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1 Introduction

A polynomial function $f : L \to L$ of a lattice $\mathcal{L} = (L; \wedge, \vee)$ is generated by the identity function $id(x) = x$ and the constant functions $c_a(x) = a$ (for every $x \in L$), $a \in L$, by applying the operations $\wedge, \vee$ finitely often. Every polynomial function in one or also in several variables is a monotone function of $\mathcal{L}$.
If every monotone function of $\mathcal{L}$ is a polynomial function then $\mathcal{L}$ is called order-polynomially complete.
In this paper we give a new characterization of finite order-polynomially lattices. We consider doubly irreducible monotone functions and point out their relation to tolerances, especially to central relations. We introduce chain-compatible lattices and show that they have a non-trivial congruence if they contain a finite interval and an infinite chain. The consequences are two new results. A modular lattice $\mathcal{L}$ with a finite interval is order-polynomially complete if and only if $\mathcal{L}$ is finite projective geometry. If $\mathcal{L}$ is simple modular lattice of infinite length then every nontrivial interval is of infinite length and has the same cardinality as any other nontrivial interval of $\mathcal{L}$.
In the last sections we show the descriptive power of polynomial functions of lattices and present several applications in geometry.

2 Prepolynomially complete lattices

As a first step we consider order-polynomially complete lattices in a more general setting. For this we use some notions from clone theory [19]. Let A be some non empty set and ρ a relation. Then $Pol\rho$ is the clone of functions on A which preserves ρ.
Let $\mathcal{A} = (A, \Omega)$ be an algebra of type τ. The clone $P(\mathcal{A})$ of the polynomial functions of $\mathcal{A}$ consists of all n-place polynomial function, $n \in \mathbb{N}$. We define recursively

i) The projection $e_i^n : A^n \to A^n$ with $e_i^n(x_1, ..., x_n) = x_i$ and the constant function $c_a^n : A^n \to A$ with $c_a^n(x_1, ..., x_n) = a,\ a \in A$, are n-place polynomial functions.

ii) If $f \in \Omega$ is a m-place operation of the algebra $\mathcal{A} = (A, \Omega)$ and $p_1, ..., p_m$ are n-place polynomial functions then $f(p_1(x_1, ..., x_n), ..., p_m(x_1, ..., x_n))$ is a polynomial function.

One may apply (ii) only a finite number of times.
We use the following concepts.

Definition 2.1 *The algebra $\mathcal{A} = (A, \Omega)$ is polynomially complete (functionally complete) if every function $f : A^n \to A$ is a polynomial function.*

Theorem 2.2 *If $\mathcal{A}$ is polynomially complete of at most a countable type τ then A is finite [10].*

Definition 2.3 *The finite algebra $\mathcal{A} = (A, \Omega)$ is prepolynomially complete if every function $f : A^n \to A$ is a polynomial function of the algebra $\mathcal{A} = (A, \Omega \cup \{g\})$ for every $g \notin P(\mathcal{A})$.*

The prepolynomially complete algebra are described according to Rosenberg's completeness theorem in [15] and are connected to the relations ρ of following types.

(1) Type O: ρ is an order relation with a least and a greatest element.

(2) Type L: Let $|A| = p^m$, p a prime number, $m \geq 1$
$\rho = \{(a_1, a_2, a_3, a_4) \mid a_1 + a_2 = a_3 + a_4\}$ where $(A, +)$ is an abelian group with $p \cdot A = 0$.

(3) Type C: ρ is a nontrivial equivalence relation.

(4) Type Z: An n-ary relation ρ is called central if there is a non-empty proper subset Z of A such that

  (i) $(a_1, ..., a_n) \in \rho$ if at least one $a_i \in Z_i$;

  (ii) ρ is invariant under permutations of coordinates;

  (iii) $(a_1, ..., a_n) \in \rho$ if $a_i = a_j$ for some distinct i, j.

(5) Type R: Let $|A| = h^m$, $h \geq 3$, $m > l$ and let $T = \{n_1, ..., n_m\}$ be a set of equivalence relations such that

  (i) every equivalence relation n_i has h equivalence classes ε_j,

(ii) $\bigcap_{i=1}^{m} \varepsilon_i \neq \emptyset$ where ε_i is an equivalence class of n_i;
$(a_1, ..., a_h) \in \rho$ for each $i, i = 1, ..., m$, at least two of the elements $a_1, ..., a_h$ are equivalent in n_i, ρ is called a regular relation.

