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PROJECTIVE RESOLUTIONS ASSOCIATED TO PROJECTIONS

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by T. de Jong and D. van Straten

Introduction.

Let X be a d-dimensional germ of an analytic space and let $\varphi : X \longrightarrow \mathbb{C}^{d+1}$ be a finite map. Via φ, we can consider $\mathcal{O}_X$ as an $\mathcal{O} := \mathcal{O}_{\mathbb{C}^{d+1}}$ - module. If X is *Cohen - Macaulay*, then $\mathcal{O}_X$ has a free resolution as $\mathcal{O}$ -module of the form:

$$0 \longrightarrow G \xrightarrow{\Phi} F \longrightarrow \mathcal{O}_X \longrightarrow 0 \qquad (1)$$

where $F = \oplus_{k=0}^{r} \mathcal{O}.f_k$ and $G = \oplus_{k=0}^{r} \mathcal{O}.g_k$ are free $\mathcal{O}$ -modules of rank r+1. The determinant f of the matrix (Φ_{ij}) of Φ can be used as a defining equation for the image Y of X in $\mathbb{C}^{d+1}$ (see [Te]).

Now $\mathcal{O}_X$ is not only a $\mathcal{O}$ -module, but even an $\mathcal{O}$ -*algebra*, due to the fact that $\mathcal{O}_X$ is a *ring*. Let f_k be mapped to u_k in $\mathcal{O}_X$ and assume $u_0 = 1$. Then one gets a surjection:

$$\mathcal{O}[f_1, f_2, \ldots, f_r] \longrightarrow \mathcal{O}_X \longrightarrow 0 \qquad (2)$$

of $\mathcal{O}$ -algebras, or equivalently, an embedding $X \hookrightarrow \mathbb{C}^{d+1} \times \mathbb{C}^r$.

The equations of X in this embedding come into two types:

$$\sum_{i=0}^{r} \Phi_{ij} f_i = 0 \qquad (3A)$$

$$f_i f_j - \sum_{k=0}^{r} M_{ijk} . f_k = 0 \qquad (3B)$$

(3 A) are the "module-equations" between the u_i that follow from (1)

(3B) are the "multiplication- equations" . They express the product $u_i . u_j$ in the module basis. The M_{ijk} are certain elements of $\mathcal{O}$ and could be called the structure constants (cf. [Ca], [M-P]).

Another way of looking at (3A) and (3B) is to say that the left hand side of these equations generate the kernel of the surjection of (2) as a $S := \mathcal{O}[f_1, f_2, \ldots, f_r]$ -module.

In the first part of this article we will extend this to a description of a projective resolution of $\mathcal{O}_X$ as an S -module. It turns out that this resolution has the form:

$$0 \longrightarrow \mathcal{L}_{r+1} \longrightarrow \mathcal{L}_r \longrightarrow \ldots \longrightarrow \mathcal{L}_1 \longrightarrow S \longrightarrow \mathcal{O}_X \longrightarrow 0 \qquad (4)$$

where $\mathcal{L}_k$ is a free S -module of rank $k\binom{r+2}{k+1}$. Note that these are the well-known ranks occuring in the minimal resolutions of varieties of minimal multiplicity([Wa],[E-R-S]). Our complex involves Φ, certain maps L and M describing the algebra structure of $\mathcal{O}_X$ on the complex (1) and a certain homotopy H expressing the associativity of the multiplication in $\mathcal{O}_X$. The construction closely follows the steps taken in [E-R-S], where a similar complex was constructed associated to a map $X \longrightarrow \mathbb{C}^d$, representing (in the case that X is Cohen-Macaulay) $\mathcal{O}_X$ as a free $\mathcal{O}_{\mathbb{C}^d}$ -module.
In the second part of the article we treat the special case that φ is generically 1-1 and the defining function f of Y is in I^2 , where I is the ideal in $\mathcal{O}$ of the conductor of the map $X \longrightarrow Y$. In this case one can express the maps L, M and H explicitely in terms of the matrix Φ_{ij} . As a consequence, we get that in this case the resolution (4) is *minimal*.

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§1. *A Projective Resolution.*

We consider a commutative ring R with 1, and E a (finitely generated) projective R -module. We put $S := \oplus_k S_k(E)$, where S_k is the k-th symmetric power of E. The "diagonal map Δ" is the map

$$\Delta : \bigwedge^k(E) \longrightarrow \bigwedge^{k-1}(E) \otimes E$$

defined on generators by

$$\Delta(e_1 \wedge \ldots \wedge e_k) = \Sigma_i (-1)^{i-1} e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_k \otimes e_i$$

Here and in the sequel the tensor products are over the ring R.
We define for any S -module M a map

$$d_M : \bigwedge^k(E) \otimes M \longrightarrow \bigwedge^{k-1}(E) \otimes M$$

by $d_M = (1 \otimes m)(\Delta \otimes 1)$, where $m : S \otimes M \longrightarrow M$ is the multiplication map. By abuse of notation, the map $M \otimes \bigwedge^k(E) \longrightarrow M \otimes \bigwedge^{k-1}(E)$ defined by $s\, d_M\, s$, where s is the swap, is also denoted by d_M. Note that $d_M\, d_M = 0$.

