UNIVERSITÀ DEGLI STUDI
DI GENOVA

Dipartimento di Matematica
Corso di Dottorato di Ricerca
in Matematica e Applicazioni
ciclo XXX

Fachbereich Mathematik
Graduate School

Tesi di Dottorato

Dissertation

Im Rahmen einer Doppelpromotion mit der Universitá degli Studi di Genova vom Fachbereich Mathematik der Technischen Universität Kaiserslautern zur Erlangung des akademischen Grades Doktor der Naturwissenschaften (Dr. rer. nat.) genehmigte Dissertation

Two instances of duality in commutative algebra

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Defense date

December 15, 2017

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Abstract

In this thesis we address two instances of duality in commutative algebra.

In the first part, we consider value semigroups of non irreducible singular algebraic curves and their fractional ideals. These are submonoids of $\mathbb{Z}^n$ closed under minima, with a conductor and which fulfill special compatibility properties on their elements. Subsets of $\mathbb{Z}^n$ fulfilling these three conditions are known in the literature as good semigroups and their ideals, and their class strictly contains the class of value semigroup ideals. We examine good semigroups both independently and in relation with their algebraic counterpart. In the combinatoric setting, we define the concept of good system of generators, and we show that minimal good systems of generators are unique. In relation with the algebra side, we give an intrinsic definition of canonical semigroup ideals, which yields a duality on good semigroup ideals. We prove that this semigroup duality is compatible with the Cohen-Macaulay duality under taking values. Finally, using the duality on good semigroup ideals, we show a symmetry of the Poincaré series of good semigroups with special properties.

In the second part, we treat Macaulay's inverse system, a one-to-one correspondence which is a particular case of Matlis duality and an effective method to construct Artinian $\mathbf{k}$-algebras with chosen socle type. Recently, Elias and Rossi gave the structure of the inverse system of positive dimensional Gorenstein $\mathbf{k}$-algebras. We extend their result by establishing a one-to-one correspondence between positive dimensional level $\mathbf{k}$-algebras and certain submodules of the divided power ring. We give several examples to illustrate our result.

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Derzeitige Beschäftigung

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Layout of the thesis

First we introduce the notations used throughout the dissertation, with references to the definitions in the text. Then the thesis is divided in two independent parts. Each part has an introduction to the concerned topic, which contains motivation, overview of the literature, and a summary of the results.

The first part illustrates the work done in the first two years of PhD in Kaiserslautern and treats value semigroups of one-dimensional semilocal Cohen–Macaulay rings and their combinatorial counterpart, i.e. good semigroups. It contains parts of two papers, [KST17] and [DGSMT17], in which the candidate was coauthor. The proofs included in this part, unless clearly stated, are original work of the author. The main results are the existence and uniqueness of a good generating system for good semigroup ideals, the compatibility of the dual operation with taking values and the symmetry of the Poincaré series of a good semigroup.

The second part illustrates the work done in the last year of PhD in Genoa and treats a generalization of the classical Macaulay inverse system to level k-algebras of every dimension. This is a joint work with S. Masuti, [MT17], and the proofs here contained are the outcome of common efforts. The main result is a one-to-one correspondence between level local algebras and particular submodules of the divided power ring.

The appendix contains a collection of basic facts used to prove the results of the thesis, in order to make the manuscript self-contained. The appendix does not contain original work.

Notations

Throughout this thesis we will use the following notations.

- General:
  - ◇ If T is an ordered set, $\min\{T\}$ denotes the minimum element of T;
  - ◇ if T is an ordered set, $\max\{T\}$ denotes the maximum element of T;
  - ◇ i, j, k are indices in $\mathbb{N}$;
  - ◇ $\underline{i}, \underline{j}, \underline{k}, \underline{n}$ are multi-indices in $\mathbb{N}^l$ for some l.
- For commutative unitary rings and their ideals:
  - ◇ $\mathbf{k}$ is a field;
  - ◇ R is a ring. If R is local (resp. *local) then $\mathfrak{m}$ is the (resp. homogeneous) maximal ideal;
  - ◇ $I, J \subsetneq R$ are ideals of R;
  - ◇ $A = R/I$ is the quotient ring of R by the ideal I. If A is local (resp. *local), then $\mathfrak{n}$ is its (resp. homogeneous) maximal ideal;
  - ◇ $\mathrm{Max}(R)$ is the set of maximal ideals of R;
  - ◇ Q_R is the total ring of fractions of R (Definition A.3);
  - ◇ $\overline{R}$ is the integral closure of R inside Q_R (Definition A.4);
  - ◇ T^{reg} is the set of all regular elements of T for any subset T of Q_R (Notation A.1);
  - ◇ $\mathcal{E}, \mathcal{F} \subseteq Q_R$ are (regular) fractional ideals of R (Definition A.5.(a));
  - ◇ $\mathcal{C}_{\mathcal{E}} = R :_{Q_R} \mathcal{E}$ is the conductor of $\mathcal{E}$ relative to R (Definition A.5.(d));
  - ◇ $\mathfrak{R}_R$ is the set of all regular fractional ideals of R (Notation A.6);
  - ◇ $\mathfrak{R}_R^*$ is the set of invertible R-submodules of Q_R (Lemma A.11);
  - ◇ $\mathcal{D}$ is the divided power ring (Equation (6.1));
  - ◇ ω_R is the canonical module of R (Definition E.2);
  - ◇ $\mathcal{K}$ is a canonical ideal of R (Definition E.10);
  - ◇ if A is Artinian, $\mathrm{Soc}(A)$ is the socle of A, and $\mathrm{socdeg}(A)$ is the socle degree of A (Definition 6.25).
- For valuations:
  - ◇ V is a valuation ring of Q, where Q is a ring with large Jacobson radical (see Definition B.1) which is its own ring of quotients (Definition B.4);
  - ◇ $\mathfrak{m}_V$ is the regular maximal ideal of the valuation ring V (Definition B.4);

- $\diamond$ $I_V = V : Q$ is the infinite prime ideal of the valuation ring V (Definition B.12);
- $\diamond$ ν_V is the valuation associated to the valuation ring V (Definition B.7);
- $\diamond$ $\mathfrak{V}_R$ is the set of all valuation rings of Q_R over a ring R (Definition 1.1);
- $\diamond$ V_ν is the valuation ring associated to the valuation ν (Definition B.9);

- For modules over a ring R:
  - $\diamond$ M, N are R-modules;
  - $\diamond$ E is an injective R-module (Definition D.1);
  - $\diamond$ $\mathrm{Hom}_R(M, N)$ is the set of R-homomorphisms between M and N;
  - $\diamond$ $\mathrm{Ext}^k_R(M, N)$ is the k-th right derived functor of the left exact functor $T(N) = \mathrm{Hom}_R(M, N)$;
  - $\diamond$ $M \otimes_R N$ is the tensor product of M and N over R;
  - $\diamond$ If M has finite length, $\ell(M)$ is the length of M (Definition C.1);
  - $\diamond$ $HF_M(-)$ is the Hilbert function of M, and $HS_M(t)$ is the Hilbert series of M (Definition C.2);
  - $\diamond$ $\mathrm{depth}_R(M)$ is the depth of M as R-module (Definition C.6);
  - $\diamond$ $\tau(M)$ is the type of M as R-module;

- For semigroups:
  - $\diamond$ $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$;
  - $\diamond$ $\alpha, \beta, \delta, \gamma, \epsilon, \zeta$ are elements in $\mathbb{N}^s$;
  - $\diamond$ $S \subseteq \mathbb{N}^s$ is a semigroup, i.e. a subset of $\mathbb{N}^s$ closed under sum and containing 0;
  - $\diamond$ $D_S \subseteq \mathbb{Z}^s$ is the set of differences of S (Definition 1.16);
  - $\diamond$ E, F are (good) semigroup ideals of S (Definition 2.2);
  - $\diamond$ $\mathfrak{G}_S$ is the set of all good semigroup ideals of S (Notation 2.3);
  - $\diamond$ C_E is the conductor ideal of a semigroup ideal E (Definition 2.8);
  - $\diamond$ γ^E is the conductor of a semigroup ideal E (Definition 2.9);
  - $\diamond$ K is a canonical semigroup ideal of S (Definition 4.5);
  - $\diamond$ Γ_R (resp. $\Gamma_{\mathcal{E}}$) is the value semigroup of a ring R (resp. a fractional ideal $\mathcal{E}$ (Definition 1.4));

Part I

Value and good semigroups

Introduction

Value semigroups of curve singularities have been widely studied by several authors over the years. Waldi [Wal72, Wal00] showed that any plane algebroid curve is determined by its value semigroup up to equivalence in the sense of Zariski. Value semigroups do not reflect only the equivalence class of their corresponding ring, but also Gorensteinness. For this reason value semigroups are interesting objects.

We first show how value semigroups (see Definition 1.4) can be defined for *admissible* rings (see Definition 1.15), a class of rings which strictly contains algebroid curves. Then we give detailed proofs of their compatibility with localization, and results about their compatibility with completion. Afterwards, we concentrate on the axioms satisfied by value semigroups and their ideals, which define the class of *good semigroups* and their ideals (see Definition 2.1). These axioms were already considered in [BDF00b, CDGZ99, CDK94, D'A97, DdlM87, DdlM88, Gar82], but it was in [BDF00a] that the notion of good semigroup was defined and it was proved that not all good semigroups are value semigroups. Hence, such semigroups are relevant by their own and they form a natural generalization of numerical semigroups. However, they are harder to study, mainly because they are not finitely generated as monoids, and not closed under finite intersections. In spite of this, there are several approaches in the literature to describe good semigroups which are value semigroups of algebroid curves by means of a finite set of data. In [Gar82, Wal72], the authors describe the value semigroup of singularities with two branches through the finite set of maximal elements. This approach has been generalized to the case of more than two branches in [DdlM87]. An alternative can be found in [CDGZ99], where the authors introduce w-generators for planar algebroid curves: the value semigroup can be described by a finite set of these w-generators (not necessarily belonging to the semigroup) and a boolean expression. In [CF02], the authors compute the value semigroup of plane curves using Hamburger-Noether expressions. For the non planar case, we refer to [BDF00a, BDF00b, CDK94, DdlM87].

Our approach differs from the ones cited above, and takes advantage of the algebraic structure of good semigroups, therefore including the class of value semigroups. First we consider the set $\mathrm{Small}(S)$ of *small elements* of a semigroup S, that is, elements of S which are smaller or equal to the conductor of the semigroup with the usual partial order. It is easy to see that $\mathrm{Small}(S)$ determines the semigroup. Therefore, it is natural to consider subsets $G \subsetneq \mathrm{Small}(S)$, from which is possible to recover completely the semigroup S. We define such a subset G to be a *good generating system*. We call G *minimal* if none of its proper subsets is a good generating system. We prove that minimal generating systems are unique in the local case (Theorem 3.13), as happens in the setting of cancellative monoids. The same is not true in general for the non local case, but it is possible to reduce to the local case. We then prove that good semigroup ideals of good semigroups also can be minimally generated by a unique system of generators. In particular, this is true for value semigroups of fractional ideals. Also, we take inspiration from the work of Carvalho and Hernandes [CH17] to show that the closure of a good semigroup is always finitely generated as a semiring.

Good semigroups also have interesting duality properties. In the numerical case, corresponding to semigroup rings, Kunz [Kun70] was the first to show that the Gorensteinness of an analytically irreducible and residually rational local ring R corresponds to a symmetry of its numerical value semigroup Γ_R. Under the same assumptions, Jäger [Jäg77] used this symmetry to define a semigroup ideal K^0 such that (suitably normalized) canonical fractional ideals $\mathcal{K}$ of R are characterized by having value semigroup ideal $\Gamma_{\mathcal{K}} = K^0$. Waldi [Wal72] was the first to give a symmetry property for non-numerical semigroups, and he showed that value semigroups of plane curves with two branches are symmetric. García [Gar82], using a similar approach, defined the concept of symmetric points. In analogy with Kunz's result, Delgado [DdlM87] then proved that general algebroid curves are Gorenstein if and only if their (non-numerical) value semigroup is *symmetric*. Later Campillo, Delgado and Kiyek [CDK94] extended Delgado result to analytically reduced and residually rational local rings R with infinite residue field. D'Anna [D'A97] broadened Jäger's approach under the preceding hypotheses. He used the definition of symmetry given by Delgado to give an explicit formula for a semigroup ideal K^0 (see Definition 4.1) such that any (suitably normalized) fractional ideal $\mathcal{K}$ of R is canonical if and only if $\Gamma_{\mathcal{K}} = K^0$.

Afterwards, Barucci, D'Anna and Fröberg [BDF00a] included in their setup the case of semilocal rings, which are the objects considered in this manuscript. Recently Pol [Pol15, Theorem 2.4] gave an explicit formula for the value semigroup ideal of the dual of a fractional ideal for Gorenstein algebroid curves.

We extend and unify D'Anna's and Pol's results for admissible rings R. We give a simple definition of a *canonical semigroup ideal* K of a good semigroup (see Definition 4.5). It turns out that this definition is equivalent to K being a translation of D'Anna's K^0, and to K inducing a duality $E \mapsto K - E$ on good semigroup ideals, i.e. $K - (K - E) = E$ for any good semigroup ideals (see Corollary 4.13). In particular, D'Anna's characterization of canonical ideals in terms of their value semigroup ideals persists for admissible rings (see Corollary 4.17). We show that

$$\Gamma_{\mathcal{K}:\mathcal{E}} = \Gamma_{\mathcal{K}} - \Gamma_{\mathcal{E}}$$

for any regular fractional ideal $\mathcal{E}$ of R (see Theorem 4.16). This means that there is a commutative diagram

$$\begin{tikzcd}
\{\text{regular fractional ideals of } R\} \arrow[r, "\mathcal{E}\mapsto\mathcal{K}:\mathcal{E}"] \arrow[d, "\mathcal{E}\mapsto\Gamma_{\mathcal{E}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\text{regular fractional ideals of } R\} \arrow[d, "\mathcal{E}\mapsto\Gamma_{\mathcal{E}}"] \\
\{\text{good semigroup ideals of } \Gamma_R\} \arrow[r, "E\mapsto\Gamma_{\mathcal{K}}-E"'] & \{\text{good semigroup ideals of } \Gamma_R\}
\end{tikzcd}$$

relating the Cohen–Macaulay duality $\mathcal{E} \mapsto \mathcal{K} : \mathcal{E}$ on R to our good semigroup duality $E \mapsto K-E$ on Γ_R for $K = \Gamma_{\mathcal{K}}$.

Canonical ideals are not the only way to detect duality properties, for rings as well as for good semigroups. In [Sta77], the author showed that Gorenstein graded algebras have symmetric Hilbert series. In particular, this holds for semigroup rings which have symmetric value semigroup. Others studied the properties of the Hilbert series and modifications of it to understand properties of curves. Campillo, Delgado and Gusein-Zade in [CDGZ03] gave a definition of *Poincaré series* for a plane curve singularity, and they showed that it coincides with the Alexander polynomial, which is a complete topological invariant of the singularity. More recently, Poincaré series were studied in relation with value semigroups. Moyano-Fernandez in [MF15], using a definition inspired by the above, analyzed the connection between univariate and multivariate Poincaré series of curve singularities and later on, together with Tenorio and Torres [MFTT17], they showed that the Poincaré series associated with generalized Weierstrass semigroups can be used to retrieve entirely the semigroup. Then Pol [Pol16] considered a symmetry problem on Gorenstein reduced curves. She proved that the Poincaré series of the Cohen–Macaulay dual of a fractional ideal $\mathcal{E}$ is symmetric with respect to the Poincaré series of $\mathcal{E}$, therefore generalizing Stanley's result to fractional ideals of Gorenstein rings. Pol's result strongly uses the fact that it is always possible to define a filtration on value semigroups (see Definition 1.8), as done first in [CDK94]. To deal with this filtration an important tool is the *distance* $d(E\backslash F)$ between two good semigroup ideals $E \subseteq F$ (see Definition 2.23). Using the notion of distance and our new-found duality on good semigroups, we are able to show that under suitable assumptions the *Poincaré series* of the dual of a good semigroup E is symmetric to the Poincaré series of E. In particular, if $E := \Gamma_{\mathcal{E}}$ for some fractional ideal $\mathcal{E}$ of an admissible ring R, this symmetry is always true.

The contents of this part are divided as follows.

In Chapter 1 we review the definition of value semigroups and their ideals, based on the notion of valuation rings over a one-dimensional Cohen–Macaulay ring. We give a proof of the properties satisfied by value semigroups of local admissible rings (see Proposition 1.21), and we show that they are compatible with localization, i.e. for any $\mathcal{E} \in \mathfrak{R}_R$ there is a decomposition into value semigroup ideals

$$\Gamma_{\mathcal{E}} = \prod_{\mathfrak{m}\in\operatorname{Max}(R)} \Gamma_{\mathcal{E}_\mathfrak{m}}.$$

We recall results from [KST17] which show that value semigroups are also compatible with completion.

In Chapter 2 we give the definition of good semigroup, and we show that such semigroups are completely identified by the set of their small elements. Then we analyze some of their properties, also in connection with value semigroups. We define the *distance* $d(E\backslash F)$ between two good semigroup ideals $E \subseteq F$ (already introduced by D'Anna in [D'A97]). This quantity plays the role of the length $\ell(\mathcal{E}/\mathcal{F})$ of the quotient of two fractional ideals $\mathcal{E} \subseteq \mathcal{F}$ on the semigroup side. In fact, the two quantities agree in case $E = \Gamma_{\mathcal{E}}$ and $F = \Gamma_{\mathcal{F}}$ (see Proposition 2.29), that is,

$$\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}}).$$

We give a proof of the fact that $d(E\backslash F) = 0$ is equivalent to $E = F$ (see Proposition 2.28), as stated by D'Anna in [D'A97, Proposition 2.8]. In particular, this implies $\mathcal{E} = \mathcal{F}$ in case $E = \Gamma_{\mathcal{E}}$ and $F = \Gamma_{\mathcal{F}}$.

In Chapter 3 we give a definition of good generating system for good semigroups starting from the set $\operatorname{Small}(S)$, as already mentioned before, and we prove that if S is a good local semigroup, then is has a unique minimal generating system. Then we give a notion of good generating system for good semigroup ideals and we show that there is a unique minimal such generating system.

In Chapter 4 we give a new definition of canonical semigroup ideal, and we state some results regarding its duality properties and its relation with D'Anna's canonical ideal. In Section 4.2 we show that of $E := \Gamma_{\mathcal{E}}$ for some fractional ideal $\mathcal{E} \in \mathfrak{R}_R$, the dual if E with respect to a canonical semigroup ideal K is the value semigroup ideal of the Cohen–Macaulay dual $\mathcal{K} : \mathcal{E}$, where $\mathcal{K}$ is a canonical ideal of R with value semigroup K.

In Chapter 5 we give some technical results on the distance between good semigroup ideals. Then we generalize the definition of Poincaré series given in [CDGZ03] to good semigroup ideals and we show that, under suitable assumptions, if E is a good semigroup ideal, then the Poincaré series of $K - E$ is symmetric to the Poincaré series of E. In particular, the symmetry holds if $E := \Gamma_{\mathcal{E}}$ for some fractional ideal $\mathcal{E}$.

1

Value semigroups of rings and their ideals

This chapter treats the definition of value semigroups for rings and their ideals and the study of their compatibility with common algebraic operations. Any one-dimensional semilocal Cohen–Macaulay ring R has a value semigroup. In case R is also reduced, such semigroup is the direct product of the value semigroups of the localizations. If instead R is analytically reduced with large residue fields, its value semigroup coincides with the one of the completion. Furthermore, if R is admissible, i.e. it is analytically reduced and residually rational with large residue fields, then its value semigroup satisfies the same properties which are fulfilled by value semigroups of algebroid curves. All of this is shown in this chapter, which is part of a joint work with P. Korell and M. Schulze (see [KST17]).

1.1 Value semigroups of one-dimensional semilocal rings

Let R be a commutative and unitary ring, and let $\mathrm{Max}(R)$ be the set of maximal ideals of R. Assume that $\mathfrak{m} \cap R^{\mathrm{reg}} \neq \emptyset$ for all $\mathfrak{m} \in \mathrm{Max}(R)$.

We denote by Q_R the total ring of fractions of R. We assume Q_R satisfies (A.1) and abbreviate $\mathcal{F} : \mathcal{E} := \mathcal{F} :_{Q_R} \mathcal{E}$ for any subsets $\mathcal{E}, \mathcal{F} \subseteq Q_R$.

In order to give a definition of value semigroup of R, we have to deal with zero-divisors, and hence we need a general notion of valuation ring over the ring R.

Valuations and valuation rings of Q_R are defined in Appendix B. Recall that $I_V = V :_{Q_R} Q_R$ is the intersection of all regular principal fractional ideals of V.

Definition 1.1. A *valuation ring over* R is a valuation ring V of Q_R such that $R \subseteq V$. We denote by $\mathfrak{V}_R$ the set of all valuation rings of Q_R over R.

Proposition 1.2. *Let R be a Noetherian one-dimensional integrally closed local ring. Then R is a discrete valuation domain.*

Proof. See [AM69, Proposition 9.2]. □

From now on, we consider R to be a one-dimensional semilocal Cohen–Macaulay ring. In general, the set $\mathfrak{V}_R$ of valuation rings over R is described in the following theorem.

Theorem 1.3. *Let R be a one-dimensional semilocal Cohen–Macaulay ring with total ring of fractions Q_R.*

(a) The set $\mathfrak{V}_R$ is finite and non-empty, and it contains discrete valuation rings only.

(b) $\operatorname{Max}(Q_R) = \{I_V \mid V \in \mathfrak{V}_R\}$*, and for any* $I \in \operatorname{Max}(Q_R)$*, there is a bijection*

$$\begin{aligned} \{V \in \mathfrak{V}_R \mid I_V = I\} &\leftrightarrow \mathfrak{V}_{R/(I\cap R)} \\ V &\mapsto V/I, \end{aligned}$$

where $Q_{R/(I\cap R)} = Q_R/I$*.*

(c) Let $\overline{R}$ *be the integral closure of* R*. Then*

(1) $\overline{R} = \bigcap_{V\in\mathfrak{V}_R} V$*;*

(2) The set of regular prime ideals of $\overline{R}$ *agrees with* $\operatorname{Max}(\overline{R})$*;*

(3) Any regular ideal of $\overline{R}$ *is principal.*

(d) There is a bijection

$$\begin{aligned} \operatorname{Max}(\overline{R}) &\leftrightarrow \mathfrak{V}_R \\ \mathfrak{n} &\mapsto ((\overline{R}\setminus\mathfrak{n})^{\mathrm{reg}})^{-1}\overline{R} \\ \mathfrak{n}_V := \mathfrak{m}_V \cap \overline{R} &\leftarrow\!\shortmid V. \end{aligned}$$

In particular, $\overline{R}/\mathfrak{n}_V = V/\mathfrak{m}_V$ *and* $\mathfrak{m}_V \cap R = \mathfrak{n}_V \cap R \in \operatorname{Max}(R)$*.*

Proof. See [KV04, Chapter II, Theorem 2.11]. Lying over implies $\mathfrak{n}_V \cap R = \mathfrak{m}_V \cap \overline{R} \cap R = \mathfrak{m}_V \cap R \in \operatorname{Max}(R)$ in part (d). □

Recall that $\mathfrak{R}_R$ is the set of regular fractional ideals of R (see Notation A.6). By Theorem 1.3.(c).(3) and Lemma A.12 we have

$$\mathfrak{R}_{\overline{R}} = \mathfrak{R}^*_{\overline{R}}.$$

Then there is an injective group homomorphism

$$\begin{aligned} \psi = \psi_R \colon \mathfrak{R}_{\overline{R}} &\to \prod_{V\in\mathfrak{V}_R} \mathfrak{R}^*_V \\ \mathcal{E} &\mapsto (\mathcal{E}V)_{V\in\mathfrak{V}_R} \\ \bigcap_{V\in\mathfrak{V}_R} \mathcal{E}_V &\leftarrow\!\shortmid (\mathcal{E}_V)_{V\in\mathfrak{V}_R}. \end{aligned} \tag{1.1}$$

In fact, writing $\mathcal{E} = t\overline{R}$ for some $t \in Q_R^{\mathrm{reg}}$,

$$\bigcap_{V\in\mathfrak{V}_R} \mathcal{E}V = \bigcap_{V\in\mathfrak{V}_R} tV = t\bigcap_{V\in\mathfrak{V}_R} V = t\overline{R} = \mathcal{E}$$

by Theorem 1.3.(c).(1). Recall now that for any $V \in \mathfrak{V}_R$ we have a diagram (see (B.5)):

$$\begin{tikzcd} Q_R \arrow[d, two heads, "\mu_V"'] \arrow[dr, two heads, "\nu_V"] & \\ \mathfrak{R}^*_{V,\infty} \arrow[r, "\cong", "\phi_V"'] & \mathbb{Z}_\infty. \end{tikzcd}$$

where ν_V is the discrete valuation associated to V. Taking this diagram component-wise with

$$\nu = \nu_R = \prod_{V\in\mathfrak{V}_R} \nu_V \text{ and } \phi = \phi_R = \prod_{V\in\mathfrak{V}_R} \phi_V$$

gives rise to the commutative diagram

$$\begin{tikzcd}
& Q_R^{\mathrm{reg}} \arrow[dl, two heads] \arrow[d, two heads, "\mu"] \arrow[dr, two heads, "\nu"] & \\
\mathfrak{R}_{\overline{R}} \arrow[r, "\cong", "\psi"'] & \prod\limits_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* \arrow[r, "\cong", "\phi"'] & \mathbb{Z}^{\mathfrak{V}_R}.
\end{tikzcd} \tag{1.2}$$

Then surjectivity of μ, and hence of ψ, follows from the Approximation Theorem for discrete valuations B.16.(c). Thus ψ is an isomorphism and both ψ and ϕ preserve the partial orders (reverse inclusion on $\mathfrak{R}_{\overline{R}}$ and $\prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^*$ and natural partial order on $\mathbb{Z}^{\mathfrak{V}_R}$).

Hence we can give the following definition:

Definition 1.4. Let R be a one-dimensional semilocal Cohen–Macaulay ring, and let $\mathfrak{V}_R$ be the set of (discrete) valuation rings of Q_R over R with corresponding valuation

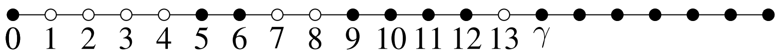
$$\nu = \nu_R = (\nu_V)_{V \in \mathfrak{V}_R} \colon Q_R \to \mathbb{Z}_\infty^{\mathfrak{V}_R}.$$

To each $\mathcal{E} \in \mathfrak{R}_R$ we associate its *value semigroup ideal*

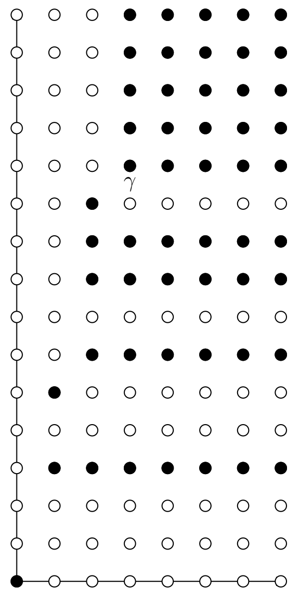
$$\Gamma_{\mathcal{E}} := \nu(\mathcal{E}^{\mathrm{reg}}) \subseteq \mathbb{Z}^{\mathfrak{V}_R}.$$

If $\mathcal{E} = R$, then the monoid Γ_R is called the *value semigroup* of R. The semigroup Γ_R is called *local* if the 0 is the only element of Γ_R with a zero component in $\mathbb{Z}^{\mathfrak{V}_R}$.

Example 1.5. Consider the irreducible curve (i.e. one-dimensional local Cohen–Macaulay ring) $R = \mathbb{C}[[x, y, z]]/(x^3 - yz, y^3 - z^2) = \mathbb{C}[[t^5, t^6, t^9]]$. The value semigroup Γ_R of R is $\Gamma_R = \langle 5, 6, 9 \rangle = \{0, 5, 6, 9, 10, 11, 12, 14 \dots\}$, like the figure below illustrates. The element γ, as we will see later, is called conductor, and is such that $\gamma + \mathbb{N} \subseteq \Gamma_R$. We will also see that it has a close relation with the conductor of the ring.



Example 1.6. Consider the curve with two branches $R = \mathbb{C}[[x, y]]/y(x^3 + y^5)$. The value semigroup Γ_R of R is $\Gamma_R = \langle (1, 5), (2, 9), (1, 3) + \mathbb{N}\mathbf{e}_1, (3, 15) + \mathbb{N}\mathbf{e}_2 \rangle$, which is illustrated in the figure below.



Remark 1.7. Let R be a one-dimensional semilocal Cohen–Macaulay ring, and let $V \in \mathfrak{V}_R$. We have

$$V^* = \{x \in Q^{\mathrm{reg}} \mid \nu_V(x) = 0\} \text{ and } \overline{R}^* = \{x \in Q^{\mathrm{reg}} \mid \nu(x) = 0\}.$$

In particular

$$R^* = \overline{R}^* \cap R = \{x \in R \mid \nu(x) = 0\}.$$

The first equality follows by definition, as $x \in V^*$ if and only if $xV = V$ if and only if $\nu_V(x) = \phi_V(\mu_V(x)) = \phi_V(V) = 0$ (see (B.1) and Diagram (B.5)). For the second, by Theorem 1.3.(c), $\overline{R} = \cap_{V\in\mathfrak{V}_R} V$. Hence $\overline{R}^* = (\cap_{V\in\mathfrak{V}_R} V)^*$. Then the claim follows directly from the fact that units commute with intersections, i.e. $(\cap_{V\in\mathfrak{V}_R} V)^* = \cap_{V\in\mathfrak{V}_R} V^*$.

Definition 1.8. We define a decreasing filtration $\mathcal{Q}^\bullet$ on Q_R by

$$\mathcal{Q}^\alpha := \{x \in Q_R \mid \nu(x) \geq \alpha\}$$

for any $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$. For any R-submodule $\mathcal{E}$ of Q_R, we denote $\mathcal{E}^\bullet = \mathcal{E} \cap \mathcal{Q}^\bullet$ the induced filtration.

Lemma 1.9. *Let R be a one-dimensional semilocal Cohen–Macaulay ring. Then*

(a) $Q^\alpha = (\phi \circ \psi)^{-1}(\alpha) = \bigcap_{V\in\mathfrak{V}_R} \mathfrak{m}_V^{\alpha_V} \in \mathfrak{R}_{\overline{R}}$ *for any* $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$.

(b) $x\overline{R} = \mathcal{Q}^{\nu(x)}$ *for any* $x \in Q_R^{\mathrm{reg}}$ *and, in particular,* $\overline{R} = \mathcal{Q}^0$.

(c) $\Gamma_{\mathcal{Q}^\alpha} = \alpha + \mathbb{N}^{\mathfrak{V}_R}$ *for any* $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$ *and, in particular,* $\Gamma_{\overline{R}} = \mathbb{N}^{\mathfrak{V}_R}$.

(d) if $\mathcal{E}$ is a (regular) fractional ideal of R, then $\mathcal{E}^\alpha$ is also a (regular) fractional ideal for any $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$.

Proof. (a) By Diagram B.5, for $x \in Q_R$, $\nu(x) \geq \alpha$ if and only if $\phi_V \circ \mu_V(x) \geq \alpha_V$ for any $V \in \mathfrak{V}_R$ if and only if, by definition of μ_V, $x \in (\phi_V \circ \mu_V)^{-1}(\alpha)$ for any $V \in \mathfrak{V}_R$, if and only if $x \in (\phi \circ \mu)^{-1}(\alpha)$. Hence the first equality is true. By definition of the isomorphism ϕ_V in (B.3)

$$\phi^{-1}(\alpha) = \prod_{V\in\mathfrak{V}_R} \phi_V^{-1}(\alpha) = \prod_{V\in\mathfrak{V}_R} \mathfrak{m}_V^\alpha.$$

Then

$$\psi^{-1}(\phi^{-1}(\alpha)) = \phi^{-1}\left(\prod_{V\in\mathfrak{V}_R} \mathfrak{m}_V^\alpha\right) = \bigcap_{V\in\mathfrak{V}_R} \mathfrak{m}_V^\alpha \in \mathfrak{R}_{\overline{R}}$$

by (1.1), and hence we have the second equality.

(b) Let $x \in Q_R^{\mathrm{reg}}$. By part (a), Diagram 1.2 and Theorem 1.3.(c).(1),

$$\begin{aligned}\mathcal{Q}^{\nu(x)} &= (\phi \circ \psi)^{-1}(\nu(x)) = \psi^{-1}(\phi^{-1}(\nu(x))) = \psi^{-1}(\mu(x)) = \psi^{-1}\left(\prod_{V\in\mathfrak{V}_R} xV\right) \\ &= \bigcap_{V\in\mathfrak{V}_R} xV = x\bigcap_{V\in\mathfrak{V}_R} V = x\overline{R}.\end{aligned}$$

In particular, by Theorem 1.3.(c).(1) we have $\overline{R} = \bigcap_{V\in\mathfrak{V}_R} V = \bigcap_{V\in\mathfrak{V}_R}\{y \in Q_R \mid \nu_V(y) \geq 0\}$, so that $\overline{R} = \mathcal{Q}^0$.

(c) Let us first prove the particular claim. By Theorem 1.3.(c).(1) and equation (B.1) $\overline{R} = \bigcap_{V \in \mathfrak{V}_R} V = \bigcap_{V \in \mathfrak{V}_R} \{y \in Q_R \mid \nu_V(y) \geq 0\}$. By Remark B.8.(b) $\nu_V(x) < \infty$ for any $V \in \mathfrak{V}_R$ and $x \in Q_R^{\mathrm{reg}}$. Thus

$$\Gamma_{\overline{R}} = \nu((\overline{R})^{\mathrm{reg}}) = \nu(\{x \in Q_R^{\mathrm{reg}} \mid \nu(x) \geq 0\}) \subseteq \mathbb{N}^{\mathfrak{V}_R}.$$

The other inclusion follows by surjectivity of ν in Diagram (1.2), and hence $\Gamma_{\overline{R}} = \mathbb{N}^{\mathfrak{V}_R}$.
Let now $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$. By surjectivity of ν in Diagram (1.2), $\alpha = \nu(x)$ for some $x \in Q_R^{\mathrm{reg}}$. Then by part (b), definition of Γ (Definition 1.4) and properties of ν (see (V1)), we have

$$\Gamma_{\mathcal{Q}^\alpha} = \Gamma_{\mathcal{Q}^{\nu(x)}} = \Gamma_{x\overline{R}} = \nu((x\overline{R})^{\mathrm{reg}}) = \nu(x) + \nu((\overline{R})^{\mathrm{reg}}) = \nu(x) + \Gamma_{\overline{R}} = \alpha + \mathbb{N}^{\mathfrak{V}_R}.$$

(d) By part (a), $\mathcal{E}^\alpha$ is an R-module. By Definition A.5.(b), $\mathcal{E}$ is a fractional ideal if there is an $r \in R^{\mathrm{reg}}$ such that $r\mathcal{E} \subseteq R$. Then clearly $r\mathcal{E}^\alpha \subseteq r\mathcal{E} \subseteq R$. If moreover $\mathcal{E}$ is regular, then there exists $x \in \mathcal{E}^{\mathrm{reg}}$. By surjectivity of ν in Diagram 1.2 and equation (B.1), there is a $y \in (R^\beta)^{\mathrm{reg}}$ for arbitrarily large $\beta \in \mathbb{Z}^{\mathfrak{V}_R}$. Then $xy \in (\mathcal{E}^\alpha)^{\mathrm{reg}}$ for $\beta \geq \alpha - \nu(x)$ and hence $\mathcal{E}^\alpha \in \mathfrak{R}_R$. □

The following result was stated without proof in [DdlM88, (1.1.1)] and [BDF00a, §2].

Proposition 1.10. *Let R be a one-dimensional semilocal Cohen–Macaulay ring with value semigroup Γ_R. Then the following are equivalent:*

(i) The ring R is local.

(ii) The semigroup Γ_R is local.

Proof.
(i) ⇒ (ii) Assume R is local, and let $\mathfrak{m}$ be its maximal ideal. Then Theorem 1.3.(d), Lying Over and Equations (B.2) and (B.3) give

$$\mathfrak{m} \subseteq \bigcap_{V \in \mathfrak{V}_R} \mathfrak{m}_V = \bigcap_{V \in \mathfrak{V}_R} \{x \in Q_R \mid \nu_V(x) > 0\}.$$

Thus $\mathfrak{m} = R \cap (\bigcap_{V \in \mathfrak{V}_R} \{x \in Q_R \mid \nu_V(x) > 0\}) = \{x \in R \mid \nu_V(x) > 0 \text{ for any } V \in \mathfrak{V}_R\}$. Now let $x \in R^{\mathrm{reg}}$ be such that $\nu_V(x) = 0$. Then $x \in R \setminus \mathfrak{m} = R^*$ and by Remark 1.7, $\nu(x) = 0$.
(ii) ⇒ (i) [KST17, Proposition 3.1.7]. □

In the following we will show that, under suitable hypotheses, semigroups $E = \Gamma_\mathcal{E}$ of fractional ideals $\mathcal{E}$ of R have certain properties. We will use these properties in order to define the notion of a good semigroup in Chapter 2.

Definition 1.11. A semilocal ring R is *analytically reduced* if its completion is reduced.

Analytically reduced rings are often referred to as *analytically unramified*.

Proposition 1.12. *Let R be analytically reduced. Then the integral closure $\overline{R}$ of R in Q_R is a finitely generated R-module.*

Proof. See [HS06, Corollary 4.6.2]. □

Remark 1.13. In the literature analytically reduced rings are usually defined in the local case. In this special case, the following are equivalent:
(i) R is analytically reduced;
(ii) $\overline{R}$ is a finitely generated R-module.
See [KV04, Chapter II, Theorem 3.22] for a proof.

Recall that the conductor of a fractional ideal $\mathcal{E}$ is the (fractional) ideal $\mathcal{C}_\mathcal{E} := \mathcal{E} : \overline{R}$ (see Definition A.5.(d)).

Lemma 1.14. *Let R be a one-dimensional semilocal Cohen–Macaulay ring. If R is analytically reduced, then $\mathcal{C}_\mathcal{E} \in \mathfrak{R}_R \cap \mathfrak{R}_{\overline{R}}$ for any $\mathcal{E} \in \mathfrak{R}_R$. In particular, $\mathcal{C}_\mathcal{E} = x\overline{R} = \mathcal{Q}^{\nu(x)}$ for some $x \in Q_R^{\mathrm{reg}}$ with $\nu(x) + \mathbb{N}^{\mathfrak{V}_R} \subseteq \Gamma_\mathcal{E}$.*

Proof. See [KST17, Lemma 3.1.9] □

Definition 1.15. Let R be a one-dimensional semilocal Cohen–Macaulay ring.

(1) R is *residually rational* if $\overline{R}/\mathfrak{n} = R/(\mathfrak{n} \cap R)$ for any $\mathfrak{n} \in \mathrm{Max}(\overline{R})$ or, equivalently (see Theorem 1.3.(d)), $V/\mathfrak{m}_V = R/(\mathfrak{m}_V \cap R)$ for any $V \in \mathfrak{V}_R$.

(2) R has *large residue fields* if $|R/\mathfrak{m}| \geq |\mathfrak{V}_{R_\mathfrak{m}}|$ for any $\mathfrak{m} \in \mathrm{Max}(R)$.

(3) R is *admissible* if it is analytically reduced and residually rational with large residue fields.

Definition 1.16. Let $S \subseteq \mathbb{Z}^I$ be a partially ordered cancellative commutative monoid. The *group of differences of S* is

$$D_S = \{\alpha - \beta \mid \alpha, \beta \in S\}.$$

We define the *difference* of two subsets $E, F \subseteq \mathbb{Z}^I$ by

$$E - F := \{\alpha + \mathbb{Z}^I \mid \alpha + F \subseteq E\}.$$

While the value semigroup operation preserves inclusions, there is no obvious counterpart of multiplication and colon operation on the semigroup side.

Remark 1.17. Let R be a one-dimensional semilocal Cohen–Macaulay ring, and let $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$.

(a) If $\mathcal{E} \subseteq \mathcal{F}$, then $\Gamma_\mathcal{E} \subseteq \Gamma_\mathcal{F}$.

This follows easily from $\mathcal{E}^{\mathrm{reg}} \subseteq \mathcal{F}^{\mathrm{reg}}$ and from ν being a group homomorphism.

(b) The inclusion $\Gamma_{\mathcal{E}\mathcal{F}} \supseteq \Gamma_\mathcal{E} + \Gamma_\mathcal{F}$ is not an equality in general.

Let $\alpha \in \Gamma_\mathcal{E} + \Gamma_\mathcal{F}$. We can write $\alpha = \nu(x) + \nu(y)$ with $x \in \mathcal{E}^{\mathrm{reg}}$ and $y \in \mathcal{F}^{\mathrm{reg}}$. Consider $xy \in (\mathcal{E}\mathcal{F})^{\mathrm{reg}}$. Then $\alpha = \nu(x) + \nu(y) = \nu(xy) \in \Gamma_{\mathcal{E}\mathcal{F}}$. Thus the inclusion. Example 1.18 shows that it is not an equality in general.

(c) The inclusion $\Gamma_{\mathcal{E}:\mathcal{F}} \subseteq \Gamma_\mathcal{E} - \Gamma_\mathcal{F}$ is not an equality in general.

Let $\nu(x) \in \Gamma_{\mathcal{E}:\mathcal{F}}$. Then $x \in Q_R^{\mathrm{reg}}$ and $x\mathcal{F} \subseteq \mathcal{E}$. Let $y \in \mathcal{F}^{\mathrm{reg}}$. Then there is a $z \in \mathcal{E}^{\mathrm{reg}}$ such that $xy = z$. In particular, $\nu(xy) = \nu(x) + \nu(y) = \nu(z)$, i.e. $\nu(x) = \nu(z) - \nu(y) \in \Gamma_\mathcal{E} - \Gamma_\mathcal{F}$. The example [BDF00a, Example 3.3] shows that it is not an equality in general.

Example 1.18. Consider the ring

$$R := \mathbb{C}[[(t_1^3, 0), (t_1^4, 0), (t_1^5, 0), (0, t_2)]] \subseteq \mathbb{C}[[t_1]] \times \mathbb{C}[[t_2]] = \overline{R}.$$

Then R is a one-dimensional complete reduced Cohen–Macaulay ring. Hence in particular it is analytically reduced. As $\mathfrak{V}_R = \{\mathbb{C}[[t_1]], \mathbb{C}[[t_2]]\}$, it is clear that R residually rational. Moreover, the residue field $\mathbb{C}$ is infinite and therefore large. Thus R is admissible. The value semigroup of R is $S := \Gamma_R$. Consider the R-submodules of Q_R

$$\mathcal{E} := \langle (t_1, 0), (t_1^2, 0), (t_1^3, t_2^2), (0, t_2^3) \rangle_R, \quad \mathcal{F} := \langle (t_1, t_2), (t_1^2, 0), (0, t_2^2) \rangle_R.$$

Then the corresponding value semigroup ideals are $E := \Gamma_\mathcal{E}$ and $F := \Gamma_\mathcal{F}$. Clearly $\mathcal{E}, \mathcal{F}, \mathcal{E}\mathcal{F} \in \mathfrak{R}_R$, and hence $E, F, \Gamma_{\mathcal{E}\mathcal{F}} \in \mathfrak{G}_S$ by Remark 2.4.(d). We show S, E, F and $E + F$ in Figure 1.1. It can be easily seen that (E2) fails for $E + F$, and hence $E + F \notin \mathfrak{G}_S$. It follows that $\Gamma_{\mathcal{E}\mathcal{F}} \subsetneq \Gamma_\mathcal{E} + \Gamma_\mathcal{F}$.

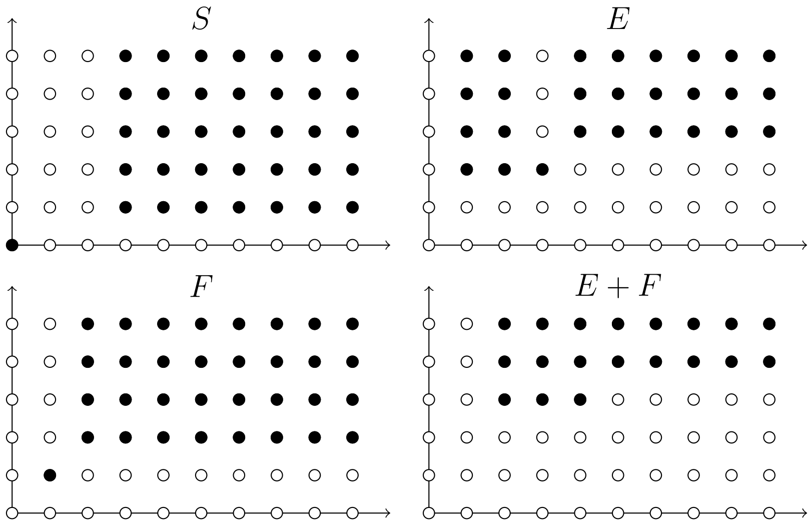


Figure 1.1: The value semigroup (ideals) in Example 1.18

The following definition was given also in [DdlM88, §1] and [D'A97, §2].

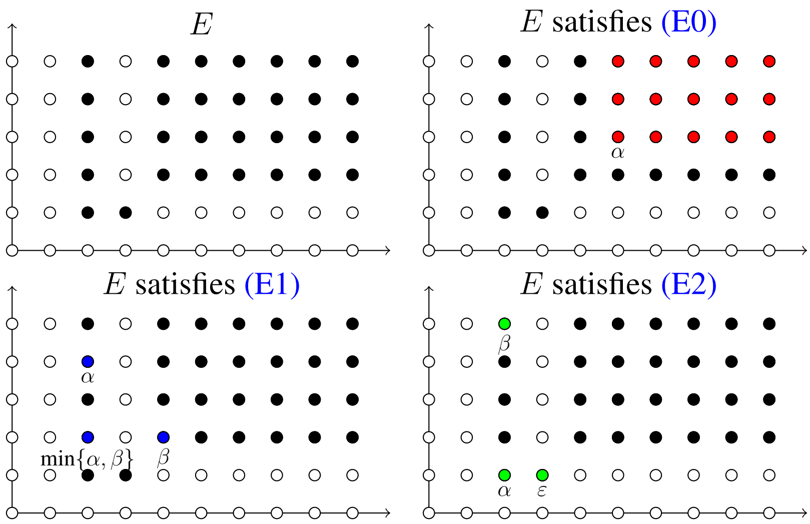
Definition 1.19. Let $\overline{S}$ be a partially ordered monoid, isomorphic to $\mathbb{N}^I$ with its natural partial order, where I is a finite set. Let $E \subseteq D_{\overline{S}} \cong \mathbb{Z}^I$. Then we consider the following properties for E:

(E0) There exists $\alpha \in D_{\overline{S}}$ such that $\alpha + \overline{S} \subseteq E$.

(E1) If $\alpha, \beta \in E$, then $\min\{\alpha, \beta\} := (\min\{\alpha_i, \beta_i\})_{i \in I} \in E$.

(E2) For any $\alpha, \beta \in E$ and $j \in I$ such that $\alpha_j = \beta_j$ there exists an $\epsilon \in E$ such that $\epsilon_j > \alpha_j = \beta_j$ and $\epsilon_i \geq \min\{\alpha_i, \beta_i\}$ for any $i \in I \setminus \{j\}$ with equality if $\alpha_i \neq \beta_i$. We call E *good* if it satisfies (E0),(E1) and (E2).

Figure 1.2: The following subset of $\mathbb{Z}^2$ satisfies (E0), (E1) and (E2):



Lemma 1.20. *Any group automorphism φ of $\mathbb{Z}^s$ preserving the partial order is defined by a permutation of the standard basis.*

Proof. See [KST17, Lemma 3.1.8]. □

In the following, we collect results from [D'A97] and provide a detailed proof.

Proposition 1.21. *Let R be a one-dimensional semilocal Cohen–Macaulay ring with value semigroup $S := \Gamma_R$, and let $E := \Gamma_{\mathcal{E}}$ for some $\mathcal{E} \in \mathfrak{R}_R$.*

(a) We have $E + S \subseteq E$.

(b) If R is analytically reduced, then E satisfies (E0) *with $\overline{S} = \Gamma_{\overline{R}}$.*

(c) If R is local analytically reduced with large residue field, then E satisfies (E1) *with $\overline{S} = \Gamma_{\overline{R}}$ and $I = \mathfrak{V}_R$.*

(d) If R is local and residually rational, then E satisfies (E2).

In particular, if R is local admissible, then E is good.

Proof.

(a) Since $\mathcal{E}$ is an R-module and $Q_R^{\mathrm{reg}} = Q_R^*$ a group, $R^{\mathrm{reg}}\mathcal{E}^{\mathrm{reg}} \subseteq \mathcal{E}^{\mathrm{reg}}$. Then since ν in Diagram (1.2) is a group homomorphism which preserves inclusions we obtain:

$$E + S = \Gamma_{\mathcal{E}} + \Gamma_R = \nu(R^{\mathrm{reg}}) + \nu(\mathcal{E}^{\mathrm{reg}}) = \nu(R^{\mathrm{reg}}\mathcal{E}^{\mathrm{reg}}) \subseteq \nu(\mathcal{E}^{\mathrm{reg}}) = \Gamma_{\mathcal{E}} = E.$$

(b) By Lemma 1.9.(c), $\Gamma_{\overline{R}} = \mathbb{N}^{\mathfrak{V}_R}$, so we need to find an α such that $\alpha + \mathbb{N}^{\mathfrak{V}_R} \subseteq E$. By Lemma 1.14 $\mathcal{C}_{\mathcal{E}} = \mathcal{E} : \overline{R} = x\overline{R} \subseteq \mathcal{E}$ for some $x \in Q_R^{\mathrm{reg}}$. Lemma 1.9.(b) yields $\mathcal{C}_{\mathcal{E}} = \mathcal{Q}^{\nu(x)}$, and Lemma 1.9.(c) gives

$$\Gamma_{\mathcal{C}_{\mathcal{E}}} = \Gamma_{\mathcal{Q}^{\nu(x)}} = \nu(x) + \mathbb{N}^{\mathfrak{V}_R}$$

As $\mathcal{C}_{\mathcal{E}} \subseteq \mathcal{E}$, $\Gamma_{\mathcal{C}_{\mathcal{E}}} \subseteq \Gamma_{\mathcal{E}} = E$. Thus $\nu(x) = \alpha \in D_{\overline{S}} = \mathbb{Z}^{\mathfrak{V}_R}$ satisfies (E0).

(c) Let $x, y \in \mathcal{E}^{\mathrm{reg}}$ with $\nu(x) = \alpha$ and $\nu(y) = \beta$. By Theorem 1.3.(c).(3) all regular ideals of $\overline{R}$ are principal, so that $\langle x, y\rangle_{\overline{R}} = z\overline{R}$ for some $z \in Q_R^{\mathrm{reg}}$. By Lemma A.18, we may assume $z \in \langle x, y\rangle_R^{\mathrm{reg}} \subseteq \mathcal{E}^{\mathrm{reg}}$. Then by Lemma 1.9 we obtain

$$\nu(\langle x, y\rangle_{\overline{R}}) = \nu(z\overline{R}) = \nu(z) + \mathbb{N}^{\mathfrak{V}_R}.$$

Now (V1) and (V2) imply $\nu(z) \geq \min\{\nu(x), \nu(y)\} \geq \nu(z)$, and hence

$$\min\{\alpha, \beta\} = \min\{\nu(x), \nu(y)\} = \nu(z) \in E.$$

(d) Denote by $\mathfrak{m}$ be the maximal ideal of R. Let $\alpha, \beta \in E$ and $W \in \mathfrak{V}_R$ such that $\alpha_W = \beta_W$. Pick $x, y \in \mathcal{E}^{\mathrm{reg}}$ such that $\nu(x) = \alpha$ and $\nu(y) = \beta$. Then $x/y \in Q_R^{\mathrm{reg}}$ and $\nu_W(x/y) = \alpha_W - \beta_W = 0$. Therefore $x/y \in W \setminus \mathfrak{m}_W$ by (B.1) and (B.2). By Theorem 1.3.(d), $\overline{R}/\mathfrak{n}_V = V/\mathfrak{m}_V$, and by hypothesis, $R/\mathfrak{m} = \overline{R}/\mathfrak{n}_V$ for any $V \in \mathfrak{V}_R$. In particular, we can consider the class $\overline{x/y} = \overline{u} \in W/\mathfrak{m}_W = R/\mathfrak{m}$ for some $u \in R \setminus \mathfrak{m}$. It follows that $\nu_W(u - x/y) > 0$ and $\nu(u) = 0$, again by (B.1) and (B.2). Then, being $\mathcal{E}$ a fractional ideal, $uy - x \in \mathcal{E}$ with

$$\nu_W(uy - x) = \nu_W(y(u - x/y)) = \nu_W(u - x/y) + \nu_W(y) > \nu_W(y) = \beta_W$$

and

$$\begin{aligned}\nu_V(uy - x) =& \nu_V(uy + (-x)) \geq \min\{\nu_V(uy), \nu_V(-x)\} = \min\{\nu_V(u) + \nu(y), \nu_V(x)\}\\ =& \min\{\alpha_V, \beta_V\}\end{aligned}$$

for any $V \in \mathfrak{V}_R \setminus \{W\}$, with equality if $\alpha_V \neq \beta_V$ (see Remark B.8.(c)). Notice that the above inequalities remain true after replacing u by any element $u' \in u + \mathfrak{m}$. It is left to show that, for some u', $\nu_V(u' - x/y) < \infty$ for any $V \in \mathfrak{V}_R$ with $\alpha_V = \beta_V$. Since R is Cohen–Macaulay, there is an $m \in \mathfrak{m}^{\mathrm{reg}} \subseteq \mathfrak{m}_W^{\mathrm{reg}}$, and hence $(\infty, \dots, \infty) > \nu(m^k) \geq k \cdot (1, \dots, 1)$. Then any $u' = u + m^k$ with $k > \max\{\nu_V(u - x/y) < \infty \mid V \in \mathfrak{V}_R \text{ with } \alpha_V = \beta_V\}$ gives

$$\begin{aligned}\nu_V(u' - x/y) &= \nu_V(u + m^k - x/y) \geq \min\{\nu_V(m^k), \nu_V(u - x/y)\} \\ &= \begin{cases} \nu_V(u - x/y) & \text{if } \nu_V(u - x/y) < \infty \\ k & \text{otherwise} \end{cases}\end{aligned}$$

for any V such that $\alpha_V = \beta_V$. Thus

$$\nu_W(u'y - x) = \nu_W(u' - x/y) + \nu_W(y) \geq \min\{\nu_W(m^k), \nu(u - x/y)\} + \nu_W(y) > \beta_W$$

and

$$\begin{aligned}\infty > \nu_V(u'y - x) &\geq \min\{\nu_V(u'y), \nu_V(-x)\} = \min\{\nu_V(u') + \nu_V(y), \nu_V(x)\} \\ &= \min\{\alpha_V, \beta_V\}.\end{aligned}$$

Hence $\epsilon = \nu(u'y - x)$ gives the claim. □

1.2 Value semigroups and localization

In the following we often identify objects which are canonically isomorphic.

Lemma 1.22. *Let R be a reduced semilocal ring. Then*

(a) $Q_R = \prod_{\mathfrak{p} \in \mathrm{Min}(R)} Q_{R/\mathfrak{p}}$, *and* $Q_{R_\mathfrak{m}} = \prod_{\mathfrak{m} \supseteq \mathfrak{p} \in \mathrm{Min}(R)} Q_{R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}}$ *for any* $\mathfrak{m} \in \mathrm{Max}(R)$.

(b) $Q_{R_\mathfrak{m}} = (Q_R)_\mathfrak{m}$ *for any* $\mathfrak{m} \in \mathrm{Max}(R)$.

(c) $\overline{R_\mathfrak{m}} = (\overline{R})_\mathfrak{m}$ *for any* $\mathfrak{m} \in \mathrm{Max}(R)$.

(d) $\overline{R} = \prod_{\mathfrak{p} \in \mathrm{Min}(R)} \overline{R/\mathfrak{p}}$.

Proof. (a) As R is reduced, the total ring of fractions Q_R is the zero-dimensional ring obtained from R by inverting all elements of R that are not in any minimal prime ideal. Thus, by the Structure Theorem for Artin rings (see [AM69, Theorem 8.7]), it is the finite direct product of the $Q_{R/\mathfrak{p}}$. For the second part, it is enough to observe that with R also $R_\mathfrak{m}$ is reduced for any $\mathfrak{m} \in \mathrm{Max}(R)$.

