{O}_Z$ of the 0–dimensional scheme $Z = Supp\,\mathcal{E}xt^1(\mathcal{F},\mathcal{O})$ of length 4 if and only if $\mathcal{F} \notin D'_4$.*

Note that by 2.4 $\mathcal{F} \notin D'_4$ if $\mathcal{F}$ has more than one singular point.

<u>Proof:</u> Let A be the presenting matrix of $\mathcal{F}$ and $p \in Z$. Then the following are equivalent:

(i) $rank\,A(p) = 0$ (iii) $\mathcal{F} \in D'_4$

(ii) $\dim \mathcal{F}_p/\mathbf{m}_p\mathcal{F}_p = 4$ (iv) The minimal number of generators of $\mathcal{E}xt^1(\mathcal{F},\mathcal{O})_p$ is > 1

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The equivalence of (i), (ii), (iv) is obvious from the presentation of $\mathcal{F}$ and the resulting dual sequence. The equivalence of (ii) and (iii) was shown in 2.4. Hence, $\mathcal{F} \notin D'_4$ if and only if for any $p \in Z$ the minimal number of generators of $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O})_p$ is 1.

3.3 Proposition: *Let $\mathcal{F} \in M(0, 2, 4)$ be reflexive and not in D'_4. Then $Z = Supp$ $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O})$ is not contained in a plane as a subscheme.*

Proof: Let Y be any zero scheme of a section of $\mathcal{F}(1)$. Then $Z \subset Y$ as a subscheme. If P is a plane with $Z \subset P$, then also $Z \subset P \cap Y$. But Y can be a non–degenerate cubic curve. Then $\ell(P \cap Y) \le 3$ whereas $\ell(Z) = 4$, contradiction.

3.3.1 Remark: In the degenerate cases this means the following:

(i) If Z has at most double points, which are in $1 : 1$ correspondence with their tangent lines, then the union of tangent lines and points of Z is not contained in a plane.

(ii) If Z consists of a triple point p and a simple point q, then (Z, p) is not contained in a line, but it is contained in a plane P as any point of length 3. Then $q \notin P$.

(iii) In the case of a point of length 4 there is the

3.3.2 Lemma: *Let Z be a point of length 4, $Supp\,(Z) = \{p\}$, such that Z is not contained in a plane and such that its Zariski tangent space $T_p Z$ is 1–dimensional. Then there is a unique plane P containing the tangent line, such that $Z \cap P$ has length 3.*

This plane is called the osculating plane of Z. Even so the lemma seems to be known, we include a proof for the convenience of the reader. Note that it is wrong, if $\dim T_p Z \ge 2$.

Proof:

1) We may assume that Z is supported at the origin $0 \in k^3$. If $E = \{x = 0\}$ is a plane such that $\mathcal{O}_{Z \cap E} = \mathcal{O}_Z / x \mathcal{O}_Z$ has length 3, then the residue class $\bar{x}$ generates a submodule of length 1 and thus $\mathfrak{m}x = 0$, where $\mathfrak{m} \subset \mathcal{O}_0$ is the maximal ideal. Moreover, x vanishes on the tangent line of Z.

2) If there were two independent linear forms x, y with this property, then the ideal $I \subset \mathcal{O}_0$ of Z contains x^2, xy, y^2, xz, yz, where z is a third coordinate. Since $T_0 Z$ has dimension 1, w.l.o.g. I also contains $x + f$, $y + g$ where the polynomials f, g do not contain linear terms. They can be reduced to polynomials in z only. Moreover, we can assume that $f = \lambda z^m$, $g = \mu z^n$, $m \le n$, $\lambda, \mu \in k$ because $zf, zg \in I$. Then $\mu x z^{m-n} - \lambda y \in I$. This means that I would contain a linear form, since $xz \in I$. (If $f = 0$ or $g = 0$ then already x or $y \in I$.) Then Z would be contained in a plane. This proves uniqueness.

3) Let now $\ell = \{x = y = 0\}$ be the geometric tangent line of Z and consider the exact sequence

$$0 \longrightarrow (\bar{x}, \bar{y})\mathcal{O}_Z \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{Z \cap \ell} \longrightarrow 0.$$

If $\ell(\mathcal{O}_{Z \cap \ell}) = 3$, then for any plane $P \supset \ell$ we had $\ell(\mathcal{O}_{Z \cap P}) \geq 3$. Since ℓ is the tangent line, it follows $\ell(\mathcal{O}_{Z \cap \ell}) = 2$. Consider now $\mathcal{C} = (x, y)\mathcal{O}_Z$ and the exact sequence

$$0 \longrightarrow \mathfrak{m}\mathcal{C} \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}/\mathfrak{m}\mathcal{C} \longrightarrow 0.$$

Then either $\mathfrak{m}\mathcal{C} = 0$ or has length 1. In the first case $\mathfrak{m}\bar{x} = 0$, $\mathfrak{m}\bar{y} = 0$, which was excluded in 2). Then $\mathcal{C}$ is generated, say, by $\bar{x}$, and there exists a scalar $\lambda \in k$ such that $\bar{y} - \lambda\bar{x} \in \mathfrak{m}\mathcal{C}$. But then $\mathfrak{m}(\bar{y} - \lambda\bar{x}) = 0$ since $\mathfrak{m}\mathcal{C}$ has length 1. This means that $(\bar{y} - \lambda\bar{x})\mathcal{O}_Z$ has length 1 and hence $\mathcal{O}_Z/(y - \lambda x)\mathcal{O}_Z$ has length 3.

3.4 Lemma: *Let $\mathcal{F} = \mathcal{F}(A) \in M(0, 2, 4)$ be reflexive and not in D_4', and let Z be its scheme of singular points. If $p \in Z$ is not a simple point (i.e. not of length 1) then:*

1) $\dim T_p Z = 1$

2) there are homogeneous coordinates $z_0, \ldots, z_3$ s.t.

   (i) $\{z_1 = z_2 = z_3 = 0\} = \{p\}$

   (ii) $\{z_2 = z_3 = 0\} = \ell$ *is the tangent line generated by $T_p Z$*

   (iii) A *is equivalent to* $\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & f_1 & f_2 & f_3 \end{pmatrix}$ *where f_i are linear forms vanishing in p.*

<u>Proof:</u> By proposition 3.2 *rank* $A(p) = 1$. We may assume that

$$A(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, we can choose the coordinates so that $z_0(p) = 1$, $z_i(p) = 0$ for $i \neq 0$, and such that A is equivalent to

$$\begin{pmatrix} g_0 & g_1 & g_2 & g_3 \\ z_0 & f_1 & f_2 & f_3 \end{pmatrix}$$

with g_i, f_j linear combinations in z_1, z_2, z_3 only. Choosing z_1, z_2, z_3 as local coordinates at p with $z_0 = 1$, the Fitting ideal of that matrix is generated by $g_1 - g_0 f_1$, $g_2 - g_0 f_2$, $g_3 - g_0 f_3$, which at the same time is the ideal of Z at p. If g_1, g_2, g_3 were linearly independent then p would be a simple point. Hence, we may assume (after going to an equivalent matrix) that $g_3 = 0$. Since $\mathcal{F}$ and A are stable,

g_0, g_1, g_2 have to be independent. It follows that the tangent space has dimension 1 with generated line $\ell = \{g_2 = g_3 = 0\}$. We can now replace g_0, g_1, g_2 by z_1, z_2, z_3.

