UNIVERSITÄT KAISERSLAUTERN

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Strictly order primal algebras

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Partial orders and the clones of functions preserving them have been thoroughly studied in recent years. The topic of this papers is strict orders which are irreflexive, asymmetric and transitive subrelations of partial orders. We call an algebra $\mathcal{A} = (A,\Omega)$ strictly order primal if for some strict order $(A;<)$ the term functions are precisely the functions which preserve this strict order. Our approach has some parallels to the theory of order primal algebras [8], [2]. We present new examples of congruence distributive varieties and of strict orders without near unanimity operations. Then we give a series of new examples showing that there are varieties which are (n+1)–permutable but not n–permutable. Furthermore the dual category of strict chains is described by the methods from B. Davey and H. Werner [3]. Throughout we use the notations of Grätzer [4] and assume a knowledge of Davey–Werner [3] for the last section.

1. Notation

Definition 1.1: A binary relation $<$ on A is called a strict order if the following properties hold.

(i) $a \nless a$ for every $a \in A$ (irreflexivity)

(ii) if $a < b$ then $b \nless a$ for all $a,b \in A$ (asymetry)

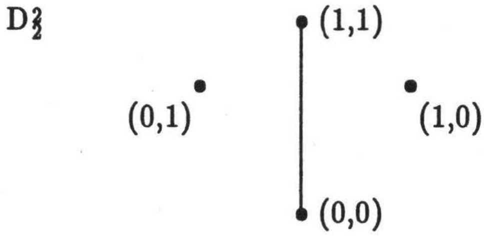
(iii) if $a < b$ and $b < c$ then $a < c$ for all $a,b,c \in A$ (transitivity).

To every strict order we can define a partial order by $a \leq b$ iff $a < b$ or $a = b$ and vice versa from every partial order we can define a strict order. But we would like to call the attention of the reader to the fact that the product of strict orders is defined according to the principles in clone theory.

[1]This work was done while the first author was visiting the University of Kaiserslautern with the financial assistance of the DAAD

Definition 1.2: Let $(A;<_A)$ and $(B;<_B)$ strict orders. Then the strict order $(A\times B;<)$ is defined componentwise in the following way. $(a_1,b_1) < (a_2,b_2)$ if and only if $a_1 <_A a_2$ and $b_1 <_B b_2$.

We present as an example the strict order $D_2^2 = (\{0,1\}^2;<)$ using a Hasse diagram



Definition 1.3: A function $f\colon A \longrightarrow B$ from a strict order $(A,<_A)$ into a strict order $(B,<_B)$ is called strictly monotone if from $a_1 <_A a_2$ it follows $f(a_1) <_B f(a_2)$.

One can observe in the above example four strictly monotone functions $f: D_2^2 \longrightarrow D_2$ which can be presented by the four term functions $x, y, x\wedge y, x\vee y$ of the lattice connected to $D_2 = (\{0,1\};<)$. We will write $a -\!\!< b$ in $(A;<)$ if $a < b$ in $(A;<)$ and there exists no $c \in A$ with $a < c < b$.

$\mathrm{Pol}_n<$ is the set of all strictly monotone functions $f\colon A^n \longrightarrow A$ and $\mathrm{Pol}< = \bigcup_{n\in\mathbb{N}} \mathrm{Pol}_n<$ is the clone of all strictly monotone functions of $(A;<)$. Given an algebra $\mathcal{A} = (A,\Omega)$ we write $T_n(\mathcal{A})$ for set of all n–place term functions of $\mathcal{A}$ and $T(\mathcal{A})$ for the clone of term functions of $\mathcal{A}$.

2. Strictly monotone functions on a chain

Notation 2.1: Assume that A has no infinite chains. The length $\ell(a,b)$ for $a,b \in A$ with $a < b$ is defined to be one less than the number of elements in a chain of maximum size from a to b. Extend ℓ to a function $\ell\colon A^2 \longrightarrow \mathbb{N}_0$ by defining $\ell(a,b) = 0$ whenever $a \nless b$. A function $f: A^n \longrightarrow A$ is called length preserving if $\ell(a,b) \leq \ell(f(a),f(b))$ for all $a,b \in A^n$.