(b) Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then $R/\mathfrak{p}$ is a domain, and hence $(R/\mathfrak{p})_\mathfrak{q} \subseteq Q_{R/\mathfrak{p}}$ for any $\mathfrak{q} \in \mathrm{Spec}(R/\mathfrak{p})$. Thus $Q_{(R/\mathfrak{p})_\mathfrak{q}} = Q_{R/\mathfrak{p}}$. In particular, for any $\mathfrak{m} \in \mathrm{Max}(R)$ with $\mathfrak{m} \supseteq \mathfrak{p}$, we have $Q_{(R/\mathfrak{p})_\mathfrak{m}} = Q_{R/\mathfrak{p}}$. From (a) it follows that

$$\begin{aligned}Q_{R_\mathfrak{m}} &= \prod_{\mathfrak{m} \supseteq \mathfrak{p} \in \mathrm{Min}(R)} Q_{R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}} = \prod_{\mathfrak{m} \supseteq \mathfrak{p} \in \mathrm{Min}(R)} Q_{(R/\mathfrak{p})_\mathfrak{m}} = \prod_{\mathfrak{m} \supseteq \mathfrak{p} \in \mathrm{Min}(R)} Q_{R/\mathfrak{p}} \\ &= \left(\prod_{\mathfrak{p} \in \mathrm{Min}(R)} Q_{R/\mathfrak{p}} \right)_\mathfrak{m} = (Q_R)_\mathfrak{m}.\end{aligned}$$ □

(c) See [HS06, Proposition 2.1.6].

(d) See [HS06, Corollary 2.1.3].

Lemma 1.23. *Let R be a reduced one-dimensional semilocal Cohen–Macaulay ring. For any $\mathfrak{m} \in \operatorname{Max}(R)$ the localization map $\pi : Q_R \to (Q_R)_\mathfrak{m} = Q_{R_\mathfrak{m}}$ induces a bijection*

$$\begin{aligned} \rho_\mathfrak{m} : \{V \in \mathfrak{V}_R \mid \mathfrak{m}_V \cap R = \mathfrak{m}\} &\to \mathfrak{V}_{R_\mathfrak{m}} \\ V &\mapsto V_\mathfrak{m} \\ \pi^{-1}(W) &\leftarrow\!\shortmid W. \end{aligned}$$

In particular, $(\mathfrak{m}_V)_\mathfrak{m} = \mathfrak{m}_W$ if $V \mapsto W$.

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$ and $V \in \mathfrak{V}_R$ with $\mathfrak{m}_V \cap R = \mathfrak{m}$. Then $R \setminus \mathfrak{m} \subseteq V \setminus \mathfrak{m}_V$. Since localization is exact (see [AM69, Proposition 3.3]), and $\mathfrak{m}_V$ is regular, $(\mathfrak{m}_V)_\mathfrak{m} \subsetneq V_\mathfrak{m}$ contains a regular non-unit and hence $V_\mathfrak{m} \subsetneq (Q_R)_\mathfrak{m}$. Thus

$$R_\mathfrak{m} \subseteq V_\mathfrak{m} \subsetneq (Q_R)_\mathfrak{m} = Q_{R_\mathfrak{m}}. \tag{1.3}$$

Let $x/y, x'/y' \in (Q_R)_\mathfrak{m} \setminus V_\mathfrak{m}$. Then $x, x \in Q_R \setminus V$, which is a multiplicatively closed set (see Theorem B.3.(ii)). Hence $xx' \in Q_R \setminus V$. As $y, y' \in (Q_R)^*_\mathfrak{m}$, also $yy' \in (Q_R)^*_\mathfrak{m}$. Thus $xx'/yy' \in (Q_R)_\mathfrak{m} \setminus V_\mathfrak{m}$ Therefore $(Q_R)_\mathfrak{m} \setminus V_\mathfrak{m}$ is multiplicatively closed. Hence, by Theorem B.3.(ii) and Definition 1.1, $V_\mathfrak{m}$ is a valuation ring, and (1.3) implies $V_\mathfrak{m} \in \mathfrak{V}_{R_\mathfrak{m}}$. Hence the map is well-defined. Moreover, since $V \subsetneq Q_R$ is a maximal subring by Theorem B.14.(d), and $\pi^{-1}(V_\mathfrak{m}) \supseteq V$, we get $V = \pi^{-1}(V_\mathfrak{m})$. Therefore the map is injective.

Let now $W \in \mathfrak{V}_{R_\mathfrak{m}}$ for $\mathfrak{m} \in \operatorname{Max}(R)$, and set $V := \pi^{-1}(W)$. Then $V_\mathfrak{m} = W \subsetneq Q_{R_\mathfrak{m}}$, and $R \subseteq V \subsetneq Q_R$. Let now $x, y \in Q_R \setminus V$. Then $\pi(x), \pi(y) \in Q_{R_\mathfrak{m}}$ and since $V = \pi^{-1}(W)$, $\pi(x), \pi(y) \notin W$. As $Q_{R_\mathfrak{m}} \setminus W$ is multiplicatively closed, $\pi(xy) = \pi(x)\pi(y) \in Q_{R_\mathfrak{m}} \setminus W$, and hence $xy \notin \pi^{-1}(W) = V$, i.e. $xy \in Q_R \setminus V$. Therefore $Q_R \setminus V = Q_R \setminus \pi^{-1}(W)$ is multiplicatively closed too. Hence, by Theorem B.14.(d) and Definition 1.1, $V \in \mathfrak{V}_R$. Consider the commutative diagram of ring homomorphisms

$$\begin{tikzcd} V \arrow[r, "\pi"] & W \\ R \arrow[u, hook] \arrow[r, "\iota"] & R_\mathfrak{m}. \arrow[u, hook] \end{tikzcd}$$

Commutativity of the diagram yields

$$\pi^{-1}(\mathfrak{m}_W) \cap R = \iota^{-1}(\mathfrak{m}_W \cap R_\mathfrak{m}) = \iota^{-1}(\mathfrak{m}_{R_\mathfrak{m}}) = \mathfrak{m} \tag{1.4}$$

where $\mathfrak{m}_W \cap R_\mathfrak{m} = \mathfrak{m}_{R_\mathfrak{m}}$ by Theorem 1.3.(d). In particular, as $\mathfrak{m}$ is regular, $\pi^{-1}(\mathfrak{m}_W)$ is too. But $\pi^{-1}(\mathfrak{m}_W)$ is a prime ideal of the discrete valuation ring V (see Theorem 1.3.(a)), which by Proposition B.3.(iii) has only one regular prime ideal, i.e. $\mathfrak{m}_V$. Hence $\pi^{-1}(\mathfrak{m}_W) = \mathfrak{m}_V$ and by (1.4) $\mathfrak{m}_V \cap R = \mathfrak{m}$. Thus the map is surjective. □

By Theorem 1.3.(d), the sets $\{V \in \mathfrak{V}_R \mid \mathfrak{m}_V \cap R = \mathfrak{m}\}$, with $\mathfrak{m} \in \operatorname{Max}(R)$, form a partition of $\mathfrak{V}_R$. By Lemma 1.23, there is a bijection

$$\begin{aligned} \rho \colon \mathfrak{V}_R &\to \bigsqcup_{\mathfrak{m} \in \operatorname{Max}(R)} \mathfrak{V}_{R_\mathfrak{m}} \\ V &\mapsto \rho_{\mathfrak{m}_V \cap R}(V) = V_{\mathfrak{m}_V \cap R}. \end{aligned}$$

Using this, we define an order preserving group isomorphism

$$\begin{aligned}\xi\colon \prod_{V\in\mathfrak{V}_R}\mathfrak{R}_V^* &\to \prod_{\mathfrak{m}\in\operatorname{Max}(R)}\prod_{W\in\mathfrak{V}_{R_\mathfrak{m}}}\mathfrak{R}_W^*\\ (\mathcal{E}_V)_{V\in\mathfrak{V}_R} &\mapsto ((\mathcal{E}_{\rho^{-1}(W)})_\mathfrak{m})_{\mathfrak{m}\in\operatorname{Max}(R),W\in\mathfrak{V}_R}\end{aligned}$$

Since it maps $(\mathfrak{m}_V^{k_V})_{V\in\mathfrak{V}_R}\mapsto(\mathfrak{m}_W^{k_{\rho^{-1}(W)}})_{\mathfrak{m}\in\operatorname{Max}(R),W\in\mathfrak{V}_R}$, it is an isomorphism thanks to the map ϕ_V of (B.3).

Combined with Diagram (1.2) for R and $R_\mathfrak{m}$ for $\mathfrak{m}\in\operatorname{Max}(R)$, it fits into a commutative diagram

$$\begin{tikzcd}
Q_R^{\mathrm{reg}} \arrow[dr, two heads] \arrow[drrr, bend left, two heads, "\nu"] \arrow[ddd] & & & \\
& \mathfrak{R}_{\overline{R}} \arrow[r, "\psi", "\cong"'] \arrow[d, "\xi"', "\cong"] & \prod_{V\in\mathfrak{V}_R}\mathfrak{R}_V^* \arrow[r, two heads, "\phi", "\cong"'] \arrow[d, "\cong"] & \mathbb{Z}^{\mathfrak{V}_R} \arrow[d, "\cong"] \\
& \prod_{\mathfrak{m}\in\operatorname{Max}(R)}\mathfrak{R}_{\overline{R_\mathfrak{m}}} \arrow[r, "\prod_\mathfrak{m}\psi_{R_\mathfrak{m}}", "\cong"'] & \prod_{\mathfrak{m}\in\operatorname{Max}(R)}\prod_{W\in\mathfrak{V}_{R_\mathfrak{m}}}\mathfrak{R}_W^* \arrow[r, "\prod_\mathfrak{m}\phi_{R_\mathfrak{m}}", "\cong"'] & \prod_{\mathfrak{m}\in\operatorname{Max}(R)}\mathbb{Z}^{\mathfrak{V}_{R_\mathfrak{m}}} \\
\prod_{\mathfrak{m}\in\operatorname{Max}(R)}Q_{R_\mathfrak{m}}^{\mathrm{reg}} \arrow[ur, two heads] \arrow[urrr, bend right, two heads, "\prod_\mathfrak{m}\nu_{R_\mathfrak{m}}"'] & & &
\end{tikzcd}$$

where $\xi(\mathcal{E})=\prod_{\mathfrak{m}\in\operatorname{Max}(R)}\mathcal{E}_\mathfrak{m}$ for any $\mathcal{E}\in\mathfrak{R}_{\overline{R}}$. Observe that if $\mathcal{E}\in\mathfrak{R}_{\overline{R}}$, then by Lemma A.14 and since localization and integral closure commute (see Lemma 1.22.(c)), $\mathcal{E}_\mathfrak{m}\in\mathfrak{R}_{(\overline{R})_\mathfrak{m}}=\mathfrak{R}_{\overline{R_\mathfrak{m}}}$. Hence ξ is well-defined. This implies

$$\nu(x)=\left(\nu_{R_\mathfrak{m}}\left(\frac{x}{1}\right)\right)_{\mathfrak{m}\in\operatorname{Max}(R)} \tag{1.5}$$

for any $x\in Q_R^{\mathrm{reg}}$.

The first part of the following theorem was stated and partly proved in [BDF00a, § 1.1].

Theorem 1.24. *Let R be a one-dimensional reduced semilocal Cohen–Macaulay ring. Then there is a decomposition into local value semigroups*

$$\Gamma_R=\prod_{\mathfrak{m}\in\operatorname{Max}(R)}\Gamma_{R_\mathfrak{m}}.$$

for any $\mathcal{E}\in\mathfrak{R}_R$ there is a decomposition into value semigroup ideals

$$\Gamma_\mathcal{E}=\prod_{\mathfrak{m}\in\operatorname{Max}(R)}\Gamma_{\mathcal{E}_\mathfrak{m}}.$$

Proof. By Proposition 1.10, $\Gamma_{R_\mathfrak{m}}$ is local for $\mathfrak{m}\in\operatorname{Max}(R)$. Hence we can prove directly the second statement. By equation (1.5), we have

$$
\begin{aligned}
\Gamma_{\mathcal{E}} &= \nu(\mathcal{E}^{\mathrm{reg}}) = \{\nu(x) \mid x \in \mathcal{E}^{\mathrm{reg}}\} \\
&= \left\{ \left(\nu_{R_\mathfrak{m}}\left(\frac{x}{1}\right)\right)_{\mathfrak{m}\in\mathrm{Max}(R)} \,\middle|\, x \in \mathcal{E}^{\mathrm{reg}} \right\} \\
&= \left\{ \left(\nu_{R_\mathfrak{m}}\left(\frac{x}{1}\right)\right)_{\mathfrak{m}\in\mathrm{Max}(R)} \,\middle|\, \frac{x}{1} \in \mathcal{E}_\mathfrak{m}^{\mathrm{reg}} \text{ for any } \mathfrak{m} \in \mathrm{Max}(R) \right\} \\
&\subseteq \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \nu_{R_\mathfrak{m}}(\mathcal{E}_\mathfrak{m}^{\mathrm{reg}}) = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \Gamma_{\mathcal{E}_\mathfrak{m}}.
\end{aligned}
$$

For the other inclusion, let $\alpha = (\alpha_\mathfrak{m})_{\mathfrak{m}\in\mathrm{Max}(R)} \in \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \Gamma_{\mathcal{E}_\mathfrak{m}}$ (each of the $\alpha_\mathfrak{m}$ is a vector in general). Then there exists elements $x_\mathfrak{m}/y_\mathfrak{m} \in \mathcal{E}_\mathfrak{m}$, $\mathfrak{m} \in \mathrm{Max}(R)$, such that $\nu_{R_\mathfrak{m}}(x_\mathfrak{m}/y_\mathfrak{m}) = \alpha_\mathfrak{m}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. By equations (V1) and Remark 1.7, if $y_\mathfrak{m} = u \in R_\mathfrak{m}^*$, then $\nu_{R_\mathfrak{m}}(x_\mathfrak{m}/y_\mathfrak{m}) = \nu_{R_\mathfrak{m}}(x') - \nu_{R_\mathfrak{m}}(u) = \nu_{R_\mathfrak{m}}(x') - 0 = \nu_{R_\mathfrak{m}}(x')$. Hence we may clear denominators and assume $y_\mathfrak{m} = 1$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. for any $\mathfrak{m} \in \mathrm{Max}(R)$ pick an element $z_\mathfrak{m} \in (\cap_{\mathfrak{n}\in\mathrm{Max}(R)\setminus\{\mathfrak{m}\}}\mathfrak{n}) \setminus \mathfrak{m}$. Note that such a $z_\mathfrak{m}$ exists by Chinese Remainder Theorem. Then by Theorem 1.3.(d) the sets $\{V \in \mathfrak{V}_R \mid \mathfrak{m}_V \cap R = \mathfrak{m}\}$ form a partition, i.e. $\nu_{R_\mathfrak{m}}(z_\mathfrak{m}/1) = \prod_{V\in\mathfrak{V}_{R_\mathfrak{m}}} \nu_V(z_\mathfrak{m}/1)$ and, as $z_\mathfrak{m} \in R \setminus \mathfrak{m} \subseteq V \setminus \mathfrak{m}_V$ for any V such that $\mathfrak{m}_V \cap R = \mathfrak{m}$, by equations (B.1) and (B.2),

$$\nu_V(z_\mathfrak{m}/1) = 0 \text{ for any } V \in \mathfrak{V}_{R_\mathfrak{m}} \tag{1.6}$$

and hence $\nu_{R_\mathfrak{m}}(z_\mathfrak{m}/1) = 0$. Using the same tools, we obtain

$$\nu_V(z_\mathfrak{m}/1) > 0 \text{ for any } V \in \mathfrak{V}_{R_\mathfrak{n}} \text{ for any } \mathfrak{n} \in \mathrm{Max}(R) \setminus \{\mathfrak{m}\}. \tag{1.7}$$

Let

$$k_\mathfrak{m} > \max\{\nu_V(x_\mathfrak{n}/1) - \nu_V(x_\mathfrak{m}/1) \mid V \in \mathfrak{V}_{R_\mathfrak{n}}, \mathfrak{n} \in \mathrm{Max}(R) \setminus \{\mathfrak{m}\}\}.$$

Then $z = \sum_{\mathfrak{m}\in\mathrm{Max}(R)} x_\mathfrak{m} z_\mathfrak{m}^{k_\mathfrak{m}} \in \mathcal{E}$ since $x_\mathfrak{m} \in \mathcal{E}$ and $z_\mathfrak{m} \in R$ for any $\mathfrak{m} \in \mathrm{Max}(R)$. By choice of $k_\mathfrak{m}$ and (1.7), we have inequalities

$$\nu_V(x_\mathfrak{m}/1) + k_\mathfrak{m}\nu_V(z_\mathfrak{m}/1) > \nu_V(x_\mathfrak{m}/1) + k_\mathfrak{m} > \nu_V(x_\mathfrak{n}/1) \tag{1.8}$$

for any $V \in \mathfrak{V}_{R_\mathfrak{n}}$ for any $\mathfrak{n} \in \mathrm{Max}(R) \setminus \{\mathfrak{m}\}$. Therefore, using (1.6) and (1.8),

$$
\begin{aligned}
\nu_V(z/1) &= \nu_V\left(\frac{\sum_{\mathfrak{m}\in\mathrm{Max}(R)} x_\mathfrak{m} z_\mathfrak{m}^{k_\mathfrak{m}}}{1}\right) \\
&\geq \min\left\{\nu_V\left(\frac{x_\mathfrak{m} z_\mathfrak{m}^{k_\mathfrak{m}}}{1}\right) \,\middle|\, \mathfrak{m} \in \mathrm{Max}(R)\right\} \\
&= \min\{\nu_V(x_\mathfrak{m}/1) + k_\mathfrak{m}\nu_V(z_\mathfrak{m}/1) \mid \mathfrak{m} \in \mathrm{Max}(R)\} \\
&= \min\{\nu_V(x_\mathfrak{n}/1) + k_\mathfrak{n}\nu_V(z_\mathfrak{n}/1), \nu_V(x_\mathfrak{m}/1) + k_\mathfrak{m}\nu_V(z_\mathfrak{m}/1) \mid \mathfrak{n} \in \mathrm{Max}(R) \setminus \{\mathfrak{m}\}\} \\
&= \min\{\nu_V(x_\mathfrak{n}/1), \nu_V(x_\mathfrak{m}/1) + k_\mathfrak{m}\nu_V(z_\mathfrak{m}/1) \mid \mathfrak{n} \in \mathrm{Max}(R) \setminus \{\mathfrak{m}\}\} \\
&= \nu_V(x_\mathfrak{n}/1).
\end{aligned}
$$

for any $V \in \mathfrak{V}_{R_\mathfrak{n}}$ for any $\mathfrak{n} \in \mathrm{Max}(R)$. As $\nu_V(x_\mathfrak{n}/1) = \nu_V\left(\frac{x_\mathfrak{n} z_\mathfrak{n}^{k_\mathfrak{n}}}{1}\right) \neq \nu_V(x_\mathfrak{m}/1) + k_\mathfrak{m}\nu_V(z_\mathfrak{m}/1) = \nu_V\left(\frac{x_\mathfrak{m} y_\mathfrak{m}^{k_\mathfrak{m}}}{1}\right)$ for $\mathfrak{n} \neq \mathfrak{m} \in \mathrm{Max}(R)$, the inequality is actually an equality. Thus

$$\nu_{R_\mathfrak{n}}(z/1) = \prod_{V\in\mathfrak{V}_{R_\mathfrak{n}}} \nu_V(z/1) = \prod_{V\in\mathfrak{V}_{R_\mathfrak{n}}} \nu_V(x_\mathfrak{n}/1) = \nu_{R_\mathfrak{n}}(x_\mathfrak{n}/1) = \alpha_\mathfrak{n}.$$

Thus $\nu(z) = \alpha$ by equation (1.5), and $\alpha \in \Gamma_{\mathcal{E}}$ as $z \in \mathcal{E}$. Hence the claim. $\square$

Corollary 1.25. *Let R be a one-dimensional reduced semilocal Cohen–Macaulay ring with large residue fields, and let $E := \Gamma_{\mathcal{E}}$ for some $\mathcal{E} \in \mathfrak{R}_R$.*

(a) If R is analytically reduced, then E satisfies (E1).

(b) If R is residually rational, then E satisfies (E2).

In particular, if R is admissible, then E is good.

Proof. Using Theorem 1.24, this follows from Proposition 1.21.(c) and (d). Note that to prove property (E2) for elements $\alpha, \beta \in \Gamma_{\mathcal{E}}$ which are different in all components in $\Gamma_{\mathcal{E}_\mathfrak{m}}$ for some $\mathfrak{m} \in \operatorname{Max}(R)$ we need to apply (E1) in $\Gamma_{\mathcal{E}_\mathfrak{m}}$. □

1.3 Value semigroups and completion

For some results in this section we refer to [KST17], as the proofs are not original work of the author.

The compatibility of value semigroup ideals with completion is due to D'Anna (see [D'A97, §1]). We give results including the semilocal case.

In the following, $\widehat{-}$ stands for the completion at the Jacobson radical of R.

Lemma 1.26. *With R also $\widehat{R}$ is a one-dimensional semilocal Cohen–Macaulay ring.*

Proof. By Lemma A.16.(e), we can reduce to the local case. Then the claim follows from [BH93, Corollary 2.1.8]. □

Theorem 1.27. *Let R be a one-dimensional local Cohen–Macaulay ring with total ring of fractions Q_R. Then there is a bijection (see Lemma A.17)*

$$\begin{aligned} \mathfrak{V}_R &\to \mathfrak{V}_{\widehat{R}} \\ V &\mapsto V\widehat{R} \\ W \cap Q_R &\leftarrow\!\shortmid W. \end{aligned}$$

In particular, $\mathfrak{m}_V \widehat{R} = \mathfrak{m}_W$ if $V \mapsto W$.

Proof. See [KST17, Theorem 3.3.2]. □

Corollary 1.28. *Let $R = (R, \mathfrak{m})$ be a one-dimensional local Cohen–Macaulay ring. Then $\overline{\widehat{R}} = \overline{R}\widehat{R}$. In particular, $\overline{\widehat{R}} = \widehat{\overline{R}}$ if $\overline{R}$ is finite over R.*

Proof. From Lemma 1.26, $\widehat{R}$ is also a one-dimensional Cohen–Macaulay ring. Then by Theorem 1.3.(c) $\overline{\widehat{R}} = \bigcap_{W \in \mathfrak{V}_{\widehat{R}}} W$ and and $\overline{R} = \bigcap_{V \in \mathfrak{V}_R} V$, by Theorem 1.27 $\{W \in \mathfrak{V}_{\widehat{R}}\} = \{V\widehat{R} \mid V \in \mathfrak{V}_R\}$ and by Lemma A.16.(d) intersection commutes with completion. Hence we can write

$$\overline{\widehat{R}} = \bigcap_{W \in \mathfrak{V}_{\widehat{R}}} W = \bigcap_{V \in \mathfrak{V}_R} (V\widehat{R}) = \left(\bigcap_{V \in \mathfrak{V}_R} V \right) \widehat{R} = \overline{R}\widehat{R}.$$

If $\overline{R}$ is finite over R, then by Lemma A.16.(f) $\overline{R}\widehat{R} = \widehat{\overline{R}}$ (see also [KV04, Chapter II, Theorem (3.19).(3)]), and hence the claim. □

Let R be a one-dimensional local Cohen–Macaulay ring. By Theorem 1.27, there is an order preserving group homomorphism

$$\begin{aligned} \prod_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* &\to \prod_{W \in \mathfrak{V}_{\widehat{R}}} \mathfrak{R}_W^* \\ (\mathcal{E}_V)_{V \in \mathfrak{V}_R} &\mapsto (\mathcal{E}_{\sigma^{-1}(W)} \widehat{R})_{W \in \mathfrak{V}_{\widehat{R}}} \end{aligned}$$

mapping $(\mathfrak{m}_V^{k_V})_{V \in \mathfrak{V}_R} \mapsto (\mathfrak{m}_W^{k_{\sigma^{-1}(W)}})_{W \in \mathfrak{V}_{\widehat{R}}}$, which is an isomorphism with (B.3). Combined with Diagram (1.2) for R and $\widehat{R}$ (see Lemma 1.26), it fits into a commutative diagram

$$\begin{tikzcd} Q_R^{\mathrm{reg}} \arrow[dr, two heads] \arrow[ddd] \arrow[rrr, bend left, two heads, "\nu"] & & & \\ & \mathfrak{R}_{\overline{R}} \arrow[r, "\psi", "\cong"'] \arrow[d, "\eta"', "\cong"] & \prod\limits_{V \in \mathfrak{V}_R} \mathfrak{R}_V^* \arrow[r, two heads, "\phi", "\cong"'] \arrow[d, "\cong"] & \mathbb{Z}^{\mathfrak{V}_R} \arrow[d, "\cong"] \\ & \mathfrak{R}_{\overline{\widehat{R}}} \arrow[r, "\psi_{\widehat{R}}", "\cong"'] & \prod\limits_{W \in \mathfrak{V}_{\widehat{R}}} \mathfrak{R}_W^* \arrow[r, two heads, "\phi_{\widehat{R}}", "\cong"'] & \mathbb{Z}^{\mathfrak{V}_{\widehat{R}}} \\ Q_{\widehat{R}}^{\mathrm{reg}} \arrow[ur, two heads] \arrow[rrru, bend right, two heads, "\nu_{\widehat{R}}"'] & & & \end{tikzcd} \tag{1.9}$$

where $\eta \colon \mathcal{E} \mapsto \mathcal{E}\widehat{R}$ and $\eta^{-1} \colon \mathcal{F} \cap Q_R \leftarrow\!\shortmid \mathcal{F}$. The homomorphisms η and η^{-1} are well-defined due to Lemma A.16.(b) and (c).

The following lemma relates value semigroup ideals to jumps in the filtration induced by $\mathcal{Q}^\bullet$ (see also [CDK94, Remark (4.3)]).

Lemma 1.29. *Let R be a one-dimensional analytically reduced local Cohen–Macaulay ring with large residue fields. Let $\alpha \in \mathbb{Z}^{\mathfrak{V}_R}$. Then the following are equivalent:*

(i) $\alpha \in \Gamma_{\mathcal{E}}$;

(ii) $\mathcal{E}^\alpha / \mathcal{E}^{\alpha + \mathbf{e}_V} \neq 0$ *for any $V \in \mathfrak{V}_R$, where $\mathbf{e}_V$ is an element of the canonical base of* $\mathbb{N}^{\mathfrak{V}_R}$.

If R is residually rational, then $\ell_R(\mathcal{E}^\alpha / \mathcal{E}^{\alpha + \mathbf{e}_V}) \leq 1$ for any $V \in \mathfrak{V}_R$.

Proof. Assume $\alpha \in \Gamma_{\mathcal{E}}$. Then there exists $x \in \mathcal{E}^{\mathrm{reg}}$ such that $\nu(x) = \alpha < \alpha + \mathbf{e}_V$ for any $V \in \mathfrak{V}_R$. Then $x \in \mathcal{E}^\alpha \setminus \mathcal{E}^{\alpha + \mathbf{e}_V}$, and hence $\mathcal{E}^\alpha / \mathcal{E}^{\alpha + \mathbf{e}_V} \neq 0$.

Conversely, assume $\mathcal{E}^\alpha / \mathcal{E}^{\alpha + \mathbf{e}_V} \neq 0$. Then by definition of $\mathcal{E}^\alpha$ (see Definition 1.8) and by Lemma 1.9.(d), $\mathcal{E}^\alpha \in \mathfrak{R}_R$. Since R is a Marot ring by Lemma B.2, $\mathcal{E}^\alpha$ is generated by regular elements. Thus there is an $x_V \in \mathcal{E}^\alpha \setminus \mathcal{E}^{\alpha + \mathbf{e}_V} \subseteq \mathcal{E}$ such that $\alpha + \mathbf{e}_V > \nu(x_V) \geq \alpha$. Since $\Gamma_{\mathcal{E}}$ satisfies property (E1) by Proposition 1.21.(c), there exists an element $z \in \mathcal{E}$ such that $\nu(z) = \min\{\nu(x_V) \mid V \in \mathfrak{V}_R\} = \alpha$. Hence $\alpha \in \Gamma_{\mathcal{E}}$.

Let us prove now the second statement. By Diagram (1.2), the map ν is surjective, so that there exists $x \in Q_R^{\mathrm{reg}}$ such that $\nu(x) = \alpha$. Then Lemma 1.9.(c) yields $\mathcal{Q}^\alpha = x\overline{R}$ and $\mathcal{Q}^0 = \overline{R}$. By Lemma 1.9.(a) $\mathcal{Q}^{\mathbf{e}_V} = \cap_{V \in \mathfrak{V}_R} \mathfrak{m}_V^{\mathbf{e}_V} = \mathfrak{m}_V \cap \overline{R}$ and by Theorem 1.3.(d), $\overline{R}/(\mathfrak{m}_V \cap \overline{R}) = V/\mathfrak{m}_V$. Thus there is an isomorphism

$$\mathcal{E}^\alpha / \mathcal{E}^{\alpha + \mathbf{e}_V} \subseteq \mathcal{Q}^\alpha / \mathcal{Q}^{\alpha + \mathbf{e}_V} \xleftarrow[\cong]{\cdot x} \mathcal{Q}^0 / \mathcal{Q}^{\mathbf{e}_V} = \overline{R}/(\mathfrak{m}_V \cap \overline{R}) = V/\mathfrak{m}_V$$

for any $V \in \mathfrak{V}_R$. If R is residually rational, then $V/\mathfrak{m}_V = R/\mathfrak{m}$, and hence

$$\mathcal{E}^\alpha/\mathcal{E}^{\alpha+\mathbf{e}_V} \hookrightarrow R/\mathfrak{m}.$$

Thus $\ell_R(\mathcal{E}^\alpha/\mathcal{E}^{\alpha+\mathbf{e}_V}) \leq \ell_R(R/\mathfrak{m}) = 1$. $\square$

We obtain the following theorem:

Theorem 1.30. *Let R be a one-dimensional analytically reduced semilocal Cohen–Macaulay ring with large residue fields. Then*

$$\Gamma_{\mathcal{E}} = \Gamma_{\widehat{\mathcal{E}}}$$

for any $\mathcal{E} \in \mathfrak{R}_R$.

Proof. See [KST17, Theorem 3.3.5].

$\square$

Remark 1.31. Let R be an analytically reduced one-dimensional local Cohen–Macaulay ring. Then Lemma 1.26 gives $\widehat{R}$ is a one-dimensional reduced local Cohen–Macaulay ring. By Theorem 1.27, $\mathfrak{V}_R$ is in one-to-one correspondence with $\mathfrak{V}_{\widehat{R}}$, and by Theorem 1.3.(d), $\mathfrak{V}_{\widehat{R}}$ is in one-to-one correspondence with $\operatorname{Max}(\overline{\widehat{R}})$. Moreover, Corollary 1.28 yields $\operatorname{Max}(\overline{\widehat{R}}) \leftrightarrow \operatorname{Max}(\widehat{\overline{R}})$, and Lemma A.16.(e) gives $\operatorname{Max}(\widehat{\overline{R}}) \leftrightarrow \operatorname{Min}(\widehat{\overline{R}})$. Since $\widehat{\overline{R}}_\mathfrak{m}$ is a domain for any $\mathfrak{m} \in \operatorname{Max}(\overline{R})$ (see Proposition 1.2) we get a sequence of bijections

$$\mathfrak{V}_R \leftrightarrow \mathfrak{V}_{\widehat{R}} \leftrightarrow \operatorname{Max}(\overline{\widehat{R}}) \leftrightarrow \operatorname{Max}(\widehat{\overline{R}}) \leftrightarrow \operatorname{Min}(\widehat{\overline{R}}) \leftrightarrow \operatorname{Min}(\overline{\widehat{R}}) \leftrightarrow \operatorname{Min}(\widehat{R})$$

sending V to $\mathfrak{q}_{\widehat{V}}$. If in addition $R = \widehat{R}$, then $\overline{R/\mathfrak{p}}$ is a one-dimensional local integrally closed Cohen–Macaulay ring, and hence by Proposition 1.2 it is a discrete valuation domain. By Theorem 1.3.(b) then it has to be $V/I_V = \overline{R/\mathfrak{p}}$ with $\mathfrak{p} = I_V \cap R$. Moreover, $\nu_V = \nu_{\overline{R/\mathfrak{p}}} \circ \pi_V$, where $\pi_V \colon Q_R \twoheadrightarrow Q_{R/\mathfrak{p}} = Q_R/I_V$ (see Theorem 1.3.(b)) for any $\mathfrak{p} \in \operatorname{Min}(R)$. Since R is complete, it is reduced, and therefore by Lemma 1.22.(a) we can write Q_R as a product: $Q_R = \prod_{\mathfrak{p} \in \operatorname{Min}(R)} Q_{R/\mathfrak{p}} = \prod_{V \in \mathfrak{V}_R} Q_{R/(I_V \cap R)}$. Thus, the map

$$(\nu_{\overline{R/\mathfrak{q}_V}})_{V \in \mathfrak{V}_R} \colon Q_R \to \mathbb{Z}_\infty^{\mathfrak{V}_R}$$

yields the same semigroup as in Definition 1.4. This approach is often used in the literature (see [KW84, DdlM87, DdlM88, D'A97]).

2

Good semigroups and their ideals

Our interest in this chapter is the class of objects which contains value semigroups and their ideals, i.e. good semigroups and good semigroup ideals. If S is a good semigroup, the set $\mathrm{Small}(S)$ of *small elements* of S, that is, elements of S which are smaller or equal to the conductor with the usual partial order, determines the semigroup. A similar statement is true for any E good semigroup ideal of S. We will see in the next chapter that this property can be used to define a good system of generators. Another interesting property satisfied by good semigroups and their ideals is the fact that the distance between two elements is well-defined, i.e. it doesn't change following different paths. This allows to define a concept of *distance* $d(F\backslash E)$ between two good semigroup ideals $E \subseteq F$. We give a proof of the fact that this distance detects equality, that is, $d(E\backslash F) = 0$ is equivalent to $E = F$. Not only, but in case $E := \Gamma_{\mathcal{E}}$ and $F := \Gamma_{\mathcal{F}}$ for some $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$ and some admissible ring R,

$$\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}}).$$

In particular, if $\mathcal{E} \subseteq \mathcal{F}$, then $\mathcal{E} = \mathcal{F}$ if and only if $E = F$.

The contents of this chapter are partly contained in [KST17] and partly in [DGSMT17].

2.1 Good properties

Let S be a cancellative commutative monoid. Then S embeds into its (free abelian) group of differences D_S (see Definition 1.16). If S is partially ordered, then D_S carries a natural induced partial order.

Definition 2.1. Let S be a partially ordered cancellative commutative monoid such that $\alpha \geq 0$ for any $\alpha \in S$. We always consider $S \neq \emptyset$. Let S be of finite rank. Then D_S is generated by a finite set I such that the isomorphism $D_S \cong \mathbb{Z}^I$ preserves the natural partial orders. Note that I is unique and contains only positive elements by Lemma 1.20. If $|I| = 1$, such an S is called *numerical semigroup*. We set

$$\overline{S} := \{\alpha \in D_S \mid \alpha \geq 0\} \cong \mathbb{N}^I.$$

We call S a *good semigroup* if properties (E0), (E1) and (E2) hold for $E = S$ (see also Definition 1.19). If 0 is the only element of S with a zero component in D_S, then we call S *local*.

Definition 2.2. A *semigroup ideal* of a good semigroup S is a subset $E \subseteq D_S$ such that

$$E + S \subseteq E.$$

We always require that it is *finitely generated*, that is there exists $\alpha \in D_S$ such that

$$\alpha + E \subseteq \overline{S}.$$

If E satisfies (E1), then its *minimum* is denoted by

$$\mu^E := \min E.$$

If E satisfies (E1) and (E2), then we call E a *good semigroup ideal* of S. The following lemma clarifies why we do not require (E0).

Notation 2.3. The set of good semigroup ideals of S is denoted by $\mathfrak{G}_S$.

Lemma 2.4. *Let S be a good semigroup.*

(a) Any semigroup ideal E of S satisfies property (E0).

(b) If $S \subseteq S' \subseteq \overline{S}$ are good semigroups, then $D_{S'} = D_S$ and hence $\overline{S'} = \overline{S}$. It follows that $\mathfrak{G}_{S'} \subseteq \mathfrak{G}_S$. In particular, $S' \in \mathfrak{G}_S$.

(c) For any semigroup ideal E of S satisfying (E1)*, $\mu^E = 0$ is equivalent to $S \subseteq E \subseteq \overline{S}$.*

(d) Let R be an admissible (local) ring. Then $S := \Gamma_R$ is a good (local) semigroup, and $\Gamma_{\mathcal{E}} \in \mathfrak{G}_S$ for any $\mathcal{E} \in \mathfrak{R}_R$

Proof. (a) Since S satisfies (E0), there is an $\alpha \in D_{\overline{S}}$ such that $\alpha + \overline{S} \subseteq S$. Then $\beta + \alpha + \overline{S} \subseteq \beta + S \subseteq E + S \subseteq E$ for any $\beta \in E$.

(b) Let $E \in \mathfrak{G}_{S'}$. Then $E \subseteq D_{S'} = D_S$ and $E + S \subseteq E + S' \subseteq E$. Moreover, $\alpha + E \subseteq \overline{S'} = \overline{S}$. Hence E is a finitely generated semigroup ideal of S. To prove that it is a good semigroup ideal, consider first property (E0). If E satisfies it for S', as $\overline{S} = \overline{S'}$, E satisfies it for S too. Property (E1) does not depend on the semigroup, and the same holds for property (E2). Hence $E \in \mathfrak{G}_S$. As $S' + S \subseteq S'$ and $S' \subseteq \overline{S'} = \overline{S}$, S' is also a semigroup ideal of S, and as it is a good semigroup, it belongs to $\mathfrak{G}_S$.

(c) If $\mu^E = 0$, then $S = 0 + S = \mu^E + S \subseteq E$, and $\alpha \geq \mu^E = 0$ for all $\alpha \in E$ implies $E \subseteq \overline{S}$. Conversely, if $S \subseteq E \subseteq \overline{S}$, then $0 = \mu^S \geq \mu^E \geq \mu^{\overline{S}} = 0$.

(d) By Proposition 1.10 if R is local Γ_R is local too. Then the statement follows from Proposition 1.21 and Corollary 1.25. ∎

Lemma 2.5. *Let S be a good semigroup, $\alpha \in D_S$ and E, E', F, F' be semigroup ideals of S. Then*

(a) For any $E \in \mathfrak{G}_S$, $E - S = E$.

(b) If $E \in \mathfrak{G}_S$, $\alpha + E \in \mathfrak{G}_S$.

(c) $(\alpha + E) - F = \alpha + (E - F) = E - (-\alpha + F)$.

(d) For any two inclusions $E \subseteq E'$ and $F \subseteq F'$, we have $E - F' \subseteq E - F \subseteq E' - F$.

Proof. (a) As $E + S \subseteq E$ by definition of semigroup ideal, clearly $E \subseteq E - S$. On the other hand, if $\alpha \in D_S$ is such that $\alpha + S \subseteq E$, then in particular $\alpha + 0 = \alpha \in E$.

(b) If E satisfies (E0),(E1) and (E2), then $\alpha + E$ satisfies them too.

(c) $(\alpha + E) - F = \{\beta \in D_S \mid \beta + F \subseteq \alpha + E\} = \{\gamma \in D_S \mid \gamma + \alpha + F \subseteq E\}$
$= \alpha + \{\gamma \in D_S \mid \gamma + F \subseteq E\} = \alpha + (E - F) = \{\beta \in D_S \mid \beta - \alpha + F \subseteq E\} = E - (-\alpha + F)$.

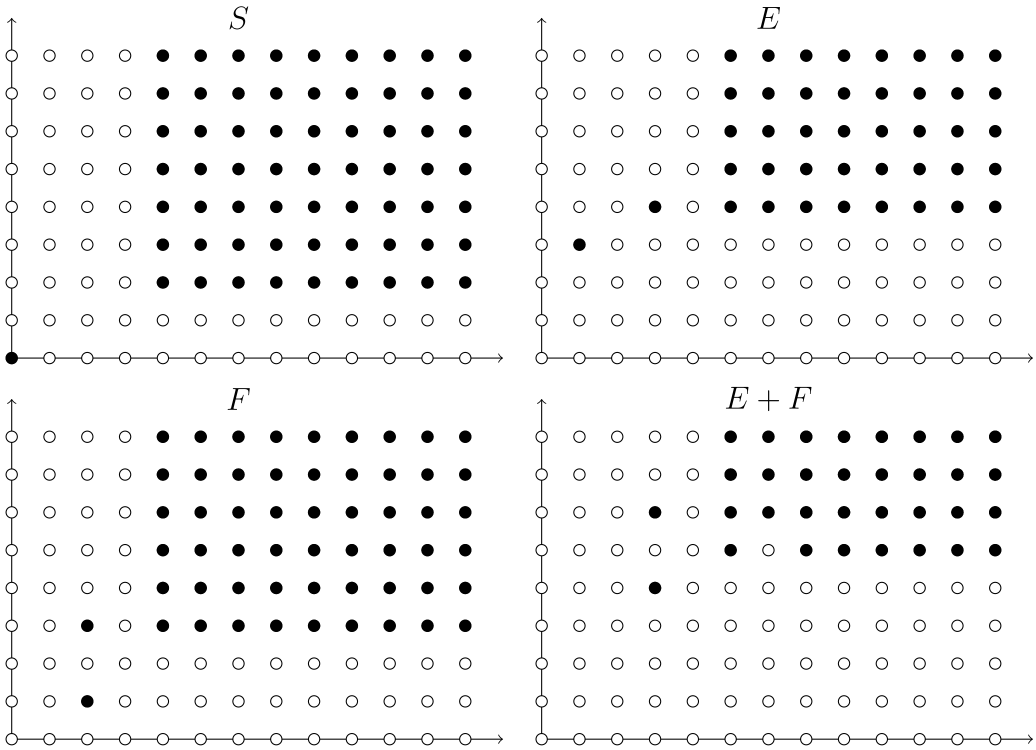
(d) $E - F' = \{\alpha \in D_S \mid \alpha + F' \subseteq E\} \subseteq \{\alpha \in D_S \mid \alpha + F \subseteq E\} = E - F$
$\subseteq \{\alpha \in D_S \mid \alpha + F \subseteq E'\} = E' - F$. □

Although $\mathfrak{G}_S$ is neither a monoid nor closed under difference (see Remark 1.17), the following result gives some positive properties.

Lemma 2.6. *For any two semigroup ideals E and F of S also $E - F$ is a semigroup ideal of S. If E satisfies* (E1)*, so does $E - F$, and $C_E \in \mathfrak{G}_S \cap \mathfrak{G}_{\overline{S}}$.*

Proof. See [KST17, Lemma 4.1.4]. □

Remark 2.7. Observe that for two semigroup ideals E and F of a good semigroup S satisfying (E1), the sum $E + F$ does not even need to satisfy (E1).



Definition 2.8. Let E and F be semigroup ideals of a good semigroup S. We write

$$E - F := \{\alpha \in D_S \mid \alpha + F \subseteq E\},$$

and we call

$$C_E := E - \overline{S} = \{\alpha \in D_S \mid \alpha + \overline{S} \subseteq E\}$$

the *conductor (semigroup) ideal* of E. We set $C := C_S$.

Definition 2.9. Let S be a good semigroup, and let E be a semigroup ideal of S satisfying (E1). Then

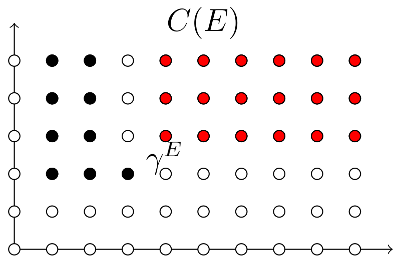
$$\gamma^E := \mu^{C_E} = \min\{\alpha \in D_S \mid \alpha + \overline{S} \subseteq E\}.$$

is called the *conductor* of E. Equivalently (see Lemma 2.6),

$$C_E = \gamma^E + \overline{S}.$$

We abbreviate $\tau^E := \gamma^E - \mathbf{1}$, $\gamma := \gamma^S$ and $\tau := \tau^S$, where $\mathbf{1} = (1, \ldots, 1) \in D_S$.

Figure 2.1: Let E be the semigroup ideal in Example 1.18. The following figure illustrates the conductor of E.



Lemma 2.10. *Let E and F be semigroup ideals of a good semigroup S satisfying property* (E1). *Then $\gamma^{E-F} = \gamma^E - \mu^F$.*

Proof. See [KST17, Lemma 4.1.9]. □

The following result decomposes good semigroups and their ideals into local components as we proved already for value semigroups in Theorem 1.24.

Theorem 2.11. *Every good semigroup S decomposes uniquely as a direct product*

$$S = \prod_{m \in M} S_{I_m}$$

of good local semigroups S_m, where $\{I_m \mid m \in M\}$ is a partition of I. Every semigroup ideal E of S satisfying (E1) *decomposes as*

$$E = \prod_{m \in M} E_{I_m}$$

where E_{I_m} is the image of $E \subseteq D_S = \mathbb{Z}^I$ under projection to $D_{S_{I_m}} = \mathbb{Z}^{I_m}$. In particular, if $E \in \mathfrak{G}_S$, then $E_{I_m} \in \mathfrak{G}_{S_{I_m}}$ for any $m \in M$.

Let R be an admissible ring. Then there is a bijection $\varphi\colon \operatorname{Max}(R) \to M$ such that

$$(\Gamma_{\mathcal{E}})_{\varphi(\mathfrak{m})} = \Gamma_{\mathcal{E}_\mathfrak{m}}$$

for any $\mathcal{E} \in \mathfrak{R}_R$.

Proof. In [BDF00a, Theorem 2.5] they prove that every good semigroup is a direct product of good local semigroups, and such representation is unique (see [BDF00a, Remark 2.6]). Moreover, by [BDF00a, Proposition 2.12], the representation of S as product of good local semigroups induces a representation of every semigroup ideal satisfying (E1) as a product. If R is an admissible ring, by Proposition 1.10 $\Gamma_{R_\mathfrak{m}}$ is a local semigroup. Hence, the unique decomposition given by Theorem 1.24, i.e. $\Gamma_R = \prod_{\mathfrak{m} \in \operatorname{Max}(R)} \Gamma_{R_\mathfrak{m}}$ has to coincide with the decomposition $\prod_{m \in M} (\Gamma_R)_{I_m}$ up to a rearrangement of the coordinates (see Lemma 1.20). Thus for any $\mathcal{E} \in \mathfrak{R}$ there is a bijection $\varphi\colon \operatorname{Max}(R) \to M$ such that

$$(\Gamma_{\mathcal{E}})_{\varphi(\mathfrak{m})} = \Gamma_{\mathcal{E}_\mathfrak{m}}.$$ □

The following objects were introduced by Delgado [DdlM87, DdlM88] to investigate the Gorenstein symmetry. They measure jumps in the fitration $\mathcal{Q}^\alpha$ (see Definition 1.8) from the proof of Theorem 1.30 (see [CDK94, Remark 4.6]).

Definition 2.12. Let S be a good semigroup, and E a semigroup ideal of S. Let $\alpha \in D_S$ and $J \subseteq I$. We define:

(a) $\Delta_J(\alpha) := \{\beta \in D_S \mid \alpha_j = \beta_j \text{ for } j \in J \text{ and } \alpha_i < \beta_i \text{ for } i \in I \setminus J\}$.
If $J = \{i\}$, then $\Delta_J(\alpha) =: \Delta_i(\alpha)$.

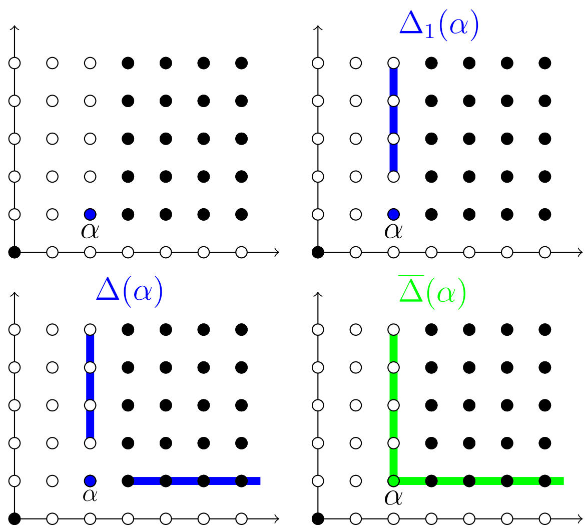
(b) $\overline{\Delta}_J(\alpha) := \{\beta \in D_S \mid \alpha_j = \beta_j \text{ for } j \in J \text{ and } \alpha_i \leq \beta_i \text{ for } i \in I \setminus J\}$.
If $J = \{i\}$, then $\overline{\Delta}_J(\alpha) =: \overline{\Delta}_i(\alpha)$.

(c) $\Delta(\alpha) := \bigcup_{i \in I} \Delta_i(\alpha)$, and $\Delta^E(\alpha) := \Delta(\alpha) \cap E$.

(d) $\overline{\Delta}(\alpha) := \bigcup_{i \in I} \overline{\Delta}_i(\alpha)$.

Notice that $\Delta_I(\alpha) = \overline{\Delta}_I(\alpha) = \alpha$.

Figure 2.2: The figure gives an example of the sets Δ in $\mathbb{Z}^2$.



We provide now some technical preliminaries which will be used later. The statement of the following lemma was proved in [DdlM88, Corollary 1.9] in case $E = S$.

Lemma 2.13. *Let S be a good semigroup. Then $\Delta^E(\tau^E) = \emptyset$ for any $E \in \mathfrak{G}_S$.*

Proof. See [KST17, Lemma 4.1.8]. □

2.2 Small elements

From now on we assume $|I| = s$ and we fix an order preserving isomorphism $D_S \cong \mathbb{Z}^s$.

Let $S \subseteq \mathbb{N}^s$ be a good semigroup and let $E \in \mathfrak{G}_S$. The set of *small elements* is defined as

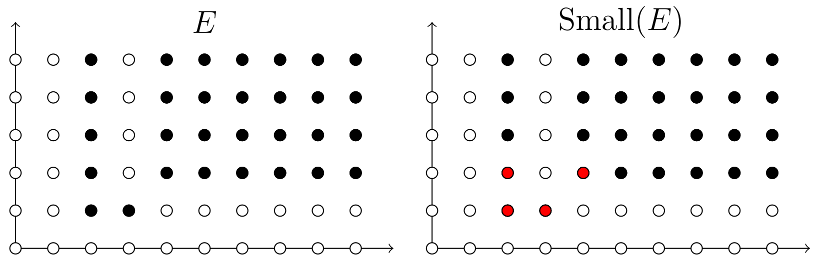
$$\mathrm{Small}(E) := \{\alpha \in E \mid \alpha \leq \gamma^E\}.$$

In particular, if $E = S$, we have

$$\mathrm{Small}(S) := \{\alpha \in S \mid \alpha \leq \gamma\}.$$

Clearly, $\gamma^E \in \mathrm{Small}(E)$ for any E.

Figure 2.3: Let S be the good semigroup $\{(0,0)\} \cup \{(2,1)+\mathbb{N}^2\}$. The figure illustrates the set of small elements of a good semigroup ideal E of S.



Notation 2.14. Let $J \subseteq I$. Then we denote

$$H_J := \{(\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s \mid \alpha_i = 0 \text{ for } i \in I \setminus J\}.$$

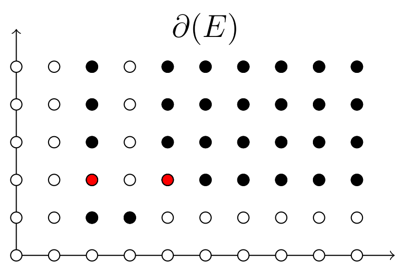
In particular, when $J = \{j\}$, H_J coincides with the j-th semiaxes.

Notation 2.15. Let S be a good semigroup and let E be a good semigroup. Let $\emptyset \neq J \subseteq I$.

(1) $\partial_J(E) = \{\alpha \in \mathrm{Small}(E) \mid \alpha_j = \gamma_j^E \text{ for any } j \in J\}$.

(2) $\partial(E) = \bigcup_{\emptyset \neq J \subseteq I} \partial_J(E)$.

Figure 2.4: The following figure illustrates the notation $\partial(E)$, for E as in Figure 2.3.



Notice that

$$(\alpha + H_J) = \{\beta \in \mathbb{Z}^s \mid \beta_j \geq \alpha_j \text{ for } j \in J, \beta_i = \alpha_i \text{ for } i \in I \setminus J\} = \overline{\Delta}_{I \setminus J}(\alpha).$$

The following Lemma was proven in case $E = S$ in [DdlM88, Lemma 1.8]. It can be found in a slightly different fashion in [KST17, Lemma 4.1.7].

Lemma 2.16. *Let S be a good semigroup and let $E \in GSI_S$. Let $\alpha \in E$. If $\alpha \in \partial_J(E)$ for some $J \subseteq I$, then $\alpha + H_J \subseteq E$.*

Proof. Choose $\delta \in \alpha + H_J$. Then $\delta \in \mathbb{Z}^s$ with

$$\delta_j \geq \alpha_j = \gamma_j^E \text{ for any } j \in J,$$
$$\delta_i = \alpha_i \text{ for any } i \in I \setminus J$$

by definition of H_J and $\partial_J(E)$.

Let us now choose a $\beta \in \mathbb{Z}^s$ such that

$$\begin{aligned}
&\beta_j = \alpha_j \text{ for any } j \in J,\\
&\beta_i > \max\{\gamma_i^E, \alpha_i\} \text{ for any } i \in I \setminus J.
\end{aligned}$$

Then $\beta \geq \gamma^E$, and hence $\beta \in E$. Now applying property (E2) to α and β we obtain for any $j \in J$ an $\alpha' \in E$ with $\alpha' \geq \alpha + \mathbf{e}_j$. Therefore, repeating the process substituting α with α' and taking again a β with the above properties, we obtain an element $\overline{\alpha}$ such that

$$\begin{aligned}
&\overline{\alpha}_j > \alpha_j^{(n)} \geq \alpha_j \text{ for any } j \in J,\\
&\overline{\alpha}_i = \min\{\beta_i, \max\{\gamma_i^E, \alpha_i\}\} = \max\{\gamma_i^E, \alpha_i\} \geq \delta_i \text{ for any } i \in I \setminus J.
\end{aligned}$$

For n big enough, we can suppose $\overline{\alpha} \geq \delta$.

Pick $\epsilon \in \mathbb{Z}^s$ such that

$$\begin{aligned}
&\epsilon_j = \delta_j \geq \gamma_j^E \text{ for any } j \in J,\\
&\epsilon_i > \max\{\gamma_i^E, \delta_i\} \text{ for any } i \in I \setminus J.
\end{aligned}$$

In particular, $\epsilon \geq \gamma^E$, and hence $\epsilon \in E$. Thus, $\delta = \min\{\epsilon, \alpha\} \in E$ since E satisfies (E1). $\square$

Once we know γ^E and $\mathrm{Small}(E)$ we can easily check membership to E.

Proposition 2.17. *Let S be a good semigroup and $E \in \mathfrak{G}_S$. Let $\alpha \in \mathbb{N}^s$. Then $\alpha \in E$ if and only if* $\min\{\alpha, \gamma^E\} \in \mathrm{Small}(E)$.

Proof. First, there are the two easy cases. If $\alpha > \gamma^E$, then clearly $\alpha \in E$, by definition of conductor. On the other hand, if $\alpha < \gamma^E$ then $\alpha = \min\{\alpha, \gamma^E\} \in \mathrm{Small}(E)$ implies $\alpha \in E$. If none of the two is the case, then let $\beta = \min\{\alpha, \gamma^E\}$. Then $\beta \in \partial_J(E)$ for some $J \subseteq I$ and $\alpha \in \beta + H_J$. By Lemma 2.16, we have $\alpha \in E$. $\square$

From this follows that a good semigroup is fully determined by its small elements.

Corollary 2.18. *Let S and S' be two good semigroups. Then $S = S'$ if and only if $\gamma^S = \gamma^{S'}$ and* $\mathrm{Small}(S) = \mathrm{Small}(S')$.

The same is true for good semigroup ideals.

Corollary 2.19. *Let S be a good semigroup and $E, E' \in \mathfrak{G}_S$. Then $E = E'$ if and only if* $\gamma^E = \gamma^{E'}$ *and* $\mathrm{Small}(E) = \mathrm{Small}(E')$.

As a consequence, we can see a good semigroup ideal as the union of its small elements, its conductor, and then a finite number of quadrants starting from points that have at least one coordinate equal to the conductor:

$$E = \mathrm{Small}(E) \cup (\gamma^E + \mathbb{N}^s) \cup \bigcup_{\alpha \in \partial_J(E), J \subseteq I} (\alpha + H_J). \tag{2.1}$$

Notice that this notation is redundant, since if $J' \subseteq J \subseteq I$ and $\alpha \in \partial_J(E)$, then $\alpha \in \partial_{J'}(E)$.

2.3 Length and distance

As a combinatorial counterpart of the relative length of two fractional ideals, we describe the *distance* of two good semigroup ideals.

Definition 2.20. Let S be a good semigroup, and let $E \subseteq \mathbb{Z}^s$ be a subset. Let $\alpha, \beta \in E$ with $\alpha \leq \beta$. Then α and β are *consecutive in* E if $\alpha < \delta < \beta$ implies $\delta \notin E$ for any $\delta \in \mathbb{Z}^s$.

A chain

$$\alpha = \alpha^{(0)} < \cdots < \alpha^{(n)} = \beta \tag{2.2}$$

with $\alpha^{(i)} \in E$ is said to be *saturated of length n* if $\alpha^{(i)}$ and $\alpha^{(i+1)}$ are consecutive in E for any $i \in \{0, \ldots, n-1\}$.

Let us now consider the following property of a subset $E \subseteq \mathbb{Z}^s$:

(E4) For fixed and comparable $\alpha, \beta \in E$, any two saturated chains (2.2) in E have the same length n.

