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ON THE CONVERGENCE AT INFINITY OF SOLUTIONS WITH FINITE DIRICHLET INTEGRAL TO THE EXTERIOR DIRICHLET PROBLEM FOR THE STEADY PLANE NAVIER-STOKES SYSTEM OF EQUATIONS

Dan Socolescu

1. INTRODUCTION

Let Ω be a two-dimensional domain exterior to a compact set Δ with smooth boundary $\partial\Delta$ and assume for the sake of simplicity and without loss of generality that diam $\Delta \geqq 2$. The steady flow in Ω of a viscous incompressible fluid past the obstacle Δ with uniform velocity $\underset{\sim}{v}_\infty$ at infinity is described by the Navier-Stokes equations and the continuity equation

$$(1) \qquad \Omega : \begin{cases} \nu\nabla^2\underset{\sim}{v} - (\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v} - \rho_\infty^{-1}\nabla p = \underset{\sim}{0} , \\ \nabla\cdot\underset{\sim}{v} = 0 , \end{cases}$$

with the boundary conditions

$$(2) \qquad \partial\Delta : \quad \underset{\sim}{v} = \underset{\sim}{0} ,$$

$$(3) \qquad [0,2\pi) : \quad \lim_{r\to\infty} \underset{\sim}{v}(r,\theta) = \underset{\sim}{v}_\infty .$$

Here ν is the coefficient of viscosity, $\nabla := (\partial_x,\partial_y)$ is the Nabla differential operator, $\underset{\sim}{v} = (u,v)$ is the velocity vector, ρ_∞ is the constant fluid density, which we take in the following without loss of generality to be equal to one, p is the pressure, r is the radius vector, i.e. the distance from the given point to the origin, taken interior to Δ, and θ is the polar angle.

In his study on this exterior Dirichlet problem in 1933, Leray [15] constructed a certain solution $(\underset{\sim}{v}_L,p_L)$ satisfying (1) and (2) and having a velocity with finite Dirichlet integral

$$(4) \qquad \int_\Omega |\nabla\underset{\sim}{v}|^2 dxdy < \infty .$$

Whether this solution had the desired limit behaviour (3) was left open. Leray's construction went as follows. Let Ω_R be the set of points in Ω of radius vector $r < R$. He first proved that for every $R > \max_{\Delta} r =: r_\Delta \geqq 1$ and every constant vector $\underset{\sim}{v}_\infty$ there is at least one solution $(\underset{\sim}{v}_R, p_R)$ of (1) in Ω_R satisfying the boundary conditions on $\partial\Omega_R$

$$(5) \qquad \underset{\sim}{v}_R = \begin{cases} \underset{\sim}{0} & \text{on } \partial\Delta\,, \\ \underset{\sim}{v}_\infty & \text{for } r = R\,. \end{cases}$$

Concerning all such (velocity) solutions $\underset{\sim}{v}_R$ Leray proved the existence of a uniform bound for the Dirichlet integral, namely for some positive constant C independent of R and ν

$$(6) \qquad \int_\Omega |\nabla \underset{\sim}{v}_R|^2 dxdy \leqq C^2(1+\nu^{-1})^2 .$$

He then showed that a sequence $R_i \to \infty$ exists, such that the solutions $(\underset{\sim}{v}_{R_i}, p_{R_i})$ of (1) and (5) in Ω_{R_i} converge uniformly together with all their first order derivatives in any compact subset of $\overline{\Omega}$ to a solution $(\underset{\sim}{v}_L, p_L)$ satisfying (1), (2) and (4) - cf. also [3], [5], [8], [10], [14] -. The further behaviour of $\underset{\sim}{v}_L$ and p_L as $r \to \infty$ was left unsettled and remained so for more than four decades.

In 1974 Gilbarg and Weinberger [11] proved that

(i) the Leray solution $(\underset{\sim}{v}_L, p_L)$ is bounded,

(ii) the velocity $\underset{\sim}{v}_L$ has a limit in the mean at infinity

$$(7) \qquad \lim_{r\to\infty} \int_0^{2\pi} |\underset{\sim}{v}_L(r,\theta) - \underset{\sim}{v}_0|^2 d\theta = 0\;, \quad \lim_{r\to\infty} \hat{\underset{\sim}{v}}_L(r) = \underset{\sim}{v}_0\;,$$

where

$$\hat{\underset{\sim}{v}}_L(r) = \frac{1}{2\pi}\int_0^{2\pi} \underset{\sim}{v}_L(r,\theta)\,d\theta\;, \quad |\underset{\sim}{v}_0| = \lim_{r\to\infty} \max_{[0,2\pi]} |\underset{\sim}{v}_L|\;,$$

(iii) the pressure p_L converges pointwise and in the mean to the same limit at infinity

$$(8_1) \qquad [0,2\pi) : \lim_{r\to\infty} p_L(r,\theta) = p_\infty ,$$

$$(8_2) \qquad \lim_{r\to\infty} \int_0^{2\pi} |p_L(r,\theta) - p_\infty|^2 d\theta = 0 .$$

The questions of whether $\underset{\sim}{v}_L$ tends pointwise to its asymptotic value $\underset{\sim}{v}_0$ and whether $\underset{\sim}{v}_0$ is equal to the prescribed asymptotic value $\underset{\sim}{v}_\infty$ were however left open.
Four years later the same authors [12] investigated the asymptotic behaviour of an arbitrary solution with finite Dirichlet integral $(\underset{\sim}{v}_D, p_D)$ of (1) and (2) and showed that

(i) the velocity $\underset{\sim}{v}_D$ has at infinity the behaviour

$$(9) \qquad \lim_{r\to\infty} |\underset{\sim}{v}_D(r,\theta)|^2/\ln r = 0 , \quad \text{uniformly in } \theta ,$$

$$(10) \qquad \begin{cases} \lim_{r\to\infty} \int_0^{2\pi} |\underset{\sim}{v}_D(r,\theta) - \hat{\underset{\sim}{v}}_D(r)|^2 d\theta = 0 , \\ \\ \lim_{r\to\infty} |\hat{\underset{\sim}{v}}_D(r)| = \lim_{r\to\infty} \max_{\theta\in[0,2\pi]} |\underset{\sim}{v}_D(r,\theta)| . \end{cases}$$

If furthermore $0 < \lim_{r\to\infty} |\hat{\underset{\sim}{v}}_D(r)| < \infty$, then there exists a constant vector $\underset{\sim}{v}_0$ such that, by denoting $i = \sqrt{-1}$,

$$(11) \qquad \begin{cases} \lim_{r\to\infty} |\hat{\underset{\sim}{v}}_D(r)| = |\underset{\sim}{v}_0| , \\ \\ \lim_{r\to\infty} \arg (\hat{u}_D(r) + i\hat{v}_D(r)) = \arg (u_0 + iv_0) , \end{cases}$$

$$(12) \qquad \lim_{r\to\infty} \int_0^{2\pi} |\underset{\sim}{v}_D(r,\theta) - \underset{\sim}{v}_0|^2 d\theta = 0 .$$

(ii) the pressure p_D converges pointwise and in the mean to the same limit at infinity

$$(13_1) \qquad [0,2\pi) : \lim_{r\to\infty} p_D(r,\theta) = p_\infty ,$$

$$(13_2) \qquad \lim_{r\to\infty} \int_0^{2\pi} |p_D(r,\theta) - p_\infty|^2 d\theta = 0 .$$

The questions of whether every bounded $\underset{\sim}{v}_D$ converges pointwise to its asymptotic mean value $\underset{\sim}{v}_0$ and whether the boundedness condition can be dropped were however left open.

In 1988 Amick [2] showed that

(i) every $\underset{\sim}{v}_D$ is bounded. If furthermore the flow is "symmetric" (i.e. if $\partial\Delta$ is symmetric about the x-axis and $\underset{\sim}{v}_\infty = (u_\infty, 0)$, then one can find at least one solution $(\underset{\sim}{v}_D, p_D)$ with p_D and u_D even in y and v_D odd in y), then $\underset{\sim}{v}_D$ tends pointwise to its asymptotic mean value $\underset{\sim}{v}_0$.

The questions of whether every "general" $\underset{\sim}{v}_D$ converges pointwise to its asymptotic mean value $\underset{\sim}{v}_0$ and whether $\underset{\sim}{v}_0 = \underset{\sim}{v}_\infty$ in the particular case of the Leray solution were however left open.

The present paper is concerned with these open problems. In fact we prove that

(i) every $\underset{\sim}{v}_D$ converges pointwise to its asymptotic mean value $\underset{\sim}{v}_0$; in the particular case of the Leray solution, $\underset{\sim}{v}_0$ is equal to the prescribed asymptotic value $\underset{\sim}{v}_\infty$ - Theorem 2 and respectively Theorem 3 -.

(ii) the exterior Dirichlet problem to the steady two-dimensional Navier-Stokes equations possesses the Liouville property, i.e. if the asymptotic mean value $\underset{\sim}{v}_0$ of the velocity $\underset{\sim}{v}_D$ is equal to zero, then $\underset{\sim}{v}_D$ is identically zero - Theorem 1 -.

(iii) the Leray sequence of solutions $(\underset{\sim}{v}_{R_i}, p_{R_i})$, $i \in N$, of (1) and (5) in Ω_{R_i} converges quasi-uniformly on $\overline{\Omega}$ to $(\underset{\sim}{v}_L, p_L)$ - Theorem 3 -.

These results were announced in [18], [19], [20], [21], [22]. However, since some proofs presented there were only sketched or incomplete, we give here all the details of the revised proofs.

2. CONVERGENCE AT INFINITY OF A SOLUTION WITH FINITE DIRICHLET INTEGRAL OF THE VELOCITY

Theorem 1 [21]. *Every solution $(\underset{\sim}{v}_D, p_D)$ of (1), (2) and (4) with zero asymptotic mean value, i.e. $(\underset{\sim}{v}_0, p_\infty) = (\underset{\sim}{0}, 0)$, is identically zero.*

Remark 1. *Since the pressure p_D is defined up to an additive constant, its asymptotic (mean) value p_∞ can always be taken to be equal to zero.*

For the proof of Theorem 1 we need the following results:

Theorem of Gilbarg and Weinberger [12, pp 384, 396, 399, 400]. *The vorticity $\omega_D := \partial_y u_D - \partial_x v_D$ of the velocity $\underset{\sim}{v}_D$, its gradient $\nabla\omega_D$ as well as $r^{1/2} ln^{-1/4} r \, \nabla\omega_D$ are square integrable in Ω, i.e.*

$$(14) \qquad \int_\Omega r \ln^{-1/2} r |\nabla\omega_D|^2 dxdy \leq C \int_\Omega \omega_D^2 dxdy \leq 2C \int_\Omega |\nabla \underset{\sim}{v}_D|^2 dxdy \ ,$$

where C is a positive constant. Moreover

$$(15) \qquad \lim_{r\to\infty} r^{3/4} \ln^{-1/8} r |\omega_D(r,\theta)| = 0 \ , \text{ uniformly in } \theta \ ,$$

$$(16) \qquad |\omega_D(z_2) - \omega_D(z_1)| \leq C_1 \mu(R) |z_2 - z_1|^{1/2} \ , \ |z_j| > R + 2 \ , \ |z_2 - z_1| \leq 1 \ , \ z_j = x_j + iy_j \ , \ j = 1,2 \ ,$$

where C_1 is a positive constant independent of R and

$$(17) \qquad \lim_{R\to\infty} R^{3/4} \ln^{-3/8} R \mu(R) = 0 \ .$$

Lemma 1 [20]. *The derivatives of the Helmholtz-Bernoulli function $H_D := \frac{1}{2}|\underset{\sim}{v}_D|^2 + p_D$ have at infinity the behaviour*

$$(18) \qquad \lim_{r\to\infty} |\nabla H_D(r,\theta)| r^{3/4} \ln^{-1} r = 0 \ , \text{ uniformly in } \theta \ .$$

Lemma 2 [20], [21]. *There exists a positive constant λ, such that*

$$(19_1) \qquad \int_0^{2\pi} (\underset{\sim}{v}_D(r,\theta) - \hat{\underset{\sim}{v}}_D(r))^2 d\theta = O(r^{-2\lambda}) \ ,$$

$$\text{for } r \to \infty \ ,$$

$$(19_2) \qquad \hat{\underset{\sim}{v}}_D(r) - \underset{\sim}{v}_o = O(r^{-\lambda}) \ .$$

where "O" is the Landau symbol. Furthermore, if $\underset{\sim}{v}_0 = \underset{\sim}{0}$, then $\lambda = 1$.