Proposition (1.1) (Scheja and Storch)

Let M be an S -module which is finitely generated as an R -module.
Put $K_k = S \otimes \bigwedge^k(E) \otimes M$ and $d = d_S \otimes 1 - 1 \otimes d_M : K_k \longrightarrow K_{k-1}$
Then $d^2 = 0$ and

$$\mathbf{K}(M) : \quad 0 \longrightarrow K_r \longrightarrow K_{r-1} \longrightarrow \ldots \longrightarrow K_1 \longrightarrow K_0 = S \otimes M \longrightarrow 0$$

is a resolution of M as S -module.

proof: For a proof see [E-R-S], theorem 1.1. (In this theorem it is assumed that M is projective, but this is not needed in the proof of the above statement.) ⊠

So in the case that M is a projective R -module, the above complex **K**(M) is a S -projective resolution of M. Special such S -modules M arize as R -algebras of the form R ⊕ E as considered in [E-R-S]. We will consider the case of R -algebras A given by an exact sequence of projective R -modules:

Diagram (1.2):

$$0 \longrightarrow G \xrightarrow{\Phi} R\oplus E \longrightarrow A \longrightarrow 0$$

where $rk(E) = r$ and $rk(G)=r+1$. We abbreviate $R\oplus E$ to F.

Because now A is (in general) no longer a projective R -module, the resolution (1.1) with M = A does not give us a projective resolution of A as an S -module. We will replace A by "$G \xrightarrow{\Phi} F$" in (1.1), but the differential needs special care. In order to define this differential we introduce some maps expressing the commutatitity and associativity of A. Consider the following commutative diagram:

Diagram (1.3):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^2(G) & \longrightarrow & F\otimes G & \longrightarrow & S_2(F) & \longrightarrow & S_2(A) & \longrightarrow 0 \\ & & \downarrow & & \downarrow m_2 & & \downarrow m_1 & & \downarrow m & \\ & & 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & A & \longrightarrow 0 \end{array}$$

The first row is a projective resolution of the second symmetric power $S_2(A)$ of A, m is the multiplication map of the algebra structure of A, which is lifted to maps m_1 and m_2 of complexes. Because $F = R \oplus E$, we have decompositions

$$S_2(F) = F \oplus S_2(E) \text{ and}$$
$$F\otimes G = G\oplus E\otimes G.$$

So we can decompose m_1 and m_2 as follows

$$m_1 = Id_F \oplus M \quad \text{where } M : S_2(E) \longrightarrow F \quad \text{and}$$
$$m_2 = Id_G \oplus L \quad \text{where } L : E\otimes G \longrightarrow G$$

By composition we get a map $E\otimes E \longrightarrow S_2(E) \longrightarrow F$ that we also denote by M.

In order to express the associativity of the multiplication of A, we consider the following commutative diagram:

Diagram (1.4):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^2(E)\otimes G & \xrightarrow{1\otimes\Phi} & \wedge^2(E)\otimes F & \longrightarrow & \wedge^2(E)\otimes A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [L,L]} & & \downarrow{\scriptstyle [M,M]} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & G & \xrightarrow{\Phi} & F & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Here $[M,M]$ is the map $M(1\otimes M)(\Delta\otimes 1)$, so

$$[M,M](e_1\wedge e_2\otimes f) = M(e_1\otimes M(e_2\otimes f)) - M(e_2\otimes M(e_1\otimes f))$$

The map $[L,L]$ is defined similarly.

The commutativity of the left hand square follows from the commutativity of diagram (1.3), whereas the commutativity of the right hand square expresses the associativity and commutativity of the algebra A. It follows that there exists a homotopy

$$H: \wedge^2(E)\otimes F \longrightarrow G$$

I.e., we have $\Phi H = [M,M]$ and $H(1\otimes\Phi) = [L,L]$.

Proposition / Definition (1.5):

Let $\mathcal{A}_k = S\otimes\wedge^k(E)\otimes F \oplus S\otimes\wedge^{k-1}(E)\otimes G$

and $\partial = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} : \mathcal{A}_k \longrightarrow \mathcal{A}_{k-1}$

with

$$d_1 = d_S\otimes 1 - (1\otimes M)(\Delta\otimes 1)$$
$$d_2 = 1\otimes\Phi$$
$$d_3 = (1\otimes H)(\Delta\otimes 1)(\Delta\otimes 1)$$
$$d_4 = -d_S\otimes 1 + (1\otimes L)(\Delta\otimes 1).$$

One has: $\partial\partial = 0$, i.e. $\mathbf{A} = (\mathcal{A}_\cdot, \partial)$ is a complex.

