Convex Operators in Vector Optimization: Directional Derivatives and the Cone of Decrease Directions

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Abstract

The paper is devoted to the investigation of directional derivatives and the cone of decrease directions for convex operators on Banach spaces. We prove a condition for the existence of directional derivatives which does not assume regularity of the ordering cone *K*. This result is then used to prove that for continuous convex operators the cone of decrease directions can be represented in terms of the directional derivatives. Decrease directions are those for which the directional derivative lies in the negative interior of the ordering cone *K*. Finally, we show that the continuity of the convex operator can be replaced by its *K*-boundedness.

Key words: Vector optimization, convex operator, directional derivative, decrease direction, normal cone,

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1. Introduction

In optimization theory it is well known that convex functions as well as convex sets in play an important role. Convexity assumptions lead to significantly stronger reults than hold for general optimizaion problems. We refer to [1] and [24] for a thorough treatment of convex analysis and to [24] and [25] for a summary of important results in real-valued convex optimization. Furthermore, in recent years interest in optimization of vector valued functions has grown. Vector optimization theory has been widely developed in recent years, optimization problems with respect to vector-valued functions have been explored, and various solution approaches have been characterized and computed (see, for example, [2-12] and references therein).

In 1963 Zadeh (see [13]) presented the need to design control systems in a multiple objective optimization framework. He observed that it is impossible to describe the quality of a control system by one criterion only. For example, in economic systems one has to estimate obvious parameters (reliability, prime cost payoff, quality of the product, etc.) as well as ecological damages, optimal connections between suppliers, etc.

In such vector optimization problems the function to be minimized, can obtain its values not only in a finite dimensional vector space but also in an infinite dimensional one. The preference order in the space is assumed to be given by a convex cone, which is referred to as a domination cone.

The theory of vector optimization is in a close relation with the theory of setvalued mappings, ory, which is a natural generalization of the classical analysis, see [14]. At the present the theory of set-valued mappings is widely used in many branches of science and technique. The nonlinear analysis constructed on the basis of this theory has become a rather convenient tool to solve many problems. It has been especially productive in mathematical economics, game theory, variation calculus, optimal control theory, etc. In [15, 16] application of setvalued mapping theory has been explored as an instrument for nonscalar optimization problems (see also [19]) and for general vector optimization problems

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Thus optimization problems in infinite dimensional spaces arise in various fields. Therefore development of techniques for solving such problems is a worthwhile research topic. One of the question is the generalization of the well developed theory of directional derivatives to infinite dimensional spaces. Research in that area has been started by the work of Valdier (see [22]). Various conditions for the existence of directional derivatives have been developed, see e.g. [23].

Our paper is devoted to the further investigation of the properties of convex operators in ordered spaces. In Section 3 we prove a new condition for the existence of directional derivatives, which does not assume regularity properties of the ordering cone. This section is proceed by some initial results presented in Section 2. In Section 4 we study the structure of the cone of decrease directions for convex operators in Banach spaces. An analythical representation of this cone is obtained. In this way corresponding results of Miliutin and Dubovtskiy, developed for singlecriteria optimization problems, are generalized for the vector valued case. Finally, in Section 5 we prove a sufficient condition for a convex operator to be continuous, which is the main assumption in Thorem 2 of Section 4.

Throughout the paper, we denote the positive orthant of the Euclidian space R^n by R^n_+.

2. Initial Facts and Findings

Suppose that Y is a Banach space and that a convex cone K in Y is given. Assume that the cone K contains the zero element of the space Y as its vertex.

Definition. A cone K is called a normal cone if there exists a positive number $\delta>0$ such that for any two elements $x,y\in K$, both with nonzero norms, the inequality

$$\|x+y\|\geq\delta$$

holds. From now on by the notion of a normal cone we understand a normal cone with a nonempty interior.

Lemma 1. The normal cone K is a pointed cone in the space Y.

Proof. Suppose that the cone K is a normal cone but that it is not pointed. Then there exists an element $y\in K\cap(-K)$ such that $y\neq 0$. As $y\neq 0$, then $\|y\|>0$ and $y/\|y\|\in K,\ -y/\|y\|\in K$. Obviously, the relation $\|y/\|y\|\|=\|-y/\|y\|\|=1$ holds. Normality of the cone K implies that there exists a positive number $\delta>0$ which satisfies the inequality

$$\|(y/\|y\|)+(-y/\|y\|)\|=\|0\|=0>\delta,$$

a contradiction. q.e.d.