Proof of Lemma 1. Let us first give the Navier-Stokes equations (1_1) the equivalent form

$$(20) \qquad \Omega : \begin{cases} \dfrac{\partial H}{\partial x} = \nu\dfrac{\partial \omega}{\partial y} - v\omega , \\ \dfrac{\partial H}{\partial y} = -\nu\dfrac{\partial \omega}{\partial x} + u\omega . \end{cases}$$

From (20) it follows immediately that H_D and ω_D are solutions of

$$(21) \qquad \Omega : \quad \nu\nabla^2 H - \underset{\sim}{v}_D\cdot\nabla H = \nu\omega_D^2 ,$$

and respectively

$$(22) \qquad \Omega : \quad \nu\nabla^2\omega - \underset{\sim}{v}_D\cdot\nabla\omega = 0 .$$

Next we write the continuity equation (1_2), the vorticity function ω_D and the equations (20), (21) and (22) in polar coordinates

$$(1_2') \qquad \Omega : \quad \frac{1}{r}\frac{\partial}{\partial r}[r(u\cos\theta + v\sin\theta)] - \frac{1}{r}\frac{\partial}{\partial\theta}(u\sin\theta - v\cos\theta) = 0 ,$$

$$(23) \qquad \Omega : \quad \frac{1}{r}\frac{\partial}{\partial r}[r(u\sin\theta - v\cos\theta)] + \frac{1}{r}\frac{\partial}{\partial\theta}(u\cos\theta + v\sin\theta) = \omega ,$$

$$(20') \qquad \Omega : \begin{cases} \dfrac{\partial H}{\partial r} = \dfrac{\nu}{r}\dfrac{\partial \omega}{\partial \theta} + (u\sin\theta - v\cos\theta)\omega , \\ \dfrac{1}{r}\dfrac{\partial H}{\partial \theta} = -\nu\dfrac{\partial \omega}{\partial r} + (u\cos\theta + v\sin\theta)\omega , \end{cases}$$

$$(21') \qquad \Omega : \quad \nu\left(\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial\theta^2}\right) - \frac{1}{r}\frac{\partial}{\partial r}[r(u_D\cos\theta + v_D\sin\theta)H]$$
$$+ \frac{1}{r}\frac{\partial}{\partial\theta}[(u_D\sin\theta - v_D\cos\theta)H] = \nu\omega_D^2 ,$$

$$(22') \qquad \Omega : \quad \nu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial\theta^2}\right) - \frac{1}{r}\frac{\partial}{\partial r}[r(u_D\cos\theta + v_D\sin\theta)\omega]$$
$$+ \frac{1}{r}\frac{\partial}{\partial\theta}[(u_D\sin\theta - v_D\cos\theta)\omega] = 0 .$$

Using (23) we give (20') another equivalent form

$$(20'')\qquad \Omega : \begin{cases} \dfrac{\partial H}{\partial r} = \dfrac{\nu}{r}\dfrac{\partial \omega}{\partial \theta} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 F) + \dfrac{1}{r}\dfrac{\partial G}{\partial \theta} , \\ \dfrac{1}{r}\dfrac{\partial H}{\partial \theta} = -\nu\dfrac{\partial \omega}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 G) - \dfrac{1}{r}\dfrac{\partial F}{\partial \theta} , \end{cases}$$

where

$$(24)\qquad \Omega : \begin{cases} F(r,\theta) := \frac{1}{2}[(u\sin\theta - v\cos\theta)^2 - (u\cos\theta + v\sin\theta)^2] , \\ G(r,\theta) := (u\sin\theta - v\cos\theta)(u\cos\theta + v\sin\theta) . \end{cases}$$

Let us write now (20) as an inhomogeneous Cauchy-Riemann equation

$$(20''')\qquad \Omega : \quad \frac{\partial(H + i\nu\omega)}{\partial \bar{z}} = \frac{i}{2}\bar{w}\omega ,$$

where $z = x + iy$ and $z = x - iy$ are the complex variables, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, $w := u - iv$ is the complex velocity. According to the Pompeiu formula [17], [23, p.22], the solution of (20''') in the disc $D(z,R) = \{\zeta \in C \mid |\zeta - z| < R, |z| > R + 2 \geqq r_\Delta + 2\}$ is given by

$$(25)\qquad D(z,R) : \quad H_D(\tilde{z}) + i\nu\omega_D(\tilde{z}) = \frac{1}{2\pi i}\{\int_{\partial D} \frac{H_D(\zeta) + i\nu\omega_D(\zeta)}{\zeta - \tilde{z}} d\zeta + PV \int_D (\zeta - \tilde{z})^{-1} \bar{w}_D(\zeta)\omega_D(\zeta)\, d\xi d\eta ,$$

where $\zeta = \xi + i\eta$ and PV denotes the Cauchy principal value. By differentiating (25) with respect to z , $\partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y)$, we obtain

$$(26)\qquad D(z,R) : \quad \frac{\partial(H_D + i\nu\omega_D)}{\partial z}(\tilde{z}) = \frac{1}{2\pi i}\{\int_{\partial D} \frac{H_D(\zeta) + i\nu\omega_D(\zeta)}{(\zeta - \tilde{z})^2} d\zeta + PV \int_D (\zeta - \tilde{z})^{-2}[w_D(\zeta)\omega_D(\zeta) - w_D(\tilde{z})\omega_D(\tilde{z})]\, d\xi d\eta ,$$

where we have used on one hand the equality

$$(27)\qquad D(z,R) : \quad \bar{\tilde{z}} - \bar{z} = -\frac{1}{\pi} PV \int_D \frac{d\xi d\eta}{\zeta - \tilde{z}} ,$$

and on the other hand the analyticity in $z \in D(z,R)$ of the line integral in (25) and respectively that of the modified area integral, i.e.

$$(28) \qquad D(z,R) : \quad PV \int_D \frac{\bar{w}_D(\zeta)\omega_D(\zeta)}{\zeta - \tilde{z}} d\xi d\eta + \bar{w}_D(z_0)\omega_D(z_0)(\bar{\tilde{z}} - \bar{z}_0)$$

in a fixed point $\tilde{z}_0 \in D(z,R)$, $\tilde{z}_0 \neq \tilde{z}$. We estimate next the two integrals in (26), taking without loss of generality $\tilde{z} = z$. From (15) as well as from the boundedness of H_D it follows then

$$(29) \qquad \int_{\partial D} \frac{H_D(\zeta) + i\nu\omega_D(\zeta)}{(\zeta - z)^2} d\zeta = O(R^{-1}) \quad , \quad \text{for } R \to \infty .$$

On the other hand using (16) as well as the boundedness of w_D we get

$$(30) \qquad \left| PV \int_D (\zeta - z)^{-2} [\bar{w}_D(\zeta)\omega_D(\zeta) - \bar{w}_D(z)\omega_D(z)] d\xi d\eta \right|$$

$$\leq \Big(PV \int_{|\zeta - z| \leq 1} + \int_{1 < |\zeta - z| < R}\Big) |\zeta - z|^{-2} \times$$

$$|\bar{w}_D(\zeta)\omega_D(\zeta) - \bar{w}_D(z)\omega_D(z)| d\xi d\eta \leq C\{\mu(R) +$$

$$\max_{\bar{D}} |\omega_D| \max_{\bar{D}} \Big[\Big|\frac{\partial \bar{w}_D}{\partial z}\Big| + \Big|\frac{\partial \bar{w}_D}{\partial \bar{z}}\Big| \Big] + \sup_{r \geq R + 2} |\omega_D| \ln R\} .$$

Using now the estimate [12, p 402]

$$(31) \qquad \lim_{r \to \infty} r^{3/4} \ln^{-1} r \, |\nabla \underset{\sim}{v}_D(r,\theta)| = 0 \text{ , uniformly in } \theta \text{ ,}$$

as well as (15), (17) and the boundedness of $\underset{\sim}{v}_D$, from (20'''), (26), (29) and (30) we infer

$$(32_1) \qquad \lim_{r \to \infty} r^{3/4} \ln^{-1} r \, \Big|\frac{\partial}{\partial z}(H_D + i\nu\omega_D)\Big| = 0 \text{ , uniformly in } \theta \text{ ,}$$

$$(32_2) \qquad \lim_{r \to \infty} r^{3/4} \, \Big|\frac{\partial}{\partial \bar{z}}(H_D + i\nu\omega_D)\Big| = 0 \text{ , uniformly in } \theta \text{ ,}$$

and the assertion of Lemma 1 then follows.

Proof of Lemma 2. By integrating (20'') with respect to θ between 0 and 2π we get

$$(33)\qquad (r_\Delta,\infty)\ :\ \begin{cases} \dfrac{d\hat{H}}{dr} = \dfrac{d\hat{F}}{dr} + \dfrac{2}{r}\hat{F}(r)\ , \\[2ex] \nu\dfrac{d\hat{\omega}}{dr} = \dfrac{d\hat{G}}{dr} + \dfrac{2}{r}\hat{G}(r)\ , \end{cases}$$

where, according to (24),

$$(34)\qquad (r_\Delta,\infty)\ :\ \begin{cases} \hat{F}(r) = \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi}\{[(u-\hat{u})\sin\theta - (v-\hat{v})\cos\theta]^2 - \\ \qquad [(u-\hat{u})\cos\theta + (v-\hat{v})\sin\theta]^2\}d\theta - \\ \qquad \hat{u}(\hat{u}^{c2} + \hat{v}^{s2}) - \hat{v}(\hat{u}^{s2} - \hat{v}^{c2})\ , \\[2ex] \hat{G}(r) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}[(u-\hat{u})\sin\theta - (v-\hat{v})\cos\theta]\times \\ \qquad [(u-\hat{u})\cos\theta + (v-\hat{v})\sin\theta]d\theta + \hat{u}(\hat{u}^{s2} - \\ \qquad \hat{v}^{c2}) - \hat{v}(\hat{u}^{c2} + \hat{v}^{s2})\ . \end{cases}$$

By

$$(35)\qquad (r_\Delta,\infty)\ :\quad \hat{f}^{cn(sn)}(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\cos n\theta(\sin n\theta)\,d\theta\ ,$$

$$n \in N \cup \{0\}\ ,\ \hat{f} := \hat{f}^{c0}\ ,$$

we denote here the Fourier coefficients of the periodic, smooth function $f(r,\theta)$, i.e.

$$(36)\qquad \Omega\ :\quad f(r,\theta) = \hat{f}(r) + 2\sum_{n=1}^{\infty}[\hat{f}^{cn}(r)\cos n\theta + \hat{f}^{sn}(r)\sin n\theta]\ .$$

Multiplying ($1_2'$) and (23) by $\sin\theta$ and respectively $\cos\theta$ and integrating with respect to θ we get

$$(37_1)\qquad (r_\Delta,\infty)\ :\quad \frac{d\hat{u}}{dr} = -\left(\frac{d}{dr} + \frac{2}{r}\right)(\hat{u}^{c2} + \hat{v}^{s2}) = \hat{\omega}^{s1}\ ,$$

$$(37_2)\qquad (r_\Delta,\infty)\ :\quad \frac{d\hat{v}}{dr} = -\left(\frac{d}{dr} + \frac{2}{r}\right)(\hat{u}^{s2} - \hat{v}^{c2}) = -\hat{\omega}^{c1}\ .$$

Using the Parseval equality, from (10) we infer

(38) $$\lim_{r\to\infty} \hat{u}_D^{cn(sn)}(r) = \lim_{r\to\infty} \hat{v}_D^{cn(sn)}(r) = 0 \ , \ n \in N \ .$$

Noting now that by the integral theorem of the mean there exist $\theta_1(r)$ and $\theta_2(r)$, such that

(39) $$[r_\Delta,\infty) : \quad \underset{\sim}{v}_D(r,\theta_1(r)) = \hat{\underset{\sim}{v}}_D(r) \ , \ H_D(r,\theta_2(r)) = \hat{H}_D(r) \ ,$$

and applying the Cauchy-Schwarz inequality we get

(40_1) $$\Omega : \quad |\underset{\sim}{v}_D(r,\theta) - \hat{\underset{\sim}{v}}_D(r)|^2 = |\int_0^{2\pi} \partial_\phi \underset{\sim}{v}_D(r,\phi)d\phi|^2 \leq 2\pi \int_0^{2\pi} |\partial_\phi \underset{\sim}{v}_D|^2 d\phi \ ,$$

(40_2) $$\Omega : \quad |\underset{\sim}{v}_D(r,\tilde{\theta}_2) - \underset{\sim}{v}_D(r,\tilde{\theta}_1)|^2 \leq 2\pi \int_0^{2\pi} |\partial_\phi \underset{\sim}{v}_D|^2 d\phi \ ,$$

and similarly for H_D, and hence , by taking account of (4), (14), (20) as well as of the boundedness of $\underset{\sim}{v}_D$

(41_1) $$[0,2\pi) : \quad \int_{r_\Delta}^{\infty} r^{-1}|\underset{\sim}{v}_D(r,\theta) - \hat{\underset{\sim}{v}}_D(r)|^2 dr \leq 2\pi \int_\Omega |\nabla \underset{\sim}{v}_D|^2 dxdy < \infty \ ,$$

(41_2) $$[0,2\pi) : \quad \int_{r_\Delta}^{\infty} r^{-1}|\underset{\sim}{v}_D(r,\tilde{\theta}_2) - \underset{\sim}{v}_D(r,\tilde{\theta}_1)|^2 dr < \infty \ ,$$

(42_1) $$[0,2\pi) : \quad \int_{r_\Delta}^{\infty} r^{-1}|H_D(r,\theta) - \hat{H}_D(r)|^2 dr \leq 2\pi \int_\Omega |\nabla H_D|^2 dxdy \leq 4\pi(\nu^2 \int_\Omega |\nabla \omega_D|^2 dxdy + \int_\Omega |\underset{\sim}{v}_D|^2 \omega_D^2 dxdy) < \infty \ ,$$

(42_2) $$[0,2\pi) : \quad \int_{r_\Delta}^{\infty} r^{-1}|H_D(r,\tilde{\theta}_2) - H_D(r,\tilde{\theta}_1)|^2 dr < \infty \ .$$

