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0.1. Introduction

The present work is a further connecting link betweeen the theory of tensor norms introduced by A. Grothendieck ([Gr]) and the theory of operator ideals established by A. Pietsch ([P1],[P3]). We focus on a further developing of the axiomatic setting, practised by Grothendieck, Lotz, Defant and Floret ([Gr],[Lz],[D],[D–F1],[D–F2]) and more abstractly in categorical terms by Cigler, Losert, Michor and Pelletier ([C–Lo–M],[M],[Pe]). Our aim is, to investigate some more aspects concerning the *inner structure* of operator ideals (and their corresponding tensor norms) which is independent of concrete realizations like series–representations (e.g. $(\mathcal{N},\|\cdot\|_{\mathcal{N}})$), measuretheoretical arguments (e.g. $(\mathcal{P}_2,\|\cdot\|_{\mathcal{P}_2})$) or factorization–properties (e.g. $(\mathcal{W},\|\cdot\|)$).

For instance we want to know sufficient and necessary conditions for an arbitrarily given maximal Banach ideal $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ such that $\mathcal{A}(M,F'') \cong \mathcal{A}(M,F)''$ does hold if M is a finite dimensional and F an arbitrary Banach space. Especially for $\mathcal{A} := \mathcal{L}$ – the ideal of all continuous operators – we recognize the weak version of the prinicple of local reflexivity ([Li–Rt],[Dn]) and therefore a transition to the local theory of Banach spaces. It will be seen that the above problem involves interesting connections with the so–called accessible operator ideals introduced by A. Defant ([D]) and the conjugated Banach ideals which appear first in the paper of Gordon, Lewis and Retherford ([G–L–R]). It seems to be useful to look on operator ideals *and* tensornorms *simultaneously.* For instance, up to now, we do not know another proof of theorem 2.9. which does not need the language of tensornorms.

We now describe the contents of this paper in more detail.

In the first paragraph we generalize the notion of conjugated operator ideals which also allows us to characterize every *maximal* Banach ideal in such a way. This can be realized by using the calculus of maximal operator ideals associated to finitely generated tensor norms which was introduced by A. Defant ([D], 2.5.).

In the second chapter we investigate the "local" structure of Banach ideals of type $(\mathcal{A}^{\Delta},\|\cdot\|_{\mathcal{A}^{\Delta}})$ where $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is a given maximal Banach ideal.

We recognize that such $\mathcal{A}^{\Delta}$ are always right–accessible (even that $(\mathcal{A}^{\Delta})^{dd}$ is accessible), but in general we are not able to show their left–accessibility.

Until a few days ago it was still unknown if there exists a (maximal) Banach ideal which is not accessible, but recently on the Oberwolfach–meeting "Geometrie der Banachräume" in September 1991, Pisier gave such a counterexample, answering a question of Floret and Defant ([D–F2], 31.6.).
However we can show that for a given maximal Banach ideal $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$, $(\mathcal{A}^{\Delta},\|\cdot\|_{\mathcal{A}^{\Delta}})$ is left–accessible iff the weak version of the principle of local reflexivity can be transformed from the canonical operator norm $\|\cdot\|$ to the ideal norm $\|\cdot\|_{\mathcal{A}}$ or equivalently if $(\mathcal{A}^*)^{\min}(E,F)$ is a subspace of $\mathcal{A}^{\Delta}(E,F)$ for all Banach spaces F with the metric approximation property and for all Banach spaces E. In that case we call the above generalization the weak $\mathcal{A}$–local reflexivity principle (short: (w) $\mathcal{A}$–ℓ.r.p.) and since especially the maximal Banach ideal $(\mathcal{L}_{p,q},\|\cdot\|_{\mathcal{L}_{p,q}})$ of all (p,q)–factorable operators $(1 \le p, q \le \infty, \frac{1}{p}+\frac{1}{q} \ge 1)$ is (left–) accessible, so is $(\mathcal{L}^*_{p,q})^{\Delta}$ and therefore the (w) $\mathcal{L}^*_{p,q}$–ℓ.r.p. holds.
This paper is a summary of some sections of the author's dissertation ([Oe]); I am very grateful to Prof. Dr. E. Schock and Prof. Dr. A. Defant for their motivating discussions.

0.2. Notation and terminology

We shall use the common notations of Banach–space–theory; in particular B_E denotes the closed unit ball of a normed space E (over $\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$), E' the dual space of E and $\mathcal{L}(E,F)$ is the class of all (continuous) operators between the normed spaces E and F. Given $T \in \mathcal{L}(E,F)$, the dual operator of T is denoted by T'. NORM, BAN and FIN denotes the class of all normed spaces, Banach spaces and finite dimensional spaces respectively. FIN(E) is the class of all finite dimensional subspaces of a normed space E and COFIN(E) is the class of all finite codimensional subspaces of E. Concerning operator ideals we follow Pietsch's book ([P1]). If $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ are both normed operator ideals, we sometimes use the abbreviation $\mathcal{A} \mathrel{\hat{=}} \mathcal{B}$ to indicate the equality $(\mathcal{A},\|\cdot\|_{\mathcal{A}}) = (\mathcal{B},\|\cdot\|_{\mathcal{B}})$ and we write $\mathcal{A}^d$ instead of $\mathcal{A}^{\text{dual}}$. If $T : E \to F$ is an operator, we indicate that it is a metric injection ($\|Tx\| = \|x\|$) by writing

$$T : E \overset{1}{\hookrightarrow} F$$

and that it is a metric surjection (F has the quotient norm of E via T) by

$$T : E \overset{1}{\longrightarrow}\!\!> F .$$

If there exists an isometric isomorphism between the spaces E and F, we write $E \cong F$. For $G \in FIN(E)$, $J_G^E : G \overset{1}{\hookrightarrow} E$ denotes the canonical metric injection and for $G \in COFIN(E)$, G closed, $Q_G^E : E \overset{1}{\longrightarrow}\!\!> E/G$ denotes the canonical metric surjection. We asssume the reader to be familiar with the basics of the general theory of tensor norms as they are presented in [D], [D–F1] or [D–F2]. Another important tool to describe local properties of ideal components is given by the *trace* on a normed space E which is the linearization of the duality bracket

$$\begin{aligned} E' \times E &\longrightarrow \mathbf{K} \\ (a,x) &\longmapsto \langle x,a\rangle \quad , \end{aligned}$$

whence

$$\begin{aligned} tr : E' \otimes E &\longrightarrow \mathbf{K} \\ \sum_{i=1}^{n} a_i \otimes x_i &\longmapsto \sum_{i=1}^{n} \langle x_i,a_i\rangle \ . \end{aligned}$$

