A Multiple Objective Planar Location Problem with a Line Barrier

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Abstract

The Multiple Objective Median Problem involves locating a new facility so that a vector of performance criteria is optimized over a given set of existing facilities. A variation of this problem is obtained if the existing facilities are situated on two sides of a linear barrier. Such barriers like rivers, highways, borders, or mountain ranges are frequently encountered in practice. In this paper, theory of the Multiple Objective Median Problem with line barriers is developed. As this problem is nonconvex but specially-structured, a reduction to a series of convex optimization problems is proposed. The general results lead to a polynomial algorithm for finding the set of efficient solutions. The algorithm is proposed for bi-criteria problems with different measures of distance.

1 Introduction

Planar location problems have been intensively studied over the last two decades due to their increasing importance in modern life. Growing population and increased economic demand gave rise to studies on choosing an

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optimal site for such facilities as shopping centers, schools, fire stations, etc. Development of personal computers required higher integration of electronic circuits which led to a similar problem of optimal locating of electronic elements. Those problems, formulated with single performance criterion as well as with multiple criteria have been studied by many authors and efficient algorithms have been proposed. For an overview see e.g. [6, 7, 18, 26].

However, as modern life encounters an ever growing concentration in many areas and aspects, more recent location models often deal with obstacles or barriers. The literature on restricted location problems is very limited and focused on some particular types of distance metrics and barrier shapes, all considered for the single criterion case. See e.g. [19] for an introduction to location problems with barriers. One circle as a barrier and the Euclidean distance were studied in [15] while closed polygons as barriers and the l_p-metric were examined in [1, 4]. Line barriers with passages have been treated in the case of the Manhattan metric l_1 [17, 2] for which arbitrarily shaped barriers can be handled, and for arbitrary metrics derived from norms in [16].

The authors believe that this paper is the first to study multiple objective restricted location problems.

The problem we consider is based on the *Multiple Objective Median Problem*, also referred to as the *Multiple Objective Weber Problem* or the *Multiple Objective Minisum Problem*. It can be formulated as

$$\min_{X \in F} [f_1(X), \ldots, f_Q(X)] \qquad (Q \geq 2). \tag{1}$$

The Q individual criteria measure the performance of a locational decision in the feasible region $F \subseteq \mathbb{R}^2$ with respect to a finite set of existing facilities $\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\}$ represented by points in $\mathbb{R}^2$. Each objective is given as a Median function, i.e. the weighted sum of distances from the new facility to the existing facilities in $\mathcal{E}x$. Thus

$$f_q(X) = \sum_{m=1}^{M} w_{q,m} d_q(X, Ex_m), \qquad q = 1, \ldots, Q, \tag{2}$$

with positive weigths $w_{q,m}$, $q = 1, \ldots, Q$, $m = 1, \ldots, M$. As each decision maker may consider different ways of transportation, distances may be measured differently in each objective. Thus for each criterion $q \in \{1, \ldots, Q\}$, d_q is an arbitrary distance function derived from a norm.

Solving (1) is understood as generating its efficient (Pareto) solutions. A feasible point $X_E \in F$ is said to be an efficient solution of (1) if there is no other point $X \in F$ such that $f(X) \leq f(X_E)$, i.e.:

$$\begin{array}{lll} & \forall q \in \{1,\ldots,Q\} & & f_q(X) \leq f_q(X_E) \\ \text{and} & \exists q \in \{1,\ldots,Q\} & s.t. & f_q(X) < f_q(X_E). \end{array} \tag{3}$$

Let $\mathcal{X}_E$ denote the set of efficient solutions of (1) and let $\mathcal{Y}_E$ denote the image of $\mathcal{X}_E$ in the objective space, that is $\mathcal{Y}_E = f(\mathcal{X}_E)$, where $f = [f_1,\ldots,f_Q]$. $\mathcal{Y}_E$ is referred to as the set of nondominated solutions of (1).

When each objective function of (2) is minimized individually over F, the set of optimal solutions, denoted by $\mathcal{X}_q$, is found:

$$\mathcal{X}_q = \{\arg\min_{X \in F} f_q(X)\}, \qquad q = 1,\ldots,Q.$$

We also define the utopia point $U = [U_1,\ldots,U_Q]$, where $U_q = \min_{X \in F} f_q(X)$, i.e. $U_q = f_q(\mathcal{X}_q)$, $q = 1,\ldots,Q$.

With respect to the classification scheme for location problems proposed in [7, 11] this problem has the classification $1/P/\bullet/d/Q{-}\sum$. This is the classification of a single-facility location problem (1 in the first position) in the plane (P in the second position) with no special assumptions and constraints ($\bullet$ in the third position), d as a vector of distance functions $d_1,\ldots,d_Q$ (d in the fourth position) and Q criteria which can all be given as Median functions ($Q - \sum$ in the fifth position). We will use this classification scheme in the following to achieve a simple description of the different problems mentioned.

In almost all models known in the literature, the feasible region F covers the complete $\mathbb{R}^2$ which is a simplification of many real-life situations. Consider various applications with areas where positioning of a new facility is not allowed (see e.g. [5, 9, 10]) or with regions where trespassing is prohibited. Such barriers may be for example determined by buildings, lakes, or mountain ranges. The idealized case that the barriers are linear and have only a finite set of passages is a special case frequently encountered in practice. Line barriers with passages may be rivers, border lines, highways, mountain ranges or, on a smaller scale, conveyer belts in an industrial plant. Here trespassing is only allowed through a finite set of passages. In this paper, the Multiple Objective Median Problem is extended by the concept of barriers, which significantly increases the complexity of the problem but makes the model a more realistic representation for many applications.

For a given finite set of closed barrier sets

$$\mathcal{B} = \{B_1, B_2, \ldots, B_b\} \subset I\!R^2,$$

let $F := I\!R^2 \setminus int(\bigcup_{i=1}^{b} B_i)$ be the feasible region where new facilities can be located. Furthermore, let $d_{\mathcal{B}}(X, Y)$ be the length of a shortest path (with respect to d) from X to Y not crossing a barrier.

Thus the Multiple Objective Median Problem can now be restated as the *Multiple Objective Median Problem with Barriers* $1/P/\mathcal{B}/d_{\mathcal{B}}/Q - \sum$:

$$\begin{array}{ll} \min & [f_1(X), \ldots, f_Q(X)] \\ \text{s.t.} & X \in F \end{array} \tag{4}$$

with the individual objective functions given by

$$f_q(X) = \sum_{m=1}^{M} w_{q,m}\, d_{q,\mathcal{B}}(X, Ex_m), \qquad q = 1, \ldots, Q, \tag{5}$$

where

$$d_{q,\mathcal{B}}(X, Y) := \inf_{\substack{r \in I\!N \\ T_1, \ldots, T_r \in F}} \sum_{i=1}^{r-1} d_q(T_i, T_{i+1}), \qquad X, Y \in F, \tag{6}$$

with $T_1 = X$, $T_r = Y$ and r intermediate points $T_i \in F$ $(i = 1, \ldots, r)$ such that there exists a feasible path (not crossing $\mathcal{B}$) from T_i to T_{i+1} with length $d_q(T_i, T_{i+1})$.

The set of efficient solutions of (4) is denoted by $\mathcal{X}_{E,\mathcal{B}}$ and the set of nondominated solutions of (4) is denoted by $\mathcal{Y}_{E,\mathcal{B}}$.

Note that $1/P/\mathcal{B}/d_{\mathcal{B}}/Q - \sum$ has a solution only if all existing facilities are located in *connected* components of F.