3.4.1 Remark: The condition dim $T_p Z = 1$ implies that Z is curvilinear at p, i.e. there is a regular coordinate system x, y, z at p such that $\mathcal{O}_{Z,p} = \mathcal{O}_p / (x, y, z^m) \mathcal{O}_p$ for $m \geq 2$.

3.4.2 Corollary: *With the same assumption as in lemma 3.4: let $p, q \in Z$ or (p, ℓ) a double point with tangent line ℓ, $q \in \ell$. Then*

$$rank \begin{pmatrix} A(p) \\ A(q) \end{pmatrix} \leq 2$$

<u>Proof</u>: In the first case $rank\ A(p) = rank\ A(q) = 1$ and hence the result. In the second case there are coordinates such that (i), (ii), (iii) are satisfied. Then, since $z_2(q) = z_3(q) = 0$, we get

$$\begin{pmatrix} A(p) \\ A(q) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ z_1(q) & 0 & 0 & 0 \\ * & * & * & * \end{pmatrix},$$

which is a matrix of $rank \leq 2$.

Notation: In the following we shall give normal forms for the presenting matrices of the reflexive sheaves in $M(0, 2, 4)$, depending on the structure of the 0–dimensional schemes of singular points. We use the following notation: If $z_0, \ldots, z_3 \in V^*$ is a basis (homogeneous coordinates) and $e_0, \ldots, e_3 \in V$ the dual basis, $p_i = < e_i > \in \mathbb{P}V = \mathbb{P}_3$ are the corresponding points, or vice versa. We write

$$Z(\mathcal{F}) = Supp\ \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}),$$

and endow it with the structure given by the Fitting ideal of A.

3.5 The case of 4 simple points.

Let $Z = \{p_0, \ldots, p_3\}$. By remark 3.3.1 we can assume that Z is not contained in a plane. Let $z_0, \ldots, z_3 \in V^*$ be a fixed basis corresponding to $p_0, \ldots p_3$. The following lemma is a result of Chang [C], lemma 2.14.

3.5.1 Lemma: *For any reflexive sheaf $\mathcal{F}$ in $M(0, 2, 4)$ with $Z(\mathcal{F}) = Z$ there is a coefficient $\lambda \in k \backslash \{0, 1\}$ such that $\mathcal{F}$ is presented by the matrix*

$$A_\lambda = \begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & \lambda z_1 & z_2 & z_3 \end{pmatrix} \qquad \lambda \in k \backslash \{0, 1\}.$$

3.5.2 Moreover, it is easy to check that $A_\lambda \sim A_\mu$ if and only if $\lambda = \mu$. Thus, the isomorphism classes of μ–stable reflexive sheaves $\mathcal{F}$ with given 4 singular points are classified by $\lambda \in k \backslash \{0,1\}$. The geometric meaning of λ is the following.

3.5.3 Let C be any smooth rational cubic curve containing Z. Then for any isomorphism $\mathbb{P}_1 \simeq C$ the 4 points on $\mathbb{P}_1$ define a j–invariant

$$j(C) = j_C(p_0, \ldots, p_3)$$

independent of the isomorphism. Chang proved in [C], proposition 2.15, that for two such cubics C, C' the following are equivalent:

(i) $j(C) = j(C')$

(ii) C and C' belong to the same reflexive sheaf $\mathcal{F}$ in $M(0,2,4)$ (as zero schemes of sections of $\mathcal{F}(1)$).

We prove in addition the more precise

3.5.4 Proposition: *Let A_λ be the above normal form w.r.t the 4 points and with $\lambda \neq 0, 1$. Then a smooth rational cubic curve C through the 4 points belongs to $\mathcal{F}(A_\lambda)$ if and only if*

$$j(C) = j(\lambda) = 2^8(\lambda^2 - \lambda + 1)^3/\lambda^2(1-\lambda)^2.$$

3.5.5 Remark: Such a cubic belongs to different non-isomorphic sheaves $\mathcal{F}(A_\lambda), \mathcal{F}(A_\mu)$ in case $\lambda \neq \mu$ but $j(\lambda) = j(\mu)$.

<u>Proof:</u> For given λ we choose a cubic curve belonging to $\mathcal{F}(A_\lambda)$ and calculate its j–invariant as a function in λ. The rest follows from Chang's result. By the diagram in 3.1 any such curve is defined by the Fitting ideal of a 2×3 matrix $A_\lambda \circ S$ where S is a 4×3 matrix of scalars.

We choose the curve C defined by the matrix

$$A_\lambda \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} z_0 + z_2 & z_0 + z_1 & z_1 + z_2 \\ z_2 + z_3 & \lambda z_1 + z_3 & \lambda z_1 + z_2 + z_3 \end{pmatrix}.$$

It is easy to verify that $< s, t > \overset{f}{\mapsto} < f_0, f_1, f_2, f_3 >$ with

$$
\begin{aligned}
f_0 &= t(s-t)(s-\lambda t) \\
f_1 &= st(s-t) \\
f_2 &= st(s-\lambda t) \\
f_3 &= 2s(s-t)(s-\lambda t)
\end{aligned}
$$

gives an isomorphism $\mathbb{P}_1 \xrightarrow{\approx} C$ with $f<0,1>=p_0$, $f<1,1>=p_2$, $f<1,0>=p_3$, $f<\lambda,1>=p_1$. By definition the j–invariant of C does not depend on the order of the points and thus is just $j(\lambda) = 2^8(\lambda^2-\lambda+1)^3/\lambda^2(\lambda-1)^2$.

3.6 The case of a double point and 2 simple points

In this case we choose and fix coordinates $z_0, \ldots, z_3$ such that $Supp\ Z = \{p_0, p_2, p_3\}$ with double point p_0 and tangent line $\ell = \{z_2 = z_3 = 0\}$. Note that by remark 3.3.1 the line and the points are not contained in a plane.