We recall that a Banach space E has the *metric approximation property* (short: m.a.p.) if for all compact sets $K \subseteq E$ and for all $\varepsilon > 0$ there is a finite dimensional operator $L \in \mathcal{F}(E,F)$ with $\|L\| \le 1$ such that $\|Lx-x\| \le \varepsilon$ for all $x \in K$.
Finally we remember the important

WEAK PRINCIPLE OF LOCAL REFLEXIVITY:

Let M and F be normed spaces, M finite dimensional and $T \in \mathcal{L}(M,F'')$. Then for every $\varepsilon > 0$ and $N \in FIN(F')$ there is an $R \in \mathcal{L}(M,F)$ such that

$$\|R\| \le (1+\varepsilon)\|T\|$$

and

$$\langle Rx,b\rangle = \langle b,Tx\rangle$$

for all $(x,b) \in M \times N$.

1. α–CONJUGATED OPERATORS

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be an arbitrarily chosen normed operator ideal. According to [G–L–R] the conjugated operator ideal $(\mathcal{A}^{\Delta},\|\cdot\|_{\mathcal{A}^{\Delta}})$ is given by all operators $T \in \mathcal{L}(E,F)$ ($E,F \in BAN$) for which there exists a number $\rho \geq 0$ such that for all $L \in \mathcal{F}(F,E) = F' \otimes E$ the following inequality holds:

$$|tr(TL)| \leq \rho\|L\|_{\mathcal{A}} .$$

By definition $\|T\|_{\mathcal{A}^{\Delta}} := \inf(\rho)$ where the infimum is formed by all such $\rho \geq 0$.

We generalize this concept as follows:

1.1. DEFINITION:

Let α be a tensor norm (on BAN). Let E,F be Banach spaces. An operator $T \in \mathcal{L}(E,F)$ is called *α–conjugated* if

$$\exists\ \rho \geq 0\ \ \forall\ \ L \in \mathcal{F}(F,E) : |tr(TL)| \leq \rho\alpha^t(L;F',E) .$$

We put $\|T\|_{(\alpha)} := \inf(\rho)$ where the infimum is formed by all such $\rho \geq 0$. The class of all α–conjugated operators is denoted by $\mathcal{D}_{\alpha}$.

A straight forward calculation shows that $(\mathcal{D}_{\alpha},\|\cdot\|_{(\alpha)})$ is always a regular Banach ideal (cf. [P1], 6.4.2.) where the regularity follows directly by using the tensor–norm–property of α^t.

If α is finitely generated, then $(\mathcal{D}_{\alpha},\|\cdot\|_{(\alpha)})$ is a maximal Banach ideal.

Our next aim is to characterize maximal or conjugated Banach ideals respectively by suitable chosen $(\mathcal{D}_{\alpha},\|\cdot\|_{(\alpha)})$.

To beginn with, we have the following

1.2. PROPOSITION:

Let α be a tensor norm. Then

$$(\mathcal{D}_{\overleftarrow{\alpha}},\|\cdot\|_{(\overleftarrow{\alpha})}) = (\mathcal{D}^{\Delta}_{\alpha^*},\|\cdot\|^{\Delta}_{(\alpha^*)}) .$$

Proof:

Let $L \in \mathcal{F}(F,E) = F' \otimes E$ $(E,F \in BAN)$. Since $(\mathcal{D}_{\alpha^*}, \|\cdot\|_{(\alpha^*)})$ and $(\alpha^*)' = \overleftarrow{\alpha^t}$ are associated (see [D–F1], 4.2.) it follows by [D–F1], 4.4. that

$$F' \otimes_{\overleftarrow{\alpha^t}} E \overset{1}{\hookrightarrow} \mathcal{D}_{\alpha^*}(F,E) .$$

Hence

$$\overleftarrow{\alpha^t}(L;F',E) = \|L\|_{(\alpha^*)} .$$

□

The investigation of maximal Banach ideals need a momentous

1.3. DEFINITION:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a p–Banach operator ideal $(0 < p \leq 1)$ and E,F be Banach spaces. Let $z \in E\otimes F$, $z = \sum_{i=1}^{n} x_i \otimes y_i$, $\Lambda_z := \sum_{i=1}^{n} j_F\, y_i \otimes x_i$. We put

$$\alpha_{\mathcal{A}}(z;E,F) := \sup \left\{ \left| \sum_{i=1}^{n} \langle y_i, Tx_i \rangle \right| : T \in B_{\mathcal{A}(E,F')} \right\}$$
$$= \sup \{ |tr(\Lambda_z T)| : T \in B_{\mathcal{A}(E,F')} \} .$$

Obviously $\alpha_{\mathcal{A}}$ is a tensor norm on BAN for which

$$E \otimes_{\alpha_{\mathcal{A}}} F \overset{1}{\hookrightarrow} \mathcal{A}(E,F')'$$

holds canonically. For associated maximal Banach ideals $(\mathcal{A},\|\cdot\|_{\mathcal{A}}) \sim \alpha$ ([D–F1],4.2.) we have the following

1.4. PROPOSITION:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal and α be a finitely generated tensor norm. Then the following are equivalent:

(1) $(\mathcal{A},\|\cdot\|_{\mathcal{A}}) \sim \alpha$

(2) $\alpha_{\mathcal{A}} = \alpha'$.