Observe that as a result of this change of the distance function, the objective functions of (4) may not be convex since in general the distance measures $d_{q,\mathcal{B}}$ are not positively homogeneous ($q \in \{1, \ldots, Q\}$). Consequently, the multiple objective problem may not have features possessed by convex multiple objective programs. In general, the efficient set $\mathcal{X}_{E,\mathcal{B}}$ may not be connected, and the set $\mathcal{Y}_{E,\mathcal{B}} + I\!R^2_{\geq}$ may be neither convex (that is, one may encounter nondominated solutions in a duality gap) nor the set $\mathcal{Y}_{E,\mathcal{B}}$ may be connected. Here connectedness of the set is understood as defined in [3]. When $Q = 2$, we may apply to (4) the following general result for bicriteria problems [22].

Theorem 1 *Let $Q = 2$. If $\mathcal{X}_{E,\mathcal{B}}$ is connected, then $\mathcal{Y}_{E,\mathcal{B}}$ is connected.*

As (4) is a nonconvex multiple objective program, it may feature globally as well as locally efficient solutions that can be found by means of some suitable scalarizations specially developed to handle nonconvexity. All the globally efficient solutions can be found by means of the lexicographic weighted Tchebycheff approach (see [20]) while the locally efficient solutions can be generated using the augmented Lagrangian approach (see [21]). In order to avoid treating (4) in this general methodological framework and to obtain specific and more effective approaches, we focus on the special case of line barriers with passages but still consider a large class of metrics including the class of l_p metrics, which transforms (4) to problem $1/P/\mathcal{B}_L/(l_p)_{\mathcal{B}_L}/Q - \sum$. In Section 2, we show that $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$ has a special structure that allows to develop conceptual results and specific approches to finding the efficient solutions. In Section 3, an algorithm is proposed for the bi-criteria case, i.e. for the problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2 - \sum$. Section 4 includes an illustrative example and the paper is concluded in Section 5.

2 General Results

The following mathematical model will be used for the Multiple Objective Median Problem with line barriers $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$:

Let $L := \{(x, y) \in I\!R^2 \mid y = ax + b\}$ be a linear barrier and let $\{P_n \in L \mid n \in \mathcal{N} := \{1, \ldots, N\}\}$ be a set of points on L, i.e. the set of passages through L. Then

$$\mathcal{B}_L := L \setminus \{P_1, \ldots, P_N\}$$

is called a *line barrier with passages* or shortly *line barrier*. (The case that the barrier is a vertical line, which is not included in this description, can be easily transformed to this definition.)

The feasible region F for new locations is defined as the union of the two closed half-planes F^1 and F^2 on both sides of $\mathcal{B}_L$. Here $F^1 \cup F^2 = I\!R^2$ because the line $y = ax + b$ belongs to both half-planes F^1 and F^2. As all results can be easily transferred to the case that the line barrier has a finite width, for simplification, this model will be used in the following although a new location placed directly on the barrier is not allowed in reality. In the case that $\mathcal{B}_L$ is of a finite width, only the boundary of $\mathcal{B}_L$ (except the passages) belongs to F.

Furthermore, a finite number of existing facilities $Ex_m^i \in F^i$, $m \in \mathcal{M}^i := \{1, \ldots, M^i\}$ are given in each half-plane F^i, $i = 1, 2$, represented by points in $I\!R^2$. A vector of positive weights $w_{q,m}^i := w_q(Ex_m^i) \in I\!R_+$, $q = 1, \ldots, Q$, is associated with each existing facility Ex_m^i representing the demand of Ex_m^i in the individual criterion. As in the more general problem formulation (4), different distance functions derived from norms are permitted for the individual criteria.

The distance measure in $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$ with respect to the given distance functions $d_1, \ldots, d_Q$ is strongly influenced by $\mathcal{B}_L$. Given a distance function d_q (for criterion q) and the barrier model as above, the distance function $d_{q,\mathcal{B}_L}$ results from (6), where the infimum can be replaced by the minimum.

$$d_{q,\mathcal{B}_L}(X,Y) := \min_{\substack{r \in I\!N \\ T_1,\ldots,T_r \in F}} \sum_{i=1}^{r-1} d_q(T_i, T_{i+1}), \qquad X, Y \in F, \tag{7}$$

with intermediate points T_i, $i = 1, \ldots, r$ defined as in case of (6). This leads to the following description of $d_{q,\mathcal{B}_L}$ (see [16]):

Lemma 1 *Let d_q be a metric derived from a norm and $i, j \in \{1, 2\}$, $i \neq j$. Then for every $q \in \{1, \ldots, Q\}$*

$$d_{q,\mathcal{B}_L}(X,Y) = \begin{cases} d_q(X,Y) & \text{if} \quad X, Y \in F^i \\ d_q(X, P_{n(q,X,Y)}) + d_q(P_{n(q,X,Y)}, Y) & \text{if} \quad X \in F^i,\ Y \in F^j, \end{cases}$$

where $n(q, X, Y)$ denotes the index of a passage located on a shortest path from X to Y with respect to criterion q.

Note that the triangle inequality holds for $d_{q,\mathcal{B}_L}$ even though $d_{q,\mathcal{B}_L}$ is not positively homogenous. Consequently, in general $d_{q,\mathcal{B}_L}$ is not a distance function derived from a norm.

As shown in [16] for the corresponding single objective problem, Lemma 1 can be used to rewrite the vector objective function evaluated at a point $X \in F^i$ with respect to each criterion $q \in \{1, \ldots, Q\}$:

Lemma 2 *Let $d = [d_1, \ldots, d_Q]$ be a vector of metrics derived from norms, $X \in F^i$ and $i, j \in \{1, 2\}$, $i \neq j$. Then for each existing facility Ex^j_m there exist passages $P_{n(q,X,Ex^j_m)}$ such that*

$$\begin{pmatrix} f_1(X) \\ \vdots \\ f_Q(X) \end{pmatrix} = \begin{pmatrix} f^i_{1,X}(X) \\ \vdots \\ f^i_{Q,X}(X) \end{pmatrix} + \begin{pmatrix} g^j_{1,X} \\ \vdots \\ g^j_{Q,X} \end{pmatrix}, \tag{8}$$

where

$$f^i_{q,X}(Y) = \sum_{m=1}^{M^i} w^i_{q,m} d_q(Y, Ex^i_m) + \sum_{m=1}^{M^j} w^j_{q,m} d_q(Y, P_{n(q,X,Ex^j_m)}), Y \in F^i, \tag{9}$$

$$g^j_{q,X} = \sum_{m=1}^{M^j} w^j_{q,m} d_q(P_{n(q,X,Ex^j_m)}, Ex^j_m) \tag{10}$$

for $q = 1, \ldots, Q$.

Lemma 2 reveals that the Multiple Objective Median Problem with line barriers $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$ is closely related to the unrestricted Median Problem. Observe also that the right hand side of (8) takes on different values depending on what passage points have been used to evaluate the distance from a point X to the existing facilities located in the opposite half-plane while passing through those passage points. Due to the definition of $n(q, X, Ex^j_m)$, we have that $f^i_{q,Y}(Y) + g^j_{q,Y} \leq f^i_{q,X}(Y) + g^j_{q,X}$ for all $X, Y \in F^i$ and $q = 1, \ldots, Q$. Corollary 1 summarizes the discussion above.