Integrating next (41_1) and (42_1) with respect to θ and the Parseval equality we obtain

(43) $$\int_{r_\Delta}^{\infty} r^{-1}|\hat{\underset{\sim}{v}}^{cn(sn)}(r)|^2 dr < \infty \ , \ n \in N \ ,$$

(44) $$\int_{r_\Delta}^{\infty} r^{-1}|\hat{H}^{cn(sn)}(r)|^2 dr < \infty \ , \ n \in N \ .$$

From the first equation (33) we get now

$$(45)\qquad \lim_{r\to\infty}\left(\frac{d\hat{F}_D}{dr}\Big/\frac{d\hat{H}_D}{dr} + \frac{2}{r}\hat{F}_D\Big/\frac{d\hat{H}_D}{dr}\right) = 1 \ .$$

On the other hand by integrating (33_1) with respect to r we obtain

$$(46)\qquad [r_\Delta,\infty) \ : \quad \hat{F}_D(r)/(\hat{H}_D(r) - H_0) - 2\int_r^\infty \frac{\hat{F}_D(\rho)}{\rho}d\rho/(\hat{H}_D(r) - H_0) = 1 \ ,$$

where, according to (10), (11), (13_1), (34) and (38),

$$(47)\qquad \lim_{r\to\infty}\hat{H}_D(r) := \lim_{r\to\infty}\left[\frac{1}{2}\hat{\underset{\sim}{v}}_D^2(r) + \frac{1}{4\pi}\int_0^{2\pi}|\underset{\sim}{v}_D(r,\theta) - \hat{\underset{\sim}{v}}_D(r)|^2 d\theta + \hat{p}_D(r)\right] = \frac{1}{2}\underset{\sim}{v}_0^2 + p_\infty =: H_0 \ ,$$

$$(48)\qquad \lim_{r\to\infty}\hat{F}_D(r) = 0 \ ,$$

$$(49)\qquad \lim_{r\to\infty}\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho = 0 \ .$$

Consequently we have to consider the following possibilities:

a) $\lim_{r\to\infty}\hat{F}_D(r)/(\hat{H}_D(r) - H_0)$ and $\lim_{r\to\infty} 2\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho/(\hat{H}_D(r) - H_0)$ do not exist,

b) the two limits exist but they are infinite and the limit of the ratio $2\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho/\hat{F}_D(r)$ is one, i.e. either

$$(50_1)\qquad \begin{cases} \lim\limits_{r\to\infty}\dfrac{\hat{F}_D(r)}{\hat{H}_D(r) - H_0} = \infty \ , \quad \lim\limits_{r\to\infty}\dfrac{2\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho}{\hat{H}_D(r) - H_0} = -\infty \ , \\ \lim\limits_{r\to\infty} 2\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho/\hat{F}_D(r) = 1 \ , \end{cases}$$

or

$$(50_2)\qquad \begin{cases} \lim\limits_{r\to\infty}\dfrac{\hat{F}_D(r)}{\hat{H}_D(r) - H_0} = -\infty \ , \quad \lim\limits_{r\to\infty}\dfrac{2\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho}{\hat{H}_D(r) - H_0} = \infty \ , \\ \lim\limits_{r\to\infty} 2\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho/\hat{F}_D(r) = 1 \ , \end{cases}$$

c) the two limits exist and they are finite, i.e.

(51) $$\lim_{r\to\infty} \hat{F}_D(r)/(\hat{H}_D(r) - H_0) = \alpha \ , \ \lim_{\to\infty} 2\int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho \times 1/(\hat{H}_D(r) - H_0) = \beta \ , \ \alpha + \beta = 1 \ ,$$

where either

c_1) $\alpha = 1$, $\beta = 0$,

or

c_2) $\alpha = 0$, $\beta = 1$,

or

c_3) $\alpha \neq 0$, $\beta = 1 - \alpha \neq 0$.

Let us study at first the case a). Taking into account (47), (48) and (49) we infer that, for a suitably chosen neighbourhood U_∞ of infinity

(52) $$U_\infty : \begin{cases} \hat{F}_D(r) = f(r)g(r) \ , \ \hat{H}_D(r) - H_0 = h(r)j(r) \ , \\ -\int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho = k(r)l(r) \ , \end{cases}$$

where without loss of generality we can assume that f, g, h, j, k, l are bounded, at least C^2-functions, such that

(53_1) $$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} h(r) = \lim_{r\to\infty} k(r) = 0 \ ,$$

(53_2) $$U_\infty : \begin{cases} f(r) > 0 \ , \ \frac{df}{dr} < 0 \ , \ \frac{d^2f}{dr^2} > 0 \ , \ h(r) > 0 \ , \ \frac{dh}{dr} < 0 \ , \\ \frac{d^2h}{dr^2} > 0 \ , \ k(r) \geq 0 \ , \ \frac{dk}{dr} \leq 0 \ , \ \frac{d^2k}{dr^2} \geq 0 \ , \end{cases}$$

(54) $\lim_{r\to\infty} g(r)$, $\lim_{r\to\infty} j(r)$ and $\lim_{r\to\infty} l(r)$ do not exist.

Moreover, from (33_1), (46) and (52) it follows

(55) $$U_\infty : \begin{cases} f(r)g(r) = r\frac{dk}{dr}l(r) + rk(r)\frac{dl}{dr} \ , \\ \{[r\frac{dk}{dr} + 2k(r)]l(r) + rk(r)\frac{dl}{dr}\}/h(r)j(r) = 1 \ . \end{cases}$$

Let us give now (46) the equivalent form

(46') $[r_\Delta,\infty)$: $(\hat{H}_D(r) - H_0)/\frac{d}{dr}[-r^2 \int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho] = \frac{1}{r}$.

But (46') implies the existence of a neighbourhood V_∞ of infinity, such that either

(56_1) V_∞ : $\hat{H}_D(r) - H_0 > 0$,

or

(56_2) V_∞ : $\hat{H}_D(r) - H_0 < 0$.

Indeed, otherwise there exists a sequence $\{r_n\}_{n\in N}$, $r_n\to\infty$, such that

(57) $\hat{H}_D(r_n) - H_0 = 0$, $n \in N$.

From (46') it follows then

(58) $\frac{d}{dr}[-r^2 \int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho](r_n) = 0$, $n \in N$,

or, equivalently, by using the last equation (52),

(59) $\frac{d\ln|l|}{dr}(r_n) = -\frac{d\ln k}{dr}(r_n) - \frac{2}{r_n}$, $n \in N$.

Since on one hand, according to (54), the left hand-side of (59) does not have a limit at infinity, on the other hand, taking into account (53_1) and (53_2), the right hand-side of (59) has a limit at infinity, (57) can not hold and, consequently, either (56_1) or (56_2) follows. Next we give (33_1) the equivalent form

($33_1'$) (r_Δ,∞) : $\frac{d\hat{H}_D}{dr}/\frac{d}{dr}(r^2\hat{F}_D) = \frac{1}{r^2}$.

Using a similar argument to the above one we infer the existence of a neighbourhood W_∞ of infinity, such that either

(60_1) W_∞ : $\frac{d\hat{H}_D}{dr} < 0$, or

$$(60_2) \qquad W_\infty : \quad \frac{d\hat{H}_D}{dr} > 0 .$$

By derivating now $(33_1')$ with respect to r we get

$$(61) \qquad (r_\Delta,\infty) : \quad \frac{d^2\hat{H}_D}{dr^2} = \frac{d}{dr}\left[\frac{1}{r^2}\frac{d}{dr}(r^2\hat{F}_D)\right] .$$

Using again a similar argument to the above ones we obtain that in a suitably chosen neighbourhood Z_∞ of infinity either

$$(62_1) \qquad Z_\infty : \quad \frac{d^2\hat{H}_D}{dr^2} > 0 ,$$

or

$$(62_2) \qquad Z_\infty : \quad \frac{d^2\hat{H}_D}{dr^2} < 0 .$$

But (56_1), (60_1) and (62_1), respectively (56_2), (60_2) and (62_2), i.e the convexity, respectively the concavity, of $\hat{H}_D(r) - H_0$ in $V_\infty \cap W_\infty \cap Z_\infty$, are in contradiction with (52), (53_1), (53_2) and (54). Consequently the case a) can not occur.

Consider next the case b). Without loss of generality we restrict ourselves to (50_1). But (33_1) and the third relation (50_1), i.e. $\lim_{r\to\infty} \frac{d}{dr}(\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho)/\frac{2}{r}\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho = -1$, imply then (after integration !)

$$(63) \qquad \hat{F}_D(r) = O(r^{-2}) \ , \ \hat{H}_D(r) - H_0 = O(r^{-2}) \ , \text{ for } r\to\infty .$$

Let us investigate now the case c_1). Since in this case

$$(64) \qquad \lim_{r\to\infty}\left|\frac{\frac{d}{dr}(\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho}{\frac{2}{r}\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho}\right| = \infty ,$$

we get that $\forall\, n \in N$, $\exists\, U_\infty(n)$, such that

$$(65) \qquad U_\infty(n) : \quad \frac{d}{dr}(\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho)/\frac{2}{r}\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho < -\frac{n}{2} .$$

From (65) it follows then

(66) $$\int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho = O(r^{-n}) \quad , \ \forall\ n \in N \ , \text{ for } r\to\infty \ .$$

For the sake of completeness we note that an inequality of the type

(67) $$U_\infty(n) \ : \quad \frac{d}{dr}(\int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho)/\frac{2}{r}\int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho > \frac{n}{2}$$

would lead to

(68) $$r^n = O(\int_r^\infty (\hat{F}_D(\rho)/\rho)d\rho) \quad , \ \forall\ n \in N \ , \text{ for } r\to\infty \ ,$$

in contradiction with (49). Applying now the same argument to (33_1), we obtain instead of (64), and respectively (65)

(69) $$\lim_{r\to\infty} \left| \frac{\frac{d\hat{F}_D}{dr}}{\frac{2}{r}\hat{F}_D} \right| = \infty \ ,$$

and respectively

(70) $$U_\infty(n) \ : \quad \frac{d\hat{F}_D}{dr}/\frac{2}{r}\hat{F}_D < -\frac{n}{2} \ .$$

Integrating now (70) and taking account of (46) and (66) we get finally

(71) $$\left\{ \begin{array}{l} \hat{F}_D(r) = O(r^{-n}) \ , \\ \hat{H}_D(r) - H_0 = O(r^{-n}) \ . \end{array} \right. \quad \forall\ n \in N \ , \text{ for } r\to\infty \ ,$$

The case c_2) leads to a contradiction. Indeed, in this case we have on one hand

(72) $$\lim_{r\to\infty} \frac{\hat{H}_D(r) - H_0}{\hat{F}_D(r)} = \pm\infty \ .$$

On the other hand (33_1) can be given after integration another equivalent form, namely

$$(73) \qquad [r_\Delta,\infty) : \quad r^2(\hat{H}_D(r) - H_0) - 2\int_{r_\Delta}^{r} \rho(\hat{H}_D(\rho) - H_0)d\rho - r_\Delta^2(\hat{H}_D(r_\Delta) - H_0) + r_\Delta^2\hat{F}_D(r_\Delta) = r^2\hat{F}_D(r) ,$$

and, consequently, by taking account of (71) we infer

$$(74) \qquad \lim_{r\to\infty} r^2(\hat{H}_D(r) - H_0)/(2\int_{r_\Delta}^{r} \rho(\hat{H}_D(\rho) - H_0)d\rho + C_\Delta) = 1 ,$$

where

$$(75) \qquad C_\Delta := r_\Delta^2(\hat{H}_D(r_\Delta) - H_0 - \hat{F}_D(r_\Delta) .$$

By integrating (74) we get then

$$(76) \qquad r^2(\hat{H}_D(r) - H_0) = Cr^2 + o(r^2) , \; C \neq 0 , \text{ for } r\to\infty ,$$

in contradiction with (47).
It remains now to consider the case c_3). Since in this case

$$(77) \qquad \lim_{r\to\infty} \frac{\hat{F}_D(r)}{-2\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho} = \lim_{r\to\infty} \frac{\frac{d\ln I}{dr}}{\frac{d\ln r^2}{dr}} = \frac{\alpha}{\beta} =: -\lambda ,$$

where

(78)

$$I := \left|-\int_r^\infty(\hat{F}_D(\rho)/\rho)d\rho\right|$$

and λ is neither zero nor infinite, after integration we obtain

$$(79) \qquad \hat{F}_D(r) = O(r^{-2\lambda}) , \text{ for } r\to\infty ,$$

where, due to (48), λ must be positive. Using now (46) and (77) we infer finally

$$(80) \qquad \hat{H}_D(r) - H_0 = O(r^{-2\lambda}) \ , \text{ for } r\to\infty \ .$$

Let us multiply now the continuity equation ($1_2'$) by $\cos n\theta$ and respectively $\sin n\theta$ and integrate with respect to θ to get, $\forall\, n \in N$,

$$(81) \qquad (r_\Delta,\infty) : \begin{cases} (\frac{d}{dr} + \frac{n+1}{r})(\hat{u}_D^{cn+1} + \hat{v}_D^{sn+1}) = -(\frac{d}{dr} - \frac{n-1}{r})(\hat{u}_D^{cn-1} \\ \qquad - \hat{v}_D^{sn-1}) \ , \\ (\frac{d}{dr} + \frac{n+1}{r})(\hat{u}_D^{sn+1} - \hat{v}_D^{cn+1}) = -(\frac{d}{dr} - \frac{n-1}{r})(\hat{u}_D^{sn-1} \\ \qquad + \hat{u}_D^{cn-1}) \ . \end{cases}$$