Proof:

Let (1) be valid. Since $\mathcal{A}(E,F') \cong (E\otimes_{\alpha'}F)'$ ([D–F1],4.3.), it follows by the Hahn–Banach–theorem that (2) holds. Conversely let (2) be valid. Let $M,N \in$ FIN. By [P1], 9.2.1., 9.2.2. it follows for arbitrary $L \in \mathcal{F}(M,N) = M'\otimes N$, that

$$\begin{aligned}\alpha(L;M',N) &= \alpha'_{\mathcal{A}}(L;M',N)\\ &= \sup\{|\mathrm{tr}(L'\theta)| : \theta \in M''\otimes_{\alpha_{\mathcal{A}}} N',\ \alpha_{\mathcal{A}}(\theta;M'',N') \le 1\}\\ &= \sup\{|\mathrm{tr}(L''\theta')| : \theta' \in B_{\mathcal{A}^*(N'',M'')}\}\\ &= \sup\{|\langle L'',\varphi\rangle| : \varphi \in B_{\mathcal{A}(M'',N'')'}\}\\ &= \|L''\|_{\mathcal{A}} = \|L\|_{\mathcal{A}}\,.\end{aligned}$$

Hence

$$(\mathcal{A},\|\cdot\|_{\mathcal{A}}) \sim \alpha$$

□

Our aim is now to show that *each* maximal Banach ideal $\mathcal{A}$ can be represented by $\mathcal{D}_\beta$ with the finitely generated tensornorm $\beta := \alpha_{\mathcal{A}}$. Therefore we need the following

1.5. LEMMA:

Let β be a tensor norm and $E,F \in$ BAN.

If β is right–finitely generated (e.g. right–injective) *or* if F is reflexive, then we have the canonical isometric isomorphism

$$\mathcal{D}_\beta(E,F') \cong (E\otimes_\beta F)'\,.$$

In particular

$$\alpha_{\mathcal{D}_\beta}(\cdot;E,F) = \beta(\cdot;E,F).$$

Proof:

We have to show that $\Phi : T \longmapsto (z \mapsto \mathrm{tr}(\Lambda_z T))$ is an isometric isomorphism from $\mathcal{D}_\beta(E,F')$ to $(E\otimes_\beta F)'$, Λ_z defined as in 1.3.. Obviously, Φ is well–defined and $\|\Phi(T)\| \le \|T\|_{(\beta)}$. Let now $\psi \in (E\otimes_\beta F)'$ be arbitrarily given. We put $\langle y,Tx\rangle := \langle x\otimes y,\psi\rangle$. First let $\varepsilon > 0$, $L \in \mathcal{F}(F',E) = F''\otimes E$ and β be right–finetely generated. Then there exist $M \in FIN(F'')$ and $z_L \in M \otimes E$ such that $L = (J_M^{F''} \otimes Id_E) z_L$ and

$$\beta^t(z_L;M,E) < (1+\varepsilon)\beta^t(L;F'',E) .$$

Let $N := [Tx_1,\dots,Tx_n] \in FIN(F')$ and $z_L = \sum_{i=1}^{n} \eta_i \otimes x_i$ be a representation of z_L in $M \otimes E$. Then by the weak principle of local reflexivity (applied to the operator $J_M^{F''} : M \hookrightarrow F''$) there exists $S_o \in \mathcal{L}(M,F)$ with $\|S_o\| \le 1+\varepsilon$ such that

$$|tr(TL)| = |<\sum_{i=1}^{n} x_i \otimes S_o \eta_i, \psi>| \le \|\psi\| \cdot (1+\varepsilon)^2 \cdot \beta^t(L;F'',E) .$$

Hence $T \in \mathcal{D}_\beta(E,F')$, $\psi = \Phi(T)$ and $\|T\|_{(\beta)} \le \|\psi\|$.

If β is arbitrary and F reflexive, then $L = \Lambda_w$ with $w \in E \otimes F$ suitable chosen. Hence

$$|tr(TL)| = |<w,\psi>| \le \|\psi\| \cdot \beta^t(L;F'',E) .$$

□

We have now prepared for the

1.6. THEOREM

(1) Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal. Then there is a uniquely defined finitely generated tensor norm β on BAN such that $(\mathcal{A},\|\cdot\|_{\mathcal{A}}) = (\mathcal{D}_\beta,\|\cdot\|_{(\beta)})$. β is given by $\alpha_{\mathcal{A}}$.

(2) Let β be a finitely generated tensor norm on BAN. Then there is a uniquely defined maximal operator ideal $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ such that $\beta = \alpha_{\mathcal{A}}$. $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is given by $(\mathcal{D}_\beta,\|\cdot\|_{(\beta)})$.

Proof:

To prove (1), we look at $(\mathcal{B},\|\cdot\|_{\mathcal{B}}) := (\mathcal{D}_{\alpha_{\mathcal{A}}},\|\cdot\|_{(\alpha_{\mathcal{A}})})$. Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}}) \sim \alpha$ with finitely generated α. Then $(\mathcal{A}^*,\|\cdot\|_{\mathcal{A}^*}) \sim \alpha^*$ ([D–F1], 4.5.). By 1.4. this implies

$$\alpha_{\mathcal{A}}^* = (\alpha')^* = (\alpha^*)' = \alpha_{\mathcal{A}^*} .$$

Consequently $\alpha_{\mathcal{A}}$ is finitely generated and $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ maximal. Hence there is a finitely generated tensor norm β such that $(\mathcal{B},\|\cdot\|_{\mathcal{B}}) \sim \beta$. This implies (by considering 1.4. and 1.5.) $\beta' = \alpha_{\mathcal{B}} = \alpha_{\mathcal{A}}$. Hence

$$(\mathcal{B},\|\cdot\|_{\mathcal{B}}) \sim \beta = \alpha'_{\mathcal{A}} \sim (\mathcal{A},\|\cdot\|_{\mathcal{A}}) .$$

Since $\mathcal{A}$ and $\mathcal{B}$ are maximal Banach ideals, the claim follows. Part (2) of this theorem follows directly by the previous lemma.

