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1 Introduction

Derived varieties play an essential role in the theory of hyperidentities. In [11] we have shown that derivation diagrams are a useful tool in the analysis of derived algebras and varieties. In this paper this tool is developed further in order to use it for algebraic constructions of derived algebras. Especially the operator S of subalgebras, H of homomorphic images and P of direct products are studied. Derived groupoids from the groupoid $Nor(x,y) = x' \wedge y'$ and from abelian groups are considered. The latter class serves as an example for fluid algebras and varieties. A fluid variety V has no derived variety as a subvariety and is introduced as a counterpart for solid varieties. Finally we use a property of the commutator of derived algebras in order to show that solvability and nilpotency are preserved under derivation.

2 Derivation diagrams of algebras

Definition 2.1 *Let V be a variety of a fixed type $\tau = \langle n_0, n_1, ..., n_\gamma, ...\rangle$, with fundamental operations $F = \{f_0, f_1, ..., f_\gamma, ...\}$. Let $\sigma = \langle t_0, t_1, ..., t_\gamma, ...\rangle$ be a fixed choice of terms of V, with t_i having arity $n_i, i \geq 0$. For any algebra $\mathcal{A} = \langle A; F\rangle$ in F, the algebra $A = \langle A; \sigma\rangle$ is called a derived algebra of A (corresponding to σ), and will be denoted by $\mathcal{A}_\sigma$. The variety derived from V using σ, is the variety V or $V\sigma$ generated by $\{\mathcal{A}\sigma : \mathcal{A} \in V\}$. A variety $V\sigma$ is called a derived variety of V. For a class K of algebras of type τ we denote by $D(K)$ the class of all derived algebras for all choices of σ of type τ from K. D is a class operator.*

Example 2.2 *Let $\mathcal{C} = (C_3, +)$ be the cyclic group of order 3. Then $\mathcal{B} = (C_3, \oplus)$ with $x \oplus y := x + 2y$ is a derived algebra. $\mathcal{B}$ is a quasigroup.*

Lemma 2.3 *If $\mathcal{C}$ is a derived algebra of $\mathcal{B}$ and $\mathcal{B}$ a derived algebra of $\mathcal{A}$ then $\mathcal{C}$ is a derived algebra of $\mathcal{B}$.*

Proof. Every operation of $\mathcal{C}$ is a term operation of $\mathcal{B}$ which again is composed by fundamental operations of $\mathcal{A}$. Hence every operation of $\mathcal{C}$ is a term operation of $\mathcal{B}$. □

Definition 2.4 *The algebra $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ are called mutually derived from each others if $\mathcal{A}$ is a derived algebra of $\mathcal{B}$ and $\mathcal{B}$ a derived algebra of $\mathcal{A}$.*

Example 2.5 *C_3 and $\mathcal{B}$ in the above example are mutually derived. If we consider $x + y := (x \oplus y) \oplus y$ as a term operation of $\mathcal{B}$ we have*

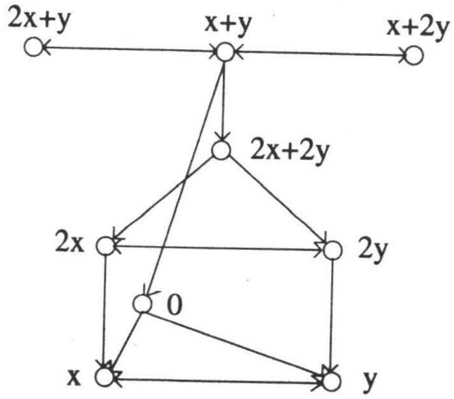
$$(x \oplus y) \oplus y = (x + 2y) + 2y = x + y.$$

(Mutually derived algebra are equivalent in.)

