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Order–semi–primal lattices

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A lattice $(L;\wedge,\vee)$ is called order–polynomially complete if every order–preserving function $f:L \longrightarrow L$ is a polynomial function. Order–polynomially complete lattices can be characterized in many ways. They are lattices with only trivial tolerances. Finite geometric lattices are order–polynomially complete if and only if they are projective geometries. [Sch74]
In this note we analyze the term functions of a (finite) order–polynomially complete lattice using the concept of semi–primality due to Pixley [P70].

Throughout we consider a finite lattice $\underset{\sim}{L} = (L;\wedge,\vee)$ with $L = \{a_1,\ldots,a_n\}$.

<u>Definition</u> <u>1.</u> Let $\underset{\sim}{A} = (A,\Omega)$ be a finite nontrivial algebra. A function $f:A^k \longrightarrow A$, $k \in \mathbb{N}$, is conservative if every subalgebra $\underset{\sim}{S} = (S,\Omega)$ of $\underset{\sim}{A}$ is preserved by f; i.e. $f[S] \subseteq S$. $\underset{\sim}{A}$ is called semi–primal if every conservative function is a term function of A.

We slightly modify this concept

<u>Definition</u> <u>2.</u> Let $\underset{\sim}{L} = (L;\wedge,\vee)$ be a finite lattice. $\underset{\sim}{L}$ is called order–semi–primal if every monotone conservative function of L is a term function of $\underset{\sim}{L}$.

<u>Notation.</u> Let $b \in L$ be a fixed element. We write $\bar{b} := (a_1,\ldots,a_n,b)$. If $\bar{c} = (c_1,\ldots,c_n,c_{n+1}) \geq \bar{b}$ then $a_i \geq c_i$ for $i = 1,\ldots,n$ and $c_{n+1} \geq b$. Then at least one of the components of $\bar{c}$ has to be 1. A tolerance ρ of a lattice $(L;\wedge,\vee)$ is a binary relation $\rho \subseteq L^2$ which is reflexive, symmetric and compatible with the operations $\wedge$ and $\vee$. A lattice L is called tolerance simple [Ch91] if L has only the trivial tolerance namely the identity relation and the all relation.

<u>Lemma</u> <u>3.</u> The following function $d_b : L^{n+1} \longrightarrow L$ for $b \in L$ is monotone and preserves all sublattices.

$$d_b(x_1,\ldots,x_n,x_{n+1}) = \begin{cases} 1 & \text{for } \bar{x} \geq \bar{b} \\ x_1\wedge\ldots\wedge x_n\wedge x_{n+1} & \text{else} \end{cases}.$$

Proof. d_b is monotone.
Let $\bar{a} \leq \bar{c}$ and let $d_b(\bar{a}) = 1$. Then $\bar{a} \geq \bar{b}$ and hence $\bar{c} \geq \bar{b}$. Therefore $d_b(\bar{c}) = 1$.
d_b preserves the sublattice $\underset{\sim}{S}$ of $\underset{\sim}{L}$.

Let $\bar{c} = (c_1,\dots,c_n,c_{n+1}) \in S$.
Case 1. $\bar{c} \geq \bar{b}$.
Hence $1 \in S$ and $d_b(\bar{c}) \in S$.
Case 2. $\bar{c} \not\geq \bar{b}$.
Now $\bar{c} = c_1 \wedge \dots \wedge c_n \wedge c_{n+1} \in S$ and hence $d_b(\bar{c}) \in S$.

Lemma 4. A finite order–semi–primal lattice is tolerance simple.

Proof. Let ρ be a non trivial tolerance with $(a,b) \in \rho$ such that $a < b$. We consider $(a_1,a_1) \in \rho,\dots,(a_n,a_n) \in \rho$, $(a,b) \in \rho$. Hence we have $d_b(a_1,\dots,a_n,a) = 0$. As d_b is a term function we have $(0,1) \in \rho$. Hence ρ is the all relation.

Lemma 5. A finite tolerance simple lattice $\underset{\sim}{L} = (L;\wedge,\vee)$ is order–semi–primal.

Proof. In [W77] it is shown that a monotone, conservative function $f:L^k \longrightarrow L$ is a term function if $\alpha(f(x_1,\dots,x_k) \leq f(\alpha(x_1),\dots,\alpha(x_k))$ for every $\vee$–preserving map $\alpha:L \longrightarrow L$ with $\alpha(x) \leq x$ for $x \in L$. For a tolerance simple lattice the only $\vee$–preserving maps α with $\alpha(x) \leq x$ are the identity and $\alpha(x) \equiv 0$ for $x \in L$. Hence L is order–semi–primal.

Together we have the following

Theorem 6. A finite lattice L is order–semi–primal if and only if L is tolerance simple.

In special cases one can also use other than conservative functions to describe the term functions of tolerance simple lattices. Let $\underset{\sim}{L} = (L;\wedge,\vee)$ be a distributive lattice with $|L| \geq 2$. L is semi–order–primal if and only if every idempotent function $f:L^2 \longrightarrow L$ is a term function. It should be also noted that there exists a local version of our results.

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