Definition 2.4 *Let ρ be of the above list. The algebra $\mathcal{A} = (A, \Omega)$ is ρ–polynomially complete if $P(\mathcal{A}) = Pol\rho$.*

We do not assume that $\mathcal{A}$ is finite and hence we have the following

Problem 1 : Is every ρ-polynomially complete algebra finite for ρ of type O, L, C, Z, R?

Definition 2.5 *Let $\mathcal{L} = (L, \wedge, \vee)$ be a lattice. $\mathcal{L}$ is called order-polynomially complete if every order-preserving map $f : L^n \to L$, $n \in \mathbb{N}$, is a polynomial function.*

A finite order-polynomially complete lattice is prepolynomially complete.

Problem 2 : Is every order-polynomially complete lattice finite?

This problem was presented by the author at the annual meeting of the Austrian Mathematical Society in Vienna 1973.
The usual approach to attack such problems is the comparison of cardinalities [10]. This was less successful in the case of lattices. But for ρ-polynomially complete algebras of some types one can show with this method that they are finite.

3 Doubly irreducible monotone functions

Definition 3.1 *An element a is join-irreducible if from $a = b \vee c$ it follows $a = b$ or $a = c$. An element a is meet irreducible if from $a = b \wedge c$ it follows $a = b$ or $a = c$. An element a is doubly irreducible [4]p.49 if it is both join- and meet-irreducible.*

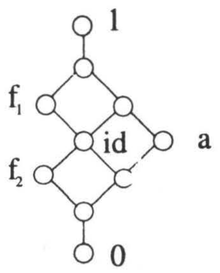
Lemma 3.2 *A finite lattice $\mathcal{L} = (L; \wedge, \vee)$ is order-polynomially complete if and only if the lattice $Mon(L, L)$ does not contain any doubly irreducible elements besides the identity or constant.*

Proof. Every polynomial function of $\mathcal{L}$ is generated from the identity and constants by the use of the operators meet $\wedge$ and join $\vee$. □

Example 3.3 *Let $\mathcal{L} = (L; \wedge, \vee)$ be a bounded lattice such that 0 and 1 are doubly irreducible and $L = [0, b] \cup [a, 1]$ for some $a, b \in L$. Then the monotone function* $f(x) = \begin{cases} 1 & \textit{for} \quad x \in [a, 1] \\ 0 & \textit{for} \quad x \in [0, b] \end{cases}$ *is doubly irreducible.*

Proof. Assume that $f(x) = g(x) \vee h(x)$. Hence we have $f(a) = g(a) \vee h(a) = 1$. As 1 is $\vee$-irreducible we may conclude without loss of generality that $g(a) = 1$. As $f(b) = g(b) \vee h(b) = 0$ we have $g(b) = 0$ and hence $f = g$. For $f(x) = g(x) \wedge h(x)$ we can use similar arguments. □

Example 3.4 *We consider the lattice $Mon(K_3, K_3)$ where K_3 is 3-element chain with $0 < a < 1$. This lattice has the following Hasse diagram.*



The monotone function f_1 with $f_1(0) = 0$, $f_1(a) = 1$, $f_1(1) = 1$ is doubly irreducible and so is the dual function $f_2, f_2(0) = 0 f_2(a) = 0, f_2(1) = 1$. Both functions do not preserve the central relation [17] of K_3 with center a. (A central relation is a special tolerance).

From the above lemma it follows.

Proposition 3.5 *Let $\mathcal{L}$ be a not necessarily finite lattice. If the lattice $Mon(L, L)$ contains a doubly irreducible element other than the identity or constant then $\mathcal{L}$ is not order-polynomially complete.*

Later we will use the following well known fact.

Proposition 3.6 *If the not necessarily finite lattice $\mathcal{L}$ contains a nontrivial tolerance (not the all relation or the identity relation) then $\mathcal{L}$ is not order-polynomially complete.*

Proof. Let θ be a nontrivial tolerance of $\mathcal{L}$ and $(a, b) \notin \theta$ with $a < b$. Let $(c, d) \in \theta$ with $c < d$. We consider $f(x) = \begin{cases} b & \text{for } x \geq d \\ a & \text{else} \end{cases}$ which is a monotone function. f cannot be a polynomial function because f does not preserve θ. □

Remark. If there exists a nontrivial tolerance θ on $\mathcal{L}$ then there may be a very large set of monotone functions which do not preserve θ. But it seems to be difficult to determine a doubly irreducible monotone function among them.