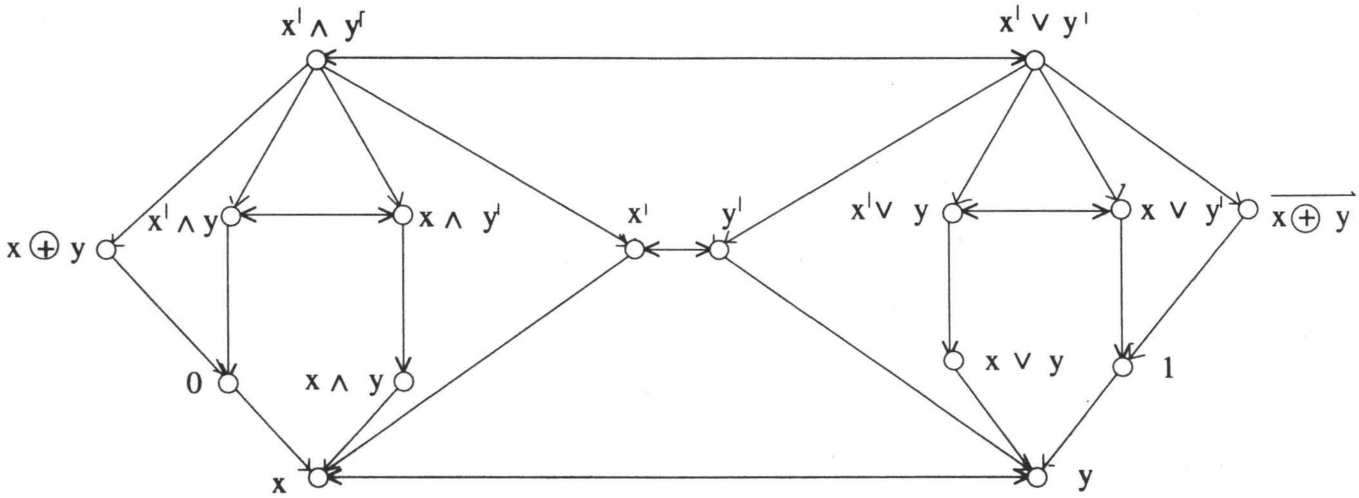
Definition 2.6 *Let $\mathcal{B}$ be a derived algebra of $\mathcal{A}$ but $\mathcal{A}$ not a derived algebra of $\mathcal{B}$.*
$\mathcal{B}$ is called a derivative of order 1 if for every algebra $\mathcal{C}$ the following implication holds: If $\mathcal{B}$ is a derivative of $\mathcal{C}$ and $\mathcal{C}$ a derivative of $\mathcal{A}$ then $\mathcal{C}$ is mutually derived from either $\mathcal{A}$ or $\mathcal{B}$.
A series of derived algebras $\mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3$ is a sequence of algebras, each of which is a derivative of order 1 of the previous one.

Notation. Given an algebra $\mathcal{A}$ the derivation diagram of $\mathcal{A}$ is a directed graph whose vertices are the derived algebras of $\mathcal{A}$. Two vertices $\mathcal{A}$ and $\mathcal{B}$ are joined by an arc $\mathcal{A} \to \mathcal{B}$ if $\mathcal{A}_\sigma = \mathcal{B}$ and $\mathcal{B}$ is a derivative $\mathcal{A}$ of order 1 or is mutually derived from $\mathcal{A}$. A vertex $\mathcal{B}$ also has a loop $\mathcal{B} \to \mathcal{A}$ if $\mathcal{B}_\sigma = \mathcal{B}$. Usually we label the vertices also with the choice σ of terms from $\mathcal{A}$.
For practical reasons it may be necessary to simplify a derivation diagram if no important information is lost.
Instead of $\circ \overset{\rightarrow}{\underset{\leftarrow}{}} \circ$ we draw $\circ \leftrightarrow \circ$.

Example 2.7 *The derivation diagram of the cyclic group* $C_3 = (\{0,1,2\}, +)$.



Example 2.8 *It is wellknown that the operation* $Nor(x,y) = x' \wedge y'$ *of the Boolean algebra* $\mathcal{B}=(\{0,1\}; \wedge, \vee, ', 0, 1)$ *generates every term function of* $\mathcal{B}$. *The groupoid* $(\{0,1\}; \circ)$ *is primal where* $x \circ y := Nor(x,y)$. *This groupoid has the following diagram where the binary operations are presented by their terms of the Boolean algebra.*



Here we have

$$x \oplus y := (x' \wedge y) \vee (x \wedge y')$$

and

$$\overline{x \oplus y} := (x' \vee y) \wedge (x \vee y').$$

3 Algebraic constructions

Lemma 3.1 *If $f : A \to B$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ then f is also a homomorphism from $\mathcal{A}_\sigma$ to $\mathcal{B}_\sigma$. If f is an isomorphism then f is also an isomorphism from $\mathcal{A}_\sigma$ to $\mathcal{B}_\sigma$.*

Proof. We only consider a binary operation and a choice σ consisting of a term $t(x, y)$. Then we have $f(t(a, b)) = t(f(a), f(b))$ as f is a homomorphism. □

