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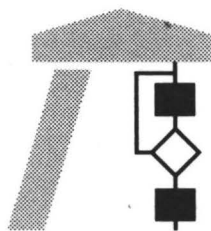
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318/02

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Januar 2002

# Quantum Integration in Sobolev Classes

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**Abstract**

We study high dimensional integration in the quantum model of computation. We develop quantum algorithms for integration of functions from Sobolev classes $W_p^r([0,1]^d)$ and analyze their convergence rates. We also prove lower bounds which show that the proposed algorithms are, in many cases, optimal within the setting of quantum computing. This extends recent results of Novak on integration of functions from Hölder classes.

## 1  Introduction

Since Shor's (1994) discovery of a polynomial factoring algorithm on a quantum computer, the question of the potential power of quantum computing was posed and studied for many problems of computer science. Most of these are of discrete type, while so far little was done for numerical problems of analysis. This field contains a variety of intrinsically difficult problems. One of them is high dimensional integration.

To judge possible gains by a quantum computer, one first of all needs to know the complexity of the respective problem in the classical settings. The complexity of many basic numerical problems in the classical deterministic and randomized setting is well understood due to previous efforts in information-based complexity theory (see Traub, Wasilkowski, and Woźniakowski, 1988, Novak, 1988, and Heinrich, 1993). This theory established precise complexity rates by developing optimal algorithms, on one hand, and proving matching lower bounds, on the other.

Based on such grounds, it is a challenging task to study these problems in the quantum model of computation and compare the results to the known

classical complexities, this way locating problems where quantum computing could bring essential speedups, and moreover, quantitatively assessing the reachable gain.

In a series of papers, Novak and the author started to investigate this field. Their research dealt with summation of sequences and integration of functions. So Novak (2001) studies integration of functions from Hölder spaces, using the algorithm of Brassard, Høyer, Mosca, and Tapp (2000) for approximating the mean of uniformly bounded sequences. Heinrich (2001a) and Heinrich and Novak (2001b) developed quantum algorithms for the mean of $p$-summable sequences and proved their optimality. Moreover, such an approach required a formal model of quantum computation for numerical problems, which was developed and studied in Heinrich (2001a). This way the basic elements of a quantum setting of information-based complexity theory were established. First ideas about path integration are discussed in Traub and Woźniakowski (2001).

Integration of functions from Sobolev spaces is one of the basic numerical problems for which we know the complexity both in the classical deterministic and randomized setting. In the present paper we study this question in the quantum setting. We develop a quantum integration algorithm by splitting the problem into levels, using a hierarchy of quadrature formulas, and this way reducing it to computing the mean of families of $p$-summable sequences. This enables us to apply the results of Heinrich (2001a) and Heinrich and Novak (2001b), and shows that the investigation of $p$-summable sequences was an important prerequisite to handle functions from Sobolev classes. We also prove lower bound which show the optimality (up to logarithmic factors) of the proposed algorithms.

The contents of the paper is as follows. In section 2 we recall some notation from the quantum setting for numerical problems as developed in Heinrich (2001a). In section 3 we add some new results of general type which will be needed later on. Section 4 recalls known facts about summation of sequences and provides some refinements of estimates. The main result about quantum integration of functions from Sobolev classes is stated and proved in section 5. The paper concludes with section 6 containing comments on the quantum bit model and a summary including comparisons to the classical deterministic and randomized setting.

For more details on the quantum setting for numerical problems we refer to Heinrich (2001a), also to the survey by Heinrich and Novak (2001a), and to an introduction by Heinrich (2001b). Furthermore, for general background on quantum computing we refer to the surveys Aharonov (1998), Ekert, Hayden, and Inamori (2000), Shor (2000), and to the monographs

Pittenger (1999), Gruska (1999), and Nielsen and Chuang (2000).

## 2  Notation

For nonempty sets $D$ and $K$, we denote by $\mathcal{F}(D, K)$ the set of all functions from $D$ to $K$. Let $F \subseteq \mathcal{F}(D, K)$ be a nonempty subset. Let $\mathbf{K}$ stand for either $\mathbf{R}$ or $\mathbf{C}$, the field of real or complex numbers, let $G$ be a normed space over $\mathbf{K}$, and let $S : F \to G$ be a mapping. We seek to approximate $S(f)$ for $f \in F$ by means of quantum computations. Let $H_1$ be the two-dimensional complex Hilbert space $\mathbf{C}^2$, with its unit vector basis $\{e_0, e_1\}$, let

$$H_m = \underbrace{H_1 \otimes \cdots \otimes H_1}_{m},$$

equippeded with the tensor Hilbert space structure. Denote

$$\mathbf{Z}[0, N) := \{0, \ldots, N - 1\}$$

for $N \in \mathbf{N}$, where we agree to write, as usual, $\mathbf{N} = \{1, 2, \ldots\}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Let $\mathcal{C}_m = \{|i\rangle : i \in \mathbf{Z}[0, 2^m)\}$ be the canonical basis of $H_m$, where $|i\rangle$ stands for $e_{j_0} \otimes \cdots \otimes e_{j_{m-1}}$, $i = \sum_{k=0}^{m-1} j_k 2^{m-1-k}$ the binary expansion of $i$. Let $\mathcal{U}(H_m)$ denote the set of unitary operators on $H_m$.

A quantum query on $F$ is given by a tuple

$$Q = (m, m', m'', Z, \tau, \beta),$$

where $m, m', m'' \in \mathbf{N}, m' + m'' \leq m, Z \subseteq \mathbf{Z}[0, 2^{m'})$ is a nonempty subset, and

$$\tau : Z \to D$$

$$\beta : K \to \mathbf{Z}[0, 2^{m''})$$

are arbitrary mappings. We let $m(Q) := m$ be the number of qubits of $Q$.

Given a query $Q$, we define for each $f \in F$ the unitary operator $Q_f$ by setting for $|i\rangle |x\rangle |y\rangle \in \mathcal{C}_m = \mathcal{C}_{m'} \otimes \mathcal{C}_{m''} \otimes \mathcal{C}_{m-m'-m''}$:

$$Q_f |i\rangle |x\rangle |y\rangle = \begin{cases} |i\rangle |x \oplus \beta(f(\tau(i)))\rangle |y\rangle & \text{if } i \in Z \\ |i\rangle |x\rangle |y\rangle & \text{otherwise,} \end{cases}$$

where $\oplus$ means addition modulo $2^{m''}$.

A quantum algorithm on $F$ with no measurement is a tuple

$$A = (Q, (U_j)_{j=0}^n).$$

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Here $Q$ is a quantum query on $F$, $n \in \mathbf{N}_0$ and $U_j \in \mathcal{U}(H_m)$ $(j = 0, \ldots, n)$, with $m = m(Q)$. Given $f \in F$, we define $A_f \in \mathcal{U}(H_m)$ as

$$A_f = U_n Q_f U_{n-1} \ldots U_1 Q_f U_0.$$

We denote by $n_q(A) := n$ the number of queries and by $m(A) = m = m(Q)$ the number of qubits of $A$. Let $(A_f(x, y))_{x,y \in \mathbf{Z}[0,2^m)}$ be the matrix of the transformation $A_f$ in the canonical basis $\mathcal{C}_m$.

A quantum algorithm from $F$ to $G$ with $k$ measurements is a tuple

$$A = ((A_\ell)_{\ell=0}^{k-1}, (b_\ell)_{\ell=0}^{k-1}, \varphi),$$

where $k \in \mathbf{N}$, $A_\ell$ $(\ell = 0, \ldots, k-1)$ are quantum algorithms on $F$ with no measurements,

$$b_0 \in \mathbf{Z}[0, 2^{m_0}),$$

$$b_\ell : \prod_{i=0}^{\ell-1} \mathbf{Z}[0, 2^{m_i}) \to \mathbf{Z}[0, 2^{m_\ell}) \quad (1 \le \ell \le k-1),$$

where $m_\ell := m(A_\ell)$, and

$$\varphi : \prod_{\ell=0}^{k-1} \mathbf{Z}[0, 2^{m_\ell}) \to G.$$

The output of $A$ at input $f \in F$ will be a probability measure $A(f)$ on $G$, defined as follows: First put

$$
\begin{aligned}
p_{A,f}(x_0, \ldots, x_{k-1}) &= |A_{0,f}(x_0, b_0)|^2 |A_{1,f}(x_1, b_1(x_0))|^2 \ldots \\
&\quad \ldots |A_{k-1,f}(x_{k-1}, b_{k-1}(x_0, \ldots, x_{k-2}))|^2.
\end{aligned}
$$

Then define $A(f)$ by setting for any subset $C \subseteq G$

$$A(f)(C) = \sum_{\varphi(x_0, \ldots, x_{k-1}) \in C} p_{A,f}(x_0, \ldots, x_{k-1}).$$

Let $n_q(A) := \sum_{\ell=0}^{k-1} n_q(A_\ell)$ denote the number of queries used by $A$. For more details and background see Heinrich (2001a). Note that we often use the term 'quantum algorithm' (or just 'algorithm'), meaning a quantum algorithm with measurement(s).