4 Chain-compatible lattices

Definition 4.1 *Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a lattice $a, b \in L$ $a \leq b$. The codimension of b in respect to a is the maximal cardinality of the chains from a to b. $codim(a, b) := \max\{|C| \mid C$ chain from a to $b\}$.*

Definition 4.2 *Let $\mathcal{L} = (L; \wedge, \vee)$ be a lattice and λ a cardinal number. $\mathcal{L}$ is called λ-chain-compatible if for all $a, b, c \in L$ with $a \leq b$ the following holds:*

$$\begin{aligned} co\dim(a, b) < \lambda \quad &\textit{implies} \quad co\dim(a \wedge c, b \wedge c) < \lambda \\ &\textit{and} \quad co\dim(a \vee c, b \vee c) < \lambda \end{aligned}$$

$\mathcal{L}$ is called chain-compatible if $\mathcal{L}$ is λ-chain-compatible for every $\lambda \leq |L|$.

Lemma 4.3 *Let $\mathcal{L} = (L; \wedge, \vee)$ be λ-chain-compatible with $\omega \leq \lambda \leq |L|$ and let $a, b, c \in L$ with $a \leq b \leq c$. If $codim(a, b) < \lambda$ and $codim(b, c) < \lambda$ then $codim(a, c) < \lambda$.*

Proof. Let C be a chain of maximal cardinality from a to c. We consider the maps $f_\vee : C \to [b, c]$, $f_\vee(x) = b \vee x$ and $f_\wedge : C \to [b, c]$, $f_\wedge(x) = b \wedge x$. Both maps are obviously monotone. Furthermore we consider the following equivalence relations $\theta_\vee := \ker f_\vee$, $\theta_\wedge := \ker f_\wedge$ and $\theta := \theta_\wedge \cap \theta_\vee$. As there exists an injective map ${}^{C}/_{\theta_\vee} \to [b, c]$ we have $\left|{}^{C}/_{\theta_\vee}\right| < \lambda$ and likewise $\left|{}^{C}/_{\theta_\wedge}\right| < \lambda$. We conclude that $\left|{}^{C}/_{\theta}\right| \leq \left|{}^{C}/_{\theta_\wedge}\right| \cdot \left|{}^{C}/_{\theta_\vee}\right| < \lambda \cdot \lambda = \lambda$. Now it remains to show that vor every equivalence class $[c]_\theta = \{x \mid x\theta c,\ x \in L\}$ we have $|[c]_\theta| < \lambda$.
From $codim(a, b) < \lambda$ it follows $codim(c, b \vee c) < \lambda$ by hypothesis and similiarly $codim(b \wedge c, c) < \lambda$. If we denote $\downarrow c = \{x \mid x \leq c, x \in L\}$ and dually $\uparrow c$ we have $[c]_\theta = \{[c]_\theta \cap \downarrow c\} \cup \{[c]_\theta \cap \uparrow c\}$. Now $[c]_\theta \cap \downarrow c$ is a chain in $[b \wedge c, c]$ and hence $|[c]_\theta \cap \downarrow c| < \lambda$. Similiarly we have $|[c]_\theta \cap \uparrow c| < \lambda$ and finally $|[c]_\theta| < \lambda$. Altogether we have $|C| < \lambda$. □

Lemma 4.4 *Let $\mathcal{L}$ have a finite interval $[k, \ell]$ and an infinite chain C. If $\mathcal{L}$ is λ-chain-compatible for $\omega \leq \lambda \leq |C|$ then $\mathcal{L}$ contains a nontrivial congruence.*

Proof. We consider the congruence θ which is generated by all pairs (u, v) such that $codim(u \wedge v, v) < \lambda$. θ is not the identity relation as $\mathcal{L}$ contains a finite interval. We will show that θ fulfills the condition

