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Introduction.

Some aspects of order–polynomially complete lattices have been discussed in detail: characterization by maps (R. Wille) and tolerances (M. Kindermann), finiteness (H. Kaiser, N. Sauer), duality and categorical properties (B. Davey, R. Quakenbush, et. al) and questions of spectra (R. McKenzie, R. Quakenbush). In this paper we are concerned with the representation of monotone functions $f:L \longrightarrow K$ of an order–polynomially complete lattice L into lattices K, where L is a sublattice of K. We prove that the finite lattice L is order–polynomially complete if and only if $Mon(L,K) \simeq P(K)$ for every $K \in HSP(L_c)$ where L_c is the lattice together with the constants as operations.

We are indebted to Magdalena Szymanska for a decisive hint. For the convenience of the reader most of the proofs are given in details.

1. Notations

For a given algebra $\underset{\sim}{A} = (A,\Omega)$ the term functions of $\underset{\sim}{A}$ are generated by the projections on A and the operations $\omega \in \Omega$. The polynomial functions of A are generated by the term functions and the constant functions $c_a(x) \equiv a$. An n–ary relation ρ on A is called compatible if $(a_{11},\dots,a_{1n}) \in \rho,\dots,(a_{k1},\dots,a_{kn}) \in \rho$ implies $(t(a_{11},\dots,a_{k1}),\dots,t(a_{1n},\dots,a_{kn})) \in \rho$ for every k–place term function of A. Our main topic concerns lattices and the compatible relation in question is the order relation. The order relation of a lattice L is not only preserved by its term functions but also by its polynomial functions.

For every subalgebra $\underset{\sim}{B} = (B,\Omega)$ of an algebra $\underset{\sim}{A} = (A,\Omega)$ we consider the algebra $F_\rho(B,A)$ of all functions $f:B \longrightarrow A$ which preserve the compatible relation ρ of A. The operations $\omega \in \Omega$ are defined pointwise for the functions $f_1,\dots,f_k \in F_\rho(B,A)$ namely $\omega(f_1,\dots,f_k)(x) = \omega(f_1(x),\dots,f_k(x))$ in $\underset{\sim}{A} = (A,\Omega)$. If ρ is the all relation we just denote by

F(B,A) the algebra of all functions $f:B \longrightarrow A$. In the case of lattices we write Mon(B,A) for the order–preserving, i.e. monotone, functions $f:B \longrightarrow A$ from the sublattice B into the lattice A.

We compare the algebra $F_\rho(B,A)$ with the algebra P(A) of all 1–place polynomial functions of $\underset{\sim}{A}$. The algebra $\underset{\sim}{A}$ is called polynomially complete if F(A,A) = P(A). The lattice $(L;\wedge,\vee)$ is called order–polynimially complete if Mon(L,L) = P(L). For functions in k variables, respectively for polynomial functions with k variables we use $F_\rho(B^k,A)$ and $P^k(A)$ respectively. For more details one should consult Lausch–Nöbauer [L–N 73].

2. Embeddings

Throughout the following we assume that order–polynomially complete lattices are finite.

Theorem 2.1. Let L be a sublattice of K such that $1_L = 1_K$ and $0_L = 0_K$ for the one element and the zero element respectively. If L is order–polynomially complete then there exist an injective mapping

$$\varphi : \mathrm{Mon}(L,K) \longrightarrow P(K).$$

Proof. We define a monotone function χ_a for every $a \in L$ with $\chi_a(x) = 1$ for $x \geq a$ and $\chi_a(x) = 0$ else. As L is order–polynomially complete χ_a can be represented as a polynomial over L and furthermore as L is a sublattice also over K. For a given function $f:L \longrightarrow K$ we have hence a polynomial function p_f presented by $p_f(x) = \bigvee_{a \in L} (\chi_a(x) \wedge f(a))$ a polynomial over K. If we put $\varphi(f) = p_f$ then φ is well defined. For $\varphi(f) = \varphi(g)$ we conclude $f(a) = g(a)$ for every $a \in L$ and hence φ is injective.