3.6.1 Lemma: *For any reflexive sheaf $\mathcal{F} \in M(0,2,4)$ with $Z(\mathcal{F}) = Z$ (as zero-dimensional schemes) there is a coefficient $\lambda \in k\backslash\{0\}$ such that $\mathcal{F}$ is presented by the matrix*

$$
A_\lambda = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & 0 & \lambda z_3 & z_1 \end{pmatrix}.
$$

Moreover, $A_\lambda \sim A_\mu$ if and only if $\lambda = \mu$.

<u>Proof:</u> By lemma 3.4 there are coordinates $w_0, \ldots, w_3$ with $p_0 = \{w_1 = w_2 = w_3 = 0\}$, $\ell = \{w_2 = w_3 = 0\}$ and

$$
A \sim \begin{pmatrix} w_1 & w_2 & w_3 & 0 \\ w_0 & f_1 & f_2 & f_3 \end{pmatrix}, \quad f_i(e_0) = 0.
$$

Then the coordinate transformation has the form

$$
\begin{aligned}
w_0 &= \alpha z_0 + \alpha_{01} z_1 + \ldots \\
w_1 &= \beta z_1 + \ldots \\
w_2 &= \alpha_{22} z_2 + \alpha_{23} z_3 \\
w_3 &= \alpha_{32} z_2 + \alpha_{33} z_3
\end{aligned}
$$

We may assume that $\alpha = 1$ and by dividing the first row of A by β, also $\beta = 1$. Furthermore by taking linear combinations of the second and third column of A, we can assume that $w_2 = z_2$, $w_3 = z_3$, and that $w_1 = z_1$. Thus, we arrive at

$$
A \sim \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 + az_1 + bz_2 + cz_3 & f_1 & f_2 & f_3 \end{pmatrix}.
$$

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By adding a multiple of the first row, we can assume $a = 0$. Furthermore, the conditions $rk\ A(e_2) = rk\ A(e_3) = 1$ imply that $b = c = 0$, $f_2(e_2) = f_3(e_2) = 0$ and $f_1(e_3) = f_3(e_3) = 0$.

Hence, we arrive at

$$A \sim \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & \alpha z_1 + \beta z_2 & \gamma z_1 + \delta z_3 & \rho z_1 \end{pmatrix}.$$

Here we can cancel ρ (which must be non-zero) and then $\alpha z_1, \gamma z_1$. After subtracting β times the first row and cancelling again $-\beta z_1$, we obtain the above normal form. Clearly $\lambda \neq 0$, otherwise A could not be stable. The second statement is an easy exercise.

3.7   The case of two double points

In this case again we choose and fix coordinates $z_0, \ldots, z_3$ such that $Supp\ Z = \{p_0, p_3\}$ and the tangent lines ℓ_0, ℓ_3 through p_0, p_3 respectively are given by $\ell_0 = \{z_2 = z_3 = 0\}$, $\ell_3 = \{z_0 = z_1 = 0\}$. Note that by remark 3.3.1 the lines must be skew.

3.7.1   Lemma: *For any reflexive sheaf $\mathcal{F} \in M(0, 2, 4)$ with $Z(\mathcal{F}) = Z$ there exists $\lambda \in k \backslash \{0\}$ such that $\mathcal{F}$ is presented by the matrix*

$$A_\lambda = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & 0 & \lambda z_2 & z_1 \end{pmatrix}.$$

Moreover, $A_\lambda \sim A_\mu$ if and only if $\lambda = \mu$.

The proof goes along the same way as in 3.6 by using lemma 3.4 and corollary 3.4.2 for the second double point.

3.8   The case of a triple and a singular point

Let $Supp\ Z = \{p_0, p_3\}$ where p_0 is a point of length 3 with 1–dimensional tangent space. Let $\ell \subset P$ be the tangent line and the plane which contains the triple point, and assume that $p_3 \notin P$, see. proposition 3.3. We choose and fix coordinates $z_0, \ldots, z_3$ such that p_0, p_3 are the corresponding coordinate points, $\ell = \{z_2 = z_3 = 0\}$ and $P = \{z_3 = 0\}$.

3.8.1   Lemma: *For any reflexive sheaf $\mathcal{F}$ in $M(0, 2, 4)$ with $Z(\mathcal{F}) = Z$ there are coefficients $\lambda \neq 0$, α in k such that $\mathcal{F}$ is presented by*

$$A_{\lambda,\alpha} = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & \lambda z_1 & \alpha z_3 & z_2 \end{pmatrix}$$

Moreover, two such matrices are equivalent if and only if the coefficients coincide.

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<u>Proof</u>: If we proceed as in the proof of lemma 3.6.1 using only $rk\ A(e_3) = 1$, we arrive at a presenting matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 + az_2 & bz_1 + cz_2 & f & \alpha z_1 + \beta z_2 \end{pmatrix}$$

where f is arbitrary in z_1, z_2, z_3. If $\alpha \neq 0$ we can assume $\alpha = 1$ and $b = 0$ and $f = dz_2 + ez_3$. But in this case the computation of the Fitting ideal at p_0 shows that p_0 could only be a point of length 2. Hence, $\alpha = 0$ and we can assume $\beta = 1, a = 0, c = 0$, and $f = \gamma z_1 + \delta z_3$. Furthermore, $b \neq 0$, for otherwise the matrix would be singular along a conic in $\{z_2 = 0\}$. Finally $\gamma = 0$, because the Fitting ideal of the matrix now contains $bz_0 z_3 - \gamma z_0 z_2$, so that the equation of P becomes $bz_3 - \gamma z_2$. So we have reduced the matrix to the desired form.

Again the second statement follows from a straightforward computation.

3.8.2 Remark:
The Fitting ideal of the normal form in lemma 3.8.1 is generated by $z_0 z_2 - \lambda z_1^2$, $z_0 z_3$, $z_1 z_3$, $z_2 z_3$, $z_1 z_2$, z_2^2 which does not contain the parameter α. We denote it by I_λ. However, λ is determined uniquely by this ideal. With a little more effort we can prove:

Any 0–dimensional scheme Z with the given geometric data as in 3.8 (and fixed coordinates) is given by the ideal I_λ for some $\lambda \neq 0$. We omit the proof but use this result in theorem 4.5.

3.9   The case of a 4–fold point

Let Z be a 4–fold structure on p. By lemma 3.3.2 and lemma 3.4 we assume that Z is not contained in a plane, has a tangent line ℓ and an osculating plane P. We fix coordinates so that $p \in \ell \subset P$ have ideals (z_1, z_2, z_3), (z_2, z_3), (z_3) respectively.

3.9.1 Lemma:
For any reflexive sheaf $\mathcal{F}$ in $M(0, 2, 4) \backslash D_4'$ with $Z(\mathcal{F}) = Z$ there are coefficients $\lambda, \mu \neq 0$ and α, β such that $\mathcal{F}$ is presented by the matrix

$$A_{\alpha,\beta,\gamma,\mu} = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 + \alpha z_1 + \beta z_2 & \lambda z_1 & \mu z_2 & z_3 \end{pmatrix}.$$

Two such matrices are equivalent if and only if the quadruples of coefficients are the same.