Corollary 1

(i) $[f^i_{1,X}, \ldots, f^i_{Q,X}]$ *is the objective function of the unrestricted Median Problem $1/P/\bullet/d/Q - \sum$ in the half-plane F^i with existing facilities $Ex^i_1, \ldots, Ex^i_{M^i}, P_1, \ldots, P_N$.*

(ii) *For all points $X, Y \in F^i$*

$$\begin{pmatrix} f_1(Y) \\ \vdots \\ f_Q(Y) \end{pmatrix} = \begin{pmatrix} f^i_{1,Y}(Y) \\ \vdots \\ f^i_{Q,Y}(Y) \end{pmatrix} + \begin{pmatrix} g^j_{1,Y} \\ \vdots \\ g^j_{Q,Y} \end{pmatrix} \leqq \begin{pmatrix} f^i_{1,X}(Y) \\ \vdots \\ f^i_{Q,X}(Y) \end{pmatrix} + \begin{pmatrix} g^j_{1,X} \\ \vdots \\ g^j_{Q,X} \end{pmatrix}. \tag{11}$$

Consequently, the Multiple Objective Median Problem with Line Barriers can be decomposed into unrestricted Multiple Objective Median Problems with respect to the facilities in one half-plane and the passage points connecting the two half-planes. Note that the second term in the right-hand-side of (8) denoted by $[g^j_{1,X}, \ldots, g^j_{Q,X}]$ is only implicitly dependent on the location of a new facility and does not directly influence the minimization of the objective $[f_1, \ldots, f_Q]$.

The relationship between the unrestricted problems and the restricted problem suggests that the properties of the former could be helpful in solving the latter. One of the most important properties of the Multiple Objective Median Problem is the inclusion of its efficient set in the convex hull of the existing facilities. In order to be able to apply this fact to the restricted problem we make the following assumption.

Assumption 1 *Let $d = [d_1, \ldots, d_Q]$ denote a vector of metrics derived from norms such that at least one of the following three statements holds for the unrestricted Multiple Objective Median Problem $1/P/\bullet/d/Q - \sum$:*

(a)

$$\mathcal{X}_E \subseteq \operatorname{conv}\{Ex_m \mid m \in \mathcal{M}\}, \tag{12}$$

(b)

$$\mathcal{X}_E \subseteq \operatorname{rect}_{\text{parallel}}\{Ex_m \mid m \in \mathcal{M}\}, \tag{13}$$

where $\operatorname{rect}_{\text{parallel}}\{Ex_m \mid m \in \mathcal{M}\}$ is the smallest rectangle containing all $\{Ex_m \mid m \in \mathcal{M}\}$ whose sides are parallel to the coordinate axes, and $\mathcal{B}_L$ is a horizontal line,

(c)

$$\mathcal{X}_E \subseteq \operatorname{rect}_{\text{diagonal}}\{Ex_m \mid m \in \mathcal{M}\}, \tag{14}$$

where $\operatorname{rect}_{\text{diagonal}}\{Ex_m \mid m \in \mathcal{M}\}$ is the smallest rectangle containing all $\{Ex_m \mid m \in \mathcal{M}\}$ whose sides are diagonal to the coordinate axes, and $\mathcal{B}_L$ is a line diagonal to the coordinate axes.

We emphasize that Assumption 1 is not very constraining since part (a) does hold e.g. for l_p distance functions where $p \in (1, \infty)$ [14, 25]. Part (b) holds for the l_1 distance function and part (c) holds for the l_∞ distance function. All of the following results hold as long as Assumption 1 is satisfied.

Under Assumption 1(a), the efficient set of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$ is contained in the union of the convex hulls of all the existing facilities and the passage points on both half-planes. Assumptions 1(b) and (c) have analogous implications.

Theorem 2 *Consider $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$ and the related problem $1/P/\bullet/d/Q - \sum$ as indicated in Lemma 2.*

(a) If the vector of distance functions d satisfies Assumption 1a, then

$$\begin{aligned}\mathcal{X}_{E,\mathcal{B}_L} \subseteq\ & \text{conv}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\} \\ & \cup\ \text{conv}\{Ex_m^2, P_n \mid m \in \mathcal{M}^2,\ n \in \mathcal{N}\}.\end{aligned}$$

(b) If the vector of distance functions d satisfies Assumption 1b, then

$$\begin{aligned}\mathcal{X}_{E,\mathcal{B}_L} \subseteq\ & \text{rect}_{\text{parallel}}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\} \\ & \cup\ \text{rect}_{\text{parallel}}\{Ex_m^2, P_n \mid m \in \mathcal{M}^2,\ n \in \mathcal{N}\}.\end{aligned}$$

(c) If the vector of distance functions d satisfies Assumption 1c, then

$$\begin{aligned}\mathcal{X}_{E,\mathcal{B}_L} \subseteq\ & \text{rect}_{\text{diagonal}}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\} \\ & \cup\ \text{rect}_{\text{diagonal}}\{Ex_m^2, P_n \mid m \in \mathcal{M}^2,\ n \in \mathcal{N}\}.\end{aligned}$$

Proof: Since the proof for parts (b) and (c) is similar to the proof of part (a), we only give the proof for this part of the theorem:

Without loss of generality let $X^* \in F^1$ be an efficient solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$ such that $X^* \notin \text{conv}\{Ex_m^1, P_n \mid m \in \mathcal{M}^1\ ,\ n \in \mathcal{N}\}$. From Lemma 2 we know that the objective function evaluated at the point X^* can be written as

$$f(X^*) = [f^1_{1,X^*}(X^*), \dots, f^1_{Q,X^*}(X^*)] + [g^2_{1,X^*}, \dots, g^2_{Q,X^*}]$$

where $[f^1_{1,X^*}, \dots, f^1_{Q,X^*}]$ is the objective function of the problem of type $1/P/\bullet/d/Q - \sum$ with existing facilities $\{Ex_m^1, P_n \mid m \in \mathcal{M}^1,\ n \in \mathcal{N}\}$. Consequently, Assumption 1 implies that $X^* \notin \mathcal{X}_E$ of this problem. Therefore there must exist a point $X^\circ \in \mathcal{X}_E$, $X^\circ \neq X^*$, of $1/P/\bullet/d/Q - \sum$ such that

$$[f^1_{1,X^*}(X^\circ), \dots, f^1_{Q,X^*}(X^\circ)] \leq [f^1_{1,X^*}(X^*), \dots, f^1_{Q,X^*}(X^*)]. \tag{15}$$

Adding $[g^2_{1,X^*},\ldots,g^2_{Q,X^*}]$ to both sides of (15) and using Corollary 1(ii), we obtain that

$$[f_1(X^\circ),\ldots,f_Q(X^\circ)] \le [f_1(X^*),\ldots,f_Q(X^*)],$$

which contradicts the assumption that $X^* \in \mathcal{X}_{E,\mathcal{B}_L}$.

□

As a result of Lemma 2 and Theorem 2, problem (4) with line barriers can be decomposed into a finite series of subproblems (P^i_k) $(i \in \{1,2\},\ k \in \mathbb{N})$ of the following form:

$$\begin{array}{ll} \min & [f_1(X),\ldots,f_Q(X)] \\ \text{s.t.} & X \in F^i_k, \end{array} \tag{16}$$

where F^i_k is a part of the half-plane F^i $(i,j \in \{1,2\},\ i \neq j)$ such that the passage points $P_{n(q,X,Ex^j_m)}$ located on a shortest path from a point $X \in F^i_k$ to a facility $Ex^j_m \in F^j$ are the same for all points $X \in F^i_k$ $(q = 1,\ldots,Q,$ $m = 1,\ldots,M^j)$. Observe that the objective functions of problem (16) are identical with those of problem (4), however the feasible set F^i_k may not be convex.