If $A$ is an algorithm with one measurement, the above definition simplifies essentially. Such an algorithm is given by

$$A = (A_0, b_0, \varphi), \quad A_0 = (Q, (U_j)_{j=0}^n).$$

The quantum computation is carried out on $m := m(Q)$ qubits. For $f \in F$ the algorithm starts in the state $|b_0\rangle$ and produces

$$|\psi_f\rangle = U_n Q_f U_{n-1} \ldots U_1 Q_f U_0 |b_0\rangle .$$

Let

$$|\psi_f\rangle = \sum_{i=0}^{2^m-1} a_{i,f} |i\rangle$$

(referring to the notation above, we have $a_{i,f} = A_{0,f}(i, b_0)$). Then $A$ outputs the element $\varphi(i) \in G$ with probability $|a_{i,f}|^2$. It is shown in Heinrich (2001a), Lemma 1, that for each algorithm $A$ with $k$ measurements there is an algorithm $\widetilde{A}$ with one measurement such that $A(f) = \widetilde{A}(f)$ for all $f \in F$ and $\widetilde{A}$ uses just twice the number of queries of $A$, that is, $n_q(\widetilde{A}) = 2n_q(A)$. Hence, as long as we are concerned with studying minimal query error and complexity (see below) up to the order, that is, up to constant factors, we can restrict ourselves to algorithms with one measurement.

Let $\theta \geq 0$. For a quantum algorithm $A$ we define the (probabilistic) error at $f \in F$ as follows. Let $\zeta$ be a random variable with distribution $A(f)$. Then

$$e(S, A, f, \theta) = \inf \left\{ \varepsilon \geq 0 \mid \mathbf{P}\{\|S(f) - \zeta\| > \varepsilon\} \leq \theta \right\}$$

(note that this infimum is always attained). Hence $e(S, A, f, \theta) \leq \varepsilon$ iff the algorithm $A$ computes $S(f)$ with error at most $\varepsilon$ and probability at least $1 - \theta$. Trivially, $e(S, A, f, \theta) = 0$ for $\theta \geq 1$. We put

$$e(S, A, F, \theta) = \sup_{f \in F} e(S, A, f, \theta)$$

(we allow the value $+\infty$ for this quantity). Furthermore, we set

$$e_n^q(S, F, \theta)$$
$$= \inf\{e(S, A, F, \theta) \mid A \text{ is any quantum algorithm with } n_q(A) \leq n\}.$$

It is customary to consider these quantities at a fixed error probability level: We denote

$$e(S, A, f) = e(S, A, f, 1/4)$$

and similarly,

$$e(S, A, F) = e(S, A, F, 1/4), \quad e_n^q(S, F) = e_n^q(S, F, 1/4).$$

The choice $\theta = 1/4$ is arbitrary - any fixed $\theta < 1/2$ would do. The quantity $e_n^q(S, F)$ is central for our study - it is the $n$-th minimal query error, that is, the smallest error which can be reached using at most $n$ queries. Note that it essentially suffices to study $e_n^q(S, F)$ instead of $e_n^q(S, F, \theta)$, since with $\mathcal{O}(\nu)$ repetitions, the error probability can be reduced to $2^{-\nu}$ (see Lemma 3 below).

The query complexity is defined for $\varepsilon > 0$ by

$$\text{comp}_\varepsilon^q(S, F) =$$
$$\min\{n_q(A) \mid A \text{ is any quantum algorithm with } e(S, A, F) \le \varepsilon\}$$

(we put $\text{comp}_\varepsilon^q(S, F) = +\infty$ if there is no such algorithm). It is easily checked that these functions are inverse to each other in the following sense: For all $n \in \mathbf{N}_0$ and $\varepsilon > 0$, $e_n^q(S, F) \le \varepsilon$ if and only if $\text{comp}_{\varepsilon_1}^q(S, F) \le n$ for all $\varepsilon_1 > \varepsilon$. Hence it suffices to determine one of them. We shall principally choose the first one.

## 3  Some General Results

Let $\emptyset \ne F \subseteq \mathcal{F}(D, K)$ and $\emptyset \ne \widetilde{F} \subseteq \mathcal{F}(\widetilde{D}, \widetilde{K})$, where $D, \widetilde{D}, K, \widetilde{K}$ are nonempty sets. Suppose we want to construct an algorithm $A$ on $F$ by the help of some reduction to an already known algorithm $\widetilde{A}$ on $\widetilde{F}$ in the following form: For $f \in F$ we construct a function $\widetilde{f} = \Gamma(f) \in \widetilde{F}$ and apply $\widetilde{A}$ to it. When does this indeed give an algorithm on $F$? To clarify the problem, note that by definition, an algorithm $A$ on $F$ can only use queries $Q$ on $F$ itself, while in the approach above we use $\widetilde{Q}_{\Gamma(f)}$ instead, where $\widetilde{Q}$ is a query on $\widetilde{F}$. The way out is to simulate $\widetilde{Q}_{\Gamma(f)}$ as $B_f$, where $B$ is an algorithm without measurement on $F$. The following result contains sufficient conditions and is a generalization of Lemma 5 of Heinrich (2001a).

Assume that we are given a mapping $\Gamma : F \to \widetilde{F}$ of the following type: There are $\kappa, m^* \in \mathbf{N}$ and mappings

$$\begin{aligned} \eta_j &: \quad \widetilde{D} \to D \quad (j = 0, \ldots, \kappa - 1) \\ \beta &: \quad K \to \mathbf{Z}[0, 2^{m^*}) \\ \varrho &: \quad \widetilde{D} \times \mathbf{Z}[0, 2^{m^*})^\kappa \to \widetilde{K} \end{aligned}$$

such that for $f \in F$ and $s \in \widetilde{D}$

$$(\Gamma(f))(s) = \varrho(s, \beta \circ f \circ \eta_0(s), \ldots, \beta \circ f \circ \eta_{\kappa-1}(s)). \tag{1}$$

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**Lemma 1.** *For each quantum query $\widetilde{Q}$ on $\widetilde{F}$ and each mapping $\Gamma$ of the above form (1) there is a quantum algorithm without measurement $B$ on $F$ such that $n_q(B) = 2\kappa$ and for all $f \in F$, $x \in \mathbf{Z}[0, 2^{\widetilde{m}})$,*

$$(\widetilde{Q}_{\Gamma(f)} \, |x\rangle) \, |0\rangle_{m-\widetilde{m}} = B_f \, |x\rangle \, |0\rangle_{m-\widetilde{m}} \,,$$

*where $\widetilde{m} = m(\widetilde{Q})$, $m = m(B) > \widetilde{m}$ and $|0\rangle_{m-\widetilde{m}}$ stands for the zero state of the last $m - \widetilde{m}$ qubits.*

*Proof.* Let

$$\widetilde{Q} = (\widetilde{m}, \widetilde{m}', \widetilde{m}'', \widetilde{Z}, \widetilde{\tau}, \widetilde{\beta}),$$

and put

$$\kappa_0 = \lceil \log \kappa \rceil, \quad m = \widetilde{m} + \kappa_0 + \kappa m^*, \quad m' = \widetilde{m}' + \kappa_0, \quad m'' = m^*,$$
$$Z = \widetilde{Z} \times [0, \kappa), \quad \tau(i, j) = \eta_j(\widetilde{\tau}(i)) \quad \text{for} \quad (i, j) \in Z,$$

let $\beta$ be as above, and define

$$Q = (m, m', m'', Z, \tau, \beta).$$

We represent

$$H_m = H_{\widetilde{m}'} \otimes H_{\widetilde{m}''} \otimes H_{\widetilde{m}-\widetilde{m}'-\widetilde{m}''} \otimes H_{\kappa_0} \otimes H_{m^*}^{\otimes \kappa},$$

a basis state of which will be written as

$$|i\rangle \, |x\rangle \, |y\rangle \, |j\rangle \, |z_0\rangle \ldots |z_{\kappa-1}\rangle \,.$$