Definition 2.21. Let S be a good semigroup and E a semigroup ideal satisfying (E4). Assume there is a saturated chain of length n between α and β with $\alpha \leq \beta \in E$. We call

$$d_E(\alpha, \beta) := n$$

the *distance of* α *and* β *in* E.

Proposition 2.22. *Let S be a good semigroup. Then any $E \in \mathfrak{G}_S$ satisfies property* (E4).

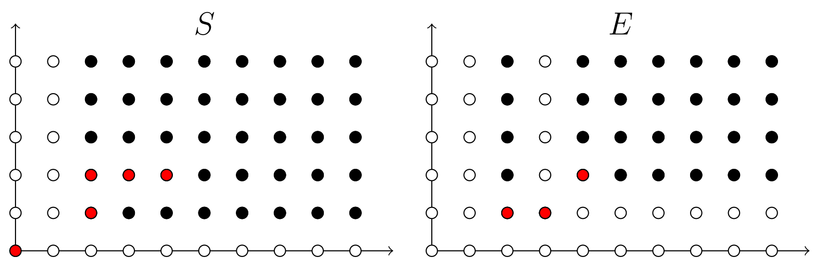
Proof. See [D'A97, Proposition 2.3]. □

Definition 2.23. Let S be a good semigroup, and let $E \subseteq F$ be two semigroup ideals of S satisfying properties (E1) and (E4). Then we call

$$d(F \backslash E) := d_F(\mu^F, \gamma^E) - d_E(\mu^E, \gamma^E)$$

the *distance between* E *and* F.

Example 2.24. In this example the figures illustrate a good semigroup ideal E, contained in the good semigroup S. The red points indicate chains of consecutive points in S (resp. E), going from $0 = \mu^S$ to γ^E (resp. from μ^E to γ^E).



Then

$$d(S \backslash E) = d_S(0, \gamma^E) - d_E(\mu^E, \gamma^E) = 4 - 2 = 2.$$

Remark 2.25. Let S be a good semigroup and let $E \subseteq F$ be two semigroup ideals satisfying properties (E1) and (E4).

(a) By (2.2), d_E is additive with respect to composition of chains.

(b) for any $\alpha, \beta \in E$ with $\alpha \leq \beta$, we have $d_E(\alpha, \beta) \leq d_F(\alpha, \beta)$.

(c) $d(F \backslash E) = d(\alpha + F \backslash \alpha + E)$ for any $\alpha \in \mathbb{Z}^I$.

(d) Using notations from Theorem 2.11

$$d(F \backslash E) = \sum_{m \in M} d(F_{I_m} \backslash E_{I_m}).$$

See [BDF00a, Proposition 2.12.(iii)].

(e) If $\epsilon \geq \gamma^E$, then

$$\begin{aligned} d(F \backslash E) &= d_F(\mu^F, \gamma^E) - d_E(\mu^E, \gamma^E) \\ &= d_F(\mu^F, \gamma^E) + d_F(\gamma^E, \epsilon) - d_E(\mu^E, \gamma^E) - d_E(\gamma^E, \epsilon) \\ &= d_F(\mu^F, \epsilon) - d_E(\mu^E, \epsilon) \end{aligned}$$

by additivity of $d(-,-)$ and since $d_F(\gamma^E, \epsilon) = d_E(\gamma^E, \epsilon)$.

In the following, we collect the main properties of the distance function $d(- \backslash -)$. We begin with additivity.

Lemma 2.26. *Let $E \subseteq F \subseteq G$ be semigroup ideals of a good semigroup S satisfying properties* (E1) *and* (E4). *Then*

$$d(G \backslash E) = d(G \backslash F) + d(F \backslash E).$$

Proof. This can be seen using Remark 2.25.(e), but it was already proven by D'Anna in [D'A97, Proposition 2.7]. □

The following lemma is needed to prove that the distance function detects equality as formulated in [D'A97, Proposition 2.8].

Lemma 2.27. *Let $E \subseteq F$ be two semigroup ideals of a good semigroup S, where $E \in \mathfrak{G}_S$ and F satisfies property* (E1). *Let $\alpha \in F \backslash E$ be minimal. Then any $\beta \in E$ maximal with $\beta < \alpha$ and $\beta' \in E$ minimal with $\alpha < \beta'$ are consecutive in E.*

Proof. Suppose $\beta < \epsilon < \beta'$ for some $\epsilon \in E$. By maximality of β and minimality of β', $\alpha \not\leq \epsilon \not\leq \alpha$, and hence $\min\{\alpha, \epsilon\} < \alpha$. By property (E1) of F, $\min\{\alpha, \epsilon\} \in F$. Thus $\min\{\alpha, \epsilon\} \in E$ by minimality of $\alpha \in F \setminus E$. Then it has to be $\beta = \min\{\alpha, \epsilon\}$ by maximality of β. In particular,

$$\beta_j = \epsilon_j < \alpha_j \leq \beta'_j$$

for some $j \in I$. As $E \in \mathfrak{G}_S$, we can apply property (E2) to $\beta, \epsilon \in E$. This yields an $\epsilon' \in E$ with $\beta_j = \epsilon_j < \epsilon'_j$ and $\beta < \epsilon'$. The element ϵ' may not be comparable with β'. We may however replace ϵ' by $\min\{\epsilon', \beta'\} \in E$ using property (E1) of E, and keep the above properties. Moreover, after this substitution, $\beta < \epsilon' < \beta'$. Hence again by maximality of β, $\beta = \min\{\alpha, \epsilon'\}$. But this is a contradiction since $\beta_j < \alpha_j$ and $\beta_j < \epsilon'_j$. Thus β and β' must be consecutive. □

Proposition 2.28. *Let S be a good semigroup, and let $E, F \in \mathfrak{G}_S$ with $E \subseteq F$. Then $E = F$ if and only if $d(F \backslash E) = 0$.*

Proof. For the non-trivial implication, assume that $d(F\backslash E) = 0$ but $E \subsetneq F$. As $d(F\backslash E) = 0$, by Definition 2.23 $d_E(\mu^E, \gamma^E) = d_F(\mu^F, \gamma^E)$. Since $E \subsetneq F$, $\mu^F \leq \mu^E$. Then Remark 2.25.(a) yields

$$d_E(\mu^E, \gamma^E) = d_F(\mu^F, \mu^E) + d_F(\mu^E, \gamma^E) \geq d_F(\mu^F, \mu^E) + d_E(\mu^E, \gamma^E).$$

Thus $d_F(\mu^F, \mu^E) \leq 0$ and $\mu^E = \mu^F$. Pick $\alpha \in F\backslash E$ minimal. In particular, $\mu^E < \alpha < \gamma^E$. In fact, assume that $\alpha \not\leq \gamma^E$. Then applying property (E1) of F to α and γ^E yields a $\delta \in F$ with $\delta < \alpha, \delta < \gamma^E$, and hence $\delta \in E$ by minimality of α. But then there is an $i \in I$ such that $\delta_i = \gamma_i < \alpha_i$, so that $\delta \in \partial_i(E)$ and $\alpha \in \delta + H_i$. Then Lemma 2.16 implies $\alpha \in E$, contradicting the assumption on α.

By Lemma 2.27 there are $\beta, \beta' \in E$ which are consecutive in E but not in F such that $\mu^E \leq \beta < \alpha < \beta' \leq \gamma^E$. Since E satisfies property (E4) (see Proposition 2.22) and $E \subseteq F$, by additivity of the distance we obtain

$$\begin{aligned} d_E(\mu^E, \gamma^E) &= d_E(\mu^E, \beta) + d_E(\beta, \beta') + d_E(\beta', \gamma^E) \\ &\leq d_F(\mu^E, \beta) + d_F(\beta, \beta') + d_F(\beta', \gamma^E) \\ &= d_F(\mu^E, \gamma^E) = d_F(\mu^F, \gamma^E) \end{aligned}$$

But $d_E(\beta, \beta') = 1$, while $d_F(\beta, \beta') = d_F(\beta, \alpha) + d_F(\alpha, \beta') \geq 2$. Hence $d_E(\mu^E, \gamma^E) < d_F(\mu^F, \gamma^E)$, contradicting the assumptions. $\square$

Finally, we show that the distance function coincides with the relative length of fractional ideals when evaluated on their value semigroup ideals.

Proposition 2.29. *Let R be an admissible ring. If $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$ such that $\mathcal{E} \subseteq \mathcal{F}$, then*

$$\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_\mathcal{F}\backslash\Gamma_\mathcal{E}).$$

Proof. See [D'A97, Proposition 2.2] for part of the following proof in the local case. By Corollary 1.25, $E := \Gamma_\mathcal{E}$ and $F := \Gamma_\mathcal{F}$ are good semigroup ideals of Γ_R and hence satisfy property (E4) by Proposition 2.22.

Let $\mathfrak{r}$ be the Jacobson radical of R. By Theorem 1.3.(d), $\mathfrak{m}_V \cap R\,\mathrm{Max}(R)$ for any $V \in \mathfrak{V}_R$. Thus $\mathfrak{r} \subseteq \bigcap_{\mathfrak{m}\in\mathrm{Max}(R)} \mathfrak{m} \subseteq \bigcap_{V\in\mathfrak{V}_R} \mathfrak{m}_V$ and hence $\nu(x) \geq (1, \ldots, 1)$ for any $x \in \mathfrak{r}$ by equation (B.2). By Lemma 1.14 $\mathcal{C}_\mathcal{E} = x\overline{R}$ for some $x \in Q_R^{\mathrm{reg}}$, and by Lemma 1.9.(b) and (c), $x\overline{R} = \mathcal{Q}^{\nu(x)}$ and $\Gamma_{\mathcal{Q}^{\nu(x)}} = \nu(x) + \mathbb{N}^{\mathfrak{V}_R}$. Hence $\mathcal{C}_\mathcal{E} = \mathcal{Q}^\epsilon$ for some $\epsilon \in \mathbb{Z}^{\mathfrak{V}_R}$ with $\epsilon \geq \gamma^E$ It follows that, for sufficiently large $k \in \mathbb{N}$, $\mu^F + k \cdot (1, \ldots, 1) \geq \epsilon$ and so

$$\mathfrak{r}^k\mathcal{F} \subseteq \left(\bigcap_{V\in\mathfrak{V}_R} \mathfrak{m}_V^k\right)\mathcal{F} \subseteq \mathcal{Q}^{\mu^F + k\cdot(1,\ldots,1)} \subseteq \mathcal{Q}^\epsilon = \mathcal{C}_\mathcal{E} \subset \mathcal{E}.$$

This turns $\mathcal{F}/\mathcal{E}$ into a module over the ring $R/\mathfrak{r}^k$. The power of the Jacobian ideal can be written $\mathfrak{r}^k = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \mathfrak{m}^k$. As any two maximal ideals are coprime, by [Mat89, Theorem 1.4] the ring $R/\mathfrak{r}^k$ can be written as a product

$$R/\mathfrak{r}^k = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} R/\mathfrak{m}^k = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} R_\mathfrak{m}/\mathfrak{m}^k = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} (R/\mathfrak{r}^k)_\mathfrak{m}.$$

where $R/\mathfrak{m}^k = R_\mathfrak{m}/\mathfrak{m}^k$ as $R/\mathfrak{m}^k$ is already local (see proof of [Mat89, Thm. 8.15]). It follows that $\mathcal{F}/\mathcal{E} = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} (\mathcal{F}/\mathcal{E})_\mathfrak{m}$, and hence

$$\ell_R(\mathcal{F}/\mathcal{E}) = \sum_{\mathfrak{m}\in\mathrm{Max}(R)} \ell_{R_\mathfrak{m}}(\mathcal{F}_\mathfrak{m}/\mathcal{E}_\mathfrak{m}).$$

Due to Theorem 1.24, $\Gamma_{\mathcal{F}} = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \Gamma_{\mathcal{F}_{\mathfrak{m}}}$. By Theorem 2.11, this is equal to $\prod_{m\in M}(\Gamma_{\mathcal{F}})_{I_m}$. And the same holds for $\Gamma_{\mathcal{E}}$. Thus, by Remark 2.25.(d), $\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}})$ if and only if $\ell_{R_{\mathfrak{m}}}(\mathcal{F}_{\mathfrak{m}}/\mathcal{E}_{\mathfrak{m}}) = d((\Gamma_{\mathcal{F}})_{I_m}\backslash(\Gamma_{\mathcal{E}})_{I_m})$. We may therefore assume that R is local.

Let $\alpha, \beta \in E$ be consecutive in E. Then $d_E(\alpha,\beta) = 1$ by definition. For any $\delta \in \mathbb{Z}^{\mathfrak{V}_R}$ with $\alpha < \delta < \beta$, $\delta \notin E$ as α and β are consecutive, and hence $\ell_R(\mathcal{E}^\delta/\mathcal{E}^{\delta+\mathbf{e}_V}) = 0$ for some $V \in \mathfrak{V}_R$ by Lemma 1.29. If $\delta_W = \beta_W$ for some $W \in \mathfrak{V}_R$, then $\mathcal{E}^\beta/\mathcal{E}^{\beta+\mathbf{e}_W} \subseteq \mathcal{E}^\delta/\mathcal{E}^{\delta+\mathbf{e}_W}$ and hence $\ell_R(\mathcal{E}^\delta/\mathcal{E}^{\delta+\mathbf{e}_W}) \geq \ell_R(\mathcal{E}^\beta/\mathcal{E}^{\beta+\mathbf{e}_W}) = 1$ since $\beta \in E$, again by Lemma 1.29. Since $\ell_R(\mathcal{E}^\delta/\mathcal{E}^{\delta+\mathbf{e}_V}) = 0$ for some $V \in \mathfrak{V}_R$, it has to be $\delta_V < \beta_V$ for some $V \in \mathfrak{V}_R$. Thus by additivity of length

$$\ell_R(\mathcal{E}^\alpha/\mathcal{E}^\beta) = \sum_{\alpha<\delta<\beta} \ell_R(\mathcal{E}^\delta/\mathcal{E}^{\delta+\mathbf{e}_V}) = 1.$$

By additivity of length and distance it follows that

$$\begin{aligned} d_E(\mu^E,\epsilon) &= \sum_{\substack{\mu^E\leq\alpha<\beta\leq\epsilon\\ \alpha,\beta \text{ consec.}}} d_E(\alpha,\beta) = \sum_{\substack{\mu^E\leq\alpha<\beta\leq\epsilon\\ \alpha,\beta \text{ consec.}}} \ell_R(\mathcal{E}^\alpha/\mathcal{E}^\beta) = \ell_R(\mathcal{E}^{\mu^E}/\mathcal{E}^\epsilon) \\ &= \ell_R(\mathcal{E}/\mathcal{E}^\epsilon), \end{aligned}$$

Recall that $\mathcal{C}_{\mathcal{E}} = \mathcal{Q}^\epsilon \subseteq \mathcal{E} \subseteq \mathcal{F}$, so that $\mathcal{C}_{\mathcal{E}} = \mathcal{E}\cap\mathcal{Q}^\epsilon = \mathcal{E}^\epsilon = \mathcal{F}\cap\mathcal{Q}^\epsilon = \mathcal{F}^\epsilon$. Hence, using Remark 2.25.(e),

$$\begin{aligned} d(F\backslash E) &= d_F(\mu^F,\epsilon) - d_E(\mu^E,\epsilon) \\ &= \ell_R(\mathcal{F}/\mathcal{F}^\epsilon) - \ell_R(\mathcal{E}/\mathcal{E}^\epsilon) \\ &= \ell_R(\mathcal{F}/\mathcal{E}^\epsilon) - \ell_R(\mathcal{E}/\mathcal{E}^\epsilon) = \ell_R(\mathcal{F}/\mathcal{E}). \end{aligned}$$ □

As a consequence, the value semigroup ideals detect equality of regular fractional ideals (as stated already by D'Anna in [D'A97, Corollary 2.5]).

Corollary 2.30. *Let R be an admissible ring, and let $\mathcal{E},\mathcal{F} \in \mathfrak{R}_R$ be such that $\mathcal{E} \subseteq \mathcal{F}$. Then $\mathcal{E} = \mathcal{F}$ if and only if $\Gamma_{\mathcal{E}} = \Gamma_{\mathcal{F}}$.*

Proof. Since $\mathcal{E} \subseteq \mathcal{F}$, also $\Gamma_{\mathcal{E}} \subseteq \Gamma_{\mathcal{F}}$ by Remark 1.17. The equality $\mathcal{E} = \mathcal{F}$ holds if and only if $\ell_R(\mathcal{F}/\mathcal{E}) = 0$. Due to Proposition 2.29 this is true if and only if $d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}}) = 0$ which, by Propositions 2.28, is equivalent to $\Gamma_{\mathcal{F}} = \Gamma_{\mathcal{E}}$. □

3
Good generating systems

An abelian semigroup A is finitely generated if there exists a finite set $G = \{a_1, \ldots, a_N\}$ of elements of A such that each element of A is a sum of elements of G (with repeated summands). This is true if A is the image of the semigroup $\mathbb{N}^N$ under a semigroup homomorphism.

Campillo, Delgado and Gusein-Zade in [CDGZ99] prove that plane curves have a value semigroup which is *w-finitely generated*, and they give a correspondence between the *w-generators* and the components of the exceptional divisor. For a reducible curve, they also find a minimal set of generators. For non-plane curves, it is not known a correspondence between w-generators of the value semigroup and objects related to the curve.

In [CDGZ99, Statement 1], the authors state without a proof that a semigroup is w-finitely generated if and only if it is the image of a coordinate semigroup under a semigroup homomorphism $\mathbb{N}^N \to \mathbb{N}^s$.

If the statement is true, then it is not difficult to see that every good semigroup is w-generated. However, even if so, it is not possible to choose a unique minimal system of generators. In fact, one can define different systems of w-generators which are minimal with respect to inclusion.

For this reason we give a different definition of generating system. Taking advantage of the fact that $\mathrm{Small}(S)$ determines a good semigroup S, and analogously $\mathrm{Small}(E)$ determines a good semigroup ideal E of S, we define good generating systems as sets of elements which generate $\mathrm{Small}(S)$ through sums and minima. Then $\mathrm{Small}(S)$ (resp. $\mathrm{Small}(E)$) is always a good generating system of S (resp. E) according to our definition, but it does not need to be minimal. We develop techniques to reduce any good generating system to a minimal one, and then we show that, in case S is local, such minimal system of generators is unique for S (resp. for E). This is part of a joint work with M. D'Anna, P. Garcia-Sanchez and V. Micale [DGSMT17]. All the proofs are original work, as the author generalized results by D'Anna, Garcia-Sanchez and Micale in the two-dimensional case to any dimension.

3.1 Good generating systems of good semigroups

For a subset A of a monoid M, we denote by

$$\langle A \rangle = \{a_1 + \cdots + a_n \mid n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$$

the submonoid of M generated by A.

Let $s \geq 1$. For a set $G \subseteq \mathbb{N}^s$ let $[G]$ be the smallest submonoid of $\mathbb{N}^s$ containing G which is closed under addition and minima (i.e. $[G] \supseteq \langle G \rangle$ and $[G]$ satisfies (E1)). Such a $[G]$ exists. In fact, the set $\mathscr{G} = \{$submonoids of $\mathbb{N}^s$ containing G which are closed under addition and minima$\} \supseteq \{\mathbb{N}^s\} \neq \emptyset$ and, moreover, the intersection of submonoids of $(\mathbb{N}^s, +)$ closed under minima is again a submonoid of $(\mathbb{N}^s, +)$ closed under minima. Thus $[G] = \bigcap \mathscr{G}$ satisfies the requirements.

Proposition 3.1. *Let $G \subseteq \mathbb{N}^s$. Then*

$$[G] = \{\min\{g_1, \ldots, g_s\} \mid g_i \in \langle G \rangle\}.$$

Proof. First of all, let us prove that

$$[G] = \{\min\{g_1, \ldots, g_n\} \mid n \in \mathbb{N}, g_i \in \langle G \rangle\}.$$

The inclusion $\{\min\{g_1, \ldots, g_n\} \mid n \in \mathbb{N}, g_i \in \langle G \rangle\} \subseteq [G]$ is clear by definition of $[G]$. So let $g \in [G]$ and assume

$$g = \sum_i g_i, \text{ with } g_i = \min\{h_j^{(i)}\}_{j \in J_i}.$$

Then, since for $\alpha, \beta, \gamma \in \mathbb{N}^s$

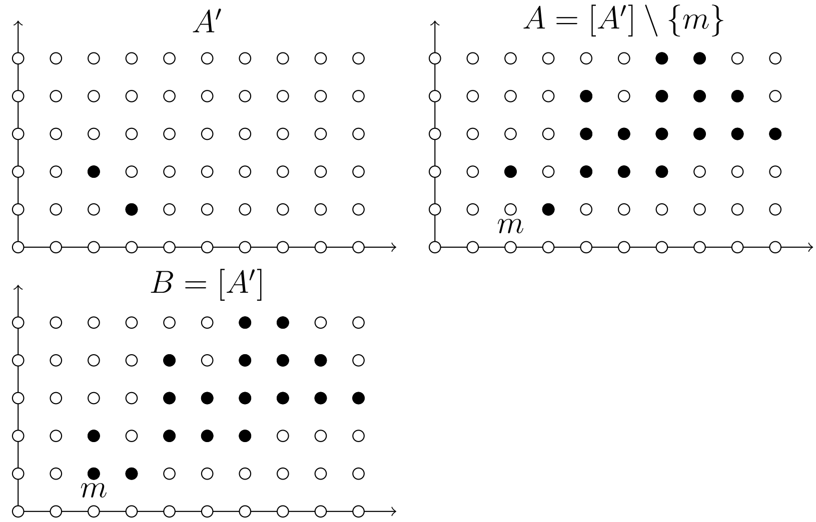
$$\min\{\alpha, \beta\} + \gamma = \min\{\alpha + \gamma, \beta + \gamma\}.$$

we have

$$\begin{aligned} g &= \sum_{i=1}^{r} \min\{h_j^{(i)}\}_{j \in J_i} = \sum_{i=1}^{r-2} \min\{h_j^{(i)}\}_{j \in J_i} + \min\{h_j^{(r-1)}\}_{j \in J_{r-1}} + \min\{h_{j'}^{(r)}\}_{j' \in J_r} \\ &= \sum_{i=1}^{r-2} \min\{h_j^{(i)}\}_{j \in J_i} + \min\{\min\{h_j^{(r-1)} + h_{j'}^{(r)}\}_{j' \in J_r}\}_{j \in J_{r-1}} \\ &= \sum_{i=1}^{r-2} \min\{h_j^{(i)}\}_{j \in J_i} + \min\{h_j^{(r-1)} + h_{j'}^{(r)}\}_{j \in J_{r-1}, j' \in J_r} \\ &= \cdots = \min\left\{\sum_i h_{j(i)}^{(i)}\right\}_{j(i) \in J_i}. \end{aligned}$$

Thus $g \in \{\min\{g_1, \ldots, g_n\} \mid n \in \mathbb{N}, g_i \in \langle G \rangle\}$. Moreover, the intersection of submonoids of $(\mathbb{N}^s, +)$ closed under minima are again submonoids of $(\mathbb{N}^s, +)$ closed under minima. Since the minimum of two elements is taken component-wise, for any set $A = \{\alpha_1, \ldots, \alpha_n\}$ in $\mathbb{N}^s$, the minimum $\min A = \min\{\alpha_1, \ldots, \alpha_n\}$ is the minimum of at most s elements of A. Hence the claim. □

Remark 3.2. Observe that $[A] = [B]$ does not imply $\langle A \rangle = \langle B \rangle$. In fact, let $A = [A'] \setminus \{m\}$, where A' is a subset of $\mathbb{N}^s$ and m is a smallest element of $[A']$ obtained as a minimum of other elements. Consider $B = [A']$. Then clearly $[B] = [A] = [A']$, but $\langle A \rangle = A \subsetneq B = \langle B \rangle$. The following figure shows an example of this fact.



Notation 3.3. Given a $\delta \in \mathbb{N}^s$ and a set $B \subseteq \mathbb{N}^s$ we denote

$$[G]_\delta := \{\min\{\delta, g\} \mid [g \in G]\}$$

and

$$B(\delta) = \{\alpha \in \mathbb{N}^s \mid \alpha \leq \delta\}.$$

We are interested in in finding out when $[G]$ covers $\mathrm{Small}(S)$ for a good semigroup S.

Definition 3.4. Let $G \subseteq \mathbb{N}^s$, and S a good semigroup with conductor γ. Then G is said to be a *good generating system* for S if

$$[G]_\gamma \cup \{0\} = \mathrm{Small}(S).$$

We say that G is *minimal* if no proper subset of G is a good generating system of S. In particular, $0 \notin G$.

Remark 3.5. Let S be a good semigroup with conductor γ. Since $[G]_\gamma = [B(\gamma)]_\gamma$, we can always assume that good systems of generators are contained in $B(\gamma)$. In particular, $[\mathrm{Small}(S)]_\gamma = \mathrm{Small}(S)$, so that $\mathrm{Small}(S)$ is always a good generating system. Therefore a good generating system always exists.

Taking Remark 3.5 into account, our goal is to remove redundant elements in $\mathrm{Small}(S)$ in order to find a minimal system.

The following lemma is trivial, considering the definitions.

Lemma 3.6. *Let S be a good semigroup. Let G be a good generating system for S and let $\alpha \in G$. If $\alpha \in [G \setminus \{\alpha\}]_\gamma$, then $G \setminus \{\alpha\}$ is a good generating system for S.*

We now give some technical lemmas o characterize elements belonging to $[G]$ for some $G \subseteq \mathbb{N}^s$.

Lemma 3.7. *Let $G \subseteq \mathbb{N}^s$ and $\alpha \in \mathbb{N}^s$. Then $\alpha \in [G]$ if and only if $\overline{\Delta}_i(\alpha) \cap \langle G \rangle \neq \emptyset$ for any $i \in I$.*

Proof. Suppose $\alpha \in [G]$. If $\alpha \in \langle G \rangle$, then $\overline{\Delta}_i(\alpha) \cap \langle G \rangle \ni \alpha$ for any $i \in I$. So let us suppose $\alpha \in [G] \setminus \langle G \rangle$. By Proposition 3.1 we know that $\alpha \in [G]$ if and only if it is the minimum of s elements of $\langle G \rangle$, $\{\beta^{(i)}\}_{i \in I}$, and since we are assuming $\alpha \notin \langle G \rangle$ we have $\alpha \neq \beta^{(i)}$ for any $i \in I$. Since α is the minimum of the βs, we have $\alpha_j \leq \beta_j^{(i)}$ for any $j \in I$. Moreover, for any $j \in I$ there exists an $i_j \in I$ such that $\alpha_j = \beta_j^{(i_j)}$. This means that for any $i \in I$ there is an index j_i with $\beta^{(j_i)}$ belonging to $\overline{\Delta}_i(\alpha)$. In particular, for any $i \in I$, $\beta^{(j_i)} \in \overline{\Delta}_i(\alpha) \cap \langle G \rangle$.

Conversely, suppose $\overline{\Delta}_i(\alpha) \cap \langle G \rangle \neq \emptyset$ for any $i \in I$. If $\alpha \in \langle G \rangle$ then of course $\alpha \in [G]$. Hence suppose $\alpha \notin \langle G \rangle$, and let $\beta^{(i)} \in \overline{\Delta}_i(\alpha) \cap \langle G \rangle$ for any $i \in I$. Then clearly $\alpha = \min\{\beta^{(i)}\}_{i \in I}$. Therefore $\alpha \in [G]$. □

Substituting $[G]$ with $[G]_\delta$ we can be more precise.

Lemma 3.8. *Let $\delta \in \mathbb{N}^s$ and $G \subseteq B(\delta)$. Let $\alpha \in B(\delta) \setminus \{\delta\}$ and let J be maximal (w.r.t. inclusion) with the property $\alpha \in \Delta_J(\delta)$ (J can also be empty). Then $\alpha \in [G]_\delta$ if and only if $\overline{\Delta}_i(\alpha) \cap \langle G \rangle \neq \emptyset$ for any $i \in I \setminus J$.*

Proof. Suppose $\alpha \in [G]_\delta$. Then $\alpha = \min\{\alpha', \delta\}$ for some $\alpha' \in [G]$. By Lemma 3.7, $\overline{\Delta}_i(\alpha') \cap \langle G \rangle \neq \emptyset$ for any $i \in I$. From the assumptions we have $\alpha_j = \delta_j$ for $j \in J$ and $\alpha_i < \delta_i$ for $i \in I \setminus J$. Therefore is has to be $\alpha'_j \geq \alpha_j$ for $j \in J$ and $\alpha'_i = \alpha_i$ for $i \in I \setminus J$. So $\overline{\Delta}_i(\alpha') \subseteq \overline{\Delta}_i(\alpha)$ for any $i \in I \setminus J$, and $\overline{\Delta}_i(\alpha) \cap \langle G \rangle \neq \emptyset$ for any $i \in I \setminus J$.

Conversely, suppose $\overline{\Delta}_i(\alpha) \cap \langle G \rangle \neq \emptyset$ for any $i \in I \setminus J$. Let $\beta^{(i)} \in \overline{\Delta}_i(\alpha) \cap \langle G \rangle$. Then $\alpha = \min\{\delta, \{\beta^{(i)}\}_{i \in I \setminus J}\} \in [G]_\delta$. □

3.1.1 The local case

From now on we will assume S to be a good local semigroup, i.e. zero is the only element with zero components (see Definition 2.1). Theorem 2.11 tells us that every good semigroup can be decomposed uniquely as a product of good local semigroups. We will observe later how we can use this decomposition to generalize our results to non-local good semigroups.

Lemma 3.9. *Let S be a good local semigroup. Let $G \subseteq \mathrm{Small}(S)$ be a good generating system for S and let $\alpha \neq \beta \in \mathbb{Z}^s$ be such that $\alpha \in \overline{\Delta}_i(\beta) \cap \langle G \rangle$ for some $i \in I$. Then $\alpha \in \langle G \setminus \{\beta\} \rangle$.*

Proof. Assume

$$\alpha = \beta + \beta^{(1)} + \cdots + \beta^{(l)}$$

for some $\beta^{(k)} \in G \subseteq S$. As $\alpha_i = \beta_i$, we have

$$\beta_i^{(1)} + \cdots + \beta_i^{(l)} = 0$$

Since $\beta^{(k)} \geq 0$, this implies $\beta_i^{(k)} = 0$ for any $k \in \{1, \ldots, l\}$. But $\alpha \neq \beta$, so there exists k such that $\beta^{(k)} \neq 0$. This is a contradiction to S being local. Hence the claim. □

The following lemma will be used to prove Theorem 3.12.

Lemma 3.10. *Let S be a good local semigroup. Let G be a good generating system for S and $\alpha \in G \setminus \partial(S)$. Assume $G \subseteq \mathrm{Small}(S)$. If there exists $\beta \in \overline{\Delta}(\alpha) \cap \langle G \setminus \{\alpha\} \rangle$, then $G \setminus \{\alpha\}$ is a good generating system for S.*

Proof. Since $\alpha \in G \setminus \partial(S) \subseteq \mathrm{Small}(S) \setminus \partial(S)$, we have $\alpha_i < \gamma_i$ for any $i \in I$. If $\alpha = \beta < \gamma$, then we are done, since then clearly $\alpha \in [G \setminus \{\alpha\}]_\gamma$.

So let us suppose $\alpha \neq \beta$. Since $\beta \in \overline{\Delta}(\alpha)$, there exists an $i \in I$ such that $\beta_i = \alpha_i$. Then, applying (E2), one can find a $\delta \in S \setminus \{\alpha, \beta\}$ such that $\alpha = \min\{\beta, \delta\}$.

Eventually substituting δ with $\min\{\delta, \gamma\}$ we can assume $\delta \in \mathrm{Small}(S) = [G]_\gamma$. Since $\alpha \notin \partial(S)$ after this substitution we still have $\delta \neq \alpha \neq \beta$.

By Proposition 3.1, we can write

$$\delta = \min\{\gamma, \{\delta^{(i)}\}_{i \in I}\}, \text{ with } \delta^{(i)} \in \langle G \rangle.$$

As $\delta \neq \alpha$ and $\alpha < \delta \leq \delta^{(i)}$, we also have $\alpha \neq \delta^{(i)}$ for any $i \in I$. Let $J \subseteq I$ be the maximal set of indices such that $\delta \in \overline{\Delta}_J(\alpha)$ (which implies $\beta \in \overline{\Delta}_{I \setminus J}(\alpha)$). Then for any $j \in J$ there exists an $i_j \in I$ such that $\delta_j^{(i_j)} = \delta_j = \alpha_j$. Hence for any $j \in J$, $\delta^{(i_j)} \in \overline{\Delta}_j(\alpha)$ and moreover

$$\alpha = \min\{\beta, \{\delta^{(i_j)}\}_{j \in J}\}.$$

Now

$$\delta^{(i_j)} \in \overline{\Delta}_j(\alpha) \cap \langle G \rangle$$

but $\delta^{(i_j)} \neq \alpha$. Thus by Lemma 3.9, for any $j \in J$,

$$\delta^{(i_j)} \in \langle G \setminus \{\alpha\} \rangle$$

and $\alpha = \min\{\beta, \{\delta^{(i_j)}\}_{j \in J}\} \in [G \setminus \{\alpha\}]_\gamma$. □

Remark 3.11. Let S be a good local semigroup. Let G be a good generating system such that $G \neq \{\gamma\}$. Then G contains an element α with $\gamma > \alpha > 0$. Thus there is a positive integer k such that $\gamma \leq k\alpha$. Hence $\gamma \in [G]_\gamma$. Therefore, we can always assume $\gamma \notin G$.

Due to Remark 3.11, unless $G = \{\gamma\}$, from now on we always assume $\gamma \notin G$.

The next theorem provides a characterization of good minimal generating systems for good local semigroups.

Theorem 3.12. *Let $S \neq \emptyset$ be a good local semigroup and let G be a good generating system for S. For $\alpha \in G$, let J_α be the maximal set of indices (w.r.t. inclusion) with the property $\alpha \in \partial_{J_\alpha}(S)$. Then G is a minimal good generating system if and only if for any $\alpha \in G$*

$$\emptyset = \begin{cases} \overline{\Delta}(\alpha) \cap \langle G \setminus \{\alpha\} \rangle & \text{if } J_\alpha = \emptyset \\ \overline{\Delta}_k(\alpha) \cap \langle G \setminus \{\alpha\} \rangle \text{ for some } k \in I \setminus J_\alpha & \text{if } J_\alpha \neq \emptyset. \end{cases}$$

Proof. In order to simplify notation, if there is no possible misunderstanding with α, let us write J instead of J_α.

Necessity. Assume that G is a minimal good generating system for S and let $\alpha \in G$. If $J = \emptyset$, that is, $\alpha \in G \setminus \partial(S)$, then the claim follows by Lemma 3.10. Now assume that J is not empty and that for any $i \in I \setminus J$ there exists an

$$\alpha^{(i)} \in \overline{\Delta}_i(\alpha) \cap \langle G \setminus \{\alpha\} \rangle.$$

Then

$$\alpha = \min\{\gamma, \{\alpha^{(i)}\}_{i \in I \setminus J}\},$$

and consequently $\alpha \in [G \setminus \{\alpha\}]_\gamma$, which is a contradiction.

Sufficiency. If $G = \{\alpha\}$, then G is minimal. Assume therefore $|G| \geq 2$ and G not minimal. Then there exist an $\alpha \in G$ such that $\alpha \in [G \setminus \{\alpha\}]_\gamma$. By Proposition 3.1, there exist $\{\alpha^{(i)}\}_{i\in I} \subseteq \langle G \setminus \{\alpha\}\rangle$ such that

$$\alpha = \min\{\gamma, \{\alpha^{(i)}\}_{i\in I}\}.$$

Since S is local, α is a positive element in $\mathbb{N}^s$. Let J be the set of indexes maximal with the property $\alpha \in \partial_J(S)$. Then $\alpha_k < \gamma_k$ for any $k \in I \setminus J$. Hence for any $k \in I \setminus J$ there is an i_k with $\alpha_k^{(i_k)} = \alpha_k$ and $\alpha^{(i_k)} \geq \alpha$, i.e. $\alpha^{(i_k)} \in \overline{\Delta}_k(\alpha) \cap \langle G \setminus \{\alpha\}\rangle$ for any $k \in I \setminus J$. This is a contradiction. □

Minimal good generating systems are unique for good local semigroups.

Theorem 3.13. *Let S be a good local semigroup. Then S has a unique minimal good generating system.*

Proof. Let γ be the conductor of S. Let A and B be two minimal good generating systems for S, and let β be minimal in $(A \cup B) \setminus (A \cap B)$. Without loss of generality, we can assume $\beta \in B$.

Let us prove that

$$\beta \notin \{\min\{\gamma, \alpha\} \mid \alpha \in \langle A\rangle\}. \tag{3.1}$$

Assume there are $\{\alpha^{(l)}\}_l \subseteq A$ with $\beta = \min\{\gamma, \sum_l \alpha^{(l)}\}$. The sum has more than one term. Otherwise, $\beta = \min\{\gamma, \alpha^{(1)}\} = \alpha^{(1)} \in A$ which is a contradiction to $\beta \notin A$. In particular, $\alpha^{(l)} \neq \beta$ for any l. As $\alpha^{(l)} \leq \sum_l \alpha^{(l)}$ and $\alpha^{(l)} \leq \gamma$, we have $\alpha^{(l)} \leq \min\{\gamma, \sum_l \alpha^{(l)}\} = \beta$. Together with the considerations above, this gives $\alpha^{(l)} < \beta$ for any l. But then $\alpha^{(l)} \in B$ for any l by minimality of β and thus $\beta \in [B \setminus \{\beta\}]_\gamma$, which contradicts the minimality of B. Thus we obtain (3.1).

Let now $J \subseteq I$ be maximal (possibly empty) such that $\beta \in \partial_J(S)$. As $\beta \in B \subseteq [A]_\gamma$, by Lemma 3.8 there exist $\{\epsilon^{(i)}\}_{i\in I\setminus J} \in \langle A\rangle$ such that

$$\epsilon^{(i)} \in \overline{\Delta}_i(\beta) \cap \langle A\rangle \text{ for any } i \in I \setminus J$$

such that

$$\beta = \min\{\gamma, \{\epsilon^{(i)}\}_{i\in I\setminus J}\} \text{ with } \epsilon_i^{(i)} = \beta_i \text{ for } i \in I \setminus J.$$

As $\epsilon^{(i)}$ do not need to be in $[B]_\gamma$, let us consider $\zeta^{(i)} = \min\{\gamma, \epsilon^{(i)}\}$ for any $i \in I \setminus J$. Then

$$\zeta^{(i)} \in \overline{\Delta}_i(\beta) \cap \{\min\{\gamma, \alpha\} \mid \alpha \in \langle A\rangle\} \text{ for any } i \in I \setminus J$$

and

$$\beta = \min\{\gamma, \{\zeta^{(i)}\}_{i\in I\setminus J}\} \text{ with } \zeta_i^{(i)} = \beta_i \text{ for } i \in I \setminus J. \tag{3.2}$$

Let $K_i \subseteq I$ be the maximal set of indices such that $\zeta^{(i)} \in \partial_{K_i}(S)$ for any $i \in I \setminus J$. As $\zeta^{(i)} \in [A]_\gamma = [B]_\gamma$, again by Lemma 3.8 there exist $\{\delta^{(i,j)}\}_{j\in I\setminus K_i} \in \langle B\rangle$ such that

$$\delta^{(i,j)} \in \overline{\Delta}_j(\zeta^{(i)}) \cap \langle B\rangle \text{ for any } j \in I \setminus K_i$$

such that

$$\zeta^{(i)} = \min\{\gamma, \{\delta^{(i,j)}\}_{j\in I\setminus K_i}\} \text{ with } \delta_j^{(i,j)} = \zeta_j^{(i)} \text{ for } j \in I \setminus K_i.$$

for any $i \in I \setminus J$. Since $\zeta_i^{(i)} = \beta_i < \gamma_i$ and hence $i \notin K_i$ for any $i \in I \setminus J$, and $\delta^{(i,j)} \geq \zeta^{(i)} \geq \beta$ for any $j \in I \setminus K_i$ and any $i \in I \setminus J$, for any $i \in I \setminus J$ there exists j_i such that $\delta_i^{(i,j_i)} = \zeta_i^{(i)}$. Thus

$$\delta^{(i,j_i)} \in \overline{\Delta}_i(\zeta^{(i)}) \cap \langle B\rangle \subseteq \bar{\Delta}_i(\beta) \cap \langle B\rangle.$$

So by (3.2) we can write

$$\beta = \min\{\gamma, \{\delta^{(i,j_i)}\}_{i \in I \setminus J}\} \text{ with } \delta_i^{(i,j_i)} = \beta_i \text{ for } i \in I \setminus J.$$

By Theorem 3.12, this implies that there exists a $i \in I \setminus J$ such that

$$\delta := \delta^{(i,j_i)} \in \langle B \rangle \setminus \langle B \setminus \{\beta\} \rangle.$$

This means $\delta = \beta + \eta$, with $\eta \in \langle B \rangle$. Since $\delta_i = \beta_i$, $\eta_i = 0$. But S is local, and this forces $\eta = 0$. So $\delta = \beta$. But by (3.1), $\zeta^{(i)} \neq \beta$, so in particular $\delta \geq \zeta^{(i)} > \beta$. This is a contradiction. So the claim is proved. $\square$

3.1.2 The non-local case

We already remarked that every good semigroup is a product of good local semigroups (see Theorem 2.11). The good generating system given by the product of the minimal good generating systems of the single components is uniquely determined, but in general it will not be a minimal generating system of the semigroup according to Definition 3.4.

Unfortunately, there is no analogous of Theorem 3.13 in the non-local case, as minimal good generating systems of non-local semigroups do not need to be unique.

Example 3.14 ([DGSMT17, Example 8]). Let S be the numerical semigroup $S := \langle 3, 5, 7 \rangle$, and let T be the numerical semigroup $T := \langle 2, 5 \rangle$. Consider their cartesian product $W = S \times T$. It is easy to verify that

$$\mathrm{Small}(W) = \{(0,0), (0,2), (0,4), (3,0), (3,2), (3,4), (5,0), (5,2), (5,4)\}.$$

Then both

$$\{(0,4), (3,2), (5,0)\}$$

and

$$\{(0,4), (3,4), (5,0), (5,2)\}$$

are minimal good generating systems for W.

3.1.3 Good semigroups as semirings

In [CH17], the authors show that value semigroups of algebroid curves determine (and are determined) by semirings, called *semiring of values*, which are finitely generated.

While their discussion makes use of the algebraic structure coming from the algebroid curve, and hence is particular to the case of value semigroups, we want to show that the "closure" of a good semigroup is always finitely generated as semiring.

In the following we will use the following concept of semiring.

Definition 3.15. A *semiring with respect to the operations* $\min$ *and* $+$ is a set $T \subseteq (\mathbb{N} \cup \{\infty\})^s$ equipped with two binary operations:

$$\min\{\alpha, \beta\} = (\min\{\alpha_1, \beta_1\}, \ldots, \min\{\alpha_s, \beta_s\}) \in T \text{ for any } \alpha, \beta \in (\mathbb{N} \cup \{\infty\})^s$$

where $\min\{n, \infty\} = n$ for any $n \in \mathbb{N}$, and

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_s + \beta_s) \in T \text{ for any } \alpha, \beta \in (\mathbb{N} \cup \{\infty\})^s$$

where $n+\infty=\infty$ for any $n\in\mathbb{N}$. Then $(R,\min)$ is a commutative monoid with identity element $\underline{\infty}:=(\infty,\dots,\infty)$, and $(T,+)$ is a commutative monoid with identity element 0. Moreover, $+$ distributes over $\min$, i.e.

$$\alpha+\min\{\beta,\gamma\}=\min\{\beta,\gamma\}+\alpha=\min\{\alpha+\beta,\alpha+\gamma\}.$$

A semiring T is *finitely generated* if there exists a finite set $G\subseteq(\mathbb{N}\cup\{\infty\})^s$ such that for any $\alpha\in T$

$$\alpha=\min\left\{\sum_{g\in G}\alpha_{1g}g,\dots,\sum_{g\in G}\alpha_{rg}g\right\} \tag{3.3}$$

with $\alpha_{ig}\in\mathbb{N}$ and $r\leq s$ (see Proposition 3.1).

Notation 3.16. We denote:

$$\mathbb{N}_\infty:=\mathbb{N}\cup\{\infty\}.$$

Definition 3.17. For a semiring T as in Definition 3.15, consider the following property:

($\tilde{\text{E}}$2) For any $\alpha,\beta\in T$ and $j\in I$ such that $\alpha_j=\beta_j$ there exists an $\epsilon\in E$ such that $\epsilon_j>\alpha_j=\beta_j$ or $\epsilon_j=\alpha_j=\beta_j=\infty$ and $\epsilon_i\geq\min\{\alpha_i,\beta_i\}$ for any $i\in I\setminus\{j\}$ with equality if $\alpha_i\neq\beta_i$.

Definition 3.18. Let S be a good semigroup. For any $\alpha\in\partial_J(S)$, let $\tilde{\alpha}$ be such that

$$\tilde{\alpha}_j=\infty \text{ for any } j\in J \text{ and } \tilde{\alpha}_i=\alpha_i \text{ for any } i\in I\setminus J.$$

Then the *closure of* S is

$$\tilde{S}:=S\cup\{\tilde{\alpha}\mid\alpha\in\partial_J(S)\}.$$

It is clear that $\tilde{S}$ is a semiring according to Definition 3.15, and in particular it satisfies property ($\tilde{\text{E}}$2).

We want to show that $\tilde{S}$ is a finitely generated semiring.

In the following we give results analogous to the ones given in [CH17], translating their operations in the value semigroup of a curve to operations on good semigroups. For this reason, we keep notations and definitions similar to their.

We will use the following notations:

- for any $\alpha\in\tilde{S}$, we denote
$$I_\alpha=\{i\in I\mid\alpha_i\neq\infty\}.$$
- $M=\tilde{S}\setminus\{0\}$.
- for any $\alpha,\beta\in\tilde{S}$ with $\alpha_i=\beta_i$ for some $i\in I$,
$$\mathscr{E}_i(\alpha,\beta)=\{\zeta\in\tilde{S}\mid\zeta_i>\alpha_i=\beta_i \text{ or } \zeta_i=\alpha_i=\infty,\zeta_j\geq\min\{\alpha_j,\beta_j\} \text{ for } j\in I\setminus\{i\}\}.$$
- $\tilde{S}_i:=\pi_i(\tilde{S})$.
- $Q^i=\pi_i(\{\alpha\in\tilde{S}\mid\alpha_j=\infty \text{ for any } j\in I\setminus\{i\}\})$.

Definition 3.19. Let S be a good semigroups, and let $\emptyset\neq G\subseteq M$. We call *G-sum* an element of $\tilde{S}$ of the type

$$\underline{n}G:=\underline{n}(G)=\sum_{g\in G}n_g g,\ \text{where } \underline{n}\in\mathbb{N}^G.$$

Observe that, in our notation, a G-sum is just an element belonging to $\langle G\rangle$.

Definition 3.20. Let S be a good semigroup, $\alpha \in \tilde{S} \setminus \{\infty\}$, and $k \in I_\alpha$. Let also $G \subseteq M$. We say that $\beta \in \tilde{S}$ is a *k-reduction of α modulo G* if there exists a G-sum $\underline{n}G$ such that

$$\alpha_k = \underline{n}G_k \text{ and } \alpha_i \leq \underline{n}G_i \text{ for any } i \in I \setminus \{k\},$$

and $\beta \in \mathrm{E2}_k(\alpha, \underline{n}G)$. In particular,

$$\beta_k > \alpha_k \text{ and } \beta_i \geq \min\{\alpha_i, \underline{n}G_i\} = \alpha_i \text{ for any } i \in I \setminus \{k\}$$

where the equality holds if $\alpha_i \neq \underline{n}G_i$. We say that β is a *reduction of α modulo G* if β is a k-reduction of α modulo G for some $k \in I_\alpha$.

Remark 3.21. If $\alpha_i = \infty$ for any $i \in I \setminus \{k\}$ and α has a reduction modulo G, then $I_\alpha = \{k\}$ and the reduction has to be a k-reduction. Then there exists $\underline{n}G$ such that $\alpha_k = \underline{n}G_k$ and $\infty = \alpha_i \leq \underline{n}G_i$ for any $i \in I \setminus \{k\}$. Hence $\underline{n}G_i = \infty$ for any $i \in I \setminus \{k\}$, and $\alpha = \underline{n}G$.

Definition 3.22. Let S be a good semigroup and $G \subseteq M$. We say that G is a *standard basis* for $\tilde{S}$ if for any $\alpha \in \tilde{S} \setminus \{\underline{\infty}\}$, α has a reduction modulo G.

Proposition 3.23. *Let S be a good semigroup and let G be a non-empty and finite subset of $M \setminus \{\underline{\infty}\}$. The following are equivalent:*

(i) Every $\alpha \in \tilde{S} \setminus \{\underline{\infty}\}$ has a k-reduction modulo G for any $k \in I_\alpha$;

(ii) Every $\alpha \in \tilde{S} \setminus \{\underline{\infty}\}$ has a reduction modulo G.

Proof. Clearly (i) implies (ii). Hence let us prove the converse. Let $\alpha \in \tilde{S} \setminus \{\underline{\infty}\}$, and $k \in I_\alpha$- Let J_α be the maximal subset of $I_\alpha \setminus \{k\}$ such that α has a j-reduction for any $j \in J$, and assume $J \neq \emptyset$. Let β be obtained from α via a finite chain of reductions modulo G. For any $j \in J_\alpha$, we can assume that either β has no j-reduction or $\gamma_j \leq \beta_j < \infty$, where γ is the conductor of S. Note that $\beta \neq \underline{\infty}$ because $k \notin J_\alpha$ and $\beta_i = \alpha_i$ for any $i \in I \setminus J_\alpha$ since an i-reduction of β is an i-reduction of α. Then, by maximality of J_α, also β does not admit an i-reduction for any $i \in I_\alpha \setminus J_\alpha$. Let $L \subseteq J_\alpha$ be minimal such that β does not admit an i-reduction for any $i \in I_\alpha \setminus L \supseteq I_\alpha \setminus J_\alpha$. Then $\gamma_l \leq \beta_l < \infty$ (otherwise $k \notin L$ by choice of β and L) for any $l \in L$. If $L = \emptyset$, then β does not have a reduction modulo G, which is a contradiction to G being a standard basis, as $\beta \neq \underline{\infty}$. On the other hand, if $L \neq \emptyset$, since $\min\{\beta, \gamma\} \in \partial_L(S)$, the definition of $\tilde{S}$ implies that there is $\tilde{\beta} \in \tilde{S} \setminus \{\underline{\infty}\}$ ($k \notin J$ so $k \notin L$ and $\beta_k = \alpha_k < \infty$) such that $\tilde{\beta}_i = \beta_i$ for any $i \in I \setminus L$ and $\tilde{\beta}_l = \infty$ for any $l \in L$. But then $\tilde{\beta}$ does not admit a reduction modulo G, which is a contradiction to G being a standard basis. □

Corollary 3.24. *Let G be a non-empty and finite subset of $M \setminus \{\underline{\infty}\}$. The following statements are equivalent:*

(i) G is a standard basis for $\tilde{S}$.

(ii) for any $\alpha \in \tilde{S} \setminus \{\underline{\infty}\}$ and for some $k \in I_\alpha$, there exists a G-sum $\underline{n}G$ (which depends on k), such that $\alpha_i \leq \underline{n}G_i$ for any $i \in I$ and $\alpha_k = \underline{n}G_k$.

(iii) for any $\alpha \in \tilde{S} \setminus \{\underline{\infty}\}$ and for any $k \in I_\alpha$, there exists a G-sum $\underline{n}G$ (which depends on k), such that $\alpha_i \leq \underline{n}G_i$ for any $i \in I$ and $\alpha_k = \underline{n}G_k$.

Theorem 3.25. *$\tilde{S}$ admits a standard basis.*

Proof. Let

$$B_0 = \{\alpha \in M \mid \alpha_i \leq \gamma_i \text{ if } i \in I_\alpha\}.$$

In our notations, $B_0 = \widetilde{\mathrm{Small}(S)}$. for any $i \in I$, let $B'_i, B''_i \subseteq \tilde{S}$ such that $\pi_i(B'_i)$ and $\pi_i(B''_i)$ are respectively standard bases for $\tilde{S}_i$ and for Q^i, which can be computed because they are numerical semigroups. Since $\pi_i(B''_i)$ is a standard basis of Q^i, we can assume $\alpha_j = \infty$ for any $j \in I \setminus \{i\}$ for any $\alpha \in B''_i$. Setting $B_i = B'_i \cup B''_i$, we want to show that $G = \cup_{i\in I} B_i$ is a standard basis for $\tilde{S}$. Let $\alpha \in \tilde{S} \setminus \{\underline{\infty}\}$. If $\alpha_i < \gamma_i$ for any $i \in I_\alpha$, then there is a G-sum $\underline{n}G$ (more specifically a B_0-sum) such that $\alpha = \underline{n}G$. If $\gamma_k \leq \alpha_k$ for some $k \in I_\alpha$, then $\alpha_k \in Q^k$. As the projections of B'_k, B''_k are standard bases of $\tilde{S}_k$ and Q^k respectively, there exists a G-product $\underline{n}G$ (indeed, a B'_k-product $B^{\underline{n}'}$ and a $\beta \in B''_k$ with $\underline{n}G = B^{\underline{n}'}\beta$) such that $\alpha_k = \underline{n}G_k$ and $\alpha_i \leq \underline{n}G_i = \infty$ for any $i \in I \setminus \{k\}$. By the above corollary, we conclude that G is a standard basis for $\tilde{S}$. $\square$

Theorem 3.26. *$\tilde{S}$ is generated by G as semiring.*

Proof. First of all, notice that $0 = \sum_{g\in G} 0 \cdot g = 0G$. By Remark 3.21, for any $\alpha^{(k)} \neq \underline{\infty} \in Q^k$, there exists a G-sum such that $\alpha = \underline{n}(k)G$. Hence

$$\underline{\infty} = \alpha^{(1)} + \alpha^{(2)} = \underline{n}(1)G + \underline{n}(2)G = (\underline{n}(1) + \underline{n}(2))G.$$

Now, given $\alpha \in M \setminus \{\infty\}$, by Corollary 3.24 for any $k \in I_\alpha$ there is a G-sum $\underline{n}(k)G$ such that

$$\alpha_k = \underline{n}(k)G_k \text{ and } \alpha_i \leq \underline{n}(k)G_i \text{ for any } i \in I \setminus \{k\}.$$

In this way, for $k \in I_\alpha$ we have

$$\alpha_k = \min_{i\in I_\alpha}\{\underline{n}(i)G_k\}.$$

Therefore

$$\alpha = \min_{i\in I_\alpha}\{\underline{n}(i)G\}$$

that is, the semiring $\tilde{S}$ is finitely generated by G. $\square$

3.2 Good generating systems of good semigroup ideals

Let S be a good local semigroup. For a generic $G \subseteq \mathbb{N}^s$ let $\{G\}$ be the smallest semigroup ideal of S containing G which is closed under minimums (i.e. $\{G\} \supseteq G + S$ and $\{G\}$ satisfies (E1)).

The proof of the following proposition is analogous as the proof of Proposition 3.1 (substituting "submonoids" with "semigroup ideals").

Proposition 3.27. *Let $G \subseteq \mathbb{N}^s$. Then*

$$\{G\} = \{\min\{g_1, \ldots, g_s\} \mid g_i \in G + S\}.$$

Notation 3.28. Given a $\delta \in \mathbb{N}^s$ we denote

$$\{G\}_\delta := \{\min\{\delta, g\} \mid g \in \{G\}\}.$$

Definition 3.29. Let $G \subseteq \mathbb{N}^s$, and let S be a good semigroup. Let $E \in \mathfrak{G}_S$ with conductor γ^E. Then G is said to be a *good generating system* for E if

$$\{G\}_{\gamma^E} = \mathrm{Small}(E).$$

We say that G is *minimal* if no proper subset of G is a good generating system of E.

Since $\{G\}_{\gamma^E} = \{B(\gamma^E)\}_{\gamma^E}$, we can always assume that good systems of generators are contained in $B(\gamma^E)$.

Remark 3.30. $[G]$ is quite different from $\{G\}$. In fact, $G = \{0\}$ is a good generating system for S as good relative ideal of itself (i.e. $\{\{0\}\}_\gamma = \mathrm{Small}(S)$) but it is clearly not a good generating system for S as a good semigroup.

Remark 3.31. Notice that with this definition we only consider relative ideals contained in $\mathbb{N}^s$. However this is not restrictive, since by definition of relative ideal there is always an α such that $\alpha + E \subseteq S \subseteq \mathbb{N}^s$. Moreover, if we consider S to be local, we can consequently assume that the relative ideal so generated does not have any element on the axes. An alternative would be to consider $E \subseteq \mathbb{Z}^s$ bounded from below.

The following are the exact equivalent of Lemmas 3.7 and 3.8.