The mapping above fullfills the following equation

$$\varphi(f)(a) = f(a) \qquad \text{for all } a \in L.$$

The proof of theorem 2.1 shows that under the above conditions for every monotone function $f:L \longrightarrow K$ there exists a polynomial of K in a canonical normal form. This extends a result on normal forms for every polynomial (also in several variables) over an order–polynomially complete lattice as already pointed out in Schweigert [Sch 74]. In this respect order–polynomially complete lattices behave very much like distributive lattices which have disjunctive normal forms.

Under which conditions is the above mapping an isomorphism? Considering the proof we would need for the polynomial functions $p,q \in P(K)$ that $p(b) = q(b)$ for every $b \in K$ if and only if $p(a) = q(a)$ for every $a \in L$. With other words a polynomial function of K should already be determined by its values of L.

Proposition 2.2. Every polynomial function p of a bounded distributive lattice is already determined by the value $p(0)$ and $p(1)$.

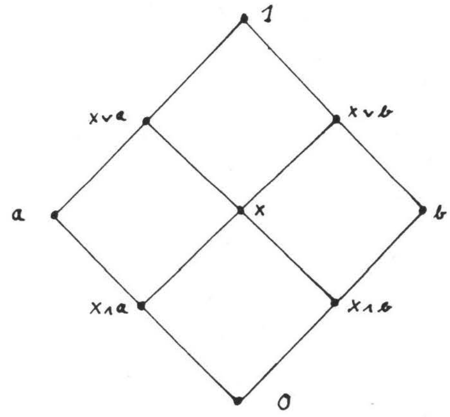
Proof. Every polynomial $p(x)$ of a distributive lattice can be put in the form $p(x) = (x \wedge c) \vee d$, with $d \leq c$. We have $p(0) = d$ and $p(1) = c$.

The immediate consequence is the

Theorem 2.3. Let D_2 be the two element lattice and D any bounded distributive lattice

$$Mon(D_2,D) \simeq P(D).$$

For the convenience of the reader we illustrate the result for a small example. Let $D_2 = \{0,1\}$ the two–element chain. We denote D_2^2 by $D = \{0,a,b,1\}$. The monotone functions in $Mon(D_2,D)$ are presented by the image of (0,1) as (0,0), (a,a), (b,b), (1,1), (0,1), (0,a), (0,b), (a,1), (b,1). P(D) consists of all polynomial functions of the form $f_{cd}(x) = (x \wedge c) \vee d$ with $d \leq c$. The corresponding values of (d,c) are (0,0), (a,a), (b,b), (1,1), (0,1), (0,a), (0,b), (a,1), (b,1). Using polynomials of K we have that $Mon(D_2,D)$ is isomorphic to the following lattice $P(D_2^2)$.



3. Representation theorems

Proposition 3.1. Let the polynomial function p_i of K_i be determined by the values of the sublattice L of K_i. Then every polynomial function p of $\prod_{i \in I} K_i$ is determined by its values on L.

Proof. Let Π_i be the ith projection of $\prod_{i \in I} K_i$. Every polynomial p(x) of $\prod_{i \in I} K_i$ has a polynom $p_i(x)$ of K_i which is the ith projection of p(x). By hypothesis all p_i are determined by the values of L in K_i. Hence also p is determined by the values of L.

Proposition 3.2. Let the polynomial functions p of K are determined by the sublattice L of K. Let M be a sublattice of K containing the sublattice L. Then every polynomial function of M is determined by the values of L.

Proof. A polynomial p(x) of M is also polynomial of K. Hence the statement holds.

Proposition 3.3. Let M be an isomorphic image of K such that L is a sublattice of M as well as of K. If every polynomial function p of K is determined by its value on L then also every polynomial function of M by its value on L.

Proof. If $q_1(x)$ and $q_2(x)$ are polynomials of M then there exists $p_1(x)$ and $p_2(x)$ of K with φ':K[x] $\longrightarrow$ M[x] $\varphi'(p_i(x)) = q_i(x)$, i = 1,2 a surjective lattice homomorphism with $\varphi|_L = \mathrm{id}$. Assume that $q_1(a) = q_2(a)$ for every $a \in L$. Then $p_1(a) = p_2(a)$ for every $a \in L$ and hence $\varphi'(p_1(x)) = \varphi'(p_2(x))$ and $q_1(x) = q_2(x)$.