Lemma 1 guarantees that the Banach space Y is a partially ordered space by means of the cone K. Note that, taking into account Krein's Theorem (see, for example, [17, 18] or [21]), we can make the following observation.

Conclusion. Suppose that the Banach space Y is partially ordered by means of the normal cone K. Then the equality

$$Y^{*}=K^{*}-K^{*}, \tag{1}$$

holds, where Y^{*} is a conjugate space for the space Y, and K^{*} is a conjugate (dual) cone of the cone K. (If (1) holds, K^{*} is called reproducing.)

Suppose that X and Y are linear topological spaces such that the space Y is partially ordered by means of a pointed cone K, and $\Omega\subseteq X$ is some convex subset in the X space.

Definition2. An operator $F:\Omega\rightarrow Y$ is referred to as a convex operator in Ω if for arbitrary two elements $x,y\in\Omega$ and an arbitrary number $\lambda\in[0,1]$ the inclusion

$$\lambda F(x)+(1-\lambda)F(y)-F(\lambda x+(1-\lambda)y)\in K$$

is satisfied. If $\Omega\equiv X$ then we say that F is a convex operator.

Lemma 2. Assume that $F:X\to Y$ is a convex operator. Then for arbitrary two elements x_o, $h\in X$ the operator $\varphi:R\to Y$, defined by means of the following equality

$$\varphi(t) = F(x_o+th), \forall t\in R,$$

is convex.

Proof. The following equalities

$$\varphi(\lambda t_1 + (1-\lambda)t_2) = F(x_0 + (\lambda t_1 + (1-\lambda))t_2 h) = F(\lambda x_0 + (1-\lambda)x_0 + \lambda t_1 h + (1-\lambda)t_2 h),$$
$$\lambda \varphi(t_1) = \lambda F(x_0 + t_1 h),$$
$$(1-\lambda)\varphi(t_2) = (1-\lambda)F(x_0 + t_2 h),$$

hold for arbitrary $t_1,\ t_2 \in R,\ \lambda \in [0,1]$.
From these equalities we get that

$$\lambda \varphi(t_1) + (1-\lambda)\varphi(t_2) - \varphi(\lambda t_1 + (1-\lambda)t_2) =$$
$$= \lambda F(x_0 + t_1 h) + (1-\lambda)F(x_0 + t_2 h) - F(\lambda(x_0 + t_1 h) + (1-\lambda)(x_0 + t_2 h).$$

Now suppose that $x=x_o+t_1h$, $y=x_o+t_2h$, and take into account the convexity of the operator F. Then we can directly conclude that the inclusion

$$\lambda \varphi(t_1) + (1-\lambda)\varphi(t_2) - \varphi(\lambda t_1 + (1-\lambda)t_2) \in K, \forall t_1, t_2 \in R, \lambda \in [0,1],$$

holds q.e.d.

Lemma 3. Suppose that the space Y is a partially ordered Banach space by means of the normal cone K, and that the operator $\varphi: R_+\to Y$ possesses the following three properties:

1. $\varphi(t)\in K$ for all $t\in R_+$;
2. $t_1 \le t_2$ implies $\varphi(t_2) - \varphi(t_1)\in K$;
3. the sequence $\{\varphi(t)\}\subset Y$ weakly converges to zero when $t\to+0$.

Then $\|\varphi(t)\| \to 0$ when $t\to+0$.

Proof. Suppose that all conditions of the Lemma are satisfied, but that $\|\varphi(t)\| \not\to 0$ when $t\to+0$. Then there exist two positive numbers $\varepsilon>0$ and $t_o>0$ such that for an arbitrary number $t\in(0,t_o)$ the inequality

$$\|\varphi(t)\| > \varepsilon$$

holds.