Defining $n(q,k,Ex^j_m)$ to be $n(q,X,Ex^j_m)$ where X is an arbitrary point in F^i_k, we observe that there exist passages $P_{n(q,k,Ex^j_m)}$ depending only on F^i_k such that for all $X \in F^i_k$

$$\begin{aligned} d_{q,\mathcal{B}_L}(X,Ex^j_m) &= d_q(X,P_{n(q,k,Ex^j_m)}) + d_q(P_{n(q,k,Ex^j_m)},Ex^j_m), \\ &\qquad q = 1,\ldots,Q,\ m = 1,\ldots,M^j. \end{aligned}$$

Consequently, the term $[g^j_{1,k},\ldots,g^j_{Q,k}] := [g^j_{1,X},\ldots,g^j_{Q,X}]$ (with an arbitrary point $X \in F^i_k$) is constant for all $X \in F^i_k$. Furthermore we have that $\bigcup_k F^i_k = F^i$. Note that the number of regions F^i_k is finite because there exists only a finite number of possible combinations of passage points $P_{n(q,\bullet,Ex^j_m)}$ as we have that $n(q,\bullet,Ex^j_m) \in \{1,\ldots,N\}$ $(q = 1,\ldots,Q,\ m = 1,\ldots,M^j)$.

Using this partitioning of the feasible region F, the vector objective function of (16) is given by

$$[f_1(X),\ldots,f_Q(X)] = [f^i_{1,k}(X) + g^j_{1,k},\ldots,f^i_{Q,k}(X) + g^j_{Q,k}],\quad X \in F^i_k, \tag{17}$$

where for $q = 1, \dots, Q$

$$f^i_{q,k}(X) = \sum_{m=1}^{M^i} w^i_{q,m} d_q(X, Ex^i_m) + \sum_{m=1}^{M^j} w^j_{q,m} d_q(X, P_{n(q,k,Ex^j_m)}), \; X \in F^i_k, \quad (18)$$

$$g^j_{q,k} = \sum_{m=1}^{M^j} w^j_{q,m} d_q(P_{n(q,k,Ex^j_m)}, Ex^j_m). \quad (19)$$

Solving problem (16) is still a complex task since finding the feasible sets F^i_k is computationally expensive. Therefore we relax the constraint $X \in F^i_k$ to $X \in F^i$ which makes every subproblem (P^i_k) a convex multiple objective problem for which connectedness of $\mathcal{X}^i_{E,k}$ is a well known result from the literature [23].

Let $\mathcal{X}^i_{E,k}$ and $\mathcal{Y}^i_{E,k}$ denote the set of efficient solutions and nondominated solutions of the relaxed problem (P^i_k), respectively.

Individual minimization of each objective function $f^i_{q,k}(X)$ over the feasible set F^i produces the set of optimal solutions:

$$\mathcal{X}^i_{q,k} := \{\arg \min_{X \in F^i} f^i_{q,k}(X)\}, \qquad q = 1, \dots, Q,$$

and the optimal solution value:

$$y^i_{q,k} = \min_{X \in F^i} f^i_{q,k}(X) + g^j_{q,k}, \qquad q = 1, \dots, Q.$$

Having the efficient set of each convex subproblem available we can specify their relationship with the efficient set of the nonconvex problem (4) with line barriers. Similarly, the nondominated set of this problem can be described by means of the nondominated set of the convex problems.

Theorem 3

(i)

$$\mathcal{X}_{E,\mathcal{B}_L} \subseteq \bigcup_{\substack{i=1,2;\\ k}} \mathcal{X}^i_{E,k}$$

(ii)

$$\mathcal{Y}_{E,\mathcal{B}_L} = \min \bigcup_{\substack{i=1,2;\\ k}} \mathcal{Y}^i_{E,k}.$$

Proof:

(i) Let $X^* \in F^i$ $(i, j \in \{1, 2\},\ i \neq j)$ be an efficient solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$. From Lemma 2 and (17) we have that there exists a $k \in I\!N$ such that

$$[f_1(X^*), \ldots, f_Q(X^*)] = [f^i_{1,k}(X^*) + g^j_{1,k}, \ldots, f^i_{Q,k}(X^*) + g^j_{Q,k}].$$

Assume that $X^* \notin \mathcal{X}^i_{E,k}$. Then there is a point $X^\circ \in \mathcal{X}^i_{E,k}$, $X^\circ \neq X^*$, such that

$$[f^i_{1,k}(X^\circ), \ldots, f^i_{Q,k}(X^\circ)] \leq [f^i_{1,k}(X^*), \ldots, f^i_{Q,k}(X^*)].$$

Adding $[g^j_{1,k}, \ldots, g^j_{Q,k}]$ to both sides of this inequality and using Corollary 1(ii), we therefore obtain

$$[f_1(X^\circ), \ldots, f_Q(X^\circ)] \leq [f_1(X^*), \ldots, f_Q(X^*)],$$

contradicting that $X^* \in \mathcal{X}_{E,\mathcal{B}_L}$.

(ii) Part (ii) results from part (i) and the definition of efficient solutions.

□

Theorem 3 provides the new information about the efficient sets and non-dominated sets of problem (4) with line barriers and of the subproblems (P^i_k), which will be used in the next section in the development of an algorithm for finding these sets in the bicriteria case.

3 Methodology for the case of two criteria

In this section we study the bi-objective Median Problem with a line barrier which we formulate as

$$\min_{X \in F} [f_1(X), f_2(X)], \tag{20}$$

where f_1 and f_2 are defined by (2). Furthermore the distance functions of both criteria are identical throughout this section, i.e. $d_1 = d_2$.

Using Theorem 3, a very simple but not efficient algorithm to find the efficient set $\mathcal{X}_{E,\mathcal{B}_L}$ can be proposed. The algorithm first checks for all existing facilities in either half-plane F^i and for all possible passages to the opposite half-plane F^j, and then determines the set of efficient solutions of the corresponding relaxed problems (P^i_k). From the union of all the efficient sets $\mathcal{X}^i_{E,k}$ of the subproblems (P^i_k), the efficient solutions of the original problem, referred to as *globally efficient solutions*, have to be determined. This can be done by constructing the *lower envelope* of all the nondominated solutions of the subproblems in the objective space.

However, a polynomial algorithm for $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2{-}\sum$ can be proposed if the idea of reducing the nonconvex original problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2-\sum$ to a finite set of the relaxed problems (P^i_k) is used more efficiently. Due to the definition of the relaxed problems (P^i_k), their number depends upon the number of the passages and existing facilities and in total there are $O(N^{2M})$ subproblems, where $M := M^1 + M^2$. We will show that considering a smaller number of the subproblems is sufficient to find the globally efficient set $\mathcal{X}_{E,\mathcal{B}_L}$. This smaller number will be additionally reduced by applying a reduction procedure eliminating subproblems whose nondominated sets are dominated by nondominated sets of other subproblems. We now discuss the details of this approach.

Without loss of generality we assume that the passages are in consecutive order, i.e. there is no passage between P_n and P_{n+1} for $1 \leq n \leq N-1$. Let $D^j_n(m)$ denote the *difference of distances* between an existing facility Ex^j_m and every two adjacent passages P_n and P_{n+1} defined as follows: for $q \in \{1,2\}$, $i,j \in \{1,2\}$, $i \neq j$, and $n = 1, \ldots, N-1$:

$$D^j_n(m) := d(Ex^j_m, P_n) - d(Ex^j_m, P_{n+1}), \qquad m \in \mathcal{M}^j.$$

Since d is a metric derived from a norm, a shortest path SP from an existing facility $Ex^j_m \in F^j$ to a point $X \in F^i$ has to pass through one of the passages $P_1, \ldots, P_N$ depending on the following condition:

$$\begin{aligned}
P_1 \in SP &\Leftrightarrow D_1^j(m) < d(P_2, X) - d(P_1, X) \\
P_n \in SP &\Leftrightarrow (d(P_n, X) - d(P_{n-1}, X) < D_{n-1}^j(m)) \\
&\qquad \wedge (D_n^j(m) < d(P_{n+1}, X) - d(P_n, X)), \quad n = 2, \ldots, N-1, \\
P_N \in SP &\Leftrightarrow d(P_N, X) - d(P_{N-1}, X) < D_{N-1}^j(m).
\end{aligned}$$

This analysis leads to the following observations. If a shortest path from a point $X \in F^i$ to an existing facility $Ex_m^j \in F^j$ passes through the passage point P_n, then for the shortest path from any existing facility $Ex_{\bar{m}}^j$ to X with $D_{n-1}^j(\bar{m}) \geq D_{n-1}^j(m)$ a passage $P_{\bar{n}}$ with $\bar{n} < n$ cannot be optimal. Analogously, the shortest path from any existing facility $Ex_{\hat{m}}^j$ to X with $D_n^j(\hat{m}) \leq D_n^j(m)$ through a passage $P_{\hat{n}}$ with $\hat{n} > n$ cannot be optimal. We conclude that not all of the $O(N^{2M})$ possible combinations of existing facilities and passage points have to be considered because a majority of these combinations will not lead to efficient solutions. In fact, the number of subproblems (P_k^i) can be reduced to $O(\binom{M+N-1}{N-1}^2)$. This is polynomial in the number of existing facilities M if the number N of passage points is constant, which is a realistic assumption.