$(*)$ $(a,b) \in \theta$ if and only if $codim(a \wedge b, b) < \lambda$.
This condition holds for the generators. Obviously $(*)$ holds for $(c,c) \in \theta$ then also for $(b,a) \in \theta$. Now we assume that $(*)$ holds for $(a,b) \in \theta$ and $(b,c) \in \theta$.Then we have $co\dim(a \wedge c, b \wedge c) < \lambda$ by λ-chain-compatibility and $codim(b \wedge c, b \wedge c) < \lambda$ by hypothesis. As we have $codim(a \wedge c, c) < \lambda$ also $(a,c) \in \theta$ fulfills the condition $(*)$.
Now assume that $(a,b) \in \theta$ and $(c,d) \in \theta$ fulfil $(*)$. We assume furthermore that $a \leq b$ and $c \leq d$. By the hypothesis we have $codim(a \vee d, b \vee d) < \lambda$ and $codim(a \vee c, a \vee d) < \lambda$ and hence $codim(a \vee c, b \vee d) < \lambda$. Therefore $(a \vee c, b \vee d) \in \theta$ fulfills $(*)$. If we have $a \not\leq b$ and $c \not\leq d$ we put $a \wedge b$ instead of a and $c \wedge d$ instead of c. Then $(a \wedge b \wedge c \wedge d, b \vee d) \in \theta$ fulfills $(*)$ and hence also

$$((a \wedge b \wedge c \wedge d) \vee ((a \vee c) \wedge (b \wedge d), (b \vee d) \vee (b \vee d))) \in \theta$$

fulfills $(*)$. This is nothing else than $((a \wedge c) \wedge (b \vee d), b \vee d) \in \theta$ fulfills $(*)$. Because of $(*)$ θ is not the all relation. □

Theorem 4.5 *Let $\mathcal{L}$ be a λ-chain-compatible lattice of infinite length. If $\mathcal{L}$ is simple then every nontrivial interval $[a,b]$ is of infinite length and has the same cardinality as any other nontrivial interval $[c,d]$ of $\mathcal{L}$.*

Proof. We change the proof of 4.4 in the following way. Let L have at least two types of infinite interval namely $||[a,b]|| = \lambda_1$ and $||[c,d]|| = \lambda_2$. $\lambda_1 < \lambda_2$. Then the intervals of cardinality λ_1 define a congruence as above and L is not simple. □

Proposition 4.6 *If $\mathcal{L}$ is chain-compatible and of finite length then* L *is modular.*

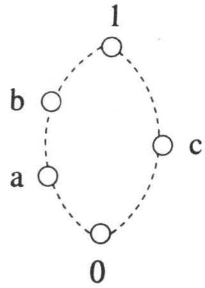
Proof. If $\mathcal{L}$ is not modular the $\mathcal{L}$ has a sublattice which is isomorphic to N_5.

We have
$$\begin{aligned} &co\dim(r,c) \geq co\dim(r \vee b, c \vee b) \\ &= co\dim(b,t) \geq co\dim(b \wedge c, t \wedge c) \\ &= co\dim(r,c)) \end{aligned}$$

Similarly we have $co\dim(r,c) = co\dim(a,t)$ and $co\dim(a,t) = co\dim(b,t)$. □

Remark. The following lattice is chain-compatible but not modular. We assume that the intervals are of cardinality ω_0. For $\lambda = \omega_0$ the following holds.

$$\begin{aligned} co\dim\ &(a \vee c, b \vee c) = co\dim(1,1) = 0 \leq \lambda \\ &(a \vee c, b \vee c) = co\dim(1,1) = 0 \leq \lambda \end{aligned}$$



Theorem 4.7 *Every bounded modular lattice is chain-compatible.*

Proof. According to Dedekind's principle [6], page 162, for all elements a, b the intervals $[a, a \vee b]$ and $[a \wedge b, b]$ are order isomorphic. If we put $b \wedge c$ instead of b we have

$$[a, a \vee (b \wedge c)] \simeq [a \wedge (b \wedge c)],\ b \wedge c.$$

If we assume $a \leq b$ it follows

$$[a \wedge c, b \wedge c] \simeq [a, a \vee (b \wedge c)].$$

Furthermore we have

$$[a, a \vee\ b \wedge c] \subseteq [a, b]$$

and hence

$$h(b \wedge c) - h(a \wedge c) \leq h(b) - h(a) < \lambda.$$

For the other case we put $a \vee c$ instead of a and we use similar arguments together with $(a \vee c) \wedge b = a \vee (c \wedge b)$ for $a \leq b$. □

5 Order-polynomially complete modular lattices.

Theorem 5.1 *A modular lattice $\mathcal{L}$ with a finite interval is order-polynomially complete if and only if $\mathcal{L}$ is a finite projective geometry.*

Proof. In [13] it is shown that every finite projective geometry is order-polynomially complete.