□

2. CONJUGATED OPERATOR IDEALS AND THE $\mathcal{A}$–LOCAL REFLEXIVITY PRINCIPLE

We have observed that each maximal Banach ideal can be represented by an Banach ideal of type $\mathcal{D}_\alpha$ with suitable tensor norm α. On the other hand there are tensor norms β such that $\mathcal{D}_\beta$ is not maximal; since there are Banach spaces which do not have the m.a.p. ([E]), $(\mathcal{D}_{\overleftarrow{\pi}},\|\cdot\|_{(\overleftarrow{\pi})}) = (\mathcal{I}^\Delta,\|\cdot\|_{\mathcal{I}^\Delta}) \neq (\mathcal{L},\|\cdot\|)$ can't be maximal ([J–O]). In this connection, interesting relations to geometric structural properties of Banach ideals appear. Especially we remember the idea of "accessibility" which was first noted – in the context of tensor norms – in the famous paper [Gr] of A. Grothendieck and transformed to the language of operator ideals by A. Defant ([D]): Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a p–Banach operator ideal $(0 < p \leq 1)$. $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is called *right–accessible* (short: r.a.) if for all $(M,F) \in FIN \times BAN$, $T \in \mathcal{L}(M,F)$ and $\varepsilon > 0$ there are $N \in FIN(F)$, $S \in \mathcal{L}(M,N)$ such that $T = J_N^F S$ and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$.
$(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is called *left–accessible* (short: l.a.) if for all $(E,N) \in BAN \times FIN$, $T \in \mathcal{L}(E,N)$ and $\varepsilon > 0$ there are $L \in COFIN(E)$, $S \in \mathcal{L}(E/L,N)$ such that $T = SQ_L^E$ and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$. $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is called *accessible* if it is r.a. and l.a. . $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is called *totally accessible* if for all $T \in \mathcal{F}(E,F)$ $(E,F \in BAN)$ and $\varepsilon > 0$ there are $L \in COFIN(E)$, $N \in FIN(F)$ and $S \in \mathcal{L}(E/L,N)$ such that $T = J_N^F S Q_L^E$ and $\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$.

Combining the previous observation with the following statement, it follows that the maximal Banach ideal $(\mathcal{I},\|\cdot\|_{\mathcal{I}})$ of the class of integral operators is not totally accessible:

2.1. PROPOSITION:
Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a Banach ideal.

(1) If $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is totally accessible then $(\mathcal{A}^*,\|\cdot\|_{\mathcal{A}^*}) = (\mathcal{A}^\Delta,\|\cdot\|_{\mathcal{A}^\Delta})$

(2) If $(\mathcal{A}^\Delta,\|\cdot\|_{\mathcal{A}^\Delta}) = (\mathcal{A}^*,\|\cdot\|_{\mathcal{A}^*})$ *and* $\mathcal{A}$ is maximal, then $\mathcal{A}$ is totally accessible.

Proof:
(1) follows directly by assumption and the definition of adjoint Banach ideals. To prove (2), let $\mathcal{A}$ be maximal with $\mathcal{A}^\Delta \stackrel{\frown}{=} \mathcal{A}^*$. By 1.6. there is a f.g. tensor norm $\alpha = \vec{\alpha}$ such that $\mathcal{A}^* \stackrel{\frown}{=} \mathcal{D}_\alpha$. Hence $\mathcal{D}_\alpha \stackrel{\frown}{=} \mathcal{D}_{\overleftarrow{\alpha}}$ (1.2.). By 1.5. this implies that $\alpha = \overleftarrow{\alpha}$ is totally accessible and so is $\mathcal{A} \stackrel{\frown}{=} \mathcal{A}^{**} \sim \alpha^t$ ([D–F1], 9.2.).

□

In contrast to the total accessibility of maximal Banach ideals, Pisier's contruction of a counterexample of a nonaccessible maximal Banach ideal is more delicate ([D–F2], 31.6.). However, if $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is a p–Banach ideal ($0 < p \leq 1$), then $(\mathcal{A}^{min},\|\cdot\|_{\mathcal{A}^{min}})$ in general is accessible ([D–F2], 25.3.). We will see that right–accessibility can be transferred to the larger Banach ideal $((\mathcal{A}^*)^\Delta,\|\cdot\|_{(\mathcal{A}^*)^\Delta})$.
To investigate such operator ideals of type $(\mathcal{A}^*)^\Delta$, it is very helpful to introduce a generalization of the cofinite hull $\overleftarrow{\alpha}$ (cf. [H] and [D–F1]):

2.2. DEFINITION:
Let α be a tensor norm on BAN and E,F be Banach spaces. Let $z \in E\otimes F$,
$z = \sum_{i=1}^n x_i \otimes y_i$, $\Lambda_z = \sum_{i=1}^n j_F y_i \otimes x_i$. We set

$$\alpha^x(z;E,F) := \sup \{ |<z,w>| : w \in B_{E'\otimes_\alpha F'} \}$$
$$= \sup \{ |tr(\Lambda_z L)| : L \in \mathcal{F}(E,F'),\ \alpha(L;E',F') \leq 1 \}.$$

By definition, α^x is itself again a tensor norm on BAN such that $\alpha^x \leq \alpha'$ and

$$E\otimes_{\alpha^x} F \xrightarrow{1} (E'\otimes_\alpha F')' .$$

Especially for $\alpha = \beta'$ we get $\alpha^{x} = \overleftarrow{\beta}$ (by ([D–F1], 3.4.)). A further important property of α^{x} – which lead us to a sort of "weak accessibility" – is given by the following

2.3. LEMMA:
Let α be a tensor norm on BAN and E,F be Banach spaces. Then

$$E'\otimes_{\alpha^{x}} F \overset{1}{\hookrightarrow} \mathcal{D}_{\alpha}^{dd}(E,F) \; .$$

In other words:

$$\alpha^{x}(L;E',F) = \|L''\|_{(\alpha)} \quad \forall\, L \in \mathcal{F}(E,F) = E'\otimes F .$$

Proof:
By definition, $\beta^{x}(A;\cdot,\cdot) = \|A'\|_{(\beta^{t})}$ for all tensor norms β and for all $A \in \mathcal{F}$. In particular, this equality holds for α^{t} and L'. Hence

$$\alpha^{x}(L;E',F) = (\alpha^{t})^{x}(L';F'',E') = \|L''\|_{(\alpha)} \, .$$

□

2.4. PROPOSITION:
Let α be a finitely generated tensor norm on BAN and E,F be Banach spaces. If α is totally accessible or if *one* of those Banach spaces has m.a.p., then

$$\alpha(z;E,F) = \alpha^{xx}(z;E,F)$$

for all $z \in E\otimes F$.