The selection procedure suggested here is given in the appendix; for a more detailed discussion about the selection of the individual subproblems (P_k^i) we refer to [16].

After the selection of an appropriate set of subproblems (P_k^i) is completed, the set of globally efficient solutions has to be determined from the sets $\mathcal{X}_{E,k}$ of efficient solutions of the selected subproblems.

Let $List(P_k^i)$ be a list of all currently selected subproblems. If M is the overall number of existing facilities, and N is the total number of passages, then $List(P_k^i)$ contains up to $L := \binom{M+N-1}{N-1}$ selected subproblems. Since only a small number of these subproblems contribute to the globally nondominated solutions, a reduction procedure is developed which reduces the number of subproblems a second time before the globally nondominated solutions are finally determined as the lower envelope of the remaining sets $\mathcal{Y}_{E,k}$. We now turn our attention to the reduction procedure.

Consider a problem (P_k^i) and its efficient and nondominated sets $\mathcal{X}_{E,k}^i$, $\mathcal{Y}_{E,k}^i$. Since (P_k^i) is a convex problem, $\mathcal{Y}_{E,k}^i$ is a curve spanned between the points A_k^i and B_k^i where

$$A_k^i = (a_{1,k}^i, a_{2,k}^i) \qquad \text{and} \qquad B_k^i = (b_{1,k}^i, b_{2,k}^i)$$

and

$$\begin{aligned}
a_{1,k}^i &= \min_{X \in F^i} f_1(X) \\
a_{2,k}^i &= f_2\left(\arg\left(\text{lex} \min_{X \in F^i} [f_1(X), f_2(X)]\right)\right) \\
b_{2,k}^i &= \min_{X \in F^i} f_2(X) \\
b_{1,k}^i &= f_1\left(\arg\left(\text{lex} \min_{X \in F^i} [f_2(X), f_1(X)]\right)\right)
\end{aligned}$$

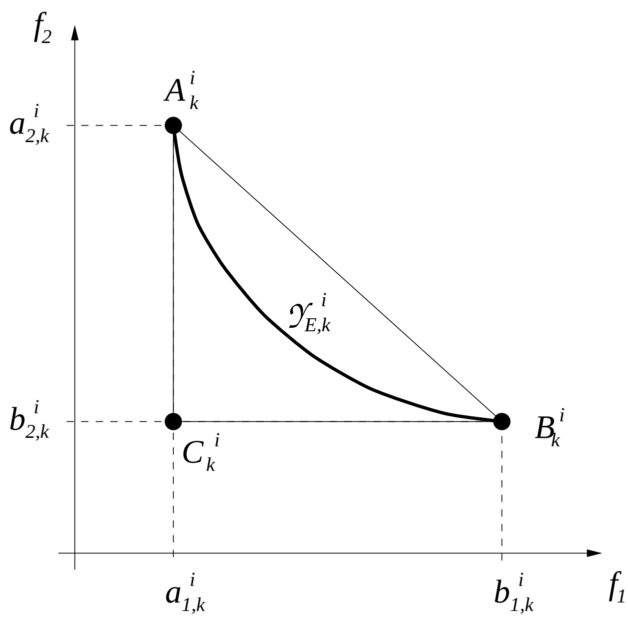

Figure 1: The nondominated set $\mathcal{Y}_{E,k}^i$ of a convex problem (P_k^i).

As illustrated in Figure 1, the nondominated curve is contained in the triangle T_k^i with vertices A_k^i, B_k^i, C_k^i, where $C_k^i = (a_{1,k}^i, b_{2,k}^i)$. Observe that the examination of the mutual location of the triangles T_k^i will help eliminate those problems (P_k^i) whose nondominated sets are dominated by nondominated sets of other subproblems.

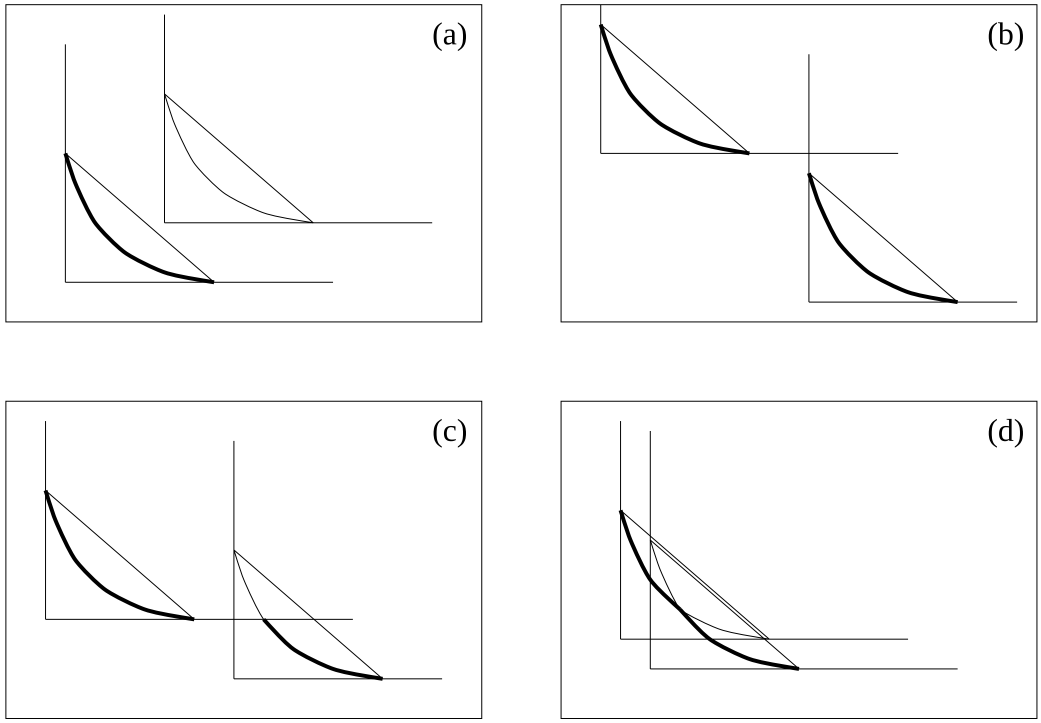

Figure 2: Some examples for possible locations of the triangles A_k^i, B_k^i, C_k^i and $A_k^{\bar{i}}, B_k^{\bar{i}}, C_k^{\bar{i}}$ for two different subproblems (P_k^i) and $(P_k^{\bar{i}})$ in the objective space $(i, \bar{i} \in \{1, 2\})$. The bold curves represent the set of globally nondominated solutions, respectively.

Figure 2 shows four of many possible locations of the nondominated curves for two arbitrary problems (P_k^i) and $(P_k^{\bar{i}})$, $i, \bar{i} \in \{1, 2\}$. In particular, Figure 2a shows that one of the two problems can be eliminated while Figure 2b presents an irreducible case. Figure 2c and d show that only subsets of the two nondominated sets may be in the globally nondominated set.