proof : This is a straightforward calculation, and is an expression of the various commutations of maps. We only indicate what is involved:

For $d_1^2 + d_2 d_3 = 0$, use $\Phi H = [M,M]$

For $d_3 d_2 + d_4^2 = 0$, use $H(1\otimes\Phi) = [L,L]$

For $d_1 d_2 + d_2 d_4 = 0$, use $M(1\otimes\Phi) = \Phi L$

For $d_3 d_1 + d_4 d_3 = 0$, one has to use the commutativity of the following diagram:

$$\begin{array}{ccccccc}
\bigwedge^3(E)\otimes F & \xrightarrow{\Delta\otimes 1} & \bigwedge^2(E)\otimes E\otimes F & \xrightarrow{s\otimes 1} & E\otimes\bigwedge^2(E)\otimes F & \xrightarrow{1\otimes H} & E\otimes G \\
\downarrow{\scriptstyle \Delta\otimes 1} & & & & & & \downarrow{\scriptstyle L} \\
\bigwedge^2(E)\otimes E\otimes F & & \xrightarrow{\quad 1\otimes m_1 \quad} & \bigwedge^2(E)\otimes F & & \xrightarrow{\quad H \quad} & G
\end{array}$$

This commutativity can be checked by composing with the injective map Φ. Then it comes down to the relations $\Phi H = [M,M]$ and $\Phi L = M(1\otimes\Phi)$, together with the equality of maps $\bigwedge^3(E)\otimes F \longrightarrow F$:

$$[M,M](1\otimes m_1)(\Delta\otimes 1) = m_1([M,M]\otimes 1)(s\otimes 1)(\Delta\otimes 1),$$

which is checked by direct computation. ⊠

Lemma (1.6) :

Let $B = \oplus B_k$ be a $\mathbb{Z}$-graded abelian group with map δ of degree -1, not necessarily with $\delta\delta = 0$.

Consider the "mapping cone" $\mathbf{C} = (C_\cdot, d)$ where

$C_k = B_k \oplus B_{k-1}$ and $d = \begin{pmatrix} \delta & Id \\ -\delta\delta & -\delta \end{pmatrix}$.

Then $d^2 = 0$, and $\mathbf{C}$ is an exact complex.

proof : A simple calculation shows that indeed $d^2 = 0$, and a homotopy between the zero map and the identity map of $\mathbf{C}$ is given by $\begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix} : C_k \longrightarrow C_{k+1}$ ⊠

Proposition (1.7) :

The complex

$$\mathbf{A} : \quad 0 \longrightarrow \mathcal{A}_{r+1} \xrightarrow{\partial} \mathcal{A}_r \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0 = S\otimes F \longrightarrow 0$$

is a S - projective resolution of A.

proof : We apply lemma (1.6) with $B_k = S\otimes\bigwedge^k(E)\otimes G$ and $\delta = d_S\otimes 1 - (1\otimes L)(\Delta\otimes 1)$ and get an exact mapping cone complex $\mathbf{C}$.

There is an injective map of complexes $\mathbf{C} \hookrightarrow \mathbf{A}$, given in degree k by $(1\otimes\Phi)\oplus Id$:

$$S\otimes\bigwedge^k(E)\otimes G \oplus S\otimes\bigwedge^{k-1}(E)\otimes G \longrightarrow S\otimes\bigwedge^k(E)\otimes F \oplus S\otimes\bigwedge^{k-1}(E)\otimes G$$

The cokernel of this map can be identified with the complex $\mathbf{K}(A)$ of (1.1). Hence we have a short exact sequence of complexes

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathbf{A} \longrightarrow \mathbf{K}(A) \longrightarrow 0$$

Because $\mathbf{C}$ is exact by (1.6) and $\mathbf{K}(A)$ is a resolution of A by (1.1) it follows from the long exact homology sequence that $\mathbf{A}$ is a S -projective resolution of A. ⊠

§ 2. *A Smaller Resolution.*

Although the complex $\mathbf{A}$ has the "right" length, it is usually not minimal. In [E-R-S] it is described how to obtain from $\mathbf{K}(A)$ a smaller complex. We will apply their ideas to prune our complex $\mathbf{A}$ in a similar way.

Definition / Notation (2.1) : (see also [B-E], [E-R-S])

Let $\pi : F = R \oplus E \longrightarrow E$ the cartesian projection and define maps

$$\text{in} := (\wedge^k \pi \otimes 1)\Delta : \wedge^{k+1}(F) \longrightarrow \wedge^k(E) \otimes F$$

as the composition of the diagonal map and the induced projection.

The commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^k(E) & \longrightarrow & \wedge^{k+1}(F) & \longrightarrow & \wedge^{k+1}(E) & \longrightarrow & 0 \\ & & = \downarrow & & \text{in} \downarrow & & \Delta \downarrow & & \\ 0 & \longrightarrow & \wedge^k(E) & \longrightarrow & \wedge^k(E) \otimes F & \longrightarrow & \wedge^k(E) \otimes E & \longrightarrow & 0 \end{array}$$

shows that $\text{Coker}(\text{in}) \approx \text{Coker}(\Delta)$. We denote this common cokernel by $L^k := L^k_2(E) := \text{Coker}(\Delta : \wedge^{k+1}(E) \longrightarrow \wedge^k(E) \otimes E)$.

L^k is a projective R -module of rank $k\binom{r+1}{k+1}$.

Consider the inclusion $F = R \oplus E \hookrightarrow S$ and the induced map $S \otimes F \longrightarrow S$

The Koszul complex $\mathbf{P} := (P., \delta)$ on this map with terms $P_k = S \otimes \wedge^k(F)$ and the usual differential, is *exact*.

Proposition (2.2) : The diagram

$$\begin{array}{ccc} P_{k+1} & \xrightarrow{\delta} & P_k \\ j \downarrow & & j \downarrow \\ \mathcal{A}_k & \xrightarrow{\partial} & \mathcal{A}_{k-1} \end{array}$$

where

$$j := (1\otimes in)\oplus 0 : S\otimes\wedge^{k+1}(F) \longrightarrow S\otimes\wedge^{k}(E)\otimes F \oplus S\otimes\wedge^{k-1}(E)\otimes G$$

is anti commutative.

Hence there is an induced differential $\partial : \mathcal{L}_k \longrightarrow \mathcal{L}_{k-1}$ $(k\geq 2)$ where $\mathcal{L}_k$ = Coker(j)= $S\otimes L^k \oplus S\otimes\wedge^{k-1}(E)\otimes G$.

proof: The anti -commutativity of the diagram

$$\begin{array}{ccc} S\otimes\wedge^{k+1}(F) & \xrightarrow{\delta} & S\otimes\wedge^{k}(F) \\ 1\otimes in \downarrow & & 1\otimes in \downarrow \\ S\otimes\wedge^{k}(E)\otimes F & \xrightarrow{d_1} & S\otimes\wedge^{k-1}(E)\otimes F \end{array}$$

can be proved as in [E--R--S], lemma(3.1).

To prove the statement of the propostion one has to show that the compostion

$$S\otimes\wedge^{k+1}(F) \xrightarrow{1\otimes in} S\otimes\wedge^{k}(E)\otimes F \xrightarrow{d_3} S\otimes\wedge^{k-2}(E)\otimes G$$

is the zero map.

A direct computation shows that the composition of this map with the injective map $1\otimes\Phi$ is equal to:

$$(1\otimes\Phi)d_3(1\otimes in)(s\otimes e_1\wedge e_2\wedge\ldots\wedge e_{k+1}) = \sum_{i<j<k}(-1)^{i+j+k} s\otimes(e_i\wedge e_j\wedge e_k)\otimes\gamma$$

where $\gamma = -M(e_i\otimes M(e_j\otimes e_k)) + M(e_i\otimes M(e_k\otimes e_j))$ + cyclic, which is zero due to the symmetry of M. ⊠

Theorem (2.3):

The complex $\mathbf{L} = (\mathcal{L}_\cdot, \partial)$

$$\mathbf{L} : 0 \longrightarrow \mathcal{L}_{r+1} \xrightarrow{\partial} \mathcal{L}_r \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{L}_2 \xrightarrow{\partial} \mathcal{L}_1 \xrightarrow{\partial} \mathcal{L}_0 = S \longrightarrow 0$$

with $\partial : \mathcal{L}_1 = S\otimes S_2(E)\oplus S\otimes G \longrightarrow \mathcal{L}_0 = S$ given by

$$\partial(s\otimes e_1\otimes e_2 \oplus t\otimes g) = s(e_1e_2 - M(e_1\otimes e_2)) + t\Phi(g)$$

and $\partial : \mathcal{L}_k \longrightarrow \mathcal{L}_{k-1}$ $(k\geq 2)$ as in (2.2), is an S -projective resolution of A. Furthermore, if the ring R is local with maximal ideal m, then the resolution is minimal (after localization at $(m,E)\subset S$) if $\Phi(G)\subset mF$ $L(E\otimes G)\subset mG$, $M(E\otimes E)\subset mF$ and $H(\wedge^2(E)\otimes E)\subset mG$.

proof: As in [E-R-S], theorem 3.2. ⊠

Remark that the rank of $\mathcal{L}_k$ is equal to $k\binom{r+1}{k+1} + (r+1)\binom{r}{k-1} = k\binom{r+2}{k+1}$

§ 3. *Generic Projections.*

In [J-S] the following situation was studied:

$$\begin{array}{ccc} & & X \\ & & \downarrow \rho \\ \Sigma & \xrightarrow{i} & Y \subset Z \end{array}$$

Here Y=Spec(B)is a hypersurface in a smooth ambient space Z=Spec(R). If $\rho : X \longrightarrow Y$ is a generically 1-1 map from a Cohen-Macaulay space X=Spec(A) to Y, then the conductor $I = Hom_B(A,B)$ defines a subspace Σ = Spec(C); C=B/I of Y. From the inclusion $i:\Sigma \longrightarrow Y$ one can reconstruct A as a B-module via $A = Hom_B(I,B)$. The *ring structure* on A is translated into the fact that the ideal I satisfies the *ring condition* (R.C.)