<u>Proof</u>: As in the proof of lemma 3.6.1 we can assume that the presenting matrix of $\mathcal{F}$ is

$$A = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 + f_0 & f_1 & f_2 & f_3 \end{pmatrix}$$

where $f_0, \ldots, f_3$ are combinations of z_1, z_2, z_3 only. Since $z_i f_3$ is in the ideal of Z, it follows that $\mathfrak{m}_p \bar{f}_3 = 0$ in $\mathcal{O}_{Z,p}$, where $\mathfrak{m}_p$ is the maximal ideal. It follows from the

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proof of lemma 3.3.2 that f_3 is the equation of the osculating plane. Hence we can assume that $f_3 = z_3$. Now we can reduce the matrix to the form

$$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 + \alpha z_1 + \beta z_2 & \lambda z_1 & \gamma z_1 + \mu z_2 & z_3 \end{pmatrix}$$

as in the previous cases. Here $\lambda, \mu \neq 0$, for otherwise we would get a second point in $Supp\ Z$ by putting the last row to zero.

Furthermore $\gamma = 0$. For if $\gamma \neq 0$, then a computation shows that $< \lambda\mu^2 + \alpha\gamma\mu - \beta\gamma^2, -\mu\gamma, \gamma^2, 0 >$ is a second point in $Supp\ Z$. This proves the first statement of the lemma. The second is again straightforward.

3.9.2 Remark: The Fitting ideal of the normal form in lemma 3.9.1 is generated by $z_0 z_2 - \lambda z_1^2 + \alpha z_1 z_2$, $z_0 z_3 - \mu z_1 z_2$, $z_1 z_3$, $z_2 z_3$, z_3^2, z_2^2 and does not contain the parameter β. We denote it by $I_{\lambda,\mu,\alpha}$. The three coefficients are uniquely determined by the ideal.

As in remark 3.8.2 we can prove: For any 4–fold point Z with the geometric data as in 3.9 (with fixed coordinates) there are coefficients $\lambda, \mu \neq 0, \alpha$ such that Z is given by the ideal $I_{\lambda,\mu,\alpha}$. This will be used in theorem II, 4.5.

3.9.3 Remark: If the reflexive sheaf belongs to D_4' then $\mathcal{O}_Z = \mathcal{O}/\mathfrak{m}(p)^2$ where $\mathfrak{m}(p)$ is the ideal sheaf of the point p.

3.10 Divisors of jumping lines

We consider now the incidence projections for points in lines from the flag variety **F**:

$$\mathbb{P}_3 \xleftarrow{p} \mathbf{F} \xrightarrow{q} G_2 V =: G,$$

and denote the direct image functor by $R^* = R^* q_* \circ p^*$. The tautological sub– and quotient bundle is denoted by S and Q, coming together with the exact sequence $0 \to S \to V \otimes \mathcal{O}_G \to Q \to 0$. If one identifies the fibre $q^{-1}(U)$ of a 2–subspace $U \in G$ with the line $\mathbb{P}U \subset \mathbb{P}V$, one easily finds that $R^1\mathcal{O}(-2) = \Lambda^2 S$, $R^1\mathcal{O}(-3) = S \otimes \Lambda^2 S$, etc. Applying this to a presentation

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{A^\bullet} 4\mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$$

we get $R^\circ \mathcal{F} = 2\Lambda^2 S (= \mathcal{O}_G(-1) = $ the restriction of $\mathcal{O}_{\mathbb{P}\Lambda^2 V}(-1)$ to G), $R^\circ \mathcal{F}(-1) = 0$, and

$$0 \longrightarrow 2S \otimes \Lambda^2 S \xrightarrow{\tilde{A}} 4\Lambda^2 S \longrightarrow R^1 \mathcal{F}(-1) \longrightarrow 0.$$

Since for a line ℓ the group $H^1(\mathcal{F} \otimes \mathcal{O}_\ell(-1))$ is zero or not in case $\mathcal{F} \otimes \mathcal{O}_\ell \simeq 2\mathcal{O}_\ell$ or $\simeq \mathcal{O}_\ell(-\nu) \oplus \mathcal{O}_\ell(\nu)$ with $\nu > 0$, we call

$$J(\mathcal{F}) = Supp\ R^1 \mathcal{F}(-1)$$

the variety of jumping lines of $\mathcal{F}$.

3.10.1 Lemma: *For any $\mathcal{F}$ in $M(0,2,4)$ the variety $J(\mathcal{F})$ of jumping lines is a quadric hypersurface in the Grassmannian G.*

<u>Proof</u>: Both sheaves in the presentation of $R^1\mathcal{F}(-1)$ have rank 4. Therefore, the equation of $J(\mathcal{F})$ is given by det $\tilde{A} : \mathcal{O}_G(-2) = (\Lambda^2 S)^{\otimes 2} = \Lambda^4(2S) \rightarrow \Lambda^4(4\mathcal{O}_G) = \mathcal{O}_G$, which is a quadric form on G. (Note that in all the cases $J(\mathcal{F})$ is not smooth.)

3.10.2 The quadric form det $\tilde{A}$ can be computed as follows. Let $e_0,\ldots,e_3 \in V$ be a basis and $p_{01},\ldots,p_{23} \in \Lambda^2 V^*$ the basis dual to $e_i \wedge e_j$, i.e. a set of (Plücker) coordinates for $\mathbb{P}_5 = \mathbb{P}\Lambda^2 V$. Let $G_{01} \subset G$ be the open set $\{p_{01} \neq 0\}$. Then in local coordinates $p_{02}, p_{03}, p_{12}, p_{13}$ of G_{01}, where $p_{01} = 1$, the two sections $x = e_0 - p_{12}e_2 - p_{13}e_3$ and $y = e_1 + p_{02}e_2 + p_{03}e_3$ give a trivialization $2\mathcal{O}_G \simeq S$ over G_{01}. The composed homomorphism $\hat{A}$

$$4\mathcal{O}_G \simeq 2S \xrightarrow{\tilde{A}} 4\mathcal{O}_G$$

over G_{01} can now be expressed by

$$\hat{A} = \begin{pmatrix} f_1(x) & \cdots & f_4(x) \\ g_1(x) & \cdots & g_4(x) \\ f_1(y) & \cdots & f_4(y) \\ g_1(y) & \cdots & g_4(y) \end{pmatrix}$$

if $A^* = \begin{pmatrix} f_1,\ldots,f_4 \\ g_1,\ldots,g_4 \end{pmatrix}$ since it is induced by $k^2 \otimes S \hookrightarrow k^2 \otimes V \otimes \mathcal{O}_G \xrightarrow{A^*} k^4 \otimes \mathcal{O}_G$.