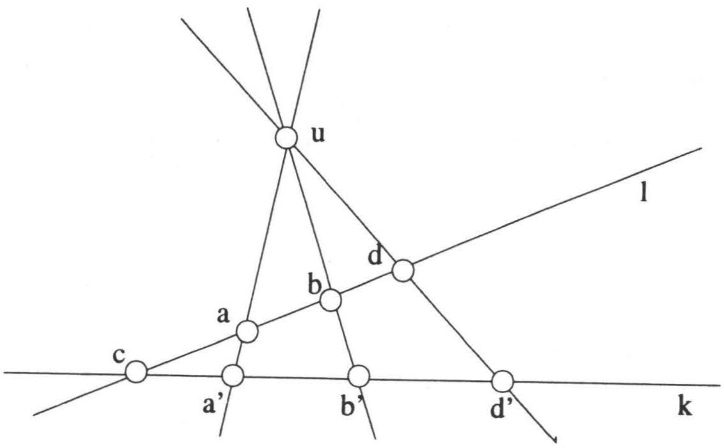
On the other hand let $\mathcal{L}$ be order-polynomially complete. Then $\mathcal{L}$ is bounded [5], [8]. If $\mathcal{L}$ contains an infinite chain then by theorem 4.7 and lemma 4.4 $\mathcal{L}$ has a nontrivial congruence relation. But this is contradiction that $\mathcal{L}$ is simple (Prop. 3.5). Therefore $\mathcal{L}$ can have only finite chains.

If $\mathcal{L}$ is infinite and of cardinality α then $\mathcal{L}$ has a antichain of the same cardinality. But according to Erné-Schweigert [5] $\mathcal{L}$ cannot contain a forest of the same cardinality as $\mathcal{L}$. Hence $\mathcal{L}$ is finite. A finite modular lattice $\mathcal{L}$ is order-polynomially complete if and only if $\mathcal{L}$ is a projective geometry. □

6 Perspectivities and projectivities

In the following we assume that $\mathcal{P} = (P; \wedge, \vee)$ is the lattices of subspaces of the $\mathbb{R}^2$. Hence $\mathcal{P}$ is the real projective plane.
A range of a line ℓ is the set of all points on the line ℓ. A pencil is the set of all line through a given point u.

Definition 6.1 *[4] A perspectivity is a bijection between the ranges of two lines defined by the sections of one pencil.*



The perspectivity maps a into a', b into b' d into d', and so on.
From the drawing we immediately get

Proposition 6.2 *Every perspectivity p can be presented by a polynomial function*

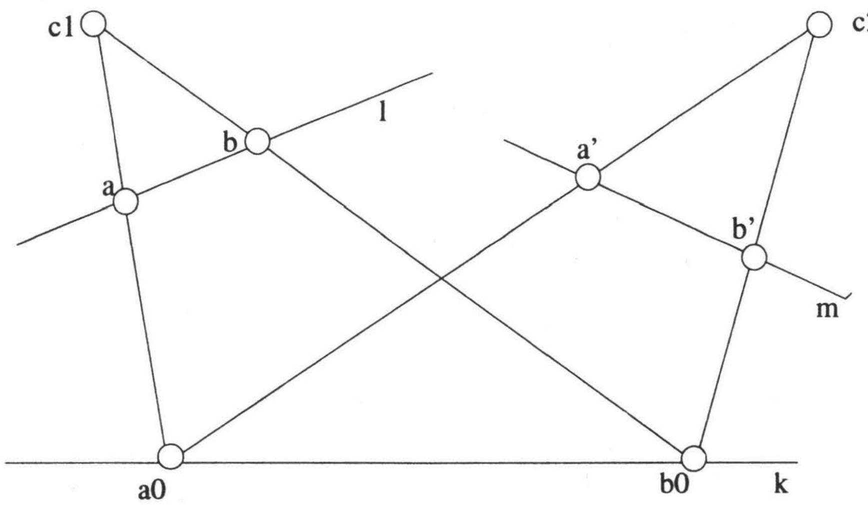
$$p(x) = ((x \wedge \ell) \vee u) \wedge k$$

Remark. $p(x)$ is a lattice homomorphism for the points x on the line ℓ.
Remark. For the two lines k, ℓ we have the property

$$\begin{array}{l} k \leq \ell \vee u \\ k \wedge u = 0 \end{array}.$$

This properties for lines coincides with the definition of perspectivity for points in [12].