Lemma 3.2 *If $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ then $\mathcal{A}_\sigma$ is a subalgebra of B_σ. The lattice sub$\mathcal{B}$ of all subalgebras of $\mathcal{B}$ is a sub-meet lattice of sub$\mathcal{B}_\sigma$.*

Proof. If $\mathcal{A}_1, \mathcal{A}_2$ are subalgebras of $\mathcal{B}$ then $\mathcal{A}_1 \wedge \mathcal{A}_2$ consists of the set-theoretic meet and $(A_1 \cap A_2)_\sigma = A_{1\sigma} \cap A_{2\sigma}$. □
We call a choice σ surjective if there exists a choice σ' such that

$$\mathcal{A}_{\sigma\sigma'} = \mathcal{A}.$$

Remark. If σ is surjective then

$$\mathrm{Sub}(\mathcal{A}) \simeq \mathrm{Sub}(\mathcal{A}_\sigma).$$

Lemma 3.3 *Let $Rel(\mathcal{A})$ be the lattice of all compatible n-ary relations of the algebra A. Then $Rel(\mathcal{A})$ is a sub-meet-lattice of $Rel(\mathcal{A}_\sigma)$.*
Especially $Con\mathcal{A}$ is a sublattice of $Con(\mathcal{A}_\sigma)$.

Proof. We consider congruences $\theta, \eta \in Con(\mathcal{A})$. As $\theta \wedge \eta = \theta \cap \eta$ we have like above that $\theta \wedge \eta = \theta \cap \eta \in Con\mathcal{A}_\sigma$. □

Corollary 3.4 *If $\mathcal{A}_\sigma$ is simple then $\mathcal{A}$ is simple. If $\mathcal{A}_\sigma$ is subdirectly irreducible then $\mathcal{A}$ is subdirectly irreducible.*

Proof. If $\theta \wedge \eta = \triangle$ in $\mathcal{A}_\sigma$ then $\theta \wedge \eta = \triangle$ also in $\mathcal{A}$ by lemma 3.3. □

Lemma 3.5 *If $\mathcal{A}, \mathcal{B}$ are algebras of type τ and σ is a choice of terms then*

$$(\mathcal{A} \times \mathcal{B})_\sigma = \mathcal{A}_\sigma \times \mathcal{B}_\sigma.$$

Proof. The carrier sets on both sides are the same and the operation coincide on both sides in the following sense. $\mathcal{A}$ and $\mathcal{B}$ generate subvarieties of $HSP(\mathcal{A} \times \mathcal{B})$. Hence there exist congruences θ_1, θ_2 such that $T(\mathcal{A}) \simeq T(\mathcal{A} \times \mathcal{B})/\theta_1$ and $T(\mathcal{B}) \simeq T(\mathcal{A} \times \mathcal{B})/\theta_2$ for the algebras. Let t be a term of the choice σ then there exist $t_1 \in T(A)$ with $t_1 = t \; mod \; \theta_1$ and $t_2 \in T(B)$ with $t_{21} = t \; mod \; \theta_2$ and t corresponds to (t_1, t_2). □
Clearly lemma 3.5 holds also for arbitrary direct products.
If we consider the step from an algebra $\mathcal{A}$ to a derived algebra $\mathcal{A}_\sigma$ the class operators H, S, P behave well. The reverse step is more complicated and deserves more analysis.

Remark. The derived algebra $\mathcal{A}_\sigma = (A, \Omega)$ induces a subalgebra $\mathcal{B}_\sigma = (B, \Omega)$ if for every operation $f \in \Omega$ and every $a_1, ..., a_n \in A$ we have $f(a_1, ..., a_n) \in \mathcal{B}$ where $\mathcal{B}$ is a subset of A. If $\mathcal{B}$ is a proper subset of A we call $\mathcal{B}$ an induced subalgebra of $\mathcal{A}_\sigma$.

Example 3.6 *For the term $t(x, y) = 2x + 2y$ of the cyclic group C_6 we have an induced subalgebra of $C_{6\sigma}$ with $\sigma = \{2x + 2y\}$.*

4 Fluid varieties

Let us call a derived algebra $\mathcal{A}_\sigma$ of $\mathcal{A}$ non-trivial if $\mathcal{A}_\sigma$ is not isomorphic to $\mathcal{A}$.
A variety V of type τ is solid if for every algebra $\mathcal{A} \in V$ all derived algebras $\mathcal{A}_\sigma$ (for every choice σ) are of the variety V.

Definition 4.1 *A variety V of type τ is fluid if for every algebra $\mathcal{A} \in V$ and for every choice σ all non-trivial algebras $\mathcal{A}_\sigma$ are not of the variety V.*

Furthermore we declare that every trivial variety is fluid (and solid).