Lemma 3.32. *Let $G \subseteq \mathbb{N}^s$ and $\alpha \in \mathbb{N}^s$. Then $\alpha \in \{G\}$ if and only if $\overline{\Delta}_i(\alpha) \cap (G + S) \neq \emptyset$ for any $i \in I$.*

Proof. Suppose $\alpha \in \{G\}$. If $\alpha \in (G + S)$, then $\overline{\Delta}_i(\alpha) \cap (G + S) \ni \alpha$ for any $i \in I$. So let us suppose $\alpha \in \{G\} \setminus (G + S)$. By Proposition 3.27 we know that $\alpha \in \{G\}$ if and only if it is the minimum of s elements of $(G + S)$, $\{\beta^{(i)}\}_{i \in I}$, and since we are assuming $\alpha \notin (G + S)$ we have $\alpha \neq \beta^{(i)}$ for any $i \in I$. Since α is the minimum of the βs, we have $\alpha_j \leq \beta_j^{(i)}$ for any $j \in I$. Moreover, for any $j \in I$ there exists an $i_j \in I$ such that $\alpha_j = \beta_j^{(i_j)}$. This means that for any $i \in I$ there is an index i_j with $\beta^{(i_j)}$ belonging to $\overline{\Delta}_i(\alpha)$. In particular

$$\beta^{(i_j)} \in \overline{\Delta}_i(\alpha) \cap (G + S).$$

for any $i \in I$.

Conversely, suppose $\overline{\Delta}_i(\alpha) \cap (G + S) \neq \emptyset$ for any $i \in I$. If $\alpha \in (G + S)$ then of course $\alpha \in \{G\}$. Hence suppose $\alpha \notin (G + S)$, and let for any $i \in I$, $\beta^{(i)} \in \overline{\Delta}_i(\alpha) \cap (G + S)$. Then clearly $\alpha = \min\{\beta^{(i)}\}_{i \in I}$. Therefore $\alpha \in \{G\}$. □

Substituting $\{G\}$ with $\{G\}_\delta$ we can be more precise.

Lemma 3.33. *Let $\delta \in \mathbb{N}^s$ and $G \subseteq B(\delta)$. Let $\alpha \in B(\delta) \setminus \{\delta\}$ and let J be maximal with the property $\delta \in \Delta_J(\alpha)$. Then $\alpha \in \{G\}_\delta$ if and only if $\overline{\Delta}_i(\alpha) \cap (G + S) \neq \emptyset$ for any $i \in I \setminus J$.*

Proof. Suppose $\alpha \in \{G\}_\delta$. Then $\alpha = \min\{\alpha', \delta\}$ for some $\alpha' \in \{G\}$. By Lemma 3.32, $\overline{\Delta}_i(\alpha') \cap (G + S) \neq \emptyset$ for any $i \in I$. From the assumptions we have $\alpha_j = \delta_j$ for $j \in J$ and $\alpha_i < \delta_i$ for $i \in I \setminus J$. Therefore is has to be $\alpha'_j \geq \alpha_j$ for $j \in J$ and $\alpha'_i = \alpha_i$ for $i \in I \setminus J$. So $\overline{\Delta}_i(\alpha') \subseteq \overline{\Delta}_i(\alpha)$ for any $i \in I \setminus J$, and so $\overline{\Delta}_i(\alpha) \cap (G + S) \neq \emptyset$ for any $i \in I \setminus J$.

Conversely, suppose $\overline{\Delta}_i(\alpha) \cap (G + S) \neq \emptyset$ for any $i \in I \setminus J$. Let $\beta^{(i)} \in \overline{\Delta}_i(\alpha) \cap (G + S)$. Then $\alpha = \min\{\delta, \{\beta^{(i)}\}_{i \in I \setminus J}\} \in \{G\}_\delta$. □

Remark 3.34. Let S be a good local semigroup. If $E \in \mathfrak{G}_S$ is generated by a good generating system G, and we suppose $E \subseteq S$, as remarked in 3.31, then G contains a positive element α. Then there is a positive integer k such that $\gamma \leq k\alpha$. Hence $\gamma^E \in \{G\}_{\gamma^E}$. We can then assume that unless $G = \{\gamma^E\}$, the conductor is never in a good generating system.

Taking into account Remark 3.34, from now on we assume that, unless If we assume this, Lemma 3.10 and Theorems 3.12 and 3.13 can be rewritten as follows and the proof works in the exact same way.

Lemma 3.35. *Let S be a good local semigroup and $E \in \mathfrak{G}_S$. Let G be a good generating system for E and $\alpha \in G \setminus \partial(E)$. If there exists $\beta \in \overline{\Delta}(\alpha) \cap (G \setminus \{\alpha\} + S)$, then $G \setminus \{\alpha\}$ is a good generating system for E.*

Proof. Since $\alpha \notin \partial(E)$ we have $\alpha < \gamma^E$. If $\alpha = \beta < \gamma^E$, then we are done, since then clearly $\alpha \in \{G \setminus \{\alpha\}\}_{\gamma^E}$. So let us suppose $\alpha \neq \beta$. By assumption we have $G \subseteq \mathrm{Small}(E)$ and $\{G\}_{\gamma^E} = \mathrm{Small}(E)$. Since $\beta \in \overline{\Delta}(\alpha)$, there exists an $i \in I$ such that $\beta_i = \alpha_i$. Then, applying (E2), one can find a $\delta \in E \setminus \{\alpha, \beta\}$ such that

$$\alpha = \min\{\beta, \delta\}.$$

Eventually substituting δ with $\min\{\delta, \gamma^E\}$ we can assume $\delta \in \mathrm{Small}(E) = \{G\}_{\gamma^E}$. Since $\alpha \notin \partial(E)$, after this substitution we still have $\delta \neq \alpha \neq \beta$.

By Proposition 3.27, we can write

$$\delta = \min\{\gamma^E, \{\delta^{(i)}\}_{i \in I}\}, \text{ with } \delta^{(i)} \in (G + S).$$

As $\delta \neq \alpha$ and $\alpha < \delta \leq \delta^{(i)}$, we also have $\alpha \neq \delta^{(i)}$ for any $i \in I$. Let $J \subseteq I$ be the maximal set of indices such that $\delta \in \overline{\Delta}_J(\alpha)$, which implies $\beta \in \overline{\Delta}_{I \setminus J}(\alpha)$. for any $j \in J$ there exists an $i_j \in I$ such that $\delta_j^{(i_j)} = \delta_j = \alpha_j$. Hence for any $j \in J$, $\delta^{(i_j)} \in \overline{\Delta}_j(\alpha)$ and moreover

$$\alpha = \min\{\beta, \{\delta^{(i_j)}\}_{j \in J}\}.$$

Since $\delta^{(i_j)} \in (G + S)$, for any $j \in J$ we can write

$$\delta^{(i_j)} = g^{(i_j)} + s^{(i_j)}$$

where $g^{(i_j)} \in G$ and $s^{(i_j)} \in S$ for any $j \in J$. This yields

$$\delta^{(i_j)} = g^{(i_j)} + s^{(i_j)} \neq \alpha$$

and

$$\delta_j^{(i_j)} = \alpha_j.$$

This implies one of the three following cases: either $s_j^{(i_j)} = \alpha_j$ and $g^{(i_j)}$ belongs to the axes, and this is impossible since we assume the elements of G to be strictly positive; or $g_j^{(i_j)}, s_j^{(i_j)} < \alpha_j$ and $\delta^{(i_j)} \in (G \setminus \{\alpha\} + S)$; or, as S is local, $s^{(i_j)} = 0$ and $\delta^{(i_j)} = g_j^{(i_j)} \in G \setminus \{\alpha\}$.

Therefore $\delta^{(i_j)} \in (G \setminus \{\alpha\} + S)$ for any $j \in J$ and $\alpha = \min\{\beta, \{\delta^{(i_j)}\}_{j \in J}\} \in \{G \setminus \{\alpha\}\}_{\gamma^E}$. □

The following proof is the analogous to the proof of Theorem 3.12.

Theorem 3.36. *Let S be a good local semigroup, $E \in \mathfrak{G}_S$ and G a good positive generating system for E. For $\alpha \in G$, let J_a be the set of indices maximal with the property $\alpha \in \partial_{J_\alpha}(E)$ (J_α can be empty). Then G is a minimal good generating system if and only if for any $\alpha \in G$*

$$\emptyset = \begin{cases} \overline{\Delta}(\alpha) \cap (G \setminus \{\alpha\} + S) & \text{if } J_\alpha = \emptyset \\ \overline{\Delta}_k(\alpha) \cap (G \setminus \{\alpha\} + S) \text{ for some } k \in I \setminus J_\alpha & \text{if } J_\alpha \neq \emptyset. \end{cases}$$

Proof. Assume G is a minimal good generating system for E. If $J = \emptyset$, i.e. $\alpha \in G \setminus \partial(E)$ and $\overline{\Delta}_i(\alpha) \cap (G \setminus \{\alpha\} + S) \neq \emptyset$ for $i \in I \setminus J$, then $\overline{\Delta}(\alpha) \cap (G \setminus \{\alpha\} + S) \neq \emptyset$ and Lemma 3.35 gives directly a contradiction.

Therefore let $\alpha \in \partial_J(E)$ for some J (maximal), and suppose that for any $i \in I \setminus J$ there exists an $\alpha^{(i)} \in \overline{\Delta}_i(\alpha) \cap (G \setminus \{\alpha\} + S)$. Then $\alpha = \min\{\gamma^E, \alpha^{(i)} \mid i \in I \setminus J\}$, and consequently $\alpha \in \{G \setminus \{\alpha\}\}_{\gamma^E}$, which is a contradiction.

Let us now prove the converse. If $G = \{\alpha\}$, then G is minimal. Suppose therefore $|G| \geq 2$ and G not minimal. Then there exist an $\alpha \in G$ such that $\alpha \in \{G \setminus \{\alpha\}\}_{\gamma^E}$. Since S is local, α is a positive element in $\mathbb{N}^s$. By Remark 3.34, it follows $\alpha \neq \gamma^E$.

Let J be the maximal set such that $\alpha \in \partial_J(E)$. Then $\alpha_j < \gamma^E_j$ for any $j \in I \setminus J$. By Proposition 3.1, there exist $\{\alpha^{(i)}\}_{i \in I} \subseteq (G \setminus \{\alpha\} + S)$ such that

$$\alpha = \min\{\gamma^E, \{\alpha^{(i)}\}_{i \in I}\}.$$

Hence for any $j \in I \setminus J$ there is i_j with $\alpha^{(i_j)}_j = \alpha_j$ and $\alpha^{(i_j)} \geq \alpha$. Then for any $j \in I \setminus J$ there is an $\alpha^{(i_j)} \in \overline{\Delta}_j(\alpha) \cap \langle G \setminus \{\alpha\} \rangle$ and this gives a contradiction. □

Theorem 3.37. *Let S be a good local semigroup and $E \in \mathfrak{G}_S$. Then E has a unique minimal good generating system.*

Proof. Let γ^E be the conductor of E. Let A and B be two minimal good generating systems for E. Let β be minimal in $(A \cup B) \setminus (A \cap B)$. Without loss of generality, we can assume $\beta \in B$.

Let us prove that $\beta \notin \{\min\{\gamma^E, \alpha + \delta\} \mid \alpha + \delta \in (A + S)\}$. Assume there are $\alpha \in A$ and $\delta \in S$ such that $\beta = \min\{\gamma^E, \alpha + \delta\}$. As $\beta \notin A$, $\alpha \neq \beta$ and hence $\delta \neq 0$. As $\alpha \leq \alpha + \delta$ and $\alpha \leq \gamma^E$, we have $\alpha \leq \min\{\gamma^E, \alpha + \delta\} = \beta$. Together with the considerations above, this gives $\alpha < \beta$. But then $\alpha \in B$ by minimality of β and thus $\beta \in \{B \setminus \{\beta\}\}_{\gamma^E}$, which contradicts the minimality of B. Thus

$$\beta \notin \{\min\{\gamma^E, \alpha + \delta\} \mid \alpha + \delta \in (A + S)\}. \tag{3.4}$$

Let now $J \subseteq I$ be maximal such that $\beta \in \partial_J(S)$ (J can also be empty). As $\beta \in B \subseteq \{A\}_{\gamma^E}$, by Lemma 3.33 there exist

$$\{\epsilon^{(i)}\}_{i \in I \setminus J} \subseteq \overline{\Delta}_i(\beta) \cap (A + S)$$

such that

$$\beta = \min\{\gamma^E, \{\epsilon^{(i)}\}_{i \in I \setminus J}\} \text{ with } \epsilon^{(i)}_i = \beta_i \text{ for } i \in I \setminus J.$$

As the $\epsilon^{(i)}$ do not need to be in $\{B\}_{\gamma^E}$, let us consider $\zeta^{(i)} = \min\{\gamma^E, \epsilon^{(i)}\}$ for any $i \in I \setminus J$. Then

$$\{\zeta^{(i)}\}_{i \in I \setminus J} \subseteq \overline{\Delta}_i(\beta) \cap \{\min\{\gamma^E, \alpha + \delta\} \mid \alpha + \delta \in (A + S)\}$$

and

$$\beta = \min\{\gamma^E, \{\zeta^{(i)}\}_{i \in I \setminus J}\} \text{ with } \zeta^{(i)}_i = \beta_i \text{ for } i \in I \setminus J. \tag{3.5}$$

Let $K_i \subseteq I$ be the maximum set of indices such that $\zeta^{(i)} \in \partial_{K_i}(S)$ for any $i \in I \setminus J$. As $\zeta^{(i)} \in \{A\}_{\gamma^E} = \{B\}_{\gamma^E}$, again by Lemma 3.33 there exist

$$\{\delta^{(i,j)}\}_{j \in I \setminus K_i} \subseteq \overline{\Delta}_j(\zeta^{(i)}) \cap (B + S)$$

such that

$$\zeta^{(i)} = \min\{\gamma^E, \{\delta^{(i,j)}\}_{j \in I \setminus K_i}\} \text{ with } \delta^{(i,j)}_j = \zeta_j \text{ for } j \in I \setminus K_i.$$

for any $i \in I \setminus J$. Since $\zeta_i^{(i)} = \beta_i < \gamma_i^E$ (i.e. $i \notin K_i$) for any $i \in I \setminus J$, and $\delta^{(i,j)} \geq \zeta^{(i)} \geq \beta$, for any $i \in I \setminus J$ there exists j_i such that

$$\delta^{(i,j_i)} \in \overline{\Delta}_i(\zeta^{(i)}) \cap (B+S) \subseteq \bar{\Delta}_i(\beta) \cap (B+S).$$

So by 3.2 we can write

$$\beta = \min\{\gamma^E, \{\delta^{(i,j_i)}\}_{i \in I \setminus J}\} \text{ with } \delta_i^{(i,j_i)} = \beta_i \text{ for } i \in I \setminus J.$$

By Theorem 3.12, this implies that there exists (at least) a $i \in I \setminus J$ such that

$$\delta := \delta^{(i_{j_i})} \in (B+S) \setminus (B \setminus \{\beta\} + S).$$

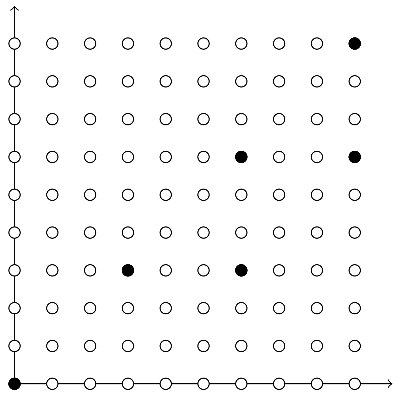
This means $\delta = \beta + \eta$, with $\eta \in S$. Since $\delta_i = \beta_i$, $\eta_i = 0$. But S is local, and this forces $\eta = 0$. So $\delta = \beta$. But by (3.1), $\zeta^{(i)} \neq \beta$ for any $i \in I \setminus J$, so in particular $\delta \geq \zeta^{(i)} > \beta$. This is a contradiction. So the claim is proved. □

3.3 Examples

While finding a minimal good generating system of a given semigroup is possible, we don't know how to characterize sets of data as good generating systems of good semigroups.

In fact, as the following examples show, not every set of elements with a whatever conductor gives a good semigroup.

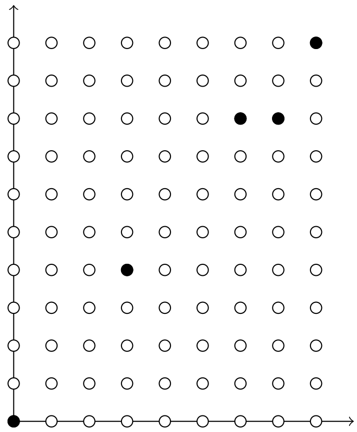
Example 3.38. Let $G = \{(3,3),(6,3)\}$ and $\gamma = (9,9)$. Then $[G]_\gamma$ looks like:



Condition (E2) does not hold for $(3,3)$ and $(6,3)$.

Even if G agrees with the conditions of Theorem 3.12, the resulting semigroup might not be good.

Example 3.39 ([DGSMT17, Example 7]). Let $G = \{(3,4),(7,8)\}$ and $\gamma = (8,10)$. Then $[G]_\gamma$ is



Clearly, condition (E2) is again not satisfied.

For more examples see [DGSMT17, Examples 5,6].

4 Duality

Let R be an admissible local ring with value semigroup $S := \Gamma_R$. A canonical semigroup ideal K_S^0 of a good semigroup S was already defined by D'Anna in [D'A97] in purely combinatoric terms. In the same paper it was shown that fractional ideals of R are (normalized) canonical ideals $\mathcal{K}$ if and only if their value semigroup ideal is $\Gamma_{\mathcal{K}} = K_S^0$. In this chapter we give a more general definition of canonical (semigroup) ideals of a good semigroup. Such canonical ideals satisfy three equivalent definitions. One of them gives a duality on good semigroup ideals which corresponds to the Cohen–Macaulay duality, i.e. if K is a canonical ideal of a good semigroup S, then $E = K - (K - E)$ for any $E \in \mathfrak{G}_S$. We only state the results regarding the equivalence on these three conditions, as they are not original work of the author, but the proofs can be found in [KST17]. Later, in Section 4.2, we show that value semigroup ideals are compatible with dualizing in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{R}_R \arrow[r, "\mathcal{E}\mapsto\mathcal{K}:\mathcal{E}"] \arrow[d, "\mathcal{E}\mapsto\Gamma_{\mathcal{E}}"'] & \mathfrak{R}_R \arrow[d, "\mathcal{E}\mapsto\Gamma_{\mathcal{E}}"] \\
\mathfrak{G}_{\Gamma_R} \arrow[r, "E\mapsto K-E"'] & \mathfrak{G}_{\Gamma_R},
\end{tikzcd}$$

where $\mathcal{K}$ is a canonical ideal of R and $K = \Gamma_{\mathcal{K}}$. The original work contained in Section 4.2 is again part of [KST17].

4.1 Duality on good semigroups

The new results of this section are part of [KST17], but are original work of P. Korell.

Definition 4.1. For any good semigroup S, we call

$$K_S^0 := \left\{\alpha \in \mathbb{Z}^s \mid \Delta^S(\tau - \alpha) = \emptyset\right\}$$

the (normalized) canonical (semigroup) ideal of S.

Lemma 4.2. *Let S be a good semigroup. Then the set K_S^0 is a semigroup ideal of S satisfying property* (E1) *with minimum $\mu^{K_S^0} = \mu^S = 0$ and conductor $\gamma^{K_S^0} = \gamma$.*

Proof. See [D'A97, Proposition 3.2], Lemma 2.13 and Notation 2.9. □

Lemma 4.3. *Let S be a good semigroup. Then the semigroup ideal K_S^0 of S has the following properties:*

(a) $\Delta^{K_S^0}(\tau) = \emptyset$.

(b) If E is a semigroup ideal of S, then

$$K_S^0 - E = \{\alpha \in \mathbb{Z}^s \mid \Delta^E(\tau - \alpha) = \emptyset\}.$$

Proof. See [KST17, Lemma 5.2.9]. □

The following Theorem 4.4 due to D'Anna characterizes the canonical ideals by having a value semigroup ideal equal to K_S^0.

Theorem 4.4. *Let R be an admissible local ring with value semigroup $S := \Gamma_R$. Then for any $\mathcal{K} \in \mathfrak{R}_R$ such that $R \subseteq \mathcal{K} \subseteq \overline{R}$ the following are equivalent:*

(i) $\mathcal{K}$ is a canonical ideal of R;

(ii) $\Gamma_{\mathcal{K}} = K_S^0$ (see Definition 4.1).

Proof. See [D'A97, Theorem 4.1]. □

Definition 4.5. Let S be a good semigroup (see Definition 2.1). We call $K \in \mathfrak{G}_S$ a *canonical (semigroup) ideal* of S if $K \subseteq E$ implies $K = E$ for any $E \in \mathfrak{G}_S$ with $\gamma^K = \gamma^E$.

Remark 4.6. If K is a canonical ideal of S, then $\alpha + K$ is a canonical ideal of S for any $\alpha \in \mathbb{Z}^s$. In fact, this follows immediately from Definition 4.5 and Lemma 2.5.(b).

The following proposition was stated in [BDF00a, Proposition 2.15].

Proposition 4.7. *Let $S = \prod_{m \in M} S_{I_m}$ be the decomposition of the good semigroup S into good local semigroups S_{I_m}. Then*

$$K_S^0 = \prod_{m \in M} K_{S_{I_m}}^0.$$

Proof. See [KST17, Proposition 5.2.3]. □

Moreover, dualizing with K_0^S preserves the property of being a good semigroup ideal.

Proposition 4.8. *Let S be a good semigroup. Then $K_S^0 - E \in \mathfrak{G}_S$ for any $E \in \mathfrak{G}_S$ and, in particular, $K_S^0 \in \mathfrak{G}_S$.*

Proof. See [KST17, Proposition 5.2.10]. □

Our definition of a canonical semigroup ideal allows shifts of K_0^S.

Proposition 4.9. *Let S be a good semigroup, and let $K \in \mathfrak{G}_S$. Then K is a canonical ideal of S if and only if $K = \alpha + K_S^0$ for some $\alpha \in \mathbb{Z}^s$. In particular, for any $\delta \in \mathbb{Z}^s$ there is a unique canonical ideal K of S with $\gamma^K = \delta$.*

Proof. See [KST17, Proposition 5.2.11]. □

As a consequence we deduce the counterpart of the push-forward formula for canonical ideals (see Lemma E.16) on the semigroup side.

Corollary 4.10. *Let $S \subseteq S' \in \overline{S}$ be good semigroups. If K is a canonical ideal of S then $K' = K - S'$ is a canonical ideal of S'.*

Proof. See [KST17, Corollary 5.2.12]. □

The following two propositions establish an equivalent definition of canonical semigroup ideals analogous to that of canonical fractional ideals (see Definition E.10).

Theorem 4.11. *Let S be a good semigroup, and let $K \in \mathfrak{G}_S$ such that $K - (K - E) = E$ for any $E \in \mathfrak{G}_S$. Then K is a canonical ideal of S.*

Proof. See [KST17, Proposition 5.2.14]. □

Theorem 4.12. *Let S be a good semigroup. Then $K_S^0 - (K_S^0 - E) = E$ for any $E \in \mathfrak{G}_S$.*

Proof. See [KST17, Proposition 5.2.16]. □

Corollary 4.13. *Any good semigroup S has a canonical ideal. Moreover, for any $K \in \mathfrak{G}_S$, the following are equivalent:*

(i) K is a canonical ideal of S;

(ii) There is an $\alpha \in \mathbb{Z}^s$ such that $\alpha + K = K_S^0$;

(iii) for any $E \in \mathfrak{G}_S$ we have $K - (K - E) = E$.

Proof. See [KST17, Theorem 5.2.7]. □

4.2 Duality of fractional ideals

In this section we show that taking values behaves well with respect to the Cohen–Macaulay duality and the semigroup duality.

The following result was stated by Waldi in case $\mathcal{E} = R$ and $\mathcal{F} = \overline{R}$ (see [Wal72, Bemerkung 1.2.21]).

Lemma 4.14. *Let R be an admissible ring and $\mathcal{E} \in \mathfrak{R}_R$ and $\mathcal{F} \in \mathfrak{R}_{\overline{R}}$. Set $E := \Gamma_{\mathcal{E}}$ and $F := \Gamma_{\mathcal{F}}$. Then $\mathcal{E} : \mathcal{F} = \mathcal{Q}^{\gamma^E - \mu^F}$ and hence $\Gamma_{\mathcal{E}:\mathcal{F}} = E - F$. In particular, $\mathcal{C}_{\mathcal{E}} = \mathcal{Q}^{\gamma^E}$ and hence $\Gamma_{\mathcal{C}_{\mathcal{E}}} = C_E$.*

Proof. $\mathcal{C}_{\mathcal{E}} = \mathcal{E} : \overline{R}$ by Definition A.5.(d). If $\mathcal{F} \in \mathfrak{R}_{\overline{R}}$, then $\mathcal{F}$ is principal by Theorem 1.3.(c).(3). Hence by Lemma 1.9.(b) $\mathcal{F} = x\overline{R} = \mathcal{Q}^{\nu(x)}$ for some $x \in Q_R^{\mathrm{reg}}$. Then by Lemma 1.9.(c), $\Gamma_{\mathcal{F}} = F = \nu(x) + \mathbb{N}^{\mathfrak{V}_R}$, which implies $\nu(x) = \mu_F$. By Lemma A.2.(b) $\mathcal{E} : \mathcal{F} = \mathcal{E} : x\overline{R} = x^{-1}(\mathcal{E} : \overline{R}) = x^{-1}\mathcal{C}_{\mathcal{E}}$. By Lemma 1.9.(b), if the particular claim is true, we have

$$\mathcal{E} : \mathcal{F} = x^{-1}\mathcal{C}_{\mathcal{E}} = x^{-1}\overline{R}\mathcal{C}_{\mathcal{E}} = \mathcal{Q}^{\nu(x^{-1})}\mathcal{C}_{\mathcal{E}} = \mathcal{Q}^{-\nu(x)}Q^{\gamma^E} = \mathcal{Q}^{-\mu^F}Q^{\gamma^E} = \mathcal{Q}^{\gamma^E - \mu^F}$$

where $\nu(x^{-1}) = -\nu(x)$ by Remark B.8.(b). Then using the definition of C_E we have

$$\Gamma_{\mathcal{E}:\mathcal{F}} = \Gamma_{x^{-1}\mathcal{C}_{\mathcal{E}}} = \nu(x^{-1}) + C_E = -\nu(x) + C_E = -\mu^F + E - \overline{S} = E - (\mu^F + \overline{S}) = E - F$$

where $-\mu^F + E - \overline{S} = E - (\mu^F + \overline{S})$ thanks to Lemma 2.5.(c). Thus it is enough to prove the particular claim. By Lemma 1.14, $\mathcal{C}_{\mathcal{E}} = \mathcal{Q}^{\nu(x)}$ for some $x \in Q_R^{\mathrm{reg}}$ such that $\nu(x) + \mathbb{N}^{\mathfrak{V}_R} \subseteq \Gamma_{\mathcal{E}}$. Then, by Definition 2.9, it has to be $\nu(x) \geq \gamma^E$ and hence $\mathcal{C}_{\mathcal{E}} \subseteq \mathcal{Q}^{\gamma^E}$. By Lemma 1.9.(c) and definition of γ^E, $\Gamma_{\mathcal{Q}^{\gamma^E}} = \gamma^E + \mathbb{N}^{\mathfrak{V}_R} \subseteq \Gamma_{\mathcal{E}} = E$, and hence $\Gamma_{\mathcal{E}^{\gamma^E}} = \Gamma_{\mathcal{Q}^{\gamma^E}} = C_E$. By Corollary 2.30, $\mathcal{E}^{\gamma^E} \subseteq \mathcal{Q}^{\gamma^E}$ and $\Gamma_{\mathcal{E}^{\gamma^E}} = \Gamma_{\mathcal{Q}^{\gamma^E}}$ imply $\mathcal{E}^{\gamma^E} = \mathcal{Q}^{\gamma^E} \subseteq \mathcal{E}$. As $\mathcal{Q}^{\gamma^E}$ is an $\overline{R}$-module, from Lemma 1.14 we get $\mathcal{Q}^{\gamma^E} \subseteq \mathcal{C}_{\mathcal{E}}$. □

By Lemma 4.14, value semigroup ideals commute with conductors in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{R}_R \arrow[r, "\mathcal{E}\mapsto\mathcal{C}_\mathcal{E}"] \arrow[d, "\mathcal{E}\mapsto\Gamma_\mathcal{E}"'] & \mathfrak{R}_{\overline{R}} \arrow[d, "\mathcal{E}\mapsto\Gamma_\mathcal{E}"] \\ \mathfrak{G}_{\Gamma_R} \arrow[r, "E\mapsto C_E"'] & \mathfrak{G}_{\Gamma_{\overline{R}}}, \end{tikzcd}$$

The following proposition allows us to reduce to the case of normalized canonical ideals.

Proposition 4.15. *Any one-dimensional analytically reduced local Cohen–Macaulay ring with large residue field R has a canonical ideal $\mathcal{K}$ such that $R \subseteq \mathcal{K} \subseteq \overline{R}$. It is unique up to multiplication by $\overline{R}^*$ with unique Γ_K.*

Proof. By Theorem E.15 there is a canonical ideal $\mathcal{E}$ of R. Since $\overline{R}$ is finite over R by Proposition 1.12, every regular ideal of $\overline{R}$ is principal by Theorem 1.3.(c).(2), and since $|\mathrm{Max}(\overline{R})| = |\mathfrak{V}_R| \leq |R/\mathfrak{m}|$ by Theorem 1.3.(d) and large residue field assumption, Lemma A.18 applies to $R' := \overline{R}$. It yields a $y \in Q_R^{\mathrm{reg}}$ such that $\mathcal{K} := y\mathcal{E}$ satisfies $R \subseteq \mathcal{K} \subseteq \overline{R}$. Hence $\mathcal{K}\overline{R} = \overline{R}$. By Proposition E.13 the canonical ideals of R are of the form $\mathcal{K}' = x\mathcal{K}$ with $x \in Q_R^{\mathrm{reg}}$. If $R \subseteq \mathcal{K}' = x\mathcal{K} \subseteq \overline{R}$, then $x \in \overline{R}^*$. By Remark 1.7, $\nu(x) = 0$ and hence $\Gamma_{K'} = \Gamma_{\mathcal{K}}$. □

We now show the compatibility of value semigroup ideals with dualizing.

Theorem 4.16. *Let R be an admissible ring with canonical ideal $\mathcal{K}$ with $\Gamma_\mathcal{K} = K$. Let $\mathcal{F} \in \mathfrak{R}_R$ and $\mathcal{E} \in \mathfrak{R}_R$ such that $\mathcal{E} \subseteq \mathcal{F}$. Then*

(a) $\Gamma_{\mathcal{K}:\mathcal{F}} = K - \Gamma_\mathcal{F}$.

(b) $d(\Gamma_\mathcal{F} \backslash \Gamma_\mathcal{E}) = d(\Gamma_{\mathcal{K}:\mathcal{E}} \backslash \Gamma_{\mathcal{K}:\mathcal{F}}) = d(K - \Gamma_\mathcal{E} \backslash K - \Gamma_\mathcal{F})$.

Proof. Set $S := \Gamma_R$ and $K = \Gamma_\mathcal{K}$. From Theorem 1.24 we have a decomposition $\Gamma_{\mathcal{K}:\mathcal{F}} = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \Gamma_{(\mathcal{K}:\mathcal{F})_\mathfrak{m}}$ for any $\mathcal{F} \in \mathfrak{R}_R$, and by Lemma A.14, we have an equality $(\mathcal{K} : \mathcal{F})_\mathfrak{m} = \mathcal{K}_\mathfrak{m} : \mathcal{F}_\mathfrak{m}$. Moreover, Remark E.11.(a) yields $\mathcal{K}$ canonical ideal of R if and only if $\mathcal{K}_\mathfrak{m}$ canonical ideal of $R_\mathfrak{m}$. Thus if we prove (a) for local rings, we have a sequence of equalities

$$\Gamma_{\mathcal{K}:\mathcal{F}} = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \Gamma_{(\mathcal{K}:\mathcal{F})_\mathfrak{m}} = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} \Gamma_{\mathcal{K}_\mathfrak{m}:\mathcal{F}_\mathfrak{m}} = \prod_{\mathfrak{m}\in\mathrm{Max}(R)} (K_\mathfrak{m} - (\Gamma_{\mathcal{F}_\mathfrak{m}})) = K - \Gamma_\mathcal{F}$$

where the last equality holds by Theorem 2.11. For (b), Remark 2.25.(d) gives the compatibility of the distance function on semigroups with the decomposition in local semigroups (see Theorem 2.11). Therefore if we prove it for local rings we have a sequence of equalities

$$\begin{aligned} d(\Gamma_\mathcal{F} \backslash \Gamma_\mathcal{E}) &= \sum_{m\in M} d((\Gamma_\mathcal{F})_{I_m} \backslash (\Gamma_\mathcal{E})_{I_m}) \\ &= \sum_{m\in M} d((\Gamma_{\mathcal{K}:\mathcal{E}})_{I_m} \backslash (\Gamma_{\mathcal{K}:\mathcal{F}})_{I_m}) = d((\Gamma_{\mathcal{K}:\mathcal{E}}) \backslash (\Gamma_{\mathcal{K}:\mathcal{F}})) \\ &= \sum_{m\in M} d((K - \Gamma_\mathcal{E})_{I_m} \backslash (K - \Gamma_\mathcal{F})_{I_m}) = d(K - \Gamma_\mathcal{E} \backslash K - \Gamma_\mathcal{F}). \end{aligned}$$

Hence we may assume that R is local.

Now assume $\mathcal{K}$ to be such that $R \subseteq \mathcal{K} \subseteq \overline{R}$. Then $\mathcal{K}$ is unique up to multiplication by $\overline{R}^*$, with unique $\Gamma_\mathcal{K}$ by Proposition 4.15. Moreover, by Theorem 4.4, in this case $\Gamma_\mathcal{K} = K_S^0$. We want to show that this assumption is not restrictive. In fact, let $\mathcal{K}' \neq \mathcal{K}$ be any canonical ideal of R. Then $\mathcal{K}'$ is of the form $x\mathcal{K}$ for some $x \in Q_R^{\mathrm{reg}}$ by Proposition E.13. By Lemma A.2.(b), $\mathcal{K}' : \mathcal{F} = (x\mathcal{K}) : \mathcal{F} = x(\mathcal{K} : \mathcal{F})$, and by Proposition 4.9, $K' := \Gamma_{\mathcal{K}'} = \alpha + K_S^0$, where $\alpha = \nu(x)$. Now suppose (a) is true for $\mathcal{K}$. Then we have a sequence of equalities

$$\Gamma_{\mathcal{K}':\mathcal{F}} = \Gamma_{(x\mathcal{K}):\mathcal{F}} = \Gamma_{x(\mathcal{K}:\mathcal{F})} = \nu(x)+\Gamma_{\mathcal{K}:\mathcal{F}} = \nu(x)+(K_S^0-\Gamma_{\mathcal{F}}) = (\nu(x)+K_S^0)-\Gamma_{\mathcal{F}} = K'-\Gamma_{\mathcal{F}}.$$

Similarly, if (b) is true for $\mathcal{K}$ with $\Gamma_{\mathcal{K}} = K_S^0$, by Remark 2.25.(c) for any other canonical ideal $\mathcal{K}' = x\mathcal{K}$ we have a sequence of equalities:

$$\begin{aligned} d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}}) &= d(\Gamma_{\mathcal{K}:\mathcal{E}}\backslash\Gamma_{\mathcal{K}:\mathcal{F}}) \\ &= d(\nu(x) + \Gamma_{\mathcal{K}:\mathcal{E}}\backslash\nu(x) + \Gamma_{\mathcal{K}:\mathcal{F}}) = d(\Gamma_{x(\mathcal{K}:\mathcal{E})}\backslash\Gamma_{x(\mathcal{K}:\mathcal{F})}) = d(\Gamma_{(x\mathcal{K}):\mathcal{E}}\backslash\Gamma_{(x\mathcal{K}):\mathcal{F}}) \\ &= d(\Gamma_{\mathcal{K}':\mathcal{E}}\backslash\Gamma_{\mathcal{K}':\mathcal{F}}) \end{aligned}$$

and

$$\begin{aligned} d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}}) &= d(K_S^0 - \Gamma_{\mathcal{E}}\backslash K_S^0 - \Gamma_{\mathcal{F}}) \\ &= d(\alpha + K_S^0 - \Gamma_{\mathcal{E}}\backslash\alpha + K_S^0 - \Gamma_{\mathcal{F}}) = d((\alpha + K_S^0) - \Gamma_{\mathcal{E}}\backslash(\alpha + K_S^0) - \Gamma_{\mathcal{F}}) \\ &= d(K' - \Gamma_{\mathcal{E}}\backslash K' - \Gamma_{\mathcal{F}}). \end{aligned}$$

Thus it is not restrictive to assume $K = K_S^0$.

We now prove both claims simultaneously. Proposition 2.29 gives $\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}})$, and dualizing with a canonical ideal preserves lengths due to Remark E.11.(b). Hence

$$d(\Gamma_{\mathcal{K}:\mathcal{E}}\backslash\Gamma_{\mathcal{K}:\mathcal{F}}) = \ell_R((\mathcal{K} : \mathcal{E})/(\mathcal{K} : \mathcal{F})) = \ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}}\backslash\Gamma_{\mathcal{E}}) =: n.$$

In particular, since $\mathcal{C}_{\mathcal{E}} \in \mathfrak{R}_R$ by Lemma 1.14,

$$d(\Gamma_{\mathcal{K}:\mathcal{C}_{\mathcal{E}}}\backslash\Gamma_{\mathcal{K}:\mathcal{F}}) = \ell_R(\mathcal{F}/\mathcal{C}_{\mathcal{E}}) =: l + n. \tag{4.1}$$

By [AM69, Proposition 6.8], there is a composition series in $\mathfrak{R}_R$ (see Definition C.1). We pick one in $\mathfrak{R}_R$:

$$\mathcal{C}_{\mathcal{E}} = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_l = \mathcal{E} \subsetneq \mathcal{E}_{l+1} \subsetneq \cdots \subsetneq \mathcal{E}_{l+n} = \mathcal{F}.$$

By Corollary 2.30, applying Γ preserves the strict inclusions. Since $\Gamma_{\mathcal{E}_i}$ are good semigroup ideals for any $i = 0, \ldots, l+n$ by Corollary 1.25, applying Γ yields a chain in $\mathfrak{G}_{\Gamma_R}$

$$\Gamma_{\mathcal{C}_{\mathcal{E}}} = \Gamma_{\mathcal{E}_0} \subsetneq \Gamma_{\mathcal{E}_1} \subsetneq \cdots \subsetneq \Gamma_{\mathcal{E}_l} = \Gamma_{\mathcal{E}} \subsetneq \Gamma_{\mathcal{E}_{l+1}} \subsetneq \cdots \subsetneq \Gamma_{\mathcal{E}_{l+n}} = \Gamma_{\mathcal{F}}.$$

Let $\mathcal{E} \in \mathfrak{R}_R$. Then $\Gamma_{\mathcal{K}:\mathcal{E}} \subseteq \Gamma_{\mathcal{K}} - \Gamma_{\mathcal{E}}$ by Remark 1.17, and since $\Gamma_{\mathcal{E}} \in \mathfrak{G}_S$ by Proposition 1.21, we get $\Gamma_{\mathcal{K}:\mathcal{E}} \subseteq \Gamma_{\mathcal{K}} - \Gamma_{\mathcal{E}} = K - \Gamma_{\mathcal{E}} \in \mathfrak{G}_S$ by Proposition 4.8. As dualizing with K reverses inclusions by Lemma 2.5.(d) and preserves strict inclusions by Theorem 4.12 (since $K - (K - E) = E$), applying $K - (-)$ to the previous chain yields a chain in $\mathfrak{G}_{\Gamma_R}$

$$\begin{aligned} \Gamma_{\mathcal{K}:\mathcal{C}_{\mathcal{E}}} = \Gamma_{\mathcal{K}} - \Gamma_{\mathcal{C}_{\mathcal{E}}} = K - \Gamma_{\mathcal{E}_0} \supsetneq \cdots \supsetneq K - \Gamma_{\mathcal{E}_l} = K - \Gamma_{\mathcal{E}} \\ \supsetneq K - \Gamma_{\mathcal{E}_{l+1}} \supsetneq \cdots \supsetneq K - \Gamma_{\mathcal{E}_{l+n}} = K - \Gamma_{\mathcal{F}} \supseteq \Gamma_{\mathcal{K}:\mathcal{F}} \end{aligned} \tag{4.2}$$

and $d(K - \Gamma_{\mathcal{E}_i}\backslash K - \Gamma_{\mathcal{E}_{i+1}}) \geq 1$ for any $i = 0, \ldots, l+n-1$ by Proposition 2.28 (distance is zero if and only if the two good semigroup ideals are the same).

Applying Lemma 2.26 (additivity of distance) to the chain (4.2) it follows with equation (4.1) that

$$d(K - \Gamma_{\mathcal{E}_i}\backslash K - \Gamma_{\mathcal{E}_{i+1}}) = 1$$

for any $i = 0, \dots, l+n-1$, and hence

$$d(K - \Gamma_{\mathcal{E}} \backslash K - \Gamma_{\mathcal{F}}) = \sum_{i=l}^{l+n-1} d(K - \Gamma_{\mathcal{E}_i} \backslash K - \Gamma_{\mathcal{E}_{i+1}}) = n = d(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{E}})$$

and that

$$d(K - \Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{K}:\mathcal{F}}) = 0.$$

By Proposition 2.28, the latter is equivalent to

$$\Gamma_{\mathcal{K}:\mathcal{F}} = K - \Gamma_{\mathcal{F}}. \qquad \square$$

We can finally generalize D'Anna's Theorem 4.4 to the semilocal case based on our Definition 4.5 of canonical semigroup ideals.

Corollary 4.17. *Let R be an admissible ring. Then for any $\mathcal{K} \in \mathfrak{R}_R$ the following are equivalent:*

(i) $\mathcal{K}$ is a canonical ideal of R;

(ii) $\Gamma_{\mathcal{K}}$ is a canonical ideal of Γ_R.

Proof. (i) $\Rightarrow$ (ii) Let $\mathfrak{m} \in \operatorname{Max}(R)$. As $\mathcal{K}$ is a canonical ideal of R, $\mathcal{K}_{\mathfrak{m}}$ is a canonical ideal of $R_{\mathfrak{m}}$ by Remark E.11.(a). Hence $\Gamma_{\mathcal{K}_{\mathfrak{m}}} = \alpha_{\mathfrak{m}} + K^0_{R_{\mathfrak{m}}}$ for some $\alpha \in D_{\Gamma_{R_{\mathfrak{m}}}}$ by Proposition 4.9. Setting $\alpha := (\alpha_{\mathfrak{m}})_{\mathfrak{m} \in \operatorname{Max}(R)}$, Theorem 1.24 and Proposition 4.7 yield

$$\Gamma_{\mathcal{K}} = \prod_{\mathfrak{m} \in \operatorname{Max}(R)} \Gamma_{\mathcal{K}_{\mathfrak{m}}} = \prod_{\mathfrak{m} \in \operatorname{Max}(R)} (\alpha_{\mathfrak{m}} + K^0_{\Gamma_{R_{\mathfrak{m}}}}) = \alpha + \prod_{\mathfrak{m} \in \operatorname{Max}(R)} K^0_{\Gamma_{R_{\mathfrak{m}}}} = \alpha + K^0_{\Gamma_R}.$$

Thus, $\Gamma_{\mathcal{K}}$ is a canonical ideal of Γ_R again by Proposition 4.9.

(ii) $\Rightarrow$ (i) Let $\mathcal{E} \in \mathfrak{R}_R$. By Corollary 1.25, Γ_R is a good semigroup and $\Gamma_E \in \mathfrak{G}_{\Gamma_R}$. By assumption and Theorems 4.16 $\Gamma_{\mathcal{K}:(\mathcal{K}:\mathcal{E})} = \Gamma_{\mathcal{K}} - (\Gamma_K - \Gamma_{\mathcal{E}})$ and as $\Gamma_{\mathcal{K}}$ is a canonical ideal, by Corollary 4.13.(4.13) , $\Gamma_{\mathcal{K}} - (\Gamma_K - \Gamma_{\mathcal{E}}) = \Gamma_{\mathcal{E}}$. Hence $\Gamma_{\mathcal{K}:(\mathcal{K}:\mathcal{E})} = \Gamma_{\mathcal{E}}$, which implies $\mathcal{K} : (\mathcal{K} : \mathcal{E}) = \mathcal{E}$ by Corollary 2.30. It follows that $\mathcal{K}$ is a canonical ideal of R according to Definition E.10.

$\square$

5
Poincaré series

In this chapter, we show some symmetry property between the Poincaré series associated to a good semigroup ideal and the one associated to its dual. The name *Poincaré series* is taken from the literature. Observe that in an algebraic context this would be called Hilbert-Samuel series. This work is a generalization of [Pol16, §5.2.8]. In her work, Pol considers value semigroup ideals associated to Gorenstein rings. Instead, our approach takes into account the more ample class of good semigroups.

5.1 Distance and Δ-sets

Definition 5.1. Let S be a good semigroup. For any $E \in \mathfrak{G}_S$, we define a decreasing filtration $E^\bullet$ on E by semigroup ideals

$$E^\alpha := \{\beta \in E \mid \beta \geq \alpha\}$$

for any $\alpha \in \mathbb{Z}^s$. The semigroup ideals E^α satisfy (E1) and (E4) for all $\alpha \in \mathbb{Z}^s$.

Remark 5.2. Let R be an admissible ring and $S = \Gamma_R$. Let $\mathcal{E}$ be a fractional ideal of R and $\mathcal{E}^\alpha$ as in Definition 1.8. If $E = \Gamma_{\mathcal{E}}$, then $E^\alpha = \Gamma_{\mathcal{E}^\alpha}$ for any $\alpha \in \mathbb{Z}^s$.

We now give some technical results about distances, and their relation with the Δ-sets (see Definition 2.12).

Lemma 5.3. *Let S be a good semigroup. Let $E \in \mathfrak{G}_S$, and $\alpha \in \mathbb{Z}^s$. Then*

$$d(E^\alpha \backslash E^{\alpha+\mathbf{e}_i}) \leq 1.$$

Proof. Since E^α satisfies (E1) and (E4) for all α, the distance is well-defined (see Definition 2.23). We have the following:

$$\begin{aligned} d(E^\alpha \backslash E^{\alpha+\mathbf{e}_i}) &= d_{E^\alpha}(\mu^{E^\alpha}, \gamma^{E^{\alpha+\mathbf{e}_i}}) - d_{E^{\alpha+\mathbf{e}_i}}(\mu^{E^{\alpha+\mathbf{e}_i}}, \gamma^{E^{\alpha+\mathbf{e}_i}}) \\ &= d_{E^\alpha}(\mu^{E^\alpha}, \gamma^{E^{\alpha+\mathbf{e}_i}}) - d_{E^\alpha}(\mu^{E^{\alpha+\mathbf{e}_i}}, \gamma^{E^{\alpha+\mathbf{e}_i}}) \end{aligned} \tag{5.1}$$

where the first equality is the definition of distance, and the second equality holds because a saturated chain between $\mu^{E^{\alpha+\mathbf{e}_i}}$ and $\gamma^{E^{\alpha+\mathbf{e}_i}}$ is contained in E^α. Now observe that μ^{E^α} and $\mu^{E^{\alpha+\mathbf{e}_i}}$ are always comparable. In fact, by minimality it has to be $\mu^{E^\alpha} = \min\{\mu^{E^\alpha}, \mu^{E^{\alpha+\mathbf{e}_i}}\} \leq \mu^{E^{\alpha+\mathbf{e}_i}}$. So (5.1) becomes

$$d(E^\alpha \backslash E^{\alpha+\mathbf{e}_i}) = d_{E^\alpha}(\mu^{E^\alpha}, \mu^{E^{\alpha+\mathbf{e}_i}}).$$

Now let $\mu^{E^\alpha} = \alpha^{(0)} < \cdots < \alpha^{(m)} = \mu^{E^{\alpha+\mathbf{e}_i}}$ be a saturated chain in E. Suppose $m \geq 2$. By minimality, we have that $\alpha^{(k)} \in \overline{\Delta}_i^E(\alpha) \setminus E^{\alpha+\mathbf{e}_i}$ for any $k \leq m$. Consider $\alpha^{(0)}, \alpha^{(1)} \in E$. They have $\alpha_i^{(0)} = \alpha_i^{(1)} = \alpha_i$ and there exists a $j \neq i$ such that $\alpha_j^{(0)} < \alpha_j^{(1)} \leq \alpha_j^{(m)} = \mu_j^{E^{\alpha+\mathbf{e}_i}}$. We can apply property (E2) to $\alpha^{(0)}, \alpha^{(1)} \in E$ and obtain a $\beta \in E$ with $\beta_i > \alpha_i$ and $\beta_j = \min\{\alpha_j^{(0)}, \alpha_j^{(1)}\} = \alpha_j^{(0)}$. In particular, $\beta \in E^{\alpha+\mathbf{e}_i}$. Thus, by minimality, it has to be $\min\{\beta, \mu^{E^{\alpha+\mathbf{e}_i}}\} = \mu^{E^{\alpha+\mathbf{e}_i}}$. Thus $\mu_j^{E^{\alpha+\mathbf{e}_i}} = \min\{\beta_j, \mu_j^{E^{\alpha+\mathbf{e}_i}}\} = \min\{\alpha_j^{(0)}, \mu_j^{E^{\alpha+\mathbf{e}_i}}\} = \alpha_j^{(0)} < \mu_j^{E^{\alpha+\mathbf{e}_i}}$. This is a contradiction. Thus the claim. □

Lemma 5.4. *Let S be a good semigroup and let $E \in \mathfrak{G}_S$. Then $d(E^\alpha \backslash E^{\alpha+\mathbf{e}_i}) = 1$ if and only if $\overline{\Delta}_i^E(\alpha) \neq \emptyset$.*

Proof. Observe that by definition $E^\alpha = E^{\alpha+\mathbf{e}_i} \cup \overline{\Delta}_i^E(\alpha)$. By Proposition 2.28, $d(E^\alpha \backslash E^{\alpha+\mathbf{e}_i}) = 0$ if and only if $E^\alpha = E^{\alpha+\mathbf{e}_i}$, i.e. if and only if $\overline{\Delta}_i^E(\alpha) = \emptyset$. So the claim follows by Lemma 5.3. □

Proposition 5.5. *Let S be a good semigroup and let $E \in \mathfrak{G}_S$. Let $\alpha \leq \beta \in \mathbb{Z}^s$. Then $E^\beta \subseteq E^\alpha$. Let $\alpha = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(n)} = \beta$ be a saturated chain in $\mathbb{Z}^s$, with $\alpha^{(j+1)} = \alpha^{(j)} + \mathbf{e}_{i(j)}$ for any $j \in \{0, \ldots, n-1\}$. We have:*

$$d(E^\alpha \backslash E^\beta) = \mathrm{Card}\{j \in \{0, \ldots, n-1\} \mid \overline{\Delta}_{i(j)}^E(\alpha^{(j)}) \neq \emptyset\}$$

Proof. Using the additivity of the distance, our assumptions and Lemma 5.4, we get the following equalities:

$$\begin{aligned} d(E^\alpha \backslash E^\beta) &= \sum_{j=0}^{n-1} d(E^{\alpha^{(j)}} \backslash E^{\alpha^{(j+1)}}) \\ &= \sum_{j=0}^{n-1} d(E^{\alpha^{(j)}} \backslash E^{\alpha^{(j)}+\mathbf{e}_{i(j)}}) \\ &= \mathrm{Card}\{j \in \{0, \ldots, n-1\} \mid \overline{\Delta}_{i(j)}^E(\alpha^{(j)}) \neq \emptyset\}. \end{aligned}$$

□

Corollary 5.6. *Let S be a good semigroup and let $E, F \in \mathfrak{G}_S$ with $E \subseteq F$. Let $\mu^F = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(m)} = \mu^E < \cdots < \alpha^{(n)} = \gamma^E$ be a saturated chain in $\mathbb{Z}^s$. In particular, $\alpha^{(j+1)} = \alpha^{(j)} + \mathbf{e}_{i(j)}$ for any $j \in \{0, \ldots, n-1\}$. Then*

$$\begin{aligned} d(F \backslash E) = &\mathrm{Card}\{j \in \{0, \ldots, n-1\} \mid \overline{\Delta}_{i(j)}^F(\alpha^{(j)}) \neq \emptyset\} \\ &- \mathrm{Card}\{j \in \{m, \ldots, n-1\} \mid \overline{\Delta}_{i(j)}^E(\alpha^{(j)}) \neq \emptyset\} \end{aligned}$$

Proof. By additivity of the distance we have:

$$\begin{aligned} d(F \backslash E) = &d(F \backslash C_E) - d(E \backslash C_E) \\ = &d(F^{\mu^F} \backslash F^{\gamma^E}) - d(E^{\mu^E} \backslash E^{\gamma^E}). \end{aligned}$$

The claim follows by Proposition 5.5. □

Lemma 5.7. *Let S be a good semigroup and let $E \in \mathfrak{G}_S$. Let $K := K_S^0$ be the normalized canonical ideal of S. If $\overline{\Delta}_i^{K-E}(\alpha) \neq \emptyset$, then $\Delta_i^E(\tau - \alpha) = \emptyset$.*

Proof. Let $\beta \in \overline{\Delta}_i^{K-E}(\alpha)$. Then

$$\begin{aligned} \beta_i =& \alpha_i, \\ \beta_j \geq& \alpha_j \text{ for any } j \neq i, \end{aligned}$$

and $\Delta^E(\tau - \beta) = \emptyset$ by Lemma 4.3.(b). Therefore $(\tau - \beta)_i = (\tau - \alpha)_i$ and $(\tau - \beta)_j \leq (\tau - \alpha)_j$. Hence $\Delta_i^E(\tau - \alpha) \subseteq \Delta_i^E(\tau - \beta) = \emptyset$. □

The following Proposition proves that the converse of Lemma 5.7 is always true when $s = 2$.

Proposition 5.8. *Let S be a good semigroup, and assume $S \subseteq \mathbb{Z}^2$. Let $E \in \mathfrak{G}_S$, and $K := K_S^0$ the normalized canonical ideal of S. If $\Delta_i^E(\tau - \alpha) = \emptyset$ then $\overline{\Delta}_i^{K-E}(\alpha) \neq \emptyset$.*

Proof. In the following we will call i and j the two directions of $\mathbb{Z}^2$. First assume $\tau - \alpha \in E$. If $\Delta_j^E(\tau - \alpha) \neq \emptyset$, then we can apply property (E2) to $\tau - \alpha$ and $\beta \in \Delta_j^E(\tau - \alpha)$ and obtain an element in $\Delta_i^E(\tau - \alpha) = \emptyset$. Thus $\Delta^E(\tau - \alpha) = \emptyset$. Therefore $\alpha \in \overline{\Delta}_i^{K-E}(\alpha)$ by definition of $K - E$, and the claim holds trivially.

So suppose $\tau - \alpha \notin E$. Let $\beta > \alpha$, with $\beta_i = \alpha_i$. If $\Delta^E(\tau - \beta) = \emptyset$, then $\beta \in \overline{\Delta}_i^{K-E}(\alpha)$, and the claim is true. Therefore assume that for any $\beta > \alpha$, with $\beta_i = \alpha_i$, $\Delta^E(\tau - \beta) \neq \emptyset$. This is equivalent to $\Delta_j^E(\tau - \beta) \neq \emptyset$. In particular, we can choose $\beta_j \geq (\gamma - \mu^E)_j$. Let $\tau - \delta \in \Delta_j^E(\tau - \beta)$. Then $\tau - \delta \in E = K - (K - E)$, i.e. $\Delta^{K-E}(\delta) = \emptyset$. But $(\tau - \delta)_j = (\tau - \beta)_j$, i.e. $\delta_j = \beta_j \geq (\gamma - \mu^E)_j = \gamma_j^{K-E}$. So $\Delta_j^{K-E}(\delta) = \emptyset$. Since we had $\Delta^{K-E}(\delta) = \emptyset$, this is a contradiction. □

Lemma 5.9. *Let S be a good semigroup and let $E \in \mathfrak{G}_S$. Let $K := K_S^0$ be the normalized canonical ideal of S, and let $\alpha, \beta \in \mathbb{Z}^s$ with $\alpha \leq \beta$. Then:*

$$d(E^\alpha \backslash E^\beta) \leq d(\overline{S}^\alpha \backslash \overline{S}^\beta) - d((K - E)^{\gamma - \beta} \backslash (K - E)^{\gamma - \alpha}).$$

Proof. Let

$$\alpha = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(n)} = \beta$$

be a saturated chain in $\mathbb{Z}^s$, with $\alpha^{(j+1)} = \alpha^{(j)} + \mathbf{e}_{i(j)}$ for any $j \in \{0, \ldots, n-1\}$. Let us denote $J = \{0, \ldots, n-1\}$.