Define the permutation operator $P_0$ by

$$P_0 \, |i\rangle \, |x\rangle \, |y\rangle \, |j\rangle \, |z_0\rangle \ldots |z_{\kappa-1}\rangle = |i\rangle \, |j\rangle \, |z_0\rangle \ldots |z_{\kappa-1}\rangle \, |x\rangle \, |y\rangle \,,$$

another permutation operator

$$P \, |i\rangle \, |j\rangle \, |z_0\rangle \ldots |z_j\rangle \ldots |z_{\kappa-1}\rangle \, |x\rangle \, |y\rangle = |i\rangle \, |j\rangle \, |z_j\rangle \ldots |z_0\rangle \ldots |z_{\kappa-1}\rangle \, |x\rangle \, |y\rangle \,,$$

the following counting operators

$$
\begin{aligned}
C_0 \, |i\rangle \, |j\rangle \ldots |y\rangle &= |i\rangle \, |j \oplus \kappa\rangle \ldots |y\rangle \\
C \, |i\rangle \, |j\rangle \ldots |y\rangle &= |i\rangle \, |j \oplus 1\rangle \ldots |y\rangle \,,
\end{aligned}
$$

where $\oplus$ is addition modulo $2^{\kappa_0}$, and the operator of sign inversion

$$J \, |i\rangle \, |j\rangle \, |z_0\rangle \ldots |y\rangle = |i\rangle \, |j\rangle \, |\ominus z_0\rangle \ldots |y\rangle \,,$$

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where $\ominus$ is subtraction modulo $2^{m^*}$ and $\ominus z$ stands for $0 \ominus z$. Finally, let

$$
\begin{aligned}
T \left| i \right\rangle &\left| j \right\rangle \left| z_0 \right\rangle \ldots \left| z_{\kappa-1} \right\rangle \left| x \right\rangle \left| y \right\rangle \\
&= \left| i \right\rangle \left| j \right\rangle \left| z_0 \right\rangle \ldots \left| z_{\kappa-1} \right\rangle \left| x \oplus \widetilde{\beta} \circ \varrho(\widetilde{\tau}(i), z_0, \ldots, z_{\kappa-1}) \right\rangle \left| y \right\rangle
\end{aligned}
$$

if $i \in \widetilde{Z}$, and

$$
T \left| i \right\rangle \left| j \right\rangle \left| z_0 \right\rangle \ldots \left| z_{\kappa-1} \right\rangle \left| x \right\rangle \left| y \right\rangle = \left| i \right\rangle \left| j \right\rangle \left| z_0 \right\rangle \ldots \left| z_{\kappa-1} \right\rangle \left| x \right\rangle \left| y \right\rangle
$$

if $i \notin \widetilde{Z}$. We define $B$ by setting for $f \in F$,

$$
B_f = P_0^{-1} C_0^{-1} (CQ_f JP)^\kappa T (PQ_f C^{-1})^\kappa C_0 P_0.
$$

Let us trace the action of $B_f$ on

$$
\left| i \right\rangle \left| x \right\rangle \left| y \right\rangle \left| 0 \right\rangle \left| 0 \right\rangle \ldots \left| 0 \right\rangle.
$$

First we assume $i \in \widetilde{Z}$. The application of $P_0$, followed by $C_0$, gives

$$
\left| i \right\rangle \left| \kappa \bmod 2^{\kappa_0} \right\rangle \left| 0 \right\rangle \ldots \left| 0 \right\rangle \left| x \right\rangle \left| y \right\rangle.
$$

The transformation $PQ_f C^{-1}$ leads to

$$
\left| i \right\rangle \left| \kappa - 1 \right\rangle \left| 0 \right\rangle \ldots \left| \beta \circ f \circ \eta_{\kappa-1} \circ \widetilde{\tau}(i) \right\rangle \left| x \right\rangle \left| y \right\rangle,
$$

and after the remaining $\kappa - 1$ applications of $PQ_f C^{-1}$ we get

$$
\left| i \right\rangle \left| 0 \right\rangle \left| \beta \circ f \circ \eta_0 \circ \widetilde{\tau}(i) \right\rangle \ldots \left| \beta \circ f \circ \eta_{\kappa-1} \circ \widetilde{\tau}(i) \right\rangle \left| x \right\rangle \left| y \right\rangle.
$$

Then the above is mapped by $T$ to

$$
\begin{aligned}
\left| i \right\rangle &\left| 0 \right\rangle \left| \beta \circ f \circ \eta_0 \circ \widetilde{\tau}(i) \right\rangle \ldots \left| \beta \circ f \circ \eta_{\kappa-1} \circ \widetilde{\tau}(i) \right\rangle \\
&\left| x \oplus \widetilde{\beta} \circ \varrho(\widetilde{\tau}(i), \beta \circ f \circ \eta_0 \circ \widetilde{\tau}(i), \ldots, \beta \circ f \circ \eta_{\kappa-1} \circ \widetilde{\tau}(i)) \right\rangle \left| y \right\rangle \\
= \left| i \right\rangle &\left| 0 \right\rangle \left| \beta \circ f \circ \eta_0 \circ \widetilde{\tau}(i) \right\rangle \ldots \left| \beta \circ f \circ \eta_{\kappa-1} \circ \widetilde{\tau}(i) \right\rangle \\
&\left| x \oplus \widetilde{\beta}((\Gamma(f))(\widetilde{\tau}(i))) \right\rangle \left| y \right\rangle.
\end{aligned}
$$

The transformation $(CQ_f JP)^\kappa$ produces

$$
\left| i \right\rangle \left| \kappa \bmod 2^{\kappa_0} \right\rangle \left| 0 \right\rangle \ldots \left| 0 \right\rangle \left| x \oplus \widetilde{\beta}((\Gamma(f))(\widetilde{\tau}(i))) \right\rangle \left| y \right\rangle,
$$

and finally $P_0^{-1} C_0^{-1}$ gives

$$
\left| i \right\rangle \left| x \oplus \widetilde{\beta}((\Gamma(f))(\widetilde{\tau}(i))) \right\rangle \left| y \right\rangle \left| 0 \right\rangle \left| 0 \right\rangle \ldots \left| 0 \right\rangle = (\widetilde{Q}_{\Gamma(f)} \left| i \right\rangle \left| x \right\rangle \left| y \right\rangle) \left| 0 \right\rangle_{m - \widetilde{m}}.
$$

The case $i \notin \widetilde{Z}$ is checked analogously. $\qquad \square$

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**Corollary 1.** *Given a mapping* $\Gamma : F \to \widetilde{F}$ *as in (1), a normed space* $G$ *and a quantum algorithm* $\widetilde{A}$ *from* $\widetilde{F}$ *to* $G$, *there is a quantum algorithm* $A$ *from* $F$ *to* $G$ *with*

$$n_q(A) = 2\kappa \, n_q(\widetilde{A})$$

*and for all* $f \in F$

$$A(f) = \widetilde{A}(\Gamma(f)).$$

*Consequently, if* $\widetilde{S} : \widetilde{F} \to G$ *is any mapping and* $S = \widetilde{S} \circ \Gamma$, *then for each* $n \in \mathbf{N}_0$

$$e^q_{2\kappa n}(S, F) \le e^q_n(\widetilde{S}, \widetilde{F}).$$

The proof is literally the same as that of Corollary 1 in Heinrich (2001a). We omit it here.