Lemma 2.2: Assume that A has no infinite chains. The function $f: A^n \longrightarrow A$ is strictly monotone on $(A;<)$ if and only if f is length preserving.

Proof. Let $f \in \mathrm{Pol}<$, $a < b$ and $\ell(a,b) = r$. Then there exists a maximal chain $a = a_0 -\!< a_1 -\!< \ldots -\!< a_{r+1} = b$ implying $f(a) < f(a_1) < \ldots < f(b)$. Hence $\ell(f(a),f(b)) \geq r$. On the other hand if f is length preserving and $a < b$ then $1 \leq \ell(a,b) \leq \ell(f(a),f(b))$ and hence $f(a) < f(b)$.

Throughout in the following we assume that $(A;<)$ is a finite chain where $A = \{0,1,\ldots,k\}$ and $0 -\!< 1 -\!< \ldots -\!< k$.

Proposition 2.3: The length function ℓ on A^n satisfies $\ell(0,a) = \min\{a_1,\ldots,a_n\}$ where $a = (a_1,\ldots,a_n)$.

Proof. One observes that for $a_m = \min\{a_1,\ldots,a_n\}$ the chain $a_m >\!- a_m - 1 >\!- \ldots >\!- 0$ coresponds to a maximal chain for $0 < a$.

Notation 2.4: Denote the top element of A^n by $\bar{k}$. Let $c \in A^n$, $c = (c_1,\ldots,c_n)$ and $r \in A$ with $\ell(0,c) \leq r \leq k - \ell(c,\bar{k})$. We define the function $g_c^r : A^n \longrightarrow A$ by

i) $g_c^r(x) = \max\{r + \ell(c,x), \ell(0,x)\}$ whenever $x > c$ or $x = c$

ii) $g_c^r(x) = \ell(0,x)$ whenever $x \not> c$ and $x \neq c$.

Note that $g_c^r(x) : A^n \longrightarrow A$ is strictly monotone.

Lemma 2.5: Every strictly monotone function $f : A^n \longrightarrow A$ can be obtained by componing the functions max and g_c^r for $c \in A^n$ and $r \in A$.

Proof. We consider $c = \{c \in A^n \mid f(c) > \ell(0,c)\}$ as an index set to define

$$h(x) = \max\{g_c^r(x) \mid c \in C, r = f(x)\}.$$

We have to show that $h(x) = f(x)$. We have $\ell(0,x) \leq f(x)$ and hence $g_c^r(x) \leq f(x)$ whenever $x \not> c$ and $x \neq c$. For $x = c$ we have $g_c^r(x) = f(x)$ and for $x > c$ we have

$$\begin{aligned} g_c^r(x) &= \max\{r+\ell(c,x),\ \ell(0,x)\} \\ &= \max\{f(c)+\ell(c,x),\ \ell(0,x)\} \\ &\leq \max\{f(c)+\ell(f(c),f(x)),\ \ell(0,f(x))\} \\ &\leq f(x) \end{aligned}$$

as f is length preserving.

Altogether we have $g_c^r(x) \leq f(x)$ and for some c we have $g_c^r(x) = f(x)$.

Lemma 2.6. Every function $g_c^r : A^n \longrightarrow A$ can be composed by functions $g_d^r : A^3 \longrightarrow A$ and min.