Set $\beta^{(j)} = \gamma - \alpha^{(n-j)}$. Then

$$\gamma - \beta = \beta^{(0)} < \beta^{(1)} < \cdots < \beta^{(n)} = \gamma - \alpha$$

is a saturated chain in $\mathbb{Z}^s$, and

$$\beta^{(j+1)} = \gamma - \alpha^{(n-(j+1))} = \gamma - (\alpha^{(n-j))} - \mathbf{e}_{i(n-(j+1))}) = \beta^{(j)} + \mathbf{e}_{i(n-(j+1))}.$$

By Proposition 5.5 we have $d(E^\alpha \backslash E^\beta) = \mathrm{Card}\{j \in J \mid \overline{\Delta}_{i(j)}^E(\alpha^{(j)}) \neq \emptyset\}$. Recall that

$E = K - (K - E)$ by Corollary 4.13. Therefore we can apply Lemma 5.7 and obtain

$$\begin{aligned} d(E^\alpha \backslash E^\beta) &= \mathrm{Card}\{j \in J \mid \overline{\Delta}^E_{i(j)}(\alpha^{(j)}) \neq \emptyset\} \\ &\leq \mathrm{Card}\{j \in J \mid \Delta^{K-E}_{i(j)}(\tau - \alpha^{(j)}) = \emptyset\} \\ &= \mathrm{Card}\{j \in J \mid \Delta^{K-E}_{i(j)}(\gamma - \alpha^{(j)} - 1) = \emptyset\} \\ &= \mathrm{Card}\{j \in J \mid \Delta^{K-E}_{i(j)}(\beta^{(n-j)} - 1) = \emptyset\} \\ &= \mathrm{Card}\{j \in J \mid \overline{\Delta}^{K-E}_{i(j)}(\beta^{(n-(j+1))}) = \emptyset\} \\ &= n - \mathrm{Card}\{j \in J \mid \overline{\Delta}^{K-E}_{i(j)}(\beta^{(n-(j+1))}) \neq \emptyset\} \\ &= n - \mathrm{Card}\{j \in J \mid \overline{\Delta}^{K-E}_{i(n-(j+1))}(\beta^{(j)}) \neq \emptyset\} \\ &= n - d((K-E)^{\gamma-\beta} \backslash (K-E)^{\gamma-\alpha}) \\ &= d(\overline{S}^\alpha \backslash \overline{S}^\beta) - d((K-E)^{\gamma-\beta} \backslash (K-E)^{\gamma-\alpha}). \end{aligned} \tag{5.2}$$

□

Proposition 5.10. *Let S be a good semigroup. Let $E \in \mathfrak{G}_S$ and $\alpha, \beta \in \mathbb{Z}^s$ with $\alpha \leq \beta$. Then the following are equivalent:*

(i) $d(E^\alpha \backslash E^\beta) = d(\overline{S}^\alpha \backslash \overline{S}^\beta) - d((K-E)^{\gamma-\beta} \backslash (K-E)^{\gamma-\alpha})$;

(ii) For any $\delta \in \mathbb{Z}^s$ such that $\alpha \leq \delta \leq \beta$ and for any $i \in \{1, \ldots, s\}$ such that $\delta + \mathbf{e}_i \leq \beta$,

$$\overline{\Delta}^E_i(\delta) \neq \emptyset \iff \Delta^{K-E}_i(\tau - \delta) = \emptyset;$$

(iii) For any $\delta \in \mathbb{Z}^s$ such that $\alpha \leq \delta \leq \beta$ and for any $i \in \{1, \ldots, s\}$ such that $\delta - \mathbf{e}_i \geq \alpha$,

$$\overline{\Delta}^{K-E}_i(\tau - \delta) \neq \emptyset \iff \Delta^E_i(\delta) = \emptyset.$$

Proof. Let

$$\alpha = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(n)} = \beta$$

and

$$\gamma - \beta = \beta^{(0)} < \beta^{(1)} < \cdots < \beta^{(n)} = \gamma - \alpha$$

be as in Lemma 5.9. Let us denote again $J = \{0, \ldots, n-1\}$. Then, from the proof of Lemma 5.9 (see (5.2)) we have that

$$d(E^\alpha \backslash E^\beta) = d(\overline{S}^\alpha \backslash \overline{S}^\beta) - d((K-E)^{\gamma-\beta} \backslash (K-E)^{\gamma-\alpha})$$

if and only if

$$\mathrm{Card}\{j \in J \mid \overline{\Delta}^E_{i(j)}(\alpha^{(j)}) \neq \emptyset\} = \mathrm{Card}\{j \in J \mid \Delta^{K-E}_{i(j)}(\tau - \alpha^{(j)}) = \emptyset\}.$$

Since the first set is contained in the second by Lemma 5.7, we obtain

$$\{j \in J \mid \overline{\Delta}^E_{i(j)}(\alpha^{(j)}) \neq \emptyset\} = \{j \in J \mid \Delta^{K-E}_{i(j)}(\tau - \alpha^{(j)}) = \emptyset\}$$

In particular

$$\overline{\Delta}^E_{i(j)}(\alpha^{(j)}) \neq \emptyset \iff \Delta^{K-E}_{i(j)}(\tau - \alpha^{(j)}) = \emptyset.$$

Now let $\delta \in \mathbb{Z}^s$ be such that $\alpha \leq \delta \leq \beta$ and for any $i \in \{1, \ldots, s\}$, $\delta + \mathbf{e}_i \leq \beta$. Then it is always possible to find a saturated chain in $\mathbb{Z}^s$ between α and β such that $\delta = \alpha^{(j)}$ and $i = i(j)$. Thus

$$\overline{\Delta}_i^E(\delta) \neq \emptyset \Longleftrightarrow \Delta_i^{K-E}(\tau - \delta) = \emptyset.$$

Finally, observing that, by Theorem 4.12, $E = K - (K - E)$, this is also equivalent to

$$\overline{\Delta}_i^{K-E}(\tau - \delta) \neq \emptyset \Longleftrightarrow \Delta_i^E(\delta) = \emptyset.$$

if $\delta - \mathbf{e}_i \geq \alpha$ (i.e. $(\tau - \delta) + \mathbf{e}_i \leq \tau - \alpha$). □

Remark 5.11. For any $\alpha \in \mathbb{Z}^s$ we have $K - \overline{S}^\alpha = \overline{S}^{\gamma - \alpha}$. In fact, due to Lemmas 2.5 and 4.3.(b) and Definition 5.1, we have:

$$K - \overline{S}^\alpha = K - (\alpha + \overline{S}) = -\alpha + (K - \overline{S}) = -\alpha + \gamma + \overline{S} = \overline{S}^{\gamma - \alpha}.$$

Corollary 5.12. *Let S be a good semigroup, and $K := K_S^0$ the normalized canonical ideal of S. Let $E \in \mathfrak{G}_S$ and $\alpha \in \mathbb{Z}^s$ with $\mu^E \leq \alpha \leq \gamma^E$. Then the following are equivalent:*

(i) $d(\overline{S}^{\mu^E} \backslash E) = d((K - E) \backslash \overline{S}^{\gamma - \mu^E})$;

(ii) $d(E \backslash E^{\gamma^E}) = d(\overline{S}^{\mu^E} \backslash \overline{S}^{\gamma^E}) - d((K - E) \backslash (K - E)^{\gamma - \mu^E})$;

(iii) for any $i \in \{1, \ldots, s\}$ such that $\alpha + \mathbf{e}_i \leq \gamma^E$,

$$\overline{\Delta}_i^E(\alpha) \neq \emptyset \iff \Delta_i^{K-E}(\tau - \alpha) = \emptyset;$$

(iv) for any $i \in \{1, \ldots, s\}$ such that $\alpha - \mathbf{e}_i \geq \mu^E$,

$$\overline{\Delta}_i^{K-E}(\tau - \alpha) \neq \emptyset \iff \Delta_i^E(\alpha) = \emptyset.$$

Proof. First of all observe that by additivity

$$d(\overline{S}^{\mu^E} \backslash E) = d(\overline{S}^{\mu^E} \backslash \overline{S}^{\gamma^E}) - d(E \backslash \overline{S}^{\gamma^E})$$

so (i) is equivalent to (ii). Now observe that by Lemma 4.3.(b) and Remark 5.11, (ii) is the same as

$$d(E^{\mu^E} \backslash E^{\gamma^E}) = d(\overline{S}^{\mu^E} \backslash \overline{S}^{\gamma^E}) - d((K - E)^{\gamma - \gamma^E} \backslash (K - E)^{\gamma - \mu^E}).$$

The claim then follows trivially from Proposition 5.10. □

Remark 5.13. Notice that, if R is an admissible ring and $\mathcal{E}$ is a regular fractional ideal of R, condition (i) of Corollary 5.12 is always true. In fact, denoting $S := \Gamma_R$ and $E := \Gamma_{\mathcal{E}}$, by Theorem 4.16 and Remark 5.11 we have:

$$\begin{aligned} d(\overline{S}^{\mu^E} \backslash E) &= \ell(x\overline{R}/\mathcal{E}) = \ell(\mathcal{K} : \mathcal{E} / \mathcal{K} : x\overline{R}) = d(K - E \backslash K - \overline{S}^{\mu^E}) \\ &= d(K - E \backslash \overline{S}^{\gamma - \mu^E}) \end{aligned}$$

where $x \in R$ is an element of valuation μ^E, K is the normalized canonical ideal of S and $\mathcal{K}$ is a normalized canonical ideal of R.

5.2 Duality of the Poincaré series

For any $J \subseteq I$ we denote $\mathbf{e}_J = \sum_{j \in J} \mathbf{e}_j$.

The following definition is analogous to the one given in [Pol16, § 5.2.8]:

Definition 5.14. Let R be an admissible ring, and let $\mathcal{E} \in \mathfrak{R}_R$. Set $t = (t_1, \dots, t_s)$. We define

$$\ell_{\mathcal{E}}(\alpha) = \ell(\mathcal{E}^{\alpha}/\mathcal{E}^{\alpha+1}), \quad L_{\mathcal{E}}(t) = \sum_{\alpha \in \mathbb{Z}^s} \ell_{\mathcal{E}}(\alpha) t^{\alpha},$$

and the *Poincaré series of* $\mathcal{E}$ is

$$P_{\mathcal{E}}(t) = L_{\mathcal{E}}(t) \prod_{i=1}^{s} (t_i - 1).$$

We now give the analogous definition for good semigroup ideals.

Definition 5.15. Let S be a good semigroup, and let $E \in \mathfrak{G}_S$. We define

$$d_E(\alpha) = d(E^{\alpha} \backslash E^{\alpha+1}), \quad L_E(t) = \sum_{\alpha \in \mathbb{Z}^s} d_E(\alpha) t^{\alpha},$$

and the *Poincaré series of* E is

$$P_E(t) = L_E(t) \prod_{i=1}^{s} (t_i - 1).$$

Remark 5.16. Due to Remark 5.2 and Proposition 2.29, if $S = \Gamma_R$ and $E = \Gamma_{\mathcal{E}}$, we have $L_E(t) = L_{\mathcal{E}}(t)$, and in particular $P_E(t) = P_{\mathcal{E}}(t)$.

Lemma 5.17. *Let S be a good semigroup ideal, and $E \in \mathfrak{G}_S$. Let us define*

$$c_E(\alpha) = \sum_{J \subseteq \{1, \dots, s\}} (-1)^{\mathrm{Card}(J^c)} d_E(\alpha - \mathbf{e}_J)$$

then the Poincaré series can be written as

$$P_E(t) = \sum_{\alpha \in \mathbb{Z}^s} c_E(\alpha) t^{\alpha}.$$

Proof. Consider $I = \{1, \dots, s\}$. Observe that

$$\begin{aligned}\prod_{i=1}^{s}(t_i - 1) &= t_1 \cdots t_s + (-1)^1 \sum_{i_1 < \cdots < i_{s-1}} t_{i_1} \cdots t_{i_{s-1}} + \cdots + (-1)^{s-1} \sum_{i=1}^{s} t_i + (-1)^s \\ &= \sum_{J \subseteq I} (-1)^{\mathrm{Card}(J^c)} t^{\mathbf{e}_J}.\end{aligned}$$

where J^c denotes the complement of J in I. Hence

$$\begin{aligned}P_E(t) &= \sum_{\alpha \in \mathbb{Z}^s} d_E(\alpha) t^{\alpha} \prod_{i=1}^{s} (t_i - 1) = \sum_{\alpha \in \mathbb{Z}^s} d_E(\alpha) t^{\alpha} \sum_{J \subseteq I} (-1)^{\mathrm{Card}(J^c)} t^{\mathbf{e}_J} \\ &= \sum_{\alpha \in \mathbb{Z}^s} \sum_{J \subseteq I} (-1)^{\mathrm{Card}(J^c)} d_E(\alpha) t^{\alpha + \mathbf{e}_J} = \\ &= \sum_{\alpha \in \mathbb{Z}^s} \sum_{J \subseteq I} (-1)^{\mathrm{Card}(J^c)} d_E(\alpha - \mathbf{e}_J) t^{\alpha} \\ &= \sum_{\alpha \in \mathbb{Z}^s} c_E(\alpha) t^{\alpha}.\end{aligned}$$

□

Lemma 5.18. *Let S be a good semigroup and let $E \in \mathfrak{G}_S$. Let $\beta \in \mathbb{Z}^s$. If $\beta_i + 1 < \mu_i^E$ or $\beta_i > \gamma_i^E$, then $d_E(\beta) = d_E(\beta + \mathbf{e}_i)$.*

Proof. Consider again $I = \{1, \dots, s\}$. Let $\beta = \beta^{(0)} < \beta^{(1)} = \beta + \mathbf{e}_i < \cdots < \beta^{(s)} = \beta + 1 < \beta^{(s+1)} = \beta + \mathbf{e}_i + 1$ be a saturated chain in $\mathbb{Z}^s$, where $\beta^{(j+1)} = \beta^{(j)} + \mathbf{e}_j$ for any $j \in I \setminus \{i\}$. Then by definition of $d_E(\beta)$ and by Proposition 2.28 we have

$$d_E(\beta) = d_E(E^\beta \backslash E^{\beta+1}) = \sum_{j=0}^{s-1} d_E(E^{\beta^{(j)}} \backslash E^{\beta^{(j+1)}}).$$

On the other hand we have

$$d_E(\beta + \mathbf{e}_i) = d_E(E^{\beta+\mathbf{e}_i} \backslash E^{\beta+\mathbf{e}_i+1}) = \sum_{j=1}^{s} d_E(E^{\beta^{(j)}} \backslash E^{\beta^{(j+1)}}).$$

Therefore

$$\begin{aligned} d_E(\beta + \mathbf{e}_i) - d_E(\beta) =& d_E(E^{\beta^{(s)}} \backslash E^{\beta^{(s+1)}}) - d_E(E^{\beta^{(0)}} \backslash E^{\beta^{(1)}}) \\ =& d_E(E^{\beta+1} \backslash E^{\beta+\mathbf{e}_i+1}) - d_E(E^{\beta} \backslash E^{\beta+\mathbf{e}_i}). \end{aligned}$$

By Lemma 5.4 we know that

$$d_E(E^\beta \backslash E^{\beta+\mathbf{e}_i}) = 1 \iff \overline{\Delta}_i^E(\beta) \neq \emptyset.$$

and

$$d_E(E^{\beta+1} \backslash E^{\beta+\mathbf{e}_i+1}) = 1 \iff \overline{\Delta}_i^E(\beta+1) \neq \emptyset.$$

When $\beta_i + 1 < \mu_i^E$, then also $\beta_i < \mu_i^E$, and therefore $\overline{\Delta}_i^E(\beta) = \overline{\Delta}_i^E(\beta + 1) = \emptyset$. This yields $d_E(\beta + \mathbf{e}_i) - d_E(\beta) = 0$. On the other hand, when $\beta_i > \gamma_i^E$, then also $\beta_i + 1 > \gamma_i^E$ and $\overline{\Delta}_i^E(\beta) \neq \emptyset, \overline{\Delta}_i^E(\beta+1) \neq \emptyset$. This implies $d_E(E^\beta \backslash E^{\beta+\mathbf{e}_i}) = d_E(E^{\beta+1} \backslash E^{\beta+\mathbf{e}_i+1}) = 1$, and thus once again $d_E(\beta + \mathbf{e}_i) - d_E(\beta) = 0$. □

Proposition 5.19. *Let S be a good semigroup and let $E \in \mathfrak{G}_S$. Then $P_E(t)$ is a polynomial.*

Proof. The goal is to prove that $c_E(\alpha) \neq 0$ only if $\mu^E \leq \alpha \leq \gamma^E$. Suppose there exists an i such that $\alpha_i < \mu_i^E$. Consider $J \subseteq I = \{1, \dots, s\}$. It is not restrictive to consider $i \notin J$ (otherwise we can consider $J \setminus \{i\}$). Notice that if $\alpha - \mathbf{e}_{J\cup\{i\}} = \beta$, then $\alpha - \mathbf{e}_J = \beta + \mathbf{e}_i$. Since $\alpha_i < \mu_i^E$, then $\mu_i^E > (\alpha - \mathbf{e}_J)_i = (\beta + \mathbf{e}_i)_i = \beta_i + 1$. So by Lemma 5.18, we have

$$d_E(\alpha - \mathbf{e}_{J\cup\{i\}}) = d_E(\alpha - \mathbf{e}_J)$$

The same is true if i is such that $\alpha_i > \gamma_i^E$. Therefore when $\alpha \notin \{\beta \mid \mu^E \leq \beta \leq \gamma^E\}$, for any $J \subseteq I$ there exists a $J' \subset I$ (it can be either $J \cup \{i\}$ or $J \setminus \{i\}$) such that

$$d_E(\alpha - \mathbf{e}_{J'}) = d_E(\alpha - \mathbf{e}_J)$$

and $\mathrm{Card}(J) = \mathrm{Card}(J') \pm 1$. Hence this terms annihilate each other in the sum

$$\sum_{J \subseteq I} (-1)^{\mathrm{Card}(J^c)} d_E(\alpha - \mathbf{e}_J),$$

so that $c_E(\alpha) = 0$ for any $\alpha \notin \{\beta \mid \mu^E \leq \beta \leq \gamma^E\}$.

Thus $P_E(t)$ is a polynomial. □

Proposition 5.20. *Let S be a good semigroup, $K := K_S^0$ the normalized canonical ideal of S and $E \in \mathfrak{G}_S$. If one of the equivalent conditions of Corollary 5.12 holds, then the Poincaré polynomials of E and $K - E$ are symmetric:*

$$P_{K-E}(t) = (-1)^{s+1} t^\gamma P_E\left(\frac{1}{t}\right).$$

Proof. By Lemma 5.17, $P_{K-E}(t) = \sum_{\alpha \in \mathbb{Z}^s} c_{K-E}(\alpha) t^\alpha$. On the other hand

$$\begin{aligned}
(-1)^{s+1} t^\gamma P_E\left(\frac{1}{t}\right) &= (-1)^{s+1} t^\gamma \sum_{\beta \in \mathbb{Z}^s} c_E(\beta) t^{-\beta} \\
&= \sum_{\beta \in \mathbb{Z}^s} (-1)^{s+1} c_E(\beta) t^{\gamma - \beta} \\
&= \sum_{\alpha \in \mathbb{Z}^s} (-1)^{s+1} c_E(\gamma - \alpha) t^\alpha.
\end{aligned}$$

Therefore the claim is equivalent to

$$c_{K-E}(\alpha) = (-1)^{s+1} c_E(\gamma - \alpha).$$

If $\alpha \notin \{\beta \mid \mu^E \leq \gamma - \beta \leq \gamma^E\} = \{\beta \mid \gamma - \gamma^E \leq \beta \leq \gamma - \mu^E\}$ then $c_{K-E}(\alpha) = c_E(\gamma - \alpha) = 0$ by the proof of Proposition 5.19. So we can assume $\gamma - \gamma^E \leq \alpha \leq \gamma - \mu^E$.

By Corollary 5.12.(ii), we have $d_{K-E}(\alpha) = d_{\overline{S}}(\alpha) - d_E(\gamma - \alpha - 1) = s - d_E(\gamma - \alpha - 1)$. Then

$$\begin{aligned}
c_{K-E}(\alpha) = & \sum_{J \subseteq \{1,\ldots,s\}} (-1)^{\mathrm{Card}(J^c)} d_{K-E}(\alpha - \mathbf{e}_J) \\
=& (-1)^s \sum_{J \subseteq \{1,\ldots,s\}} (-1)^{\mathrm{Card}(J)} (s - d_E(\gamma - \alpha - 1 + \mathbf{e}_J)) \\
=& (-1)^s s \sum_{J \subseteq \{1,\ldots,s\}} (-1)^{\mathrm{Card}(J)} + (-1)^{s+1} \sum_{J \subseteq \{1,\ldots,s\}} (-1)^{\mathrm{Card}(J)} d_E(\gamma - \alpha - 1 + \mathbf{e}_J) \\
=& (-1)^s s \sum_{i=0}^{s} (-1)^i \binom{s}{i} + (-1)^{s+1} \sum_{J \subseteq \{1,\ldots,s\}} (-1)^{s + \mathrm{Card}(J^c)} d_E(\gamma - \alpha - \mathbf{e}_{J^c}) \\
=& (-1)^s (1-1)^s + (-1)^{s+1} c_E(\gamma - \alpha) \\
=& (-1)^{s+1} c_E(\gamma - \alpha).
\end{aligned}$$

Hence the claim. ☐

So we obtain Pol's result [Pol16, Proposition 5.2.28] as a corollary:

Corollary 5.21. *Let R be an admissible ring and $\mathcal{E} \in \mathfrak{R}_R$. If $\mathcal{E}$ is a regular fractional ideal of an admissible ring R, and $E = \Gamma_\mathcal{E}$, then:*

$$P_{K-E}(t) = (-1)^{s+1} t^\gamma P_E\left(\frac{1}{t}\right).$$

Question 5.22. Are the equivalent conditions of Corollary 5.12 always true? Proposition 5.8 and Remark 5.13 show that they are true in the 2-dimensional case and in the value semigroup case.

Part II

Inverse system

Introduction

Let $\mathbf{k}$ be an algebraic closed field and let A be a (*)local $\mathbf{k}$-algebra (see §E.1 for a definition of *local). Then $A = R/I$ with R either the power series ring $\mathbf{k}[[x_1, \dots, x_n]]$ or the polynomial ring $k[x_1, \dots, x_n]$ and I either an ideal (not necessarily homogeneous) of $R = \mathbf{k}[[x_1, \dots, x_n]]$, or I homogeneous ideal of $R = k[x_1, \dots, x_n]$. As an effective consequence of Matlis duality, it is known that A is Artinian if and only if the dual module

$$I^\perp := (R/I)^\vee = \mathrm{Hom}_R(R/I, E_R(\mathbf{k}))$$

is a finitely generated R-module, where $E_R(\mathbf{k})$ is the injective hull of $\mathbf{k}$. The module $I^\perp$ is called *inverse system of* I, and it can be embedded into the *divided power ring* $\mathcal{D} = \mathbf{k}_{DP}[X_1, \dots, X_n] \cong E_R(\mathbf{k})$, which has a structure of R-module via the contraction action (see Definition 6.6).

It is in particular well-known that A is Gorenstein if and only if $I^\perp$ is a cyclic R-submodule of $\mathcal{D}$. These cyclic R-submodules were called by Macaulay *principal systems* (see [Mac94]), and contain the same information of the original ideals. In the last twenty years several authors have explored this topic. Among others: [CENR13], [ER93], [ER12], [ER15], [EM07], [EI95], [Ger96], [Iar95], [Iar97], [Iar84], [IK99], [MS05]. Principal systems generated by certain forms associated to partitions occurred in the *n-factorial conjecture* in combinatorics and geometry, see [Hai94]. They are also related to constant-coefficient partial differential equations and to dualizing modules. There has been recent interest in non-homogeneous principal inverse systems in connection with the study of "cactus" forms, see [RS13],[BR13],[BB14]. The inverse system viewpoint can be used to parametrize Artinian Gorenstein quotients of R having a given Hilbert function, see [Wat89], [Iar97] and, more in general, to study the properties of the punctual Hilbert Scheme. The inverse system of zero-dimensional schemes was described in [Ger96], [CI12], [Kle07], [GS98], [Ems78]. In general, Macaulay's inverse system allows to construct the inverse system of any ideal I (need not be Gorenstein) when $A = R/I$ is Artinian. In particular, it allows to construct level rings in the zero-dimensional case (see for example [Ems78] and [Iar94]).

Level rings are Cohen–Macaulay rings with "minimal" socle type, and in particular Gorenstein rings are level rings of type 1. Level rings have been studied in several different contexts. One of the first to observe their properties was Stanley in [Sta77], but later many others took an interest in them: Iarrobino [Iar84], Geramita, Harima, Migliore, Shin [GHMS07], Boij [Boi94] and more recently Bertella [Ber09] and De Stefani [DS14], to name a few. Although in the literature level rings are most studied in the Artinian case, there are many examples of positive dimensional level rings: Stanley-Reisner rings of matroid simplicial complexes (see [Sta77]), semigroup rings corresponding to arithmetic sequences (see [MT95]), and generic determinantal rings. However, an inverse system for level rings of positive dimension is not known. An important point is that for non- Artinian $\mathbf{k}$-algebras, the inverse system is not finitely generated.

In this part of the thesis, extending a recent result by Elias and Rossi [ER17], we give the structure of the inverse system of local and graded level $\mathbf{k}$-algebras of positive dimension, and we describe the global generators of the R-submodules W of $\mathcal{D} = \mathbf{k}_{DP}[X_1, \dots, X_n]$ corresponding to d-dimensional level $\mathbf{k}$-algebras.

In Chapter 6 we recall how a polynomial ring S can be seen as an R-module via the derivation or contraction actions. We then define the divided power ring, and we state results about its structure as module and as ring. Later we recall known facts about Macaulay's Inverse System for Artinian algebras, and in particular Macaulay's one-to-one correspondence between level $\mathbf{k}$-algebras and same degree polynomials with linearly independent forms (see Proposition 6.31).

In Chapter 7 we recall that a homogeneous $\mathbf{k}$-algebra A is *level* if the canonical module ω_A of A is a free R-module generated by elements of same degree. We then give a definition in the local case: a local $\mathbf{k}$-algebra A is *level* if A/J is Artinian level for some minimal general reduction J of the maximal ideal (see Definition 7.21). Defining local level K-algebras of positive dimension is non-trivial. In fact, contrary to the graded case, where any Artinian reduction of the maximal ideal has the same socle type, in the local case Artinian reductions of the maximal ideal given by non-minimal general reductions may have different socle type.

Given a notion of local $\mathbf{k}$-algebra in any dimension, we then investigate the structure of $I^{\perp}$ when R/I is a positive dimensional level $\mathbf{k}$-algebra. Generalizing the result of [ER17], we give a notion of L_d^{τ}*-admissible* submodules of $\mathcal{D}$ (see Definition 7.37). and we establish a one-to-one correspondence between level $\mathbf{k}$-algebras R/I of positive dimension d (see Theorem 7.50). Observe that our L_d^{τ}-admissibility is not merely the "union" of the conditions given in [ER17]. Also, while in the Artinian case the intersection of Gorenstein ideals of same socle degree is always level, as a trivial consequence of Macaulay's inverse system, the analogous is not true in positive dimension (see Example 8.6). Our correspondence is therefore an important tool, as it provides an effective method to construct level $\mathbf{k}$-algebras. In the graded case, we can retrieve the level ring with just a finite number of generators of the inverse system. Moreover, we can read both multiplicity and regularity of a level graded $\mathbf{k}$-algebra in the dual module.

In Chapter 8, we collect some applications of our main result, and we give constructive examples. In particular we show inverse systems of semigroup rings defined by arithmetic sequences and of matroids coming from simplicial complexes.

6
Inverse system of Artinian rings

In this chapter we give the definition of derivation and contraction action on a polynomial ring, and we then introduce the divided power ring in order to avoid characteristic problems. The divided power ring $\mathcal{D}$ is actually a ring, but we will not consider it as such in the following. However, we give its ring structure for completeness, and we then concentrate on its structure as R-module, were R is either a polynomial or a power series ring. Using the divided power ring we then recall what is the Macaulay's inverse system of an ideal of R and the annihilator of a submodule M of $\mathcal{D}$. This two operations (inverse system and annihilator) give rise to a specialization of Matlis duality, i.e. a one-to-one correspondence between Artinian algebras of type R/I and finitely generated R-submodules of $\mathcal{D}$. In particular, we recall the one-to-one correspondence between level zero-dimensional algebras and R-submodules of $\mathcal{D}$ generated by polynomials of same degree with linearly independent forms. The contents of this chapter are taken from the literature, and are not original work.

6.1 Divided powers

Let $\mathbf{k}$ be an arbitrary field. Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$ be the ring of formal power series with maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ (or $R = \mathbf{k}[x_1, \ldots, x_n]$ the polynomial ring with maximal homogeneous ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$) and let $S = \mathbf{k}[y_1, \ldots, y_n]$ be a polynomial ring, with homogeneous maximal ideal $(y_1, \ldots, y_n)$. It is well known that R is an S-module with the induced product. Also, S can be considered as R-module with two different external products: derivation and contraction. If $\mathrm{char}(\mathbf{k}) = 0$, the R-module structure of S by derivation is defined by

$$\begin{aligned} &R \times S \to S \\ &(x^\alpha, y^\beta) \mapsto x^\alpha \circ y^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where for any $\alpha, \beta \in \mathbb{N}^n$, $\alpha! = \prod_{i=1}^n \alpha_i!$.

In any characteristic, the R-module structure of S by contraction is defined by

$$\begin{aligned} &R \times S \to S \\ &(x^\alpha, y^\beta) \mapsto x^\alpha \circ y^\beta = \begin{cases} y^{\beta-\alpha} & \text{if } \beta \geq \alpha \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We denote by (S, der) the R-module S with the derivation action and (S, cont) the R-module S with the contraction action.

The following is [Eli13, Proposition 2.1]:

Proposition 6.1. *For any field* $\mathbf{k}$ *there is a* R*-module homomorphism*

$$\begin{aligned} \sigma : (S, \mathrm{der}) &\to (S, \mathrm{cont}) \\ y^\alpha &\mapsto \alpha! y^\alpha. \end{aligned}$$

If $\mathrm{char}(\mathbf{k}) = 0$*, then* σ *is an isomorphism of* R*-modules.*

Proof. The first statement follows from the following equalities:

$$\begin{aligned} \sigma(x^\alpha \circ y^\beta) &= \sigma\left(\frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha}\right) \\ &= \frac{\beta!}{(\beta-\alpha)!}((\beta-\alpha)! y^{\beta-\alpha}) \\ &= \beta! y^{\beta-\alpha} = x^\alpha \circ \sigma(y^\beta). \end{aligned}$$

If $\mathrm{char}(\mathbf{k}) = 0$ then the inverse of σ is the map

$$y^\alpha \mapsto \frac{1}{\alpha!} y^\alpha.$$

□

The following proposition relates the injective hull of $\mathbf{k}$ (see Definition D.7) with the modules defined above.

Proposition 6.2. *Let* $R = \mathbf{k}[[x_1, \dots, x_n]]$*. If* $\mathbf{k}$ *is of characteristic zero then* $E_R(\mathbf{k}) \cong (S, \mathrm{der}) \cong (S, \mathrm{cont})$*. If* $\mathbf{k}$ *is of positive characteristic then* $E_R(\mathbf{k}) \cong (S, \mathrm{cont})$*.*

Proof. See [Eli13, Theorem 2.2]. □

To avoid the distinction based on the characteristic in Proposition 6.2, we introduce the divided powers ring.

Definition 6.3. Let $\mathbf{k}$ be an infinite field of arbitrary characteristic. Let $R = \mathbf{k}[x_1, \dots, x_n]$. Then the *divided power ring* $\mathcal{D}$ is the dual of R, i.e.

$$\mathcal{D} := \mathrm{Hom}_{\mathbf{k}}(R, \mathbf{k}). \tag{6.1}$$

Since $R = \bigoplus_{j \geq 0} \mathrm{Sym}^j R_1$, where the $\mathbf{k}$-vector space R_1 has basis $x_1, \dots, x_n$, then $\mathcal{D}$ is graded, i.e.

$$\mathcal{D} = \mathrm{Hom}_{\mathbf{k}}(\bigoplus_{j \geq 0} \mathrm{Sym}^j R_1, \mathbf{k}) = \bigoplus_{j \geq 0} \mathrm{Hom}_{\mathbf{k}}(\mathrm{Sym}^j R_1, \mathbf{k}) =: \bigoplus_{j \geq 0} \mathcal{D}_j.$$

The action of $Gl_n(\mathbf{k})$ on R_1 by

$$Ax_i = \sum_{j=1}^{n} A_{ji} x_j, \text{ with } A \in Gl_n(\mathbf{k}),$$

can be extended to an action of $Gl_n(\mathbf{k})$ on R. Thus by duality $Gl_n(\mathbf{k})$ transposed on monomials induces an action on $\oplus_{j \geq 0} \mathcal{D}_j$. We denote $\mathcal{D} := \mathbf{k}_{DP}[x_1, \dots, x_n]$.

Remark 6.4. Let $R = \mathbf{k}[x_1, \ldots, x_n]$, with unique homogeneous maximal ideal $\mathfrak{m}$. Then the completion of R is $\widehat{R} = \mathbf{k}[[x_1, \ldots, x_n]]$. Thus

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{k}}(\widehat{R}, \mathbf{k}) &= \operatorname{Hom}_{\mathbf{k}}(\varinjlim R/\mathfrak{m}^j, \mathbf{k}) \\
&= \varprojlim_j \operatorname{Hom}_{\mathbf{k}}(R/\mathfrak{m}^j, \mathbf{k}) \\
&= \varprojlim_j \prod_{i \geq 0} \operatorname{Hom}_{\mathbf{k}}((R/\mathfrak{m}^j)_i, \mathbf{k}) \\
&= \prod_{i \geq 0} \varprojlim_j \operatorname{Hom}_{\mathbf{k}}((R/\mathfrak{m}^j)_i, \mathbf{k}).
\end{aligned}$$

If $j \geq i$, then $(R/\mathfrak{m}^j)_i = R_i$, and hence

$$\operatorname{Hom}_{\mathbf{k}}(\widehat{R}, \mathbf{k}) = \prod_{i \geq 0} \operatorname{Hom}_{\mathbf{k}}(R_i, \mathbf{k}) = \prod_{i \geq 0} \mathcal{D}_i.$$

In the following chapters we will consider R to be either the polynomial ring or the power series ring. Thanks to Remark 6.4, in both cases we can consider $\mathcal{D}$ to be the dual of R.

Notation 6.5. Let $R = \mathbf{k}[x_1, \ldots, x_n] = \bigoplus_{j \geq 0} \operatorname{Sym}^j R_1$, and let $R_j = \operatorname{Sym}^j R_1$. For any $j \in \mathbb{N}$, $B(j) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha \in \mathbb{N}^n, |\alpha| = \alpha_1 + \cdots + \alpha_n = j\}$ is the *standard monomial basis* of R_j.

Definition 6.6. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ and let $X_1, \ldots, X_n$ be the basis of $\mathcal{D}_1$ such that $X_j(x_i) = \delta_i^j$ for any $i, j \in \{1, \ldots, n\}$, where δ_i^j is the Kronecker delta. Then the *divided power monomials* of $\mathcal{D}_j$ are:

$$X^{(\alpha)} = X_1^{(\alpha_1)} \cdots X_n^{(\alpha_n)},$$

for any $\alpha \in \mathbb{Z}^n$. They are a basis of $\mathcal{D}_j$ dual to $B(j)$. We put $X^{[\alpha]} = 0$ if $\alpha_i < 0$ for some i.

For any j, we call the elements of $\mathcal{D}_j$ *divided power forms* and the elements of $\mathcal{D}$ *divided power polynomials*.

Definition 6.7. Let $R = \mathbf{k}[x_1, \ldots, x_n]$. For any i, j there is a (surjective) *contraction map*

$$\begin{aligned}
R_i \times \mathcal{D}_j &\longrightarrow \mathcal{D}_{j-i} \\
(f, F) &\longmapsto f \circ F
\end{aligned} \tag{6.2}$$

where $f \circ F = 0$ if $j < i$, or otherwise it is defined recursively through the formula

$$(f' \circ F)(f) = F(ff') \text{ with } f' \in R_{j-i}.$$

In particular, this gives

$$(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \circ (X_1^{(\beta_1)} \cdots X_n^{(\beta_n)}) = \begin{cases} 0 & \text{if } \beta_i - \alpha_i < 0 \text{ for some } i \\ X_1^{(\beta_1 - \alpha_1)} \cdots X_n^{(\beta_n - \alpha_n)} & \text{otherwise} \end{cases}$$

for any $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| = i$ and $|\beta| = j$. These maps can be extended by linearity to a contraction map $R \times \mathcal{D} \to \mathcal{D}$, which makes $\mathcal{D}$ into a graded R-module.

Observe that 6.2 implies

$$R_i \circ \mathcal{D}_j = \mathcal{D}_{j-i}. \tag{6.3}$$

Definition 6.8. Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$. Then we can define also in this case a *contraction map*

$$\begin{aligned} R \times \mathcal{D} &\longrightarrow \mathcal{D} \\ (h, F) &\longmapsto h^* \circ F \end{aligned} \tag{6.4}$$

where h^* is the image of h in $\mathbf{k}[x_1, \ldots, x_n] = \mathrm{gr}_{\mathfrak{m}}(R)$, where $\mathfrak{m}$ is the maximal ideal of R and $\mathrm{gr}_{\mathfrak{m}}(R) := \oplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. This makes $\mathcal{D}$ an R-module. In the following, we will abuse the notation and write $h \circ F$ instead of $h^* \circ F$. Let $R_i = \{h \in R \mid \deg(h) = i\}$. As a consequence of (6.3) we obtain

$$R_i \circ \bigoplus_{k \leq j} \mathcal{D}_k \subseteq \bigoplus_{k \leq j-i} \mathcal{D}_k.$$

In particular, as the action of R on $\mathcal{D}$ lowers degrees, $\mathcal{D}$ is not a finitely generated R-module.

Remark 6.9. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$. Let $f, g \in R$ and $F, G \in \mathcal{D}$. The contraction action on $\mathcal{D}$ has the following properties:

(a) $f \circ (F + G) = f \circ F + f \circ G$.
(b) $(fg) \circ F = f \circ (g \circ F)$.
(c) $(f + g) \circ F = f \circ F + g \circ F$.
(d) $f \circ (cF) = (cf) \circ F = c(f \circ F)$.

Proposition 6.10. *Let* $\mathbf{k}$ *be of any characteristic. Then* $E_R(\mathbf{k}) \cong \mathcal{D}$ *as* R*-modules.*

Proof. See [Gab58, §3.f] and [Eis95, Example A3.4.(b)]. □

Remark 6.11. Let now $\mathbf{k}$ be of characteristic 0. One considers the differentiation action of $R = \mathbf{k}[x_1, \ldots, x_n]$ on $S := \mathbf{k}[y_1, \ldots, y_n]$ given by

$$f \circ h = f\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)(h), \qquad f \in R, h \in S.$$

This action yields a duality between R_j and $S_j = \{h \in S \mid \deg(h) = j, h \text{ homogeneous }\}$ for any $j \geq 0$, and the basis dual to $B(j)$ is

$$\frac{1}{\alpha_1! \cdots \alpha_n!} y_1^{\alpha_1} \ldots y_n^{\alpha_n} = X^{(\alpha)}$$

So that the map

$$\begin{aligned} \varphi : \mathbf{k}[y_1, \ldots, y_n] &\longrightarrow \mathcal{D} \\ y_i &\longmapsto X_i \end{aligned} \tag{6.5}$$

is an isomorphism of R-modules sending

$$\frac{1}{\alpha_1! \cdots \alpha_n!} y_1^{\alpha_1} \ldots y_n^{\alpha_n} \mapsto X^{(\alpha)} \tag{6.6}$$

where X^α is the one in Definition 6.6. The isomorphism (6.5) is compatible with the action of $Gl_n(\mathbf{k})$.

In particular, in this case the isomorphism of $\mathbf{k}[y_1, \ldots, y_n]$ with $\mathcal{D}$ induces a multiplication on $\mathcal{D}$, thus making $\mathcal{D}$ into a ring. The multiplication is defined on monomials as follows

$$X^{(\alpha)} \cdot X^{(\beta)} = \frac{(\alpha + \beta)!}{\alpha! \beta!} X^{(\alpha+\beta)} \tag{6.7}$$

where

$$\frac{(\alpha + \beta)!}{\alpha! \beta!} = \frac{(\alpha_1 + \beta_1)! \cdots (\alpha_n + \beta_n)!}{\alpha_1! \cdots \alpha_n! \beta_1! \cdots \beta_n!}.$$

This is extended by linearity, and therefore gives a structure of $\mathbf{k}$-algebra on $\mathcal{D}$.

Remark 6.12. In general, for any R-module M, $\mathrm{Sym}(M)^*$ has a natural divided power structure (see for example [Eis95, §A2.4]) coming from a multiplication with $d!$ dividing x^d. Hence it makes sense to set $x^{(d)} := \frac{x^d}{d!}$.

In the following we will often denote the power monomials in $\mathcal{D}$ without the brackets, to make notation easier.

6.2 Macaulay Inverse System

In this chapter we always consider $\mathbf{k}$ to be an infinite field.

Definition 6.13. Let $R = \mathbf{k}[x_1, \dots, x_n]$ or $R = \mathbf{k}[[x_1, \dots, x_n]]$. For any ideal I of R, the R-module

$$I^\perp := \{G \in \mathcal{D} \mid I \circ G = 0\} \cong \mathrm{Hom}_R(R/I, \mathbf{k})$$

is the *Macaulay's inverse system of* I.

Definition 6.14. Let $\mathcal{D}$ be the divided power ring. For any R-submodule M of $\mathcal{D}$, the *annihilator of* M is

$$\mathrm{Ann}_R(M) := \{f \in R \mid f \circ G = 0 \text{ for any } G \in M\} \cong \mathrm{Hom}_R(M, \mathbf{k}).$$

Proposition 6.15 (Macaulay's duality)**.** *Let $R = \mathbf{k}[x_1, \dots, x_n]$ or $R = \mathbf{k}[[x_1, \dots, x_n]]$. There is a order-reversing bijection:*

$$\left\{\begin{matrix} I \text{ ideal of } R \text{ such that} \\ I \text{ is } \mathfrak{m}\text{-primary} \end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix} M \text{ finitely generated} \\ R\text{-submodule of } \mathcal{D} \end{matrix}\right\}$$

given by $I \mapsto I^\perp$ and $\mathrm{Ann}_R(M) \leftarrow\!\shortmid M$.

Proof. See [Eli13, Proposition 2.3]. □

It is clear that Proposition 6.15 is a particular case of Theorem D.13.

Remark 6.16. Proposition 6.15 implies that for any finite R-submodule M of $\mathcal{D}$

$$M = (\mathrm{Ann}_R(M))^\perp$$

and for any Artinian I (i.e. R/I is Artinian)

$$I = \mathrm{Ann}_R(I^\perp).$$

Remark 6.17. Let $R = \mathbf{k}[x_1, \dots, x_n]$ or $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \dots, x_n \rangle$. Let $A = R/I$ be an Artinian quotient of R, and let $\mathfrak{n} = \mathfrak{m}/I$ be the (homogeneous) maximal ideal of A. Let

$$\mathrm{HF}_A(i) = \dim_{\mathbf{k}} \left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}} \right) \text{ for } i \geq 0$$

be the Hilbert function of A (see Definition C.2). By Proposition 6.15, the multiplicity of A is the integer

$$e(A) := \dim_{\mathbf{k}}(A) = \dim_{\mathbf{k}}(I^\perp).$$

Since A is Artinian, $s(A)$ is the last integer such that $\mathrm{HF}_A(i) \neq 0$ and $e(A) = \sum_{i=0}^{s} \mathrm{HF}_A(i)$. By definition, the embedding dimension of A is $\mathrm{HF}_A(1)$.

Example 6.18 ([Eli13]). Let $F = X_2^3 + X_1X_2 + X_1^2 \in \mathcal{D}$ be a polynomial. Then the submodule of $\mathcal{D}$ generated by F is

$$\langle F\rangle = \langle F, X_2^2 + X_1, X_2 + X_1, X_1, 1\rangle_k$$

and $\dim_{\mathbf{k}}(\langle F\rangle) = 5$. The annihilator of F is $I := \mathrm{Ann}_R(F) = \langle x_1x_2 - x_2^3, x_1^2 - x_1x_2\rangle$ is a complete intersection ideal of R. The Hilbert function of $A = R/I$ is $\mathrm{HF}_A = \{1, 2, 1, 1\}$, and hence $e(A) = 5$ and $s(A) = 3$.

Definition 6.19. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \ldots, x_n\rangle$. Let $A = R/I$ be an Artinian quotient of R, and let $\mathfrak{n} = \mathfrak{m}/I$ be the (homogeneous) maximal ideal of A.

The *associated graded ring of* A is the standard graded $\mathbf{k}$-algebra

$$\mathrm{gr}_{\mathfrak{n}}(A) = \bigoplus_{i\geq 0} \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}.$$

If A is a standard graded $\mathbf{k}$-algebra and $\mathfrak{n}$ is its homogeneous maximal ideal, then $\mathrm{gr}_{\mathfrak{n}}(A) \cong A$ as graded algebras. Let I^* be the homogeneous ideal of $\mathbf{k}[x_1, \ldots, x_n]$ generated by the initial forms of the elements in I. Then $\mathrm{gr}_{\mathfrak{n}}(A)$ and $\mathbf{k}[x_1, \ldots, x_n]/I^*$ are isomorphic as graded $\mathbf{k}$-algebras.

Notation 6.20. Let $\mathcal{D}$ be the divided power ring. In the following we denote

$$\mathcal{D}_{\leq j} := \bigoplus_{i\leq j} \mathcal{D}_i$$

Let us define the $\mathbf{k}$-vector space

$$(I^\perp)_i := \frac{I^\perp \cap \mathcal{D}_{\leq i} + \mathcal{D}_{\leq i-1}}{\mathcal{D}_{\leq i-1}}.$$

The following proposition relates the graded parts in Notation 6.20 to the Hilbert function. The proof of this fact is taken from [Eli13, Proposition 2.6].

Proposition 6.21. *Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A = R/I$ be a $\mathbf{k}$-algebra. For any $i \geq 0$ there is an equality*

$$\mathrm{HF}_A(i) = \dim_{\mathbf{k}}(I^\perp)_i.$$

Proof. Let us consider the exact sequence of R-modules

$$0 \to \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}} \to \frac{A}{\mathfrak{n}^{i+1}} \to \frac{A}{\mathfrak{n}^i} \to 0.$$

Dualizing this sequence we get:

$$0 \to (I + \mathfrak{m}^i)^\perp \to (I + \mathfrak{m}^{i+1})^\perp \to \mathrm{Hom}_R\left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}, \mathbf{k}\right) \to 0.$$

Hence:

$$\mathrm{Hom}_R\left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}, \mathbf{k}\right) \cong \frac{(I + \mathfrak{m}^{i+1})^\perp}{(I + \mathfrak{m}^i)^\perp} = \frac{I^\perp \cap \mathcal{D}_{\leq i}}{I^\perp \cap \mathcal{D}_{\leq i-1}} \cong \frac{I^\perp \cap \mathcal{D}_{\leq i} + \mathcal{D}_{\leq i-1}}{\mathcal{D}_{\leq i-1}}.$$

The claim follows from linear algebra. □

Proposition 6.22. *Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathcal{D}$ be the divided power ring. Consider the $\mathbf{k}$-bilinear pairing*

$$\begin{aligned}(-,-) : R_j \times \mathcal{D}_j &\longrightarrow \mathcal{D}_0 \cong \mathbf{k} \\ (f, G) &\longmapsto (f \circ G)(0)\end{aligned}$$

for any $j \in \mathbb{N}$. Let I be an ideal of R, homogeneous if $R = \mathbf{k}[x_1, \ldots, x_n]$. Then:

(a) The map $(-,-)$ extends to a non-degenerate pairing $R \times \mathcal{D} \to \mathbf{k}$ of $\mathbf{k}$-vector spaces.

(b) $I^\perp = \{f \in R \mid (I, f) = 0\}$.

(c) The map $(-,-)$ induces a bilinear non-degenerate map of $\mathbf{k}$-vector spaces

$$\overline{(-,-)} : R/I \times I^\perp \to \mathbf{k}.$$

(d) For any $i \geq 0$, there is an isomorphism of $\mathbf{k}$-vector spaces:

$$((R/I^*)_i)^\perp \cong (I^\perp)_i,$$

where I^* is the initial ideal of I, and $I^* = I$ if I homogeneous.

Proof. See [Eli13, Proposition 2.7]. □

Remark 6.23. The pairing $(-,-)$ is perfect in case $R = \mathbf{k}[[x_1, \ldots, x_n]]$. In fact in this case

$$R = \prod_{j \geq 0} \hookrightarrow \mathrm{Hom}_{\mathbf{k}}(\mathcal{D}, \mathbf{k}) = \mathrm{Hom}(\bigoplus_{j \geq 0} \mathcal{D}_j, \mathbf{k}) = \prod_{j \geq 0} \mathrm{Hom}_{\mathbf{k}}(\mathcal{D}_j, \mathbf{k}) = \prod_{j \geq 0} R_j^{\vee\vee}$$

where the map is injective because $R_j \hookrightarrow R_j^{\vee\vee}$.

Using this bilinear pairing we can deduce the following property of the annihilators.

Proposition 6.24. *Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathcal{D}$ be the divided power ring. Let $W_1,\ W_2$ be finitely generated R-submodules of $\mathcal{D}$. Then*

$$\mathrm{Ann}_R(W_1 \cap W_2) = \mathrm{Ann}_R(W_1) + \mathrm{Ann}_R(W_2).$$

Proof. See [Cil94, Proposition 12.9]. □

Definition 6.25. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \ldots, x_n \rangle$. Let $A = R/I$ be an Artinian quotient of R, and let $\mathfrak{n} = \mathfrak{m}/I$ be the (homogeneous) maximal ideal of A. The *socle of* A is defined as:

$$\mathrm{Soc}(A) = (0 :_A \mathfrak{n}),$$

and it is a finite $\mathbf{k}$-vector space. We denote by $\mathrm{socdeg}(A)$ the *socle degree of* A, i.e. the maximum positive integer j such that $\mathfrak{n}^j \neq 0$.

Remark 6.26. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A = R/I$ be an Artinian quotient of R. Then the Cohen–Macaulay type of A (see Definition E.1) can be defined as follows:

$$\tau(A) := \dim_{\mathbf{k}}(\mathrm{Soc}(A)).$$

Definition 6.27. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \ldots, x_n \rangle$. Let $A = R/I$ be an Artinian quotient of R, and let $\mathfrak{n} = \mathfrak{m}/I$ be the (homogeneous) maximal ideal of A.

The *socle type of* A is the sequence

$$\sigma(A) = (0, \ldots, \sigma_{r-1}, \sigma_r, \ldots, \sigma_s, 0, \ldots, 0)$$

where s is the socle degree of A and

$$\sigma_i := \dim_{\mathbf{k}} \left(\frac{\mathrm{Soc}(A) \cap \mathfrak{n}^i}{\mathrm{Soc}(A) \cap \mathfrak{n}^{i+1}} \right).$$

Clearly $\sigma_s > 0$ and $\sigma_j = 0$ for $j > s$.

Definition 6.28. Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \ldots, x_n \rangle$. Let $A = R/I$ be an Artinian quotient of R, and let $\mathfrak{n} = \mathfrak{m}/I$ be the (homogeneous) maximal ideal of A. Let s be the socle degree of A. Then A is said to be *level* if

$$\mathrm{Soc}(A) = \mathfrak{n}^s.$$

Equivalently,

$$\sigma_j = 0 \text{ for } j \neq s \text{ and } \sigma_s = \tau$$

where τ is the Cohen–Macaulay type of A.

If A is level of type 1, then A is Gorenstein (see Proposition E.5).

Definition 6.28 implies that level Artinian $\mathbf{k}$-algebras have ”minimal” socle type.

Proposition 6.29. *Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \ldots, x_n \rangle$. Let $A = R/I$ be an Artinian quotient of R, and let $\mathfrak{n} = \mathfrak{m}/I$ be the (homogeneous) maximal ideal of A. Then*

$$\mathrm{Hom}_R(\mathrm{Soc}(A), \mathbf{k}) \cong \frac{I^\perp}{\mathfrak{m} \circ I^\perp},$$

and

$$\tau(A) = \mu(I^\perp),$$

where $\mu(I^\perp)$ denotes the minimum number of generators of $I^\perp$ as R-module.

Proof. Consider the exact sequence

$$0 \to \mathrm{Soc}(A) \to A \xrightarrow{(x_1, \ldots, x_n)} A^n.$$

Dualizing this sequence we obtain

$$I^\perp \times \cdots \times I^\perp \xrightarrow{(x_1, \ldots, x_n) \circ (-)} I^\perp \to \mathrm{Hom}_R(\mathrm{Soc}(A), \mathbf{k}) \to 0 \tag{6.8}$$

where $\langle x_1, \ldots, x_n \rangle \circ (f_1, \ldots, f_n) = \sum_{i=1}^n x_i \circ f_i$. By Nakayama's Lemma, and Remark 6.26, $\tau(A) = \dim_{\mathbf{k}}(\mathrm{Soc}(A)) = \dim_{\mathbf{k}}(\mathrm{Hom}_R(\mathrm{Soc}(A), \mathbf{k})) = \mu(I^\perp)$. Hence by (6.8) we get

$$\mathrm{Hom}_R(\mathrm{Soc}(A), \mathbf{k}) \cong \frac{I^\perp}{(x_1, \ldots, x_n) \circ I^\perp} = \frac{I^\perp}{\mathfrak{m} \circ I^\perp}. \qquad \square$$

Let $F \in \mathcal{D}$ be such that $F \in \mathcal{D}_{\leq r}$. Then

$$F = \sum_{i=0}^{r} F_i \text{ with } F_i \in \mathcal{D}_i \text{ and } F_i \text{ homogeneous.}$$

We denote by $\operatorname{top}(F)$ the degree r form of F, i.e. $\operatorname{top}(F) := F_r$.

Lemma 6.30. *Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \ldots, x_n \rangle$. Let $M \subseteq \mathcal{D}$ be an finitely generated R-submodule such that $M \subseteq \mathcal{D}_{\leq s}$ for some s. Then*

$$\mathfrak{m} \circ M = M \cap \mathcal{D}_{\leq s-1}.$$

Proof. By Definitions 6.7 and 6.8 we have

$$\mathfrak{m} \circ M = \left(\bigoplus_{i \geq 1} R_i \right) \circ M \subseteq \left(\bigoplus_{i \geq 1} R_i \right) \circ \mathcal{D}_{\leq s} \subseteq \mathcal{D}_{\leq s-1}.$$

As M is an R-module, it is clear that $\mathfrak{m} \circ M \subseteq M$. Hence

$$\mathfrak{m} \circ M \subseteq M \cap \mathcal{D}_{\leq s-1}.$$

Conversely, let $G \in M \cap \mathcal{D}_{\leq s-1}$, and assume $M = \langle F_1, \ldots, F_r \rangle$. Then $G = \sum_i f_i \circ F_i$ and $G \in \mathcal{D}_{\leq s-1}$. As $M \subseteq \mathcal{D}_{\leq s}$, we have $F_i \in \mathcal{D}_{\leq s}$. Hence $G \in \mathcal{D}_{\leq s-1}$ if and only if $\deg(f_i) > 0$ (see also Remark 6.9). $\square$

Proposition 6.31 (Macaulay's Inverse System for level algebras)**.** *Let $R = \mathbf{k}[x_1, \ldots, x_n]$ or $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $\mathfrak{m}$ denote the ideal $\langle x_1, \ldots, x_n \rangle$. Let $A = R/I$ be an Artinian quotient of R, and let $\mathfrak{n} = \mathfrak{m}/I$ be the (homogeneous) maximal ideal of A. Let I be an $\mathfrak{m}$-primary ideal of R. Then A is level of socle degree s and type τ if and only if $I^{\perp}$ is generated by τ polynomials $F_1, \ldots, F_\tau \in \mathcal{D}$ such that $\deg(F_i) = s$ for any $i \in \{1, \ldots, \tau\}$ and $\operatorname{top}(F_1), \ldots, \operatorname{top}(F_\tau)$ are $\mathbf{k}$-linear independent forms of degree s.*

In particular, A is Gorenstein of socle degree s if and only if $I^{\perp}$ is a cyclic R-module generated by a polynomial of degree s.

Proof. Assume A is an Artinian level algebra of socle degree s and type τ. Then $\operatorname{Soc}(A) = \mathfrak{n}^s = (\mathfrak{m}^s + I)/I$, and by the proof of Proposition 6.21

$$\operatorname{Hom}_R(\operatorname{Soc}(A), \mathbf{k}) \cong \operatorname{Hom}_R\left(\frac{\mathfrak{n}^s}{\mathfrak{n}^{s+1}}, \mathbf{k} \right) \cong \frac{I^{\perp} \cap \mathcal{D}_{\leq s}}{I^{\perp} \cap \mathcal{D}_{\leq s-1}} = \frac{I^{\perp}}{I^{\perp} \cap \mathcal{D}_{\leq s-1}}, \tag{6.9}$$

where the last equality holds since $I^{\perp} \subseteq \mathcal{D}_{\leq s}$, as I is generated by elements of degree at maximum s. Since the isomorphism in (6.9) and the one in Proposition 6.29 are compatible, this implies

$$\mathfrak{m} \circ I^{\perp} = I^{\perp} \cap \mathcal{D}_{\leq s-1}.$$

Hence $I^{\perp}$ is generated by τ polynomials $F_1, \ldots, F_\tau$ of degree s and $\operatorname{top}(F_1), \ldots, \operatorname{top}(F_\tau)$ are necessarily $\mathbf{k}$-linear independent.