Notation. L_c is the lattice $(L;\wedge,\vee,c_a(a \in L))$ where every constant function c_a is added as operation. Especially we require that $c_1 \equiv 1$ and $c_0 \equiv 0$.

From the above proposition it follows

Proposition 3.4. Let L be a sublattice of K and φ the following homomorphism

$$\varphi: P(K) \longrightarrow \mathrm{Mon}(L,K)$$
$$p \longmapsto p|_L$$

Then φ is injective for every $K \in ISP(L_c)$.

Theorem 3.5. Let L be a sublattice of K and let φ be the above homomorphism. If φ is injective then $K \in HSP(L_c)$.

Proof. If φ is injective then we will show by induction that $\varphi_n: P^n(K) \longrightarrow \mathrm{Mon}(L^n,K)$ is injective for every $n \in \mathbb{N}$. Let us assume that we have $p(a_1,\dots,a_n) = q(a_1,\dots,a_n)$ for $a_i \in L$,

$i = 1,\dots,n$ and $p,q \in P^n(K)$. We conclude $p(a_1,b_2,\dots,b_n) = q(a_1,b_2,\dots,b_n)$ for every $b_i \in K$, $i = 1,\dots,n$ and furthermore $p(b_1,\dots,b_n) = q(b_1,\dots,b_n)$ for every $b_1 \in K$ using induction. Hence every equation which holds for L does so for K.

Theorem 3.6. Let L be a simple sublattice of K and let φ be the above homomorphism

$$\varphi{:}P(K) \longrightarrow Mon(L,K)$$
$$P \longmapsto p|_L.$$

Then φ is injective if and only if $K \in HSP(L_c)$.

Proof. Obviously $HSP(L_c)$ is congruence distributive and hence we have according to [D–Du–Qu–R 78] that $HSP(L_c) = ISPHS(L_c)$. As L_c is simple and has no proper subalgebra it follows $HSP(L_c) = ISP(L_c)$.

As order–polynomially complete lattices are simple we have the following result.

Theorem 3.7. Let L be a finite lattice. L is order–polynomially complete if and only if

$$Mon(L,K) \simeq P(K)$$

for every $K \in HSP(L_c)$.

In our proofs we have used no properties which are specific only for lattices.

Theorem 3.8. Let L be a finite algebra. L is polynomially complete if and only if

$$F(L,K) \simeq P(K)$$

for every $K \in HSP(L_c)$.

H. Werner shows in his paper [W 70] p.382 that any function $f{:}L \longrightarrow L$ of a polynomially complete algebra L can be presented by the characteristic function χ_a with $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise and a multiplication. The analogue arguments prove now the theorem 3.8.

4. Monotone functions of several variables

To extend the results for more than one variable we need some technical propositions.

Proposition 4.1. Let $(P,\leq)$, $(P_1,\leq)$, $(P_2,\leq)$ be ordered sets.

(4.1.1) $\quad Mon(P,P_1{\times}P_2) \simeq Mon(P,P_1) \times Mon(P,P_2)$

(4.1.1) $\quad \mathrm{Mon}(P,P_1\times P_2) \simeq \mathrm{Mon}(P,P_1) \times \mathrm{Mon}(P,P_2)$

(4.1.2) $\quad \mathrm{Mon}(P_1\times P_2,P) \simeq \mathrm{Mon}(P_1,\mathrm{Mon}(P_2,P))$

If these ordered sets are lattices then the order–isomorphism is also a lattice–isomorphism.

The proof is by direct calculation.