Proof. We assume $n \geq 3$ and we put for $c = (c_1,\dots,c_n)$

$$D := \{d \in \{c_1,\dots,c_n\}^3 \mid \ell(0,d) \leq r \leq k-\ell(d,\bar{k})\}$$

writing $d = (c_i,c_j,c_k)$ and $\bar{x} = (x_i,x_j,x_k) \in X$

$$X = \{(x_i,x_j,x_k) \mid i,j,k \in \mathbb{N}\ 1 \leq i < j < k\}.$$

We define

$$h(x) = \min\{g_d^r(\bar{x}) \mid d \in D,\ \bar{x} \in X\}.$$

If we have $\bar{x} > d$ or $\bar{x} = d$ then we have

$$\begin{aligned} g_d^r(\bar{x}) &= \max\{r+\ell(d,\bar{x}),\ \ell(0,\bar{x})\} \\ &\geq \max\{r+\ell(c,x),\ \ell(0,x)\} \\ &\geq g_c^r(x) \end{aligned}$$

If we have $\bar{x} \not> d$ and $\bar{x} \neq d$ then we have $x \not> c$ and $x \neq c$ and we get

$$g_d^r(\bar{x}) = \ell(0,\bar{x}) \geq \ell(0,x) = g_c^r(x).$$

It remains to show that for some d we have

$$g_d^r(\bar{x}) = g_c^r(x).$$

Case: $x = c$.

In this case we have $\bar{x} = d$ and so $g_d^r(\bar{x}) = g_c^r(x)$.

Case: $x > c$.

Let $c = a_0 -\!< a_1 -\!< \ldots -\!< a_k = x$ be a maximal chain from c to x. Then for some component m we have a chain $c_m = a_{0m} -\!< a_{1m} -\!< \ldots -\!< a_{km} = x_m$ and for all other components there exist chains of equal or larger length. Let $x_i = \ell(0,x)$. If $c_i \geq r$ then we take $c_j = \ell(0,c)$ else $c_j = k - \ell(c,k)$. Obviously $d = (c_i,c_j,c_m) \in D$. We have

$$\begin{aligned} g^r_d(\bar{x}) &= \max\{r+\ell(d,\bar{x}),\ \ell(0,\bar{x})\} \\ &= \max\{r+\ell(c,x),\ \ell(0,x)\} \\ &= g^r_c(x). \end{aligned}$$

Case: $x \not> c$ and $x \neq c$.

In this case there exist $h,k \in \{1,\ldots,n\}$ such that $x_h \not> c_h$ and $x_k \neq c_k$. Let $x_s = \min\{x_1,\ldots,x_n\}$. For $c_s \geq r$ let $c_i = \min\{c_1,\ldots,c_n\}$ else $c_i = \max\{c_1,\ldots,c_n\}$. If $c_s \neq x_s$ then we choose $d = (c_s,c_h,c_i)$ else $d = (c_s,c_k,c_i)$. We have also $d \in D$, $d \not< \bar{x}$ and $d \neq \bar{x}$. Hence

$$g^r_d(\bar{x}) = x_s = g^r_c(x).$$

3. Strictly order primal algebras

Definition 3.1: The algebra $\mathcal{A} = (A,\Omega)$ is called n–sop for $n \in \mathbb{N}$ if $T_n(\mathcal{A}) = \mathrm{Pol}_n<$ for some strict order $<$. We call $\mathcal{A}$ a sop (:= strictly order primal) algebra if $\mathcal{A}$ is n–sop for every $n \in \mathbb{N}$.

Examples.

3.2: Every lattice L where L is a chain is 1–sop. Observe that the identity function id is the only function which preserves this strict order.

3.3: A non trivial lattice L is 2–sop if and only if L is isomorphic to the two–element distributive lattice D_2. D_2 is not 3–sop.

3.4: Consider the algebra $G = (\{0,1\};\wedge,\vee,q)$ where $q(x_1,x_2,x_3) = x_1+x_2+x_3$ with the addition mod 2. The algebra G is 3–sop.

The following lemma can be proved in a similar way as the analogous lemma in [7].

Lemma 3.5: If $\mathcal{A}$ is n–sop then $\mathcal{A}$ is k–sop for $1 \leq k \leq n$.

Using lemma 2.6 above we have the following result:

Theorem 3.6: Let $\mathcal{A}$ be an algebra with $T(\mathcal{A}) \subseteq \mathrm{Pol}<$ where $<$ is a finite chain. Then $\mathcal{A}$ is sop if and only if $\mathcal{A}$ is 3–sop.