3.10.3 Example: Let $A^* = \begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & \lambda z_1 & z_2 & z_3 \end{pmatrix}$ be the normal form of 3.5. in case of 4 singular points. Then

$$\det \hat{A} = \det \begin{pmatrix} 1 & 0 & -p_{12} & 0 \\ 0 & 0 & -p_{12} & -p_{13} \\ 0 & 1 & p_{02} & 0 \\ 0 & \lambda & p_{02} & p_{03} \end{pmatrix} = \lambda p_{02}p_{13} + p_{03}p_{12} - p_{02}p_{13}.$$

Going back to homogeneous coordinates (we put $p_{01} = 1$ in 3.10.2) and reducing modulo the equation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$ of G, we get here

$$\det \hat{A} = \lambda p_{02}p_{13} - p_{01}p_{23}.$$

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Furthermore, it is now easy to determine the singular locus of $J(\mathcal{F})$ by checking the Jacobi matrix of $\det \hat{A}$ and the equation of G. Here $Sing\, J(\mathcal{F})$ consists of 6 isolated points, namely the 6 lines $\ell_{ij} = <e_i \wedge e_j> \in G$, which join the 4 points of $\mathcal{F}(A)$. In general we have:

3.11 Lemma: *Let $\mathcal{F}$ be a reflexive sheaf in $M(0,2,4) \backslash D_4'$. Then*

(i) *if p is a singular point of $\mathcal{F}$ then the α-plane P_p of lines through p is contained in $J(\mathcal{F})$.*

(ii) *If p, q are two singular points of $\mathcal{F}$ or p is a singular point of length ≥ 2 and q is on the tangent line of p, then the line ℓ joining p, q or the tangent line, is a singular point of $J(\mathcal{F})$.*

(iii) *If $\mathcal{F} \in D_4'$ with centre p, then $P_p \subset Sing\, J(\mathcal{F})$.*

The proof of (i), (ii) follows easily from the description of $\hat{A}$ and lemma 3.4.2. (iii) follows from $A(p) = 0$.

3.12 In the following **table** we collect the normal forms and the statements on $J(\mathcal{F})$, which are obtained as in example 3.10.3. We use the following **notations**:

$e_0, \ldots, e_3 \in V$ and $z_0, \ldots, z_3 \in V^*$ are dual bases and $p_i = <e_i> \in \mathbb{P}V$ are the corresponding points. ℓ_{ij} (resp. P_{ijk}) is the line (resp. plane) spanned by p_i, p_j (resp. p_i, p_j, p_k).

If Z is a 0–dimensional subscheme with 1–dimensional tangent space at $p \in Z$, we denote by $t_p Z$ the line in $\mathbb{P}_3$ spanned by it and we write length (p) for length $(\mathcal{O}_{Z,p})$.

Let us recall that $Z(\mathcal{F}) = Supp\, \mathcal{E}xt^1(\mathcal{F}, \mathcal{O})$ is the singular locus of the sheaf $\mathcal{F}$.

$Z = Z(\mathcal{F})$ or $\mathcal{F}$	normal form	$Sing\ J(\mathcal{F})$
$p_0, \ldots, p_3$ 3.5	$\begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & \lambda z_1 & z_2 & z_3 \end{pmatrix}$ $\lambda \neq 0, 1$	6 points $\ell_{01}, \ldots, \ell_{23}$
$Supp\ Z = \{p_0, p_1, p_2\}$ $t_{p_0} Z = \ell_{01}$ 3.6	$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & 0 & \lambda z_3 & z_1 \end{pmatrix}$ $\lambda \neq 0$	ℓ_{01}, ℓ_{23} and pencil of lines through p_0 in P_{012}
$Supp\ Z = \{p_0, p_3\}$ $t_{p_0} Z = \ell_{01}$ $t_{p_3} Z = \ell_{23}$ 3.7	$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & 0 & \lambda z_2 & z_1 \end{pmatrix}$ $\lambda \neq 0$	ℓ_{01}, ℓ_{23} and pencils of lines through p_0 (resp.p_3) in P_{023} (resp.P_{013})
$Supp\ Z = \{p_0, p_3\}$ $t_{p_0} Z = \ell_{01}$ length $(p_0) = 3$ 3.8	$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & \lambda z_1 & \alpha z_3 & z_2 \end{pmatrix}$ $\lambda \neq 0$	pencil of lines through $p_0 \in P_{013}$
$Supp\ Z = \{p_0\}$ $t_{p_0} Z = \ell_{01}$ oscul. plane P_{012} 3.9	$\begin{pmatrix} z_1 & & z_2 & z_3 & 0 \\ z_0 + \alpha z_1 + \beta z_2 & \lambda z_1 & \mu z_2 & z_3 \end{pmatrix}$ $\lambda, \mu \neq 0$	pencil of lines through p_0 in oscul. plane
$Supp\ Z = \{p_0\}$ $\mathcal{F} \in D_4'$ 2.4	$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}$ coordinates depend on $\mathcal{F}$	net P_{p_0} of all lines through p_0
$\mathcal{F} \in S_1 \setminus S_0$ 1.5	$\begin{pmatrix} z_0 & z_2 & z_3 & 0 \\ z_1 & 0 & z_2 & z_3 \end{pmatrix}$ coordinates depend on $\mathcal{F}$	all jumping lines of $\mathcal{E}$ meeting the line ℓ
$\mathcal{F} \in S_2 \setminus S_0$ 1.6	$\begin{pmatrix} z_0 & z_1 & z_3 & 0 \\ z_1 & z_2 & 0 & z_3 \end{pmatrix}$ coordinates depend on $\mathcal{F}$	net of all lines in the plane of the conic of $\mathcal{F}$

4 Comparison with the Hilbert scheme of four points

The map $[\mathcal{F}] \to Z(\mathcal{F})$ for reflexive sheaves gives us a rational morphism from $M(0,2,4)$ to the Hilbert scheme $Hilb^4(\mathbb{P}_3)$ of 0–dimensional subschemes of length 4 in $\mathbb{P}_3$. Using the normal forms of the last section we describe the fibres in more detail and show that the generic fibre is $\mathbb{P}_1$. Moreover, we study the subvarieties of reflexive sheaves whose singular locus of length 4 has a multiple structure.

4.1 Some facts about $\mathbf{H} = \mathbf{Hilb^4(\mathbb{P}_3)}$.

Let $H'_4 \subset H$ be the subvariety of 4–fold points p with structure sheaf $\mathcal{O}_p/\mathfrak{m}_p^2$. It is known, see [LB], [I], [F] that

- (i) H is irreducible of dimension 12,

- (ii) H'_4 is the singular locus of H.