Conversely, assume that $I^{\perp} = \langle F_1, \ldots F_\tau \rangle$ such that $F_i \in \mathcal{D}_{\leq s}$ for $i = 1, \ldots, \tau$ and that $\operatorname{top}(F_1), \ldots, \operatorname{top}(F_\tau)$ are $\mathbf{k}$-linear independent forms of degree s. Then by Lemma 6.30, $\mathfrak{m} \circ I^{\perp} = I^{\perp} \cap \mathcal{D}_{\leq s-1}$, and

$$\frac{I^{\perp}}{\mathfrak{m} \circ I^{\perp}} = \frac{I^{\perp}}{I^{\perp} \cap \mathcal{D}_{\leq s-1}}$$

is generated as $\mathbf{k}$-vector space by the linearly independent $\operatorname{top}(F_1), \dots, \operatorname{top}(F_\tau)$. Thus $F_1, \dots F_\tau$ is a minimal system of generators of $I^\perp$ by Nakayama. Then Proposition 6.29 yields that the type of A is equal to $\tau = \mu(I^\perp)$. Furthermore, since $F_i \in \mathcal{D}_{\leq s}$ for $i \in \{1, \dots, \tau\}$, we have $I = \operatorname{Ann}_R(I^\perp) \subseteq \operatorname{Ann}_R(\mathcal{D}_{\leq s}) = \mathfrak{m}^{s+1}$, i.e. A has socle degree s. Finally, Proposition 6.29 yields $\operatorname{Soc}(A) = \mathfrak{n}^s$, i.e. A is Artinian level of socle degree s. $\square$

7

Inverse system of level algebras of positive dimension

This chapter contains the main result of this part of the thesis. We first recall the definition of level graded algebra. Then we introduce the concept of level local $\mathbf{k}$-algebras in positive dimension, whose definition is not trivial, and we prove some general results on these algebras. Afterwards, we consider R to be either the polynomial ring or the power series ring, and we consider Cohen–Macaulay $\mathbf{k}$-algebras given by quotients of type R/I, where I is homogeneous in case R is graded. In [ER17] the authors proved that $I^\perp$ is *G_d-admissible* if R/I is Gorenstein of dimension d and, conversely, for any G_d-admissible submodule W of $\mathcal{D}$, $\operatorname{Ann}_R(W)$ is a d-dimensional Gorenstein $\mathbf{k}$-algebra. Generalizing the result of [ER17], we establish a one-to-one correspondence between level $\mathbf{k}$-algebras R/I of positive dimension d and particular submodules of $\mathcal{D}$, which we call L_d^τ-admissible (see Definition 7.37).

The content of this chapter is part of a joint work with Shreedevi Masuti (see [MT17]).

7.1 Level $\mathbf{k}$-algebras of positive dimension

Throughout this chapter we will consider

$$R = \mathbf{k}[x_1, \dots, x_n] \text{ or } R = \mathbf{k}[[x_1, \dots, x_n]],$$

where $n \geq 1$ and $\mathbf{k}$ is an infinite field. The unique (homogeneous) maximal ideal of R will be denoted by

$$\mathfrak{m} = \langle x_1, \dots, x_n \rangle$$

In the graded case, the definition of level $\mathbf{k}$-algebra is well-known:

Definition 7.1. Let $R = \mathbf{k}[x_1, \dots, x_n]$, and let $A := R/I$ be Cohen–Macaulay for some homogeneous ideal $I \subseteq R$. Then A is called *level* if there is an homogeneous homomorphism under which all elements in a minimal set of generators of the canonical module ω_A have the same degree.

Remark 7.2. If A is Artinian, then the minimal number of generators of ω_A coincides with the dimension of $\operatorname{Soc}(A)$, and therefore A is a level ring if and only if the homogeneous socle is equal to ω_A. Hence Definition 7.1 coincides with Definition 6.28.

Proposition 7.3. *Let $R = \mathbf{k}[x_1, \dots, x_n]$, and let $A := R/I$ be Cohen–Macaulay for some homogeneous ideal $I \subseteq R$. Let $a \in A$ be a homogeneous non-zero divisor. Then A is level if and only if $A/\langle a \rangle$ is level.*

Proof. By [BH93, Corollary 3.6.14], if ω_A is the canonical module of A, then

$$\omega_{A/\langle a\rangle} \cong (\omega_A/a\omega_A)(\deg(a)).$$

Hence it is clear that ω_A is generated by elements of same degree if and only if $\omega_{A/\langle a\rangle}$ is. □

In the local case, a definition of level $\mathbf{k}$-algebra is well-known in the Artinian case (see Definition 6.28). One idea to define local level in positive dimension is to take an Artinian reduction (defined below) of a ring A and define A to be level if so is the Artinian reduction. But unlike in the graded case (see Proposition 7.3), in the local case two different Artinian reductions can have different socle type (see Definition 6.27). We show this fact in Example 7.13.

First, let us recall the definition of reduction.

Definition 7.4. Let $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. An ideal $J \subseteq \mathfrak{n}$ is said to be a *reduction* of $\mathfrak{n}$ if there exists a non-negative integer n such that $\mathfrak{n}^{n+1} = J\mathfrak{n}^n$. Then $\mathfrak{n}^{n+1} \equiv 0$ modulo J, A/J is Artinian. If J does not contain properly any other reduction, then J is *minimal*.

Remark 7.5. By [HS06, Proposition 8.3.7], for any local Noetherian ring with infinite residue field, there exist minimal reductions of the maximal ideal minimally generated by $l(\mathfrak{n})$ elements, where $l(\mathfrak{n})$ is the analytic spread of $\mathfrak{n}$. Since $\mathfrak{n}$ is $\mathfrak{n}$-primary, $l(\mathfrak{n})$ is actually the dimension of the ring (see [BH93, Exercise 4.6.13]). Then a reduction J is generated by a regular system of parameters for $\mathfrak{n}$, which is just a regular sequence in the Cohen–Macaulay case.

Definition 7.6. Let $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. The *reduction number of* $\mathfrak{n}$ *with respect to* J, where J is an ideal of A, is

$$r_J(\mathfrak{n}) := \min\{k \mid \mathfrak{n}^{k+1} = J\mathfrak{n}^k\}.$$

The *reduction number of* $\mathfrak{n}$ is

$$r(\mathfrak{n}) := \min\{r_J(\mathfrak{n}) \mid J \text{ is a minimal reduction of } \mathfrak{n}\}.$$

Remark 7.7. Observe that the Cohen–Macaulay type of a ring is independent from the minimal reduction, i.e. for a local ring A, $\tau(A) = \dim_{\mathbf{k}}(\mathrm{Soc}(A/J))$ for any J minimal reduction (see [BH93, Lemma 1.2.19]).

Definition 7.8. Let $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$ with maximal ideal $\mathfrak{n}$. An element $a \in \mathfrak{n}$ is *A-superficial for* $\mathfrak{n}$ if there exists a non-negative integer c such that

$$(\mathfrak{n}^{j+1} :_A a) \cap \mathfrak{n}^c = \mathfrak{n}^j$$

for any $j \geq c$. A sequence of elements $a_1, \dots, a_r$ is called *A-superficial for* $\mathfrak{n}$ if a_i is an $A/\langle a_1, \dots, a_{i-1}\rangle$-superficial elements for $\mathfrak{n}$ for any $i \in \{1, \dots, r\}$.

Definition 7.9. Let $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$ with maximal ideal $\mathfrak{n}$. Let $G := \mathrm{gr}_{\mathfrak{n}}(A)$ be the associated graded ring of A. Each element $a \in A$ has a natural image in G, denoted by a^*, which is called *initial form of a with respect to* $\mathfrak{n}$. If $a = 0$ then $a^* = 0$, otherwise $a^* = \overline{a} \in \mathfrak{n}^t/\mathfrak{n}^{t+1}$ where t is the unique integer such that $a \in \mathfrak{n}^t \setminus \mathfrak{n}^{t+1}$.

Remark 7.10. Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$ with maximal ideal $\mathfrak{n}$. Let $G := \mathrm{gr}_{\mathfrak{n}}(A)$ be the associated graded ring of A, with maximal ideal G^+. Let $N := \{x \in \mathfrak{n} \mid \mathfrak{n}^n x = 0 \text{ for some } n\}$ and $H = \{y \in G \mid (G^+)^n y = 0 \text{ for some } n\}$. If $a \in \mathfrak{n} \setminus \mathfrak{n}^2$, the following conditions are equivalent (see [RV10, Theorem 1.2]):

(i) a is A-superficial for $\mathfrak{n}$;

(ii) $a^* \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(G/H)} \mathfrak{p}$;

(iii) $(0 :_G a^*)_j = 0$ for $j \gg 0$;

(iv) $N : a = N$ and $\mathfrak{n}^{j+1} \cap a\mathfrak{n} = a\mathfrak{n}^j$ for $j \gg 0$.

(v) $\mathfrak{n}^{j+1} : a = \mathfrak{n}^j + (0 :_A a)$ and $\mathfrak{n}^j \cap (0 :_A a) = 0$ for $j \gg 0$.

Since $\mathbf{k}$ is infinite, condition (ii) ensures the existence of A-superficial elements for $\mathfrak{n}$.

Lemma 7.11. *Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay of dimension d for some ideal $I \subseteq R$ with maximal ideal $\mathfrak{n}$. Let $G := \mathrm{gr}_{\mathfrak{n}}(A)$ be the associated graded ring of A, and let G^+ be the maximal homogeneous ideal of G, i.e. $G^+ = \oplus_{j \geq 1} \mathfrak{n}^j/\mathfrak{n}^{j+1}$. Let $a_1, \ldots, a_r$ be an A-superficial sequence for $\mathfrak{n}$. Then $a_1^*, \ldots, a_r^*$ is a G-regular sequence if and only if $\mathrm{depth}_{G^+}(G) \geq r$.*

In particular, since $\dim(G) = \dim(A)$, if G is Cohen–Macaulay, then $a_1^*, \ldots, a_r^*$ is a G-regular sequence if and only if $d \geq r$.

Proof. See [RV10, Lemma 1.3]. □

Proposition 7.12. *Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay of dimension d for some ideal $I \subseteq R$ with maximal ideal $\mathfrak{n}$. Then any reduction of $\mathfrak{n}$ can be generated by A-superficial elements for $\mathfrak{n}$ and conversely the ideal generated by some maximal A-superficial sequence for $\mathfrak{n}$ is a minimal reduction of $\mathfrak{n}$.*

Proof. See [HS06, Theorem 8.6.3]. □

Example 7.13. Let $A = \mathbb{Q}[[t^6, t^7, t^{11}, t^{15}]]$. Then A has type 2. It can be checked via computer algebra systems that the two ideals $J = \langle t^6 \rangle$ and $J' = \langle t^6 + t^7 \rangle$ are two minimal reductions of $\mathfrak{m} = \langle t^6, t^7, t^{11}, t^{15} \rangle$. Moreover, it can be verified that A/J has Hilbert function $(1, 3, 2)$. Thus $\mathrm{type}(A) = 2 = \dim_{\mathbf{k}}(\mathrm{Soc}(A/J)) = \mathrm{HF}_{A/J}(2)$, i.e. $\mathrm{Soc}(A/J) = (\mathfrak{n}/J)^2$, which means A/J is level. On the other hand, A/J' has Hilbert function $(1, 3, 1, 1)$, and hence is not level.

The following lemma shows that if the associated graded ring $gr_{\mathfrak{n}}(A) = \oplus_{i \geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}$ of a Cohen–Macaulay local ring A is Cohen–Macaulay as well, then Artinian reductions of A have same socle type. This need not be true in general, as Example 7.13 shows.

Proposition 7.14. *Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. Assume that $G := gr_{\mathfrak{n}}(A) = \oplus_{i \geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}$ is Cohen–Macaulay. Then:*

(a) The socle degree of A/J is the same for any minimal reduction J of $\mathfrak{n}$.

(b) If A/J is level for some minimal reduction J of $\mathfrak{n}$, then A/J' is level for any minimal reduction J' of $\mathfrak{n}$.

In particular, if A is level, then A/J is level for any minimal reduction J of $\mathfrak{n}$.

Proof. Let $J \subseteq \mathfrak{n}$ be a minimal reduction of $\mathfrak{n}$. Then by Proposition 7.12 there exist $\underline{a} := a_1, \ldots, a_d \in \mathfrak{n}$ superficial sequence such that $J = \langle \underline{a} \rangle$. Since G is Cohen–Macaulay, by Lemma 7.11 $a_1^*, \ldots, a_d^*$ is G-regular. Let

$$\mathrm{HS}_A(t) = \frac{1 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

be the Hilbert series of A, with $h_s \neq 0$. Since $\mathrm{HS}_{A/xA}(t) = \mathrm{HS}_A(t)(1-t)$ for any $x^* \in G^{\mathrm{reg}}$, we obtain

$$\mathrm{HS}_{A/J}(t) = 1 + h_1 t + \cdots + h_s t^s.$$

Thus the socle degree of A/J equals s and $\dim_{\mathbf{k}}((\mathfrak{n}^s + J)/J) = h_s = \tau(A)$ do not depend on the minimal reduction J of A since the type of A is an invariant. Since $(\mathfrak{n}^s + J)/J \subseteq \mathrm{Soc}(A/J)$ and $\dim_{\mathbf{k}}((\mathfrak{n}^s + J)/J) = \tau(A) = \dim_{\mathbf{k}}(\mathrm{Soc}(A/J))$, we have actually an equality $\mathrm{Soc}(A/J) = (\mathfrak{n}^s + J)/J$ for any minimal reduction J. Hence the claim. □

Remark 7.15. In Example 7.13 not only J and J' are both Artinian reductions, but they are also superficial sequences (see Definition 7.8), so also regular elements. This explains the difficulty in defining level local algebras through quotients of regular elements, even if such elements are superficial. For this reason, we use general reductions.

Theorem 7.16. *Let $(A, \mathfrak{n})$ be a Noetherian local ring with infinite residue field, and let I be an ideal of analytic spread at most l. Then there exists a non-empty Zariski-open subset $U(I)$ of $(I/\mathfrak{n}I)^l$ such that whenever $a_1, \ldots, a_l \in I$ with $(a_1 + \mathfrak{n}I, \ldots, a_l + \mathfrak{n}I) \in U(I)$, then $(a_1, \ldots, a_l)$ is a reduction of I.*

Furthermore, if there exists a reduction of I with reduction number n, then there exists a non-empty Zariski-open subset $U(I, n)$ of $(I/\mathfrak{n}I)^l$ such that whenever $a_1, \ldots, a_l \in I$ with $(a_1 + \mathfrak{n}I, \ldots, a_l + \mathfrak{n}I) \in U(I, n)$, then $(a_1, \ldots, a_l)$ is a reduction of I with reduction number at most n.

Proof. See [HS06, Theorem 8.6.6]. □

Definition 7.17. Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay of dimension d for some ideal $I \subseteq R$ with maximal ideal $\mathfrak{n}$. We say that an ideal $\langle \underline{a} \rangle$ generated by a sequence $\underline{a} := a_1, \ldots, a_d$ is a *general reduction of* $\mathfrak{n}$ if $\underline{a}$ belongs to the non-empty Zariski-open $U(\mathfrak{n}, r(\mathfrak{n}))$ of Theorem 7.16. If this is the case, we call $\underline{a} := a_1, \ldots, a_d \in \mathfrak{n}$ a *general sequence*.

Remark 7.18. Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay of dimension d for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. Let $\underline{a} := a_1, \ldots, a_d$ be a general sequence in $\mathfrak{n}$.

(a) $\langle \underline{a} \rangle$ is a minimal reduction of $\mathfrak{n}$ (see [Xie12, Corollary 2.5]).
(b) $a_1, \ldots, a_d$ form a superficial sequence for $\mathfrak{n}$ (see [Xie12, Corollary 2.5]).
(c) If $a_i = z_i + I$ for any $i = 1, \ldots, d$, then $z_i \in \mathfrak{m}$ and $z_1, \ldots, z_d$ are general in $\mathfrak{m}$.

Conversely, if $z_1, \ldots, z_d$ are general in $\mathfrak{m}$, then $z_1 + I, \ldots, z_d + I$ are general in $\mathfrak{n}$.
In particular, this means that $A/\langle \underline{a} \rangle$ is Artinian for any general sequence $\underline{a}$.

Notice that by Definition 7.17 and Remark 7.18.(a) there always exist a general reduction which is a minimal reduction of $\mathfrak{n}$. We call such a reduction *minimal general reduction*.

The following proposition guarantees that the socle degree and the Hilbert function are independent of the chosen minimal general reduction.

Proposition 7.19. *Let $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. For any two minimal general reductions J and J' of $\mathfrak{n}$,*

$$\operatorname{socdeg}(A/J) = \operatorname{socdeg}(A/J')$$

and

$$\dim_{\mathbf{k}}((\mathfrak{n}^i + J)/J) = \dim_{\mathbf{k}}((\mathfrak{n}^i + J')/J')$$

for any $i \geq 0$.

Proof. See [MX16, Proposition 3.2]. □

More generally, Elias and Iarrobino proved the following:

Lemma 7.20. *Let $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. Let $\underline{a}$ be a general sequence for $\mathfrak{n}$. Then there are integers $c_A(i, s)$ such that*

$$c_A(i, s) = \ell\left(\operatorname{Soc}(A/\langle\underline{a}\rangle) \cap (\mathfrak{n}/\langle\underline{a}\rangle)^i\right).$$

In other words, the socle type of A/J is the same for any minimal general reduction J.

Proof. See [EI87, Lemma 1.1]. □

Motivated by this, we define level local $\mathbf{k}$-algebras as follows:

Definition 7.21. Let $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A = R/I$ be Cohen–Macaulay of positive dimension for some ideal $I \subseteq R$. We say that A is *level* if A/J is an Artinian level $\mathbf{k}$-algebra for some minimal general reduction J of $\mathfrak{n}$. Due to Proposition 7.19, this is equivalent to require A/J is an Artinian level $\mathbf{k}$-algebra for any minimal general reduction J of $\mathfrak{n}$.

In particular, if A is Gorenstein, then A is level.

Example 7.22. This example is [RV00, Example 3, p. 125]. Consider the semigroup ring

$$A = \mathbf{k}[[t^6, t^8, t^{10}, t^{13}]] \cong \mathbf{k}[[x, y, z, w]]/(y^2 - xz, yz - x^3, z^2 - x^2y, w^2 - x^3y).$$

Then A is Cohen–Macaulay of type 2 and $gr_{\mathfrak{n}}(A)$ is Cohen–Macaulay. Let $J = (t^6)$. Since A/J has Hilbert function $(1, 3, 2)$, it is level. Thus Proposition 7.14 implies A is level.

In general, if $\operatorname{gr}_{\mathfrak{n}}(A)$ is not Cohen–Macaulay, it is not known whether A/J being level for a minimal general reduction implies A/J level for any minimal reduction.

Notation 7.23. Let now $R = \mathbf{k}[x_1, \dots, x_n]$ or $R = \mathbf{k}[[x_1, \dots, x_n]]$, and let $A = R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n} = \mathfrak{m}/I$ be the maximal (homogeneous) ideal of A. Let $\underline{a} := a_1, \dots, a_d$ be a regular linear sequence if A is graded, or let $\langle\underline{a}\rangle$ be a minimal general reduction of $\mathfrak{n}$ if A is local. Moreover, let $\underline{z} := z_1, \dots, z_d$ be a linear regular sequence of R if R is the polynomial ring or a minimal general reduction of $\mathfrak{m}$ if R is the power series ring.

We will use the following notations:

$$\begin{aligned}
\underline{a}^{\underline{n}} &:= a_1^{n_1}, \dots, a_d^{n_d}, \text{ with } \underline{n} \in \mathbb{N}^d,\\
\underline{z}^{\underline{n}} &:= z_1^{n_1}, \dots, z_d^{n_d}, \text{ with } \underline{n} \in \mathbb{N}^d,\\
|\underline{n}| &:= n_1 + \cdots + n_d, \text{ with } \underline{n} \in \mathbb{N}^d\\
\mathbf{e}_i &:= (0, \dots, 1, \dots, 0) \in \mathbb{N}^d,\\
\mathbf{t} &:= (t, \dots, t) \in \mathbb{N}^d.
\end{aligned}$$

In order to construct the inverse system of a level $\mathbf{k}$-algebra A, we need $A/\langle \underline{a}^{\underline{n}} \rangle$ to be level for any $\underline{n} \in \mathbb{N}^d$ and some minimal reduction $\langle \underline{a} \rangle$. This follows immediately from the definition in the graded case. In the local case we need to assume further that $\underline{a}$ is a general sequence. Thus the aim for the remainder of this section is to analyze the structure of quotients of level $\mathbf{k}$-algebras by ideals of type $\langle \underline{a}^{\underline{n}} \rangle$.

Definition 7.24. Let A be a local ring with maximal ideal $\mathfrak{n}$. The *index of nilpotency of A with respect to a reduction J of* $\mathfrak{n}$ is

$$s_J(A) := \operatorname{socdeg}(A/J).$$

Proposition 7.25. *Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. Then there is an integer $s(A)$ such that*

$$s(A) = s_J(A)$$

for any minimal general reduction J of $\mathfrak{n}$*. Moreover,*

$$s_{J'}(A) \leq s(A)$$

for any minimal reduction J' of $\mathfrak{n}$.

Proof. See [MX16, Proposition 3.2] and [Fou06, 5.3.3]. $\square$

Remark 7.26. From Proposition 7.25 it follows that $\langle t^6 \rangle$ is not a minimal general reduction in Example 7.13. In fact, $s_{\langle t^6 \rangle}(A) < s_{\langle t^6, t^7 \rangle}(A) \leq s(A)$, where $s(A)$ is the index of nilpotency of A with respect to any general reduction.

Definition 7.27. Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$ and $A = R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. The *index of nilpotency of A* is the number

$$s(A) := s_J(A)$$

where J is a minimal general reduction of $\mathfrak{n}$.

The index of nilpotency $s(A)$ is well-defined by Proposition 7.25.

Definition 7.28. Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. The *core* of $\mathfrak{n}$ is

$$\operatorname{core}(\mathfrak{n}) := \bigcap_{J \text{ red. of } \mathfrak{n}} J = \bigcap_{J \text{ min. red. of } \mathfrak{n}} J.$$

We recall the following theorem, which gives an explicit formula to compute the core. The result is true more generally for equimultiple ideals, but we state it only for $\mathfrak{n}$.

Theorem 7.29. *Let $R = \mathbf{k}[[x_1, \ldots, x_n]]$ with $\operatorname{char}(\mathbf{k}) = 0$, and let $A := R/I$ be Cohen–Macaulay for some ideal $I \subseteq R$. Let $\mathfrak{n}$ be the maximal ideal of A. Let J be a minimal reduction of $\mathfrak{n}$ and $r := r(\mathfrak{n})$ the reduction number of $\mathfrak{n}$ (see Definition 7.6). Then*

$$\operatorname{core}(\mathfrak{n}) = J^{r+1} : \mathfrak{n}^r.$$

Equivalently, $\operatorname{core}(\mathfrak{n}) = J^{n+1} : \mathfrak{n}^n$ *for any $n \geq r$.*

Proof. See [HT05, Theorem 3.7]. $\square$

Theorem 7.29 is not true in positive characteristic, as [PU05, Example 4.9] shows. Thus hereafter we need to assume $\operatorname{char}(\mathbf{k}) = 0$ when considering the local case $R = \mathbf{k}[[x_1, \ldots, x_n]]$.

We thank Alessandro De Stefani for providing a proof of the following proposition (through private communication) in the one-dimensional case.

Proposition 7.30. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \ldots, x_n]$*, and* $A := R/I$ *is Cohen–Macaulay of dimension* $d \geq 1$ *for some ideal* $I \subseteq R$*. We denote by* $\mathfrak{n}$ *the homogeneous maximal ideal of* A*. Let* $\underline{a} = a_1, \ldots, a_d \in \mathfrak{n}$ *be a linear regular sequence in* A*. Let* s *be the Castelnuovo-Mumford regularity of* A*.*

(2) $R = \mathbf{k}[[x_1, \ldots, x_n]]$ *with* $\operatorname{char}(\mathbf{k}) = 0$*, and* $A := R/I$ *is Cohen–Macaulay of dimension* $d \geq 1$ *for some ideal* $I \subseteq R$*. We denote by* $\mathfrak{n}$ *the maximal ideal of* A*. Let* $\langle \underline{a} \rangle$ *be a minimal general reduction of* $\mathfrak{n}$*. Let* s *be the index of nilpotency of* A*.*

Then for any $\underline{n} \in \mathbb{N}_+^d := (\mathbb{N} \setminus \{0\})^d$*,*

$$\operatorname{socdeg}(A/\langle \underline{a}^{\underline{n}} \rangle) = s + |\underline{n}| - d. \tag{7.1}$$

Proof. (1) We prove (7.1) by induction on $|\underline{n}|$. The Castelnuovo-Mumford regularity of an Artinian graded ring coincides with its socle degree. Since the Castelnuovo-Mumford regularity of A, $\operatorname{reg}(A)$, and the regularity of the Artinian reduction $A/\langle \underline{a} \rangle$ are the same, we have

$$\operatorname{socdeg}(A/\langle \underline{a} \rangle) = \operatorname{reg}(A/\langle \underline{a} \rangle) = \operatorname{reg}(A) = s.$$

Thus the assertion is clear for $|\underline{n}| = d$, which is the base case as $\underline{n} \in \mathbb{N}_+^d$. Let $|\underline{n}| > d$. It is not restrictive to assume that $\underline{n} = (n_1, \ldots, n_d)$ with $n_1 \geq 2$. By induction $A/\langle \underline{a}^{\underline{n}-\mathbf{e}_1} \rangle$ has socle degree $s + |\underline{n}| - d - 1$. Let $f \in \mathfrak{n}^{s+|\underline{n}|-d+1}$ be a homogeneous polynomial. Since $\mathfrak{n}^{s+|\underline{n}|-d+1} \subseteq \langle \underline{a}^{\underline{n}-\mathbf{e}_1} \rangle$,

$$f = a_1^{n_1-1} f_1 + a_2^{n_2} f_2 + \cdots + a_d^{n_d} f_d$$

where $f_i \in A$ are homogeneous polynomials. Thus $\deg(f_1) \geq s + |\underline{n}| - d + 1 - (n_1 - 1)$, and so $f_1 \in \mathfrak{n}^{s+|\underline{n}|-d+1-(n_1-1)}$. By induction hypothesis, $s + |\underline{n}| - d + 1 - (n_1 - 1) = \operatorname{socdeg}(A/\langle a_1^1, a_2^{n_2}, \ldots, a_d^{n_d} \rangle)$, and hence $\mathfrak{n}^{s+|\underline{n}|-d+1-(n_1-1)} \subseteq \langle a_1^1, a_2^{n_2}, \ldots, a_d^{n_d} \rangle$. Hence $f_1 \in \langle a_1^1, a_2^{n_2}, \ldots, a_d^{n_d} \rangle$ and $f \in \langle \underline{a}^{\underline{n}} \rangle$. This yields $\mathfrak{n}^{s+|\underline{n}|-d+1} \subseteq \langle \underline{a}^{\underline{n}} \rangle$. On the other hand, assume $f \in \mathfrak{n}^{s+|\underline{n}|-d} \cap$. Then $a_1 f \in \mathfrak{n}^{s+|\underline{n}|-d} \setminus \langle \underline{a}^{\underline{n}-\mathbf{e}_1} \rangle$ is a homogeneous polynomial, then $a_1 f \in \mathfrak{n}^{s+|\underline{n}|-d} \setminus \langle \underline{a}^{\underline{n}} \rangle$. Hence $\mathfrak{n}^{s+|\underline{n}|-d} \nsubseteq \langle \underline{a}^{\underline{n}} \rangle$. This proves (7.1).

(2) Under the assumptions, $s = s(A)$ is the index of nilpotency of A. Let

$$V = \{J \mid J \text{ is a minimal reduction of } \mathfrak{n}\}.$$

By Proposition 7.25, $\operatorname{socdeg}(A/J) \leq s(A) = s$ for any minimal reduction J of $\mathfrak{n}$. Therefore

$$\mathfrak{n}^{s+1} \subseteq \bigcap_{J \in V} J = \operatorname{core}(\mathfrak{n}). \tag{7.2}$$

We again prove (7.1) by induction on $|\underline{n}|$. Let $J := \langle \underline{a} \rangle$. The assertion is clear if $|\underline{n}| = d$. Let $|\underline{n}| \geq d + 1$. Set $k := \max\{0, r - |\underline{n}| + d\}$, where $r := r(\mathfrak{n})$ is the reduction number of $\mathfrak{n}$. Since $k + |\underline{n}| - d \geq r$, by Theorem 7.29 we have

$$\operatorname{core}(\mathfrak{n}) = J^{k+|\underline{n}|-d+1} : \mathfrak{n}^{k+|\underline{n}|-d}. \tag{7.3}$$

Therefore, using (7.2) and (7.3), we get

$$J^k \mathfrak{n}^{s+|\underline{n}|-d+1} \subseteq \mathfrak{n}^{k+|\underline{n}|-d} \mathfrak{n}^{s+1} \subseteq \mathfrak{n}^{k+|\underline{n}|-d} \operatorname{core}(\mathfrak{n}) \tag{7.4}$$

Moreover, there is also an inclusion $J^{|\underline{n}|-d+1} \subseteq \langle \underline{a}^{\underline{n}} \rangle$. Indeed for any $\underline{a}^{\underline{n}'} \in J^{|\underline{n}|-d+1}$ with $\underline{n}' = (n'_1, \ldots, n'_d) \in \mathbb{N}^d$ and $|\underline{n}'| \geq |\underline{n}| - d + 1$, there exists an $i \in \{1, \ldots, d\}$ such that $n'_i \geq n_i$ and $a_1^{n'_1} \cdots a_d^{n'_d} \subseteq \langle \underline{n}^{\underline{n}} \rangle$. So if $k = 0$ (7.4) yields

$$\mathfrak{n}^{s+|\underline{n}|-d+1} \subseteq J^{|\underline{n}|-d+1} \subseteq \langle \underline{a}^{\underline{n}} \rangle.$$

Assume now that $k \geq 1$. Since $\bigoplus_{i \geq 0} J^i/J^{i+1}$ is Cohen–Macaulay (see Sally's machine [RV10, Lemma 1.4]) and a_1 is a superficial element, then $J^{i+1} :_A a_1 = J^i$ for any $i \geq 0$.Hence again by (7.4) we get

$$\mathfrak{n}^{s+|\underline{n}|-d+1} \subseteq J^{k+|\underline{n}|-d+1} : J^k \subseteq J^{k+|\underline{n}|-d+1} : \langle z_1^k \rangle = J^{|\underline{n}|-d+1} \subseteq \langle \underline{a}^{\underline{n}} \rangle.$$

Thus $\operatorname{socdeg}(A/\langle \underline{a}^{\underline{n}} \rangle) \leq s + |\underline{n}| - d$. On the other hand, if $x \in \mathfrak{n}^s \setminus \langle \underline{a} \rangle$, then $(a_1^{n_1-1} \cdots a_d^{n_d-1})x \in \mathfrak{n}^{s+|\underline{n}|-d} \setminus \langle \underline{a}^{\underline{n}} \rangle$. Thus $\mathfrak{n}^{s+|\underline{n}|-d} \not\subseteq \langle \underline{a}^{\underline{n}} \rangle$. Hence $\operatorname{socdeg}(A/\langle \underline{a}^{\underline{n}} \rangle) = s + |\underline{n}| - d$. □

Proposition 7.31. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \ldots, x_n]$, and $A := R/I$ is graded level of type τ for some ideal $I \subseteq R$. We denote by $\mathfrak{n}$ the homogeneous maximal ideal of A. Let $\underline{a} = a_1, \ldots, a_d \in \mathfrak{n}$ be a linear regular sequence in A.

(2) $R = \mathbf{k}[[x_1, \ldots, x_n]]$ with $\operatorname{char}(\mathbf{k}) = 0$, and $A := R/I$ is local level for some ideal $I \subseteq R$. We denote by $\mathfrak{n}$ the maximal ideal of A. Let $\langle \underline{a} \rangle$ be a minimal general reduction of $\mathfrak{n}$.

Then $A/\langle \underline{a}^{\underline{n}} \rangle$ is an Artinian level $\mathbf{k}$-algebra of type τ and socle degree $s(A) + |\underline{n}| - d$ for any $\underline{n} \in \mathbb{N}^d$, where $s(A) = \operatorname{socdeg}(A/\langle \underline{a} \rangle)$ is the index of nilpotency in case A is local.

In particular, if A satisfies (1), then A is level if and only if $A/\langle \underline{a} \rangle$ is level.

Proof. (1) Follows from Propositions 7.3 and 7.30.

(2) Since A is level of type τ, $A/\langle \underline{a} \rangle$ is an Artinian local level $\mathbf{k}$-algebra of type $\tau(A/\langle \underline{a} \rangle) =: \tau$. By Proposition 7.25, $s := \operatorname{socdeg}(A/\langle \underline{a} \rangle) = s(A)$. Let $t := \operatorname{socdeg}(A/\langle \underline{a}^{\underline{n}} \rangle)$. By Proposition 7.30(2), $t = s + |\underline{n}| - d$. Consider the homomorphism

$$\begin{aligned} \mu : \frac{\mathfrak{n}^s + \langle \underline{a} \rangle}{\langle \underline{a} \rangle} &\xrightarrow{\cdot z_1^{n_1-1} \cdots z_d^{n_d-1}} \frac{\mathfrak{n}^t + \langle \underline{a}^{\underline{n}} \rangle}{\langle \underline{a}^{\underline{n}} \rangle} \\ \overline{x} &\longmapsto \overline{z_1^{n_1-1} \cdots z_d^{n_d-1} x}. \end{aligned}$$

Clearly μ is well-defined. Since $\underline{a}$ is a regular sequence in A, it is easy to verify that μ is injective. Hence

$$\dim_{\mathbf{k}} \frac{\mathfrak{n}^t + \langle \underline{a}^{\underline{n}} \rangle}{\langle \underline{a}^{\underline{n}} \rangle} \geq \frac{\mathfrak{n}^s + \langle \underline{a} \rangle}{\langle \underline{a} \rangle} = \tau.$$

On the other hand, since

$$\frac{\mathfrak{n}^t + \langle \underline{a}^{\underline{n}} \rangle}{\langle \underline{a}^{\underline{n}} \rangle} \subseteq \operatorname{Soc}(A/\langle \underline{a}^{\underline{n}} \rangle)$$

and $A/\langle \underline{a}^{\underline{n}} \rangle$ is an Artinian local ring of type τ (because the type is independent from the Artinian reduction), we always have

$$\dim_{\mathbf{k}} \frac{\mathfrak{n}^t + \langle \underline{a}^{\underline{n}} \rangle}{\langle \underline{a}^{\underline{n}} \rangle} \leq \tau.$$

Therefore $\dim_{\mathbf{k}} \frac{\mathfrak{n}^t + \langle \underline{a}^{\underline{n}} \rangle}{\langle \underline{a}^{\underline{n}} \rangle} = \tau$ and hence $A/\langle \underline{a}^{\underline{n}} \rangle$ is level. □

Remark 7.32. There may be an alternative proof of Propositions 7.30 and 7.31 using Rees Theorem [BH93, Theorem 1.1.8]. This would make possible to avoid results on the core, and therefore make the assumption on characteristic 0 not necessary.

7.2 Inverse system of level $\mathbf{k}$-algebras

In this section we give the structure of the Inverse System of level $\mathbf{k}$-algebras. We always assume $\mathbf{k}$ is an infinite field.

Definition 7.33. Let $R = \mathbf{k}[x_1, \dots, x_n]$. Let $A = R/I$ be Cohen–Macaulay of dimension d for some ideal I of R. Then there exist elements $\underline{z} := z_1, \dots, z_d \in R_1$ such that $\underline{z} + I := z_1 + I, \dots, z_d + I$ form a regular sequence in A. We call $\underline{z}$ as a *regular linear sequence* for R/I.

Definition 7.34. Let $R = \mathbf{k}[[x_, \dots, x_n]]$. Let $A = R/I$ be Cohen–Macaulay of dimension d for some ideal I of R. Then there exists a sequence of general elements $\underline{z} := z_1, \dots, z_d \in \mathfrak{m}$. From the definition of general elements we get that $\underline{z} \in \mathfrak{m} \setminus \mathfrak{m}^2$, and hence is a part of a regular system of parameters of R. Thus, by Remark 7.18.(c), $\underline{z} + I := z_1 + I, \dots, z_d + I$ forms a sequence of general elements in R/I and hence $\langle \underline{z} + I \rangle$ is a minimal general reduction of $\mathfrak{n}$ by Remark 7.18.(a). We call $\underline{z}$ a *regular sequence of general linear forms* for R/I.

Remark 7.35. Whether $R = \mathbf{k}[x_1, \dots, x_n]$ or $R = \mathbf{k}[[x_1, \dots, x_n]]$, the sequence $\underline{z}$ of Definition 7.33 or Definition 7.34 can be extended to a minimal system of linear generators of $\mathfrak{m}$, say $z_1, \dots, z_d, \dots, z_n$. Let $Z_1, \dots, Z_n$ be the corresponding dual basis in $\mathcal{D}$, i.e. elements such that $z_i \circ Z_j = \delta_{ij}$. Then $\mathcal{D}_1 = \langle Z_1, \dots, Z_n \rangle_{\mathbf{k}}$.

Notation 7.36. In the following we denote $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$.

The following definition will be motivated by Proposition 7.46, as inverse systems of level algebras satisfy the properties listed.

Definition 7.37. Assume $R = \mathbf{k}[x_1, \dots, x_n]$, or $R = \mathbf{k}[[x_1, \dots, x_n]]$. Let d and τ be positive integers. An R-submodule W of $\mathcal{D}$ is called *L_d^τ-admissible* for some $\tau, d > 0$ (where L stays for "level") if it admits a system of generators $\{H_{\underline{n}}^j \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}_+^d\}$ satisfying the following conditions:

(1) For any $\underline{n} \in \mathbb{N}_+^d$ there exist an integer $s_{\underline{n}}$ such that
$$s_{\underline{n}} = \deg H_{\underline{n}}^1 = \deg H_{\underline{n}}^2 = \cdots = \deg H_{\underline{n}}^\tau$$
and $\operatorname{top}(H_{\underline{n}}^1), \dots, \operatorname{top}(H_{\underline{n}}^\tau)$ are linearly independent.

(2) There exists a regular sequence $z_1, \dots, z_d \in R$ of linear forms in case $R = \mathbf{k}[x_1, \dots, x_n]$ or of general linear forms in case $R = \mathbf{k}[[x_1, \dots x_n]]$ such that
$$z_i \circ H_{\underline{n}}^j = \begin{cases} H_{\underline{n}-\mathbf{e}_i}^j & \text{if } \underline{n} - \mathbf{e}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$
for any $\underline{n} \in \mathbb{N}_+^d$, $j \in \{1, \dots, \tau\}$ and $i \in \{1, \dots, d\}$.

(3) For any $\underline{n} \in \mathbb{N}_+^d$, the submodules of $\mathcal{D}$
$$W_{\underline{n}} = \langle H_{\underline{n}}^j \mid j \in \{1, \dots, \tau\} \rangle_R$$
and
$$V_{\underline{n}}^i = \langle Z_1^{k_1} \cdots Z_n^{k_n} \mid \underline{k} = (k_1, \dots, k_n) \in \mathbb{N}^n \text{ with } k_i \leq n_i - 2 \text{ and } |\underline{k}| \leq s_{\underline{n}} \rangle_R,$$
satisfy
$$W_{\underline{n}} \cap V_{\underline{n}}^i \subseteq W_{\underline{n}-\mathbf{e}_i} \tag{7.5}$$
for any $i \in \{1, \dots, d\}$ and $\underline{n} \in \mathbb{N}_+^d$ such that $\underline{n} - \mathbf{e}_i > 0$.

We say that W is *graded* if $R = \mathbf{k}[x_1, \dots, x_n]$ and $H^j_{\underline{n}}$ is homogeneous for any $j \in \{1, \dots, \tau\}$ and for any $\underline{n} \in \mathbb{N}^d$.

Example 7.38. In Chapter 8 we give some explicit examples of L^τ_d-admissible submodules of $\mathcal{D}$. In particular, in Proposition 8.2 we will show that the cone is always L^τ_d-admissible. Indeed, let $H^1, \dots, H^\tau \in \mathbf{k}[Z_{d+1}, \dots, Z_n]$ be (homogeneous) polynomials of same degree with $\operatorname{top}(H^1), \dots, \operatorname{top}(H^\tau)$ linearly independent. Let W be the R-submodule of $\mathcal{D}$ generated by the polynomials

$$H^j_{\underline{n}} = Z_1^{n_1-1} \cdots Z_d^{n_d-1} H^j$$

for any $j \in \{1, \dots, \tau\}$ and $\underline{n} \in \mathbb{N}^d$, $\underline{n} \geq \mathbf{1}$. Then W is L^τ_d-admissible.

Remark 7.39. From Definition 7.37 we obtain the following facts.

(a) Definition 7.37.(2) implies $s_{\underline{n}'} \geq s_{\underline{n}}$ for any $\underline{n}' \geq \underline{n}$.

(b) $1 \in W_{\underline{n}}$ for any $W_{\underline{n}} \neq 0$. More in general, this holds for any non-zero R-submodule of $\mathcal{D}$.

(c) If W is an R-submodule of $\mathcal{D}$ satisfying Definition 7.37. (2), then

$$W_{\underline{n}-\mathbf{e}_i} \subseteq \cap V^i_{\underline{n}} \text{ for } \underline{n} - \mathbf{e}_i > \mathbf{0}.$$

Indeed, $z_i^{n_i-1} \circ H^j_{\underline{n}} = 0$ for any $j \in \{1, \dots, \tau\}$. So $W_{\underline{n}-\mathbf{e}_i} \subseteq V^i_{\underline{n}}$ for any $i \in \{1, \dots, d\}$. In particular,

$$\begin{aligned} W_{\mathbf{1}} \subseteq V^i_{\mathbf{1}+\mathbf{e}_i} &= \bigcap_i \langle Z^{k_1} \cdots Z^{k_n} \mid \underline{k} \in \mathbb{N}^n \text{ with } k_i = 0 \text{ and } |\underline{k}| < s_{\mathbf{1}+\mathbf{e}_i} \rangle_R \\ &= \langle Z_{d+1}^{k_{d+1}} \cdots Z_n^{k_n} \mid k_j \in \mathbb{N} \text{ for } j \in \{d+1, \dots, n\} \rangle_R. \end{aligned}$$

(d) If W is is L^τ_d-admissible, then equality holds in Definition 7.37.(7.5). In fact, by (c), $W_{\underline{n}-\mathbf{e}_i} \subseteq V^i_{\underline{n}}$. And by Definition 7.37.(2) $W_{\underline{n}-\mathbf{e}_i} \subseteq W_{\underline{n}}$. Hence

$$W_{\underline{n}-\mathbf{e}_i} \subseteq W_{\underline{n}} \cap V^i_{\underline{n}} \text{ for } \underline{n} - \mathbf{e}_i > \mathbf{0}.$$

(e) Let $W_{\underline{n}}$ and $V^i_{\underline{n}}$ be as in Definition 7.37. Then

$$z_j \circ W_{\underline{n}} \subseteq W_{\underline{n}-\mathbf{e}_j}$$

and

$$z_j \circ V^i_{\underline{n}} \subseteq V^i_{\underline{n}-\mathbf{e}_j}$$

for any $j \in \{1, \dots, d\}$.

We now want to confront L^τ_d-admissibility with the definition of G_d-admissibility given in [ER17, Definition 3.6], in order to show that our conditions are not the "union" of the ones imposed by the authors in [ER17].

Definition 7.40. Assume $R = \mathbf{k}[x_1, \dots, x_n]$, or $R = \mathbf{k}[[x_1, \dots, x_n]]$. Let d be a positive integer. An R-submodule W of $\mathcal{D}$ is called *G_d-admissible* for some $d > 0$ (where G stays for "Gorenstein") if it admits a system of generators $\{H_{\underline{n}} \mid \underline{n} \in \mathbb{N}^d_+\}$ satisfying Definition 7.37.(2) and

$$\operatorname{Ann}_R(\langle H_{\underline{n}-\mathbf{e}_i} \rangle) \circ H_{\underline{n}} = \langle H_{\underline{n}-(n_i-1)\mathbf{e}_i} \rangle \text{ for any } i \in \{1, \dots, d\}, \text{ and } \underline{n} - \mathbf{e}_i > 0. \qquad (7.6)$$

In the following proposition we show that Definition 7.40 coincides with Definition 7.37 if $\tau = 1$.

Proposition 7.41. *Assume $R = \mathbf{k}[x_1, \dots, x_n]$, or $R = \mathbf{k}[[x_1, \dots, x_n]]$. Let d be a positive integer and W a R-submodule of $\mathcal{D}$.*

(a) If W is L_d^τ-admissible, then

$$\operatorname{Ann}_R(W_{\underline{n}-\mathbf{e}_i}) \circ W_{\underline{n}} = W_{\underline{n}-(n_i-1)\mathbf{e}_i} \tag{7.7}$$

for any $i \in \{1, \dots, d\}$ and $\underline{n} - \mathbf{e}_i > 0$. The converse is not true in general.

(b) If $\tau = 1$, then W is L_d^1-admissible if and only if W is G_d-admissible.

Proof. (a) Assume W is a (graded) L_d^τ-admissible submodule of $\mathcal{D}$ generated by $\{H_{\underline{n}}^j \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}_+^d\}$, and let $W_{\underline{n}} = \langle H_{\underline{n}}^j \mid j \in \{1, \dots, \tau\}\rangle$. Let $j \in \{1, \dots, \tau\}$, and $f \in \operatorname{Ann}_R(W_{\underline{n}-\mathbf{e}_i})$. Then

$$z_i \circ (f \circ H_{\underline{n}}^j) = f \circ (z_i \circ H_{\underline{n}}^j) = f \circ H_{\underline{n}-\mathbf{e}_i}^j = 0.$$

Hence

$$f \circ H_{\underline{n}}^j \in W_{\underline{n}} \cap V_{\underline{n}}^i \cap V_{\underline{n}-\mathbf{e}_i}^i \subseteq W_{\underline{n}-\mathbf{e}_i} \cap V_{\underline{n}-\mathbf{e}_i}^i.$$

Now (7.5) implies $f \circ H_{\underline{n}}^j \in W_{\underline{n}-2\mathbf{e}_i}$. Repeating the same argument, we get $f \circ H_{\underline{n}}^j \in W_{\underline{n}-(n_i-1)\mathbf{e}_i}$. The fact that the converse does not hold can be checked in Example 8.6.

(b) It is clear that condition (7.6) in Definition 7.40 coincides with condition (7.7). So if W is L_d^1-admissible, part (a) yields directly that W is G_d-admissible.

Conversely, assume $W = \langle H_{\underline{n}} \mid \underline{n} \in \mathbb{N}^d\rangle$ is G_d-admissible. Then W satisfies (7.6). Let $W_{\underline{n}} := \langle H_{\underline{n}}\rangle$. We claim that for $i \in \{1, \dots, d\}$,

$$W_{\underline{n}} \cap \ker_{\mathcal{D}}(Z_i) \subseteq W_{\underline{n}-(n_i-1)\mathbf{e}_i} \text{ for any } \underline{n} \in \mathbb{N}^d. \tag{7.8}$$

Indeed, consider $f \circ H_{\underline{n}} \in W_{\underline{n}} \cap \ker_{\mathcal{D}}(Z_i)$. Then $z_i \circ (f \circ H_{\underline{n}}) = 0$. Hence

$$f \circ H_{\underline{n}-\mathbf{e}_i} = f \circ (z_i \circ H_{\underline{n}}) = z_i \circ (f \circ H_{\underline{n}}) = 0.$$

This implies that $f \in \operatorname{Ann}_R(\langle H_{\underline{n}-\mathbf{e}_i}\rangle)$. Therefore by (7.6) $f \circ H_{\underline{n}} \in \langle H_{\underline{n}-(n_i-1)\mathbf{e}_i}\rangle = W_{\underline{n}-(n_i-1)\mathbf{e}_i}$. To prove that W is L_d^1-admissible it is enough to prove (7.5). We prove (7.5) by induction on n_i. Let $n_i = 2$. Let $f \in R$ such that $f \circ H_{\underline{n}} \in W_{\underline{n}} \cap V_{\underline{n}}^i$. Then $z_i \circ (f \circ H_{\underline{n}}) \in V_{\underline{n}-\mathbf{e}_i}^i = 0$ by Remark 7.39.(e) and Definition 7.37. So $f \circ H_{\underline{n}} \in W_{\underline{n}} \cap \ker_{\mathcal{D}}(Z_i) \subseteq W_{\underline{n}-\mathbf{e}_i}$ by (7.8).

Assume now that (7.5) is true for $\underline{n}$ with $n_i > 2$, and let $f \circ H_{\underline{n}+\mathbf{e}_i} \in W_{\underline{n}+\mathbf{e}_i} \cap V_{\underline{n}+\mathbf{e}_i}^i$. Then by Remark 7.39.(e), we have

$$z_i \circ (f \circ H_{\underline{n}+\mathbf{e}_i}) \in W_{\underline{n}} \cap V_{\underline{n}}^i$$

and hence $z_i \circ (f \circ H_{\underline{n}+\mathbf{e}_i}) \in W_{\underline{n}-\mathbf{e}_i}$ by induction. Thus there exists $g \in R$ such that

$$z_i \circ (f \circ H_{\underline{n}+\mathbf{e}_i}) = g \circ H_{\underline{n}-\mathbf{e}_i} = g \circ (z_i \circ H_{\underline{n}}) = z_i \circ (g \circ H_{\underline{n}}).$$

This gives $z_i \circ (f \circ H_{\underline{n}+\mathbf{e}_i} - g \circ H_{\underline{n}}) = 0$ and hence

$$f \circ H_{\underline{n}+\mathbf{e}_i} - g \circ H_{\underline{n}} = (f - g z_i) \circ H_{\underline{n}+\mathbf{e}_i} \in W_{\underline{n}+\mathbf{e}_i} \cap \ker_{\mathcal{D}}(Z_i) \subseteq W_{\underline{n}+\mathbf{e}_i - n_i\mathbf{e}_i} = W_{\underline{n}-(n_i-1)\mathbf{e}_i}$$

where the inclusion $W_{\underline{n}+\mathbf{e}_i} \cap \ker_{\mathcal{D}}(Z_i) \subseteq W_{\underline{n}+\mathbf{e}_i-n_i\mathbf{e}_i}$ follows from (7.8). As $W_{\underline{n}-(n_i-1)\mathbf{e}_i} \subseteq W_{\underline{n}}$, we get $f \circ H_{\underline{n}+\mathbf{e}_i} \in W_{\underline{n}}$. Hence the claim. $\square$

Lemma 7.42. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \dots, x_n]$*, and* $W = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}^d_+ \rangle$ *is a non-zero graded* L^τ_d*-admissible* R*-submodule of* $\mathcal{D}$ *with respect to a regular sequence of linear forms* $\underline{z}$*.*

(2) $R = \mathbf{k}[[x_1, \dots, x_n]]$*, and* $W = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}^d_+ \rangle$ *is a non-zero* L^τ_d*-admissible* R*-submodule of* $\mathcal{D}$ *with respect to a regular sequence of general linear forms* $\underline{z}$*.*

Set $W_{\underline{n}} := \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\} \rangle$*. Then* $W_{\mathbf{1}} = 0$ *if and only if* $W_{\underline{n}} = 0$ *for any* $\underline{n} \in \mathbb{N}^d$*.*

Proof. Suppose $W_{\mathbf{1}} = 0$. We use induction on $t = |\underline{n}|$. If $t = d$, then $\underline{n} = \mathbf{1}$ and by assumption $W_{\mathbf{1}} = 0$. Assume $t > d$ and $W_{\underline{n}} = 0$ for any $\underline{n}$ with $|\underline{n}| \leq t$. It suffices to show that $W_{\underline{n}+\mathbf{e}_i} = 0$ for any $i \in \{1, \dots, d\}$. From (7.5),

$$W_{\underline{n}+\mathbf{e}_i} \cap V^i_{\underline{n}+\mathbf{e}_i} \subseteq W_{\underline{n}} = 0.$$

If $W_{\underline{n}+\mathbf{e}_i} \neq 0$, then $1 \in W_{\underline{n}+\mathbf{e}_i} \cap V^i_{\underline{n}+\mathbf{e}_i}$ by Remark 7.39.(b), which is a contradiction. Hence $W_{\underline{n}+\mathbf{e}_i} = 0$ for any $i \in \{1, \dots, d\}$. The converse is trivial. □

Remark 7.43. Lemma 7.42 works more in general. If $N \subseteq M$ are two R-submodules $\mathcal{D}$, then $N \neq 0$ if and only if $1 \in N$ (see also Remark 7.39.(b)). But this is equivalent to $1 \in M$, which is true if and only if $M \neq 0$.

Lemma 7.44. *Assume* $R = \mathbf{k}[x_1, \dots, x_n]$*, or* $R = \mathbf{k}[[x_1, \dots, x_n]]$*. Let* $f : M \to N$ *be an epimorphism between two non-zero (graded)* R*-modules* M *and* N*, both minimally generated by* ν *elements, i.e* $\mu(M) = \mu(N) = \nu$*. Let* $m_1, \dots, m_\nu$ *be such that* $f(m_1), \dots, f(m_\nu)$ *generate* N*. Then* $m_1, \dots, m_\nu$ *generate* M*.*

Proof. Since $f : M \to N$ is surjective, as R is graded or local, $\overline{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is well-defined and also surjective. As $N/\mathfrak{m}N$ is generated by $f(m_1) + \mathfrak{m}N, \dots, f(m_\nu) + \mathfrak{m}N$ as a $\mathbf{k}$-vector space and $\dim_{\mathbf{k}} N/\mathfrak{m}N = \nu$, the elements $f(m_1) + \mathfrak{m}N, \dots, f(m_\nu) + \mathfrak{m}N$ are linearly independent. By linearity of the homomorphism $\overline{f}$, this yields $m_1 + \mathfrak{m}M, \dots, m_\nu + \mathfrak{m}M$ linearly independent. Hence $m_1 + \mathfrak{m}M, \dots, m_\nu + \mathfrak{m}M$ generate $M/\mathfrak{m}M$. By Nakayama's Lemma (which holds both in the graded and in the local case), $m_1, \dots, m_\nu$ generate M. □

Proposition 7.45. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \dots, x_n]$*, and* R/I *is Cohen–Macaulay of dimension* d *for some homogeneous ideal* $I \subseteq R$*. Let* $\underline{z} = z_1, \dots, z_d \in R$ *be such that* $\underline{z} + I$ *is a regular sequence of linear forms for* R/I*.*

(2) $R = \mathbf{k}[[x_1, \dots, x_n]]$ *with* $\mathrm{char}(\mathbf{k}) = 0$*, and* R/I *is Cohen–Macaulay of dimension* d *for some ideal* $I \subseteq R$*. Let* $\underline{z} = z_1, \dots, z_d \in R$ *be such that* $\underline{z} + I$ *is a regular sequence of general linear forms for* R/I*.*

For any $\underline{n} \in \mathbb{N}^d$*, set* $T_{\underline{n}} := (I + \langle \underline{z}^{\underline{n}} \rangle)^\perp$*.*

(a) If $d = 1$*, then there is an exact sequence of finitely generated* R*-submodules of* $\mathcal{D}$

$$0 \longrightarrow T_1 \longrightarrow T_n \xrightarrow{z_1 \circ} T_{n-1} \longrightarrow 0$$

for any $n \geq 2$*.*

(b) *If $d \geq 2$, there is an exact sequence of finitely generated R-submodules of $\mathcal{D}$*

$$0 \to T_{\mathbf{1}} \to T_{\underline{n}} \to \bigoplus_{k=1}^{d} T_{\underline{n}-\mathbf{e}_k} \to \bigoplus_{1 \leq i < j \leq d} T_{\underline{n}-\mathbf{e}_i-\mathbf{e}_j}$$

for any $\underline{n} \in \mathbb{N}^d$ such that $\underline{n} \geq \mathbf{2}$.

Proof. See [ER17, Proposition 3.2]. □

Proposition 7.46. *Assume one of the following:*

(1) *$R = \mathbf{k}[x_1, \ldots, x_n]$, and R/I is graded level of dimension d and type τ for some homogeneous ideal $I \subseteq R$. Let $\underline{z} = z_1, \ldots, z_d \in R$ be such that $\underline{z} + I$ is a regular sequence of linear forms for R/I.*

(2) *$R = \mathbf{k}[[x_1, \ldots, x_n]]$ with $\operatorname{char}(\mathbf{k}) = 0$, and R/I is local level of dimension d and type τ for some ideal $I \subseteq R$. Let $\underline{z} = z_1, \ldots, z_d \in R$ be such that $\underline{z} + I$ is a regular sequence of general linear forms for R/I.*

Then there exist a L_d^τ-admissible system of generators $\mathcal{H} := \{H_{\underline{n}}^j \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}_+^d\} \subseteq \mathcal{D}$ such that

$$(I + \langle \underline{z}^{\underline{n}} \rangle)^\perp = \langle H_{\underline{n}}^j \mid j \in \{1, \ldots, \tau\} \rangle$$

for any $\underline{n} \in \mathbb{N}^d$. The system $\mathcal{H}$ is graded in case (1).