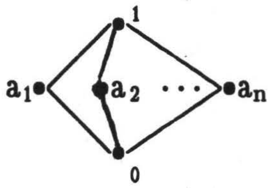
Examples.

3.7: The algebra $G = (\{0,1\};\wedge,\vee,q)$ is sop.

3.8: The algebra $G = (\{0,1,\dots,k\};\ \mathrm{Pol}_3<)$ is sop.

3.9: By a similar method one can show for the "projective line" $\mathcal{M}_n$; $n \geq 2$, that $\mathcal{M}_n = (M_n;\ \mathrm{Pol}_3<)$ is sop.

$(M_n;<)$



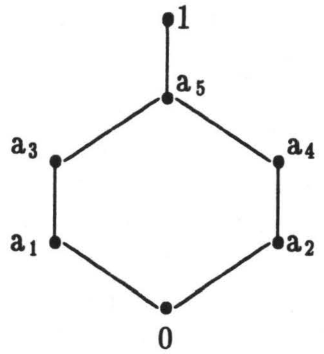
Remark 3.10: In an unpublished manuscript [6] it has been shown that for the strict orders $(\mathbb{Q};<)$ and $(\mathbb{R};<)$ the clone Pol< is locally preprimal. Pol< includes the following operations: $x+y$, $c\cdot x$ with $c \in \mathbb{Q}^+$ or $c \in \mathbb{R}^+$ respectively, and $\min\{x,y\}$, $\max\{x,y\}$. These algebras can be called locally sop.

4. Congruence distributivity and n–permutability

In our examples the existence of near unanimity operations play an important role. Therefore we would like to present two examples of strict orders which cause major obstacles to proofs like those above.

Examples.

4.1: The strict order which is induced from the following lattice order does not admit a majority function. This is the least smallest example with this property.



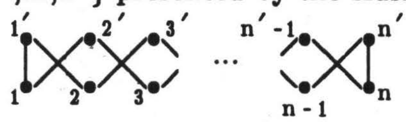
Observe that for a majority function h we would have that

$$h(a_3,a_4,a_5) < h(a_5,a_5,1) = a_5$$
$$h(a_3,a_4,a_5) > h(a_1,a_2,a_1) = a_1$$
$$h(a_3,a_4,a_5) > h(a_1,a_2,a_2) = a_2$$

in contradiction to the fact that there is no element x with $a_1,a_2 < x < a_5$.

4.2: Let $<$ be the strict order induced by the lattice order $(D_3;<)$ where D_3 is the Boolean lattice with 8 elements. Then there exist no near unanimity function in Pol<. This again is the smallest example with this property.

Notation 4.3: We consider the following strict order "zig zag" (D,D') $D = \{1,\dots,n\}$, $D' = \{1',\dots,n'\}$ presented by the Hasse diagram



A zig zag line is a sequence $a_0 < a_1 > a_2 < a_3 \dots \lessgtr a_n$ where $a_i < a_{i+1}$ is a lower respectively $a_i > a_{i+1}$ an upper neighbor of a_{i+1} in the zig zag (D,D'), $i = 1,\dots,n-1$.

Theorem 4.4: If **A** is a sop algebra with a zigzag (D,D'), $n \geq 3$ as a subalgebra then **A** generates a variety which is not congruence distributive.

Proof: Without loss of generality we may assume that n is even. Then a shortest zig zag line from 1 to n' has n elements. We show the following conditions for terms d, $t \in T(\mathbf{A})$.