We consider the following subvarieties of H. Let $H_{t2} \subset H_{t1}$ be the subvarieties of all $Z \in H$ such that there is a point $p \in Z$ with $\dim T_p Z \geq 2$ (resp. ≥ 1), and let $H_{p\ell}$ be the subvariety of $Z \in H$ which are contained in a plane. Finally, let H_2^0, $H_{2,2}^0$, H_3^0, $H_4^0 \subset H \backslash H_{t2} \cup H_{p\ell}$ be the subvarieties with exactly one point of length 2 (resp. two points of length 2, resp. one point of length 3, resp. one point of length 4), and let H_2, $H_{2,2}, H_3$, H_4 be their closures in H.

4.1.1 Proposition: $H'_4 \subset H_4 \subset H_3 \subset H_2$ and $H_4 \subset H_{2,2} \subset H_2$.

Proof: The inclusions $H_3 \subset H_2$ and $H_{2,2} \subset H_2$ follow from the fact that 2– and 3–fold points can be split under deformation in their planes. $H'_4 \subset H_4$ and $H_4 \subset H_3$ is easy by using the normal forms of remarks 3.8.2 and 3.9.2, or follow from $D'_4 \subset D_4 \subset D_3$ below.

We are left to prove $H_4 \subset H_{2,2}$. Let $Z \in H_4^0$. By remark 3.9.2 we can assume that coordinates are chosen such that the ideal sheaf $\mathcal{I}_Z$ of Z is generated in degree 2 by the six forms as in remark 3.9.2, and that Z is supported on p_0. It is easy to see that then also z_1^4 is a section of $\mathcal{I}_Z(4)$ and that, moreover, $\mathcal{I}_{Z,p_0}$ is generated by

$$x_2 - \lambda x_1^2 + \alpha x_1 x_2, \quad x_3 - \mu x_1 x_2, \quad x_1^4,$$

where $x_i = z_i/z_0$ are the local coordinates, $\mu\lambda \neq 0$. Let $Z_t \subset \{z_0 \neq 0\} \subset \mathbb{P}_3$ be the family defined by the first two equations and $x_1^2(x_1 - t)^2$. This is a flat family of deformations of $Z = Z_0$ and defines a germ of a curve in H. For $t \neq 0, Z_t$ consists of two double points. This proves that $H_4^0 \subset H_{2.2}$ and hence also $H_4 \subset H_{2,2}$.

4.2 The rational morphism M $\cdots$ > H

Let $M_r \subset M = M(0, 2, 4)$ the the open subscheme of reflexive sheaves $\mathcal{F}$, and let $X \subset G_4(k^2 \otimes V^*)^{ss}$ be its inverse image. The dual of the universal homomorphism (UF) in 1.1 gives us a subscheme $\tilde{Z} \subset X \times \mathbb{P}_3$ by applying the functor Λ^2:

$$\Lambda^2(\Lambda) \boxtimes \mathcal{O}_{\mathbb{P}_3}(-2) \longrightarrow \mathcal{O}_{X \times \mathbb{P}_3} \longrightarrow \mathcal{O}_{\tilde{Z}} \longrightarrow 0.$$

For each $\Lambda \in X$ the fibre $\tilde{Z}(\Lambda) \subset \mathbb{P}_3$ is the singular locus of the reflexive sheaf $\mathcal{F}(\Lambda)$ and hence a point of H. Moreover, $\tilde{Z}$ is flat over X. This gives us a unique morphism $X \to H$ such that $\tilde{Z}$ becomes the pullback of the universal scheme over H. This morphism is $SL(2)$–equivariant since Λ^2 kills the $SL(2)$–action and, therefore, we obtain a morphism

$$M_r \xrightarrow{h} H,$$

whose underlying map is $[\mathcal{F}] \to Z(\mathcal{F}) = Supp\ \mathcal{E}xt^1(\mathcal{F}, \mathcal{O})$.

We let $\tilde{M} \subset M \times H$ be the closure of the graph of h and thus get two morphisms

$$M \xleftarrow{\sigma} \tilde{M} \xrightarrow{\eta} H.$$

Clearly σ is birational and an ismorphism over M_r and h maps $M \backslash S_0 \cup S_1 \cup S_2 \cup D_4'$ into $H \backslash H_{t2} \cup H_{p\ell}$ by proposition 3.3, lemma 3.4. It follows from proposition 4.4 that this map is also surjective. D_4' is mapped onto H_4'.

4.2.1 Remark: For a point $(([\mathcal{F}]), Z) \in \tilde{M}$ we necessarily have $Z \subset Z(\mathcal{F})$ if $\mathcal{F}$ is stable and $Z(\mathcal{F})$ is given by the Fitting ideal of the matrix A. If $\mathcal{F}$ is only semistable and $[\mathcal{F}] = [\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}]$ then $Z \subset \ell \cup \ell'$. Note that for a non–trivial extension $0 \to \mathcal{I}_\ell \to \mathcal{F} \to \mathcal{I}_{\ell'} \to 0$ the variety $Z(\mathcal{F})$ consists of ℓ and two points or a double point on ℓ', see remark 1.8.

4.3 Notation: Let D_2^0, $D_{2,2}^0$, D_3^0, $D_4^0 \subset M \backslash S_0 \cup S_1 \cup S_2 \cup D_4'$ be the inverse images of H_2^0, $H_{2,2}^0$, H_3^0, H_4^0 under h respectively and let D_2, $D_{2,2}$, D_3, D_4 be their closures in M.

4.4 Proposition: *The restriction of h induces surjective morphisms*

$$D_2^0 \longrightarrow H_2^0, \ D_{2,2}^0 \longrightarrow H_{2,2}^0, \ D_3^0 \longrightarrow H_3^0, \ D_4^0 \longrightarrow H_4^0.$$

Moreover, the first and second are fibrations with fibre k^* and the third and fourth are fibrations with fibre k.

<u>Proof:</u>   The surjectivity in the first two cases is obvious by 3.6 and 3.7 since double structures on points are determined by the tangent lines. Then the normal forms give us sheaves over a given 0–dimensional scheme in H_2^0 or $H_{2,2}^0$. In the last two cases surjectivity follows from the remarks 3.8.2 and 3.9.2: the ideal of any Z is obtained by a normal form. The structure of a fibration follows directly from the statements in lemmas 3.6.1, 3.7.1, 3.8.1, 3.9.1. In the case of lemma 3.8.1 the coefficient α does not occur in the Fitting ideal and in case 3.9.1 the same is true for the coefficient β. With some more effort one should be able to verify that the fibrations are in fact locally trivial.