The equations (81) reduce in the particular case $n = 1$ to the equations (37_1) and (37_2). On the other hand, by taking into account the expression of the first Fourier coefficient of ω (see (23) !), i.e.

$$(82) \qquad (r_\Delta,\infty) : \quad \frac{1}{r}\frac{d}{dr}[r(\hat{u}_D^{s1} - \hat{v}_D^{c1})] = \hat{\omega}_D(r) \ ,$$

from the second equation (33) and (34) it follows

$$(33_2') \qquad (r_\Delta,\infty) : \quad \frac{d}{dr}[r^2(\hat{G} - \nu\hat{\omega}_D)] + 2\nu\hat{\omega}_D r = \frac{d}{dr}[r^2(\hat{G} - \nu\hat{\omega}_D) + 2\nu r(\hat{u}_D^{s1} - \hat{v}_D^{c1})] = 0 \ ,$$

or, equivalently, after integration with respect to r,

$$(83) \qquad (r_\Delta,\infty) : \quad \frac{d}{dr}(\hat{u}_D^{s1} - \hat{v}_D^{c1}) - \frac{1}{r}(\hat{u}_D^{s1} - \hat{v}_D^{c1}) = \frac{1}{2\pi\nu}\int_0^{2\pi}[(u_D - \hat{u}_D)\sin\theta - (v_D - \hat{v}_D)\cos\theta][(u_D - \hat{u}_D)\cos\theta + (v_D - \hat{v}_D)\sin\theta]d\theta + \frac{1}{\nu}[\hat{u}_D(\hat{u}_D^{s2} - \hat{v}_D^{c2}) - \hat{v}_D(\hat{u}_D^{c2} + \hat{v}_D^{s2})] + \frac{\tilde{C}}{r^2} \ .$$

By repeating now for the equations (81) and (83) the same argument as for the equation (33_1), respectively the equation (46), we infer that all the Fourier coefficients of $\underset{\sim}{v}_D$ satisfy relations of the type (80), i.e.

$$(84)\qquad \begin{cases} \hat{\underset{\sim}{v}}_D^{cn(sn)}(r) = O(r^{2\alpha_n n/\beta_n}) , \\ \hat{\underset{\sim}{v}}_D(r) - \underset{\sim}{v}_0 = O(r^{2\alpha_n n/\beta_n}) , \end{cases} \quad n \in N , \text{ for } r\to\infty ,$$

where, according to (11) and (38), and using a similar argument to that employed in the case c_2) we get

$$(85)\qquad \frac{\alpha_n}{\beta_n} < 0 , \alpha_n \neq 0 , n \in N \cup \{0\} , \alpha_n \not\to 0 \text{ for } n\to\infty .$$

From (63), (71), (80) and (84) the assertions (19_1) and (19_2) then follow (for a suitable choice of λ!). In order to prove the last assertion of Lemma 2 let us consider the particular case where $\underset{\sim}{v}_0 = \underset{\sim}{0}$. Using ($19_1$) and ($19_2$) and integrating (83) with respect to r we get , for $r\to\infty$,

$$(86_1)\qquad \hat{u}_D^{s1}(r) - \hat{v}_D^{c1}(r) + \int_r^\infty \frac{1}{\rho}[\hat{u}_D^{s1}(\rho) - \hat{v}_D^{c1}(\rho)]d\rho = O(r^{-\lambda}) ,$$

$$(86_2)\qquad \frac{1}{2\pi\nu}\int_r^\infty\int_0^{2\pi} [(u_D - \hat{u}_D)\sin\theta - (v_D - \hat{v}_D)\cos\theta][(u_D - \hat{u}_D)\cos\theta + (v_D - \hat{v}_D)\sin\theta]d\theta d\rho + \frac{1}{\nu}\int_r^\infty[\hat{u}_D(\hat{u}_D^{s2} - \hat{v}_D^{c2}) - \hat{v}_D(\hat{u}^{c2} + \hat{v}^{s2})]d\rho - \frac{\tilde{C}}{r} = \max\{O(r^{-2\lambda + 1}), O(r^{-1})\} .$$

By estimating (86_2) we have taken into account that the only term on the left hand-side which depends on $\hat{u}_D^{s1} - \hat{v}_D^{c1}$, i.e. according to the Parseval equality,

$$(87)\qquad \frac{1}{\nu}\int_r^\infty(\hat{u}_D^{s1} - \hat{v}_D^{c1})(\hat{u}_D^{c1} + \hat{v}_D^{s1})d\rho ,$$

disappears when considering the non-slip boundary condition (2). Indeed, integrate to this end ($1_2^!$) with respect to θ and obtain

$$(88)\qquad [r_\Delta,\infty) : \quad \hat{u}_D^{c1}(r) + \hat{v}^{s1}(r) = \frac{C}{r} .$$

By the flux-divergence theorem C = 0, provided (2) does hold. Comparing now (86_1) and (86_2) the last assertion of Lemma 2 then follows. For the sake of completeness we note that this assertion, i.e.

" $\lambda = 1$, provided $\underset{\sim}{v}_0 = \underset{\sim}{0}$ and (2) holds" is not only a consequence of (86_1) and (86_2), which were obtained by integrating twice (33_2), but follows also when comparing the rates of decay at infinity of all the Fourier coefficients of the Navier-Stokes equivalent system of equations (20''), i.e. by multiplying (20'') by $\cos n\theta$ ($\sin n\theta$), $n \in \mathbb{N}$, and then integrating once with respect to θ and twice with respect to r and using (19_1) and (19_2) as well as the (rate of decay of the) Fourier coefficients of ω_D obtained from (23). However, the last assertion of Lemma 2 is no longer valid if we use other boundary conditions than the non-slip boundary condition (2), namely non-homogeneous ones. In order to show this fact let us assume that the vorticity ω is independent of θ in Ω. From ($1_2'$), (22') and (23) it follows then

(89) $$\Omega : \quad u\cos\theta + v\sin\theta = \frac{C}{r} ,$$

(90) $$\Omega : \begin{cases} \omega = \dfrac{1}{r}\dfrac{d}{dr}[r(u\sin\theta - v\cos\theta)] , \\[2ex] \dfrac{1}{r}\dfrac{d}{dr}(r\dfrac{d\omega}{dr}) - \dfrac{C}{\nu r}\dfrac{d\omega}{dr} = 0 , \end{cases}$$

where C is the constant appearing in (88). Integrating (90) we get

(91) $$\Omega : \quad u\sin\theta - v\cos\theta = \frac{C_1}{2 + C/\nu}\, r^{1 + \frac{C}{\nu}} + \frac{C_2}{r} ,$$

(92) $$\Omega : \quad \omega = C_1 r^{C/\nu} ,$$

where C_1 and C_2 are also integration constants. If $\underset{\sim}{v}$ has finite Dirichlet integral, then ω is square integrable in Ω. Hence C has to satisfy the condition

(93) $$C < -\nu .$$

According to (89) and (91) the velocity $\underset{\sim}{v}$ is given by

(94) $$\Omega : \begin{cases} u = Cr^{-1}\cos\theta + (C_3 r^{1 + C/\nu} + C_2 r^{-1})\sin\theta , \\[1ex] v = Cr^{-1}\sin\theta - (C_3 r^{1 + C/\nu} + C_2 r^{-1})\cos\theta , \end{cases}$$

with $C_3 = C_1(2 + C/\nu)^{-1}$. From (20') it follows now that H is also independent of θ, and satisfies the equation

$$(95) \qquad \Omega : \quad \frac{dH}{dr} = \frac{C_1^2}{2 + C/\nu} r^{1 + 2C/\nu} + C_1 C_2 r^{C/\nu - 1} ,$$

where we have taken into account (91) and (92). After integration with respect to r we get (up to an integration constant $C_4 = p_\infty$!)

$$(96) \qquad \Omega : \quad H(r) = C_1 C_3 (2 + 2C/\nu)^{-1} r^{2 + 2C/\nu} + C_1 C_2 C^{-1} \nu r^{C/\nu} .$$

Taking account of (92), from (96) it follows then

$$(97) \qquad \Omega : \quad p(r) = \frac{C_1 C_3}{2(1 + C/\nu)(2 + C/\nu)} r^{2 + 2C/\nu} + \frac{2C_1 C_2 \nu}{C(2 + C/\nu)} r^{C/\nu} - \frac{C^2 + C_2^2}{2} r^{-2} + p_\infty .$$

Assume now for the sake of simplicity that $\partial\Delta$ is starshaped with respect to the origin. Consequently we can give the equation of the boundary $\partial\Delta$ the form

$$(98) \qquad [0,2\pi) : \quad r = r_{\partial\Delta}(\theta) .$$

Then (94) and (97) solve the following exterior Dirichlet problem for the Navier-Stokes equations (and the continuity equation!) (1)

$$(99) \qquad \partial\Delta : \quad \underset{\sim}{v} = C r_{\partial\Delta}^{-1}(\theta)(\cos\theta, \sin\theta) + [C_3 r_{\partial\Delta}^{1 + C/\nu}(\theta) + C_2 r_{\partial\Delta}^{-1}(\theta)](-\sin\theta, \cos\theta) ,$$

$$(100) \qquad [0,2\pi) : \quad \lim_{r\to\infty} \underset{\sim}{v}(r,\theta) = \underset{\sim}{0} .$$

Taking now into account the fact that the unit outward normal to $\partial\Delta$ is given by

$$(101) \qquad [0,2\pi) : \quad \underset{\sim}{n} = \frac{1}{(r'^2_{\partial\Delta} + r^2_{\partial\Delta})^{1/2}}(-r'_{\partial\Delta}\sin\theta - r_{\partial\Delta}\cos\theta, r'_{\partial\Delta}\cos\theta - r_{\partial\Delta}\sin\theta) ,$$

and using the periodicity of $r_{\partial\Delta}$ we get

$$\int_{\partial\Delta} \underset{\sim}{v}\cdot\underset{\sim}{n}ds = -C , \tag{102}$$

in accordance with (88) and (89). Choosing now $\partial\Delta$ to be the unit circle, i.e. $r_{\partial\Delta}(\theta) \equiv 1$, and taking $C_2 = -C_3$, from (94) and (97) it follows that the above exterior Dirichlet problem admits infinitely many solutions ($C \neq 0$!). Assume next $C = 0$. Since ω is square integrable in Ω, from (92) it follows that $C_1 = 0$ and, consequently, $C_3 = 0$. The solution $(\underset{\sim}{v},p)$ of the above exterior Dirichlet problem, i.e. the equations (94) and (97), then become

$$\Omega : \begin{cases} u(r,\theta) = C_2 r^{-1}\sin\theta , \\ v(r,\theta) = -C_2 r^{-1}\cos\theta , \end{cases} \tag{103}$$

$$\Omega : \quad p(r) = p_\infty - \frac{C_2^2}{2} r^{-2} . \tag{104}$$

On the other hand, according to the following

Theorem of Berker and Finn [4], [6, p. 129]. *Let* $\underset{\sim}{v}$ *be a solution of (1) and (2). If there exists a limiting velocity* $\underset{\sim}{v}_\infty$, *such that*

$$\lim_{r\to\infty} r^{1/2}|\underset{\sim}{v}(r,\theta) - \underset{\sim}{v}_\infty| = 0 , \text{ uniformly in } \theta \tag{105}$$

then $\underset{\sim}{v} \equiv \underset{\sim}{0}$ *in* Ω,

the constant C_2 must be equal to zero, and the assertion that the non-uniqueness result showed above is valid only in the case of a non-vanishing flux C then follows. Moreover, from (94) we get (for $C \neq 0$!), in accordance with (86_1), (86_2) and (87),

$$\lambda = -1 - \frac{C}{\nu} , \tag{106}$$

and, consequently, by taking account of (93),

$$\lambda \neq 1 , \text{ provided } C \neq -2\nu . \tag{107}$$

For the sake of completeness let us show why, in the case of a non-vanishing outflow C (C < 0!), the proof of the last assertion of Lemma 2 does no longer work. Indeed, from (84), (87) and (88) it follows

$$(108)\qquad \frac{1}{\nu}\int_r^\infty(\hat{u}_D^{s1} - \hat{v}_D^{c1})(\hat{u}_D^{c1} + \hat{v}_D^{s1})d\rho = \frac{C}{\nu}\int_r^\infty\frac{1}{\rho}(\hat{u}_D^{s1} - \hat{v}_D^{c1})d\rho = O(r^{-\lambda}) \ , \text{ for } r\to\infty \ .$$

On the other hand, the equation

$$(109)\qquad [r_\Delta,\infty) \ : \quad \hat{u}_D^{s1}(r) - \hat{v}_D^{c1}(r) + \int_r^\infty \frac{1 + C/\nu}{\rho}[\hat{u}_D^{s1}(\rho) - \hat{v}_D^{c1}(\rho)]d\rho = 0$$

has the solution

$$(110)\qquad [r_\Delta,\infty) \ : \quad \hat{u}^{s1}(r) - \hat{v}^{c1}(r) = C_1(2 + C/\nu)^{-1}r^{1 + C/\nu} \ ,$$

in accordance with (91). Consequently the last assertion of Lemma 2 does no longer hold.