**Lemma 2.** *Let* $D, K$ *and* $F \subseteq \mathcal{F}(D, K)$ *be nonempty sets, let* $k \in \mathbf{N}_0$ *and let* $S_l : F \to \mathbf{R}$ $(l = 0, \dots, k)$ *be mappings. Define* $S : F \to \mathbf{R}$ *by* $S(f) = \sum_{l=0}^{k} S_l(f)$ $(f \in F)$. *Let* $\theta_0, \dots, \theta_k \ge 0$, $n_0, \dots, n_k \in \mathbf{N}_0$ *and put* $n = \sum_{l=0}^{k} n_l$. *Then*

$$e^q_n\left(S, F, \sum_{l=0}^{k} \theta_l\right) \le \sum_{l=0}^{k} e^q_{n_l}(S_l, F, \theta_l).$$

*Proof.* Let $\delta > 0$ and let $A_l$ be a quantum algorithm from $F$ to $\mathbf{R}$ with $n_q(A_l) \le n_l$ and

$$e(S_l, A_l, F, \theta_l) \le e^q_{n_l}(S_l, F, \theta_l) + \delta.$$

Let $A = \sum_{l=0}^{k} A_l$ be the composed algorithm (in the sense of section 2 of Heinrich, 2001a). Then

$$n_q(A) = \sum_{l=0}^{k} n_q(A_l) \le \sum_{l=0}^{k} n_l. \tag{2}$$

Fix an $f \in F$ and let $(\zeta_{l,f})_{l=0}^{k}$ be independent random variables with distribution $A_l(f)$. It follows that with probability at least $1 - \theta_l$,

$$|S_l(f) - \zeta_{l,f}| \le e^q_{n_l}(S_l, F, \theta_l) + \delta.$$

Setting

$$\zeta_f = \sum_{l=0}^{k} \zeta_{l,f},$$

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we infer from Lemma 2 of Heinrich (2001a) that $\zeta_f$ has distribution $A(f)$. Consequently,

$$|S(f) - \zeta_f| = \left| \sum_{l=0}^{k} (S_l(f) - \zeta_{l,f}) \right| \leq \sum_{l=0}^{k} (e_{n_l}^q(S_l, F, \theta_l) + \delta)$$

with probability at least

$$1 - \sum_{l=0}^{k} \theta_l.$$

This gives

$$e(S, A, f, \sum_{l=0}^{k} \theta_l) \leq \sum_{l=0}^{k} e_{n_l}^q(S_l, F, \theta_l) + (k+1)\delta$$

for all $f \in F$, and the desired result follows from (2). $\qquad\square$

**Corollary 2.** *Let* $D, K$, $F \subseteq \mathcal{F}(D,K)$, $k \in \mathbf{N}_0$ *and* $S, S_l : F \to \mathbf{R}$ $(l = 0, \ldots, k)$ *be as in Lemma 2. Assume* $\nu_0, \ldots, \nu_k \in \mathbf{N}$ *satisfy*

$$\sum_{l=0}^{k} e^{-\nu_l/8} \leq \frac{1}{4}.$$

*Let* $n_0, \ldots, n_k \in \mathbf{N}_0$ *and put* $n = \sum_{l=0}^{k} \nu_l n_l$. *Then*

$$e_n^q(S, F) \leq \sum_{l=0}^{k} e_{n_l}^q(S_l, F).$$

This is an obvious consequence of Lemma 2 above and of Lemma 3 in Heinrich (2001a), which can be restated in the following form:

**Lemma 3.** *Let* $S$ *be any mapping from* $F \subseteq \mathcal{F}(D,K)$ *to* $\mathbf{R}$. *Then for each* $n, \nu \in \mathbf{N}$,

$$e_{\nu n}^q(S, F, e^{-\nu/8}) \leq e_n^q(S, F).$$

## 4  Summation

This section provides the prerequisites from summation needed later for the study of integration. For $N \in \mathbf{N}$ and $1 \leq p < \infty$, let $L_p^N$ denote the space of all functions $f : \mathbf{Z}[0,N) \to \mathbf{R}$, equipped with the norm

$$\|f\|_{L_p^N} = \left( \frac{1}{N} \sum_{i=0}^{N-1} |f(i)|^p \right)^{1/p}.$$

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Define $S_N : L_p^N \to \mathbf{R}$ by

$$S_N f = \frac{1}{N} \sum_{i=0}^{N-1} f(i),$$

and let

$$\mathcal{B}(L_p^N) := \{f \in L_p^N \mid \|f\|_{L_p^N} \leq 1\}$$

be the unit ball of $L_p^N$.

We need the following results about summation, where (3) and (4) are from Heinrich (2001a), Theorems 1 and 2, and (5) is from Heinrich and Novak (2001b), Corollary 2.

**Proposition 1.** *Let $1 \leq p < \infty$. There are constants $c_1, c_2, c_3 > 0$ such that for all $n, N \in \mathbf{N}$ with $n \leq c_1 N$,*

$$c_2 n^{-1} \leq e_n^q(S_N, \mathcal{B}(L_p^N)) \leq c_3 n^{-1} \quad if \quad 2 < p < \infty, \tag{3}$$

$$c_2 n^{-1} \leq e_n^q(S_N, \mathcal{B}(L_2^N)) \leq c_3 n^{-1} \log^{3/2} n \log \log n, \tag{4}$$

*and*

$$c_2 \min(n^{-2(1-1/p)}, n^{-2/p} N^{2/p-1}) \leq e_n^q(S_N, \mathcal{B}(L_p^N))$$
$$\leq c_3 \min(n^{-2(1-1/p)}, n^{-2/p} N^{2/p-1}) \max(\log(n/\sqrt{N}), 1)^{2/p-1}. \tag{5}$$

*if $1 \leq p < 2$.*

**Remark.** We often use the same symbol $c, c_1, \ldots$ for possibly different positive constants (also when they appear in a sequence of relations). These constants are either absolute or may depend only on $p, r, d$ – in all lemmas and the theorem this is precisely described anyway by the order of the quantifiers.

In the case $p = 2$ we will not use Proposition 1 alone – that would give just a logarithmic factor instead of the iterated logarithm of Theorem 1 below. In the region where $n$ is close to $N$ we use a refinement which can be obtained on the basis of the results in Heinrich and Novak (2001b). We introduce for $M \in \mathbf{N}$

$$S_{N,M} f = \frac{1}{N} \sum_{i \in \mathbf{Z}[0,N], |f(i)| < M} f(i)$$

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and

$$S'_{N,M}f = S_N f - S_{N,M}f = \frac{1}{N} \sum_{i \in \mathbf{Z}[0,N], |f(i)| \geq M} f(i).$$

Let us first recall the case $p = 2$ of Corollary 3 of Heinrich and Novak (2001b), which we will use here:

**Lemma 4.** *There is a constant $c > 0$ such that for all $n, M, N \in \mathbf{N}$,*

$$e_n^q(S'_{N,M}, \mathcal{B}(L_2^N)) = 0$$

*whenever*

$$M \geq cNn^{-1} \max(\log(n/\sqrt{N}), 1).$$

The next result can be shown by repeating the respective part of the proof of Theorem 1 of Heinrich (2001a) (for the analogous $p < 2$ case see also the proof of Proposition 2 in Heinrich and Novak, 2001b).

**Lemma 5.** *There is a constant $c > 0$ such that for all $k, n, N \in \mathbf{N}$, $k > 1$*

$$e_n^q(S_{N,2^k}, \mathcal{B}(L_2^N)) \leq c \left( n^{-1} k^{3/2} \log k + 2^k n^{-2} (k \log k)^2 \right).$$

From these we can derive the following estimate:

**Lemma 6.** *There is a constant $c > 0$ such that for all $n, N \in \mathbf{N}$ with $n \leq N$,*

$$e_n^q(S_N, \mathcal{B}(L_2^N)) \leq cn^{-1} \lambda(n, N)^{3/2} \log \lambda(n, N),$$

*where*

$$\lambda(n, N) = \log(N/n) + \log\log(n + 1) + 2.$$

**Remark.** Observe that Lemma 6 gives an improvement over Proposition 1 only for $n$ close to $N$.

*Proof.* It suffices to prove the statement for

$$N \leq n^{3/2}, \tag{6}$$

the other case follows directly from (4). Let $c_0$ denote the constant from Lemma 4 and let $k$ be the smallest natural number with $k \geq 2$ and

$$c_0 Nn^{-1} \max(\log(n/\sqrt{N}), 1) \leq 2^k. \tag{7}$$

By Lemma 4,

$$e_n^q(S'_{N,2^k}, \mathcal{B}(L_2^N)) = 0. \tag{8}$$

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This together with Lemma 5 and Corollary 2 gives

$$e_{c_1 n}^q (S_N, \mathcal{B}(L_2^N)) \leq c n^{-1} k^{3/2} \log k + c \, 2^k n^{-2} (k \log k)^2, \tag{9}$$

with a certain constant $c_1 \in \mathbf{N}$. It follows from (7) that

$$2^{k-1} \leq \max \left( c_0 N n^{-1} \max(\log(n/\sqrt{N}), 1), 2 \right), \tag{10}$$

which, in turn, implies

$$2^k \leq c N n^{-1} \log(n+1), \tag{11}$$

$$k \leq c(\log(N/n) + \log \log(n+1) + 1) = c\lambda(n, N), \tag{12}$$

and thus

$$k \leq c \log(N+1). \tag{13}$$

From (12) and $\lambda(n, N) \geq 2$ we conclude

$$\log k \leq c \log \lambda(n, N), \tag{14}$$

while (13) gives

$$\log k \leq c(\log \log(N+1) + 1). \tag{15}$$

From (11), (13), (15), and (6), we infer

$$
\begin{aligned}
2^k n^{-1} k^{1/2} (\log k)^2 &\leq cN n^{-2} \log(n+1)(\log N)^{1/2}(\log \log(N+1)+1)^2 \\
&\leq cN^{-1/3}(\log(N+1))^{3/2}(\log \log(N+1)+1)^2 \leq c.
\end{aligned}
$$