4.5   Theorem II:

  a) *The varieties* D_2, $D_{2,2}$, D_3, D_4 *are all irreducible and smooth along* D_2^0, $D_{2,2}^0$, D_3^0, D_4^0 *respectively.*

  b) $D_4' \subset D_4 \subset D_3 \subset D_2$ *and* $D_4 \subset D_{2,2} \subset D_2$.

  c) dim $D_4 = 10$, dim $D_3 =$ dim $D_{2,2} = 11$, dim $D_2 = 12$.

<u>Proof:</u>   The subvariety H_2^0 consists of schemes Z which are determined by 3 points and a line through one of them, not contained in a plane with the other two. It follows that H_2^0 is an orbit under the action of $PGL(4)$ and has dimension 11. By proposition 4.4 the statements a), c) follow for D_2. A similar argument works in the case of $D_{2,2}$. Now by remark 3.8.2 a scheme $Z \in H_3^0$ is in 1 : 1 correspondence with a tuple (p, ℓ, P, q, λ), where ℓ is the tangent line of p and P the plane containing the 3–fold point p, $\lambda \neq 0$. Again we conclude that H_3^0 is smooth, irreducible of dimension $3 + 3 + 2 + 1 + 1 = 10$. Then D_3 satisfies a), c). An analogous argument applies to H_4^0 by using 3.9.2. Here P becomes the osculating plane.

In order to derive the inclusions we consider 1–parameter deformations of the normal forms: $D_4' \subset D_4$ and $D_4 \subset D_3$ follow from the families

$$\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ tz_0 & z_1 & z_2 & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 & & z_2 & z_3 & tz_3 \\ z_0 + \alpha z_1 + \beta z_2 & \lambda z_1 & \mu z_2 & z_3 \end{pmatrix}$$

which give points in D_4^0 resp. D_3^0 for $t \neq 0$.

The inclusions $D_4 \subset D_{2,2}$ and $D_3, D_{2,2} \subset D_2$ follow from the corresponding inclusions in H and the dimension of the fibres of $M_r \overset{h}{\to} H$. Let us prove this in the less obvious case $D_4 \subset D_{2,2}$. By proposition 4.4 the fibres of h over H_4^0 and $H_{2,2}^0$ are isomorphic to k^*. Since $D_{2,2} = \overline{h^{-1}(H_{2,2}^0)}$, we consider the restriction $\alpha = h|D_{2,2}$. As for any dominant morphism we obtain for $D_{2,2} \overset{\alpha}{\to} H_{2,2}$: any component Y of $\alpha^{-1}(H_4^0)$ has

$$\dim Y \geq \dim H_4^0 + \dim D_{2,2} - \dim H_{2,2} = \dim H_4^0 + 1.$$

Now $\alpha^{-1}(H_4^0) \subset D_{2,2} \cap h^{-1}(H_4^0) \subset D_{2,2} \cap D_4 \subset D_4$, and therefore we obtain

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$$\dim H_4^0 + 1 \leq \dim Y \leq \dim D_4 = \dim H_4^0 + 1.$$

Since D_4 is irreducible, $Y = D_{2,2} \cap D_4 = D_4$, hence $D_4 \subset D_{2,2}$.

4.6 Fibres of $\tilde{M} \xrightarrow{\eta} H$

As a last part we discuss the fibres of the projection η in the different cases. It turns out that all the fibres over $H \setminus H_{t2} \cup H_{p\ell}$ are isomorphic to $\mathbb{P}_1$.

4.6.1 Case of 4 simple points

Let $Z = \{p_0, \ldots, p_3\} \in H$. Then by 3.5.1 the reflexive sheaves over Z are parametrized by $\lambda \in k \setminus \{0, 1\}$ by the normal forms $\begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & \lambda_{z_1} & z_2 & z_3 \end{pmatrix}$.

We consider now the morphism $\mathbb{P}_1 \to \eta^{-1}(Z) \subset \tilde{M}$ defined by $<\lambda, \mu> \to ([\mathcal{F}_{\lambda:\mu}], Z)$ where $\mathcal{F}_{\lambda:\mu}$ is presented by

$$\begin{pmatrix} z_0 & \mu z_1 & z_2 & 0 \\ 0 & \lambda z_1 & z_2 & z_3 \end{pmatrix}$$

and Z is the given scheme. For $<\lambda, \mu> \neq <0, 1>, <1, 0>, <1, 1>$ clearly $\varphi <\lambda, \mu> \in \eta^{-1}(Z) \cap M_r$ and φ is an isomorphism away from the three points. It follows that φ is an isomorphism at all. The three extra points in $\eta^{-1}(Z) \subset M \times H$ consist of the three possible pairs of lines through the four points, namely

$$((\ell_{01}, \ell_{23}), Z), \quad ((\ell_{02}, \ell_{13}), Z), \quad ((\ell_{12}, \ell_{03}), Z)$$

according to the classes of matrices

$$\begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & 0 & z_2 & z_3 \end{pmatrix}, \quad \begin{pmatrix} z_0 & z_1 & z_2 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}, \quad \begin{pmatrix} z_0 & 0 & z_2 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}.$$

4.6.2 Case of a double and 2 simple points

Let $Supp\, Z = \{p_0, p_2, p_3\}$ with double structure in p_0 defined by the line ℓ_{01}. In this case the isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ is defined by the normal form, see lemma 3.6.1,

$$\begin{pmatrix} z_1 & z_2 & \mu z_3 & 0 \\ z_0 & 0 & \lambda z_3 & z_1 \end{pmatrix}.$$

For the exceptional points corresponding to $<0, 1>$ and $<1, 0>$ we obtain

$$((\ell_{01}, \ell_{23}), Z) \quad \text{and} \quad ((\ell_{02}, \ell_{03}), Z).$$

4.6.3 Case of two double points

Let $Supp\ Z = \{p_0, p_3\}$ with double structures in p_0 and p_3 defined by ℓ_{01} and ℓ_{23}. Again we get an isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ by the normal form, see lemma 3.7.1,

$$\begin{pmatrix} z_1 & z_2 & \mu z_3 & 0 \\ z_0 & 0 & \lambda z_2 & z_1 \end{pmatrix}.$$

The exceptional points are now

$$((\ell_{01},\ \ell_{23}), Z) \quad \text{and} \quad ((\ell_{03},\ \ell_{03}), Z).$$

4.6.4 Case of a 3–fold and a simple point

Starting with the normal form lemma 3.8.1 for $Supp\ Z = \{p_0, p_3\}$ with triple structure in p_0, then $Z = Z(\mathcal{F}(A_{\lambda,\alpha}))$ has the ideal $\mathcal{I}_\lambda$ generated by

$$z_0 z_3,\ z_1 z_2,\ z_1 z_3,\ z_2 z_3,\ z_2^2,\ \lambda z_1^2 - z_0 z_2$$

which depends on λ but not on α. It follows that $\mathcal{I}_\lambda = \mathcal{I}_{\lambda'}$ if and only if $\lambda = \lambda'$. Therefore, if we fix λ we also get an isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ by using the parameter α. For $\alpha = \infty$ we get the only exceptional point $((\ell_{01},\ \ell_{03}), Z)$ representing the tangent line of Z at p_0 and the line connecting p_0, p_3.