Proof:
Obviously in general $\alpha^{xx} := (\alpha^{x})^{x} \leq \alpha$. If α is totally accessible, then $\alpha = (\alpha')^{x} \leq (\alpha^{x})^{x}$ and we received equality. To prove the other case, we may assume that F has m.a.p. By the approximation lemma ([D–F1], 2.2) we can select F to be finite dimensional. So let $\varepsilon > 0$ be given. Then, by 1.5., there exists a *finite* operator

$L \in \mathcal{D}_\alpha(E,F')$ with $\|L\|_{(\alpha)} \leq 1$ such that

$$\alpha(z;E,F) < (1+\varepsilon)|\mathrm{tr}(\Lambda_z L)| \leq (1+\varepsilon)\alpha^{xx}(z;E,F)\cdot\alpha^{x}(L;E',F').$$

By 2.3., $\alpha^{x}(L;E',F') = \|L\|_{(\alpha)} \leq 1$, and we have obtained the requested result.

□

2.5. COROLLARY:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a Banach ideal and $E,F \in$ BAN such that E' *or* F has the m.a.p.. Then

$$\mathcal{A}^{\min}(E,F) \overset{1}{\hookrightarrow} ((\mathcal{A}^{*})^{\Delta})^{dd}(E,F)\ .$$

In particular, $((\mathcal{A}^{*})^{\Delta})^{dd}$ is always accessible.

Proof:

We may assume that $\mathcal{A}$ is maximal. Then $\mathcal{A} \hat{=} \mathcal{D}_\alpha \sim \alpha'$ with $\alpha = \alpha_{\mathcal{A}}$ finitely generated (1.6.). By [D–F1], 7.1. and the previous considerations we obtain (by assumption):

$$\mathcal{A}^{\min}(E,F) \cong E'\tilde{\otimes}_{\alpha'}F = E'\tilde{\otimes}_{(\overleftarrow{\alpha})^{x}}F \overset{1}{\hookrightarrow} ((\mathcal{A}^{*})^{\Delta})^{dd}(E,F).$$

□

If E' has the m.a.p., even more holds:

2.6. PROPOSITION:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a Banach ideal and E,F be Banach spaces such that E' has the m.a.p.. Then:

$$(*) \qquad \mathcal{A}^{\min}(E,F) \overset{1}{\hookrightarrow} (\mathcal{A}^{*})^{\Delta}(E,F)\ .$$

In particular, $((\mathcal{A}^{*})^{\Delta},\|\cdot\|_{(\mathcal{A}^{*})^{\Delta}})$ is always right–accessible.

Proof:

We may assume that $\mathcal{A} \triangleq \mathcal{D}_\alpha \sim \alpha'$ with $\alpha = \alpha_{\mathcal{A}}$ finitely generated. First, let $E \in FIN$. Since E' has the m.a.p. and E is reflexive, we obtain (like the previous proof):

$$\mathcal{A}^{min}(E,F) \cong E'\tilde{\otimes}_{\alpha'}F = E'\tilde{\otimes}_{(\overleftarrow{\alpha})^x}F \overset{1}{\hookrightarrow} (\mathcal{A}^*)^\Delta(E,F).$$

Let now E' have m.a.p., $L \in \mathcal{F}(E,F) = E'\otimes F$ and $\varepsilon > 0$. By [P1], 10.2.6. there is an $A \in \mathcal{F}(E,E)$ such that $\|A\| \leq 1+\varepsilon$ and $LA = L$. Canonical factorization of A leads to $M \in FIN(E)$ and $A_0 \in \mathcal{L}(E,M)$ such that $A = J_M^E A_0$ and $\|A_0\| \leq 1+\varepsilon$.

By the preceding discussion we have for $L = Id_F(LJ_M^E)A_0 = (A_0'\otimes Id_F)LJ_M^E$:

$$\alpha'(L;E',F) \leq (1+\varepsilon)\alpha'(LJ_M^E;M',F) \leq (1+\varepsilon)\|L\|_{(\overleftarrow{\alpha})} \leq$$
$$\leq (1+\varepsilon)\|L''\|_{(\overleftarrow{\alpha})} = (1+\varepsilon)(\overleftarrow{\alpha})^x(L;E',F) \leq (1+\varepsilon)\alpha'(L;E',F).$$

Again we have obtained (*). Since $\mathcal{A}^{min}$ is always (right–) accessible, (*) implies the right–accessibility of $(\mathcal{A}^*)^\Delta$.

□

Using the totally accessible Banach ideal $\mathcal{A} := \mathcal{P}_2 \triangleq \mathcal{P}_2^*$, we recognize that in general (*) is not an (isometric) equality: Let $E_0 := C(K)$, $F_0 := L_2(\mu)$ where K is a compact T_2–space and μ a positive, regular Borel–measure and look at the canonical embedding $J_2 : E_0 \hookrightarrow F_0$. Then $J_2 \in \mathcal{P}_2$ (cf. [J], 19.6.3.). Since J_2 is not compact, $J_2 \notin \mathcal{P}_2^{min} \triangleq \mathcal{K}\circ\mathcal{P}_2$ ([P1], 19.1.2., 19.2.8., 24.6.3.). Hence $\mathcal{P}_2^{min}(E_0,F_0) \not\triangleq (\mathcal{P}_2^*)^\Delta(E_0,F_0) \triangleq \mathcal{P}_2(E_0,F_0)$. 2.6. leads immediately to the following question which remains still open:

2.7. PROBLEM:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal. Is $(\mathcal{A}^\Delta,\|\cdot\|_{\mathcal{A}^\Delta})$ always left–accessible ?

Therefore we are concentrating on the search for properties which are equivalent to the left–accessibility of such Banach ideals $\mathcal{A}^\Delta$, as described in 2.7., and we will recognize a surprisingly connection to a natural generalization of the weak principle of local reflexivity, which is given by the following

2.8. DEFINITION:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a p–Banach operator ideal $(0 < p \leq 1)$. We are talking about the *weak $\mathcal{A}$–local reflexivity principle* (short: (w) $\mathcal{A}$–l.r.p.), if the following property is given:
Let M ∈ FIN, F ∈ BAN and N ∈ FIN(F'). For all $\varepsilon > 0$ and $T \in \mathcal{L}(M,F'')$ there is an $S \in \mathcal{L}(M,F)$ such that

$$\|S\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$$

and

$$<b,Tx> = <Sx,b>$$

for all $(x,b) \in M \times N$.