Proof. Let $\underline{z} = z_1, \ldots, z_d \in R$ be a sequence of linear forms in case (1) or a sequence of general linear forms in case (2). Then by Remark 7.18, $\underline{z} + I := z_1 + I, \ldots, z_d + I$ is a minimal general reduction of $\mathfrak{n}$. Recall that by Remark 7.7, the type of R/I does not depend on the reduction. Then by Propositions 7.30 and 7.31, $R/(I + \langle \underline{z}^{\underline{n}} \rangle)$ is an Artinian level $\mathbf{k}$-algebra of type τ and socle degree $s_{\underline{n}} = s + |\underline{n}| - d$ for any $\underline{n} \in \mathbb{N}^d$. Set

$$W_{\underline{n}} := (I + \langle \underline{z}^{\underline{n}} \rangle)^\perp.$$

Since $R/(I + \langle \underline{z} \rangle)$ is an Artinian local level $\mathbf{k}$-algebra of type τ and socle degree $s = \operatorname{socdeg}(R_{\mathbf{1}})$, by Proposition 6.31 there exist polynomials $H_{\mathbf{1}}^1, H_{\mathbf{1}}^2, \ldots, H_{\mathbf{1}}^\tau$ of degree s such that the forms $\operatorname{top}(H_{\mathbf{1}}^1), \ldots, \operatorname{top}(H_{\mathbf{1}}^\tau)$ are linearly independent and $W_{\mathbf{1}} = \langle H_{\mathbf{1}}^j \mid j = 1, \ldots, \tau \rangle$. As $\langle \underline{z} \rangle + I$ is $\mathfrak{m}$-primary and $d \geq 1$, we have $W_{\mathbf{1}} \neq 0$.

For $\underline{n} = (n_1, \ldots, n_d)$, let

$$|\underline{n} \geq 2| = |\{n_i \mid n_i \geq 2\}|.$$

We put the lexicographic order on $\{1, \ldots, d\} \times \mathbb{N}$, i.e. $(i_1, j_1) < (i_2, j_2)$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2$. We use induction on the pair $(|\underline{n} \geq 2|, |\underline{n}| - d + |\underline{n} \geq 2|) \in \{1, \ldots, d\} \times \mathbb{N}$ to construct $\{H_{\underline{n}}^j\}_{j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}_+^d}$ such that

(1) $\deg(H_{\underline{n}}^j) = s + |\underline{n}| - d$ for any $j \in \{1, \ldots, \tau\}$ and $\operatorname{top}(H_{\underline{n}}^1), \ldots, \operatorname{top}(H_{\underline{n}}^\tau)$ are linearly independent,

(2) $\{H_{\underline{n}}^j \mid \underline{n} \in \mathbb{N}_+^d, j \in \{1, \ldots, \tau\}\}$ satisfy Definition 7.37.(2), and

(3) $(I + \langle \underline{z}^{\underline{n}} \rangle)^\perp = \langle H_{\underline{n}}^j \mid j \in \{1, \ldots, \tau\} \rangle$.

Assume that $|\underline{n} \geq 2| = 1$. Up to a permutation, we may assume $\underline{n} = (n, 1, \ldots, 1)$ with $n \geq 2$. Since $|\underline{n} - \mathbf{e}_1 \geq 2| \leq 1$, we have

$$(|\underline{n} - \mathbf{e}_1 \geq 2|, |\underline{n} - \mathbf{e}_1| - d + |\underline{n} - \mathbf{e}_1 \geq 2|) < (|\underline{n} \geq 2|, |\underline{n}| - d + |\underline{n} \geq 2|).$$

Hence by induction for any $\underline{n}' \leq \underline{n} - \mathbf{e}_1$ and $j \in \{1, \dots, \tau\}$ there exist $H^j_{\underline{n}'} \in W_{\underline{n}'}$ such that $\{H^j_{\underline{n}'} \mid \underline{n}' \in \mathbb{N}^d_+, 1 \leq j \leq \tau, \underline{n}' \leq \underline{n} - \mathbf{e}_1\}$ satisfy the required conditions. Let $J = I + (z_2, \dots, z_d)$. Now, by Proposition 7.45.(a), we get an exact sequence

$$0 \longrightarrow T_{\mathbf{1}} = (J + \langle z_1 \rangle)^\perp \longrightarrow T_{\underline{n}} = (J + \langle z_1^n \rangle)^\perp \xrightarrow{z_1 \circ} T_{\underline{n} - \mathbf{e}_1} = (J + \langle z_1^{n-1} \rangle)^\perp \longrightarrow 0. \quad (7.9)$$

Therefore for any $j \in \{1, \dots, \tau\}$, there exist polynomials $H^j_{\underline{n}} \in W_{\underline{n}}$ such that $z_1 \circ H^j_{\underline{n}} = H^j_{\underline{n} - \mathbf{e}_1}$. By Proposition 7.30, $\operatorname{socdeg}(R/(I + \langle \underline{z}^{\underline{n} - \mathbf{e}_1} \rangle)) = s + |\underline{n}| - d - 1$ and by Proposition 6.31, this implies $\deg H^j_{\underline{n} - \mathbf{e}_1} = s + |\underline{n}| - d - 1$. Hence by exactness of (7.9), $\deg H^j_{\underline{n}} = s + |\underline{n}| - d$ for any $j \in \{1, \dots, \tau\}$. Since $\operatorname{top}(H^1_{\underline{n} - \mathbf{e}_1}), \dots, \operatorname{top}(H^\tau_{\underline{n} - \mathbf{e}_1})$ are linearly independent, $\operatorname{top}(H^1_{\underline{n}}), \dots, \operatorname{top}(H^\tau_{\underline{n}})$ are linearly independent too. By Proposition 7.31, $(I + \langle \underline{z}^{\underline{n}} \rangle)^\perp$ and $(I + \langle \underline{z}^{\underline{n} - \mathbf{e}_1} \rangle)^\perp$ are both minimally generated by τ elements. Then Lemma 7.44 gives $(I + \langle \underline{z}^{\underline{n}} \rangle)^\perp = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\} \rangle$.

Now let $l := |\underline{n} \geq 2| \geq 2$. After a permutation, we can assume that $\underline{n} = (n_1, \dots, n_l, 1, \dots, 1)$ with $n_i \geq 2$ for $i = 1, \dots, l$. We set $\underline{z}' = z_1, \dots, z_l$ and $\underline{n}' = (n_1, \dots, n_l) \in \mathbb{N}^l_+$ and $J = I + \langle z_{l+1}, \dots, z_d \rangle$. By Proposition 7.45.(b) we get an exact sequence

$$0 \longrightarrow T_{\mathbf{1}_l} \longrightarrow T_{\underline{n}'} \longrightarrow \bigoplus_{k=1}^{l} T_{\underline{n}' - \mathbf{e}_k} \xrightarrow{\phi^*_{\underline{n}'}} \bigoplus_{1 \leq i < j \leq l} T_{\underline{n}' - \mathbf{e}_i - \mathbf{e}_j},$$

where $T_{\mathbf{1}} = W_{\mathbf{1}}$, $T_{\underline{n}'} = (J + \langle \underline{z}'^{\underline{n}'} \rangle)^\perp = W_{\underline{n}}$, $T_{\underline{n}' - \mathbf{e}_k} = (J + \langle \underline{z}'^{\underline{n}' - \mathbf{e}_k} \rangle)^\perp = W_{\underline{n} - \mathbf{e}_k}$ and $T_{\underline{n}' - \mathbf{e}_i - \mathbf{e}_j} = (J + \langle \underline{z}'^{\underline{n}' - \mathbf{e}_i - \mathbf{e}_j} \rangle)^\perp = W_{\underline{n} - \mathbf{e}_i - \mathbf{e}_j}$ and $\phi^*_{\underline{n}'}$ is dual to the map

$$\begin{aligned} \phi_{\underline{n}'} : \bigoplus_{1 \leq i < j \leq l} R/(I + \langle \underline{z}^{\underline{n}' - \mathbf{e}_i - \mathbf{e}_j} \rangle) &\to \bigoplus_{k=1}^{l} R/(I + \langle \underline{z}^{\underline{n}' - \mathbf{e}_k} \rangle) \\ (\overline{v_{i,j}})_{1 \leq i < j \leq l} &\mapsto \sum_{1 \leq i < j \leq l} (0, \dots, 0, z_j (\overline{v_{i,j}})_i, 0, \dots, 0, -z_i (\overline{v_{i,j}})_j, 0, \dots, 0). \end{aligned}$$

where $z_j(\overline{v_{i,j}})_i$ is in the i-th position and $-z_i(\overline{v_{i,j}})_j$ in the j-th position. By induction for any $k \in \{1, \dots, l\}$ there exist $\{H^j_{\underline{n} - \mathbf{e}_k}\}_{j \in \{1, \dots, \tau\}} \subseteq \mathcal{D}$ such that

(1) $\deg H^j_{\underline{n} - \mathbf{e}_k} = s + |\underline{n}| - d - 1$ for any $j \in \{1, \dots, \tau\}$ and the forms of higher degree of $H^1_{\underline{n} - \mathbf{e}_k}, \dots, H^\tau_{\underline{n} - \mathbf{e}_k}$ are linearly independent,

(2) $z_i \circ H^j_{\underline{n} - \mathbf{e}_k} = H^j_{\underline{n} - \mathbf{e}_k - \mathbf{e}_i}$ for any $i \in \{1, \dots, l\}$, $i \neq k$, and

(3) $(I + \langle \underline{z}^{\underline{n} - \mathbf{e}_k} \rangle)^\perp = \langle H^j_{\underline{n} - \mathbf{e}_k} \mid j \in \{1, \dots, \tau\} \rangle$.

Therefore for any $j \in \{1, \dots, \tau\}$,

$$(H^j_{\underline{n} - \mathbf{e}_1}, \dots, H^j_{\underline{n} - \mathbf{e}_l}) \in \ker(\phi^*_{\underline{n}'}).$$

Hence by the above exact sequence we conclude that there exist $H^j_{\underline{n}} \in W_{\underline{n}}$ such that $z_k \circ H^j_{\underline{n}} = H^j_{\underline{n} - \mathbf{e}_k}$ for any $k \in \{1, \dots, l\}$. By Proposition 7.30, $\operatorname{socdeg} R_{\underline{n} - \mathbf{e}_k} = s + |\underline{n}| - d - 1$ and thus $\deg H^j_{\underline{n} - \mathbf{e}_k} = s + |\underline{n}| - d - 1$ by Proposition 6.31. Therefore we conclude that $\deg H^j_{\underline{n}} = s + |\underline{n}| - d$. As $\operatorname{top}(H^1_{\underline{n} - \mathbf{e}_1}), \dots, \operatorname{top}(H^\tau_{\underline{n} - \mathbf{e}_1})$ are linearly independent, $\operatorname{top}(H^1_{\underline{n}}), \dots, \operatorname{top}(H^\tau_{\underline{n}})$ are linearly independent too. By Lemma 7.44, $\{H^1_{\underline{n}}, \dots, H^\tau_{\underline{n}}\}$ are thus generators of $(I + \langle \underline{z}^{\underline{n}} \rangle)^\perp$. Thus we have constructed $\{H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}^d\}$ satisfying conditions (1),(2) and (3). For them to be L^τ_d-admissible, we still have to verify (7.5).

Fix $i \in \{1, \dots, d\}$ and let $F \in (I + \langle \underline{z}^{\underline{n}} \rangle)^\perp \cap V^i_{\underline{n}}$, where we recall $V^i_{\underline{n}} = \langle Z_1^{k_1} \cdots Z_n^{k_n} \mid \underline{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ with $k_i \leq n_i - 2$ and $|\underline{k}| \leq s + |\underline{n}| - d \rangle$. Since $F \in (I + \langle \underline{z}^{\underline{n}} \rangle)^\perp$ we have

$(I + \langle \underline{z}^{\underline{n}} \rangle) \circ F$. This means that F is annihilated by all the elements in $(I + \langle \underline{z}^{\underline{n}} \rangle)$, so in particular by elements of I. Thus $I \circ F = 0$. As $F \in V^i_{\underline{n}}$, we have

$$z_i^{n_i - 1} \circ F = 0.$$

Hence

$$(I + \langle \underline{z}^{\underline{n} - \mathbf{e}_i} \rangle) \circ F = I \circ F + \langle \underline{z}^{\underline{n} - \mathbf{e}_i} \rangle \circ F = \langle \underline{z}^{\underline{n} - \mathbf{e}_i} \rangle \circ F = 0$$

Thus $F \in (I + \langle \underline{z}^{\underline{n} - \mathbf{e}_i} \rangle)^{\perp}$. This proves (7.5). Therefore $\langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d_+ \rangle$ is a (graded) L^τ_d-admissible submodule of $\mathcal{D}$. Since $\langle H^j_{\mathbf{1}} \mid j \in \{1, \ldots, \tau\} \rangle \neq 0$, we have that $\langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d_+ \rangle \neq 0$ by Lemma 7.42. □

The following proposition justifies better Definition 7.37.(3), and in particular it clarifies the meaning of the $V^i_{\underline{n}}$.

Proposition 7.47. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \ldots, x_n]$, *and* $W = \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle$ *is a non-zero graded* L^τ_d*-admissible* R*-submodule of* $\mathcal{D}$ *with respect to a regular sequence of linear forms* $\underline{z}$.

(2) $R = \mathbf{k}[[x_1, \ldots, x_n]]$, *and* $W = \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle$ *is a non-zero* L^τ_d*-admissible* R*-submodule of* $\mathcal{D}$ *with respect to a regular sequence of general linear forms* $\underline{z}$.

Set $W_{\underline{n}} := \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\} \rangle$ *for any* $\underline{n} \in \mathbb{N}^d$. *Then*

$$\operatorname{Ann}_R(W_{\underline{n}}) = \langle \underline{z}^{\underline{n}} \rangle + \bigcap_{\underline{n} \in \mathbb{N}^d} \operatorname{Ann}_R(W_{\underline{n}}).$$

Proof. Let us denote

$$I_{\underline{n}} := \operatorname{Ann}_R(W_{\underline{n}}).$$

Then $I_{\underline{n}}$ is an ideal of R. We denote

$$I = \bigcap_{\underline{n} \in \mathbb{N}^d} \operatorname{Ann}_R(W_{\underline{n}}) = \bigcap_{\underline{n} \in \mathbb{N}^d} I_{\underline{n}}.$$

Then I is also an ideal of R, as it is intersection of ideals.

($\subseteq$) First observe that

$$I_{\underline{n}} \subseteq I_{\underline{n}+\mathbf{1}} + \langle \underline{z}^{\underline{n}} \rangle, \tag{7.10}$$

for any $\underline{n} \in \mathbb{N}^d$. In fact, for any $i \in \{1, \ldots, d\}$ and $\underline{n} \in \mathbb{N}^d$,

$$\operatorname{Ann}_R(V^i_{\underline{n}+\mathbf{e}_i}) = \langle z_i^{n_i} \rangle + \langle z_1, \ldots, \hat{z}_i \ldots, z_n \rangle^{s_{\underline{n}+\mathbf{e}_i}+1},$$

where we recall $s_{\underline{n}} = \deg(H^j_{\underline{n}})$ for any $j \in \{1, \ldots, \tau\}$. As $W_{\underline{n}+\mathbf{e}_i} \cap V^i_{\underline{n}+\mathbf{e}_i} \subseteq W_{\underline{n}}$, Proposition 6.24 gives

$$\begin{aligned} I_{\underline{n}} &= \operatorname{Ann}_R(W_{\underline{n}}) \subseteq \operatorname{Ann}_R(W_{\underline{n}+\mathbf{e}_i} \cap V^i_{\underline{n}+\mathbf{e}_i}) \\ &= \operatorname{Ann}_R(W_{\underline{n}+\mathbf{e}_i}) + \operatorname{Ann}_R(V^i_{\underline{n}+\mathbf{e}_i}) \\ &= I_{\underline{n}+\mathbf{e}_i} + \langle z_i^{n_i} \rangle, \end{aligned}$$

where the last equality follows since $\langle z_1, \ldots, \hat{z}_i, \ldots, z_n\rangle^{s_{\underline{n}+\mathbf{e}_i}+1} \subseteq \mathrm{Ann}_R(W_{\underline{n}+\mathbf{e}_i})$. Therefore

$$\begin{aligned} I_{\underline{n}} &\subseteq I_{\underline{n}+\mathbf{e}_1} + \langle z_1^{n_1}\rangle \subseteq I_{\underline{n}+\mathbf{e}_1+\mathbf{e}_2} + \langle z_1^{n_1}, z_2^{n_2}\rangle \subseteq \cdots \subseteq I_{\underline{n}+\mathbf{e}_1+\cdots+\mathbf{e}_d} + \langle z_1^{n_1}, \ldots, z_d^{n_d}\rangle \\ &= I_{\underline{n}+\mathbf{1}} + \langle \underline{z}^{\underline{n}}\rangle \end{aligned}$$

for any $\underline{n} \in \mathbb{N}^d$.
Now fix $\underline{n} \in \mathbb{N}^d$ and consider $f \in I_{\underline{n}}$. By (7.10) there exist $f_{\underline{n}+\mathbf{1}} \in I_{\underline{n}+\mathbf{1}}$ and $g_0 \in \langle \underline{z}^{\underline{n}}\rangle$ such that

$$f = f_{\underline{n}+\mathbf{1}} + g_0.$$

Since $f_{\underline{n}+\mathbf{1}} \in I_{\underline{n}+\mathbf{1}}$, again by (7.10) there are $f_{\underline{n}+\mathbf{2}} \in I_{\underline{n}+\mathbf{2}}$ and $g_1 \in \langle \underline{z}^{\underline{n}+\mathbf{1}}\rangle$ such that

$$f_{\underline{n}+\mathbf{1}} = f_{\underline{n}+\mathbf{2}} + g_1.$$

Thus $f = f_{\underline{n}+\mathbf{2}} + g_0 + g_1$. By recurrence there are sequences $\{f_{\underline{n}+\mathbf{t}}\}_{\mathbf{t},t\geq 0}$ and $\{g_t\}_{t\geq 0}$, where $\mathbf{t} = (t, \ldots, t) \in \mathbb{N}^d$, $f_{\underline{n}+\mathbf{t}} \in I_{\underline{n}+\mathbf{t}}$, $g_t \in \langle \underline{z}^{\underline{n}+\mathbf{t}}\rangle$, such that

$$f_{\underline{n}+(\mathbf{t}-\mathbf{1})} = f_{\underline{n}+\mathbf{t}} + g_{t-1}.$$

So, for any $t \geq 0$, it holds

$$f = f_{\underline{n}+\mathbf{t}} + \sum_{i=0}^{t-1} g_i. \tag{7.11}$$

Let $g' = \sum_{i\geq 0} g_i \in \mathbf{k}[[x_1, \ldots, x_n]]$, and let $f' = \lim_{t\to\infty} f_{\underline{n}+\mathbf{t}} \in \mathbf{k}[[x_1, \ldots, x_n]]$. Taking limit as $t \to \infty$ in (7.11), we get

$$f = \lim_{t\to\infty} f = \lim_{t\to\infty} \left(f_{\underline{n}+\mathbf{t}} + \sum_{i=0}^{t-1} g_i \right) = f' + g'.$$

Since $g_t \in \langle \underline{z}^{\underline{n}+\mathbf{t}}\rangle$ for any $t \geq 0$ (in particular $g_0 \in \langle \underline{z}^{\underline{n}}\rangle$), we obtain $g' \in \langle \underline{z}^{\underline{n}}\rangle$. Now for any $\underline{k} \in \mathbb{N}^d$, there exists a positive integer N such that $\mathfrak{m}^N \subseteq I_{\underline{k}}$. Since $f_{\underline{k}+\mathbf{t}} - f' \in \mathfrak{m}^N \subseteq I_{\underline{k}}$ for any $t \gg 0$ and $I_{\underline{k}+\mathbf{t}} \subseteq I_{\underline{k}}$ for any $t \geq 0$, we get that $f' \in I_{\underline{k}}$ for any $\underline{k} \in \mathbb{N}^d$. Thus $f' \in I = \bigcap_{\underline{k}\in\mathbb{N}^d} I_{\underline{k}}$, and hence $f \in I + \langle \underline{z}^{\underline{n}}\rangle$. This gives that $I_{\underline{n}} \subseteq I + \langle \underline{z}^{\underline{n}}\rangle$.
If $R = \mathbf{k}[x_1, \ldots, x_n]$, then $f' \in I \subseteq R$. Since $f \in R$ we get that $g' = \sum_{i\geq 0} g_i \in R$.

($\supseteq$) By Definition 7.37.(2), $z_i^{n_i} \circ H_{\underline{n}}^j = 0$ for any $j \in \{1, \ldots, \tau\}$ and $i \in \{1, \ldots, d\}$. Hence $\langle \underline{z}^{\underline{n}}\rangle \subseteq I_{\underline{n}}$. Clearly, $I \subseteq I_{\underline{n}}$. Hence the claim. □

Lemma 7.48. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \ldots, x_n]$ with unique homogeneous maximal ideal $\mathfrak{m}$, and $\underline{z} = z_1, \ldots, z_l$ is a regular linear sequence of R.

(2) $R = \mathbf{k}[[x_1, \ldots, x_n]]$ with unique maximal ideal $\mathfrak{m}$, and $\underline{z} = z_1, \ldots, z_l$ is a regular general linear sequence of R.

Let $I \subseteq R$ be an ideal, homogeneous in case (1). *Then $I = \bigcap_{\underline{n}\in\mathbb{N}^d}(I + \langle \underline{z}^{\underline{n}}\rangle)$.*

Proof. Let $\overline{f} \in R/I$, and assume $\overline{f} \in J_t := \overline{\langle z_1^t, \ldots, z_l^t\rangle} \subseteq \mathfrak{m}^t/I$ for any $t \geq 1$. Then by Krull intersection theorem, $\cap_{t\geq 1} J_t \subseteq \cap_{t\geq 1}\mathfrak{m}^t/I = 0$. Thus $\overline{f} = 0$, i.e. $f \in I$. Hence $\bigcap_{\underline{n}\in\mathbb{N}^d}(I + \langle \underline{z}^{\underline{n}}\rangle) \subseteq I$. The other inclusion is trivial. □

Proposition 7.49. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \ldots, x_n]$, *and* $W = \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle$ *is a non-zero graded* L^τ_d*-admissible* R*-submodule of* $\mathcal{D}$ *with respect to a regular sequence of linear forms* $\underline{z}$.

(2) $R = \mathbf{k}[[x_1, \ldots, x_n]]$, *and* $W = \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle$ *is a non-zero* L^τ_d*-admissible* R*-submodule of* $\mathcal{D}$ *with respect to a regular sequence of general linear forms* $\underline{z}$.

Set $W_{\underline{n}} := \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\} \rangle$ *for any* $\underline{n} \in \mathbb{N}^d$. *Then*

$$I := \bigcap_{\underline{n} \in \mathbb{N}^d} \operatorname{Ann}_R(W_{\underline{n}}) = \operatorname{Ann}_R(W)$$

is an ideal of R *and* $\underline{z}$ *is a regular sequence modulo* I.

Proof. First of all, let us observe that I is an ideal of R. The annihilators $\operatorname{Ann}_R(W_{\underline{n}})$ are ideals for any $\underline{n}$. Thus I is an ideal since it is intersection of ideals.

Let $I_{\underline{n}} := \operatorname{Ann}_R(W_{\underline{n}})$. By Proposition 7.47 we have

$$I_{\underline{n}} = I + \langle \underline{z}^{\underline{n}} \rangle.$$

First we prove that z_1 is a nonzero-divisor of $A = R/I$. By (2) the action of z_1 defines an epimorphism of R-modules

$$W_{\underline{n}} \xrightarrow{z_1 \circ} W_{\underline{n}-\mathbf{e}_1} \longrightarrow 0 \tag{7.12}$$

for any $\underline{n} - \mathbf{e}_1 > 0$. Since $I_{\underline{n}} = I + \langle \underline{z}^{\underline{n}} \rangle$ by Proposition 7.47, applying $\operatorname{Hom}_R(-, \mathbf{k})$ to (7.12) yields by Proposition 6.22.(c) an exact sequence of R-modules

$$0 \longrightarrow \frac{R}{I + \langle \underline{z}^{\underline{n}-\mathbf{e}_1} \rangle} \xrightarrow{\cdot z_1} \frac{R}{I + \langle \underline{z}^{\underline{n}} \rangle}.$$

Let $f \in R$ be such that $z_1 f \in I$. Since $z_1 f \in I + \langle \underline{z}^{\underline{n}} \rangle$, from the exactness of the sequence we deduce that $f \in I + \langle \underline{z}^{\underline{n}-\mathbf{e}_1} \rangle = I_{\underline{n}-\mathbf{e}_1}$ for any $\underline{n} - \mathbf{e}_1 > 0$ and hence we conclude that $f \in I$.

Now assume that $z_1, \ldots, z_l$, $l < d$, is a regular sequence of R/I. Given $\underline{n}' = (n_{l+1}, \ldots, n_d) \in \mathbb{N}^{d-l}_+$, we take $\underline{n} = (1, \ldots, 1, n_{l+1}, \ldots, n_d) \in \mathbb{N}^d$. By Definition 7.37.(2) the derivation by z_{l+1} defines an epimorphism of R-modules for any $n_{l+1} \geq 2$

$$W_{\underline{n}} \xrightarrow{z_{l+1} \circ} W_{\underline{n}-\mathbf{e}_{l+1}} \longrightarrow 0.$$

Since $I_{\underline{n}} = I + \langle \underline{z}^{\underline{n}} \rangle$, this sequence induces an exact sequence of R-modules

$$0 \longrightarrow \frac{R}{I + \langle z_1, \ldots, z_l \rangle + \langle z_{l+1}^{n_{l+1}-1}, \ldots, z_d^{n_d} \rangle} \xrightarrow{\cdot z_{l+1}} \frac{R}{I + \langle z_1, \ldots, z_l \rangle + \langle z_{l+1}^{n_{l+1}}, \ldots, z_d^{n_d} \rangle}.$$

Let $f \in R$ be such that $z_{l+1} f \in I + \langle z_1, \ldots, z_l \rangle$. Since $z_{l+1} f \in I + \langle z_1, \ldots, z_l \rangle + \langle z_{l+1}^{n_{l+1}}, \ldots, z_d^{n_d} \rangle$ for any $n_{l+1}, \ldots, n_d$, by exactness we deduce that $f \in I + \langle z_1, \ldots, z_l \rangle + \langle z_{l+1}^{n_{l+1}-1}, \ldots, z_d^{n_d} \rangle$ for any $n_{l+1} \geq 2$. Lemma 7.48 applied to $I + \langle z_1, \ldots, z_l \rangle$ yields $f \in I + \langle z_1, \ldots, z_l \rangle$. $\square$

Now we prove the main theorem.

Theorem 7.50. *Assume one of the following:*

(1) $R = \mathbf{k}[x_1, \ldots, x_n]$ *with unique homogeneous maximal ideal* $\mathfrak{m}$, *or*

(2) $R = \mathbf{k}[[x_1, \ldots, x_n]]$ *with unique maximal ideal* $\mathfrak{m}$ *and* $\mathrm{char}(\mathbf{k}) = 0$.

Let $d \leq n$ *be a positive integer. There is a one-to-one correspondence between the following sets:*

$$\left\{ \begin{matrix} I \subseteq R \textit{ such that } R/I \textit{ is a} \\ \textit{level } \mathbf{k}\textit{-algebra with} \\ \dim(R/I) = d \textit{ and } \tau(R/I) = \tau \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} W = \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle \\ L^\tau_d\textit{-admissible submodule of } \mathcal{D} \\ \textit{for some seq. of (gen.) lin. forms } \underline{z} \in R \end{matrix} \right\}$$

$$I \longmapsto I^\perp$$

$$\textstyle\bigcap_{\underline{n} \in \mathbb{N}^d} \mathrm{Ann}_R(\langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\} \rangle) \longleftarrow W$$

In particular, to homogeneous ideals correspond graded L^τ_d*-admissible submodules of* $\mathcal{D}$ *and vice versa.*

Proof. We prove the result at the same time for graded and local case. We give different details only if necessary.

Let

$$\mathcal{C} := \{I \subseteq R \mid R/I \text{ level } \mathbf{k}\text{-algebra with } \dim(R/I) = d, \tau(R/I) = \tau\};$$
$$\mathcal{C}' := \{W \subseteq \mathcal{D} \mid 0 \neq W L^\tau_d\text{-admissible submodule with } W = \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle\}.$$

We define two maps

$$\begin{aligned} \theta : \mathcal{C} &\to \mathcal{C}' & \theta' : C' &\to \mathcal{C} \\ I &\mapsto I^\perp & W &\mapsto \bigcap_{\underline{n} \in \mathbb{N}^d} \mathrm{Ann}_R \left(\langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\} \rangle \right). \end{aligned}$$

First, let us prove that these two maps are well-defined.

Let $I \in \mathcal{C}$. Then there exists a sequence of (general) linear forms $\underline{z} = z_1, \ldots, z_d \in R$. Then by Proposition 7.46 there exists an L^τ_d-admissible system of generators $\{H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d\} \subseteq \mathcal{D}$ such that $(I + \langle \underline{z}^{\underline{n}} \rangle)^\perp = \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\} \rangle$. Hence the map θ is well defined.

Conversely, let $W \in \mathcal{C}'$ be generated by $\{H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d_+\}$ with respect to a regular sequence of (general) linear forms $\underline{z} := z_1, \ldots, z_d$ in R. Set

$$\begin{aligned} W_{\underline{n}} &:= \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\} \rangle; \\ I_{\underline{n}} &:= \mathrm{Ann}_R(W_{\underline{n}}); \\ I &:= \textstyle\bigcap_{\underline{n} \in \mathbb{N}^d} I_{\underline{n}}. \end{aligned}$$

We want to show that R/I is a level $\mathbf{k}$-algebra of dimension d and type τ.

By Proposition 7.49, I is an ideal of R and $\underline{z}$ is a regular sequence modulo I. Hence $\dim(R/I) \geq d$. On the other hand, since $W \neq 0$, by Lemma 7.42, $W_{\mathbf{1}} \neq 0$. Therefore by Proposition 7.47, $I + \langle \underline{z} \rangle = I_{\mathbf{1}} = \mathrm{Ann}_R(W_{\mathbf{1}})$. As $W_{\mathbf{1}}$ is finitely generated, $R/(I + \langle \underline{z} \rangle)$ is Artinian by Macaulay's Inverse System. This yields $\dim(R/I) \leq d$. Thus $\dim(R/I) = d$. In particular, since $\underline{z}$ is a regular sequence of length d, R/I is Cohen–Macaulay.

Let us now prove that R/I is a level $\mathbf{k}$-algebra. By Remark 7.18, $\underline{z} + I/I \subseteq \mathfrak{m}/I$ is a minimal general reduction of $\mathfrak{m}/I$. Proposition 7.47 gives $I + \langle \underline{z} \rangle = I_{\mathbf{1}}$. Since $I_{\mathbf{1}} = \mathrm{Ann}_R(W_{\mathbf{1}})$ and $W_{\mathbf{1}}$ is generated by polynomials $H^1_{\mathbf{1}}, \ldots, H^\tau_{\mathbf{1}}$ of same degree with $\mathrm{top}(H^1_{\mathbf{1}}), \ldots, \mathrm{top}(H^\tau_{\mathbf{1}})$ linearly independent, by Proposition 6.31 $R/(I + \langle \underline{z} \rangle)$ is an Artinian level $\mathbf{k}$-algebra of type τ. Since

R/I is Cohen–Macaulay, we conclude that R/I is a d-dimensional level $\mathbf{k}$-algebra according to Definition 7.21 (resp. Definition 7.1 if graded). Hence θ' is well-defined.

Finally, we prove that θ and θ' are inverses of each other. Let $I \in \mathcal{C}$ and $\underline{z} = z_1, \dots, z_d \in \mathfrak{m}$ a regular sequence of (general) linear forms. Then

$$\theta'\theta(I) = \theta'(\langle \bigcup_{\underline{n}\in\mathbb{N}^d} (I + \langle \underline{z} \rangle^{\underline{n}})^\perp \rangle) = \bigcap_{\underline{n}\in\mathbb{N}^d} \operatorname{Ann}_R((I + \langle \underline{z}^{\underline{n}} \rangle)^\perp) = \bigcap_{\underline{n}\in\mathbb{N}^d} (I + \langle \underline{z}^{\underline{n}} \rangle) = I$$

where $\operatorname{Ann}_R((I + \langle \underline{z}^{\underline{n}} \rangle)^\perp) = I + \langle \underline{z}^{\underline{n}} \rangle$ by Remark 6.16, and $\bigcap_{\underline{n}\in\mathbb{N}^d}(I + \langle \underline{z}^{\underline{n}} \rangle) = I$ by Lemma 7.48.

Conversely, let $W = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle \in \mathcal{C}'$. Then

$$\theta\theta'(W) = \theta(\bigcap_{\underline{n}\in\mathbb{N}^d} \operatorname{Ann}_R(W_{\underline{n}})) = \langle \bigcup_{\underline{n}\in\mathbb{N}^d} (\operatorname{Ann}_R(W_{\underline{n}}))^\perp \rangle = \langle \bigcup_{\underline{n}\in\mathbb{N}^d} W_{\underline{n}} \rangle = W$$

where $W_{\underline{n}} = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\} \rangle$ and $(\operatorname{Ann}_R(W_{\underline{n}}))^\perp = W_{\underline{n}}$ by Remark 6.16. □

Remark 7.51. If Propositions 7.30 and 7.31 are true also for non-general sequences, as stated in Remark 7.32, then Proposition 7.46 holds too, and as a consequence we can also state Theorem 7.50 in more generality. In particular, we have a correspondence between ideals I such that $R/I + \langle \underline{z} \rangle$ is level for some regular sequence $\underline{z}$ and L^τ_d-admissible R-submodules of $\mathcal{D}$ with respect to the regular sequence $\underline{z}$.

The following theorem shows that important information about a level $\mathbf{k}$-algebra is encoded in its inverse system.

Theorem 7.52. *Let $d \le n$ be a positive integer.*

(a) Let $R = \mathbf{k}[x_1, \dots, x_n]$. Then there is a one-to-one correspondence between d-dimensional graded level $\mathbf{k}$-algebras $A = R/I$ of Castelnuovo-Mumford regularity r and multiplicity e and non-zero graded L^τ_d-admissible R-submodules $W = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle$ of $\mathcal{D}$ such that $\deg H^j_{\mathbf{1}} = r$ and $\dim_{\mathbf{k}}(\langle H^j_{\mathbf{1}} : j \in \{1, \dots, \tau\} \rangle) = e$.

(b) Let $R = \mathbf{k}[[x_1, \dots, x_n]]$ with $\operatorname{char}(\mathbf{k}) = 0$. Then there is a one-to-one correspondence between d-dimensional local level $\mathbf{k}$-algebras $A = R/I$ of multiplicity e and non-zero L^τ_d-admissible R-submodules $W = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle$ of $\mathcal{D}$ such that $\dim_{\mathbf{k}}(\langle H^j_{\mathbf{1}} : j \in \{1, \dots, \tau\} \rangle) = e$.

Proof. (a) If $R = \mathbf{k}[x_1, \dots, x_n]$ and $A = R/I$ is a homogeneous level $\mathbf{k}$-algebra, then the multiplicity and the Castelnuovo-Mumford regularity of A coincide with those of $A/\langle \underline{z} \rangle A$ for any $\underline{z} := z_1, \dots, z_d$ regular linear sequence for R/I (see [BH93, Remark 4.1.11] for the multiplicity and [BH93, §4.3] for the regularity). Hence

$$e(A) = \dim_{\mathbf{k}}(\langle H^j_{\mathbf{1}} \mid j \in \{1, \dots, \tau\} \rangle)$$

and

$$\operatorname{reg}(A) = \deg(H^j_{\mathbf{1}}) \text{ for any} j \in \{1, \dots, \tau\}.$$

(b) Let $A = R/I$ be a d-dimensional level local ring and let $\underline{z} = z_1, \dots, z_d$ be a sequence of general linear forms in R. By Theorem 7.50, the dual module $I^\perp = W = \langle H^j_{\underline{n}} \mid j \in \{1, \dots, \tau\}, \underline{n} \in \mathbb{N}^d \rangle$ is an L^τ_d-admissible submodule of $\mathcal{D}$. As $(I + \langle \underline{z} \rangle)^\perp = \langle H^j_{\mathbf{1}_1} : j \in \{1, \dots, \tau\} \rangle$, by Proposition 6.31

$$\operatorname{socdeg}(A/\langle \underline{z} \rangle A) = \deg(H^j_{\mathbf{1}})$$

for any $j \in \{1, \dots, \tau\}$. Since $\langle \underline{z} \rangle$ is a minimal general reduction of $\mathfrak{n}$, and therefore $\underline{z}$ is a superficial sequence by Remark 7.18.(b), the multiplicity of A (see Definition C.5) coincides with the multiplicity of $A/\langle \underline{z} \rangle A$ and hence

$$e(A) = \dim_{\mathbf{k}}(A/\langle \underline{z} \rangle A) = \dim_{\mathbf{k}}(\langle H_{\mathbf{1}}^{j} \mid j \in \{1, \dots, \tau\}\rangle). \qquad \square$$

8
Examples and remarks

In this section we give examples of applications of Theorem 7.50. In general it is very difficult to verify whether a given R-submodule of $\mathcal{D}$ is L_d^τ-admissible, as one needs to check the conditions in Definition 7.37 for an infinite number of elements. However, as observed in [ER17, Proposition 4.2] for the Gorenstein case, in the graded case it suffices to verify these conditions for finitely many elements (Proposition 8.4).

In the local case, we give only examples where the graded associated ring is Cohen–Macaulay, so that, due to Proposition 7.14, we can consider any minimal reduction instead of general ones.

Definition 8.1. Let $d > 1$. An ideal $I \subseteq R = \mathbf{k}[[x_1, \dots, x_n]]$ is a *cone* with respect to an ideal $J \subseteq \mathbf{k}[[x_{d+1}, \dots, x_n]]$ if $I = JR$.

The following proposition shows that every cone constructed starting from τ suitable elements is the dual of a level $\mathbf{k}$-algebra.

Proposition 8.2. *Let $d > 1$, and let $H^1, \dots, H^\tau \in \mathbf{k}_{DP}[X_{d+1}, \dots, X_n]$ be elements of same degree with $\operatorname{top}(H^1), \dots, \operatorname{top}(H^\tau)$ linearly independent, and let $H = \langle H^1, \dots, H^\tau \rangle \subseteq \mathcal{D}$. Let W be the R-submodule of $\mathcal{D}$ generated by the elements*

$$H^j_{\underline{n}} = X_1^{n_1-1} \cdots X_d^{n_d-1} H^j = X^{\underline{n}-\mathbf{1}} H^j$$

for any $j \in \{1, \dots, \tau\}$ and $\underline{n} \in \mathbb{N}^d$, $\underline{n} \geq \mathbf{1}$. Then $R/\operatorname{Ann}_R(W)$ is a d-dimensional level $\mathbf{k}$-algebra. Moreover, if the H^j are homogeneous for all $j \in \{1, \dots, \tau\}$, then $R/\operatorname{Ann}_R(W)$ is also graded.

In particular, $\operatorname{Ann}_R(W)$ is a cone with respect to $\operatorname{Ann}_S(H)$, where $S = \mathbf{k}[[x_{d+1}, \dots, x_n]]$ with $\operatorname{char} \mathbf{k} = 0$ or $S = \mathbf{k}[x_{d+1}, \dots, x_n]$ if $R/\operatorname{Ann}_R(W)$ is graded.

Proof. We show that W is a L_d^τ-admissible R-submodule of $\mathcal{D}$ with respect to the sequence $\underline{x} = x_1, \dots, x_d$. We will denote by X_i the elements of $\mathcal{D}$ dual to the coordinates x_i, and by $\underline{X} = X_1, \dots, X_d$. It is clear that for any $\underline{n} \in \mathbb{N}^d$, $\deg H^1_{\underline{n}} = \cdots = \deg H^\tau_{\underline{n}}$ and $\operatorname{top}(H^1_{\underline{n}}), \dots, \operatorname{top}(H^\tau_{\underline{n}})$ are linearly independent. Also, for any $j \in \{1, \dots, \tau\}$, $i \in \{1, \dots, d\}$ and $\underline{n} \in \mathbb{N}^d$ with $n_i \geq 2$

$$x_i \circ H^j_{\underline{n}} = x_i \circ (X_1^{n_1-1} \cdots X_d^{n_d-1} H^j) = X_1^{n_1-1} \cdots X_i^{n_i-2} \cdots X_d^{n_d-1} H^j = \begin{cases} H^j_{\underline{n}-\mathbf{e}_i} & \text{if } n_i > 1 \\ 0 & \text{if } n_i = 1 \end{cases} \tag{8.1}$$

and hence W satisfies Definition 7.37.(2). Let us prove Definition 7.37.(3). From (8.1) we obtain that

$$x_k \circ H^j_{\underline{n}} = X^{\mathbf{n}-\mathbf{1}-\mathbf{e}_k} H^j \in V^i_{\underline{n}} = \langle \underline{X}^{\underline{k}} \mid \underline{k} \in \mathbb{N}^d, k_i \leq n_i - 2, |\underline{k}| \leq \deg(H^1) - |\underline{n}| - d \rangle$$

if and only if $k = i$ and $n_k = n_i > 1$. Moreover, in this case, by (8.1), $x_i \circ H^j_{\underline{n}} \in W_{\underline{n}-\mathbf{e}_i}$. Hence $W_{\underline{n}} \cap V^i_n \subseteq W_{\underline{n}-\mathbf{e}_i}$ for any $\underline{n} - \mathbf{e}_i > \mathbf{2}$. Hence W is a L^τ_d-admissible R-submodule of $\mathcal{D}$. Theorem 7.50 implies $R/\operatorname{Ann}_R(W)$ is a level $\mathbf{k}$-algebra of dimension d. Notice that we didn't prove that $\underline{x}$ is a general sequence. However, since the ring we find applying the correspondence has Cohen–Macaulay graded associated ring, it is level with respect to any Artinian reduction (see Proposition 7.14).

To prove that $\operatorname{Ann}_R(W)$ is a cone with respect to $\operatorname{Ann}_S(H)$, recall that

$$\operatorname{Ann}_R(W_{\underline{n}}) = \langle \underline{x} \rangle + \operatorname{Ann}_S(H),$$

by construction. Hence

$$\begin{aligned}\operatorname{Ann}_R(W) &= \bigcap_{\underline{n}\in\mathbb{N}^d} \operatorname{Ann}_R(W_{\underline{n}}) \\ &= \bigcap_{\underline{n}\in\mathbb{N}^d} (\langle x_1^{n_1}, \ldots, x_d^{n_d}\rangle + \operatorname{Ann}_S(W_{\mathbf{1}})R) \\ &= \operatorname{Ann}_S(W_{\mathbf{1}})R\end{aligned}$$

where the last equality holds thanks to Lemma 7.48. □

Definition 8.3. Let $t_0 \in \mathbb{N}_+$. We say that a family $\mathcal{H} = \{H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\},\ \underline{n} \in \mathbb{N}^d,\ |\mathfrak{n}| \le t_0\}$ of elements of $\mathcal{D}$ is *L^τ_d-admissible* if the elements $H^j_{\underline{n}}$ satisfy the conditions of Definition 7.37 up to $\underline{n}$ with $|\underline{n}| \le t_0$.

The following proposition shows that in the graded case finitely many admissible elements $\mathcal{H}$ are sufficient to recover a graded level $\mathbf{k}$-algebra.

Proposition 8.4. *Let $H^1_{\mathbf{1}}, \ldots, H^\tau_{\mathbf{1}}$ be elements of degree r with $\operatorname{top}(H^1_{\mathbf{1}}), \ldots, \operatorname{top}(H^\tau_{\mathbf{1}})$ linearly independent. Let $t_0 \ge (r+2)d$ for some $d > 0$ and let $\mathcal{H} = \{H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}, \underline{n} \in \mathbb{N}^d, |\underline{n}| \le t_0\}$ be an admissible set of homogeneous elements with respect to a regular linear sequence $\underline{z} = z_1, \ldots, z_d$ for R. Assume there exists a graded level $\mathbf{k}$-algebra $A = R/I$ such that $(I + \langle \underline{z}^{\underline{n}}\rangle)^\perp = W_{\underline{n}} := \langle H^j_{\underline{n}} \mid j \in \{1, \ldots, \tau\}\rangle$ for any $|\underline{n}| \le t_0$. Then*

$$I = \{f \in \operatorname{Ann}_R(W_{\boldsymbol{r+2}}) \mid \deg(f) \le r+1\}.$$

Proof. Recall that the Castelnuovo-Mumford regularity of A is

$$\operatorname{reg}(A) = \operatorname{reg}(A/\langle \underline{z}\rangle A) = \operatorname{reg}(R/\operatorname{Ann}_R(W_{\mathbf{1}})) = \operatorname{socdeg}(R/\operatorname{Ann}_R(W_{\mathbf{1}})) = \deg H^1_{\mathbf{1}} = r.$$

It is well known that the maximum degree of a minimal system of generators of I is at most $\operatorname{reg}(R/I) + 1$. Hence the claim follows from the identity $\operatorname{Ann}_R(W_{\mathbf{r+2}}) = I + \langle \underline{z}^{\mathbf{r+2}}\rangle$. □

8.1 Level algebras from L^τ_d-admissible systems

The following example shows how Propositions 8.2 and 8.4 can be effectively used to construct examples.

Example 8.5. Let $R = \mathbb{Q}[x, y, z]$ and $\mathcal{D} = \mathbb{Q}[X, Y, Z]$. Let

$$\begin{aligned}
H^1_1 &= Y^3 & H^2_1 &= Z^3 \\
H^1_2 &= XH^1_1 & H^2_2 &= XH^2_1 \\
H^1_3 &= X^2H^1_1 & H^2_3 &= X^2H^2_1 \\
H^1_4 &= X^3H^1_1 & H^2_4 &= X^3H^2_1 \\
H^1_5 &= X^4H^1_1 & H^2_5 &= X^4H^2_1.
\end{aligned}$$

By Proposition 8.2 the set

$$\mathcal{H} = \{H_1^1, H_1^2, H_2^1, H_2^2, H_3^1, H_3^2, H_4^1, H_4^2\}$$

is L_1^τ-admissible. The involved elements are homogeneous, and hence by Proposition 8.4 we have that

$$I = \langle y^4, yz, z^4\rangle \subseteq R = \mathbf{k}[x, y, z]$$

is a 1-dimensional level ideal of type 2.

From Proposition 6.31 it follows that the intersection of Gorenstein Artinian ideals (i.e. ideals such that the quotient is Gorenstein) of same socle degree is level. The following example shows that this is no longer true if ideals have positive dimension.

Example 8.6. Let $R = \mathbb{Q}[x, y, z]$ and $\mathcal{D} = \mathbb{Q}[X, Y, Z]$. Let

$$\begin{array}{ll}
H_1^1 = Y^3 - Z^3 & H_1^2 = Y^2Z \\
H_2^1 = XH_1^1 + YZ^3 & H_2^2 = XH_1^2 \\
H_3^1 = XH_2^1 - Y^2Z^3 & H_3^2 = XH_2^2 \\
H_4^1 = XH_3^1 + Y^3Z^3 - 4Z^6 & H_4^2 = XH_3^2 \\
H_5^1 = XH_4^1 + Y^7 - Y^4Z^3 + 4YZ^6 & H_5^2 = XH_4^2.
\end{array}$$

By using Singular or Macaulay 2 one can verify that the set $\mathcal{H}_1 = \{H_1^1, H_2^1, H_3^1, H_4^1, H_5^1\}$ is G_1-admissible and hence by Proposition 8.4 the ideal

$$I = \langle yz + xz, y^3 + z^3 - xy^2 + x^2y - x^3\rangle$$

is a 1-dimensional Gorenstein ideal (see [ER17, Example 4.4]). Similarly, the set $\mathcal{H}_2 = \{H_1^2, H_2^2, H_3^2, H_4^2, H_5^2\}$ is G_1-admissible and the corresponding 1-dimensional Gorenstein ideal is

$$J = \langle z^2, y^3\rangle.$$

In fact, both I and J are complete intersections. Using Singular or Macaulay 2, it is easy to check that

$$I \cap J = \langle xz^2 + yz^2, 4y^3z + z^4, x^3y^3 - x^2y^4 + xy^5 - y^6 - y^2z^3\rangle$$

and $R/(I \cap J)$ is a 1-dimensional ring with the following graded minimal R-free resolution:

$$0 \to R(-6) \oplus R(-7) \to R(-3) \oplus R(-4) \oplus R(-6) \to R \to 0.$$

Hence $R/(I \cap J)$ is not level. If we consider what should be the dual submodule in $\mathcal{D}$:

$$W := \{H_n^j \mid j = 1, 2 \text{ and } n \in \mathbb{N}_+^1\}$$

we have

$$Z^3 \in \langle H_2^1, H_2^2\rangle \cap \mathbb{Q}[Y, Z] \setminus \langle H_1^1, H_1^2\rangle$$

and hence W does not satisfy Definition 7.37.(3). However, it is easy to verify that W satisfies (7.7).

8.2 Inverse systems of level $\mathbf{k}$-algebras

In this section we want to show some examples of the inverse systems of special classes of level algebras.

Let $n_1, \dots, n_l$ be an arithmetic sequence, i.e.

$$n_i = n_{i-1} + q = n_1 + (i-1)q$$

for $i \in \{2, \dots, l\}$ and $q \in \mathbb{N}, q \geq 1$. Then the ring $A = \mathbf{k}[t^{n_1}, \dots, t^{n_l}]$ is a semigroup ring whose associated graded ring is level (see [MT95, Prop. 1.12]). By [Frö87, Example 1.(b)], the type of $\mathrm{gr}_{\mathfrak{n}}(A)$ is always greater or equal than the type of A. If $\mathrm{gr}_{\mathfrak{n}}(A)$ is level, then the two types coincide. Hence we can deduce that the local ring A is also level (see Definition 7.21 and Proposition 7.14).

Example 8.7. Let $A = \mathbb{Q}[[t^6, t^{10}, t^{14}, t^{18}]]$. Then A is a semigroup ring associated to an arithmetic sequence, and we know from the previous observations that it is level. It is easy to check that $A = R/I$ where $R = \mathbb{Q}[[x, y, z, w]]$ and

$$I = \langle x^3 - w, x^4 - yz, xz - y^2, x^3y - z^2 \rangle.$$

Then $\langle x \rangle$ is a minimal reduction for A, and as $gr_{\mathfrak{n}}(A)$ is Cohen–Macaulay, Proposition 7.14 tells us that we don't need to find a minimal general reduction. The following elements in $\mathcal{D}$ are the inverse system of A up to degree 5:

$$\begin{aligned}
H_1^1 &= Y & H_1^2 &= Z \\
H_2^1 &= XH_1^1 & H_2^2 &= XH_1^2 + Y^2 \\
H_3^1 &= X^2H_1^1 & H_3^2 &= X^2H_1^2 + XY^2 = XH_2^2 \\
H_4^1 &= X^3H_1^1 + YW + Z^2 & H_4^2 &= X^3H_1^2 + X^2Y^2 + ZW = XH_3^2 + ZW \\
H_5^1 &= X^4H_1^1 + XYW + XZ^2 + Y^2Z & H_5^2 &= X^4H_1^2 + X^3Y^2 + XZW + YZ^2 + Y^2W \\
&= XH_4^1 + Y^2Z & &= XH_4^2 + YZ^2 + Y^2W
\end{aligned}$$

In principle, by Theorem 7.50, we have an infinite number of elements in the inverse system. However, in this case, to recover the ideal we only need a finite number. Let $W_{(5,5)} = \{H_5^1, H_5^2\}$. Using Singular, one can verify that

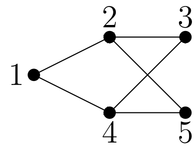
$$\begin{aligned}
I &= \mathrm{Ann}_R(W_{(5,5)})_{\leq 4} = \langle x^3 - w, xz - y^2, xw - yz, z^2 - yw, x^5 \rangle_{\leq 4} \\
&= \langle x^3 - w, xz - y^2, xw - yz, z^2 - yw \rangle.
\end{aligned}$$

Another important class of level rings are matroid simplicial complexes. By [Sta96, Proposition 3.2] all the Stanley-Reisner rings associated to matroid complexes are level. Let us describe a particular type of these matroids, coming from matrices. If $\mathbf{k}$ is a field and $m \leq n$, let $X \in \mathbf{k}^{m \times n}$. The $m \times m$ minors of X are denoted by $[i_1, \dots, i_m]$ where $1 \leq i_1 < \cdots < i_m \leq n$. Let us consider the simplicial complex Δ with vertices $\{1, \dots, n\}$ and facets $\{F = \{i_1, \dots, i_m\} \mid [i_1, \dots, i_m] \neq 0\}$. Then Δ is a matroid. Stanley's result yields that R/I_Δ is a graded level algebra, where I_Δ is the Stanley-Reisner ideal associated to Δ (i.e. the ideal of non-facets).

Example 8.8. Let

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 \end{pmatrix}$$

Then the simplicial complex Δ has facets $\{\{12\}, \{23\}, \{34\}, \{45\}, \{14\}, \{25\}\}$. The figure below illustrates the simplicial complex:



Let $R = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$. We can easily compute the Stanley-Reisner ring associated to this matroid:

$$I_\Delta = \langle x_1x_3, x_2x_4, x_1x_5, x_3x_5\rangle$$

The ring R/I_Δ is a graded level ring of dimension 2 and type 2. Observe that $x_2 + x_4$ and $x_1 + x_3 + x_5$ form a regular sequence for $A = R/I_\Delta$. In order to find its inverse system, we first operate a change of coordinates:

$$\begin{aligned}
\varphi : R &\to S = \mathbb{Q}[y_1, \dots, y_5]\\
x_2 + x_4 &\mapsto y_1\\
x_1 + x_3 + x_5 &\mapsto y_2\\
x_3 &\mapsto y_3\\
x_4 &\mapsto y_4\\
x_5 &\mapsto y_5
\end{aligned}$$

Under this change of coordinates we get the ideal

$$I = \varphi(I_\Delta) = \langle (y_2 - y_3 - y_5)y_3, (y_1 - y_4)y_4, (y_2 - y_3 - y_5)y_5, y_3y_5\rangle.$$

The ring $A = S/I$ is again a graded level ring of dimension 2 and type two,and y_1, y_2 form a regular sequence for A. Consider $\mathcal{D} = \mathbb{Q}[Y_1, \dots, Y_5]$ the divided power ring dual to S. Using Singular, we can compute the first generators of $I^\perp \subseteq \mathcal{D}$:

$$\begin{aligned}
H^1_{(1,1)} =& Y_4Y_5\\
H^1_{(1,2)} =& Y_2(Y_4Y_5) + Y_4Y_5^2\\
H^1_{(2,2)} =& Y_1H^1_{(1,2)} + Y_2Y_4^2Y_5 + Y_4^2Y_5^2\\
&\vdots\\
H^1_{(4,4)} =& Y_1^2Y_2^2H^1_{(2,2)} + Y_1Y_2^3Y_4^3Y_5+\\
& Y_2^3Y_4^4Y_5 + Y_1Y_2^2Y_4^3Y_5^2+\\
& Y_2^2Y_4^4Y_5^2 + Y_1^3Y_2Y_4Y_5^3+\\
& Y_1^2Y_2Y_4^2Y_5^3 + Y_1Y_2Y_4^3Y_5^3+\\
& Y_2Y_4^4Y_5^3 + Y_1^3Y_4Y_5^4+\\
& Y_1^2Y_4^2Y_5^4 + Y_1Y_4^3Y_5^4+\\
& Y_4^4Y_5^4
\end{aligned}$$

$$\begin{aligned}
H^2_{(1,1)} =& Y_3Y_4\\
H^2_{(1,2)} =& Y_2(Y_3Y_4) + Y_3^2Y_4\\
H^2_{(2,2)} =& Y_1H^2_{(1,2)} + Y_2Y_3Y_4^2 + Y_3^2Y_4^2\\
&\vdots\\
H^2_{(4,4)} =& Y_1^2Y_2^2H^2_{(2,2)} + Y_1^3Y_2Y_3^3Y_4+\\
& Y_1^3Y_3^4Y_4 + Y_1^2Y_2Y_3^3Y_4^2+\\
& Y_1^2Y_3^4Y_4^2 + Y_1Y_2^3Y_3Y_4^3+\\
& Y_1Y_2^2Y_3^2Y_4^3 + Y_1Y_2Y_3^3Y_4^3+\\
& Y_1Y_3^4Y_4^3 + Y_2^3Y_3Y_4^4+\\
& Y_2^2Y_3^2Y_4^4 + Y_2Y_3^3Y_4^4+\\
& Y_3^4Y_4^4.
\end{aligned}$$

The set

$$\mathcal{H}_1 = \{H^1_{(1,1)}, H^2_{(1,1)}, H^1_{(1,2)}, H^2_{(1,2)}, H^1_{(2,1)}, H^2_{(2,1)}, H^1_{(2,2)}, H^2_{(2,2)}, \dots, H^1_{(4,4)}, H^3_{(4,4)}\}$$

is L_d^τ-admissible, with $d = \dim(R/I_\Delta) = 2$.