Ring Condition (3.1):

$$Hom_B(I,I) \xrightarrow{\approx} Hom_B(I,B)$$

Conversely, any ideal $I \subset B$ that satifies this ring condition gives rize to an algebra structure on the module $Hom_B(I, B)$, which has as an R- module a projective resolution as in (1.2). The ring condition can also be interpreted saying that the hypersurface Y has to be *singular* along Σ. For example, if the local equation f of Y is in the I_R^2 , (where I_R is the ideal of Σ in R), then $\Sigma \hookrightarrow Y$ satisfies (R.C.). This particular case will be studied in some more detail in § 4.

Below we will describe how, in this situation of a "generic projection" $X \longrightarrow Y \subset Z$, the algebra structure on A is determined by the map Φ.
So we start with diagram (1.2). The ideal of the image is constructed as follows: the map Φ induces a map $\wedge^{r+1}(\Phi): \wedge^{r+1}(G) \longrightarrow \wedge^{r+1}(F)$ and by transposition an injective map $i:\mathcal{L} \hookrightarrow R$, where $\mathcal{L} := \wedge^{r+1}(G) \otimes \wedge^{r+1}(F^*)$ is an invertible module. We define $B := R/\mathcal{L}$.
Now A is a B - module. This is Cramers rule, and an intrinsic way of saying this is by looking at the map $\wedge^r(\Phi): \wedge^r(G) \longrightarrow \wedge^r(F)$, which by transposition and the natural isomorphism $\wedge^r(F^*) \approx \wedge^{r+1}(F^*) \otimes F$ gives rize to a map $\Psi : F \otimes \mathcal{L} \longrightarrow G$. One has $\Phi \Psi = (Id \otimes i)$

so Ψ is a homotopy expressing the fact that multiplication with elements of $\mathcal{L}$ is zero on A. From now on we make the following assumption:

Assumption (3.2):

The canonical map

$$\mathrm{Hom}_B(A,B) \xrightarrow{\text{can}} B\,;\ (a:A \longrightarrow B) \longmapsto a(1)$$

is injective.

So $\mathrm{Hom}_B(A,B)$ is via can an ideal I in B (and in A) and is called the conductor of the ring map $B \longrightarrow A$. The map *can* sits in a diagram with exact rows and columns.

Diagram(3.3):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & E^*\otimes\mathcal{L} & \longrightarrow & E^*\otimes\mathcal{L} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi\otimes 1} & & \downarrow & & \\
0 & \longrightarrow & F^*\otimes\mathcal{L} & \xrightarrow{\Phi^*\otimes 1} & G^*\otimes\mathcal{L} & \longrightarrow & \mathrm{Hom}_B(A,B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p\otimes 1} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \text{can}} & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & R & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & C & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The second row is a presentation of $\mathrm{Hom}_B(A,B)$ and can be obtained from (1.2) essentially by dualization. The third row is the definition of B. The map $p:F^* \longrightarrow R$ is induced by the inclusion $R \hookrightarrow F$, and the map $\Delta : G^* \otimes \mathcal{L} \longrightarrow R$ is induced by the composition:

$$R \hookrightarrow F \xrightarrow{\Psi} G\otimes\mathcal{L}^*$$

by transposition. The columns of the diagram are obtained by the snake lemma.

The decomposition $F=R\oplus E$ decomposes the map $\Phi:G \longrightarrow F$ into two maps:

$\alpha: G \longrightarrow R$

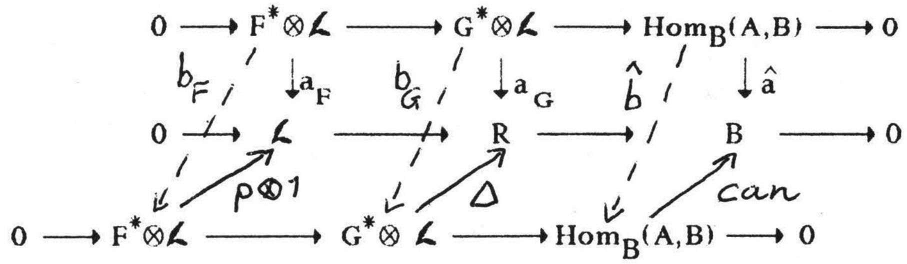
$\varphi: G \longrightarrow E$

The diagonal map $\alpha^*: \mathcal{L} \longrightarrow G^* \otimes \mathcal{L}$ is induced by α by transposition (and tensoring with $\mathcal{L}$).