Consequently

$$2^k n^{-2} (k \log k)^2 \leq c n^{-1} k^{3/2} \log k,$$

and hence, by (9), (12), and (14),

$$e_{c_1 n}^q (S_N, \mathcal{B}(L_2^N)) \leq c n^{-1} k^{3/2} \log k \leq c n^{-1} \lambda(n, N)^{3/2} \log \lambda(n, N).$$

$\square$

# 5 Integration

This section contains the main result. Let $D = [0,1]^d$ and let $C(D)$ be the space of continuous functions on $D$, equipped with the supremum norm. For $1 \leq p < \infty$, let $L_p(D)$ be the space of $p$-integrable functions, endowed with the usual norm

$$\|f\|_{L_p(D)} = \left( \int_D |f(t)|^p dt \right)^{1/p}.$$

The Sobolev space $W_p^r(D)$ consists of all functions $f \in L_p(D)$ such that for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d$ with $|\alpha| := \sum_{j=1}^d \alpha_j \leq r$, the generalized partial derivative $\partial^\alpha f$ belongs to $L_p(D)$. The norm on $W_p^r(D)$ is defined as

$$\|f\|_{W_p^r(D)} = \left( \sum_{|\alpha| \leq r} \|\partial^\alpha f\|_{L_p(D)}^p \right)^{1/p}.$$

We shall assume that $r/d > 1/p$, which, by the Sobolev embedding theorem (see Adams, 1975, or Triebel, 1995), implies that functions from $W_p^r(D)$ are continuous on $D$, and hence function values are well defined. Let $\mathcal{B}(W_p^r(D))$ be the unit ball of $W_p^r(D)$ and let $I_d : W_p^r(D) \to \mathbf{R}$ be the integration operator

$$I_d(f) = \int_D f(t) dt.$$

**Theorem 1.** *Let $r, d \in \mathbf{N}$, $1 \leq p < \infty$ and assume $r/d > 1/p$. There are constants $c_1, c_2 > 0$ such that for all $n \in \mathbf{N}$ with $n > 4$*

$$c_1 n^{-r/d-1} \leq e_n^q(I_d, \mathcal{B}(W_p^r(D))) \leq c_2 n^{-r/d-1} \quad \text{if} \quad 2 < p < \infty,$$

$$c_1 n^{-r/d-1} \leq e_n^q(I_d, \mathcal{B}(W_2^r(D))) \leq c_2 n^{-r/d-1} \lambda_0(n),$$

$$c_1 n^{-r/d-1} \leq e_n^q(I_d, \mathcal{B}(W_p^r(D))) \leq c_2 n^{-r/d-1} (\log n)^{2/p-1} \quad \text{if} \quad 1 \leq p < 2.$$

*The function $\lambda_0$ denotes an iterated-logarithmic factor:*

$$\lambda_0(n) = (\log \log n)^{3/2} \log \log \log n.$$

*Proof.* First we prepare the needed tools for the upper bound proof. For $l \in \mathbf{N}_0$ let

$$D = \bigcup_{i=0}^{2^{dl}-1} D_{li}$$

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be the partition of $D$ into $2^{dl}$ congruent cubes of disjoint interior. Let $s_{li}$ denote the point in $D_{li}$ with the smallest Euclidean norm. We introduce the following extension operator

$$E_{li} : \mathcal{F}(D, \mathbf{R}) \to \mathcal{F}(D, \mathbf{R})$$

by setting

$$(E_{li}f)(s) = f(s_{li} + 2^{-l}s)$$

for $f \in \mathcal{F}(D, \mathbf{R})$ and $s \in D$. Now let $J$ be any quadrature rule on $C(D)$,

$$Jf = \sum_{j=0}^{\kappa-1} a_j f(t_j) \quad (f \in C(D))$$

with $a_j \in \mathbf{R}$ and $t_j \in D$, which is exact on $\mathcal{P}_{r-1}(D)$, that is,

$$Jf = I_d f \quad \text{for all} \quad f \in \mathcal{P}_{r-1}(D), \tag{16}$$

where $\mathcal{P}_{r-1}(D)$ denotes the space of polynomials on $D$ of degree not exceeding $r-1$. (For example, for $d = 1$ one can take the Newton-Cotes formulas of appropriate degree and for $d > 1$ their tensor products.) Since $r > d/p$, we have, by the Sobolev embedding theorem (see Adams, 1975, or Triebel, 1995), $W_p^r(D) \subset C(D)$ and there is a constant $c > 0$ such that for each $f \in W_p^r(D)$

$$\|f\|_{C(D)} \le c\|f\|_{W_p^r(D)}. \tag{17}$$

Consequently,

$$|Jf| \le \sum_{j=0}^{\kappa-1} |a_j| |f(t_j)| \le \sum_{j=0}^{\kappa-1} |a_j| \|f\|_{C(D)} \le c\|f\|_{W_p^r(D)}. \tag{18}$$

For $f \in W_p^r(D)$ we denote

$$|f|_{r,p,D} = \left( \sum_{|\alpha|=r} \int_D |\partial^\alpha f(t)|^p \, dt \right)^{1/p}.$$

According to Theorem 3.1.1 in Ciarlet (1978), there is a constant $c > 0$ such that for all $f \in W_p^r(D)$

$$\inf_{g \in \mathcal{P}_{r-1}(D)} \|f - g\|_{W_p^r(D)} \le c|f|_{r,p,D}. \tag{19}$$

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We conclude from (16), (18) and (19),

$$
\begin{aligned}
|I_d f - Jf| &\leq \inf_{g \in \mathcal{P}_{r-1}(D)} |I_d(f-g) - J(f-g)| \\
&\leq c \inf_{g \in \mathcal{P}_{r-1}(D)} \|f-g\|_{W_p^r(D)} \leq c|f|_{r,p,D}.
\end{aligned} \tag{20}
$$

Now define for $l \in \mathbf{N}_0$

$$
J_l f = 2^{-dl} \sum_{i=0}^{2^{dl}-1} J(E_{li}f) = 2^{-dl} \sum_{i=0}^{2^{dl}-1} \sum_{j=0}^{\kappa-1} a_j f(s_{li} + 2^{-l}t_j),
$$

which is the composed quadrature obtained by scaling $J$ to the subcubes $D_{li}$. Then we have for $f \in W_p^r(D)$

$$
\begin{aligned}
|I_d f - J_l f| &= |I_d f - 2^{-dl} \sum_{i=0}^{2^{dl}-1} J(E_{li}f)| \\
&\leq 2^{-dl} \sum_{i=0}^{2^{dl}-1} |I_d(E_{li}f) - J(E_{li}f)| \\
&\leq c\, 2^{-dl} \sum_{i=0}^{2^{dl}-1} |E_{li}f|_{r,p,D} \\
&\leq c \left( 2^{-dl} \sum_{i=0}^{2^{dl}-1} |E_{li}f|_{r,p,D}^p \right)^{1/p}
\end{aligned}
$$

and

$$
\begin{aligned}
2^{-dl} \sum_{i=0}^{2^{dl}-1} |E_{li}f|_{r,p,D}^p &= 2^{-dl} \sum_{i=0}^{2^{dl}-1} \sum_{|\alpha|=r} \int_D |\partial^\alpha f(s_{li} + 2^{-l}t)|^p \, dt \\
&= 2^{-prl} \sum_{i=0}^{2^{dl}-1} \sum_{|\alpha|=r} \int_{D_{li}} |\partial^\alpha f(t)|^p \, dt \\
&= 2^{-prl} |f|_{r,p,D}^p \leq 2^{-prl} \|f\|_{W_p^r(D)}^p.
\end{aligned} \tag{21}
$$

It follows that

$$
|I_d f - J_l f| \leq c\, 2^{-rl} |f|_{r,p,D} \leq c\, 2^{-rl} \|f\|_{W_p^r(D)}. \tag{22}
$$

Let us now describe the main idea: First we approximate $I_d f$ by the quadrature $J_k f$ for some $k$, giving the desired precision, but having a number of nodes much larger than $n$. This $J_k$, in turn, will be split into the sum of a single quadrature $J_{k_0}$, with number of nodes of the order $n$, which we compute classically, and a hierarchy of quadratures (more precisely, differences of quadratures) $J_l'$ ($l = k_0, \ldots, k - 1$). It will be shown that the computation of the $J_l' f$ reduces to the computation of the mean of sequences with well-bounded $L_p^{N_l}$-norms for suitable $N_l$. This enables us to apply the results of section 4 and approximate the means by quantum algorithms. In the sequel we give the formal details, the proper balancing of parameters and the proof of the error estimates.