Denote by $A=co(\varphi(R_+))$ the convex hull of the image of φ. Obviously, A is a convex set and for arbitrary $t\in R_+$ we have that $\varphi(t)\in R_+$. Let x be an arbitrary point in A. Then we can find two sets $\{\lambda_i\}_{i=1}^n$ and $\{t_i\}_{i=1}^n$ of positive numbers which satisfy the following equalities:

$$\sum_{i=1}^n \lambda_i = 1, \quad x = \sum_{i=1}^n \lambda_i \varphi(t_i).$$

Define $t^* := \min(t_o,t_1,...,t_n)$. Taking into account Condition 2. of the Lemma we get the relations

$$\varphi(t_i) - \varphi(t) \in K, \forall i = 1,\dots,n,$$
$$\|\varphi(t)\| > \varepsilon, \forall t \in (0,t^*). \tag{2}$$

According to Condition 1. of the Lemma we have the inclusion $\varphi(t)\in K$ for arbitrary $t\in(0,t^*)$, and as K is a convex cone from inequality (2) we obtain the inclusion

$$\sum_{i=1}^{n} \lambda_i \varphi(t_i) - \varphi(t) \in K,$$

i.e., x-$\varphi(t) \in K$ for arbitrary $t \in (0,t^*)$.

As Y is a Banach space and K is a normal cone then according to the fact that K is normal if and only if the norm is semimonotone (see e.g. [17], [21]) and by taking into account the last inclusion, there exists a positive number $c>0$ such that

$$\|\varphi(t)\| \le c\|x\|, \forall t \in (0,t^*).$$

Then by taking into account (2) and denoting $\alpha = \varepsilon/c$ from the last inequality we get

$$\|x\| > \alpha. \tag{3}$$

Note that inequality (3) is true for arbitrary elements $x \in A$. Therefore, if U is an open ball with its center $0 \in Y$ and the radius α, then the following set equality

$$A \cap U = \emptyset \tag{4}$$

is valid.

As A is a convex and nonempty set, U is a convex, open and nonempty set, and by taking into account equality (4) and the Hahn-Banach Theorem for normed spaces we can conclude that there exists a nonzero functional $y^* \in Y^*$ and a positive number r such that the inequalities

$$< y^*, x > \; > r, \forall x \in A,$$
$$< y^*, x > \; \le r, \forall x \in U, \tag{5}$$

hold.

As for arbitrary $t \in R_+$ we have that $\varphi(t) \in A$, then from (5) we obtain

$$< y^*, \varphi(t) > \; > r > 0, \forall t \in R_+.$$

By passing to the limit in the last inequality we get the following:

$$\lim_{t \to +0} < y^*, \varphi(t) > \ge r > 0.$$

The last fact contradicts the assumption of weak convergence for the sequence $\{\varphi(t)\}$. q.e.d.

Note that if K is a normal cone in the Banach space Y then the cone $\overline{K}$ is also a normal one and then according to Lemma 1 this cone is a pointed one. Therefore, Krein's Theorem is valid for a closed cone K too, i.e., equality (1) holds.

3. Existence of Directional Derivatives

In this section we will use the results obatined in the previous section to prove the first main result of this paper. Theorem 1 below provides a new condition for the existence of directional derivatives, which does not use regularity properties of the cone K.

Theorem 1. Assume that X is a linear space, that Y is a weakly complete Banach space which is partially ordered by a normal closed cone K, and that $F{:}X\to Y$ is a convex operator. Then for an arbitrary point $(x_o, h)\in X\times X$ the directional derivative, i.e. the limit:

$$\lim_{t\to +0}\frac{F(x_0+th)-F(x_0)}{t}=F'(x_0,h)$$

does exist in the sense of strong convergence in Y.

Proof. Let t_o, $t_2\in R$ be such that $t_o< t_2$ and let $t_o< t_1< t_2$. Obviously, $t_1- t_o>0$ and $t_2- t_o>0$, and if we denote by $\lambda=(t_1-t_o)/(t_2- t_o)$, then $0<\lambda<1$. Also the equality $\lambda t_2+(1-\lambda)t_o=t_1$ holds. Consider an operator $\varphi:R\to Y$ given in the form $\varphi(t)=F(x_o+th)$, $\forall t\in R$. According to Lemma 2 φ is convex and therefore the inclusion

$$\frac{t_1-t_0}{t_2-t_0}\varphi(t_2)+(1-\frac{t_1-t_0}{t_2-t_0})\varphi(t_0)-\varphi(t_1)=\frac{t_1-t_0}{t_2-t_0}(\varphi(t_2)-\varphi(t_0))-(\varphi(t_1)-\varphi(t_0))=$$

$$=(t_1-t_0)(\frac{\varphi(t_2)-\varphi(t_0)}{t_2-t_0}-\frac{\varphi(t_1)-\varphi(t_0)}{t_1-t_0})\in K$$

holds.