α)
$$\left.\begin{array}{l} d(1\ ,1\ ,2\ ) = 1 \\ d(1',1',2') = 1' \\ d(x\ ,y\ ,x\ ) = x \end{array}\right\} \text{ implies } \left\{\begin{array}{l} d(1',2',2') = 1' \\ d(1\ ,2\ ,\ 2) = 1 \end{array}\right.$$

β)
$$\left.\begin{array}{l} t(1',2',2') = 1' \\ t(1\ ,2\ ,2\ ) = 1 \\ t(x\ ,y\ ,x\ ) = x \end{array}\right\} \text{ implies } \left\{\begin{array}{l} t(1\ ,1\ ,2\ ) = 1 \\ t(1',1',2') = 1' \end{array}\right.$$

From $1 = d(1,1,2) < d(2',1',3') > d(3,2,4) < d(4',3',5') > \dots$
$> d(n-1,n-2,n) < d(n',n'-1,n') = n'$ we conclude that $d(i,i-1,i+1) = i$ for $1 < i < n$ or respectively $1' < i < n'$. Especially we have $d(2',1',3') = 2'$ and furthermore $d(1,2,2) < d(2',1',3') = 2'$, $d(1',1',1') = 1'$. Hence we have $d(1,2,2) = 1$. In the same we get $d(1',2',2') = 1'$. Hence α) is proved.

Again we use that a shortest zig zag line from 1 to n' has n elements and consider $1' = t(1',1',1') > t(2,1,3) < t(3',2',4') < \dots > t(n',n'-1,n') = n'$ which implies $t(2,1,3) = 2$. Furthermore we have $t(1',1',2') > t(2,1,3) = 2$ and $t(1,1,1) = 1$. We conclude that $t(1',1',2') = 1'$ and in the same way that $t(1,1,2)=1$. Hence β) is proved.

Now we assume that the variety generated by **A** is congruence distributive. Then there are ternary terms $t_0,\dots,t_k$ with $t_0(x,y,z) = x$, $t_k(x,y,z) = z$ $t_i(x,y,z) = x$ $(0 \le i \le k)$, $t_i(x,x,z) = t_{i+1}(x,x,z)$ for i even and $t_i(x,z,z) = t_{i+1}$ for i odd. Because of the implications α) and β) we have $t_{k-1}(1,2,2) = 1$ for $k-1$ odd or respectively $t_k(1,1,2) = 1$ for $k-1$ even. This contradicts the conditions for congruence distributivity.

Corollary 4.4: If **A** is a sop algebra with a zig zag (D,D'), $n \ge 3$, as a subalgebra then the variety generated by **A** has no near unanimity term.

Lemma 4.5: Let $<$ be a bounded strict order on A with a maximal chain $0 -\!\!< 1 -\!\!< 2 \dots -\!\!< n$. If **A** generates a variety with n–permutable congruences then there exist ternary terms $p_0,p_1,\dots,p_n$ such that $p_0(x,y,z) = x$, $p_n(x,y,z) = z$ and $p_i(x,x,y) = p_{i+1}(x,y,y)$. For $x,y \in \{0,\dots,n\}$ and $n > x > y$ we have $p_i(x,x,y) = p_{i+1}(x,y,y) < p_{i+1}(x+1,x+1,y+1)$. Hence we have
$1 = p_0(1,1,0) < p_1(2,2,1) < p_2(3,3,2) < \dots < p_{n-1}(n-2,n-2,n-1) =$
$p_n(n-2,n-1,n-1) = n-1$. This contradicts $\ell(1,n-1) = n-2$.

Lemma 4.6: Let $<$ be a bounded strict order on $A = \{0,1,\dots,n\}$ with a maximal chain $0 -\!\!< 1 -\!\!< 2 -\!\!< \dots -\!\!< n$. If **A** is a sop algebra with $T(\mathbf{A}) = \mathrm{Pol} <$ then the variety generated by **A** is $(n+1)$–permutable.

Proof: Let $A_i = \{(t,s,s) \mid s = i-1, t \ge 1\}$ and $B_i = \{(u,u,w) \mid u = n-1, w \ge n-i+1\}$. By lemma 2.5 the function

$$p_i(x,y,z) = \max\{g_c^r(x,y,z) \mid c = (c_1,c_2,c_3) \in A_i \cup B_i,\ r = \max\{c_1,c_3\}\}$$

is a term function of **A** for $i = 1,\dots,n$. This function p_i has the following properties

$$p_i(x,y,y) = \begin{cases} x & x > y \geq i-1 \qquad 1) \\ y & x > x > n-1 \qquad 2) \\ \min\{x,y\} & \text{else} \qquad 3) \end{cases}$$

$$p_i(x,x,y) = \begin{cases} x & x > y > i-1 \qquad 4) \\ y & y > x \geq n-i \qquad 5) \\ \min\{x,y\} & \text{else} \qquad 6) \end{cases}$$