These observations will be incorporated into the reduction procedure as follows:

In the first part of the procedure, the *Hershberger Algorithm* [13], that finds the lower envelope of a collection of line segments in linear time, is used to determine the lower envelope of the segments $\overline{A_k^i B_k^i}$ of all subproblems in $List(P_k^i)$. Since our goal is to find a superset of the nondominated sets of the subproblems, we add an auxiliary horizontal line at point B_k^i and an auxiliary vertical line at point A^i_k (this is equivalent to finding $\partial(\overline{A^i_k B^i_k} + \mathbb{R}^2_{\geq})$) of each individual segment $\overline{A^i_k B^i_k}$ to eliminate points coming from other subproblems but dominated by the points of subproblem (P^i_k).

After the lower envelope is found, all the subproblems contributing to it are selected and stored in a second list $\underline{List}(P^i_k)$.

In the second step of the procedure all those subproblems (P^i_k) are added to the list $\underline{List}(P^i_k)$ for which at least one point (i.e. the point C^i_k) is not dominated by the lower envelope.

Summarizing, the following procedure is obtained:

Reduction Procedure:

Let $\mathbb{R}^2_{\geq} := \{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0, \ y \geq 0\}$.

Input: $List(P^i_k)$, Segments $\overline{A^i_k B^i_k}$.

Step 1 Construct $\partial(\overline{A^i_k B^i_k} + \mathbb{R}^2_{\geq})$ for all subproblems in $List(P^i_k)$.

Step 2 Apply the Hershberger Algorithm to find the lower envelope of these line segments.

Step 3 Identify those subproblems in $List(P^i_k)$ whose corresponding segments $\overline{A^i_k B^i_k}$ contribute to the lower envelope. Let $\underline{List}(P^i_k)$ be the list of these subproblems and remove them from $List(P^i_k)$.

Step 4 For every remaining subproblem $(P^i_k) \in List(P^i_k)$ check whether C^i_k is dominated by the lower envelope. If it is not dominated, add (P^i_k) to $\underline{List}(P^i_k)$.

Output: Reduced list of subproblems $\underline{List}(P^i_k)$.

The time complexity of the reduction procedure $O(r) = O(L)$ is linear in the number of subproblems L. For many location problems the savings resulting from the reduction procedure will be substantial, however they cannot be theoretically guaranteed.

The reduction procedure eliminates only those subproblems (P^i_k) whose nondominated sets are entirely dominated by the nondominated set of another subproblem (P^i_k) (see Figure 2a). Cases with partial reductions (see Figure 2c, d) are subject to further investigation.

Theorem 4

$$\mathcal{Y}_{E,\mathcal{B}_L} \subseteq \bigcup_{\underline{List}(P^i_k)} \mathcal{Y}^i_{E,k}.$$

Proof: Assume that there exists a subproblem $(P^{\bar{i}}_{\bar{k}}) \in List(P^i_k)$ such that $Y^{\bar{i}}_{E,\bar{k}}$ is globally nondominated, but $Y^{\bar{i}}_{E,\bar{k}} \notin \bigcup_{\underline{List}(P^i_k)} \mathcal{Y}^i_{E,k}$.

Since $Y^{\bar{i}}_{E,\bar{k}} \notin \bigcup_{\underline{List}(P^i_k)} \mathcal{Y}^i_{E,k}$, the corresponding point $C^{\bar{i}}_{\bar{k}}$ of the triangle $T^{\bar{i}}_{\bar{k}}$ of this subproblem is dominated by some point D in the lower envelope found by the Hershberger Algorithm. Therefore there exists a point $Y^{\hat{i}}_{E,\hat{k}} \in \bigcup_{\underline{List}(P^i_k)} \mathcal{Y}^i_{E,k}$, $Y^{\hat{i}}_{E,\hat{k}} \neq Y^{\bar{i}}_{E,\bar{k}}$, dominating D and thus dominating $Y^{\bar{i}}_{E,\bar{k}}$, which contradicts the assumption.

□

Recall that our ultimate goal is to analytically find $\mathcal{X}_{E,\mathcal{B}_L}$, while in general we do not know the analytical descriptions of $\mathcal{X}^i_{E,k}$ and $\mathcal{Y}^i_{E,k}$ for each individual subproblem. In this situation we find piecewise linear approximations of $\mathcal{Y}^i_{E,k}$ and determine the globally nondominated points, as proposed in [12], by means of the *Hershberger Algorithm* [13].

As this algorithm finds a lower envelope of a collection of line segments in the plane, we again add an auxiliary horizontal line at point B^i_k and a vertical line at point A^i_k of every triangle T^i_k to eliminate points coming from other subproblems but dominated by the points of subproblem (P^i_k). After the lower envelope is found, these auxiliary lines are eliminated.

The resulting lower envelope of the approximations of $\mathcal{Y}^i_{E,k}$ will be an approximation of the globally nondominated solutions $\mathcal{Y}_{E,\mathcal{B}_L}$.

In order to find a piecewise linear approximation of $\mathcal{X}^i_{E,k}$, we repeatedly solve the single objective Median problem $1/P/\bullet/d/\sum$ formulated as

$$\begin{array}{ll} \min & \lambda(f^i_{1,k}(X) + g^j_{1,k}) + (1-\lambda)(f^i_{2,k}(X) + g^j_{2,k}) \qquad (i,j \in \{1,2\},\ i \neq j) \\ \text{s.t.} & X \subset F^i \\ & \lambda \in (0,1). \end{array}$$

with the weights λ of our choice.

In the case that $d = l_p$, $p \in (1,\infty)$, this can be done e.g. by applying the *Weiszfeld Algorithm* [24] which determines an approximate solution of

$1/P/\bullet/l_p/\sum$. For an overview of solution procedures for various kinds of planar Median problems we refer to [6].

The Weiszfeld Algorithm as well as many other algorithms for solving single and multiple objective location problems have been implemented in LOLA [8]. In particular, the points A_i^k and B_i^k can be found by solving the corresponding single objective Median Problems.

The quality of the approximation depends on the number of repetitions. The piecewise linear curves found for every subproblem become the input to the Hershberger Algorithm [13] that produces their lower envelope in time $O(S)$, where S is the total number of line segments composing the piecewise linear curves.

The discussion above leads to a polynomial algorithm for solving the bi-objective Median Problem with a line barrier:

Algorithm for solving $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2-\Sigma$:

Step 1 Apply the selection procedure (see Appendix) and create a list $List(P_k^i)$ of selected subproblems (P_k^i).

Step 2 For every subproblem $(P_k^i) \in List(P_k^i)$: find the triangle T_k^i.

Step 3 Apply the reduction procedure and create a reduced list of subproblems $\underline{List}(P_k^i)$.

Step 4 For every subproblem $(P_k^i) \in \underline{List}(P_k^i)$ find a piecewise linear approximation of its nondominated set.

Step 5 Construct the lower envelope of the piecewise linear approximations of the subproblems $(P_k^i) \in \underline{List}(P_k^i)$.

Output: The lower envelope is an approximation of $\mathcal{Y}_{E,\mathcal{B}_L}$.

If M is the number of existing facilities, N is the number of passages, $L = \binom{M+N-1}{N-1}$ is the maximum number of subproblems in $List(P_k^i)$ after the application of the selection procedure, and S is the total number of line segments composing the piecewise linear approximations, the overall complexity of the proposed Algorithm is $O(s+r+h)$ where

$O(s) = O(NM \log M + L)$	is the complextity of the selection procedure,
$O(r) = O(L)$	is the complexity of the reduction procedure and
$O(h) = O(S)$	is the complexity of the piecewise linear approximation of the nondominated sets and of the Hershberger Algorithm.

