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ERROR ESTIMATES FOR TIKHONOV REGULARIZATION WITH UNBOUNDED REGULARIZING OPERATORS

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ABSTRACT

It is shown that Tikhonov regularization for ill–posed operator equation $Kx = y$ using a possibly unbounded regularizing operator L yields an order-optimal algorithm with respect to certain stability set when the regularization parameter is chosen according to the Morozov's discrepancy principle. A more realistic error estimate is derived when the operators K and L are related to a Hilbert scale in a suitable manner. The result includes known error estimates for ordininary Tikhonov regularization and also the estimates available under the Hilbert scale approach.

1 INTRODUCTION

Many problems in science and engineering have their mathematical formulation as an operator equation

$$Kx = y \tag{1}$$

where $K : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y with its range $R(K)$ not closed in Y (c.f. Groetsch [2]), Engl [1]). It is well known that if $R(K)$ is not closed then the equation (1) or the problem of solving (1) is ill–posed (cf. Groetsch [3]). A prototype of an ill–posed equation is the Fredholm integral equation of the first kind,

$$\int_a^b k(s,t)x(t)dt = y(s), \quad a \leq s \leq b,$$

with a non–degenerate kernel $k(.,.) \in L^2([a,b] \times [a,b])$ and $X = Y = L^2[a,b]$.

Regularization procedures are employed for obtaining stable approximate solutions of ill–posed equations of the type (1). These procedures are especially useful when the data available is inexact. That is, we may have an approximation $\tilde{y}$ of y with a known error level $\delta > 0$, $\|y - \tilde{y}\| \leq \delta$.

In this paper we consider the well known Tikhonov regularization method using a possibly unbounded regularizing operator L. In fact, we assume that $L : D(L) \subset X \to Z$ is a closed densily defined linear operator between Hilbert spaces X and Z. Then Tikhonov regularization involves minimization of the map

$$x \mapsto \|Kx - \tilde{y}\|^2 + \alpha\|Lx\|^2, \quad x \in D(L). \tag{2}$$

It is known that if K and L satisfy the relation

$$\|Kx\|^2 + \|Lx\|^2 \geq \gamma\|x\|^2, \quad x \in D(L), \tag{3}$$

for some $\gamma > 0$, then the map in (2) attains its minimum at a unique element $x_\alpha(\tilde{y}) \in D(L)$. (See e.g., Locker and Prenter [5], Morozov [9], Nair, Hegland and Anderssen [10].)

It is also known that if $y \in R(A) + R(A)^\perp$, $A = K_{|D(L)}$, then the set

$$S_y := \{x \in D(L) : \|Kx - y\| \leq \|Ku - y\|, \forall u \in D(L)\}$$

is nonempty, and there exists a unique $\hat{x}(y) \in S_y$ such that

$$\|L\hat{x}(y)\| \leq \|Lx\|, \quad \forall x \in S_y,$$

and

$$x_\alpha(y) \to \hat{x}(y) \quad \text{as} \quad \alpha \to 0.$$

(cf. [5], [9], [10].) What one would like to have is the convergence

$$x_\alpha(\tilde{y}) \to \hat{x}(y) \quad \text{as} \quad \alpha \to 0 \quad \text{and} \quad \delta \to 0. \tag{4}$$

But examples can be easily constructed where this is no longer true. Therefore a strategy has to be adopted for choosing the regularization parameter $\alpha = \alpha(\delta, \tilde{y})$ so as to have the convergeence in (4). For this purpose we consider the simple procedure suggested by Morozov ([8], [9]), namely, to choose $\alpha = \alpha(\delta, \tilde{y})$ such that

$$\|K\tilde{x}_\alpha - \tilde{y}\| = \delta, \tag{5}$$

where $\tilde{x}_\alpha = x_\alpha(\tilde{y})$. It is known that if

$$\|(I - P_L)y\| > 0 \quad \text{and} \quad \|(I - P_L)\tilde{y}\| > \delta,$$

where $P_L : Y \to Y$ is the orthogonal projection onto the closure of the set $\{Kx : x \in D(L), Lx = 0\}$, then there exists a unique α depending on δ and $\tilde{y}$ satisfying (5)(cf. Morozov [9], Section 10). Note that if L is injective, then $P_L = 0$.