0) We note that for $x \lesssim y$ we have $x \lesssim p_i(x,y,y)$, $p_i(x,x,y) \lesssim y$ as p_i is a term function or respectively for $y \lesssim x$, $y \in p_i(x,y,y)$, $p_i(x,x,y) \lesssim x$.

1) For $x > y \geq i-1$ there exists $(t,s,s) \in A_i$ such that $((t = x)$ and $(s = y))$ or $((t = x-1)$ and $(s < y))$. Hence we have $g_{(\overset{t}{t},s,s)}\,(x,y,y) = x \lesssim p_i(x,y,y)$ which implies $p_i(x,y,y) = x$ by 0).

2) For $x > y \geq n-i$ there exists $(u,u,w) \in B_i$ such that $u < x$ and $w = y-1$. Therefore $g_{(\overset{w}{u},u,w)}\,(x,y,z) = y \lesssim p_i(x,y,y)$ which implies $p_i(x,y,z) = x$ by 0).

3) If neither $x > y \geq i-1$ nor $y > x > n-1$ holds then there exists no element $c \in A_i \cup B_i$ such that $(x,y,y) \geq c$. Then $g_i^r(x,y,y) = \min\{x,y\}$ for every $c \in A_i \cup B_i$. Hence $p_i(x,x,y) = \min\{x,y\}$.

4) For $x > y \geq i-1$ there exists (t,s,s) sucht that $t = x-1$ and $s < y$. This implies $g_{(\overset{t}{t},s,s)}\,(x,x,y) = x \lesssim p_i(x,x,y)$ which implies $p_i(x,x,y) = x$ by 0).

5) For $x > y \geq n-1$ there exists $(u,u,w) \in B_i$ such that $((x = u)$ and $(y = w))$ or $((u < x)$ and $(w = y-1))$. Hence we have $g_{(\overset{w}{u},u,w)}\,(x,x,y) = y \lesssim p_i(x,x,y)$ which implies $p_i(x,x,y) = y$.

6) If neither $x > y > i-1$ nor $y > x \geq n-1$ hold then there exists no element $c \in A_i \cup B_i$ such that $(x,y,y) \geq c$. Then $g_c^r(x,y,y) = \min\{x,y\}$ for every $x \in A_i \cup B_i$. Hence $p_i(x,x,y) = \min\{x,y\}$.

Theorem 4.7: Let $<$ be a bounded strict order on $A = \{0,1,\dots,n\}$ with a maximal chain $0 -\!< 1 -\!< 2 -\!< \dots -\!< n$. If **A** is a sop algebra with $T(\mathbf{A}) = \text{Pol}<$ then the variety generated by **A** is $(n+1)$–permutable but not n–permutable.

G. Grätzer asks for examples of varieties which show that n–permutability and (n+1)–permutability are not equivalent. (Schmidt[7]) The above theorem provides a new series of such examples.

5. A duality for a finite sop algebra

In following we use notions and methods which were developed in B. Davey and H. Werner [3]. We consider the finite sop algebra $\mathcal{A}$ where $(A;\leq)$ is the strict order induced by a k–elementary chain. Since $\mathcal{A}$ has a ternary near–unanimity term, the NU–duality theorem from [3] guarantees that the prevariety generated by $\mathcal{A}$ has a duality given by relations of arity at most two. Theorem 4.1 isolates an appropriate set of relations.

Theorem 5.1: Let $<$ be the strict order which is induced from a k–element chain, $A = \{0,...,k-1\}$, $\mathcal{A} = (A,F)$ with $T(\mathcal{A}) = \mathrm{Pol}<$, $\mathcal{L} = \mathrm{ISP}(\mathcal{A})$,

$$\mathcal{A} = (A;\tau,<,<\circ<,...,\underbrace{<\circ<\circ\ldots\circ<}_{k-1},0,1,...,k-1)$$

and $\mathcal{X}_0 = \mathrm{ISP}(\mathcal{A})$. Then the protoduality is a full duality between $\mathcal{L}$ and $\mathcal{X}_0$.