We are now prepared to prove our main

2.9. THEOREM:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a *maximal* Banach ideal. Then the following statements are equivalent:

(1) $(\mathcal{A}^{\Delta},\|\cdot\|_{\mathcal{A}^{\Delta}})$ is left accessible

(2) The (w) $\mathcal{A}$–l.r.p. holds

(3) $\mathcal{A}(M,F'') \cong \mathcal{A}(M,F)''$ for all M ∈ FIN, F ∈ BAN

(4) $(\mathcal{A}^*)^{\min}(E,F) \overset{1}{\hookrightarrow} \mathcal{A}^{\Delta}(E,F)$ for all F ∈ BAN *with* m.a.p., E ∈ BAN.

Proof:

By 1.6. $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ can be represented as $\mathcal{A} \doteq \mathcal{B}^*$ with $\mathcal{B} := \mathcal{A}^* \doteq \mathcal{D}_{\beta}$, β finitely generated on BAN.

(1) ⟹ (3): Let $((\mathcal{B}^*)^{\Delta},\|\cdot\|_{(\mathcal{B}^*)^{\Delta}})$ be left–accessible and M ∈ FIN, F ∈ BAN.
Since M is reflexive, 2.3. implies that

$$\mathcal{D}_{\beta^*}(M,F) \cong M'\otimes_{(\beta)^t}F \cong F\otimes_{\beta}M'.$$

Hence, by Lemma 1.5.:

$$\mathcal{A}(M,F)' = \mathcal{D}_{\beta^*}(M,F)' \cong \mathcal{D}_{\beta}(F,M) \triangleq (\mathcal{B}^*)^{\Delta}(F,M) \triangleq \mathcal{A}^{\Delta}(F,M).$$

(This isometric isomorphism holds *in general* – without assumption (1).)

Using [D–F2], 25.2., [D–F1], 7.1. and the assumption (1), we conclude that

$$(\mathcal{B}^*)^{\Delta}(F,M) \triangleq \mathcal{B}^{min}(F,M) \cong F'\tilde{\otimes}_{\beta'}M \cong M\tilde{\otimes}_{\beta^*}F' .$$

In consideration of 1.5., dualization now yields:

$$\mathcal{D}_{\beta^*}(M,F)'' \cong (M\otimes_{\beta^*}F')' \cong \mathcal{D}_{\beta^*}(M,F''),$$

and we have obtained statement (3).

(3) $\Longrightarrow$ (2): Let $M \in FIN$, $F \in BAN$ and $N \in FIN(F')$. By assumption, the following map is an isometric isomorphism:

$$(*) \qquad \begin{array}{ccc} \mathcal{A}(M,F)'' & \xrightarrow{\cong} & \mathcal{A}(M,F'') \\ \xi & \longmapsto & T_\xi \end{array}$$

Thereby $T_\xi \in \mathcal{L}(M,F'')$ is defined by $<b,T_\xi x> := <tr((b\otimes x)\cdot),\xi>$ $(x \in M,\ b \in F')$ and $tr((b\otimes x)\cdot) \in \mathcal{A}(M,F)'$ is given by $<S,tr((b\otimes x)\cdot)> := tr((b\otimes x)S) = <Sx,b>$ $(S \in \mathcal{A}(M,F))$.

Let $\{b_1,\dots,b_m\}$ be a basis of N, $\{x_1,\dots,x_n\}$ be a basis of M and $L_{ij} := b_i \otimes x_j$ $(1 \le i \le m,\ 1 \le j \le n)$. Since

$$\begin{array}{ccc} \mathcal{A}^{\Delta}(F,M) & \xrightarrow{\cong} & \mathcal{A}(M,F)' \\ R & \longmapsto & tr(R\cdot) \end{array}$$

the linear span of $\{tr(L_{ij}\cdot) : 1 \le i \le m,\ 1 \le j \le n\}$ is a finite dimensional subspace of $\mathcal{A}(M,F)'$ on which Helly's lemma can be applied ([P1], 28.1.1.).

Now, let $T \in \mathcal{A}(M,F'')$ and $\varepsilon > 0$ be given. By (*) there is a $\xi_o \in \mathcal{A}(M,F)''$ with $\|\xi_o\| = \|T\|_{\mathcal{A}}$ and $<b_i,Tx_j> = <tr(L_{ij}\cdot),\xi_o>$ for all i and j. Hence, by Helly, there exists an $S_o \in \mathcal{A}(M,F)$ such that

$$\|S_o\|_{\mathcal{A}} \leq (1+\varepsilon)\|T\|_{\mathcal{A}}$$

and

$$<b_i,Tx_j> = <S_o x_j, b_i> \quad (1 \leq i \leq m, 1 \leq j \leq n).$$

Since $\{b_1,...,b_m\}$ and $\{x_1,...,x_n\}$ are chosen as bases of N and M, we have reached statement (2) by linearity.

(2) $\Longrightarrow$ (4): Let $M \in$ FIN, $E \in$ BAN and $\varepsilon > 0$ be given. By a similar argument as in the proof of 2.6., we only have to show that

$$E'\tilde{\otimes}_{\beta'}M \cong \mathcal{B}^{\min}(E,M) \xhookrightarrow{1} (\mathcal{B}^*)^{\Delta}(E,M) \; .$$

Let $L \in \mathcal{F}(E,M) = E'\otimes M$, $L = \sum_{i=1}^{n} a_i \otimes z_i$, $a_1,...,a_n \in E'$ and $z_1,...,z_n \in M$. By 2.3. and 2.4., there is an $A \in \mathcal{F}(M'',E'')$ such that $\|A\|_{\mathcal{B}*} = (\overleftarrow{\beta})^t(A;M''',E'') \leq 1$ and

$$\beta'(L;E',M) = (\overleftarrow{\beta})^x(L;E',M) = \|L''\|_{(\overleftarrow{\beta})} < (1+\varepsilon)|tr(L''A)| \; .$$