We show that the Tikhonov regularization together with the parameter choice strategy (5) yield an order–optimal algorithm with respect to the stabilizing set

$$M_\rho = \{x \in D(L) : \|Lx\| \leq \rho\}.$$

That is, we show that

$$\|\hat{x} - \tilde{x}_\alpha\| = O(E(M_\rho, \delta)),$$

where $\hat{x} = \hat{x}(y)$, $\tilde{x}_\alpha = x_\alpha(\tilde{y})$ and $E(M_\rho, \delta)$ is the *best possible maximal error* defined by

$$E(M_\rho, \delta) = \inf_R \sup\{\|x - Rv\| : x \in M_\rho, v \in Y, \|Kx - v\| \leq \delta\}.$$

In order to obtain more realistic error estimates, we relate the operators K and L with a Hilbert scale in a suitable manner. The resulting error estimates include known estimates for ordinary Tikhonov regularizaation, that is for the case $L = I$, and also the well known estimates available under Hilbert scales approach. Our proofs are simpler and straight forward.

2 MAIN RESULTS

Let $K : X \to Y$ and $L : D(L) \subset X \to Z$ be as in the earlier section satisfying the condition (3) and $y \in R(A)$ such that $\|(I - P_L)y\| > 0$, where $A = K_{|D(L)}$ and $P_L : Y \to Y$ is the orthogonal projection onto the closure of the set $\{Kx : x \in D(L), Lx = 0\}$. Let $\tilde{y} \in Y$ satisfy

$$\|y - \tilde{y}\| \leq \delta < \|(I - P_L)\tilde{y}\|, \tag{6}$$

and let $\alpha = \alpha(\delta, \tilde{y})$ be the unique positive real satisfying (5).

For $\rho > 0$ let

$$M_\rho = \{x \in D(L) : ||Lx|| \leq \rho\}$$

and

$$e(M_\rho, \delta) = \sup\{||x|| : x \in M_\rho, ||Kx|| \leq \delta\}.$$

It is proved in Micchelli and Rivlin [7] that

$$e(M_\rho, \delta) \leq E(M_\rho, \delta) \leq 2e(M_\rho, \delta).$$

Using the notation $\hat{x} = \hat{x}(y)$ and $\tilde{x}_\alpha = x_\alpha(\tilde{y})$, we have the following order–optimal result.

THEOREM 1 *If $\hat{x} \in M_\rho$ for some $\rho > 0$, then*

$$||\hat{x} - \tilde{x}_\alpha|| \leq 2e(M_\rho, \delta).$$

Proof. Since $\tilde{x}_\alpha$ minimizes the map (2), it follows from (5) and (6) that

$$\begin{aligned} \delta^2 + \alpha||L\tilde{x}_\alpha||^2 &= ||K\tilde{x}_\alpha - \tilde{y}||^2 + \alpha||L\tilde{x}_\alpha||^2 \\ &\leq ||K\hat{x} - \tilde{y}||^2 + \alpha||L\hat{x}||^2 \\ &\leq \delta^2 + \alpha||L\hat{x}||^2. \end{aligned}$$

Hence

$$||L\tilde{x}_\alpha|| \leq ||L\hat{x}||.$$

Using this, we obtain

$$\begin{aligned} ||L(\hat{x} - \tilde{x}_\alpha)||^2 &= \langle L(\hat{x} - \tilde{x}_\alpha), L(\hat{x} - \tilde{x}_\alpha)\rangle \\ &= \langle L\hat{x}, L\hat{x}\rangle - 2Re\langle L\hat{x}, L\tilde{x}_\alpha\rangle + \langle L\tilde{x}_\alpha, L\tilde{x}_\alpha\rangle \\ &\leq 2(\langle L\hat{x}, L\hat{x}\rangle - Re\langle L\hat{x}, L\tilde{x}_\alpha\rangle. \end{aligned}$$

Thus

$$||L(\hat{x} - \tilde{x}_\alpha)||^2 \leq 2|\langle L\hat{x}, L(\hat{x} - \tilde{x}_\alpha)\rangle|. \tag{7}$$

From this it follows that $||L(\hat{x} - \tilde{x}_\alpha)|| \leq 2\rho$. Also, since $K\hat{x} = y$, $||y - \tilde{y}|| \leq \delta$ and (5), we have

$$||K(\hat{x} - \tilde{x}_\alpha)|| \leq 2\delta. \tag{8}$$

Thus,

$$\|L\left(\frac{\hat{x}-\tilde{x}_\alpha}{2}\right)\| \leq \rho \quad \text{and} \quad \|K\left(\frac{\hat{x}-\tilde{x}_\alpha}{2}\right) \leq \delta$$

so that $(\hat{x}-\tilde{x}_\alpha)/2 \in M_\rho$ and $\|\hat{x}-\tilde{x}_\alpha\| \leq 2e(M_\rho,\delta)$. □