4 Example

In the following example we consider a location problem with the classification $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2-\sum$, where distances are measured according to the Manhattan metric l_1. For the analogous unrestricted Median problem of type $1/P/\bullet/l_1/2-\sum$ efficient algorithms are given in [11]. These algorithms are implemented in LOLA, the Library of Location Algorithms [8], which will be used to find the efficient and nondominated sets of $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2-\sum$.

Let the line barrier

$$\mathcal{B}_L := \{(x,y) \in I\!R^2 \mid y=5\} \setminus \{P_1=(4,5), P_2=(9,5)\}$$

divide the plane into the two half-planes F_1 and F_2. Furthermore four existing facilities are given on both sides of $\mathcal{B}_L$ with coordinates and weights as listed in Table 1. Thus $\mathcal{M}^1 = \mathcal{M}^2 = \{1,2\}$ and $M^1 = M^2 = 2$.

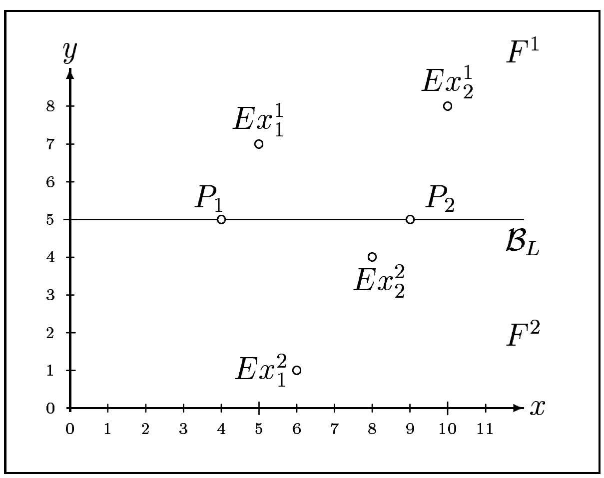

Figure 3: The example problem with the classification $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2-\sum$.

Existing facility Ex^i_m		$w^i_{1,m}$	$w^i_{2,m}$	$D^i_1(m)$
Ex^1_1	$(5,7)$	8	2	-3
Ex^1_2	$(10,8)$	5	6	5
Ex^2_1	$(6,1)$	10	1	-1
Ex^2_2	$(8,4)$	7	4	3

Table 1: Existing facilities with their weights and the values of $D^i_1(m) = d(Ex^i_m, P_1) - d(Ex^i_m, P_2)$.

In step 1 of the Algorithm presented above the selection procedure is applied and $List(P^i_k)$ includes the 6 subproblems (P^i_k) listed in Table 2 which are further investigated.

(P^i_k)	weights of P_1		weights of P_2		weights of existing facilities
	$\tilde{w}_1$	$\tilde{w}_2$	$\tilde{w}_1$	$\tilde{w}_2$	
(P^1_0)	0	0	17	5	$\tilde{w}_q(Ex^1_m) := w_q(Ex^1_m),\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex^2_m) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P^1_1)	10	1	7	4	$\tilde{w}_q(Ex^1_m) := w_q(Ex^1_m),\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex^2_m) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P^1_2)	17	5	0	0	$\tilde{w}_q(Ex^1_m) := w_q(Ex^1_m),\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex^2_m) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P^2_0)	0	0	13	8	$\tilde{w}_q(Ex^1_m) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex^2_m) := w_q(Ex^2_m),\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P^2_1)	8	2	5	6	$\tilde{w}_q(Ex^1_m) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex^2_m) := w_q(Ex^2_m),\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P^2_2)	13	8	0	0	$\tilde{w}_q(Ex^1_m) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex^2_m) := w_q(Ex^2_m),\ q \in \{1,2\},\ m \in \mathcal{M}^2$

Table 2: Weights of the existing facilities $\mathcal{E}x = \{Ex^1_1, Ex^1_2, Ex^2_1, Ex^2_2, P_1, P_2\}$ of the 6 selected subproblems (P^i_k) of type $1/P/\bullet/l_1/2-\sum$.

At the end of step 3, the reduced list $\underline{List}(P^i_k)$ includes (P^1_0), (P^2_0), (P^2_1). Due to the choice of the distance function and the fact that the nondominated sets of the subproblems are piecewise linear curves, step 4 can be omitted.

For illustrative reasons, we include the sets of efficient solutions (see Table 3) and nondominated solutions (see Figure 4) of the subproblems in $List(P^i_k)$ which were determined using LOLA [8].

sub-problem	efficient solutions of the subproblems $\mathcal{X}^i_{E,k}$
(P^1_0)	$\{(x,y) \in \mathbb{R}^2 \mid (x=9) \wedge (5 \leq y \leq 7)\}$
(P^1_1)	$\{(x,y) \in \mathbb{R}^2 \mid (5 \leq x \leq 9) \wedge (y=5)\}$ $\cup \{(x,y) \in \mathbb{R}^2 \mid (x=9) \wedge (5 \leq y \leq 7)\}$
(P^1_2)	$\{(x,y) \in \mathbb{R}^2 \mid (4 \leq x \leq 5) \wedge (5 \leq y \leq 7)\}$
(P^2_0)	$\{(x,y) \in \mathbb{R}^2 \mid (8 \leq x \leq 9) \wedge (4 \leq y \leq 5)\}$
(P^2_1)	$\{(x,y) \in \mathbb{R}^2 \mid (6 \leq x \leq 8) \wedge (y=4)\}$ $\cup \{(x,y) \in \mathbb{R}^2 \mid (x=8) \wedge (4 \leq y \leq 5)\}$
(P^2_2)	$\{(x,y) \in \mathbb{R}^2 \mid (4 \leq x \leq 6) \wedge (4 \leq y \leq 5)\}$

Table 3: Efficient solutions of the 6 subproblems in $List(P^i_k)$.

Step 5 yields the set of globally nondominated solutions $\mathcal{Y}_E$ and the set of globally efficient solutions $\mathcal{X}_E$ of this example problem:

$$\begin{aligned}
\mathcal{Y}_E &= \mathcal{Y}^1_{E,0} \cup \mathcal{Y}^2_{E,0} \cup \mathcal{Y}^2_{E,1} \\
\mathcal{X}_E &= \mathcal{X}^1_{E,0} \cup \mathcal{X}^2_{E,0} \cup \mathcal{X}^2_{E,1} \\
&= \{(x,y) \in \mathbb{R}^2 \mid ((x=9) \wedge (5 \leq y \leq 7)) \\
&\quad \vee ((8 \leq x \leq 9) \wedge (4 \leq y \leq 5)) \\
&\quad \vee ((6 \leq x \leq 8) \wedge (y=4))\}.
\end{aligned}$$

The set of globally efficient solutions $\mathcal{X}_E$ is graphed in Figure 5.

5 Conclusions

This paper studies the Multiple Objective Median Problem with a line barrier. The primary goal of this pioneering research is the analytical determination of the efficient set of the problem. The structure of the efficient set is first examined in order to motivate the design of special algorithms.