As $t_1-t_o>0$ and $1/(t_1-t_o)K\subset K$ we have

$$(\frac{\varphi(t_2)-\varphi(t_0)}{t_2-t_0}-\frac{\varphi(t_1)-\varphi(t_0)}{t_1-t_0})\in K \tag{6}$$

for arbitrary $t_1\in(t_o,t_2)$.

Choose $t_2 = t>0$, $t_1 = 0$, $t_o= -1$ to get from (6)

$$\frac{\varphi(t)-\varphi(-1)}{t+1}-(\varphi(0)-\varphi(-1))\in K,\ \forall t\in R_+. \tag{7}$$

It follows from (7) that

$$\frac{\varphi(t)-\varphi(0)}{t}-(\varphi(0)-\varphi(-1))\in K,\ \forall t\in R_+. \tag{8}$$

It is easy to show that from (6) we can conclude

$$\frac{\varphi(t)-\varphi(0)}{t}-\frac{\varphi(t_1)-\varphi(0)}{t_1}\in K \tag{9}$$

for arbitrary $t\in R_+$ and $t_1\in(0,t)$.

Now consider an arbitrary, but fixed, element $y^*\in K^*$. As $y^*\in Y^*$, and y^* is bounded in K the number $\langle y^*, \varphi(0)-\varphi(-1)\rangle = \alpha(y^*)$ is finite.

On the other hand, from (8) we get

$$< y^*, \frac{\varphi(t)-\varphi(0)}{t} > \ge \alpha(y^*), \forall t \in R_+.$$

This last inequality implies the boundness from below of the sequence of numbers $\{<y^*,(\varphi(t)-\varphi(0))/t>\}$ for $t\to+0$.

Furthermore, inclusion (9) yields the inequality

$$< y^*, \frac{\varphi(t)-\varphi(0)}{t} > \ge < y^*, \frac{\varphi(t_1)-\varphi(0)}{t_1} >, \forall t_1 \in (0,t).$$

The latter means that the sequence $\{<y^*,(\varphi(t)-\varphi(0))/t>\}$, for $t\to+0$, is nonincreasing. We can conclude that there exists a finite number $\beta(y^*)$, which satisfies the equality

$$\lim_{t\to+0} < y^*, \frac{\varphi(t)-\varphi(0)}{t} > = \beta(y^*). \tag{10}$$

As the sequence $\{<y^*,(\varphi(t)-\varphi(0))/t>\}$ with $t\to+0$ is converging, it is a fundamental (Cauchy) sequence, i.e., for an arbitrary positive number $\varepsilon>0$ there exists another positive number $\bar{t}$ such that for an arbitrary number $t_1\in(0, t)$, $t<\bar{t}$, the inequality

$$\left| < y^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right| < \varepsilon.$$

holds.

Now consider an arbitrary element $h^*\in Y^*$. According to equality (1) there exist two elements g^*, $q^*\in K^*$ such that h^* can be represented as $h^*=g^*-q^*$. Now we shall show that the sequence $\{<h^*,(\varphi(t)-\varphi(0))/t>\}$, fort $t\to+0$, is fundamental (Cauchy), too. We have

$$\left| < h^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right| = \left| < g^* - q^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right| \le$$

$$\le \left| < g^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right| + \left| < q^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right|. \tag{11}$$

As the sequences $\{<g^*,(\varphi(t)-\varphi(0))/t>\}$, $\{<q^*,(\varphi(t)-\varphi(0))/t>\}$, with $t\to+0$, are fundamental, we have that for an arbitrary positive number $\varepsilon>0$ two positive numbers $\bar{t}_1 > 0$, $\bar{t}_2 > 0$ can be found, such that the following two inequalities

$$\left| < g^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right| < \frac{\varepsilon}{2}, \forall t \in (t_1, \bar{t}_1),$$

$$\left| < q^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right| < \frac{\varepsilon}{2}, \forall t \in (t_1, \bar{t}_2),$$

hold.

By denoting $t^* = \min\{\bar{t}_1, \bar{t}_2\}$ and taking into account inequality (11) from the latter inequalities we easily obtain

$$\left| < h^*, \frac{\varphi(t)-\varphi(0)}{t} - \frac{\varphi(t_1)-\varphi(0)}{t_1} > \right| < \varepsilon, \forall t \in (t_1, t^*).$$

Thus fundamentality of the sequence $\{<h^*,(\varphi(t)-\varphi(0))/t>\}$, for $t\to+0$, is proved.