Proof. By [3] (Davey, Werner) we have to show that $\mathcal{A}$ is injective in $\mathcal{X}_0$ (INJ) and fulfill the condition (E3F).

(INJ) Let $\mathcal{X} \subseteq \mathcal{A}^n$ and $\varphi : \mathcal{X} \longrightarrow \mathcal{A}$ a morphism. Then φ preserves the relations $<,<\circ<,...,\underbrace{<\circ<\circ\ldots\circ<}_{k-1},0,1,...,k-1$. Note that $(\bar{x},\bar{y}) \in \underbrace{<\circ\ldots\circ<}_{i}$ iff $\ell(\bar{x},\bar{y}) \geq i$ in $(P^n,<)$. Hence φ has the following properties

i) for $\bar{x} = (x_1,...,x_n) \in X$ we have
if $\min\{x_1,...,x_n\} = x_i \neq 0$ then $x_i-1 = \varphi(x_i-1,\ x_i-1,...,x_i-1) < \varphi(x_1,...,x_n)$.
Hence $\min\{x_1,...,x_n\} \leq \varphi(x_1,...,x_n)$. Similarly we get $\varphi(x_1,...,x_n) \leq \max\{x_1,...,x_n\}$.

ii) for $\bar{x} = (x_1,...,x_n)$, $\bar{y} = (x_1,...,x_n) \in X$, $\bar{x} < \bar{y}$ and $\ell(\bar{x},\bar{y}) = r$ in $(P^n,<)$ we have $(\bar{x},\bar{y}) \in \underbrace{<\circ\ldots\circ<}_{r}$ and hence $\varphi(x_1,...,x_n)+r \leq \varphi(y_1,...,y_n)$.

Consequently we can extend φ by Lemma 2.5 to a length function $\psi : A^n \longrightarrow A$ in the following way

$$\psi(\bar{x}) = \begin{cases} \varphi(\bar{x}) & \text{for } \bar{x} \in X \\ \max\{\ell(\bar{y},\bar{x})+\varphi(\bar{y}),\ \min\{x_1,\dots,x_n\}\} & \text{for } \bar{x} \notin X \text{ and } \exists \bar{y} \in X : \bar{y} < \bar{x} \\ \min\{x_1,\dots,x_n\} & \text{otherwise} \end{cases}.$$

E3F) Let $X \subset Y \subseteq A^n$, $a \in Y \setminus X$. Then the following morphisms exist

$$\varphi,\psi : Y \longrightarrow A \text{ with } \varphi(\bar{x}) = \begin{cases} \min\{x_1,\dots,x_n\} & \text{for } \bar{x} \ngeq a \\ \max\{x_1,\dots,x_n\} & \text{for } \bar{x} \geq a \end{cases}$$

$$\psi(\bar{x}) = \begin{cases} \min\{x_1,\dots,x_n\} & \text{for } \bar{x} \ngtr a \\ \max\{x_1,\dots,x_n\} & \text{for } \bar{x} > a \end{cases}$$

such that $\varphi \restriction X = \psi \restriction X$ but $\varphi \neq \psi$.

Remark: Because we have a ternary near unanimity function h with $h(x,y,z) = \max\{\min\{x,y\}, \min\{x,z\}, \min\{y,z\}\}$ in $T(\mathcal{A})$ the variety $V(\mathcal{A})$ is congruence distributive. Furthermore $\mathcal{A}$ has only simple subalgebras. Hence by the theorem von B. Jonsson we obtain $V(\mathcal{A}) = ISP(\mathcal{A})$. i.e. the full duality of Theorem 4.1 for the quasi-variety generated by $\mathcal{A}$ is also a full duality for $V(\mathcal{A})$.

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