The module A can also be obtained back as $A \xrightarrow{\approx} \mathrm{Hom}_B(\mathrm{Hom}_B(A,B),B)$ and under this isomorphism the element 1 corresponds to the map can.

The ring condition (R.C.) $\mathrm{Hom}_B(I,I) \xrightarrow{\approx} \mathrm{Hom}_B\ (I,B)$ now means that every element $a \in A$, corresponding to $(\hat{a} : \mathrm{Hom}_B(A,B) \longrightarrow B\ ;\ \varphi \longmapsto \varphi(a))$ $\in \mathrm{Hom}_B(\mathrm{Hom}_B(A,B),B)$ and represented by (a_F, a_G) can be lifted to (b_F, b_G), representing $\hat{b} \in \mathrm{Hom}_B(\mathrm{Hom}_B(A,B),\mathrm{Hom}_B(A,B))$, making the following diagram commutative:

Diagram (3.5):



By transposition b_F and b_G induce maps $M(a):F \longrightarrow F$ and $L(a):G \longrightarrow G$, representing the multiplication by a on A. The maps a_F and a_G are determined by a as follows:

Proposition (3.6):

1) The transposition $a_F^* \in F$ is a lift of $a \in A$.

2) The transposition $a_G^* \in G \otimes \mathcal{L}^*$ is equal to $\tilde{\Psi}(a_F^*)$, where $\tilde{\Psi}: F \longrightarrow G \otimes \mathcal{L}^*$ is the map induced by Ψ.

The proof is left to the reader.

So in short, the maps L(a) and M(a) describing multiplication by $a \in A$ are determined by the following steps:

1) Lift a to $a_F^* \in F$ and get $a_F: F^* \otimes \mathcal{L} \longrightarrow \mathcal{L}$

2) Compute a_G^* as $\tilde{\Psi}(a_F^*) \in G \otimes \mathcal{L}^*$ and get $a_G: G^* \otimes \mathcal{L} \longrightarrow R$

3) Lift the map a_G over the map $\Delta: G^* \otimes \mathcal{L} \longrightarrow R$ to get a map $b_G: G^* \otimes \mathcal{L} \longrightarrow G^* \otimes \mathcal{L}$ and by transpostition $L(a): G \longrightarrow G$

This is the essential step, and the condition to be able to do this is of course again (R.C.)

4) Lift the compostion $b_G(\Phi^* \otimes 1)$ over $(\Phi^* \otimes 1)$ to get $b_F: F^* \otimes \mathcal{L} \longrightarrow F^* \otimes \mathcal{L}$ and by transposition $M(a): F \longrightarrow F$. As the map *can* is injective, this is possible for any choice of b_G in step 3).

§4. *A Particular Case.*

A particular case in which the ring condition (3.1) is satisfied arizes as follows. Suppose we are given the R-resolution of Σ of the form:

$$0 \longrightarrow E^* \xrightarrow{\varphi^*} G^* \xrightarrow{\Delta} R \longrightarrow C \longrightarrow 0$$

where we assume (for reasons of simplicity) that E and G are *free* R-modules. We choose bases $\{f_k\}$ and $\{g_k\}$ $(k=0,\ldots,r)$ for $F = R \oplus E$ resp. G and assume that $f_0 = 1 \in R$. The map $\varphi: G \longrightarrow E$ has as matrix φ_{ij}, i.e. $\varphi(g_j) = \varphi_{ij} f_j$. Here and in the sequel we use the Einstein summation convention: indices occuring twice are summed over.
The module $\mathcal{L}$ is trivial and the component $\Delta_i := \Delta(g_i^*)$ can be obtained as the i-th minor of φ_{ij}. Let I_R be the R ideal generated by the Δ_i.
The particular case we want to discuss in some more detail is the following:
Suppose we are given an element

$$f \in I_R^2$$

We will assume that f is a non zero-divisor in R.
Such an f can always be written as:

$$f = h_{ij} \Delta_i \Delta_j$$

where h_{ij} is a symmetric matrix of elements of R.

(If one does not want to assume E and G free, than the matrix (h_{ij}) should be considered as an element h of $S_2(G^*)\otimes\mathcal{L}$.). We now take

$$\alpha_i = h_{ij}\,\Delta_j$$

and let Φ: F $\longrightarrow$G be the map defined by the following matrix:

$$\Phi = \left(\Phi_{ij}\right) = \begin{pmatrix} \alpha_0 & \cdots & \alpha_r \\ \varphi_{10} & \cdots & \varphi_{1r} \\ \vdots & & \vdots \\ \varphi_{r0} & \cdots & \varphi_{rr} \end{pmatrix}$$

So f = det(Φ), which is a generator for the ideal of a space Y. We will determine the maps L, M and H of § 1 expressing the ring structure of A = Cok(φ). To do this we need some elementary relations between minors of matrices.