Summarizing the discussion above, we can state that the sequence { $(\varphi(t)-\varphi(0))/t$ }$\subset Y$, for $t\to+0$, is weakly fundamental because the sequence {$<h^*,(\varphi(t)-\varphi(0))/t>$}$\subset R$, with $t\to+0$, is fundamental for arbitrary $h^*\in Y^*$. As Y is a weakly complete Banach space, there exists an element $y_0\in Y$, which is a weak limit for the sequence { $(\varphi(t)-\varphi(0))/t$ }, for $t\to+0$, i.e.

$$\lim_{t\to+0} < h^*, \frac{\varphi(t)-\varphi(0)}{t} - y_0 > = 0. \tag{12}$$

Now we are going to show that $(\varphi(t)-\varphi(0))/t - y_0 \in K$ for arbitrary $t\in R_+$.
Suppose the opposite. Then there exists a number $\bar{t}\in R_+$, which satisfies the condition $(\varphi(t)-\varphi(0))/t - y_0 \notin K.$ Thus, there exists a functional $y^*\in K^*$, which satisfies the inequality

$$< y^*, \frac{\varphi(\bar{t})-\varphi(0)}{\bar{t}} - y_0 > < 0.$$

(Indeed, if there does not exist such a functional $y^*\in K^*$, then for an arbitrary $y^*\in K^*$ the following inequality

$$< y^*, \frac{\varphi(\bar{t})-\varphi(0)}{\bar{t}} - y_0 > \ \geq 0$$

should be valid. The latter one is possible only in the case when $(\varphi(\bar{t})-\varphi(0))/\bar{t} \in \overline{K}.$ As K is a closed one then $\overline{K}=K,$ but this contradicts the assumption that $(\varphi(\bar{t})-\varphi(0))/\bar{t} \notin \overline{K}.$)

On the other hand, according to (9) the inclusion

$$\frac{\varphi(\bar{t})-\varphi(0)}{\bar{t}} - y_0 - \frac{\varphi(t)-\varphi(0)}{t} + y_0 \in K, \ \forall t \in (0,\bar{t}),$$

holds, and therefore this relation implies the inclusion

$$< y^*, \frac{\varphi(\bar{t})-\varphi(0)}{\bar{t}} - y_0 > \geq < y^*, \frac{\varphi(t)-\varphi(0)}{t} - y_0 >, \ \forall t \in (0,\bar{t}).$$

Passing to the limit in the last inequality we get

$$\lim_{t\to+0} < y^*, \frac{\varphi(t)-\varphi(0)}{t} - y_0 > \leq < y^*, \frac{\varphi(\bar{t})-\varphi(0)}{\bar{t}} - y_0 > < 0.$$

So we obtain a contradiction with equality (12), and thus the inclusion

$$\frac{\varphi(t)-\varphi(0)}{t} - y_0 \in K, \ \forall t \in R_+,$$

is proved.

Now consider an operator $S:R_+\to Y$, which is defined by the equality

$$S(t) = \frac{\varphi(t)-\varphi(0)}{t} - y_0, \ \forall t \in R_+. \tag{13}$$

By taking into account inclusions (9) and (13) it is easy to show that the operator S satisfies all assumptions of Lemma 3, according to which $\|S(t)\| \to 0,$ when $t\to+0$. Note that Y is a locally convex space and therefore a weak limit y_0 for the sequence { $(\varphi(t)-\varphi(0))/t$ }, when $t\to+0$, is unique.

As the relation $(\varphi(t)-\varphi(0))/t=(F(x_0+th)-F(x_0))/t$ holds, denoting $y_0=F'(x_0, h)$ we obtain the equality

$$\lim_{t\to+0} \frac{F(x_0+th)-F(x_0)}{t} = F'(x_0,h)$$

in the sense of strong convergence in Y. q.e.d.

Corollary 1. Assume that X is a linear space, Y is a weakly complete Banach space which is partially ordered by means of the normal closed cone K, and F:$X \to Y$ is a convex operator. Then

$$F(x_0 + h) - F(x_0) - F'(x_0, h) \in K, \forall (x_0, h) \in X \times X. \tag{14}$$

Proof. Because of the inclusion $(\varphi(t)-\varphi(0))/t-y_o \in K$, $\forall t \in R_+$ and Theorem 3, we can directly show that inclusion (14) holds, by using $t=1$. q.e.d.