Remark 2. *The above counter-example to the uniqueness of the viscous fluid flow of an incompressible fluid past a circular disk* ***was given first by*** *Hamel* [13].

Proof of Theorem 1. By multiplying (1_1) by $\underset{\sim}{v}$ and integrating over Ω_r we obtain, after using the Gauß-Green theorem and (2),

$$(111)\qquad -\nu\int_{\Omega_r}|\nabla \underset{\sim}{v}_D|^2dxdy = -\frac{\nu}{2}\int_{\Omega_r}[\nabla^2(|\underset{\sim}{v}_D|^2) - \nabla\cdot(H_D\underset{\sim}{v}_D)]dxdy$$

$$= -\nu r\int_0^{2\pi}[\frac{\partial}{\partial r}(\frac{|\underset{\sim}{v}_D|^2}{2}) - \frac{1}{\nu}H_D(\underset{\sim}{v}_D\cdot\underset{\sim}{n})]d\theta \ .$$

Letting r tend to infinity and using (4) we get

$$(112)\qquad -\nu\int_{\Omega}|\nabla \underset{\sim}{v}_D|^2dxdy = \lim_{r\to\infty}[-\nu r\frac{d}{dr}\int_0^{2\pi}(|\underset{\sim}{v}_D|^2/2)d\theta + r\int_0^{2\pi}H_D(\underset{\sim}{v}_D\cdot\underset{\sim}{n})d\theta] \ .$$

On the other hand taking into account (88) we obtain the identity

(113)
$$\frac{r}{2\pi}[-\nu \frac{d}{dr}\int_0^{2\pi}(|\underset{\sim}{v}_D|^2/2)d\theta + \int_0^{2\pi} H_D(\underset{\sim}{v}_D\cdot n)d\theta] =$$
$$-\nu r\frac{d}{dr}(|\hat{\underset{\sim}{v}}_D|^2/2) - \frac{\nu r}{2\pi}\frac{d}{dr}\int_0^{2\pi}(|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2/2)d\theta +$$
$$r(\hat{u}_D\hat{H}_D^{c1} + \hat{v}_D\hat{H}_D^{s1}) + \frac{r}{2\pi}\int_0^{2\pi}[\frac{1}{2}|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|2 + \hat{\underset{\sim}{v}}_D\cdot(\underset{\sim}{v}_D -$$
$$\hat{v}_D) + p_D - \hat{p}_D][(u_D - \hat{u}_D)\cos\theta + (v_D - \hat{v}_D)\sin\theta]d\theta .$$

Next we multiply (20'') by cosθ(sinθ) and sinθ(cosθ) respectively, and after integration with respect to θ we get

(114)
$$(r_\Delta,\infty) : \begin{cases} \dfrac{d\hat{H}_D^{c1}}{dr} = \dfrac{\nu}{r}\hat{\omega}_D^{s1} + \dfrac{1}{r^2}\dfrac{d}{dr}(r^2\hat{F}^{c1}) + \dfrac{\hat{G}^{s1}}{r} , \\[2ex] -\dfrac{\hat{H}_D^{c1}}{r} = -\nu\dfrac{d\hat{\omega}_D^{s1}}{dr} + \dfrac{1}{r^2}\dfrac{d}{dr}(r^2\hat{G}^{s1}) + \dfrac{\hat{F}^{c1}}{r} , \end{cases}$$

(115)
$$(r_\Delta,\infty) : \begin{cases} \dfrac{dH_D^{s1}}{dr} = -\dfrac{\nu}{r}\hat{\omega}^{c1} + \dfrac{1}{r^2}\dfrac{d}{dr}(r^2\hat{F}^{s1}) - \dfrac{\hat{G}^{c1}}{r} , \\[2ex] \dfrac{\hat{H}_D^{s1}}{r} = -\nu\dfrac{d\hat{\omega}_D^{c1}}{dr} + \dfrac{1}{r^2}\dfrac{d}{dr}(r^2\hat{G}^{c1}) - \dfrac{\hat{F}^{s1}}{r} . \end{cases}$$

From (114) and (115) it follows then

(116) $$[r_\Delta,\infty) : \quad \hat{H}_D^{c1}(r) = \hat{F}^{c1}(r) - \hat{G}^{s1}(r) + \frac{C_1}{r} + \nu\hat{\omega}_D^{s1}(r) ,$$

(117) $$[r_\Delta,\infty) : \quad \hat{H}_D^{s1}(r) = \hat{F}^{s1}(r) + \hat{G}^{c1}(r) + \frac{C_2}{r} - \nu\hat{\omega}_D^{c1}(r) ,$$

where, according to (24),

(118_1) $$[r_\Delta,\infty) : \quad \hat{F}^{c1} - \hat{G}^{s1} = -\frac{1}{4\pi}\int_0^{2\pi}|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2\cos(\theta - 2\phi)d\theta -$$
$$(\hat{u}_D^{s1} - \hat{v}_D^{c1})\hat{v}_D ,$$

$$(118_2)\qquad [r_\Delta,\infty)\ :\qquad \hat{F}^{s1} + \hat{G}^{c1} = \frac{1}{4\pi}\int_0^{2\pi} |\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2 \sin(\theta - 2\phi)\,d\theta + (\hat{u}_D^{s1} - \hat{v}_D^{c1})\hat{u}_D \ ,$$

$$(119)\qquad \Omega\ :\qquad u_D(r,\theta) - \hat{u}_D(r) = |\underset{\sim}{v}_D(r,\theta) - \hat{\underset{\sim}{v}}_D(r)|\cos\phi\ ,$$

$$v_D(r,\theta) - \hat{v}_D(r) = |\underset{\sim}{v}_D(r,\theta) - \hat{\underset{\sim}{v}}_D(r)|\sin\phi\ .$$

Taking into account (116) and (117), from (113) it follows then

$$(113')\qquad [r_\Delta,\infty)\ :\qquad \frac{r}{2\pi}\Big[-\nu\frac{d}{dr}\int_0^{2\pi}(|\underset{\sim}{v}_D|^2/2)\,d\theta + \int_0^{2\pi} H_D(\underset{\sim}{v}_D\cdot\underset{\sim}{n})\,d\theta\Big] =$$
$$-\frac{\nu r}{2\pi}\frac{d}{dr}\int_0^{2\pi}(|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2/2)\,d\theta + C_1\hat{u}_D + C_2\hat{v}_D -$$
$$\frac{r\hat{u}_D}{4\pi}\int_0^{2\pi}|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2\cos(\theta - 2\phi)\,d\theta + \frac{r\hat{v}_D}{4\pi}\int_0^{2\pi}|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2 \times$$
$$\sin(\theta - 2\phi)\,d\theta + \frac{r}{2\pi}\int_0^{2\pi}\big[|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2/2 + \hat{\underset{\sim}{v}}_D\cdot(\underset{\sim}{v}_D -$$
$$\hat{\underset{\sim}{v}}_D) + p_D - \hat{p}_D\big]|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|\cos(\theta - \phi)\,d\theta\ .$$

Noting now that by the Cauchy-Schwarz inequality

$$(120)\qquad (r_\Delta,\infty)\ :\qquad \Big|\frac{d}{dr}\int_0^{2\pi}(|\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D|^2/2)\,d\theta\Big| \le \Big(\int_0^{2\pi}(\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D)^2 d\theta\Big)^{1/2} \times$$
$$\Big(\int_0^{2\pi}\big[\frac{\partial}{\partial r}(\underset{\sim}{v}_D - \hat{\underset{\sim}{v}}_D)\big]^2 d\theta\Big)^{1/2}\ ,$$

and using (31) and Lemma 2, from (113') we get, for $r\to\infty$,

$$(121)\qquad \frac{r}{2\pi}\Big[-\nu\frac{d}{dr}\int_0^{2\pi}(|\underset{\sim}{v}_D|^2/2)\,d\theta + \int_0^{2\pi} H_D(\underset{\sim}{v}_D\cdot\underset{\sim}{n})\,d\theta\Big] = o(r^{-3/4})$$
$$+ O(r^{-1}) + O(r^{-2}) + O\Big(\Big[\int_0^{2\pi}(p_D - \hat{p}_D)^2 d\theta\Big]^{1/2}\Big)\ .$$

Taking account of (13_2), from (112) we infer then that the Dirichlet integral of $\underset{\sim}{v}_D$ vanishes and, hence, using (2) the Liouville property stated in Theorem 1 then follows.

Theorem 2 [23]. *Every solution* $(\underset{\sim}{v}_D, p_D)$ *of (1), (2) and (4) tends pointwise at infinity to its asymptotic mean value* $(\underset{\sim}{v}_0, p_\infty)$.

Proof. We choose $(\underset{\sim}{v}_D, p_D)$ to be an arbitrary, non-identically zero solution of (1), (2) and (4), and denote by $(\underset{\sim}{v}_0, p_\infty)$ its asymptotic mean value. Choosing the point $P(R,\phi)$, $R > r_\Delta$, as the origin of a new system of polar coordinates (r',θ'), i.e. $re^{i\theta} = Re^{i\phi} + r'e^{i\theta'}$, we give the first equation $(20_1')$ the equivalent form

$$(20_1^{iv}) \qquad \Omega : \quad \frac{\partial}{\partial r'}[|\underset{\sim}{v}_D - \underset{\sim}{v}_0|^2/2 + p_D] = \frac{\nu}{r'}\frac{\partial \omega_D}{\partial \theta'} + [(u_D - u_0) \times \sin\theta' - (v_D - v_0)\cos\theta']\omega_D + \frac{u_0}{r'}\frac{\partial v_D}{\partial \theta'} - \frac{v_0}{r'}\frac{\partial u_D}{\partial \theta'} .$$

Integrating first (20_1^{iv}) with respect to r' on $[0,\frac{R}{2}]$ and then with respect to θ' on $[0,2\pi]$, and taking the absolute value we obtain

$$(122) \qquad \Omega : \quad \frac{1}{2}|\underset{\sim}{v}_D(P) - \underset{\sim}{v}_0|^2 \leq \frac{1}{4\pi}\int_0^{2\pi} |\underset{\sim}{v}_D(\tfrac{R}{2},\theta') - \underset{\sim}{v}_0|^2 d\theta' + \frac{1}{4\pi}\int_0^{R/2}\int_0^{2\pi} |[(u_D - u_0)\sin\theta' - (v_D - v_0)\cos\theta']\omega_D| d\theta' dr' + |p_D(P) - \hat{p}_D(\tfrac{R}{2})| .$$

Applying next the Cauchy-Schwarz inequality we get

$$(123) \qquad [r_\Delta,\infty) : \quad \int_1^{R/2}\int_0^{2\pi} |[(u_D - u_0)\sin\theta' - (v_D - v_0)\cos\theta']\omega_D| d\theta' dr'$$
$$\leq \sqrt{2}\left(\int_1^{R/2}\int_0^{2\pi} r'^{-1}|\underset{\sim}{v}_D - \underset{\sim}{v}_0|^2 d\theta' dr' \int_1^{R/2}\int_0^{2\pi} \omega_D^2 dx' dy'\right)^{1/2}$$
$$\leq \sqrt{2}\left(\int_1^{R/2}\int_0^{2} r'^{-1}|\underset{\sim}{v}_D - \underset{\sim}{v}_0|^2 d\theta' dr' \int_{R/2}^{3R/2}\int_0^{2\pi} \omega_D dx dy\right)^{1/2} ,$$

where we have used the fact that the disc $r' < R/2$ is contained in the circular annulus $R/2 < r < 3R/2$. On the other hand we have

$$(124) \qquad [r_\Delta,\infty) : \quad \int_0^1\int_0^{2\pi} |[(u_D - u_0)\sin\theta' - (v_D - v_0)\cos\theta'|\omega_D] d\theta' dr'$$

$$\leq C^* \max_{\overline{\Omega}_{\frac{R}{2}+1} \setminus \Omega_{\frac{R}{2}-1}} |\omega_D| ,$$

where C* depends on the bounded velocity $\underset{\sim}{v}_D$. From (41_1) and (122) we obtain then

$$(125) \qquad \Omega : \quad |\underset{\sim}{v}_D(P) - \underset{\sim}{v}_0|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\underset{\sim}{v}_D(\tfrac{R}{2},\theta') - \underset{\sim}{v}_0|^2 d\theta' +$$

$$2|p_D(P) - \hat{p}_D(\tfrac{R}{2})| + \tilde{C}^* \max_{\overline{\Omega}_{\frac{R}{2}+1} \setminus \Omega_{\frac{R}{2}-1}} |\omega_D| +$$

$$\frac{2\sqrt{2}}{\pi}(4\pi^2 \int_{R/2}^{3R/2} \int_0^{2\pi} |\nabla \underset{\sim}{v}_D|^2 dxdy + \int_1^{R/2} \int_0^{2\pi} r'^{-1} |\hat{\underset{\sim}{v}}_D -$$

$$\underset{\sim}{v}_0|^2 dr')^{1/2} (\int_{R/2}^{3R/2} \int_0^{2\pi} \omega_D^2 dxdy)^{1/2} ,$$

where again we have used that the disc $r' < R/2$ is contained in the circular annulus $R/2 < r < 3R/2$. Next we show that the first equation (10) as well as the estimates (19_1) and (19_2) are independent of the fact that the system of coordinates is fixed or moving. To this end we consider again the two systems of polar coordinates, the old, fixed one, (r,θ), and the new, moving one, (r',θ'). Using $2ab \leq a^2 + b^2$ as well as (40_1) we get

$$(126) \qquad (r_\Delta,\infty) : \quad |\frac{d}{dr'} \int_0^{2\pi} |\underset{\sim}{v}_D(r',\theta') - \hat{\underset{\sim}{v}}_D(r')|^2 d\theta'| \leq$$

$$2 \int_0^{2\pi} |\underset{\sim}{v}_D(r',\theta') - \hat{\underset{\sim}{v}}_D(r')| |\frac{\partial \underset{\sim}{v}_D}{\partial r'}| d\theta' \leq$$

$$4\pi^2 \int_0^{2\pi} r' |\nabla \underset{\sim}{v}_D|^2 d\theta' .$$

Integrating (126) with respect to r' on $[0,r^*]$, $r^* \leq R/2$, and using again the fact that the disc $r' < R/2$ is contained in the circular annulus $R/2 < r < 3R/2$, we obtain

$$(127) \qquad [r_\Delta,\infty) : \quad \int_0^{2\pi} |\underset{\sim}{v}_D(r^*,\theta') - \hat{\underset{\sim}{v}}_D(r^*)|^2 d\theta' \leq 4\pi^2 \int_{R/2}^{3R/2} \int_0^{2\pi} |\nabla \underset{\sim}{v}_D|^2 dxdy .$$