Definition (4.1):

Let $\Phi = (\Phi_{ij})$ be a square matrix of size r +1. Let $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_p)$ be strictly increasing sequences of numbers smaller than r. Then we define $\Psi_{I,J} = (-1)^k \det(\Phi^{I,J})$ where $k = i_1 + \ldots + i_p + j_1 + \ldots + j_p$. and $\Phi^{I,J}$ is obtained from the matrix Φ by deleting columns $i_1, \ldots, i_p$ and rows $j_1, \ldots, j_p$. The $\Psi_{I,J}$ for non-strictly increasing sequences of numbers are defined by making $\Psi_{I,J}$ anti-symmetric in both I and J.

Lemma (4.2):

One has the following identities:

1L) $\Phi_{ij}\Psi_{jk} = \det(\Phi)\delta_{ik}$

2L) $\Phi_{ij}\Psi_{jk\,mn} = \Psi_{kn}\delta_{im} - \Psi_{km}\delta_{in}$

3L) $\Phi_{ij}\Psi_{jkm\,npq} = \Psi_{km\,pq}\delta_{in} + \Psi_{km\,qn}\delta_{ip} + \Psi_{kmnp}\delta_{iq}$

1R) $\Psi_{kj}\Phi_{ji} = \det(\Phi)\delta_{ki}$

2R) $\Psi_{nm\,kj}\Phi_{ji} = \Psi_{nk}\delta_{mi} - \Psi_{mk}\delta_{ni}$

3R) $\Psi_{qpnmkj}\Phi_{ji} = \Psi_{qpmk}\delta_{ni} + \Psi_{nq\,mk}\delta_{pi} + \Psi_{pnmk}\delta_{qi}$

proof: 1L) is Cramers rule. 2L) is obtained by expanding the determinant obtained from Φ by deleting column k and rows m and n and concating with the i-th row of Φ with respect to its j column. 3L) is obtained similarly. The "R"-identies are obtained by "reflection" ⊠

Because of the special shape of our matrix Φ we find it useful to use the following notation.

Notation (4.3):

We put $\Delta_i = \Psi_{i0}$

$$\Delta_{ijk} = \Psi_{ij0k}$$

$$\Delta_{ijkmn} = \Psi_{ijk0mn}$$

We remark that the Δ_i are in fact the components of the map $\Delta : G^* \longrightarrow R$. The ideal generated by these Δ_i is I, and the ring condition (R.C.) is exactly that $\Psi_{ij} \in I$ for all i and j.

The identities we will use all follow from (4.2) by putting some index equal to zero and are summarized in:

Identities (4.4):

1) $\alpha_k \Delta_{kij} = \Psi_{ij} \qquad (j \geq 1)$

2) $\varphi_{ij}\Delta_{jkm} = - \Delta_k \delta_{im}$

3) $\Delta_{ijk}\varphi_{km} = \Delta_i \delta_{jm} - \Delta_j \delta_{im}$

4) $\Delta_{qpnkj}\varphi_{ji} = \Delta_{qpk}\delta_{ni} + \Delta_{nqk}\delta_{pi} + \Delta_{pnk}\delta_{qi}$

Theorem (4.5):

Matrices L^p and M^p, representing multiplication by f_p, i.e. making a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow L^p & & \downarrow M^p & & \downarrow f_p & & \\ 0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

are given by $(L^p)_{ij} = h_{jk}\Delta_{kip}$

$(M^p)_{ij} = 1$ if $j=0$ and $i=p$.

$= (1/2)\,\mathrm{trace}(L^p . L^j)$ if $i=0$, $j>0$

$= 0$ otherwise.

proof: Substituting $\alpha_k = h_{km}\Delta_m$ in (4.4) 1) we obtain:

$$\Psi_{ip} = h_{km}\Delta_m\Delta_{kip} \qquad (\, p \geq 0\,)$$

As explained in § 3 the map $(L^p)^*: G^* \longrightarrow G^*$ is a lift of Ψ_p over Δ where $\Psi_p(g_i^*) = \Psi_{ip}$, i.e. we have a commutative diagram:

$$\begin{array}{ccc} & & G^* \\ & \nearrow^{(L^p)^*} & \downarrow \Delta \\ G^* & \xrightarrow{\Psi_p} & R \end{array}$$

So, we can take $(L^p)^*_{mi} = h_{km}\Delta_{kip}$. Hence the statement.

To prove the statement about M^p we have to show the commutativity of the diagram in the statement of the theorem. Because of the special form of the M^p this is equivalent to $(\varphi L^p)(g_j) = \alpha_j f_p$ and the commutativity of the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & E \\ L^p\downarrow & & \mu\downarrow \\ G & \xrightarrow{\alpha} & R \end{array}$$

Here $\mu(f_i) = 1/2\mathrm{trace}(L^p L^i)$. Indeed $(\varphi L^p)_{ij} = \varphi_{ik}(L^p)_{kj} = \varphi_{ik}h_{jm}\Delta_{mkp}$. By (4.4) 2) this is equal to $h_{jm}\Delta_m\delta_{ip}$ which is equal to $\alpha_j\delta_{ip}$ which proves the first statement.