From Proposition 8.4 we know that $W_{(4,4)} = \langle H^1_{(4,4)}, H^2_{(4,4)}\rangle$ is enough to identify the ideal I. In fact, it can be verified that

$$\begin{aligned}
I = \operatorname{Ann}_R(W_{(4,4)})_{\leq 3} = & \langle y_3y_5, y_2y_5 - y_5^2, y_1y_4 - y_4^2, y_2y_3 - y_3^2, y_2^4y_1^4, y_5^5, y_4^5, y_3^5\rangle_{\leq 3}\\
=&\langle y_3y_5, y_2y_5 - y_5^2, y_1y_4 - y_4^2, y_2y_3 - y_3^2\rangle.
\end{aligned}$$

8.3 Questions and Remarks

Question 8.9. Is it true that every graded or local level $\mathbf{k}$-algebra of dimension $d > 0$ and type τ is intersection of τ graded or local Gorenstein $\mathbf{k}$-algebras of dimension $d > 0$?

Remark 8.10. The answer to Question 8.9 in the Artinian case is positive, and it a direct consequence of Macaulay's inverse system. However, it is not clear if its possible to use Theorem 7.50 to answer in general. In fact, for now, we are not able to prove that the module $W_j = \langle H^j_{\underline{n}} \mid \mathfrak{n} \in \mathbb{N}^d \rangle$ is G_d-admissible for any $j \in \{1, \ldots, \tau\}$, as it is not clear if it satisfies (7.6) (see Proposition 7.41).

Question 8.11. Can Theorem 7.50 be extended to characterize the inverse System of any Cohen–Macaulay $\mathbf{k}$-algebra, depending on the socle type?

Remark 8.12. Question 8.11 can be answered in the graded case following basically the same proof of Theorem 7.50. However, in the local case it doesn't seem possible to give a definition of socle type of A which is preserved by quotients of type $A/\langle \underline{z}^{\underline{n}} \rangle$ when $\underline{n}$ varies, even taking a general linear sequence.

Question 8.13. Is there an analogous of Proposition 8.4 for the local case?

Part III

Appendix

A
Fractional ideals

All rings under consideration will be commutative and unitary.

Notation A.1. Let R be a ring. We will use the following notations:

- $\mathrm{Max}(R)$ is the set of maximal ideals of R.
- R^{reg} is the set of all regular (non zero-divisors) elements of R, and $R^* \subseteq R^{\mathrm{reg}}$ the set of invertible elements of R.
- For an R-module M, $\hat{M}$ denotes the completion of M at the Jacobson radical of R.

Let Q be a ring such that

$$Q^{\mathrm{reg}} = Q^* \tag{A.1}$$

and let $\mathcal{E}, \mathcal{F} \subseteq Q$. We abbreviate

$$\mathcal{F} : \mathcal{E} := \mathcal{F} :_Q \mathcal{E} = \{x \in Q \mid x\mathcal{E} \subseteq \mathcal{F}\}.$$

Lemma A.2. *Let $x \in Q^{\mathrm{reg}}$ and $\mathcal{E}, \mathcal{E}', \mathcal{F}, \mathcal{F}', \mathcal{G}$ be R-submodules of Q. Then*

(a) $(\mathcal{G} : \mathcal{F}) : \mathcal{E} = \mathcal{G} : (\mathcal{F} \cdot \mathcal{E})$.

(b) $(x\mathcal{E}) : \mathcal{F} = x(\mathcal{E} : \mathcal{F}) = \mathcal{E} : (x^{-1}\mathcal{F})$.

(c) For any two inclusions $\mathcal{E} \subseteq \mathcal{E}'$ and $\mathcal{F} \subseteq \mathcal{F}'$, $\mathcal{E} : \mathcal{F}' \subseteq \mathcal{E} : \mathcal{F} \subseteq \mathcal{E}' : \mathcal{F}$.

(d) If $R \subseteq R' \subseteq Q$ is a ring extension and $\mathcal{F}$ and R'-module, $\mathcal{E} : \mathcal{F} = (\mathcal{E} : R') : \mathcal{F}$.

Proof. (a) We get the equalities

$$\begin{aligned}(\mathcal{G} : \mathcal{F}) : \mathcal{E} &= \{x \in Q \mid x\mathcal{E} \subseteq (\mathcal{G} : \mathcal{F})\} = \{x \in Q \mid x\mathcal{E} \subseteq \{y \in Q \mid y\mathcal{F} \subseteq \mathcal{G}\}\} \\ &= \{x \in Q \mid x\mathcal{E}\mathcal{F} \subseteq \mathcal{G}\} = \mathcal{G} : (\mathcal{F} : \mathcal{E}).\end{aligned}$$

(b) We get the equalities

$$\begin{aligned}(x\mathcal{E}) : \mathcal{F} &= \{y \in Q \mid y\mathcal{F} \subseteq x\mathcal{E}\} = x\{y \in Q \mid y\mathcal{F} \subseteq \mathcal{E}\} = x(\mathcal{E} : \mathcal{F}) \\ &= \{y \in Q \mid yx^{-1}\mathcal{F} \subseteq \mathcal{E}\} = \mathcal{E} : (x^{-1}\mathcal{F}).\end{aligned}$$

(c) Since $\mathcal{F} \subseteq \mathcal{F}'$

$$\mathcal{E} : \mathcal{F}' = \{x \in Q \mid x\mathcal{F}' \subseteq \mathcal{E}\} \subseteq \{x \in Q \mid x\mathcal{F} \subseteq \mathcal{E}\} = \mathcal{E} : \mathcal{F}$$

and, as $\mathcal{E} \subseteq \mathcal{E}'$

$$\mathcal{E} : \mathcal{F} = \{x \in Q \mid x\mathcal{F} \subseteq \mathcal{E}\} \subseteq \{x \in Q \mid x\mathcal{F} \subseteq \mathcal{E}'\}.$$

(d) We get the equalities

$$\begin{aligned}\mathcal{E} : \mathcal{F} &= \{x \in Q \mid x\mathcal{F} \subseteq \mathcal{E}\} = \{x \in Q \mid x\mathcal{F}R' \subseteq \mathcal{E}R'\} \\ &= \{x \in Q \mid x\mathcal{F} \subseteq \{y \in Q \mid yR' \subseteq \mathcal{E}\}\} = (\mathcal{E} : R') : \mathcal{F}.\end{aligned}$$

□

Definition A.3. The *total ring of fractions* Q_R of a ring R is the localization of R at R^{reg}.

Definition A.4. The *integral closure of a ring* R is the set of elements of the total ring of fractions Q_R which are integral over R, and we denote it by $\overline{R}$. If R is reduced, then $\overline{R}$ is the *normalization* of R.

For any subset $S \subseteq Q_R$, we set

$$S^{\mathrm{reg}} := S \cap Q_R^{\mathrm{reg}}.$$

Note that $R^{\mathrm{reg}} = R \cap Q_R^{\mathrm{reg}}$.

Definition A.5. Let R be a ring with Q_R satisfying (A.1).

(a) An R-submodule $\mathcal{E}$ of Q_R is called *regular* if $\mathcal{E}^{\mathrm{reg}} \neq \emptyset$ or, equivalently, $Q_R\mathcal{E} = Q_R$.

(b) An R-submodule $\mathcal{E} \subseteq Q_R$ such that $r\mathcal{E} \subseteq R$ for some $r \in R^{\mathrm{reg}}$ is called a *fractional ideal* of R. If R is Noetherian, this is equivalent to $\mathcal{E}$ being a finitely generated R-submodule of Q_R.

(c) If every regular ideal, or equivalently regular fractional ideal, I of R is generated by I^{reg}, then R is called a *Marot ring*.

(d) The *conductor* of a fractional ideal $\mathcal{E}$ of R is $\mathcal{C}_{\mathcal{E}} = \mathcal{E} : \overline{R}$.

Notation A.6. Let R be a ring with Q_R satisfying (A.1). We denote $\mathfrak{R}_R$ the set of regular fractional ideals of R.

Remark A.7. The set $\mathfrak{R}_R$ is clearly a (commutative) monoid under product of ideals. Moreover, it is closed under ideal quotient. In fact, if $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$, then $\mathcal{E} \colon \mathcal{F} \in \mathfrak{R}_R$. This follows immediately, since $\mathcal{E}^{\mathrm{reg}} \neq \emptyset$ and $\mathcal{E} \subseteq \mathcal{E} \colon \mathcal{F}$.

Definition A.8. Let R be a ring with Q_R satisfying (A.1). An R-submodule $\mathcal{E}$ of Q_R is *invertible* if $\mathcal{E}\mathcal{F} = R$ for some R-submodule $\mathcal{F}$ of Q_R.

Remark A.9. Let R be a ring with Q_R satisfying (A.1). Then $Q_R^{\mathrm{reg}} = Q_R^*$ Hence any regular element $x \in Q_R$ is invertible. In particular, xR is an invertible R-submodule of Q_R.

Remark A.10. Let R be a ring with Q_R satisfying (A.1). If $\mathcal{E}$ is invertible, then its inverse is uniquely determined as $\mathcal{F} = \mathcal{E}^{-1} = R : \mathcal{E}$. Indeed, inverses are unique when they exist, and if $\mathcal{E}$ is invertible, with $\mathcal{E}\mathcal{F} = R$, then $f\mathcal{E} \subseteq R$ for any $f \in \mathcal{F}$. Hence $\mathcal{F} \subseteq R : \mathcal{E}$. Now $R = \mathcal{E}\mathcal{F} \subseteq \mathcal{E}(R : \mathcal{E}) \subseteq R$, so that $\mathcal{F} = R : \mathcal{E}$.

Lemma A.11. *Let R be a ring with Q_R satisfying* (A.1). *Every invertible R-submodule of Q_R is regular and finitely generated.*

Proof. See [KV04, Chapter II, Remark 2.1.(3) and Proposition 2.2.(2)]. □

In particular, as all invertible ideals are regular, the (abelian) group $\mathfrak{R}_R^*$ of all invertible R-submodules of Q_R is a submonoid of $\mathfrak{R}_R$.

Lemma A.12. *If R is semilocal, then*

$$\mathfrak{R}_R^* = \{\mathcal{E} \in \mathfrak{R}_R \mid \mathcal{E} \textit{ cyclic } R-\textit{submodule of} Q_R\}.$$

Proof. See [KV04, Chapter II, Proposition 2.2.(3)]. ☐

Lemma A.13. *Let R be a ring with total ring of fractions Q_R.*
For $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$, there is a canonical isomorphism

$$\begin{aligned} \mathcal{F} : \mathcal{E} &\to \operatorname{Hom}_R(\mathcal{E}, \mathcal{F}) \\ x &\mapsto (y \mapsto xy), \end{aligned}$$

of R-modules compatible with multiplication in Q_R and composition of homomorphisms. The composed isomorphism

$$\psi : \mathcal{F} : (\mathcal{F} : \mathcal{E}) \to \operatorname{Hom}_R(\mathcal{F} : \mathcal{E}, \mathcal{F}) \to \operatorname{Hom}_R(\operatorname{Hom}_R(\mathcal{E}, \mathcal{F}), \mathcal{F})$$

fits into a commutative diagram of canonical maps

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr, "\alpha"'] & \mathcal{F} : (\mathcal{F} : \mathcal{E}) \arrow[d, "\psi"', "\cong"] \\ & \operatorname{Hom}_R(\operatorname{Hom}_R(\mathcal{E}, \mathcal{F}), \mathcal{F}). \end{tikzcd}$$

Proof. For the first isomorphism see [HK71, Lem. 2.1]. The homomorphism α is the natural double dual map defined by $x \mapsto (\varphi \mapsto \varphi(x))$, and $\mathcal{E} \to \mathcal{F} : (\mathcal{F} : \mathcal{E})$ is the inclusion map. To observe that the diagram commutes, let $x \in \mathcal{E}$. Then $x \in \mathcal{F} : (\mathcal{F} : \mathcal{E})$ and $\psi(x) = (\mathcal{F} : \mathcal{E} \ni y \mapsto xy) \cong (\operatorname{Hom}_R(\mathcal{E}, \mathcal{F}) \ni \varphi \mapsto x\varphi(1)) = (\varphi \mapsto \varphi(x)) = \alpha(x)$. See also [HK71, Lemma 2.3]. ☐

Lemma A.14. *Let R be a ring with total ring of fractions Q_R. Let $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$.*

(a) $(\mathcal{F} : \mathcal{E})_\mathfrak{m} = (\mathcal{F}_\mathfrak{m} : \mathcal{E}_\mathfrak{m})$ *for any* $\mathfrak{m} \in \operatorname{Max}(R)$.

(b) $\mathcal{E}_\mathfrak{m} = \mathcal{E} R_\mathfrak{m} \in \mathfrak{R}_{R_\mathfrak{m}}$.

Proof. (a) By Lemma A.13, compatibility of Hom with flat extensions (see [Eis95, Proposition 2.10]) and the fact that localization is flat ([Eis95, Proposition 2.5]), we have canonical isomorphisms

$$\begin{aligned} (\mathcal{F}_\mathfrak{m} : \mathcal{E}_\mathfrak{m}) &= \operatorname{Hom}_{R_\mathfrak{m}}(\mathcal{E}_\mathfrak{m}, \mathcal{F}_\mathfrak{m}) = \operatorname{Hom}_{R_\mathfrak{m}}(\mathcal{E} \otimes_R R_\mathfrak{m}, \mathcal{F} \otimes R_\mathfrak{m}) \\ &= R_\mathfrak{m} \otimes_R \operatorname{Hom}_R(\mathcal{E}, \mathcal{F}) = R_\mathfrak{m} \otimes_R (\mathcal{F} : \mathcal{E}) = (\mathcal{F} : \mathcal{E})_\mathfrak{m}. \end{aligned}$$

As $(\mathcal{F}_\mathfrak{m} : \mathcal{E}_\mathfrak{m})$ and $(\mathcal{F} : \mathcal{E})_\mathfrak{m}$ are both subsets of $Q_\mathfrak{m}$, this is in fact an equality.

(b) The flat ring homomorphism $\varphi :\to R_\mathfrak{m}$ induces a ring homomorphism $\widetilde{\varphi} : Q_R \to Q_{R_\mathfrak{m}}$. In particular, regular elements of R are regular elements of $R_\mathfrak{m}$. Hence, since $\mathcal{E} \in \mathfrak{R}_R$,

$$\mathcal{E} \otimes_R R_\mathfrak{m} = \widetilde{\varphi}(\mathcal{E}) R_\mathfrak{m} \subseteq Q_{R_\mathfrak{m}} R_\mathfrak{m} = Q_{R_\mathfrak{m}} \in \mathfrak{R}_{R_\mathfrak{m}}.$$ ☐

Lemma A.15. *Let R be a semilocal ring. Then any finite ring extension $R \subseteq R'$ is semilocal.*

Proof. As R is Noetherian and R' is R-finite, R' is Noetherian too. Hence $R'/\mathfrak{m}R'$ is Noetherian as well for any $\mathfrak{m} \in \operatorname{Max}(R)$. As R' is finite over R, it is also an integral extension. Hence all primes of R' over $\mathfrak{m}$ are maximal (see [HS06, Lemma 2.1.7]). Then $\operatorname{Max}\operatorname{Spec}(R') = \cup_{\mathfrak{m}\in\operatorname{Max}(R)} \operatorname{Max}\operatorname{Spec}(R'/\mathfrak{m}R') = \cup_{\mathfrak{m}\in\operatorname{Max}(R)} \operatorname{Min}\operatorname{Spec}(R'/\mathfrak{m}R')$. As $|\operatorname{Max}(R)| < \infty$ because R semilocal and $|\operatorname{Min}\operatorname{Spec}(R'/\mathfrak{m}R')| < \infty$ because $R'/\mathfrak{m}R'$ is Noetherian for any $\mathfrak{m} \in \operatorname{Max}(R)$, we obtain $|\operatorname{Max}\operatorname{Spec}(R')| < \infty$ and hence R' semilocal. □

Lemma A.16. *Let R be a ring, and let $\mathcal{E}$ and $\mathcal{F}$ be R-submodules of Q_R. Then*

(a) $R \subseteq \widehat{R}$ is faithfully flat and hence $Q_R \subseteq Q_R \otimes_R \widehat{R} \subseteq Q_{\widehat{R}}$.

(b) If $\mathcal{E}$ is finitely generated, then $\mathcal{E} \otimes_R \widehat{R} = \mathcal{E}\widehat{R} \subseteq Q_R \otimes_R \widehat{R}$ and $\mathcal{E}\widehat{R} = \widehat{\mathcal{E}}$.

(c) $\mathcal{E}\widehat{R} \cap Q_R = \mathcal{E}$.

(d) $(\mathcal{E} \cap \mathcal{F})\widehat{R} = \mathcal{E}\widehat{R} \cap \mathcal{F}\widehat{R}$.

(e) If R is semilocal, then $\widehat{R} = \prod_{\mathfrak{m}\in\operatorname{Max}(R)} \widehat{R_\mathfrak{m}}$ is a product of local rings $\widehat{R_\mathfrak{m}} = \widehat{R}_\mathfrak{m} = \widehat{R}_{\widehat{\mathfrak{m}}}$.

(f) If R is semilocal and $R \subseteq R'$ is a finite ring extension, then $R' \otimes_R \widehat{R} = \widehat{R}'$.

Proof. (a) By [Mat89, Theorem 8.14], the completion $\widehat{R}$ with respect to the Jacobian radical is faithfully flat over R. By [Mat89, Theorem 8.10(i)], the topology associated to the Jacobian radical is Hausdorff, i.e. $\cap_{k\in\mathbb{Z}} J^k = (0)$. But $\cap_{k\in\mathbb{Z}} J^k$ is exactly the kernel of the map $R \to \widehat{R}$. Hence the map is injective, and $R \subseteq \widehat{R}$. Since $\widehat{R}$ is faithfully flat over R, $\widehat{R} \subseteq Q_R \otimes_R \widehat{R}$. Moreover, every non zero-divisor of R is a non zero-divisor of $\widehat{R}$, so $Q_R \otimes_R \widehat{R}$ is contained in $Q_{\widehat{R}}$.

(b) Since $\widehat{R}$ is a flat R-module and $\mathcal{E}$ is finitely generated, the map $\mathcal{E} \otimes_R \widehat{R} \to Q_R \otimes_R \widehat{R}$ is injective, and therefore $\mathcal{E} \otimes_R \widehat{R}$ can be identified with its image $\mathcal{E}\widehat{R}$. Hence $\mathcal{E} \otimes_R \widehat{R} = \mathcal{E}\widehat{R} \subseteq Q_R \otimes_R \widehat{R}$. The equality $\mathcal{E}\widehat{R} = \widehat{\mathcal{E}}$ follows from [Mat89, Theorem 8.7] , as for any finite R-module M there is an isomorphism $M \otimes_R \widehat{R} = \widehat{M}$.

(c) By [Bou89, Chapter I, §3, Proposition 10(ii)], if M is an R-module and M' is a submodule of M, then

$$M \cap \widehat{R}M' = M'.$$

Taking $M = Q_R$ and $M' = \mathcal{E}$ we obtain the claim.

(d) By [Bou89, Chapter I, §3, Proposition 10(iv)] if M is an R-module and M', M'' are two submodules of M, then

$$\widehat{R}(M' \cap M'') = \widehat{M}' \cap \widehat{R}M''.$$

Taking $M = Q_R$, $M' = \mathcal{E}$ and $M'' = \mathcal{F}$ gives the claim.

(e) By [Mat89, Theorem 8.15], $\widehat{R}$ decomposes as a direct product

$$\widehat{R} = \prod_{\mathfrak{m}\in\operatorname{Max}(R)} \widehat{R_\mathfrak{m}}.$$

To see that $\widehat{R}_\mathfrak{m} = \widehat{R}_{\widehat{\mathfrak{m}}}$ note that $\mathfrak{m}\widehat{R} = \widehat{\mathfrak{m}}$ by (b) and hence $\mathfrak{m} = \widehat{\mathfrak{m}} \cap R$ by (c).

(f) Observe first that here $\widehat{R}'$ has to be intended as the completion with respect to the Jacobian radical of R'. However, as $R \subseteq R'$ is a finite extension, by Lemma A.15 R' is semilocal, and by [Nag62, Theorem (16.8)] the topology of R' as semilocal ring coincides with that of R' as a finite R-module. Then the claim follows from [Mat89, Theorem 8.7], since for any finite R-module M there is an isomorphism $M \otimes_R \widehat{R} = \widehat{M}$. □

Lemma A.17. *Let R be a one-dimensional local Cohen–Macaulay ring. Then $Q_R\widehat{R} = Q_{\widehat{R}}$ and there is an inclusion preserving group isomorphism*

$$\begin{aligned} \mathfrak{R}_R &\to \mathfrak{R}_{\widehat{R}} \\ \mathcal{E} &\mapsto \widehat{\mathcal{E}} \\ \mathcal{F} \cap Q_R &\leftarrow\!\shortmid \mathcal{F}. \end{aligned}$$

Proof. By [KV04, Chapter II, (2.4)], since $R^{\mathrm{reg}} \subseteq \widehat{R}^{\mathrm{reg}}$, for any regular element $r \in \mathfrak{m}$ we have $Q_{\widehat{R}} = \widehat{R}[1/r]$, and hence $\widehat{Q}_R = Q_{\widehat{R}}$, as $\widehat{R}Q_R$ is the smallest subring of $Q_{\widehat{R}}$ containing $\widehat{R}$ and Q_R. Let $\mathfrak{m}$ be the maximal ideal of R. Surjectivity follows from [HK71, Lemma 2.11], as it states that for any $\widehat{\mathfrak{m}}$-primary ideal $\widehat{\mathcal{E}}$ there is an $\mathfrak{m}$-primary ideal $\mathcal{E}$ with $\widehat{\mathcal{E}} = \mathcal{E}\widehat{R}$. Notice that the same can be said for fractional ideals, since if $\widehat{\mathcal{E}}$ is a fractional ideal of $\widehat{R}$ there exists an $x \in Q_{\widehat{R}}^{\mathrm{reg}} = Q_R\widehat{R}$ such that $x\widehat{\mathcal{E}} \subseteq \widehat{R}$ and hence there is an $\mathfrak{m}$-primary ideal $\mathcal{E}$ with $\widehat{\mathcal{E}} = x^{-1}\mathcal{E}\widehat{R}$. Injectivity follows from Lemma A.16.(c). □

Lemma A.18. *Let $(R, \mathfrak{m})$ be a one-dimensional local Cohen–Macaulay ring, and let $R \subseteq R' \subseteq Q_R$ be a finite extension ring with $|R/\mathfrak{m}| \geq |\mathrm{Max}(R')|$, and $\mathcal{E} \in \mathfrak{R}_R$ be such that $\mathcal{E}R'$ is a cyclic R'-module. Then $\mathcal{E}R' = xR'$ for some $x \in \mathcal{E}^{\mathrm{reg}}$. In particular, $R \subseteq y\mathcal{E} \subseteq R'$ for $y = x^{-1} \in Q_R^{\mathrm{reg}}$.*

Proof. By Lemma A.15, R' is semilocal. Let $\mathcal{E}R'$ be a cyclic R'-module. Then $\mathcal{E}R' = zR'$ for some $z \in Q_R^{\mathrm{reg}}$. The lemma is proved if we show that $z \in \mathcal{E}^{\mathrm{reg}}$. Multiplying by $z^{-1}\mathcal{E}$ we get $z^{-1}\mathcal{E}R' = R'$, and the claim becomes there exists a unit $u \in R'$ such that $u \in z^{-1}\mathcal{E}$. This is proven by Jäger in [Jäg77, Hilfssatz 2]. Thus $\mathcal{E}R' = zR' = zuR'$, and in particular $R \subseteq (zu)^{-1}\mathcal{E} \subseteq (zu)^{-1}\mathcal{E}R' = R'$. Hence $x = zu$ gives the claim. □

Lemma A.19. *Let $(R, \mathfrak{m})$ be a one-dimensional local Cohen–Macaulay ring, and let $\mathcal{E} \in \mathfrak{R}_R$. Then $\mathcal{E}$ is a faithful maximal Cohen–Macaulay module.*

Proof. Let $\mathbf{k} = R/\mathfrak{m}$. As $\mathcal{E}$ is a regular fractional ideal, there exists a $y \in \mathcal{E}^{\mathrm{reg}}$ such that $xy = 0$ implies $x = 0$. Thus the only zero multiplication map in $\mathrm{End}(\mathcal{E})$ is the one coming from zero itself. Hence the map $R \to \mathrm{End}_R(\mathcal{E})$ is injective, and $\mathcal{E}$ is a faithful module. By definition

$$\dim(\mathcal{E}) = \dim(R/\operatorname{Ann}_R(\mathcal{E})) = \dim(R/\ker(R \to \mathrm{End}_R(\mathcal{E}))) = \dim(R) = 1.$$

Hence, as $y \in \mathcal{E}^{\mathrm{reg}}$, we get

$$1 = \dim(\mathcal{E}) \geq \operatorname{depth}_R(\mathcal{E}) \geq 1.$$

Hence $\mathcal{E}$ is a maximal Cohen–Macaulay module. □

B
Valuations

In this chapter we give basic definitions and facts on valuations and valuation rings. All rings considered are Noetherian commutative and unitary. For other references see [KV04, Mat73, CDK94].

Definition B.1. A ring Q has *a large Jacobson radical* if every prime ideal of Q containing the Jacobson radical of Q is a maximal ideal (see [KV04, Chapter I, Proposition (1.9)] for equivalent characterizations).

For example, any semilocal ring has a large Jacobson radical (see [KV04, Chapter I, Remark (1.11)].

Lemma B.2. *Let Q be a ring with large Jacobson radical and which is its own ring of quotients. Then every ring having Q as ring of quotients is a Marot ring (see Definition A.5.*(c)*).*

Proof. See [KV04, Chapter I, Proposition (1.12)]. □

From now on we will always assume Q is a ring with large Jacobson radical and which is its own ring of quotients. In particular, Q satisfies (A.1).

Theorem B.3. *Let $V \subsetneq Q$ be a subring of Q having Q as its ring of quotients. The following are equivalent:*

(i) There exists a prime ideal $\mathfrak{p}$ of V such that $\mathfrak{p}W = W$ for any subring W of Q properly containing V;

(ii) $Q \setminus V$ is a multiplicatively closed set;

(iii) for any $x \in Q^{\mathrm{reg}}$, either $x \in V$ or $x^{-1} \in V$;

(iv) The set of regular V-submodules of Q is totally ordered by reverse inclusion.

If these conditions are satisfied, then there is a unique regular maximal ideal $\mathfrak{m}_V$ of V. In particular, $V^{\mathrm{reg}} \setminus V^* \subseteq \mathfrak{m}_V$.

Proof. See [KV04, Chapter I, Theorem (2.2)] □

Definition B.4. A subring $V \subsetneq Q$ having Q as ring of quotients is called a *(Manis) valuation ring of Q* or *pseudo-valuation ring of Q* if it satisfies the equivalent conditions of Theorem B.3. The maximal ideal $\mathfrak{m}_V$ of V such that $\mathfrak{m}_V \supseteq V^{\mathrm{reg}} \setminus V^*$ is called the *regular maximal ideal of V*.

In the following for simplicity we will refer to Manis valuation rings simply as valuation rings.

Proposition B.5. *Let V be a valuation of Q. Then*

(a) V is integrally closed, and every proper subring of Q containing V is a valuation ring of Q.

(b) Every finitely generated regular V-submodule of Q is cyclic.

(c) Let $\mathfrak{m}_V$ be the regular maximal ideal of V. Then the conductor $(V :_Q Q) \subseteq \mathfrak{m}_V$ of V in Q is a prime ideal of V and of Q, and it is the intersection of all regular ideals of V.

(d) If $x \in Q \setminus (V :_Q Q)$, then the intersection $M(x)$ of all regular V-submodules of Q containing x is a regular cyclic V-submodule of Q, and for any regular $z \in Q$, we have $M(x) = zV$ if and only if $xz^{-1} \in V \setminus \mathfrak{m}_V$.

Proof. See [KV04, Chapter I, Proposition (2.4)]. □

Notation B.6. Let $(\Gamma, <)$ be a totally ordered (additive) abelian group. Set $\Gamma_\infty := \Gamma \cup \{\infty\}$. We make Γ_∞ into a totally ordered monoid containing Γ by defining $\gamma < \infty$ for any $\gamma \in \Gamma$ and $\gamma + \infty = \infty$ for any $\gamma \in \Gamma_\infty$.

Definition B.7. A surjective map $\nu : Q \to \Gamma_\infty$ such that $\nu(1) = 0$, $\nu(0) = \infty$ and

(V1) $\nu(xy) = \nu(x) + \nu(y)$ and

(V2) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$

for any $x, y \in Q$ is called a *(Manis) valuation* of Q. If $\Gamma = \mathbb{Z}$, then ν is called a *discrete (Manis) valuation*.

For simplicity in the following we refer to Manis valuations simply as valuations.

Remark B.8. A valuation $\nu : Q \to \Gamma_\infty$ has the following properties:

(a) $\nu(-x) = \nu(x)$ for any $x \in Q$.

(b) $\nu(x^{-1}) = -\nu(x) \neq \infty$ for any $x \in Q^{\mathrm{reg}}$ (see [KV04, Chapter I, Remark 2.9]).

(c) If $\nu(x) \neq \nu(y)$, then $\nu(x + y) = \min\{\nu(x), \nu(y)\}$ (see [KV04, Chapter I, Proposition 2.10]).

Definition B.9. Let $\nu : Q \to \Gamma$ be a valuation. Then $V_\nu = \{x \in Q \mid \nu(x) \geq 0\}$ is a subring of Q and it is called the *ring of ν*.

Notation B.10. By [KV04, Chapter I, Lemma (2.1)], if V is a valuation ring of Q, then $Q_V = Q$. Thus, in analogy with Notation A.6, we denote by $\mathfrak{R}_V$ the set of finitely generated regular V-submodules of Q, and by $\mathfrak{R}_V^*$ the group of invertible V-submodules of Q.

Remark B.11. By Proposition B.5.(b), $\mathfrak{R}_V$ consists only of cyclic modules. Moreover, by Theorem B.3.(iv), the group $\mathfrak{R}_V^*$ is totally ordered by reverse inclusion, let us denote by $<$ the order relation.

Definition B.12. Let V be a valuation ring of Q. The *infinite prime ideal* of V

$$I_V := V :_Q Q = \bigcap_{\mathcal{E} \in \mathfrak{R}_V^*} \mathcal{E} \in \operatorname{Spec}(V) \cap \operatorname{Spec}(Q)$$

is the intersection of all regular (principal) fractional ideals of V (see Proposition B.5.(c)).

We can include $\mathfrak{R}_V^*$ into the totally ordered monoid

$$\mathfrak{R}_{V,\infty}^* = \mathfrak{R}_V^* \cup \{I_V\}\,.$$

Then, by definition of I_V, $\mathcal{E} < I_V$ for any $\mathcal{E} \in \mathfrak{R}_V^*$. Moreover, $\mathcal{E} \cdot I_V = \mathcal{E} \cap I_V = I_V$ for $\mathcal{E} \in \mathfrak{R}_{V,\infty}^*$.

Definition B.13. Let V be a valuation ring. Then we can define a map

$$\begin{aligned} \mu_V : Q &\to \mathfrak{R}_V \\ x &\mapsto \bigcap_{\substack{\mathcal{E}\in\mathfrak{R}_V \\ x\in\mathcal{E}}} \mathcal{E}. \end{aligned}$$

If $x \in Q \setminus I_V$, then by Proposition B.5.(d), $\mu_V(x) \in \mathfrak{R}_V^*$ and $\mu_V(x) = xV$. Then the map

$$\begin{aligned} \mu_V \colon Q &\to \mathfrak{R}_{V,\infty}^* \\ x &\mapsto \begin{cases} xV & \text{if } x \in Q \setminus I_V \\ I_V & \text{if } x \in I_V. \end{cases} \end{aligned}$$

is a valuation in the sense of Definition B.7 (but with multiplicative notaton for the group $\Gamma_\infty = \mathfrak{R}_{V,\infty}^*$, see Notation B.6). Moreover, μ_V is surjective as every invertible regular V-submodule of Q is finitely generated by Lemma A.11 and hence cyclic by Proposition B.5.(b).

Then definition of μ_V implies

$$V = \{x \in Q \mid \mu_V(x) \geq V\}\,. \tag{B.1}$$

By Theorem B.3, the units of V are

$$V^* = \{x \in Q^{\mathrm{reg}} \mid \mu_V(x) = V\} = (V \setminus \mathfrak{m}_V)^{\mathrm{reg}}$$

and the regular maximal ideal is

$$\mathfrak{m}_V = \{x \in Q \mid \mu_V(x) > V\}\,. \tag{B.2}$$

B.1 Discrete valuations

According to Notation B.6, we write $\mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$.

Proposition B.14. *Let V be a valuation ring of Q, and let $\mathfrak{m}_V$ be the regular maximal ideal of V. The following are equivalent:*

(i) V is the ring of a discrete valuation $\nu_V : Q \to \mathbb{Z}_\infty$;

(ii) Every regular ideal of V is finitely generated;

(iii) The ideal $\mathfrak{m}_V$ is finitely generated, and is the only regular prime ideal of V.

If these properties are satisfied, then:

(a) Every regular ideal of V is a principal ideal.

(b) *Let $\mathfrak{m}_V = tV$. Then t is a regular element, $Q = V[t^{-1}]$ and every regular element x of Q has a unique representation $x = at^k$ with $a \in V^*$ and $k \in \mathbb{Z}$. Moreover,*

$$\{\mathfrak{m}_V^k = t^k V \mid k \in \mathbb{Z}\} = \mathfrak{R}_V^*.$$

(c) *$V :_Q Q = \bigcap_{n\in\mathbb{N}} \mathfrak{m}_V^n$.*

(d) *Let W be a subring of Q containing V. Then either $W = V$ or $W = Q$.*

Proof. See [KV04, Chapter I, Proposition (2.15)]. □

Definition B.15. A valuation ring V of Q is called a *discrete valuation ring* if it satisfies the equivalent conditions of Proposition B.14.

The valuation ν_V associated to a discrete valuation ring V of Q is discrete with valuation ring $V_\nu = V$.

Theorem B.16 (Approximation theorem for discrete valuations)**.** *Let $V_1, \dots, V_n$ be pairwise distinct discrete valuation rings of Q. For any $i \in \{1, \dots, n\}$ let $\nu_i : Q \to \mathbb{Z}_\infty$ be the valuation of Q defined by V_i and let $\mathfrak{m}_{V_i}$ be the regular maximal ideal of V_i. Set $S := V_1 \cap \dots \cap V_n$ and $\mathfrak{m}_i = S \cap \mathfrak{m}_{V_i}$ for any $i \in \{1, \dots, n\}$. Then:*

(a) *The prime ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are regular maximal ideals of S which are pairwise distinct, and they are all the regular prime ideals of S. Moreover, for any $i \in \{1, \dots, n\}$, we have $S_{[\mathfrak{m}_i]} := \{x \in Q \mid sx \in S \text{ for some } s \in S \setminus \mathfrak{m}_i\} = V_i$ and $\mathfrak{m}_i V_i = \mathfrak{m}_{V_i}$.*

(b) *For any $a_1, \dots, a_n \in Q$ and $m_1, \dots, m_n \in \mathbb{Z}$ there exists an $a \in Q$ such that*

$$\nu_i(a - a_i) \geq m_i$$

for any $i \in \{1, \dots, n\}$.

(c) *For any $m_1, \dots, m_n \in \mathbb{Z}$ there exists an $a \in Q$ such that*

$$\nu_i(a) = m_i$$

for any $i \in \{1, \dots, n\}$.

Proof. See [KV04, Chapter I, Theorem 2.20]. □

The proof of Theorem B.16 makes use of the Chinese Remainder Theorem and of Proposition B.14.

Assume V is a discrete valuation ring of Q. Then by Proposition B.14 the regular maximal ideal $\mathfrak{m}_V = \min\{\mathcal{E} \in \mathfrak{R}_V^* \mid \mathcal{E} > V\}$ is the only regular finitely generated prime ideal of V. In particular, by Proposition B.14.(b), $\mathfrak{m}_V \in \mathfrak{R}_V^*$ and $\{\mathfrak{m}_V^k \mid k \in \mathbb{Z}\} = \mathfrak{R}_V^*$. Thus there is a unique order preserving group isomorphism

$$\begin{aligned} \phi_V \colon \mathfrak{R}_V^* &\xrightarrow{\cong} \mathbb{Z}, \\ \mathcal{E} &\mapsto \max\{j \in \mathbb{Z} \mid \mathfrak{m}_V^j \leq \mathcal{E}\}, \\ \mathfrak{m}_V^k &\leftarrow\!\shortmid k. \end{aligned} \tag{B.3}$$

In fact, Proposition B.14.(a) yields $\mathcal{E} \in \mathfrak{R}_V^*$ principal, i.e. $\mathcal{E} = xV$ with $x \in Q^{\text{reg}}$. Then Proposition B.14.(b) implies $x = at^k$ with t such that $\mathfrak{m}_V = tV$ and $a \in V^*$. This gives $\mathcal{E} = \mathfrak{m}_V^k$. Hence $k = \max\{j \in \mathbb{Z} \mid \mathfrak{m}_V^j \leq \mathcal{E}\} < \infty$. Clearly, ϕ_V is a surjective map.

We can extend ϕ_V to $\mathbb{Z}_\infty$ by setting

$$\phi_V(I_V) := \infty. \tag{B.4}$$

This yields a commutative diagram

$$\begin{tikzcd} Q \arrow[d, two heads, "\mu_V"'] \arrow[dr, two heads, "\nu_V"] & \\ \mathfrak{R}^*_{V,\infty} \arrow[r, "\cong", "\phi_V"'] & \mathbb{Z}_\infty \end{tikzcd} \tag{B.5}$$

where μ_V is defined as in Definition B.13.

C
Modules

Let R be a Noetherian commutative and unitary ring.

Definition C.1. A chain of R-submodules of an R-module M is a sequence $(M_i)_{i=1}^n$ of R-submodules of M such that

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0.$$

A *composition series of* M is a maximal chain, i.e. M_{i-1}/M_i is simple (has no submodules except 0 and itself) for any i. If M has a composition series, then all composition series have the same length n. In this case we say that M has finite length, and the *length of* M is $\ell(M) = n$.

Definition C.2. Let R be a graded ring, and let M be a graded R-module whose graded components M_n have finite length for any n. The numerical function

$$\begin{aligned} \mathrm{HF}_M(-) : \mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto \ell(M_n) \end{aligned}$$

is the *Hilbert function of* M. The *Hilbert series of* M is

$$\mathrm{HS}_M(t) = \sum_{n\in\mathbb{Z}} HF_M(n)t^n.$$

Definition C.3. Let $(R, \mathfrak{m})$ be a local ring. Then the *Hilbert function of* R and the *Hilbert series of* R are the ones of its associated graded ring, i.e.

$$\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i\geq 0} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}.$$

Proposition C.4. *Let $M \neq 0$ be a finite graded R-module of dimension d. Then there exists a unique $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) \neq 0$ such that*

$$HS_M(t) = \frac{Q_M(t)}{(1-t)^d}.$$

Moreover, if $Q_M(t) = \sum_i h_i t^i$ then $\min\{i \mid h_i \neq 0\}$ is the least number such that $M_i \neq 0$

Proof. See [BH93, Corollary 4.1.8]. □

Definition C.5. Let M be a finite graded R-module of dimension d. Let $Q_M(t)$ be as in Proposition C.4. The *multiplicity of* M is the integer

$$e(M) = Q_M(1).$$

If $d = 0$, then $e(M) = \ell(M)$.

Definition C.6. Let $(R, \mathfrak{m}, \mathbf{k})$ be a local ring, and M a finite R-module. Then the *depth of* M is

$$\mathrm{depth}(M) := \min\{i \mid \mathrm{Ext}^i_R(\mathbf{k}, M) \neq 0\} = \max\{\text{ length of maximal } M\text{-sequences in } \mathfrak{m}\}.$$

Definition C.7. A ring R is called *Cohen–Macaulay* if $\dim(R) = \mathrm{depth}(R)$.

Lemma C.8. *Let $(R, \mathfrak{m})$ be a local ring, M a finite R-module and $\hat{M}$ its $\mathfrak{m}$-adic completion. Then M is Cohen–Macaulay if and only if $\hat{M}$ is Cohen–Macaulay.*

Proof. See [BH93, Corollary 2.1.8]. □

D
Injective modules and Matlis duality

Let R be a commutative and unitary ring.

Definition D.1. Let R be a commutative ring and let E be an R-module. We say that E is *injective* if $\operatorname{Hom}_R(-, E)$ is an exact functor.

As a consequence of this definition, a module E is injective if and only if for any injective morphisms of R-modules $h : M \to N$ and for any morphisms $f : M \to E$, there exists a morphism $g : N \to E$ making the following diagram commutative

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "h"] & N. \arrow[ul, dashed, "g"'] \end{tikzcd}$$

Proposition D.2. *Let E, M, N be R-modules. Then:*

(a) If E is injective, then every short exact sequence of type

$$0 \to E \to M \to N \to 0$$

splits.

(b) If $E \subseteq M$ is an injective submodule of a module M, then E is a direct summand of M.

(c) If $\{E_j\}_{j \in J}$ is a family of injective R-modules, then $\prod_{j \in J} E_j$ is also an injective module.

(d) Every direct summand of an injective R-module is injective.

(e) A finite direct sum of injective R-modules is injective.

There is a criterion to check the injectivity of a module:

Proposition D.3 (Baer's criterion)**.** *An R-module E is injective if and only if for any ideal I of R, every homomorphism $f : I \to E$ can be extended to R.*

Proof. See [Eis95, Lemma A3.4]. □

Theorem D.4. *Any R-module M can be embedded as a submodule of an injective module.*

Proof. See [Eli13, Theorem 1.9]. □

Definition D.5. Let $N \subseteq M$ be R-modules. We call M an *essential extension of* N if for any non-zero submodule U of M we have $U \cap N \neq 0$. An essential extension is *proper* if $N \neq M$.

The following proposition relates essential extensions and injective modules:

Proposition D.6. *Let N and M be R-modules. Then:*

(a) The module N is injective if and only if it has no proper essential extensions.

(b) If $N \subseteq M$ is an essential extension, and E an injective R-module with $E \supseteq N$. Then there exists a monomorphism $\phi : M \to E$ extending the inclusion $N \subseteq M$.

Proof. See [Eli13, Proposition 1.11]. □

Definition D.7. Let R be a ring and M an R-module. An *injective hull of* M is an injective module $E_R(M)$ such that $M \subseteq E_R(M)$ is an essential extension.

Proposition D.8. *Let M be an R-module. Then:*

(a) M admits an injective hull.

(b) If $M \subseteq E$ and E in injective, then a maximal essential extension of M in E is an injective hull of M.

(c) Let $E_R(M)$ be an injective hull of M, and let $\alpha : M \to E$ be a monomorphism, with E injective R-module. Then there exists a monomorphism $\varphi : E_R(M) \to E$ such that the following diagram is commutative

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\alpha"] & E_R(M) \arrow[dl, "\varphi"] \\ E & \end{tikzcd}$$

i.e., the injective hulls of M are the minimal injective modules in which M can be embedded.

(d) If $E_R(M)$ and $E_R(M)'$ are injective hulls of M, then there exists an isomorphism φ : $E_R(M) \to E_R(M)'$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[d] \arrow[dr] & \\ E_R(M) \arrow[r, "\varphi"] & E_R(M)'. \end{tikzcd}$$

Proof. See [Eli13, Proposition 1.13]. □

Through this proposition, we can build an injective resolution $E^\bullet(M)$ of a module M. Let $E^0(M) = E_R(M)$ and denote the embedding by ∂^{-1}. If the injective resolution has been constructed till the i-th step:

$$0 \to E^0(M) \overset{\partial^0}{\to} E^1(M) \overset{\partial^1}{\to} \cdots \to E^{i-1}(M) \overset{\partial^{i-1}}{\to} E^i(M)$$

then we define $E^{i+1} = E_R(\operatorname{coker} \partial^{i-1})$, and ∂^i is defined as the inclusion.

Definition D.9. Let M be an R-module. The *injective dimension* of M is the smallest integer n for which there exists an injective resolution $E^\bullet$ of M with $E^m = 0$ for $m > n$. If there is no such an n, the injective dimension of M is infinite.

Definition D.10. A Noetherian local ring R is a *Gorenstein ring* if its injective dimension is finite. A Noetherian ring is a *Gorenstein ring* if its localization at every maximal ideal is a Gorenstein local ring.

Definition D.11. Let $(R, \mathfrak{m}, \mathbf{k})$ be a local ring. Given an R-module M the *Matlis dual* of M is defined as $M^\vee = \operatorname{Hom}_R(M, E_R(\mathbf{k}))$. The functor $(-)^\vee := \operatorname{Hom}_R(-, E_R(\mathbf{k}))$ is a contravariant exact functor from the category of R-modules into itself.

Proposition D.12. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a local ring. Then the functor $(-)^\vee$ is a faithful functor. Moreover, if M is a R-module of finite length, then $\ell(M^\vee) = \ell(M)$. If R is an Artinian ring, then $\ell(E_R(\mathbf{k})) = \ell(R) < \infty$.*

Proof. See [Eli13, Proposition 1.16]. □

Theorem D.13 (Matlis duality). *Let $(R, \mathfrak{m}, \mathbf{k})$ be a complete Noetherian local ring and let M be a R-module. Then:*

(a) If M is finite, then $M^\vee$ is Artinian, and vice versa.

(b) If M is either finite or Artinian then $M^{\vee\vee} \cong M$.

Proof. See [Eli13, Theorem 1.21]. □

E
Canonical modules

Definition E.1. Let $(R, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring, and M a finite non-zero R-module of depth t. The *type of* M is the number

$$\tau(M) = \dim_{\mathbf{k}}(\operatorname{Ext}_R^t(\mathbf{k}, M)).$$

Definition E.2. Let $(R, \mathfrak{m}, \mathbf{k})$ be a Cohen–Macaulay local ring. A *canonical module of* R is a maximal Cohen–Macaulay module C of type 1 and of finite injective dimension.

Theorem E.3. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a Cohen–Macaulay local ring. Then a canonical module is unique up to isomorphism. In particular, if $\dim(R) = 0$, then $E_R(\mathbf{k})$ is the uniquely determined canonical module.*

Proof. See [BH93, Theorem 3.3.4]. □

If R is a Cohen–Macaulay local ring, we denote the unique (up to isomorphism) canonical ideal with ω_R.

Theorem E.4. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring.*

(a) The following conditions are equivalent:

(i) R is Gorenstein;

(ii) ω_R exists and is isomorphic to R.

(b) Let $I \subseteq R$ be an ideal with $\operatorname{ht}(I) = k$. Then

$$\operatorname{Ext}_R^k(R/I, \omega_R) \cong \omega_{R/I}.$$

Proof. See [BH93, Theorem 3.3.7]. □

Proposition E.5. *The following conditions are equivalent:*

(i) A is a Gorenstein ring;

(ii) A is a Cohen–Macaulay ring of type 1.

Proof. See [BH93, Theorem 3.2.10]. □

E.1 Canonical modules of graded rings

We say that a graded ring is a *local ring if it has a unique graded ideal $\mathfrak{m}$ which is not properly contained in any graded proper ideal.

Definition E.6. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay *local ring of dimension d. A finite graded R-module C is a **canonical module* of R if there exist homogeneous isomorphisms

$$\operatorname{Ext}^i_R(R/\mathfrak{m}, C) \cong \begin{cases} 0 & \text{if } i \neq d \\ R/\mathfrak{m} & \text{if } i = d. \end{cases}$$

Proposition E.7. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay *local ring, and C be a *canonical module of R. Then C is a canonical module of R and, if $\mathfrak{m}$ is maximal, it is uniquely determined up to homogeneous isomorphism.*

Proof. See [BH93, Proposition 3.6.9] □

Remark E.8. Let $R = \mathbf{k}[x_1, \dots, x_n]$ be a polynomial ring over a field, with $\deg x_i = a_i > 0$ for $i = 1, \dots, n$. The *maximal ideal of R is the ideal $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$. Then the Koszul complex of $x_1, \dots, x_n$ yields a homogeneous free resolution of $R/\mathfrak{m}$ whose last term is $R(-\sum_{i=1}^n a_i)$. Hence $\operatorname{Ext}^i_R(R/\mathfrak{m}, R) = 0$ for $i \neq n$ and the *canonical module of R is $\operatorname{Ext}^n_R(R/\mathfrak{m}, R) = R(-\sum_{i=1}^n a_i)$.

Proposition E.9. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay *local ring with *canonical modules ω_R. Then the following conditions are equivalent:*

(i) R is Gorenstein;

(ii) $\omega_R \cong R(a)$ for some integer $a \in \mathbb{Z}$.

Proof. See [BH93, Proposition 3.6.11]. □

A result analogous to Theorem E.4.(b) holds also for graded rings.

E.2 Canonical ideals

Let R be a one-dimensional Cohen–Macaulay ring. In the following we recall some basics from the theory of canonical ideals of R. We begin with a definition (see [HK71, Definition 2.4]).

Definition E.10. Let R be a one-dimensional Cohen–Macaulay ring. We call $\mathcal{K} \in \mathfrak{R}_R$ a *canonical (fractional) ideal* of R if, for any $\mathcal{E} \in \mathfrak{R}_R$,

$$\mathcal{E} = \mathcal{K} : (\mathcal{K} : \mathcal{E}) \tag{E.1}$$

or, equivalently, $\mathcal{E} = \operatorname{Hom}_R(\operatorname{Hom}_R(\mathcal{E}, \mathcal{K}), \mathcal{K})$ (see Lemma A.13). In particular,

$$R = \mathcal{K} : \mathcal{K}. \tag{E.2}$$

Remark E.11. Let R be a one-dimensional Cohen–Macaulay ring. Then

(a) $\mathcal{K}$ is a canonical ideal of R if and only if $K_\mathfrak{m} = \mathcal{K}R_\mathfrak{m} \in \mathfrak{R}_{R_\mathfrak{m}}$ is a canonical ideal of $R_\mathfrak{m}$ for any $\mathfrak{m} \in \operatorname{Max}(R)$.

This fact is proven in [HK71, Lemma 2.6], but follows also from Lemma A.13 and Lemma A.14.

(b) The functor $(\mathcal{K} : -)$ preserves lengths, i.e. $\ell(\mathcal{K} : \mathcal{E}/\mathcal{K} : \mathcal{F}) = \ell(\mathcal{F}/\mathcal{E})$ for any $\mathcal{E}, \mathcal{F} \in \mathfrak{R}_R$.
This fact is proven in [HK71, Remark 2.5.(c)].

Lemma E.12. *Let R be a one-dimensional Cohen–Macaulay ring. Then a canonical ideal $\mathcal{K}$ is a canonical module.*

Proof. By [BH93, Proposition 3.3.13 and Def. 3.3.16] it is sufficient to prove that $\mathcal{K}_\mathfrak{m}$ is a faithful maximal Cohen–Macaulay module of $R_\mathfrak{m}$ of type 1 for any $\mathfrak{m} \in \operatorname{Max}(R)$. By Remark E.11.(a) $\mathcal{K}_\mathfrak{m}$ is a canonical ideal and $\mathcal{K}_\mathfrak{m} \in \mathfrak{R}_{R_\mathfrak{m}}$ for any $\mathfrak{m} \in \operatorname{Max}(R)$. Then, by Lemma A.19, $\mathcal{K}_\mathfrak{m}$ is a faithful maximal Cohen–Macaulay module of $R_\mathfrak{m}$. We only need to prove that $\mathcal{K}_\mathfrak{m}$ is of type one. Let $\mathfrak{m} \in \operatorname{Max}(R)$ and let $\mathbf{k} = R_\mathfrak{m}/\mathfrak{m}$. Let us consider the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow R_\mathfrak{m} \longrightarrow \mathbf{k} \longrightarrow 0.$$

Applying $\operatorname{Hom}(-, \mathcal{K}_\mathfrak{m})$ we obtain the exact sequence

$$0 \longleftarrow \operatorname{Ext}^1_{R_\mathfrak{m}}(\mathbf{k}, \mathcal{K}_\mathfrak{m}) \longleftarrow \mathcal{K}_\mathfrak{m} : \mathfrak{m} \longleftarrow \mathcal{K}_\mathfrak{m} : R_\mathfrak{m} \longleftarrow 0.$$

Then by Definition E.1 Remark E.11.(b)

$$\tau(\mathcal{K}_\mathfrak{m}) = \dim_\mathbf{k} \operatorname{Ext}^1_{R_\mathfrak{m}}(\mathbf{k}, \mathcal{K}_\mathfrak{m}) = \ell(\operatorname{Ext}^1_{R_\mathfrak{m}}(\mathbf{k}, \mathcal{K}_\mathfrak{m})) = \ell(\mathcal{K}_\mathfrak{m} : \mathfrak{m}/\mathcal{K}_\mathfrak{m} : R_\mathfrak{m}) = \ell(R_\mathfrak{m}/\mathfrak{m}) = 1.$$

□

As a consequence of Lemma E.12, if R is a canonical ideal, then R is Gorenstein (see Theorem E.4.(a)).

Canonical ideals are unique up to projective factors.

Proposition E.13. *Let R be a one-dimensional Cohen–Macaulay ring with a canonical ideal $\mathcal{K}$. Then $\mathcal{K}'$ is a canonical ideal of R if and only if $\mathcal{K}' = \mathcal{E}\mathcal{K}$ for some invertible ideal $\mathcal{E}$ of R. In case R is semilocal, the latter condition becomes $\mathcal{K}' = x\mathcal{K}$ for some $x \in Q_R^{\mathrm{reg}}$.*

Proof. By [HK71, Satz 2.8], if $\mathcal{K}$ is a canonical ideal of R, then $\mathcal{E}\mathcal{K}$ is a canonical ideal of R for any $\mathcal{E}$ projective (and therefore invertible) fractional ideal of R. Moreover, again by [HK71, Satz 2.8], if $\mathcal{K}$ and $\mathcal{K}'$ are two canonical ideals of R, then there exist a projective ideal $\mathcal{E}$ such that $\mathcal{K}' = \mathcal{E}\mathcal{K}$. The claim in case R is semilocal follows from Lemmas A.11 and A.12, which say that every invertible fractional ideal of R is regular and principal. □

Lemma E.14. *Let R be a one-dimensional local Cohen–Macaulay ring. Then R has a canonical ideal $\mathcal{K}$ if and only if its completion $\widehat{R}$ has a canonical ideal $\widehat{\mathcal{K}}$.*

Proof. Since by Lemma A.16.(a) $R \subseteq \widehat{R}$ is faithfully flat, and Hom is compatible with flat base change, we have

$$\operatorname{Hom}_R(\operatorname{Hom}_R(\mathcal{E}, \mathcal{K}), \mathcal{K})\widehat{R} = \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{\mathcal{E}}, \widehat{\mathcal{K}}), \widehat{\mathcal{K}}).$$

Thus, by Lemma A.13 and since the map in Lemma A.17 is an isomorphism (here we need the local assumption), it is clear that $\mathcal{K}$ is a canonical ideal for R if and only if $\mathcal{E} = \operatorname{Hom}_R(\operatorname{Hom}_R(\mathcal{E}, \mathcal{K}), \mathcal{K})$ for any $\mathcal{E} \in \mathfrak{R}_R$ if and only if $\widehat{\mathcal{E}} = (\operatorname{Hom}_{\widehat{R}}(\widehat{\mathcal{E}}, \widehat{\mathcal{K}}), \widehat{K})$ for any $\widehat{\mathcal{E}} \in \mathfrak{R}_{\widehat{R}}$ if and only if $\widehat{\mathcal{K}}$ is a canonical ideal for $\widehat{R}$ (see also [HK71, Lemma 2.10]). □

In case R is local, the existence of a canonical ideal of R can be characterized as follows.

Theorem E.15. *A one-dimensional local Cohen–Macaulay ring R has a canonical ideal if and only if $\widehat{R}$ is generically Gorenstein. In particular, any one-dimensional analytically reduced local Cohen–Macaulay ring has a canonical ideal.*

Proof. See [HK71, Korollar 2.12 and Satz 6.21]. □

Canonical ideals propagate along finite ring extensions (see [BH93, Theorem 3.3.7.(b)]).

Lemma E.16. *Let $\varphi\colon R \to R'$ be a local homomorphism of one-dimensional local Cohen–Macaulay rings such that R' is a finite R-module and $Q_R = Q_{R'}$. If $\mathcal{K}_R$ is a canonical ideal of R, then $\mathcal{K}_R : R'$ is a canonical ideal of R'.*

Proof. By Lemma A.2.(a) $(\mathcal{K}_R : R') : \mathcal{E}R' = \mathcal{K}_R : \mathcal{E}R'$ for any $\mathcal{E} \in \mathfrak{R}_R$. Hence

$$(\mathcal{K}_R : R') : ((\mathcal{K}_R : R') : \mathcal{E}R') = \mathcal{K}_R : (\mathcal{K}_R : \mathcal{E}R') = (\mathcal{K}_R : (\mathcal{K}_R : \mathcal{E}))R' = \mathcal{E}R'$$

Since for any $\mathcal{E}' \in \mathfrak{R}_{R'}$, $\mathcal{E}' = \mathcal{E}R'$, we get that $\mathcal{K}_R : R'$ is a canonical ideal according to Definition E.10. □

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