Definition 6.3 *A projectivity is a product of perspectivities.*



To avoid brackets we assume in the following that $\wedge$ binds stronger than $\vee$. We have

$$p_1(x) = [(x \wedge \ell) \vee c_1] \wedge k$$
$$p_2(x) = [(x \wedge k) \vee c_2] \wedge m$$

$$\begin{aligned} p_2(p_1(x)) &= [(p_1(x) \wedge k) \vee c_2] \wedge m \\ &= [([(x \wedge \ell) \vee c_1] \wedge k \wedge k) \vee c_2] \wedge m \\ &= [(x \wedge \ell \vee c_1) \wedge k \vee c_2] \wedge m \end{aligned}$$

Theorem 6.4 *Every projectivity relating ranges of different lines can be presented by*

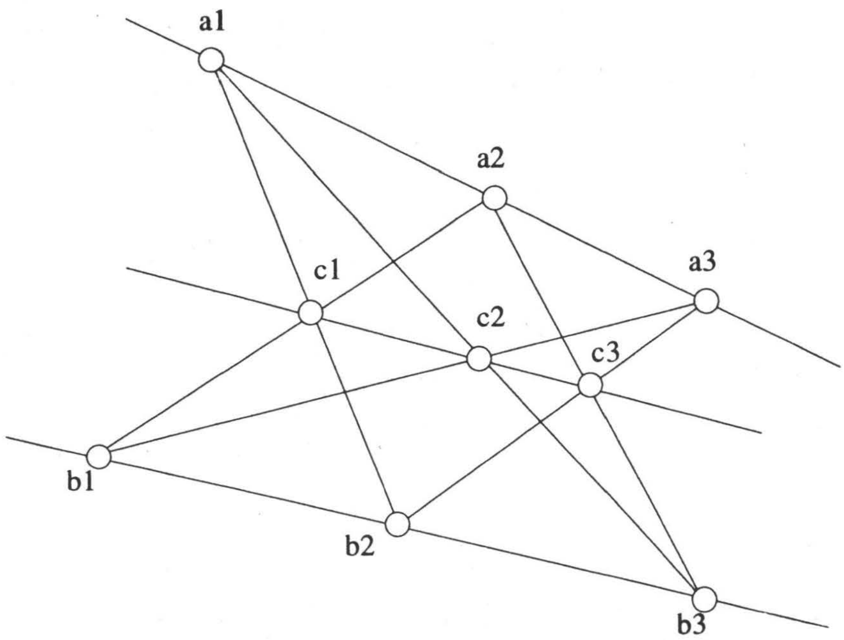
$$p(x) = p_2(p_1(x)) = [(x \wedge \ell \vee c_1) \wedge k \vee c_2] \wedge m$$

Proof. By [4], p.243, 14.51 every projectivity is a product of two perspectivities. Above we have shown that it has this form. □

Problem. Give a lattice theoretic proof of 6.4 using the theorem of Pappus in its lattice theoretic form.

Theorem 6.5 *(Fundamental theorem of projective goemetry.)*
A projectivity $p(x)$ is determined when three points of one range and the corresponding three of the other range are given.

Proof.



$$c_1 = (a_1 \vee b_2) \wedge (a_2 \vee b_1)$$
$$c_2 = (a_1 \vee b_3) \wedge (a_3 \vee b_1) \quad .$$
$$c_3 = (a_2 \vee b_3) \wedge (a_3 \vee b_{21})$$

By Pappus we have $c_1 \vee c_3 = c_1 \vee c_2 \quad (= c_2 \vee c_3)$.