4. Structure of the Cone of Decrease Directions

In this section we define the cone of decrease directions of an operator *F*. We show, using Theorem 1, that the cone of decrease directions can be represented in terms of directional derivatives.

Suppose that two linear topological spaces *X* and *Y* are given, and that $F:X\to Y$ is some operator. *K* is a cone in *Y* as introduced above.

Definition. (See, for example, [19]). Assume that x_o is a fixed point in the space *X*. A vector $h\in X$ is referred to as a decrease direction for the operator *F* in the point x_o if there exist $\varepsilon_0(h)\in R_+$, $q(h)\in \operatorname{int}K$, and a neighbourhood $U(h)$ of *h*, such that

$$< y^*, F(x_o+\varepsilon\bar{h}) - F(x_0) - \varepsilon q(h) > \le 0,\ \forall y^*\in K^*,\ \varepsilon\in(0,\varepsilon_0(h)),\ \bar{h}\in U(h).$$

We assume that the set of decrease directions for the operator *F* in the point x_o is nonempty and denote it by $K(F,x_o)$. It was shown in [19] that this set $K(F,x_o)$ is an open cone in *X*, and some additional properties have been investiagted. We will refer to $K(F,x_o)$ as the cone of decreas directions of the operator *F*.

Theorem 2. Suppose that *X* is a normed space, that *Y* is a weakly complete Banach space, which is partially ordered by means of the normal closed cone *K*, and that $F:X\to Y$ is a continuous convex operator. Then for an arbitrary point $x_0\in X$ the cone of decrease directions $K(F,x_o)$ of the operator *F* is convex and can be represented in the following way:

$$K(F,x_0)=\{h\in X \mid F'(x_0,h)\in -\operatorname{int}K\}. \qquad (15)$$

Proof. It is obvious from Theorem 1 that for an arbitrary point $(x_o,h)\in X\times X$ the directional derivative $F'(x_o,h)$ exists.

Let $h\in K(F,x_o)$. Then there exist $\varepsilon_0(h)\in R_+$, $q(h)\in R_+(-\operatorname{int}K)$, and a neighbourhood $U(h)$, which satisfy the inequality

$$< y^*, F(x_o+\varepsilon\bar{h}) - F(x_0) - \varepsilon q(h) > \le 0,\ \forall y^*\in K^*,\ \varepsilon\in(0,\varepsilon_0(h)),\ \bar{h}\in U(h). \qquad (16)$$

In particular, inequality (16) holds for $\bar{h}=h$, too, i.e.,

$$< y^*, F(x_o+\varepsilon h) - F(x_0) - \varepsilon q(h) > \le 0,\ \forall y^*\in K^*,\ \varepsilon\in(0,\varepsilon_0(h)). \qquad (17)$$

By taking into account Theorem 1, the continuity of the functional $y^*\in K^*$, and passing to the limit in (17) with $\varepsilon\to+0$, we get

$$\lim_{\varepsilon\to+0} < y^*, \frac{F(x_0+\varepsilon h)-F(x_0)}{\varepsilon} - q(h) > = < y^*, \lim_{\varepsilon\to+0}\frac{F(x_0+\varepsilon h)-F(x_0)}{\varepsilon} - q(h) > =$$
$$= < y^*, F'(x_0,h) - q(h) > \le 0,\ \forall y^*\in K^*.$$

The latter inequality holds if and only if the inclusion $q(h)-F'(x_o,h)\in \bar{K}$ holds. But as *K* is a closed cone, then we have $F'(x_o,h)\in -(-q(h)+K)$, i.e., there exists an element $y\in K$, which satisfies the equality $F'(x_o,h)\in -(-q(h)+K)$. As $-q(h)\in \operatorname{int}K$ and $y\in K$, it is easy to prove the validity of the inclusion $-q(h)+y\in \operatorname{int}K$, i.e., $F'(x_o,h)\in -\operatorname{int}K$. So the first inclusion

$$K(F,x_0) \subset \{h \in X \mid F'(x_0,h) \in -\operatorname{int} K\}. \tag{18}$$

has been proved.