4.6.5 Case of a 4–fold point

Let $Z = Z(\mathcal{F})$ be a 4–fold point in p_0 where $\mathcal{F}$ is given by the normal form in lemma 3.9.1. Then the ideal $\mathcal{I}_{\lambda,\mu,\alpha}$ is generated by

$$z_1 z_3,\ z_2 z_3,\ z_3^2,\ z_2^2,\ z_0 z_2 - \lambda z_1^2 + \alpha z_1 z_2,\ z_0 z_3 - \mu z_1 z_2$$

and does not depend on β. Again λ, μ, α is uniquely determined by $\mathcal{I}_{\lambda,\mu,\alpha}$ as it is easily verified (note that Z is not contained in a plane). Therefore, if we fix (λ, μ, α) we get an isomorphism $\mathbb{P}_1 \to \eta^{-1}(Z)$ by using the parameter β. For $\beta = \infty$ we get the only exceptional point $((\ell_{01},\ \ell_{01}), Z)$ representing the tangent line of Z at p_0.

4.6.6 If $\mathcal{F} \in D_4'$ and reflexive, then the fibre of a singular point $Z \in H_4'$ is isomorphic to the $\mathbb{P}_5$ of all conics in the plane dual to $Supp\ Z$, see 2.4.

4.6.7 Fibres over schemes $Z \in H_{p\ell}$

Let $\tilde{S}_2$ be the "proper transform" of $S_2 \subset M$, i.e. $\tilde{S}_2$ is the closure of $\tilde{S}_2^0 = \sigma^{-1}(S_2 \backslash S_0)$ in $\tilde{M}$. The points of $\tilde{S}_2^0$ consist of pairs $([\mathcal{F}], Z)$ where $\mathcal{F}$ has a representation $0 \to \mathcal{F} \to 2\mathcal{O} \to \mathcal{O}_C(1) \to 0$ with smooth conic $C = Z(\mathcal{F})$ and $Z \subset C$. Since here $[\mathcal{F}]$ is determined by C, the fibres of the morphism $\tilde{S}_2 \to H_{p\ell}$ consist of conics

through fixed 4 points in a plane, and thus are again isomorphic to $\mathbb{P}_1$.

4.6.8 Remark: The fibres over points $Z \in H_{t2}$ arise from limit points in $\tilde{M}$. We omit the details for this.

In addition to the results on the subvarieties S_i in proposition 2.3 and D_i in theorem 4.5 we have the following proposition on the relation between them.

4.7. Proposition: $S_0 \subset D_{2,2}$; $\quad$ $S_1, S_2 \subset D_4$; $\quad$ $S_0 \cap D_3 \subset S_0'$.

It follows from the last statement and from $S_0' \subset D_4'$ that $S_0 \cap D_4 = S_0 \cap D_3 = S_0'$.

Proof:

(1) For $t \neq 0$ it is easy to see that

$$
\begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ z_0 & 0 & t^2 z_2 & z_1 \end{pmatrix} \quad \sim \quad \begin{pmatrix} t z_1 & z_2 & z_3 & 0 \\ z_0 & 0 & t z_2 & z_1 \end{pmatrix}
$$

and that the matrices represent a sheaf $\mathcal{F}_t$ in $D_{2,2}^0$. For $t = 0$ we obtain a given sheaf $\mathcal{F}_0 = \mathcal{I}_{\ell_{01}} \oplus \mathcal{I}_{\ell_{23}}$ in $S_0 \backslash S_0'$. This proves $S_0 \subset D_{2,2}$.

(2) A sheaf $\mathcal{F}_0$ in $S_1 \backslash S_0$ resp. $S_2 \backslash S_0$ can be represented in normal form by

$$
\begin{pmatrix} z_0 & z_2 & z_3 & 0 \\ z_1 & 0 & z_2 & z_3 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} z_0 & z_1 & z_3 & 0 \\ z_1 & z_2 & 0 & z_3 \end{pmatrix}.
$$

The 1–parameter families

$$
\begin{pmatrix} z_0 & z_2 & z_3 & t z_1 \\ z_1 & 0 & z_2 & z_3 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} z_0 & z_1 & z_3 & 0 \\ z_1 & z_2 & t z_2 & z_3 \end{pmatrix}
$$

show that in each case $\mathcal{F}_0$ can be deformed into a reflexive sheaf $\mathcal{F}_t$ with a 4–fold point. This proves the second statement.

(3) If $m = [\mathcal{I}_\ell \oplus \mathcal{I}_{\ell'}] \in S_0 \cap D_3$ we can find a 1–parameter family A_t of matrices such that A_0 represents m and $\mathcal{F}(A_t) \in D_3^0$ for $t \neq 0$, i.e. $Z(\mathcal{F}(A_t))$ has a curvi–linear 3–fold structure in one of its points. Let $Z_0 \in H_3$ be the limit in H, so that we get a point $(m, Z_0) \in \widetilde{M}$, in particular $Z_0 \subset \ell \cup \ell'$, see remark 4.2.1. We distinguish the following cases. If $Z_0 \notin H_{t2} \cap H_{p\ell}$, then $Z_0 \in H_3^0$ or H_4^0. In both cases the lines ℓ and ℓ' of m meet by 4.6.4 and 4.6.5. If $Z_0 \in H_{t2}$ then Z_0 contains a point p with $\dim T_p Z_0 \geq 2$, and since $Z_0 \subset \ell \cup \ell'$ the two lines must meet in p. If, finally, $Z_0 \in H_{p\ell}$ then $m \in S_2$ by 4.6.7, and again the lines meet. This proves the last statement of the proposition.

4.8 Summarizing the results on the subvarieties we have

$$
\begin{array}{ccccccccc}
 & & & & & & 8 & S_0 & \\
 & & & & & & & \cap & \\
S'_0 & \subset & S_1, & S_2, & D'_4 & \subset & D_4 & \subset & D_3, & D_{2,2} & \subset & D_2 \\
7 & & 8 & 8 & 8 & & 10 & & 11 & 11 & & 12
\end{array}
$$

where the commas mean "and" for the inclusions on both sides, and the numbers are the dimensions, each subvariety being irreducible. Moreover,

$$
S'_0 = S_1 \cap S_2 = S_0 \cap S_1 = S_0 \cap S_2 = S_0 \cap D'_4 = S_0 \cap D_4 = S_0 \cap D_3.
$$

Question: It is an interesting question to construct and to describe a natural desingularization of $M(0,2,4)$.

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