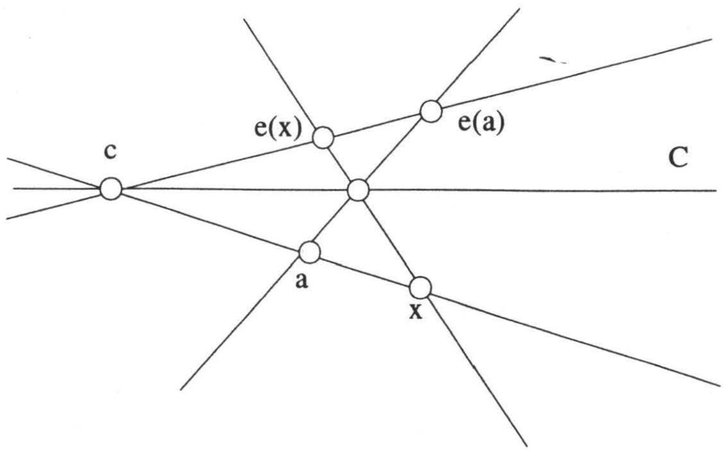
$$p(x) = [(x \wedge \ell \vee O_1) \wedge k \vee O_2] \wedge m.$$

We have

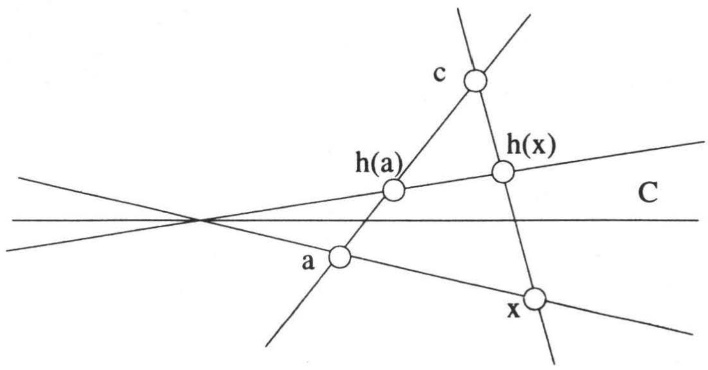
$$\ell = a_1 \vee a_2 \qquad m = b_1 \vee b_2$$
$$O_1 = b_1 \qquad k = c_1 \vee c_2 \qquad O_2 = a_1$$

7 Central collineations

A collineation τ is an order-automorphism of $\mathcal{P}$. τ maps collinear points into collinear points, ranges into ranges, pencils into pencils and so on.
A perspective collineation of $\mathcal{P}$ is a collineation with a center c and an axis C which leaves invariant all the lines through c and all points on C. A perspective collineation is either an elation or a homology according as the center and the axis are or are not incident. Throughout we assume that $a \neq c$ and $a \not\leq C$.



The elation $e(x)$ is given by C, c, a and $e(a)$.



The homology $h(x)$ is given by C, c, a and $h(a)$.

Lemma 7.1 *Let S be a finite set of points of $\mathcal{P}$. If the central collineation τ is given by C, c, a and $\tau(a)$ then we have $\tau|_S = p|_S$ where*

$$p(x) = \bigvee_{b \in S} (x \wedge b \vee c) \wedge [(x \wedge b \vee a) \wedge C \vee \tau(a)] \qquad (7.1)$$

Proof. The proof can be read from the drawings from the above examples. If $c \in C$ we have

$$p(c) = [(c \vee a) \wedge C \vee \tau(a)] = c \wedge (c \vee \tau(a)) = c.$$

If $r \in S$ with $r \leq C$ we have

$$\begin{aligned} p(r) &= (r \vee c) \wedge [(r \vee a)] \wedge C \vee \tau(a) \\ &= (r \vee c) \wedge (r \vee \tau(a)) = r \end{aligned}$$

Theorem 7.2 *If $\mathcal{P}$ is a finite projective plane and S is set of points of $\mathcal{P}$ then very central collineation τ given by C, c, a and $\tau(a)$ is of the above form (7.1).*

Remark. We observe that p is $\wedge-$homomorphism of $\mathcal{P}$ and $p(x) \leq \tau(x)$ for every $x \in P$, $\mathcal{P} = (P, \wedge, \vee)$.