Define

$$
\begin{aligned}
J' f := (J_1 - J_0) f &= 2^{-d} \sum_{i=0}^{2^d - 1} \sum_{j=0}^{\kappa - 1} a_j f(s_{1,i} + 2^{-1} t_j) - \sum_{j=0}^{\kappa - 1} a_j f(t_j) \\
&:= \sum_{j=0}^{\kappa' - 1} a_j' f(t_j'),
\end{aligned}
\tag{23}
$$

where

$$
\kappa' \leq \kappa(2^d + 1).
\tag{24}
$$

For $l \in \mathbf{N}_0$, set

$$
J_{li}' f = J'(E_{li} f) = \sum_{j=0}^{\kappa' - 1} a_j' f(s_{li} + 2^{-l} t_j'),
\tag{25}
$$

$$
J_l' = 2^{-dl} \sum_{i=0}^{2^{dl} - 1} J_{li}'.
\tag{26}
$$

It is easily checked that

$$
J_{l+1} f = 2^{-dl} \sum_{i=0}^{2^{dl} - 1} J_1(E_{li} f).
$$

and hence

$$
\begin{aligned}
J_{l+1} f - J_l f &= 2^{-dl} \sum_{i=0}^{2^{dl} - 1} (J_1(E_{li} f) - J_0(E_{li} f)) \\
&= 2^{-dl} \sum_{i=0}^{2^{dl} - 1} J_{li}' f = J_l' f.
\end{aligned}
\tag{27}
$$

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Using (22) and (21), we get

$$2^{-dl} \sum_{i=0}^{2^{dl}-1} |J'_{li}f|^p \leq 2^{-dl} \sum_{i=0}^{2^{dl}-1} |J_1(E_{li}f) - J_0(E_{li}f)|^p$$

$$\leq 2^{-dl} \sum_{i=0}^{2^{dl}-1} (|(I_d - J_1)(E_{li}f)| + |(I_d - J_0)(E_{li})f)|)^p$$

$$\leq c\, 2^{-dl} \sum_{i=0}^{2^{dl}-1} |E_{li}f|_{r,p,D}^p \leq c\, 2^{-prl} \|f\|_{W_p^r(D)}^p. \qquad (28)$$

Now we derive the upper bounds. Clearly, it suffices to prove them for

$$n \geq \max(\kappa, 5). \qquad (29)$$

Let

$$k_0 = \lfloor d^{-1} \log(n/\kappa) \rfloor. \qquad (30)$$

By the above, we have $k_0 \geq 0$. Furthermore, let

$$k = \lceil (1 + d/r)k_0 \rceil, \qquad (31)$$

hence $k > k_0$. By (27)

$$J_k = J_{k_0} + \sum_{l=k_0}^{k-1} J'_l. \qquad (32)$$

For

$$k_0 \leq l < k \qquad (33)$$

put $N_l = 2^{dl}$. We shall define mappings $\Gamma_l : \mathcal{B}(W_p^r(D)) \to L_p^{N_l}$ in order to apply Lemma 1. For this purpose we fix an $m^* \in \mathbf{N}$ with

$$2^{-m^*/2} \leq k^{-1} 2^{-rk} \qquad (34)$$

and

$$2^{m^*/2-1} \geq c, \qquad (35)$$

where $c$ is the constant from (17). Hence,

$$\|f\|_{C(D)} \leq 2^{m^*/2-1} \quad \text{for} \quad f \in \mathcal{B}(W_p^r(D)). \qquad (36)$$

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Define $\eta_{lj} : \mathbf{Z}[0, N_l) \to D \quad (j = 0, \ldots . \kappa' - 1)$ by

$$\eta_{lj}(i) = s_{li} + 2^{-l}t'_j \quad (i \in \mathbf{Z}[0, N_l)),$$

and $\beta : \mathbf{R} \to \mathbf{Z}[0, 2^{m^*})$ for $z \in \mathbf{R}$ by

$$\beta(z) = \begin{cases} 0 & \text{if} \quad z < -2^{m^*/2-1} \\ \lfloor 2^{m^*/2}(z + 2^{m^*/2-1}) \rfloor & \text{if} \quad -2^{m^*/2-1} \le z < 2^{m^*/2-1} \\ 2^{m^*} - 1 & \text{if} \quad z \ge 2^{m^*/2-1}. \end{cases} \tag{37}$$

Furthermore, let $\gamma : \mathbf{Z}[0, 2^{m^*}) \to \mathbf{R}$ be defined for $y \in \mathbf{Z}[0, 2^{m^*})$ as

$$\gamma(y) = 2^{-m^*/2}y - 2^{m^*/2-1}. \tag{38}$$

It follows that for $-2^{m^*/2-1} \le z \le 2^{m^*/2-1}$,

$$\gamma(\beta(z)) \le z \le \gamma(\beta(z)) + 2^{-m^*/2}. \tag{39}$$

Next let $\varrho : \mathbf{Z}[0, 2^{m^*})^{\kappa'} \to \mathbf{R}$ be given by

$$\varrho(y_0, \ldots, y_{\kappa'-1}) = \sum_{j=0}^{\kappa'-1} a'_j \gamma(y_j).$$

Finally, we set

$$\Gamma_l(f)(i) = \varrho((\beta \circ f \circ \eta_{lj}(i))_{j=0}^{\kappa'-1}).$$

for $f \in \mathcal{B}(W_p^r(D))$. We have

$$\Gamma_l(f)(i) = \sum_{j=0}^{\kappa'-1} a'_j \gamma(\beta(f(s_{li} + 2^{-l}t'_j))),$$

hence, by (25), (36) and (39),

$$\begin{aligned} |J'_{li}f - \Gamma_l(f)(i)| &\le \sum_{j=0}^{\kappa'-1} |a'_j||f(s_{li} + 2^{-l}t'_j) - \gamma(\beta(f(s_{li} + 2^{-l}t'_j)))| \\ &\le 2^{-m^*/2} \sum_{j=0}^{\kappa'-1} |a'_j| \le c\, 2^{-m^*/2} \le ck^{-1}2^{-rk}, \end{aligned} \tag{40}$$

and therefore, by (26), for all $f \in \mathcal{B}(W_p^r(D))$,

$$|J'_l f - S_{N_l}\Gamma_l(f)| \le 2^{-dl} \sum_{i=0}^{2^{dl}-1} |J'_{li}f - \Gamma_l(f)(i)| \le ck^{-1}2^{-rk}. \tag{41}$$

Moreover, by (28), (40), and (33),

$$
\begin{aligned}
\|\Gamma_l(f)\|_{L_p^{N_l}} &\leq \|(J'_{li}f)_{i=0}^{N_l-1}\|_{L_p^{N_l}} + \|\Gamma_l(f) - (J'_{li}f)_{i=0}^{N_l-1}\|_{L_p^{N_l}} \\
&\leq \|(J'_{li}f)_{i=0}^{N_l-1}\|_{L_p^{N_l}} + \|\Gamma_l(f) - (J'_{li}f)_{i=0}^{N_l-1}\|_{L_\infty^{N_l}} \\
&\leq c\,2^{-rl}.
\end{aligned}
$$

Consequently,

$$
\Gamma_l(\mathcal{B}(W_p^r(D))) \subseteq c\,2^{-rl}\mathcal{B}(L_p^{N_l}). \tag{42}
$$