From (127) it follows then

(128) $$\lim_{R\to\infty} \int_0^{2\pi} |\underset{\sim}{v}_D(\tfrac{R}{2},\theta') - \hat{\underset{\sim}{v}}_D(\tfrac{R}{2})|^2 d\theta' = 0 \ .$$

Writing next (33_1) and (46) in the new variables (r',θ'), and using the same argument as for proving (19_1) and (19_2), we infer that for $r'\to\infty$

($19_1'$) $$\int_0^{2\pi} |\underset{\sim}{v}_D(r',\theta') - \hat{\underset{\sim}{v}}_D(r')|^2 d\theta' = O(r'^{-2\lambda}) \ ,$$

($19_2'$) $$\hat{\underset{\sim}{v}}_D(r') - \underset{\sim}{v}_0 = O(r'^{-\lambda}) \ .$$

From ($19_2'$) it follows then

(129) $$\sup_{R\in[2,\infty)} \int_1^{R/2} r'^{-1} |\hat{\underset{\sim}{v}}_D(r') - \underset{\sim}{v}_0|^2 dr' < \infty \ .$$

Taking now account of (4), (13_1), (14), (15), ($19_1'$), ($19_2'$), (128), (129), and letting $R\to\infty$, from (125) we infer

(130) $$[0,2\pi) \ : \quad \lim_{R\to\infty} \underset{\sim}{v}_D(r,\theta) = \underset{\sim}{v}_0 \ .$$

3. CONVERGENCE AT INFINITY OF THE LERAY SOLUTION

Theorem 3 [23]. *The Leray solution* $(\underset{\sim}{v}_L, p_L)$ *of (1), (2) and (3) tends pointwise at infinity to* $(\underset{\sim}{v}_\infty, p_\infty)$. *Furthermore, the Leray sequence of solutions* $(\underset{\sim}{v}_{R_i}, p_{R_i})$, $i \in N$, *of (1) and (5) in* Ω_{R_i} *converges quasi-uniformly on* $\overline{\Omega}$ *to* $(\underset{\sim}{v}_L, p_L)$.

For the proof of Theorem 3 we need the following results:

Definition 1 [16, p. 66]. *Let X and Y be metric spaces and let* f_n, $n \in N$, *map X into Y. The sequence* $\{f_n\}_{n\in N}$ *is said to converge quasi-uniformly on X to* $f : X \to Y$ *if*

(i) $\{f_n\}$ *converges pointwise to f,*

(ii) for every $\varepsilon > 0$ *there exists a sequence* $\{n_p\}_{p\in N} \subset N$ *and a sequence* $\{D_p\}_{p\in N}$ *of open sets* $D_p \subset X$, $X = \bigcup_{p=1}^{\infty} D_p$, *such that*

$$\mathrm{dist}_Y(f(x), f_{n_p}(x)) < \varepsilon \ , \ p \in N \ , \ x \in D_p \ .$$

Theorem of Arzela, Gagaeff and Alexandrov [1], [9], [16, p. 68]. *Let X, Y be metric spaces and let f_n, $n \in N$, map X into Y continuously. The sequence $\{f_n\}$ converges on X to a continuous map $f : X \to Y$, iff the convergence is quasi-uniform.*

Lemma 3 [20]. *The Leray sequence of velocities $\{v_{R_i}\}_{i \in N}$ satisfies the estimate*

$$(131) \quad [r_\Delta, R_i] : \quad \int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r,\theta)|^2 d\theta = O(\ln r) \quad , \text{ uniformly in } R_i \ .$$

Proof. By the Cauchy-Schwarz inequality we have

$$(132) \quad (r_\Delta, R_i) : \quad \frac{d}{dr}\Big[\int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r,\theta)|^2 d\theta\Big]^{1/2} = \frac{1}{2}\Big[\int_0^{2\pi} |\underset{\sim}{v}_{R_i}|^2 d\theta\Big]^{-1/2} \times \int_0^{2\pi} \frac{\partial}{\partial r}|\underset{\sim}{v}_{R_i}|^2 d\theta \leq \Big[\int_0^{2\pi} |\frac{\partial}{\partial r}\underset{\sim}{v}_{R_i}|^2 d\theta\Big]^{1/2} \ .$$

Integrating (132) between r_Δ and r and applying again the Cauchy-Schwarz inequality we get

$$(133) \quad [r_\Delta, R_i] : \quad \Big[\int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r,\theta)|^2 d\theta\Big]^{1/2} - \Big[\int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r_\Delta,\theta)|^2\Big]^{1/2} \leq \Big(\ln\frac{r}{r_\Delta}\Big)^{1/2}\Big[\int_{r_\Delta}^{r}\int_0^{2\pi} |\frac{\partial}{\partial\rho}\underset{\sim}{v}_{R_i}(\rho,\theta)|^2 \rho\, d\theta\, d\rho\Big]^{1/2} \ .$$

Thus,

$$(134) \quad [r_\Delta, R_i] : \quad \int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r,\theta)|^2 d\theta \leq 2\int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r_\Delta,\theta)|^2 d\theta + \Big[\int_{r_\Delta}^{r}\int_0^{2\pi} |\frac{\partial}{\partial\rho}\underset{\sim}{v}_{R_i}|^2 \rho\, d\theta\, d\rho\Big] \ln\Big(\frac{r}{r_\Delta}\Big) \ .$$

Using now (6) and the fact that, according to the construction of the Leray solution,

$$(135) \quad \lim_{R_i \to \infty} \int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r_\Delta,\theta)|^2 d\theta = \int_0^{2\pi} |\underset{\sim}{v}_L(r_\Delta,\theta)|^2 d\theta \ ,$$

the assertion (131) then follows.

Lemma 4 [20]. *The gradient* $\nabla\omega_{R_i}$ *is square integrable in* Ω_{R_i}, *uniformly in* R_i.

Proof. As known ω_{R_i} is solution of the equation

$$(136) \qquad \Omega_{R_i} : \quad \nu\nabla^2\omega - \underset{\sim}{v}_{R_i}\cdot\nabla\omega = 0 .$$

Let now $\eta(r)$ be a smooth function which vanishes near $r \to \infty$ and near $r = r_\Delta$, $h(\omega_{R_i})$ be a function of one variable, which is C^1 and piecewise C^2. An easy computation which uses the fact that $\nabla\cdot\underset{\sim}{v}_{R_i} = 0$ shows that

$$(137) \qquad \Omega_{R_i} : \quad \nabla\cdot[\nu\eta(r)\nabla h(\omega_{r_i}) - \nu h(\omega_{R_i})\nabla\eta(r) - \eta(r)h(\omega_{R_i})\underset{\sim}{v}_{R_i}]$$
$$= \nu\eta(r)h''(\omega_{R_i})|\nabla\omega_{R_i}|^2 - h(\omega_{R_i})[\nu\nabla^2\eta + \underset{\sim}{v}_R\cdot\nabla\eta]$$
$$\eta(r)h'(\omega_{R_i})[\nu\nabla^2\omega_{R_i} - \underset{\sim}{v}_{R_i}\cdot\nabla\omega_{R_i}] .$$

Since ω_{R_i} satisfies (136) and η vanishes near $r = r_\Delta$ and for $r \geqq r_\infty$, integration over the domain $r > r_\Delta$ yields the identity

$$(138) \qquad \int_{\Omega_{R_i}} \nu\eta h''(\omega_{R_i})|\nabla\omega_{R_i}|^2 dxdy = \int_{\Omega_{R_i}} h(\omega_{R_i})[\nu\nabla^2\eta + \underset{\sim}{v}_{R_i}\cdot\nabla\eta]dxdy ,$$

provided we take $R_i \geqq r_\infty$. Now we choose $r_\infty/2 =: R > r_1 > r_\Delta$ and non-negative C^2 cut-off functions ξ_1 and ξ_2, such that

$$(139_1) \quad [r_\Delta, R_i] : \quad \xi_1(r) := \begin{cases} 0 , & r \leqq \frac{1}{2}(r_\Delta + r_1) , \\ 1 , & r \geqq r_1 , \end{cases}$$

$$(139_2) \quad [r_\Delta, R_i] : \quad \xi_2(r) := \begin{cases} 1 , & r \leqq 1 , \\ 0 , & r \geqq 2 , \end{cases}$$

and set

(140) $[r_\Delta, R_i]$: $\eta(r) := \xi_1(r)\xi_2(\frac{r}{R})$.

Next we choose a positive constant ω_0 and set

$$(141) \qquad h(\omega_{R_i}) := \begin{cases} \omega_{R_i}^2 , & |\omega_{R_i}| \leqq \omega_0 , \\ \omega_0(2|\omega_{R_i}| - \omega_0) , & |\omega_{R_i}| \geqq \omega_0 . \end{cases}$$

Inserting (139_1), (139_2), (140) and (141) in (138) we get

$$(142) \qquad 2\int_{\substack{|\omega_{R_i}|\leq\omega_0 \\ r_1 \leqq r \leqq R}} |\nabla\omega_{R_i}|^2 dxdy \leqq 2\int_{|\omega_{R_i}|\leq\omega_0} \eta|\nabla\omega_{R_i}|^2 dxdy$$

$$= \int_{\substack{\{r_\Delta\leqq r\leqq r_1\} \\ \cup\{R\leqq r\leqq 2R\}}} h(\omega_{R_i})[\nabla^2\eta + \nu^{-1}\underset{\sim}{v}_{R_i}\cdot\nabla\eta]dxdy .$$

Consider now the part of the right integral over the circular annulus $R \leqq r \leqq 2R$. We have on one hand

(143) $[R,2R]$: $|\nabla\eta| \leqq \frac{C}{R}$, $|\nabla^2\eta| \leqq \frac{C}{R^2}$,

where the constant C is independent of R. On the other hand $h(\omega_{R_i})$ satisfies the inequalities

$$(144) \qquad h(\omega_{R_i}) \leqq \omega_{R_i}^2 , \quad h(\omega_{R_i}) \leqq 2\omega_0|\omega_{R_i}| .$$

Therefore, using (6)

$$(145) \qquad \Big|\int_{R\leqq r\leqq 2R} h\nabla^2\eta dxdy\Big| \leqq \frac{C}{R^2}\int_{R\leqq r\leqq 2R} \omega_{R_i}^2 dxdy \leqq$$

$$\frac{C}{R^2}\int_{R\leqq r\leqq 2R} |\nabla\underset{\sim}{v}_{R_i}|^2 dxdy \leqq C_1/R^2 ,$$

where the constant C_1 is independent of R_i and R. For the other part of the integral over $R \leqq r \leqq 2R$ we have

(146)
$$\left|\int_{R\leqq r\leqq 2R} h\underset{\sim}{v}_{R_i}\cdot\nabla\eta\, dxdy\right| \leqq \int_{R\leqq r\leqq 2R} |h\hat{\underset{\sim}{v}}_{R_i}\cdot\nabla\eta|\,dxdy + \int_{R\leqq r\leqq 2R} |h(\underset{\sim}{v}_{R_i} - \hat{\underset{\sim}{v}}_{R_i})\cdot\nabla\eta|\,dxdy \leqq 2\omega_0 \int_{R\leqq r\leqq 2R} |\omega_{R_i}| \times |\nabla\eta|\,|\underset{\sim}{v}_{R_i} - \hat{\underset{\sim}{v}}_{R_i}|\,dxdy + \int_{R\leqq r\leqq 2R} \omega_{R_i}^2 |\nabla\eta|\,|\hat{\underset{\sim}{v}}_{R_i}|\,dxdy\,.$$

Using now (40_1), written for $\underset{\sim}{v}_{R_i}$, (143) and applying the Cauchy-Schwarz inequality we obtain