Let $N := [a_1,...,a_n] \in FIN(E')$. By assumption (2) there is an $B \in \mathcal{B}^*(M'',E)$ such that $\|B\|_{\mathcal{B}*} \leq (1+\varepsilon)\|A\|_{\mathcal{B}*} \leq 1+\varepsilon$ and

$$<a,(Aj_M)z> = <(Bj_M)z,a> \quad \forall\, a \in N,\, z \in M.$$

Hence

$$|tr(L''A)| = |tr((j_M L)B)| \leq \|L\|_{(\mathcal{B}^*)^{\Delta}} \cdot (1+\varepsilon) \leq (1+\varepsilon)\beta'(L;E',M)$$

and the implication follows.
Since (4) obviously implies (1), we obtain all desired equivalences, and the theorem is proven.

□

Given a maximal, right–accessible Banach ideal $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$, then $(\mathcal{A}^*,\|\cdot\|_{\mathcal{A}^*})$ is always left–accessible ([D–F1],9.2.), in particular $(\mathcal{A}^{\Delta},\|\cdot\|_{\mathcal{A}^{\Delta}})$. Hence

2.10. COROLLARY:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a maximal, right–accessible Banach ideal. Then the (w) $\mathcal{A}$–l.r.p. holds.

Especially, this statement *is true* for the (right–)accessible, maximal Banach ideal $(\mathcal{L}_{p,q},\|\cdot\|_{\mathcal{L}_{p,q}}) \sim \alpha_{p,q}$ $(1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} \geq 1)$ of (p,q)–factorable operators ([D–F1], 4.6., 9.4.).

Another important consequence is a generalization of a result of Pietsch (cf. [P1], E. 3.2.):

2.11. LEMMA:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a p–Banach ideal $(0 < p \leq 1)$ such that the (w) $\mathcal{A}$–l.r.p. holds. Let M $\in$ FIN, E,F $\in$ BAN, A $\in \mathcal{L}(E',M')$ and T $\in \mathcal{F}(F',E')$. Then, given $\varepsilon > 0$, there exists an X $\in \mathcal{L}(M,E)$ such that $\|X\|_{\mathcal{A}} \leq (1+\varepsilon)\|A'\|_{\mathcal{A}}$ and AT = X'T.

Proof:

That is the proof of [P1], E. 3.2., only using the ideal norm $\|\cdot\|_{\mathcal{A}}$ instead of $\|\cdot\|$.

□

This lemma leads us to a statement which is especially true for all maximal Banach ideals $\mathcal{A}$ with the (w) $\mathcal{A}^d$–l.r.p.:

2.12. LEMMA:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a p–Banach ideal $(0 < p \leq 1)$ such that $(\mathcal{A},\|\cdot\|_{\mathcal{A}}) \subseteq (\mathcal{A}^{dd},\|\cdot\|_{\mathcal{A}^{dd}})$ and the (w) $\mathcal{A}^d$–l.r.p. holds. Let E,F be Banach spaces such that F' has the m.a.p.. Then we obtain isometrically:

$$(\mathcal{A}^d)^\Delta(E,F) \stackrel{\wedge}{=} (\mathcal{A}^\Delta)^d(E,F).$$

This equality is *also valid* if we only assume that $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ is an arbitrarily given p–Banach ideal $(0 < p \leq 1)$ and that E' has m.a.p. .

Proof:

Let $\varepsilon > 0$, $T \in (\mathcal{A}^d)^\Delta(E,F)$ and $L \in \mathcal{F}(E',F')$. First, let F' have the m.a.p.. Then, there is an $A \in \mathcal{F}(F',F')$ such that $\|A\| \le 1+\varepsilon$ and $L = AL$. Canonical factorization of A leads to $M \in FIN$, $A_1 \in \mathcal{L}(F',M'')$, $A_2 \in \mathcal{L}(M'',F')$ such that $A = A_2A_1$, $\|A_2\| \le 1$ and $\|A_1\| \le 1+\varepsilon$. By assumption on $\mathcal{A}$ and the previous lemma, there is an $X \in \mathcal{F}(M',E)$ with $\|X\|_{\mathcal{A}^d} \le (1+\varepsilon)\|A_1L\|_{\mathcal{A}}$ and $(A_1L)(T'A_2) = X'T'A_2$. Since $A_2 \in \mathcal{F}(M'',F')$ there exists an $B_2 \in \mathcal{F}(F,M')$ such that $A_2 = B_2'$. Thus we obtain:

$$|tr(T'L)| = |tr(B_2TX)| \le (1+\varepsilon)^2\|B_2T\|_{(\mathcal{A}^d)^\Delta}\|L\|_{\mathcal{A}} .$$

Hence $T \in (\mathcal{A}^\Delta)^d(E,F)$ and $\|T\|_{(\mathcal{A}^\Delta)^d} \le \|T\|_{(\mathcal{A}^d)^\Delta}$.

The other inclusion follows directly by the definition of conjugated Banach ideals, since always

$$((\mathcal{A}^\Delta)^d,\|\cdot\|_{(\mathcal{A}^\Delta)^d}) \subseteq ((\mathcal{A}^d)^\Delta,\|\cdot\|_{(\mathcal{A}^d)^\Delta}) .$$

Now, let have E' m.a.p.. Then, there is an $B \in \mathcal{F}(E',E')$ such that $\|B\| \le 1+\varepsilon$ and $T'L = BT'L$. Again, canonical factorization (of B) leads to $M \in FIN(E')$, $C \in \mathcal{L}(E',M)$ such that $B = J_M^{E'}C$ and $\|C\| \le 1+\varepsilon$. Let $S \in \mathcal{L}(F,M')$ with $LJ_M^{E'}j_M^{-1} = S'$. Since obviously $(\mathcal{A}^d,\|\cdot\|_{\mathcal{A}^d}) \circ ((\mathcal{A}^d)^\Delta,\|\cdot\|_{(\mathcal{A}^d)^\Delta}) \subseteq (\mathcal{I},\|\cdot\|)$, we obtain the following estimation:

$$\begin{aligned}|tr(T'L)| &= |tr(T'S'j_MC)| \le \|ST\|_{\mathcal{I}}\cdot\|C\| \\ &\le \|S'\|_{\mathcal{A}}\cdot\|T\|_{(\mathcal{A}^d)^\Delta}\cdot(1+\varepsilon) \le \|L\|_{\mathcal{A}}\cdot\|T\|_{(\mathcal{A}^d)^\Delta}\cdot(1+\varepsilon) .\end{aligned}$$

Hence, $T \in (\mathcal{A}^\Delta)^d(E,F)$ and $\|T'\|_{\mathcal{A}^\Delta} \le \|T\|_{(\mathcal{A}^d)^\Delta}$.