<u>Proposition</u> <u>4.2.</u> If $(L;\wedge,\vee)$ and $(K;\wedge,\vee)$ are lattices with 0 and 1 then for $n \in \mathbb{N}$

$$P^n(L) \times P^n(K) \simeq P^n(L \times K).$$

<u>Proof.</u> We define $\varphi : P^n(L) \times P^n(K) \longrightarrow P^n(L\times K)$ by
$\varphi(p,q)((x_1,y_1),\dots,(x_n,y_n)) := (p(x_1,\dots,x_n),q(y_1,\dots,y_n))$.
If $p \in P^n(L)$ and $q \in P^n(K)$ then $(p,q) = (p,0) \vee (0,q)$. But $(p,0) \in P^n(L,K)$ in any case:
If $p \equiv a \in L$ then $(a,0) \in P^n(L\times K)$.
If $p \equiv x_i$ then $(x_i,0) = (x_i,x_i) \wedge (1,0)$ is a polynomial.
If $p = p_1 \vee p_2$ then $(p,0) = (p_1,,0) \vee (p_2,0)$ and if $p = p_1 \wedge p_2$ then $(p,0) = (p_1,0) \wedge (p_2,0)$ are polynomials over $L\times K$. It is easy to check that φ is a lattice isomorphism.

<u>Proposition</u> <u>4.3.</u> If $(L;\wedge,\vee)$ is a lattice then

$$P^n(L) \simeq P(P^{n-1}(L)).$$

<u>Proof.</u> We define $\varphi : P(P^{n-1}(L)) \longrightarrow P^n(L)$ by

$$\varphi(p)(a_1,\dots,a_n) := p(a_1)(a_2,\dots,a_n).$$

φ is well defined because $\varphi(x) = x_1 \in P^n(L)$ and for $p \in P^{n-1}(L)$ we have

$$\varphi(p)(a_1,\dots,a_n) = p(a_2,\dots,a_n)$$

and hence $\varphi(p) \in P^n(L)$. φ is a homomorphism as

$$\begin{aligned}\varphi(p\vee q)(a_1,\dots,a_n) &= (p \vee q)(a_1)(a_2,\dots,a_n)\\ &= (p(a_1) \vee q(a_1))(a_2,\dots,a_n)\\ &= (p(a_1)(a_2,\dots,a_n)) \vee (q(a_1)(a_2,\dots,a_n))\\ &= (\varphi(p) \vee \varphi(q))(a_1,\dots,a_n).\end{aligned}$$

φ is injective. Consider $\varphi(p) = \varphi(q)$ and hence $p(a_1)(a_2,\dots,a_n) = q(a_1)(a_2,\dots,a_n)$. We have that $p(a_1) = q(a_1)$ for every $a_1 \in L$. Let $r \in P^{n-1}(L)$. Then

$$\begin{aligned}p(r)(a_2,\dots,a_n) &= p(r(a_2,\dots,a_n),a_2,\dots,a_n)\\ &= q(r(a_2,\dots,a_n),a_2,\dots,a_n) = a_1(r)(a_2,\dots,a_n).\end{aligned}$$

Hence we have $p = q$. Obviously φ is surjective.

<u>Theorem</u> <u>4.4.</u> If $(L;\wedge,\vee)$ is a finite order–polynomially complete lattice then for

every $K \in HSP(L_c)$ and for every $n \in \mathbb{N}$

$$Mon(L^n,K) \simeq P^n(K).$$

Proof by induction on n.

For $n = 1$ we use theorem 3.4 and for $n > 1$ we have that

$Mon(L^n,K) \simeq Mon(L,Mon(L^{n-1},K))$ (4.1.2)

$\simeq Mon(L,P^{n-1}(K))$ by induction

$\simeq P(P^{n-1}(K))$ by theorem 3.4

$\simeq P^n(K)$ by proposition 4.3.

5. Some remarks

For which lattices L,K,M do we have $Mon(L,K) \simeq P(M)$? For instance, if C_3,C_4 is the three–element, respectively four–element chain then $Mon(C_3,C_3) \simeq P(C_4)$. This is only a lattice homomorphism whereas in the sections before also the composition of functions was preserved. In such a case it may happen that a constant function is mapped to the identity function. Of more interest is the following technical result: $Mon(P(L),P(L)) \simeq P(K)$ where $K = Mon(P(L),L)$.

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