(147)
$$\int_{R\leqq r\leqq 2R} |\omega_{R_i}|\,|\nabla\eta|\,|\underset{\sim}{v}_{R_i} - \hat{\underset{\sim}{v}}_{R_i}|\,dxdy \leqq \left[\tilde{C}\int_{R\leqq r\leqq 2R} \omega_{R_i}^2\,dxdy \times \int_R^{2R}\int_0^{2\pi} \left|\frac{\partial}{\partial\theta}\underset{\sim}{v}_{R_i}\right|^2 \frac{d\theta dr}{r}\right]^{1/2} \leqq \tilde{C}\int_{R\leqq r\leqq 2R} |\nabla \underset{\sim}{v}_{R_i}|^2 dxdy\,,$$

where the constant $\tilde{C}$ is independent of R_i and R. Applying again the Cauchy-Schwarz inequality, from Lemma 3 we infer

(148) $[r_\Delta, R_i]$: $\quad \hat{\underset{\sim}{v}}_{R_i}(r) = O(\ln^{1/2} r)$, uniformly in R_i ,

and hence,

(149)
$$\int_{R\leqq r\leqq 2R} \omega_{R_i}^2 |\nabla\eta|\,|\hat{\underset{\sim}{v}}_{R_i}|\,dxdy \leqq C_2 R^{-1}\ln^{1/2} R \int_{R\leqq r\leqq 2R} \omega_{R_i}^2\,dxdy\,,$$

where the constant C_2 is also independent of R_i and R. Fixing now R_i and letting $R \to R_i$, from (6), (142), (145), (146) and (149) it follows then

(150)
$$\lim_{R\to R_i} 2 \int_{\substack{|\omega_{R_i}|\leqq\omega_0 \\ r_1\leqq r\leqq R}} |\nabla\omega_{R_i}|^2 dxdy = 2\int_{\substack{|\omega_{R_i}|\leqq\omega_0 \\ r_1\leqq r\leqq R_i}} |\nabla\omega_{R_i}|^2 dxdy \leqq \int_{r_\Delta\leqq r\leqq r_1} \omega_{R_i}^2 \left(|\nabla^2\eta| + \nu^{-1}|\underset{\sim}{v}_{R_i}|\,|\nabla\eta|\right) dxdy\,.$$

Using now on one hand the fact that $\{\underset{\sim}{v}_{R_i}\}$ converges uniformly in any compact subset of $\overline{\Omega}$ to $\underset{\sim}{v}_L$, on the other hand the independence of the right hand integral of ω_0, and letting $\omega_0 \to \infty$, we infer that

(151) $$\int_{\Omega_{R_i}} |\nabla\omega_{R_i}|^2 dxdy \leq K \int_{\Omega_{R_i}} \omega_{R_i}^2 dxdy ,$$

where the constant K is independent of R_i. Taking account of (6), the assertion of the lemma then follows.

Lemma 5 [20]. *The sequence of vorticities* $\{\omega_{R_i}\}_{i\in N}$ *satisfies the estimate*

(152) $$\Omega_{R_i}: \quad \omega_{R_i}(r,\theta) = O(r^{-1/2}) \text{ , uniformly in } \theta \text{ and } R_i.$$

Proof. For $r_\Delta < 2^n < 2^{n+1} < R_i$, (6) and (151) imply

(153) $$\int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^{2\pi} (r^2\omega_{R_i}^2 + 2r|\omega_{R_i} \frac{\partial}{\partial\theta} \omega_{R_i}|)d\theta \leq$$

$$\int_{2^n<r<2^{n+1}} (\omega_{R_i}^2 + 2|\omega_{R_i} \nabla\omega_{R_i}|)dxdy \leq C^* ,$$

where the constant C^* is independent of R_i. Hence by the integral theorem of the mean, there is an $r_n \in (2^n, 2^{n+1})$ such that

(154) $$\int_0^{2\pi} [r_n^2\omega_{R_i}^2(r_n,\theta) + 2r_n|\omega_{R_i}(r_n,\theta)\frac{\partial}{\partial\theta} \omega_{R_i}(r_n,\theta)|]d\theta$$

$$\leq \frac{1}{\ln 2} \int_{2^n<r<2^{n+1}} (2\omega_{R_i}^2 + |\nabla\omega_{R_i}|^2)dxdy \leq \frac{C^*}{\ln 2} .$$

Taking into account the inequality

(155) $$\omega_{R_i}^2(r_n,\theta) - \int_0^{2\pi} \omega_{R_i}^2(r_n,\phi)d\phi \leq \int_0^{2\pi} |\frac{\partial}{\partial\phi} \omega_{R_i}^2(r_n,\phi)|d\phi$$

$$= 2\int_0^{2\pi} |\omega_{R_i}(r_n,\phi)\frac{\partial}{\partial\phi} \omega_{R_i}(r_n,\phi)|d\phi ,$$

from (154) we infer

(156) $$r_n \max_{\theta\in[0,2\pi]} \omega_{R_i}^2(r_n,\theta) \leq C .$$

But ω_{R_i} as solution of the elliptic equation (136) satisfies the

Hopf maximum principle. Noting that $r_{n+1} \leqq 4r_n$, we obtain that

(157) $$r \max_{[0,2\pi]} \omega^2_{R_i}(r,\theta) \leqq \max[4r_n \max_{[0,2\pi]} \omega^2_{R_i}(r_n,\theta),$$

$$r_{n+1} \max_{[0,2\pi]} \omega^2_{R_i}(r_{n+1},\theta)] \ , \text{ for } r \in (r_n, r_{n+1}) \ .$$

From (156) and (157) we conclude then (152).

Lemma 6 [20]. *The Leray sequence of velocities* $\{\underset{\sim}{v}_{R_i}\}_{i\in N}$ *satisfies the estimate*

(158) $$\Omega_{R_i} : \quad \underset{\sim}{v}_{R_i}(r,\theta) = O(\ln^{1/2} r) \ , \text{ uniformly in } \theta \text{ and } R_i \ .$$

Proof. We note at first that the complex velocity w_{R_i} is solution of the inhomogeneous Cauchy-Riemann equation

(159) $$\Omega_{R_i} : \quad \frac{\partial w_{R_i}}{\partial \bar{z}} = \frac{i}{2}\, \omega_{R_i} \ .$$

Let denote now $r = |z| \geqq 8r_\Delta$ and choose the integer n so that $r \in [2^n, 2^{n+1})$. Next we show that there is a sequence $\{r_n\}_{n\in N}$, $r_n \in (2^n, 2^{n+1})$, such that

(160) $$\Omega_{R_i} : \quad \underset{\sim}{v}_{R_i}(r_n,\theta) = O(\ln^{1/2} r_n) \ , \text{ uniformly in } \theta \text{ and } R_i \ .$$

Indeed, let $r_\Delta < 2^n < 2^{n+1} < R_i$. Using the inequality

(161) $$\int_{2^n}^{2^{n+1}} \int_0^{2\pi} \left|\frac{\partial}{\partial\theta} \underset{\sim}{v}_{R_i}(r,\theta)\right|^2 \frac{d\theta dr}{r} \leqq \int_{2^n<r<2^{n+1}} |\nabla \underset{\sim}{v}_{R_i}|^2 dxdy \ ,$$

it follows from the integral theorem of the mean that for some $r_n \in (2^n, 2^{n+1})$

(162) $$\int_0^{2\pi} \left|\frac{\partial}{\partial\theta} \underset{\sim}{v}_{R_i}(r_n,\theta)\right|^2 d\theta \leqq \frac{1}{\ln 2} \int_{2^n<r<2^{n+1}} |\nabla \underset{\sim}{v}_{R_i}|^2 dxdy \ .$$

By the Cauchy-Schwarz inequality and $(a + b)^2 \leqq 2(a^2 + b^2)$ we get

(163) $[0,2\pi)$: $|\underset{\sim}{v}_{R_i}(r_n,\theta)| \leq [2\pi \int_0^{2\pi} |\frac{\partial}{\partial\theta'} \underset{\sim}{v}_{R_i}(r_n,\theta')|^2 d\theta']^{1/2} + |\underset{\sim}{v}_{R_i}(r_n,\phi)|$,

(164) $[0,2\pi)$: $|\underset{\sim}{v}_{R_i}(r_n,\theta)|^2 \leq 4\pi \int_0^{2\pi} |\frac{\partial}{\partial\theta'} \underset{\sim}{v}_{R_i}(r_n,\theta')|^2 d\theta' + 2|\underset{\sim}{v}_{R_i}(r_n,\phi)|^2$.

Integrating (164) with respect to ϕ, we find

(165) $[0,2\pi)$: $|\underset{\sim}{v}_{R_i}(r_n,\theta)|^2 \leq \frac{1}{\pi}\int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r_n,\phi)|^2 d\phi + 4\pi \int_0^{2\pi} |\frac{\partial}{\partial\theta'} \underset{\sim}{v}_{R_i}|^2 d\theta'$.

(160) now follows from (6), (131) and (162). Next denote by A_n the circular annulus $r_{n-2} \leq |\zeta| \leq r_{n+2}$ and by $\tilde{A}_n$ the subset $\tilde{A}_n := \{z \in A_n|\ |z| \in [2^n,2^{n+1})\}$. By the Pompeiu formula we get

(166) $\tilde{A}_n$: $w_{R_i}(z) = \frac{1}{2\pi i}\{ \int_{\partial A_n} [w_{R_i}(\zeta)/(\zeta - z)\ d\zeta + PV\int_{A_n} [\omega_{R_i}(\zeta)/(\zeta - z)]d\xi d\eta\}$.

For $z \in \tilde{A}_n$, $\text{dist}(z,\partial A_n) \geqq 2^{n-1} \geqq |z|/4 = r/4$. From (160) we infer then that the line integral in (166) is $O(\sqrt{\ln r})$, uniformly in θ and R_i. To estimate the area integral, we write

(167) $\tilde{A}_n$: $|PV\int_{A_n} (\zeta - z)^{-1}\omega_{R_i}(\zeta)d\xi d\eta| \leqq [PV\int_D + \int_{A_n\setminus D}]|\omega_{R_i}(\zeta)| \times |\zeta - z|^{-1} d\xi d\eta$,

where $D := \{\zeta \in A_n||\zeta - z| < 1\}$. By Lemma 5 the first integral on the right is bounded by $Cr^{-1/2}$, where the constant C is independent of θ and R_i; and since A_n is contained in the disc $|\zeta - z| < 6r$, we have by the Cauchy-Schwarz inequality and (6)

(168) $\tilde{A}_n$: $\int_{A_n\setminus D} |\zeta - z|^{-1}|\omega_{R_i}(\zeta)|d\xi d\eta \leqq (\int_{A_n} \omega_{R_i}^2 d\xi d\eta)^{1/2} \times$

$$[\int_{1<|\zeta-z|<6r} |\zeta - z|^{-2} d\xi d\eta]^{1/2} \leq C_1 \ln^{1/2} r ,$$

where the constant C_1 is independent of θ and R_i. Combining these estimates, (158) then follows.

Lemma 7 [20]. *$r^{1/2}\ln^{-1/4} r \nabla\omega_{R_i}$ is square integrable in Ω_{R_i}, uniformly with respect to R_i.*

Proof. Choose $R_i/2 \geq R \geq r_1 \geq r_\Delta$ and two non-negative C^2 cut-off functions ξ_1 and ξ_2 with the properties (139). Setting

$$(169) \quad [r_\Delta, R_i] : \quad \eta(r) := \xi_1(r)\xi_2(\tfrac{r}{R}) \frac{r}{\ln^{1/2} r} ,$$

and inserting this function and $h(\omega_{R_i}) := \omega_{R_i}^2$ in (138) we obtain

$$(170) \quad 2\int_{r_1<r<R} |\nabla\omega_{R_i}|^2 dxdy \leq 2\int_{\Omega_R} \eta |\nabla\omega_{R_i}|^2 dxdy = \int_{\Omega_R} \omega_{R_i}^2 (\nabla^2\eta + \nu^{-1} \underset{\sim}{v}_{R_i} \cdot \nabla\eta) dxdy .$$

Using the inequalities

$$(171) \quad (r_\Delta, R_i) : \quad |\nabla\eta| \leq \frac{\overline{C}}{\ln^{1/2} r} , \ |\nabla^2\eta| \leq \overline{C} ,$$

where the constant $\overline{C}$ is independent of R and R_i, as well as (6), (158) and the fact that $\{\underset{\sim}{v}_{R_i}\}$ converges uniformly together with all their first order derivatives in any compact subset of $\overline{\Omega}$ to $\underset{\sim}{v}_L$, from (170) the assertion of the lemma then follows.

Lemma 8 [20]. *The sequence of vorticities $\{\omega_{R_i}\}_{i\in N}$ satisfies the estimate*

$$(172) \quad \Omega_{R_i} : \quad \omega_{R_i}(r,\theta) = O(r^{-3/4}\ln^{1/8} r) , \text{ uniformly in } \theta \text{ and } R_i .$$

Proof. We note that for $r_\Delta < 2^n < 2^{n+1} < R_i$

$$(173) \quad \int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^{2\pi} [r^2\omega_{R_i}^2 + 2r^{3/2}\ln^{-1/4} r |\omega_{R_i} \frac{\partial}{\partial\theta} \omega_{R_I}|] d\theta \leq$$

$$\int_{2^n<r<2^{n+1}} (\omega_{R_i}^2 + 2r^{1/2}\ln^{-1/4} r|\omega_{R_i}||\nabla\omega_{R_i}|)\,dxdy$$

$$\leq \int_{2^n<r<2^{n+1}} (2\omega_{R_i}^2 + r\ln^{-1/2} r|\nabla\omega_{R_i}|^2)\,dxdy \ .$$

Using Lemma 7 and proceeding exactly as in the proof of Lemma 5, we get (172).