Now let $h \in \{h \in X \mid F'(x_o,h) \in -\text{int}K\}$. We are going to show that there exist $\varepsilon_0(h) \in R_+$, $q(h) \in (-\text{int}K)$, and $U(h)$ satisfying inequality (16). Obviously inequality (16) holds for arbitrary $\varepsilon_0(h) \in R_+$, $U(h) \in \tau$, if $y^* = 0^*$, and therefore we shall consider only positive functionals $y_+^* \in K_+^* \subset K^*$.

We have

$$\lim_{\varepsilon \to +0} < y_+^*, \frac{F(x_0 + \varepsilon h) - F(x_0)}{\varepsilon} >= < y_+^*, F'(x_0,h) >, \forall y_+^* \in K_+^*. \tag{19}$$

Because - $F'(x_o,h) \in \text{int}K$, we get the inequality

$$< y_+^*, F'(x_0,h) > < 0, \forall y_+^* \in K_+^*. \tag{20}$$

When proving Theorem 1 we have already shown that the sequence $\{< y_+^*, (F(x_0 + \varepsilon h) - F(x_0)) / \varepsilon >\}$, with $\varepsilon \to +0$, is nonincreasing, and therefore relying on (19) and (20), we can confirm that there exists a number $\varepsilon_0(h) \in R_+$ such that

$$< y_+^*, \frac{F(x_0 + \varepsilon_0(h)h) - F(x_0)}{\varepsilon_0(h)} > < 0, \forall y_+^* \in K_+^*.$$

From this we get the inclusion

$$F(x_0) - F(x_0 + \varepsilon_0(h)h) \in \varepsilon_0(h) \operatorname{int} K \subset \operatorname{int} K,$$

which implies the inequality

$$< y_+^*, F(x_0) - F(x_0 + \varepsilon_0(h)h) >= \delta(y_+^*) > 0, \forall y_+^* \in K_+^*. \tag{21}$$

As the operator $F:X \to Y$ is continuous, the operator $y_+^* \circ F: X \to R$ will be continuous for arbitrary $y_+^* \in K_+^*$, too. Therefore, for a positive number $\delta(y_+^*)/2$ there can be found a neighbourhood $U(x_o+\varepsilon_0(h)h) = x_o + \varepsilon_0 U(h)$ of the point $x_o + \varepsilon_0(h)h$, which satisfies

$$\left|(y_+^* \circ F)(x_0 + \varepsilon_0(h)\bar{h}) - (y_+^* \circ F)(x_0 + \varepsilon_0(h)h)\right| =$$

$$= \left|< y_+^*, F(x_0 + \varepsilon_0(h)\bar{h}) - F(x_0 + \varepsilon_0(h)h) >\right| \le \frac{\delta(y_+^*)}{2}, \forall y_+^* \in K_+^*, \bar{h} \in U(h).$$

From the latter inequality we get

$$< y_+^*, F(x_0 + \varepsilon_0(h)\bar{h}) > \le < y_+^*, F(x_0 + \varepsilon_0(h)h) > + \frac{\delta(y_+^*)}{2}, \forall y_+^* \in K_+^*, \bar{h} \in U(h). \tag{22}$$

According to (21) we have

$$< y_+^*, F(x_0 + \varepsilon_0(h)h) > = < y_+^*, F(x_0) > - \delta(y_+^*).$$

Therefore (22) implies the inequality

$$< y_+^*, F(x_0 + \varepsilon_0(h)\bar{h}) - F(x_0) > \le -\frac{\delta(y_+^*)}{2}, \forall y_+^* \in K_+^*, \bar{h} \in U(h). \tag{23}$$

Assume that $\varepsilon \in (0, \varepsilon_0(h))$. Obviously $\varepsilon / \varepsilon_0(h) \in (0,1)$. Besides the equality

$$x_0 + \varepsilon \bar{h} = \frac{\varepsilon}{\varepsilon_0(h)}(x_0 + \varepsilon_0(h)\bar{h}) + (1 - \frac{\varepsilon}{\varepsilon_0(h)})x_0, \forall \bar{h} \in U(h)$$

holds.