By (30), $\kappa\,2^{dk_0} \leq n$, hence

$$
e_n^q(J_{k_0}, \mathcal{B}(W_p^r(D)), 0) = 0 \tag{43}
$$

(this just means that with $\kappa\,2^{dk_0}$ queries we can compute $J_{k_0}$, the mean of $\kappa\,2^{dk_0}$ numbers, classically, or, more precisely, up to any precision by simulating the classical computation on a suitable number of qubits). By assumption, $r/d > 1/p$ and $p \geq 1$. Hence

$$
r > \frac{d}{p} \geq \left(\frac{2}{p} - 1\right) d.
$$

Now fix any $\delta$ with

$$
0 < \delta < \min\left(r, \frac{p}{2}\left(r - \left(\frac{2}{p} - 1\right) d\right)\right), \tag{44}
$$

and put for $l = k_0, \ldots, k-1$

$$
n_l = \left\lceil 2^{dk_0 - \delta(l-k_0)} \right\rceil, \tag{45}
$$

$$
\nu_l = \left\lceil 8(2\ln(l - k_0 + 1) + \ln 8) \right\rceil. \tag{46}
$$

It follows from (46) that

$$
\sum_{l=k_0}^{k-1} e^{-\nu_l/8} \leq \frac{1}{8} \sum_{l=k_0}^{k-1} (l - k_0 + 1)^{-2} < \frac{1}{4}. \tag{47}
$$

Put

$$
\widetilde{n} = n + 2\kappa' \sum_{l=k_0}^{k-1} \nu_l n_l. \tag{48}
$$

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By (45), (46), and (31),

$$
\begin{aligned}
\widetilde{n} &\leq n + 2\kappa' \sum_{l=k_0}^{k-1} \lceil 8(2\ln(l - k_0 + 1) + \ln 8) \rceil \left\lceil 2^{dk_0 - \delta(l - k_0)} \right\rceil \\
&\leq c\, 2^{dk_0} \leq cn.
\end{aligned}
\tag{49}
$$

From (22) above and Lemma 6(i) of Heinrich (2001a),

$$
e_{\widetilde{n}}^q(I_d, \mathcal{B}(W_p^r(D))) \leq c\, 2^{-rk} + e_{\widetilde{n}}^q(J_k, \mathcal{B}(W_p^r(D))).
\tag{50}
$$

By Lemma 2 and (43),

$$
\begin{aligned}
e_{\widetilde{n}}^q(J_k, \mathcal{B}(W_p^r(D))) & \\
&\leq e_n^q(J_{k_0}, \mathcal{B}(W_p^r(D)), 0) + e_{\widetilde{n}-n}^q(J_k - J_{k_0}, \mathcal{B}(W_p^r(D))) \\
&= e_{\widetilde{n}-n}^q(J_k - J_{k_0}, \mathcal{B}(W_p^r(D))).
\end{aligned}
\tag{51}
$$

Using (32), (48), Corollary 2 and (47), we get

$$
\begin{aligned}
e_{\widetilde{n}-n}^q(J_k - J_{k_0}, \mathcal{B}(W_p^r(D))) &= e_{2\kappa' \sum_{l=k_0}^{k-1} \nu_l n_l}^q \left( \sum_{l=k_0}^{k-1} J_l', \mathcal{B}(W_p^r(D)) \right) \\
&\leq \sum_{l=k_0}^{k-1} e_{2\kappa' n_l}^q(J_l', \mathcal{B}(W_p^r(D))).
\end{aligned}
\tag{52}
$$

From (41) above and Lemma 6(i) of Heinrich (2001a) we conclude

$$
e_{2\kappa' n_l}^q(J_l', \mathcal{B}(W_p^r(D))) \leq ck^{-1}2^{-rk} + e_{2\kappa' n_l}^q(S_{N_l}\Gamma_l, \mathcal{B}(W_p^r(D))).
\tag{53}
$$

Corollary 1, relation (42) above and Lemma 6(iii) of Heinrich (2001a) give

$$
\begin{aligned}
e_{2\kappa' n_l}^q(S_{N_l}\Gamma_l, \mathcal{B}(W_p^r(D))) &\leq e_{n_l}^q(S_{N_l}, c\, 2^{-rl}\mathcal{B}(L_p^{N_l})) \\
&= c\, 2^{-rl} e_{n_l}^q(S_{N_l}, \mathcal{B}(L_p^{N_l})).
\end{aligned}
\tag{54}
$$

Joining (50)–(54), we conclude

$$
e_{\widetilde{n}}^q(I_d, \mathcal{B}(W_p^r(D))) \leq c\, 2^{-rk} + c \sum_{l=k_0}^{k-1} 2^{-rl} e_{n_l}^q(S_{N_l}, \mathcal{B}(L_p^{N_l})).
\tag{55}
$$

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Now we prove the upper bound in the case $2 < p < \infty$. Relation (55), Proposition 1, (45), and (31) give

$$
\begin{aligned}
e_{\tilde{n}}^q(I_d, \mathcal{B}(W_p^r(D))) &\leq c\, 2^{-rk} + c \sum_{l=k_0}^{k-1} 2^{-rl} n_l^{-1} \\
&\leq c\, 2^{-rk} + c\, 2^{-(r+d)k_0} \sum_{l=k_0}^{k-1} 2^{-(r-\delta)(l-k_0)} \\
&\leq c\, 2^{-(r+d)k_0} \leq c n^{-r/d-1}.
\end{aligned}
$$

Next we consider the case $1 \leq p < 2$. Observe that, by (44),

$$
\frac{2}{p}\delta < r - \left(\frac{2}{p}-1\right) d. \tag{56}
$$

It follows from (55), Proposition 1, (56), (31), and (30) that

$$
\begin{aligned}
&e_{\tilde{n}}^q(I_d, \mathcal{B}(W_p^r(D))) \\
&\leq c\, 2^{-rk} + c \sum_{l=k_0}^{k-1} 2^{-rl} n_l^{-2/p} N_l^{2/p-1} \max(\log(n_l/\sqrt{N_l}), 1)^{2/p-1} \\
&\leq c\, 2^{-rk} + c \sum_{l=k_0}^{k-1} 2^{-rl-\frac{2}{p}dk_0+\frac{2}{p}\delta(l-k_0)+(\frac{2}{p}-1)dl}(k_0+1)^{\frac{2}{p}-1} \\
&\leq c\, 2^{-rk} + c\, 2^{-(r+d)k_0}(k_0+1)^{\frac{2}{p}-1} \sum_{l=k_0}^{k-1} 2^{(-r+(\frac{2}{p}-1)d+\frac{2}{p}\delta)(l-k_0)} \\
&\leq c\, 2^{-(r+d)k_0}(k_0+1)^{\frac{2}{p}-1} \leq c n^{-r/d-1}(\log n)^{\frac{2}{p}-1}.
\end{aligned}
$$

Finally we consider the case $p = 2$. From (55) and Lemma 6 we get

$$
\begin{aligned}
&e_{\tilde{n}}^q(I_d, \mathcal{B}(W_p^r(D))) \\
&\leq c\, 2^{-rk} + c \sum_{l=k_0}^{k-1} 2^{-rl} n_l^{-1} \lambda(n_l, N_l)^{3/2} \log \lambda(n_l, N_l) \\
&\leq c\, 2^{-rk} + c\, 2^{-(r+d)k_0} \times \\
&\qquad \sum_{l=k_0}^{k-1} 2^{-(r-\delta)(l-k_0)}(l-k_0+\log\log n+1)^{3/2} \log(l-k_0+\log\log n+1) \\
&\leq c\, 2^{-(r+d)k_0}(\log\log n)^{3/2} \log\log\log n \leq c n^{-r/d-1}\lambda_0(n).
\end{aligned}
$$

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To conclude the proof of the upper bounds in all three cases, we use (49) and scale $n$.