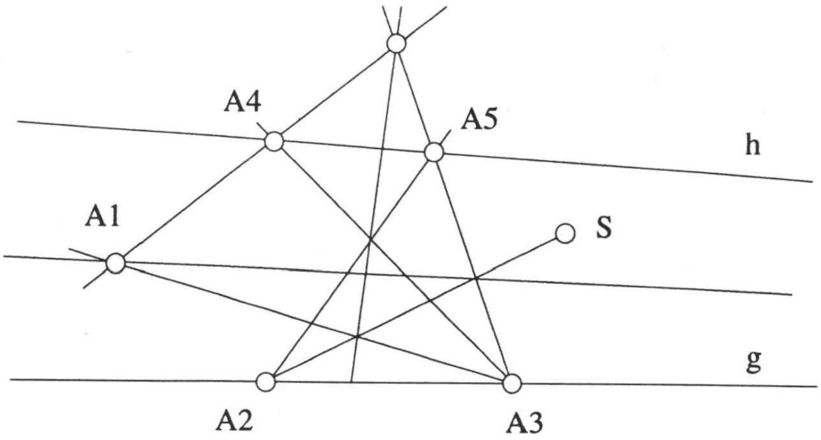
Corollary 7.3 *For every finite projective plane $\mathcal{P}$ with a set S of points, c a point, C a line, not incident with c and C the polynomial $p(x)$ of the form (7.1) is a polynomial automorphism.*

8 Geometric constructions with a straight edge

In this section we will consider geometric constructions with a straight edge on a sheet of paper. The sheet of paper will be conceived as the real projective plane under the hypothesis as given in Bieberbach §1 [2]. This projective plane will be considered according to Birkhoff p.30 as a geometric lattice $\mathcal{L} = (L; \wedge, \vee)$.
A straight edge is a ruler without measurement. The construction with a straight edge can be described by a kind of polynomial function of $\mathcal{L}$.
Every construction starts with a finite set of given points and lines. The points are denoted by $A_1, ..., A_{n-m}$ and the lines by $g_{n-m+1}, ..., g_n$.

Theorem 8.1 *Every point and every line which can be constructed by a given set $R = \{A_1, ..., A_{n-m}, g_{n-m+1}, ..., g_n\}$ of points and lines are images from R by polynomial functions of $\mathcal{L}$.*

Proof. A geometric construction with a straight edge is actually performed on a sheet of paper. If the construction lies within the boundaries of the paper the polynomial function can be found just by following the steps of the construction. A problem are points and lines outside the sheet of paper. In most cases the theorem of Desargues has to be applied. According to Bieberbach there are three fundamental constructions. We will confine us to the
First fundamental construction.
Let g, h be two lines which meet in a point A outside of the boundaries of the sheet of paper. How can one draw a line from a to a point A_1 on the paper?



We choose the points A_2, A_3 on the line g, the points A_4, A_5 on h and a point S. Then we have the line

$$A' =$$
$$(A_2 \vee S) \wedge \{[(A_1 \vee A_3) \wedge (A_2 \vee S) \vee (A_2 \vee A_5) \wedge (A_4 \vee A_3)] \wedge (A_1 \vee A_4) \vee A_5\}$$

The polynomial function is given by

$$p(x_1, ..., x_6) =$$
$$(x_2 \vee x_6) \wedge \{[(x_1 \vee x_3) \wedge (x_2 \vee x_6) \vee (x_2 \vee x_5) \wedge (x_4 \vee x_3)] \wedge (x_1 \vee x_4) \vee x_5\}$$

for the set $R = \{A_1, ..., A_5, S\}$.

Theorem 8.2 *Let the point A or respectively the line g be an image of the polynomial function $p(x_1, ..., x_n)$ from the set $R = \{A_1, ..., A_n, g_1, ..., g_m\}$. Then the point A or respectively the line g can be constructed from R finitely many steps only with a straight edge.*

Proof. Let us cosider $p(A_1, ..., A_n, g_1, ..., g_m)$. We reduce $p(x_1, ..., x_n)$ in the following way. We delete every subword $w(x_1, ..., x_n)$ of $p(x_1, ..., x_n)$ which has the property $w(A_1, ..., A_n, g_1, ..., g_m) = 1$ (the whole plane) or 0.
Furthermore we apply $x \vee x = x$ if necessary. $\overline{p}(x_1, ..., x_n)$ will still have the point A as an image for R. Now we proceed by induction. If $\overline{p}(x_1, ..., x_n) = x_i$, $i = 1, ..., n$ it can be constructed as we have the element A_i or g_i at our disposal. Now assume

$$\overline{p}(x_1, ..., x_n) = \overline{q}_1(x_1, ..., x_n) \vee \overline{q}_2(x_1, ..., x_n)$$

or

$$\overline{p}(x_1, ..., x_n) = \overline{q}_1(x_1, ..., x_n) \wedge \overline{q}_2(x_1, ..., x_n)$$

then we have again a construction by straight edge: in the first case as a line through two points and in the second case as a section of two lines.

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