To obtain a more realistic estimate for the error $\|\hat{x}-\tilde{x}_\alpha\|$, we consider a Hilbert scale $(X_s)_{s\in\mathbf{R}}$ with $X_0 = X$, and assume that there exists $a > 0$, $b \geq 0$, $c > 0$ and $d > 0$ such that

$$\|Kx\| \geq c\|x\|_{-a}, \quad \forall x \in X \tag{9}$$

and

$$\|Lx\| \geq d\|x\|_b, \quad \forall x \in D(L) \cap X_b. \tag{10}$$

Now recall the interpolation inequality (cf. Krein and Petunin [4])

$$\|x\|_s \leq \|x\|_r^\theta \|x\|_t^{1-\theta}, \forall x \in X_t,$$

where $r \leq s \leq t$ and $\theta = \frac{t-s}{t-r}$. Taking $r = -a$, $t = b$ and $s = 0$, it follows from the interpolation inequality and (9) and (10) that

$$\|x\| \leq \left(\frac{\|Kx\|}{c}\right)^\theta \left(\frac{\|Lx\|}{d}\right)^{1-\theta}, \quad \theta = \frac{b}{a+b}, \tag{11}$$

for every $x \in D(L) \cap X_b$.

THEOREM 2 *If $\hat{x} \in M_\rho \cap X_b$ for some $\rho > 0$, then*

$$\|\hat{x}-\tilde{x}_\alpha\| \leq 2\left(\frac{\rho}{d}\right)^{\frac{a}{a+b}} \left(\frac{\delta}{c}\right)^{\frac{b}{a+b}}.$$

Proof. From (11), it follows that for every $x \in M_\rho \cap D(L)$,

$$\|x\| \leq \left(\frac{\delta}{c}\right)^\theta \left(\frac{\rho}{d}\right)^{1-\theta}, \quad \theta = \frac{b}{a+b},$$

so that

$$e(M_\rho,\delta) \leq \left(\frac{\delta}{c}\right)^{\frac{a}{a+b}} \left(\frac{\rho}{d}\right)^{\frac{b}{a+b}}.$$

Now the result follows from Theorem 1. □

Next we obtain an improved estimate under stronger assumptions on $\hat{x}$. For this first we require the following result.

LEMMA 1 *If B is a bounded self adjoint operator on X and $0 \leq \tau \leq 1$, then*

$$\|B^\tau x\| \leq \|Bx\|^\tau \|x\|^{1-\tau}, \quad \forall x \in X.$$

Proof. The result is obvious if either $\tau = 0$ or $\tau = 1$. Therefore assume that $0 < \tau < 1$. As a consequence of the spectral theorem we have

$$\|B^\tau x\|^2 = \int_J \lambda^{2\tau} d\langle E_\lambda x, x\rangle, \forall x \in X,$$

where J is an open interval containing the spectrum of B and $\{E_\lambda\}_{\lambda \in J}$ is the spectral family for B. Now by Hölder's inequality we have,

$$\begin{aligned} \|B^\tau x\|^2 &\leq \left(\int_J \lambda^2 d\langle E_\lambda x, x\rangle\right)^\tau \left(\int_J d\langle E_\lambda x, x\rangle\right)^{1-\tau} \\ &= \|Bx\|^{2\tau} \|x\|^{2(1-\tau)} \end{aligned}$$

for every $x \in X$ and $0 < \tau < 1$. □

THEOREM 3 *Suppose $D(L^*L) \subset X_b$, $\hat{x} \in D(L^*L)$ and $L^*L\hat{x} = (K^*K)^\nu u$ for some $u \in X$ and $0 \leq \nu \leq 1/2$. Then*

$$\|\hat{x} - \tilde{x}_\alpha\| \leq c_0 (2\delta)^p,$$

where

$$p = \frac{2(a\nu + b)}{2(a\nu + b) + a} \quad \text{and} \quad c_0 = \left(\frac{1}{c}\right)^{\frac{2b}{2(a\nu+b)+a}} \left(\frac{\sqrt{2\|u\|}}{d}\right)^{\frac{2a}{2(a\nu+b)+a}}.$$

Proof. Since $\hat{x} - \tilde{x}_\alpha \in D(L^*L) \subset X_b$, from (11) we have

$$\|\hat{x} - \tilde{x}_\alpha\| \leq \left(\frac{\|K(\hat{x} - \tilde{x}_\alpha)\|}{c}\right)^\theta \left(\frac{\|L(\hat{x} - \tilde{x}_\alpha\|}{d}\right)^{1-\theta}, \tag{12}$$

where $\theta = b/(a+b)$. Now using the fact that $L^*L\hat{x} = (K^*K)^\nu u$, $0 \le \nu \le 1$, the relation (7) implies

$$\begin{aligned} \|L(\hat{x} - \tilde{x}_\alpha)\|^2 &\le 2|\langle (K^*K)^\nu, \hat{x} - \tilde{x}_\alpha\rangle| \\ &= 2|\langle u, (K^*K)^\nu(\hat{x} - \tilde{x}_\alpha)\rangle| \\ &\le 2\|u\|\|(K^*K)^\nu(\hat{x} - \tilde{x}_\alpha)\|. \end{aligned}$$