Now we turn to the lower bounds. Since $\mathcal{B}(W_p^r(D)) \subset \mathcal{B}(W_q^r(D))$ for $p > q$, it suffices to consider the case $2 < p < \infty$. Fix such a $p$. Let $\psi$ be a $C^\infty$ function on $\mathbf{R}^d$ with

$$\operatorname{supp}\psi \subset (0,1)^d, \quad \sigma_1 := I_d\psi > 0,$$

and denote $\|\psi\|_{W_p^r(D)} = \sigma_2$. Let $n \in \mathbf{N}$, $k = \lceil d^{-1}(\log(n/c_1)+1)\rceil$, where $c_1$ is the constant from Proposition 1, which can be assumed to satisfy $0 < c_1 \le 1$, and put $N = 2^{dk}$. It follows that

$$c_1 2^{-d} 2^{dk} \le 2n \le c_1 2^{dk} = c_1 N. \tag{57}$$

Set

$$\psi_i(t) = \psi(2^k(t - s_i)) \quad (i = 0, \dots, N-1),$$

with the $s_i$ as in the beginning of the proof. We have

$$I_d\psi_i = 2^{-dk} I_d\psi = \sigma_1 2^{-dk} = \sigma_1 N^{-1} \tag{58}$$

and

$$\|\psi_i\|_{W_p^r(D)} \le 2^{(r-d/p)k}\|\psi\|_{W_p^r(D)} = \sigma_2 2^{(r-d/p)k}.$$

Consequently, taking into account the disjointness of the supports of the $\psi_i$, for all $a_i \in \mathbf{R}$ $(i = 0, \dots, N-1)$,

$$\Big\|\sum_{i=0}^{N-1} a_i \psi_i\Big\|_{W_p^r(D)}^p = \sum_{i=0}^{N-1} |a_i|^p \|\psi_i\|_{W_p^r(D)}^p \le \sigma_2^p 2^{prk} \big\|(a_i)_{i=0}^{N-1}\big\|_{L_p^N}^p. \tag{59}$$

Fix any $m^* \in \mathbf{N}$ with

$$m^*/2 - 1 \ge dk/p. \tag{60}$$

Let $\beta : \mathbf{R} \to \mathbf{Z}[0, 2^{m^*})$ and $\gamma : \mathbf{Z}[0, 2^{m^*}) \to \mathbf{R}$ be defined as in (37) and (38). For $f \in \mathcal{B}(L_p^N)$ we have

$$|f(i)| \le N^{1/p} = 2^{dk/p} \le 2^{m^*/2-1}.$$

Hence, by (39),

$$\gamma(\beta(f(i))) \le f(i) \le \gamma(\beta(f(i))) + 2^{-m^*/2}. \tag{61}$$

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Define

$$\Gamma : \mathcal{B}(L_p^N) \to W_p^r(D) \quad \text{by} \quad \Gamma(f) = \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i)\,\psi_i.$$

By (59) and (61), for $f \in \mathcal{B}(L_p^N)$,

$$
\begin{aligned}
\|\Gamma(f)\|_{W_p^r(D)} &\leq \sigma_2 2^{rk}\|\gamma \circ \beta \circ f\|_{L_p^N} \\
&\leq \sigma_2 2^{rk}\left(\|f\|_{L_p^N} + \|f - \gamma \circ \beta \circ f\|_{L_p^N}\right) \\
&\leq \sigma_2 2^{rk}\left(1 + 2^{-m^*/2}\right).
\end{aligned}
$$

Furthermore, by (58),

$$
\begin{aligned}
I_d \circ \Gamma(f) &= \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i) I_d \psi_i \\
&= \sigma_1 N^{-1} \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i) \\
&= \sigma_1 S_N(\gamma \circ \beta \circ f).
\end{aligned}
\tag{62}
$$

Define

$$\eta : D \to \mathbf{Z}[0,N) \quad \text{by} \quad \eta(s) = \min\{i \,|\, s \in D_{ki}\},$$

with the $D_{ki}$ as in the beginning of the proof, and

$$\varrho : D \times \mathbf{Z}[0, 2^{m^*}) \to \mathbf{R} \quad \text{by} \quad \varrho(s,z) = \gamma(z)\psi_{\eta(s)}(s).$$

Then

$$
\begin{aligned}
\Gamma(f)(s) &= \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i)\,\psi_i(s) \\
&= \gamma \circ \beta \circ f(\eta(s))\,\psi_{\eta(s)}(s) \\
&= \varrho(s, \beta \circ f \circ \eta(s)).
\end{aligned}
$$

So $\Gamma$ is of the form (1) (with $\kappa = 1$) and maps

$$\mathcal{B}(L_p^N) \quad \text{into} \quad \sigma_2 2^{rk}\left(1 + 2^{-m^*/2}\right)\mathcal{B}(W_p^r(D)).$$

By Corollary 1 and Lemma 6(iii) in Heinrich (2001a),

$$
\begin{aligned}
e_{2n}^q(I_d \circ \Gamma, \mathcal{B}(L_p^N)) &\leq e_n^q\left(I_d, \sigma_2 2^{rk}\left(1 + 2^{-m^*/2}\right)\mathcal{B}(W_p^r(D))\right) \\
&= \sigma_2 2^{rk}\left(1 + 2^{-m^*/2}\right)e_n^q(I_d, \mathcal{B}(W_p^r(D))).
\end{aligned}
$$

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Using (61) again, we infer

$$\sup_{f \in \mathcal{B}(L_p^N)} |S_N f - S_N(\gamma \circ \beta \circ f)| \leq 2^{-m^*/2},$$

and hence, by Proposition 1 and Lemma 6(i) and (ii) of Heinrich (2001a), using also (62) and (57),

$$
\begin{aligned}
cn^{-1} &\leq e_{2n}^q(S_N, \mathcal{B}(L_p^N)) \\
&\leq e_{2n}^q(S_N \circ \gamma \circ \beta, \mathcal{B}(L_p^N)) + 2^{-m^*/2} \\
&= \sigma_1^{-1} e_{2n}^q(I_d \circ \Gamma, \mathcal{B}(L_p^N)) + 2^{-m^*/2} \\
&\leq \sigma_1^{-1} \sigma_2 2^{rk} \left(1 + 2^{-m^*/2}\right) e_n^q(I_d, \mathcal{B}(W_p^r(D))) + 2^{-m^*/2} \\
&\leq cn^{r/d} e_n^q(I_d, \mathcal{B}(W_p^r(D))) + 2^{-m^*/2}.
\end{aligned}
$$

Since $m^*$ can be made arbitrarily large, the desired result follows. $\quad\square$

# 6  Comments

Let us discuss the cost of the presented algorithm in the bit model of computation. The algorithm consists of quantum summations on the levels $l = k_0, \ldots, k-1$. On level $l$ we have $N_l = 2^{dl}$ and $n_l = \Theta(2^{-\delta(l-k_0)}n)$, where $\delta > 0$ does not depend on $l$ or $n$. Recall also that $2^{dk_0} = \Theta(n)$ and $k - k_0 = \Theta(\log n)$. Referring to the respective discussion of summation in the bit model in Heinrich (2001a), section 6, and Heinrich and Novak (2001b), section 5, we conclude that on level $l$ we need $\mathcal{O}(\log N_l)$ qubits, $\mathcal{O}(n_l \log N_l)$ quantum gates, $\mathcal{O}(\log n_l \log \log n_l)$ measurements in the case $p > 2$ and

$$\mathcal{O}(n_l^2 N_l^{-1}/\max(\log(n_l/\sqrt{N_l}), 1) + \log(N_l/n_l) \log \log(N_l/n_l))$$

measurements for $p \leq 2$. Summarizing, we see that altogether the algorithm needs $\mathcal{O}(\log n)$ qubits, $\mathcal{O}(n \log n)$ quantum gates, $\mathcal{O}((\log n)^2 \log \log n)$ measurements for $p > 2$ and $\mathcal{O}(n/\log n)$ measurements for $p \leq 2$. Thus the quantum bit cost differs by at most a logarithmic factor from the quantum query cost $\Theta(n)$.

In the following table we summarize the results of this paper and compare them with the respective known quantities of the classical deterministic and randomized setting. We refer to Heinrich and Novak (2001a) and the bliography therein for more information on the classical setting. The respective entries of the table give the minimal error, constants and logarithmic factors are suppressed.

| | deterministic | random | quantum |
|---|---|---|---|
| $\mathcal{B}(W^r_{p,d}(D)),\ 2 \le p < \infty$ | $n^{-r/d}$ | $n^{-r/d-1/2}$ | $n^{-r/d-1}$ |
| $\mathcal{B}(W^r_{p,d}(D)),\ 1 < p < 2$ | $n^{-r/d}$ | $n^{-r/d-1+1/p}$ | $n^{-r/d-1}$ |
| $\mathcal{B}(W^r_{1,d}(D))$ | $n^{-r/d}$ | $n^{-r/d}$ | $n^{-r/d-1}$ |

The quantum rate for $1 \le p < 2$ is a certain surprise. Previous results led one to conjecture that the quantum setting could reduce the exponent of the classical randomized setting by at most $1/2$. Now we see that in the case $p = 1$ there is even a reduction by 1.

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