Definition 4.2 *The algebra $\mathcal{A}$ is fluid if no non-trivial derived algebra $\mathcal{A}_\sigma$ is in the variety $HSP(\mathcal{A})$.*

We observe with the help of the derivation diagram of the groupoid Nor that fluid algebras may not generate a fluid variety. For instance $x' \wedge y$ and $x \wedge y'$ are fluid groupoids but the variety generated by the direct product of them is not fluid.
We write commutative groupoids additively and denote terms $(...((x+x)+x)+...)+x$ by kx if there are k variables in this term. In the following we consider the cyclic group $(\mathbb{Z}_n;+)$ of non negative integers mod n. It is easy to see that the following identities hold

i) $x + (y + z) = (x + y) + z$

ii) $x + y = y + x$

iii) $x + nx = x$

Proposition 4.3 *If $\mathcal{G} = (\mathbb{Z}_n;+)$ is the semigroup which is described as above then G is fluid.*

Proof. The binary terms $t(x,y)$ of $\mathcal{G}$ can be presented by $t(x,y) = kx + ly$; $k,l \in \mathbb{N}_0$, $0 \le k$, $l \le n-1$. Let $\mathcal{G}_\sigma = (G;\oplus)$ where $x \oplus y = kx + ly$ be a derived groupoid. Let us assume that $\mathcal{G}_\sigma \in HSP(\mathcal{G})$. Then we have $x \oplus y = y \oplus x$ and therefore $kx + ly = ky + lx$. We have $k = l$ and the operation $\oplus$ is of the form $x \oplus y = kx + ky$, $0 \le k \le n-1$.
Because of the associativity we have $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and hence $kx + k^2y + k^2z = k^2x + k^2y + kz$. If $k = 0$ then $x \oplus nx + 0$ with $x \neq 0$. Otherwise we have $x + kz = kx + z$ for every $x, y \in \mathbb{Z}_n$. From this follows $k = 1$ and $\mathcal{G}_\sigma = \mathcal{G}$. □

Theorem 4.4 *If $A = (G;+,-,0)$ is an abelian group of order n then $\mathcal{G} = (G;+)$ is a fluid semigroup.*

Proof. $\mathcal{A}$ is the direct product of cyclic groups $\mathcal{A}_i = (G_i; +, -, 0)$ with an order of prime power $p_i^{n_i}$. Therefore we have $\mathcal{G}_\sigma = \left(\prod_{i\in I} \mathcal{G}_i\right)_\sigma = \prod_{i\in I} \mathcal{G}_{i\sigma}$. If we assume that $\mathcal{G}_\sigma \in HSP(\mathcal{G})$ then it follows that $\mathcal{G}_\sigma \in HSP(\mathcal{G})$ for every $i \in I$. Especially the associative and commutative law has to hold. We have shown in 4.3 that $\mathcal{G}_{i\sigma} \in HSP(\mathcal{G})$ if and only if $\mathcal{G}_{i\sigma} \simeq \mathcal{G}_i$. Hence from our assumption it follows that $\mathcal{G}_\sigma \simeq \mathcal{G}$. □

Proposition 4.5 *Every subvariety W of a fluid variety V is fluid.*

Proof. If $\mathcal{A} \in W \subseteq V$ is an algebra such that a non-trivial derived algebra $\mathcal{A}_\sigma$ is of W then we also have the contradiction $\mathcal{A}_\sigma \in V$. □

Remark. The subvarieties of a fluid variety V of type τ are a sublattice of the lattice of all varieties of type τ.

Example 4.6 *If we consider the derivation diagram in 2.7 we have the following fluid groupoids which are presented by their Boolean terms* $\{x, y, 0, 1, x \wedge y, x \vee y, x \oplus y\}$. *Fluid varieteties are generated by* $\{x, y, 0, x \wedge y, x \oplus y, x' \wedge y, x \wedge y\}$.

In Graczyńska [5] the properties of the greatest solid subvariety $k(V)$ of a variety V are described and analyzed.

Proposition 4.7 *If the variety V is fluid then the greatest solid subvariety $k(V)$ is trivial.*

Proof. Let V be fluid and assume that $k(V)$ is not trivial. Let $\mathcal{A}$ be a non trivial algebra of $k(V)$. We have $\mathcal{A} \in V$ and also $\mathcal{A}_\sigma \in V$ because $k(V)$ is a solid subvariety of V. □

Remark. An arrow $A \to A_\sigma$ in a derivation diagram is called solid if $A_\sigma \in HSP(A)$ and fluid if $A_\sigma \notin HSP(A)$. In Plonka [7] the hypersubstitutions are also distinguished between such properties.

