UNIVERSITÄT KAISERSLAUTERN

PRE–FIXED POINTS OF POLYNOMIAL FUNCTIONS IN LATTICES

Marcel Erne and Dietmar Schweigert

STRICTLY ORDER PRIMAL ALGEBRAS

Otfried Lüders and Dietmar Schweigert

Preprint Nr 211

FACHBEREICH MATHEMATIK

PRE–FIXED POINTS OF POLYNOMIAL FUNCTIONS IN LATTICES

Marcel Erne and Dietmar Schweigert

STRICTLY ORDER PRIMAL ALGEBRAS

Otfried Lüders and Dietmar Schweigert

Preprint Nr 211

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Erwin–Schrödinger–Straße
6750 Kaiserslautern

Dezember 1991

Pre-fixed points of polynomial functions in lattices

Marcel Erné and Dietmar Schweigert

An element c of an ordered set P is called a *lower*, respectively, *upper pre-fixed point* of a function $f\colon P \to P$ if $c \le f(c)$, respectively, $f(c) \le c$. Of course, a least element 0 is always a lower pre-fixed point, and a greatest element 1 is always an upper pre-fixed point.

LEMMA 1. *If every order-preserving selfmap of an ordered set P has an upper pre-fixed point then each well-ordered subset has an upper bound, and consequently, each element of P is dominated by a maximal one.*

Indeed, if W is a well-ordered chain in P without upper bounds then the map $f\colon P \to P$ with

$$f(x) = \min\{y \in W : y \nleq x\}$$

is order-preserving but has no upper pre-fixed point.

COROLLARY 1. *A lattice L has a greatest (least) element iff each order-preserving selfmap of L has a lower (upper) pre-fixed point.*

Recall that a lattice is *order-polynomially complete* iff each order-preserving selfmap of L is a polynomial function. Up to now, only finite examples of such lattices are known (cf. [5]). Recently, H. K. Kaiser and N. Sauer [4] have established a few necessary conditions for a lattice to be order-polynomially complete; the main result was that such a lattice must be bounded and cannot be countably infinite. The crucial step is to show that *every polynomial function of a lattice has upper and lower pre-fixed points*. Below, we give a few succinct arguments for these facts.

To each polynomial function p of a lattice L, we assign a certain lattice element c_p by induction on the word representation of p: if p is the identity function, i.e. $p(x) = x$, take for c_p an arbitrary element of L; if p is a constant c, set $c_p = c$; if $p = q \vee r$, put $c_p = c_q \vee c_r$; and finally, if $p = q \wedge r$, put $c_p = c_q$. Then a straightforword induction yields

LEMMA 2. *For any polynomial function p of a lattice L, $c_p \le c$ implies $p(c) \le c$. In particular, p has an upper and, by duality, a lower pre-fixed point.*

This together with Corollary 1 immediately gives

COROLLARY 2. *Any order-polynomially complete lattice is bounded.*

Lemma 2 raises the question whether every polynomial function of a lattice has a fixed point. For large classes of lattices, the answer is affirmative, because under rather mild conditions, the existence of a lower pre-fixed point already ensures a fixed point; for example, this is the case if each ascending sequence (a_n) has a supremum and f preserves these suprema:

$$f(\bigvee a_n) = \bigvee f(a_n).$$

Indeed, if $c \le f(c)$ then the sequence $(f^n(c))$ is ascending and its supremum is the smallest fixed point of f above c (cf. [1, 4.5]). Now call a lattice L *ω-chain continuous* if each ascending sequence (a_n) in L has a supremum and satisfies the equation

$$(*)\quad \bigvee\{a_n : n \in \omega\} \wedge b = \bigvee\{a_n \wedge b : n \in \omega\} \quad \text{for all } b \in L.$$

Since in an ω-chain continuous lattice every polynomial function preserves suprema of ascending sequences, we conclude:

LEMMA 3. *In a bounded ω-chain continuous lattice, every polynomial function has a fixed point.*

R. Freese [3] has given an example of a fixed-point free polynomial in the free lattice with three generators. This is quite remarkable with regard to Lemma 3, because the above equation (*) holds in a free lattice whenever the involved sequence (a_n) has a supremum. Combining Lemma 3 and Corollary 2 with the known fact that a lattice is complete if and only if each order-preserving selfmap has a fixed point (cf. [2]), we arrive at

COROLLARY 3. *Any order-polynomially complete ω-chain continuous lattice is complete.*

However, it remains open whether order-polynomial completeness alone implies lattice-theoretical completeness (or even finiteness).

By a *forest* of an ordered set, we mean a subset whose principal ideals are well-ordered. Notice that every subset of a forest is again a forest. In a bounded ordered set P, each forest F gives rise to an order-preserving function f defined by

$$f(x) = \min\{y \in F \cup \{1\} : x \leq y\}$$

if this minimum exists, and $f(x) = 0$ in all other cases. Clearly, the range of f is $F \cup \{1\}$ or $F \cup \{0,1\}$. Thus we have:

LEMMA 4. *If a bounded ordered set contains an infinite forest of cardinality α then it admits at least 2^{α} order-preserving functions.*

Since an infinite lattice of cardinality α has α polynomial functions, Lemma 4 amounts to

COROLLARY 4. *An infinite order-polynomially complete lattice cannot contain any forest of the same cardinality.*

In particular, this shows that an order-polynomially complete lattice cannot be countably infinite, because any such lattice contains either an infinite antichain or an infinite ascending or descending chain, in any case, an infinite forest or dual forest.

REFERENCES

[1] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*. Cambridge University Press, Cambridge 1990.

[2] A. C. Davis, *A characterization of complete lattices*. Pacific J. Math. 5 (1955), 311-319.

[3] R. Freese, *Some order theoretic questions about free lattices and free modular lattices*. In: I. Rival (ed.), *Proc. of the Banff Symp. on Ordered sets*, D. Reidel, Dordrecht 1982.

[4] H. K. Kaiser and N. Sauer, *On order-polynomially complete lattices*. Preprint TU Wien 1989.

[5] D. Schweigert, *Über endliche, ordnungspolynomvollständige Verbände*. Monatsh. f. Math. **78** (1974), 68-76.

Institut für Mathematik
Universität Hannover
D–3000 Hannover

Fachbereich Mathematik
Universität Kaiserslautern
D-6750 Kaiserslautern