□

An immediate consequence of this lemma is given by

2.13. COROLLARY:
Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a *maximal* Banach ideal and E,F be Banach spaces such that F'' has m.a.p.. Then the following statements are equivalent:

(1) $(\mathcal{A}^{\Delta},\|\cdot\|_{\mathcal{A}^{\Delta}})$ is left–accessible

(2) $\mathcal{A}^{\Delta}(E,F) \triangleq (\mathcal{A}^{\Delta})^{dd}(E,F)$.

Proof:
(1) ⟹ (2): Since (1) is equivalent to the (w) $(\mathcal{A}^{d})^{d}$–l.r.p., this implication follows directly by the previous lemma.

(2) ⟹ (1): $(\mathcal{A}^{\Delta})^{dd}$ is always (left–) accessible, by 2.5.

□

Finally we show some applications of the (w) $\mathcal{A}$–l.r.p. which are dealing with a duality–theory of maximal Banach ideals in the following sense:

2.14. PROPOSITION:
Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal, M ∈ FIN and F an arbitrary Banach space. Then the following statements are equivalent:

(1) The (w) $\mathcal{A}$–l.r.p. holds
(2) $(\mathcal{A}^{*})^{min}(F,M) \cong \mathcal{A}(M,F)'$.

Thereby the isometric isomorphism (2) is given by canonical trace–duality.

Proof:
(1) ⟹ (2): The assumption implies the left–accessibility of $\mathcal{A}^{\Delta}$, by 2.9. By [D–F2], 25.2. it follows that $(\mathcal{A}^{*})^{min}(F,M) \triangleq \mathcal{A}^{\Delta}(F,M)$. The proof of theorem 2.9. shows further that $\mathcal{A}^{\Delta}(F,M) \cong \mathcal{A}(M,F)'$, given by trace–duality. Hence we obtain (2).

(2) ⟹ (1): Let (2) be valid. Then, by [D–F1], 7.2. it follows that

$$\mathcal{A}(M,F'') \cong (\mathcal{A}^{*})^{min}(F,M)' \cong \mathcal{A}(M,F)'' .$$

Hence we obtain (1), by 2.9.

□

2.15. PROPOSITION:

Let $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$ be a maximal Banach ideal and E,F be Banach spaces such that E has m.a.p.. If the (w) $\mathcal{A}$–l.r.p. holds, then

$$(\mathcal{A}^*)^{\min}(F,E) \overset{1}{\hookrightarrow} \mathcal{A}(E,F)' ,$$

given by trace–duality.

Proof:

By 1.4. and 1.6. $\mathcal{A}^* \triangleq \mathcal{D}_{\alpha_{\mathcal{A}^*}} \sim \alpha'_{\mathcal{A}^*} = \alpha^t_{\mathcal{A}} = \alpha_{\mathcal{A}^d}$. Therefore, we have only to show (by [D–F1], 7.2.) that

$$(*) \qquad \begin{aligned} \Phi : F'\otimes_{\alpha_{\mathcal{A}^d}} E &\overset{1}{\hookrightarrow} \mathcal{A}(E,F)' \\ L &\longmapsto (T \longmapsto tr(TL)) \end{aligned}$$

is a well–defined isometric isomorphism.

Since $|tr(TL))| \le \|T\|_{\mathcal{A}}\cdot \alpha_{\mathcal{A}^d}(L;F',E)$ for all $T \in \mathcal{A}(E,F)$ and $L \in \mathcal{F}(F,E)$, (*) is well–defined and $\|\Phi(L)\| \le \alpha_{\mathcal{A}^d}(L;F',E)$. To show the converse inequality, let $\varepsilon > 0$ and $L \in \mathcal{F}(F,E)$ be given. By definition of $\alpha_{\mathcal{A}^d}$, there is an $S \in \mathcal{A}^d(F',E')$, $\|S\|_{\mathcal{A}^d} \le 1$ such that

$$\alpha_{\mathcal{A}^d}(L;F',E) < (1+\varepsilon)|tr(L'S)| .$$

Since E has m.a.p., there is an $A \in \mathcal{F}(E,E)$ such that $\|A\| \le 1+\varepsilon$ and $L = AL$. Let $M \in FIN(E)$ and $A_o \in \mathcal{L}(E,M)$ such that $\|A_o\| \le 1+\varepsilon$ and $A = J^E_M A_o$. Then $tr(L'S) = tr((J^E_M)'S(A_oL)')$. Since $(J^E_M)'S \in \mathcal{F}(F',M')$ and $(A_oL)' \in \mathcal{F}(M',F')$, the (w) $\mathcal{A}$–l.r.p. implies, that there is an $X \in \mathcal{L}(M,F)$ such that $\|X\|_{\mathcal{A}} \le (1+\varepsilon)\|(J^E_M)'S\|_{\mathcal{A}^d} \le 1+\varepsilon$ and $(J^E_M)'S(A_oL)' = X'(A_oL)'$ (by 2.11.). Therefore we obtain:

$$\begin{aligned} \alpha_{\mathcal{A}^d}(L;F',E) < (1+\varepsilon)|tr(XA_oL)| &= (1+\varepsilon)|\Phi(L)(XA_o)| \\ &\le (1+\varepsilon)\|\Phi(L)\|\cdot\|XA_o\|_{\mathcal{A}} \\ &\le (1+\varepsilon)^3\|\Phi(L)\| . \end{aligned}$$

Hence we have received (*) and the proposition has been proven.

□

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