5 Preservation Properties

We like to extend our results in [10] and will use the following definition [3].

Definition 5.1 *Let α, β, δ be congruences of an algebra $\mathcal{A}$. α centralizes β modulo δ (which is denoted by $C(\alpha, \beta, \delta)$) if for every $n > 1$, for $t \in T_n(\mathcal{A})$ and for all $(u, v) \in \alpha$, $(x_2, y_2) \in \beta, ..., (x_n, y_n) \in \beta$ this equivalence holds.*

$$(5.1) \quad \begin{array}{c} t(u, x_2, ..., x_n) \stackrel{\delta}{=} t(u, y_1, ..., y_n) \Leftrightarrow \\ t(v, x_2, ..., x_n) \stackrel{\delta}{=} t(v, y_1, ..., y_n) \end{array}$$

Definition 5.2 *The commutator $[\alpha, \beta]$ in $\mathcal{A}$ is the smallest congruence δ for which $C(\alpha, \beta, \delta)$. holds.*

Proposition 5.3 *Let $\mathcal{A}$ be an algebra with $\alpha, \beta \in Con\mathcal{A}$ and let $\mathcal{A}_\sigma$ be a derived algebra of A. Let $\leq$ be the order relation of the lattice of equivalences of the set $\mathcal{A}$. For the commutator $[\alpha, \beta]_{\mathcal{A}}$ and the commutator $[\alpha, \beta]_{\mathcal{A}_\sigma}$ of the algebra A_σ the following holds*

$$(5.3) \quad [\alpha, \beta]_{\mathcal{A}_\sigma} \leq [\alpha, \beta]_{\mathcal{A}}.$$

Proof. It is obvious that every congruence of $\mathcal{A}$ is also a congruence of $\mathcal{A}_\sigma$ as $T(A_\sigma)$ is a subset of termfunctions of $T(A)$. If $\delta = [\alpha, \beta]_{\mathcal{A}}$ then we have $C(\alpha, \beta, \delta)$ in $\mathcal{A}$ and again as $T(\mathcal{A}_\sigma) \subseteq T(\mathcal{A})$ we have $C(\alpha, \beta, \delta)$ in $\mathcal{A}_\sigma$. □

Theorem 5.4 *Let $\mathcal{A}_\sigma$ be a derived algebra of $\mathcal{A}$. Then the following holds*

(5.4.1) *If $\mathcal{A}$ is abelian then $\mathcal{A}_\sigma$ is abelian.*

(5.4.2) *If $\mathcal{A}$ is solvable then $\mathcal{A}_\sigma$ is solvable.*

(5.4.3) *If $\mathcal{A}$ is nilpotent then $\mathcal{A}_\sigma$ is nilpotent.*

Proof. (5.4.1) If $\mathcal{A}$ is abelian then $[\nabla_A, \nabla_A] = \Delta_A$ in $\mathcal{A}$. As $[\nabla_A, \nabla_A]_{\mathcal{A}_\sigma} \leq [\nabla, \nabla]_{\mathcal{A}} = \Delta_A$ also A_σ is abelian.
(5.4.2) Let $\mathcal{A}$ be solvable. Then $[\nabla]^n_{\mathcal{A}} = \Delta$ for some $n \in \mathbb{N}$ where $[\nabla]^n = \left[[\nabla]^{n-1}, [\nabla]^{n-1}\right]$. By the above proposition we have for $\mathcal{A}_\sigma$ that $[\nabla]^n_{\mathcal{A}_\sigma} \leq [\nabla^n]_{\mathcal{A}} = \Delta$.
(5.4.3) If $\mathcal{A}$ is nilpotent then $(\nabla, \nabla]^n = \left[\nabla, (\nabla, \nabla]^{n-1}\right] = \Delta$ for some $n \in \mathbb{N}$. Again by proposition 5.3 A_σ is nilpotent for some $k \leq n$. □

Remark. Derived algebras behave well with respect to types of algebras. In the case of [11] it is easy to see that a subclone $p(A_\sigma)$ of polynomial function preserves all the relations which characterize the clone $p(\mathcal{A})$ of polynomial functions of $\mathcal{A}$. In the case of McKenzie's theory of types a choice σ induces an order homomorphism on the lattices of types.

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