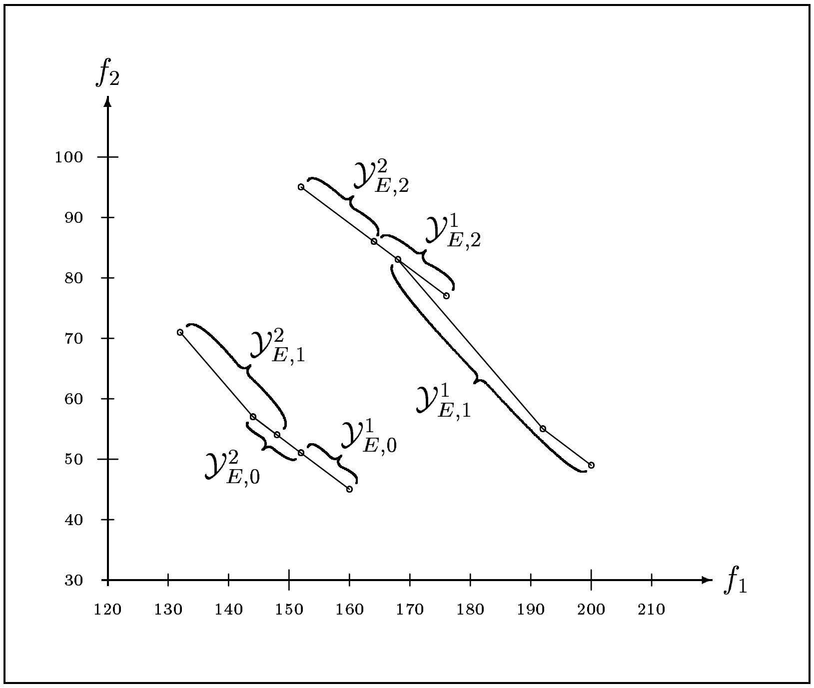

Figure 4: Nondominated solutions of the 6 subproblems in $List(P^i_k)$.

The theoretical analysis shows that the original nonconvex problem can be decomposed to a series of multiple objective convex subproblems.

A polynomial algorithm for solving bi-criteria median problems with a line barrier and different distance measures is proposed. An approximation of the nondominated set of these problems is determined as the lower envelope of the nondominated sets of the subproblems. An illustrative example is included. More research is needed to efficiently design the reduction procedure eliminating some of the subproblems. Currently, the procedure checks only for the nondominated sets that are entirely dominated by nondominated sets of other subproblems. Cases with partial domination should also be considered.

We would like to emphasize that several approaches in the literature can find an approximate description of the efficient set of the general convex multiple objective problem. None of them, however, offers an exact description neither deals with nonconvex problems. Interestingly, the approach proposed in this paper finds an approximate description of a nonconvex problem and

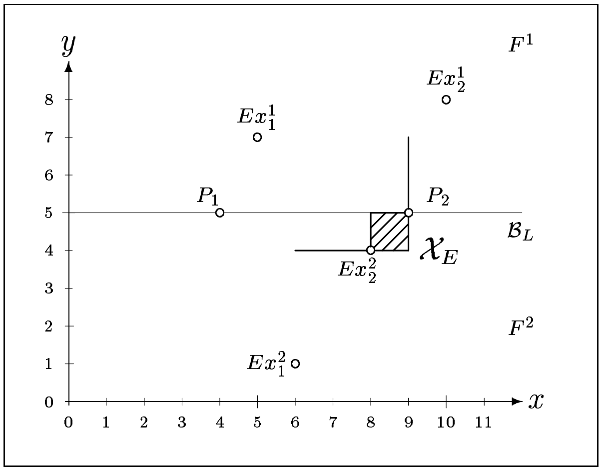

Figure 5: Efficient solution $\mathcal{X}_E$ of the example problem of type $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2 - \sum$.

an exact description for problems with specific distance functions.

Clearly, other location problems with barriers should be studied in the multiple objective framework. Complexity of those problems, however, may heavily affect the ability to analytically approximate their efficient sets. In this case, one may be interested in obtaining partial information about the efficient solutions and in designing tools for choosing a most preferred solution as the optimal one.

Furthermore, not only location problems can lead to nonconvex multiple objective problems decomposable to a series of convex problems. This class of nonconvex multiple objective problems should be explored independently of their applications.

6 Appendix

In this section we briefly sketch the selection procedure that returns a list of subproblems (P_k^i) for further consideration in the Algorithm to solve $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2 - \sum$. For a more detailed description we refer to [16].

For $n = 1, \ldots, N-1$ and $j \in \{1, 2\}$ let $\pi_n^j : \mathcal{M}^j \to \mathcal{M}^j$ be a permutation of $\mathcal{M}^j$ such that

$$D_n^j(\pi_n^j(1)) \le \cdots \le D_n^j(\pi_n^j(M^j)).$$

Furthermore define for $n = 1, \ldots, N-1$

$$k_n := \underset{m \in \mathcal{M}^j}{\operatorname{argmax}} \ \left\{0, \pi_n^j(m) \mid D_n^j(\pi_n^j(m)) < d(P_{n+1}, X) - d(P_n, X)\right\}$$

and $k_N := M^j$. Since two permutations π_n^j and $\pi_{\tilde{n}}^j$ need not be the same for $n \neq \tilde{n}$, let $\mathcal{M}_1^j := \mathcal{M}^j$ and

$$\mathcal{M}_n^j := \mathcal{M}_{n-1}^j \setminus \left\{\pi_{n-1}^j(m) \mid \pi_{n-1}^j(m) \le k_{n-1}\right\}, \quad n = 2, \ldots, N.$$

Then

$$f(X) = \underbrace{\sum_{m=1}^{M^i} w_m^i d(X, Ex_m^i) + \sum_{n=1}^{N} \Bigg(\sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \le k_n}} w_{\pi_n^j(m)}^j \Bigg) d(X, P_n)}_{=: f_{k_1,\ldots,k_N}^j(X)}$$

$$+ \underbrace{\sum_{n=1}^{N} \sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \le k_n}} w_{\pi_n^j(m)}^j d(P_n, Ex_{\pi_n^j(m)}^j)}_{=: g_{k_1,\ldots,k_N}^j}.$$

Selection Procedure:

For $i = 1, 2$ do

Step 1 Let $j \in \{1, 2\}$ with $j \neq 2$ and

$$D_n^j(m) := d(P_n, Ex_m^j) - d(P_{n+1}, Ex_m^j); \; m \in \mathcal{M}^j; \; n = 1, \ldots, N-1.$$

Step 2 For $n = 1$ to $N-1$ find a permutation $\pi_n^i : \mathcal{M}^i \to \mathcal{M}^i$ such that

$$D_n^j(\pi_n^j(1)) \le \cdots \le D_n^j(\pi_n^j(M^j)).$$

Step 3 Let $\mathcal{M}_1^j := \mathcal{M}^j$.

For $k_1 = 0$ to M^j do

Determine $\mathcal{M}_2^j$.

For $k_2 = k_1$ to M^j do

Determine $\mathcal{M}_3^j$.

...

For $k_{N-1} = k_{N-2}$ to M^j do

Determine $\mathcal{M}_N^j$ and let $k_N := M^j$.

(a) For $n = 1$ to N let

$$\tilde{w}_q(P_n) := \sum_{\substack{\pi_n^j(m) \in \mathcal{M}_n^j \\ \pi_n^j(m) \leq k_n}} w_q(Ex^j_{\pi_{n^j}(m)}), \quad q \in \{1, 2\}.$$

(b) For $m \in \mathcal{M}^i$ let

$$\tilde{w}_q(Ex_m^i) := w_q(Ex_m^i), \quad q \in \{1, 2\}.$$

(c) Add the subproblem $(P^i_{k_1,\ldots,k_N})$ of type $1/P/\bullet/d/\sum$ with the existing facilities $\mathcal{E}x = \{P_1, \ldots, P_N, Ex_1^i, \ldots, Ex^i_{M^i}\}$, the weights defined in (a) and (b), objective function $f^i_{k_1,\ldots,k_N}$ and constant term $g^j_{k_1,\ldots,k_N}$ (see (18) and (19)) to the list of subproblems.

Output: List of subproblems $List(P_k^i)$.

With $M := M^1 + M^2$, the time complexity of the selection procedure is $O(N(M \log M) + \binom{M+N-1}{N-1})$.

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