Lemma 9. *The sequence of vorticities* $\{\omega_{R_i}\}_{i\in N}$ *satisfies the Hölder condition*

(174) $$|\omega_{R_i}(z_2) - \omega_{R_i}(z_1)| \leq C\mu_{R_i}(R)|z_2 - z_1|^{1/2} \ ,$$

$$|z_1|, |z_2| > R + 2 \ , \ |z_2 - z_1| \leq 1 \ ,$$

where C *is a constant independent of* R *and* R_i *and*

(175) $$\begin{cases} \mu_{R_i}(R) = O(R^{-3/4}\ln^{3/8} r) \ , \\ \mu_{R_i}(R) = O(R^{-3/4}) \ , \text{ if } |w_{R_i}| \text{ is bounded} \ . \end{cases}$$

Proof. Let us define

(176) $$\mu_{R_i}(R) := \sup_{r\geq R} |\omega_{R_i}(r,\theta)|[1 + |w_{R_i}(r,\theta)|^{1/2}] \ .$$

so that (175) follows as a consequence of (158) , (172) and (176). It remains now to prove (174). Let $D(z_0;r')$ denote the disc of radius r' and center z_0, with $|z_0| > R + 2$. We set

(177) $$I(r';z_0) := \int_{D(z_0;r')} |\nabla\omega_{R_i}|^2 dxdy$$

and show at first that

(178) $$I(1;z_0) \leq C\mu_{R_i}^2(R) \ ,$$

where the constant C is independent of R_i and R. To this end let

η be a non-negative cut-off function such that $\eta(r) = 1$ for $r \leqq 1$, $\eta(r) = 0$ for $r \geqq 2$. Inserting $\eta = \eta(|z - z_0|)$ and $h(\omega_{R_i}) = \omega_{R_i}^2$ into (138), we obtain

$$(179) \qquad 2 \int_{D(z_0;2)} \eta |\nabla\omega_{R_i}|^2 dxdy = \int_{D(z_0;2)} \omega_{R_i}^2 [\nu\nabla^2\eta + \underset{\sim}{v}_{R_i} \cdot \nabla\eta] dxdy ,$$

provided we take $R_i \geqq 2$. Taking into account the fact that in $D(z_0;2)$ $|\nabla^2\eta| \leqq C$, $|\nabla\eta| \leqq C$, where the constant C depends on the choice of η, from (176) and (179) we get then (178). Next we derive a growth estimate for $I(r';z_0)$, from which (174) will follow. Multiplying (136) by ω_{R_i}, integrating by parts, and using the fact that $\nabla \cdot \underset{\sim}{v}_{R_i} = 0$, we find

$$(180) \qquad \int_{D(z_0;r')} |\nabla\omega_{R_i}|^2 dxdy = \int_{\partial D(z_0;r')} \omega_{R_i} \frac{\partial\omega_{R_i}}{\partial r'} r' d\theta' \; (2\nu)^{-1} \int_{\partial D(z_0;r')} \omega_{R_i}^2 \underset{\sim}{v}_{R_i} \cdot (\cos\theta', \sin\theta') r' d\theta' ,$$

where r', θ' are polar coordinates with respect to z_0 as origin. Using again the flux-divergence theorem as well as the identity $2ab \leqq a^2/r' + r'b^2$, we get the estimate

$$(181) \qquad \int_{\partial D(z_0;r')} \omega_{R_i} \frac{\partial\omega_{R_i}}{\partial r'} r' d\theta' = \frac{r'}{2} \int_{\partial D(z_0;r')} [(\frac{\partial\omega_{R_i}}{\partial r'})^2 + (\omega_{R_i} - \hat{\omega}_{R_i})^2/r'^2] r' d\theta' + \hat{\omega}_{R_i} \int_{\partial D(z_0;r')} \nabla^2\omega_{R_i} dxdy .$$

Taking now account of an inequality of type (40_1) and of (136), from (180) and (181) it follows

$$(182) \qquad I(r';z_0) \leqq \frac{r'}{2} \frac{dI(r';z_0)}{dr'} + C^* r' , \quad C^* := C\mu_{R_i}^2(R) ,$$

or, equivalently,

(182') $$\frac{d}{dr'}(I(r';z_0)/r'^2) \geq -2C^*/r'^2 \ .$$

Integrating this inequality and using (178), we obtain (r' ≦ 1!)

(183) $$I(r';z_0) \leqq I(1;z_0)r'^2 + 2C^*r' \leqq C\mu_{R_i}(R)r' \ .$$

Since (183) is valid for all discs $|z - z_0| \leqq r' \leqq 1$ contained in $|z| > R + 2$, from the Morrey lemma [7] it follows then (174).

Lemma 10. *The sequence of Helmholtz-Bernoulli functions* $\{H_{R_i}\}_{i\in N}$ *satisfies the estimate*

(184) $$\Omega_{R_i}: \quad H_{R_i}(r,\theta) = O(\ln r) \quad , \text{uniformly in } \theta \text{ and } R_i \ .$$

Proof. We note at first that the Helmholtz-Bernoulli function H_{R_i} is according to (20''') the real part of a solution to an inhomogeneous Cauchy-Riemann equation

(185) $$\Omega_{R_i}: \quad \frac{\partial(H_{R_i} + i\nu\omega_{R_i})}{\partial\bar{z}} = \frac{i}{2}\bar{w}_{R_i}\omega_{R_i} \ .$$

On the other hand from (20) it follows

(186) $$\Omega_{R_i}: \quad |\nabla H_{R_i}|^2 \leqq 2\nu^2|\nabla\omega_{R_i}|^2 + 4|\underset{\sim}{v}_{R_i} - \hat{\underset{\sim}{v}}_{R_i}|^2\omega_{R_i}^2 + 4|\hat{\underset{\sim}{v}}_{R_i}|^2\omega_{R_i}^2 \ .$$

Using now the estimate [11, pp 13, 16, 18]

(187) $$\Omega_{R_i}: \quad |\hat{\underset{\sim}{v}}_{R_i}(r)| \leqq C \ ,$$

where the constant C is independent of R_i, as well as (6), (41_1) and Lemma 4, from (186) we infer that the gradient of H_{R_i} is square integrable in Ω_{R_i}, uniformly in R_i. Proceeding now exactly as in the proofs of Lemma 3 and Lemma 6, the assertion (184) then follows.

Lemma 11. *The sequence of derivatives of the Helmholtz-Bernoulli* *functions* $\{\frac{dH_{r_i}}{dr}\}_{i\in N}$ *has at infinity the behaviour*

$$(188)\qquad \Omega_{R_i}\ :\quad \frac{dH_{R_i}}{dr} = O(r^{-3/4}\ln^{9/8} r)\ ,\ \text{uniformly in } \theta \text{ and } R_i .$$

Proof. We proceed exactly as in the proofs of Lemma 1 and Lemma 8 and get (188).

Lemma 12. *There exists a positive constant* λ, *which is independent of* R_i, *such that*

$$(189_1)\qquad \Omega_{R_i}\ :\quad \int_0^{2\pi} |\underset{\sim}{v}_{R_i}(r,\theta) - \hat{\underset{\sim}{v}}_{R_i}(r)|^2 d\theta = O(r^{-2\lambda})\ ,$$

$$(189_2)\qquad \Omega_{R_i}\ :\quad \hat{\underset{\sim}{v}}_{R_i}(r) - \underset{\sim}{v}_\infty = O(r^{-\lambda})\ ,\ \text{uniformly in } R_i\ .$$

Proof. We start again with the first equation (33), written now for the Leray sequence $(\underset{\sim}{v}_{R_i}, p_{R_i})$, i.e.

$$(190)\qquad (r_\Delta, R_i)\ :\quad \frac{d\hat{F}_{R_i}}{dr}\Big/\frac{d\hat{H}_{R_i}}{dr} + \frac{2}{r}\hat{F}_{R_i}\Big/\frac{d\hat{H}_{R_i}}{dr} = 1\ .$$

Integrating next (190) with respect to r we get

$$(191)\qquad (r_\Delta, R_i)\ :\quad \hat{F}_{R_i}(r)/(\hat{H}_{R_i}(r) - H_{R_i,\infty}) - 2\int_r^{R_i} (\hat{F}_{R_i}(\rho)/\rho)d\rho\ \times$$

$$1/(\hat{H}_{R_i}(r) - H_{R_i,\infty}) = 1\ ,$$

where, according to (5), (34) and (47), written respectively for $(\underset{\sim}{v}_{R_i}, p_{R_i})$, and the Parseval equality

$$(192)\qquad H_{R_i,\infty} := \hat{H}_{R_i}(R_i) = \frac{1}{2}\underset{\sim}{v}_\infty^2 + \hat{p}_{R_i}(R_i)\ .$$

On the other hand $\hat{p}_{R_i}$ is bounded in Ω_{R_i}, uniformly in R_i [11], i.e.

$$(193)\qquad \Omega_{R_i}\ :\quad |\hat{p}_{R_i}(r)| \leq C\ ,\ \text{uniformly in } R_i\ ,$$

and, hence, (at least for a subsequence!)

$$(194)\qquad \lim_{R_i\to\infty} \hat{p}_{R_i}(R_i) = p_\infty\ .$$

Letting now r R_i in (191) we infer that the limit exists and is equal to one. Proceeding now exactly as in the proof of Lemma 2, and using the Lemmas 3 to 12 as well as (187) and (194), the assertions (189_1) and (189_2) then follow.

Proof of Theorem 3 [22]. In order to show that in the case of the Leray solution $(\underset{\sim}{v}_L, p_L)$

(195) $$\underset{\sim}{v}_0 = \underset{\sim}{v}_\infty \ ,$$

it suffices to prove that

(196) $$\int_{r_\Delta}^{\infty} r^{-1} |\hat{\underset{\sim}{v}}_L(r) - \underset{\sim}{v}_0|^2 dr < \infty \ ,$$

(197) $$\int_{r_\Delta}^{\infty} r^{-1} |\hat{\underset{\sim}{v}}_L(r) - \underset{\sim}{v}_\infty|^2 dr < \infty \ .$$

Indeed, taking account of the inequality

(198) $$[r_\Delta, \infty) \ : \quad |\underset{\sim}{v}_\infty - \underset{\sim}{v}_0| \leqq |\underset{\sim}{v}_L(r) - \underset{\sim}{v}_\infty| + |\underset{\sim}{v}_L(r) - \underset{\sim}{v}_0| \ ,$$

and using (196) and (197), we find

(199) $$\int_{r_\Delta}^{\infty} r^{-1} |\underset{\sim}{v}_\infty - \underset{\sim}{v}_0|^2 dr < \infty \ ,$$

and (195) then follows. Noting now that $\underset{\sim}{v}_L$ has finite Dirichlet integral and taking account of (19_2), we obtain (196). For the proof of (197) we remark at first that from (189_2) it follows

(200) $$[r_\Delta, R_i] \ : \quad \int_{r_\Delta}^{R_i} \rho^{-1} |\hat{\underset{\sim}{v}}_{R_i}(\rho) - \underset{\sim}{v}_\infty|^2 d\rho \leqq C \ ,$$

where the constant C is independent of R_i, and hence, letting $R_i \to \infty$, (197) then follows. It remains now to show that the Leray sequence of solutions $(\underset{\sim}{v}_{R_i}, p_{R_i})$, $i \in N$, of (1) and (5) in Ω_{R_i} converges quasi-uniformly on $\overline{\Omega}$ to the Leray solution $(\underset{\sim}{v}_L, p_L)$ of (1), (2) and (3) in Ω. To this end we define $\underset{\sim}{v}^e_{R_i}$ and $\underset{\sim}{v}^e_L$ as follows

$$(201) \qquad \bar{\Omega} : \quad \underset{\sim}{v}^{e}_{R_i} := \begin{cases} \underset{\sim}{v}_{R_i} & \text{in } \Omega_{R_i} , \\ \underset{\sim}{v}_{\infty} & \text{in } \bar{\Omega} \setminus \Omega_{R_i} , \end{cases} \quad i \in N ,$$

$$(202) \qquad \bar{\Omega} : \quad \underset{\sim}{v}^{e}_{L} := \begin{cases} \underset{\sim}{v}_{L} & \text{in } \Omega , \\ \underset{\sim}{v}_{\infty} & \text{at infinity} . \end{cases}$$

It is easy to see on one hand that $\underset{\sim}{v}^{e}_{R_i}$, $i \in N$, and $\underset{\sim}{v}^{e}_{L}$ are continuous on $\bar{\Omega}$. On the other hand by using the stereographic projection the extended plane $\bar{R}^2$ becomes a metric space. By Definition 1 and the Theorem of Arzela, Gagaeff and Alexandrov the assertion then follows.

Remark 3. *The proofs of Lemmas 3-9 follow closely the proofs of the corresponding lemmas in* [*12, pp. 383, 384, 387, 388, 396, 399*].

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