Taking $B = (K^*K)^{1/2}$ and $\tau = 2\nu$ in Lemma 1, and using (8), we obtain

$$\begin{aligned} \|(K^*K)^\nu(\hat{x} - \tilde{x}_\alpha)\| &\le \|K(\hat{x} - \tilde{x}_\alpha)\|^{2\nu}\|\hat{x} - \tilde{x}_\alpha\|^{1-2\nu} \\ &\le (2\delta)^{2\nu}\|\hat{x} - \tilde{x}_\alpha\|^{1-2\nu}. \end{aligned}$$

Here we used the relation $\|(K^*K)^{1/2}x\| = \|Kx\|$. Thus,

$$\|L(\hat{x} - \tilde{x}_\alpha)\| \le \sqrt{2\|u\|}(2\delta)^\nu\|\hat{x} - \tilde{x}_\alpha\|^{(1-2\nu)/2}.$$

Therefore, (12) gives

$$\|\hat{x} - \tilde{x}_\alpha\| \le \left(\frac{1}{c}\right)^\theta \left(\frac{\sqrt{2\|u\|}}{d}\right)^{1-\theta} (2\delta)^{\theta+\nu(1-\theta)}\|\hat{x} - \tilde{x}_\alpha\|^{(1-\theta)(1-2\nu)/2},$$

so that

$$\|\hat{x} - \tilde{x}_\alpha\|^{1-(1-\theta)(1-2\nu)/2} \le \left(\frac{1}{c}\right)^\theta \left(\frac{\sqrt{2\|u\|}}{d}\right)^{1-\theta} (2\delta)^{\theta+\nu(1-\theta)}.$$

From this the result follows by observing that $\theta + \nu(1-\theta) = (a\nu + b)/(a+b)$ and $1 - (1-\theta)(1-2\nu)/2 = [2(a\nu + b) + a]/2(a\nu + b)$. □

COROLLARY 1 *(i). If $L = I$ and $\hat{x} = (K^*K)^\nu u$ for some $u \in X$ and $0 \le \nu \le 1/2$, then*

$$\|\hat{x} - \tilde{x}_\alpha\| \le 2\|u\|^{\frac{1}{2\nu+1}}\delta^{\frac{2\nu}{2\nu+1}}.$$

(ii). Suppose $\hat{x} \in D(L^*L)$ and $\hat{x} = L^*Lu$ for some $u \in X$. Then

$$\|\hat{x} - \tilde{x}_\alpha\| \le 2\left(\frac{1}{c}\right)^{\frac{2b}{2b+a}} \left(\frac{1}{d}\right)^{\frac{2a}{2b+a}} \|u\|^{\frac{a}{2b+a}}\delta^{\frac{2b}{2b+a}}.$$

(iii). Suppose $\hat{x} \in D(L^*L)$ and $L^*L\hat{x} = K^*u$ for some $u \in X$. Then

$$\|\hat{x} - \tilde{x}_\alpha\| \leq 2 \left(\frac{1}{c}\right)^{\frac{b}{a+b}} \left(\frac{\sqrt{2\|u\|}}{d}\right)^{\frac{a+2b}{2a+2b}}.$$

Proof. The estimates in (i), (ii) amd (iii) are obtained from Theorem 3 by taking $b = 0$, $\nu = 0$ and $\nu = 1/2$ respectively. □

REMARKS. We note that in Corollary 1, the result (i) is the well known optimal order result for ordinary Tikhonov regularization, and (iii) is the best rate obtained by Natterer [11] under the frame work of Hilbert scales. Also the estimates in (i) and (ii) are of better order than the classical result in Theorem 2. Receently Mair [6] obtained results similar to the ones in Theorems 1 and 2 with $\sqrt{2}$ in place of 2, but under the ápriori choice $\alpha = \delta^2/\rho^2$. The estimate in Theorem 3 may be compared with the one obtained by Schröter and Tautenhahn [12] for $\|.\|_r$ under the frame work of Hilbert scales. In fact, using the estimate in Theorem 3 the following estimate for the error in Hilbert scale norm $\|.\|_r$ can be deduced:

$$\|\hat{x} - \tilde{x}_\alpha\|_r \leq c_1(2\delta)^\mu$$

with

$$\mu = \frac{2(a\nu + b) - r}{2(a\nu + b) + a} \quad \text{and} \quad c_1 = [(1/c)^{2b - r(2\nu+1)}(\sqrt{2\|u\|}/d)^{2a+2r}]^{\frac{1}{2(a\nu+b)+a}}.$$

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