We now calculate $2\mu\varphi(g_j)$. This is equal to:

$$\varphi_{ij}h_{ab}\Delta_{bci}h_{cd}\Delta_{dap}$$

Using (4.4) 3) this can be rewritten as:

$$h_{jd}\Delta_{dap}h_{ab}\Delta_b - \Delta_c h_{cd}\Delta_{dap}h_{aj}$$

Because of the symmetry of h_{ij} and the anti-symmetry of Δ_{dap} in d and a we see that the above expression is equal to $2h_{jd}\Delta_{dap}h_{ab}\Delta_b = 2\alpha_a(L^p)_{aj}$ which proves the assertion. ⊠

Theorem (4.6):

The homotopy $H: \bigwedge^2(E)\otimes F \longrightarrow G$ has as matrix

$$1/2(\mathcal{E}_{pqij} - \mathcal{E}_{qpij})$$

where $\mathcal{E}_{pqij} = h_{ab}\Delta_{bcp}h_{cd}\Delta_{daiqj}$

proof:

We have to prove that $H(1\otimes\Phi) = [L,L]$ and $\Phi H = [M,M]$.

It suffices to prove the first equality because from this it follows that $\Phi H(1\otimes\Phi) = \Phi[L,L] = (1\otimes\Phi)[M,M]$. Composing this with Ψ we get the identity $f.H(1\otimes\Phi) = f[M,M]$. As we assume f to be a non-zero divisor the second equality follows.

We compute:

$$\mathcal{E}_{pqim}\Phi_{mk} = h_{ab}\Delta_{bcp}h_{cd}\Delta_{daiqm}\Phi_{mk}.$$

Using (4.4) 4) this is equal to:

$$h_{ab}\Delta_{bcp}h_{cd}\{\Delta_{daq}\delta_{ik} + \Delta_{idq}\delta_{ak} + \Delta_{aiq}\delta_{dk}\}$$

$$= (h_{ab}\Delta_{bcp}h_{cd}\Delta_{daq})\delta_{ik} + 2h_{kb}\Delta_{bcp}h_{cd}\Delta_{idq} \qquad (*)$$

by relabeling the indices in the last term and using the anti-symmetry of the deltas twice. On the other hand

$$(L^qL^p)_{ik} = L^q_{ic}L^p_{ck} = (h_{cd}\Delta_{diq})(h_{kb}\Delta_{bcp})$$

which is -1/2 times the last term of (*).

It follows that $H_{pqim}\Phi_{mk} = [L^p, L^q]_{ik}$ because the first term of (*) is symmetric in p and q. ⊠

Remark (4.7): The maps L, M and H can be described intrinsically in terms of Φ and $h \in S_2(G^*)\otimes \mathcal{L}$. However, to prove the commutativities expressed by Theorems (4.5) and (4.6) this basis free approach seems to be of no help.

Corollary (4.8): If (R,m) is a local ring, the entries of φ_{ij} are in m and $f \in I_R^2$ as above, then the complex **L.** of theorem (2.3) is a *minimal* resolution of $A = \mathrm{Cok}(\Phi)$ as S- module (after localizing at (m,E)).

proof : This follows from (2.3) and the explicit formulas for L, M and H given in (4.5) and (4.6). ⊠

References

[B-E] D. Buchsbaum and D. Eisenbud; *Generic free resolutions and a family of generically perfect ideals,* Adv. Math. 18, 245-301 (1975)

[Ca] F. Catanese; *Commutative Algebra Methods and Equations of Regular Surfaces:* in LNM 1056, 68-111, Springer Verlag, Berlin etc. 1984

[E-R-S] D. Eisenbud, O. Riemenschneider and F.-O. Schreyer; *Projective Resolutions of Cohen-Macaulay Algebras,* Math. Ann. 85-98,(1981)

[J-S] T. de Jong and D. van Straten; *Deformations of the Normalization of Hypersurfaces,* Math. Ann. 288, 527-547 (1990)

[M-P] D. Mond and R. Pellikaan; *Fitting ideals and multiple points of analytic mappings,* in: Algebraic Geometry and Complex Analysis. Proceedings 1987, Ramirez de Arelleno (ed.). LNM 1414, Springer Verlag, Berlin etc. 1990

[Te] B. Teissier; *The Hunting of Invariants in the Geometry of Discriminants,* in: Real and Complex Singularities, Oslo 1976, Proceedings of the Nordic Summer School. Sijthoff & Noordhoff, Alphen a/d Rijn 1977

[Wa] J. Wahl; *Equations Defining Rational Singularities,* Ann. Sc. Ec. Norm. Sup. 4ieme Serie, 10, 231-264 (1977)

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