According to the assumptions of the Theorem the operator F:$X\to Y$ is convex and therefore, we get the inclusion

$$\frac{\varepsilon}{\varepsilon_0(h)}F(x_0+\varepsilon_0(h)\bar{h})+(1-\frac{\varepsilon}{\varepsilon_0(h)})F(x_0)-F(x_0+\varepsilon\bar{h})\in K,\ \forall\varepsilon\in(0,\varepsilon_0(h)),\bar{h}\in U(h).$$

This relation and (23) imply the following chain of inequalities:

$$<y_+^*,F(x_0+\varepsilon\bar{h})>\le\frac{\varepsilon}{\varepsilon_0(h)}<y_+^*,F(x_0+\varepsilon_0(h)\bar{h})>+(1-\frac{\varepsilon}{\varepsilon_0(h)})<y_+^*,F(x_0)>=$$

$$=\frac{\varepsilon}{\varepsilon_0(h)}<y_+^*,F(x_0+\varepsilon_0(h)\bar{h})-F(x_0)>+<y_+^*,F(x_0)>\le$$

$$\le-\frac{\delta(y_+^*)\varepsilon}{2\varepsilon_0(h)}+<y_+^*,F(x_0)>,\ \forall y_+^*\in K_+^*,\varepsilon\in(0,\varepsilon_0(h)),\bar{h}\in U(h). \qquad (24)$$

According to (21) we have

$$-\frac{1}{2\varepsilon_0(h)}\delta(y_+^*)=<y_+^*,\frac{F(x_0+\varepsilon_0(h)h)-F(x_0)}{2\varepsilon_0(h)}>.$$

But as $F(x_o)$-$F(x_0+\varepsilon_o(h)h)\in\text{int}K$, and setting $q(h)$= -($F(x_o)$-$F(x_o+\varepsilon_o(h)h)/2\varepsilon_o(h)$, we can conclude that $q(h)\in$-int$K$.

Taking into account the above considerations it easy to obtain the following inequality from (24).

$$<y_+^*,F(x_0+\varepsilon\bar{h})>-F(x_0)-\varepsilon q(h)>\le 0,\ \forall y_+^*\in K_+^*,\ \varepsilon\in(0,\varepsilon_0(h)),\ \bar{h}\in U(h). \qquad (25)$$

Now using the remark about the zero element $0^*\in K^*$, we can conclude that (25) is valid for an arbitrary $y^*\in K^*$. Summarizing all the above considerations, we can see that $h\in K(F,x_o)$, i.e., inclusion (18) can be reversed and hence we can conclude that equality (15) holds.

Now let us show the convexity of the cone $K(F,x_o)$. Suppose that (h_1,h_2) is an arbitrary point in $K(F,x_o)\times K(F, x_o)$. Then according to (15) we get $F'(x_o,h_1)$, $F'(x_o,h_2)\in$-int$K$. As the operator F:$X\to Y$ is convex, we know that for arbitrary $\lambda\in[0,1]$ for all $\varepsilon\in R_+$

$$\lambda(F(x_0+\varepsilon h_1)-F(x_0))+(1-\lambda)(F(x_0+\varepsilon h_2)-F(x_0))-F(x_0+\varepsilon\ (\lambda h_1+(1-\lambda)h_2))+F(x_0)\in K.$$

As $(1/\varepsilon)K\subset K$ we then have for all $\varepsilon\in R_+$

$$\lambda\frac{F(x_0+\varepsilon h_1)-F(x_0)}{\varepsilon}+(1-\lambda)\frac{F(x_0+\varepsilon h_2)-F(x_0)}{\varepsilon}-\frac{F(x_0+\varepsilon(\lambda h_1+(1-\lambda)h_2))-F(x_0)}{\varepsilon}\in K,$$

As K is a closed cone and the sequence defined by the left side of the latter inclusion with $\varepsilon\to+0$, is a sequence in K, according to Theorem 1   a limit exists in K for $\varepsilon\to+0$, i.e.

$$F'(x_0,\lambda h_1+(1-\lambda)h_2)-\lambda F'(x_0,h)-(1-\lambda)F'(x_0,h)\in -K.$$

It is obvious that for an arbitrary $\lambda\in[0,1]$ the following inclusion

$$\lambda F'(x_0,h_1)+(1-\lambda)F'(x_0,h_2)\in\text{int}\,K$$

holds, and therefore the inclusion

$$F'(x_0,\lambda h_1+(1-\lambda)h_2)\in-\text{int}\,K,\ \forall h_1,h_2\in K(F,x_0),\ \forall\lambda\in[0,1],$$

holds.

The latter fact means that the cone $K(F,x_o)$ is a convex one. q.e.d.

5. *K*-Bounded Operators

In this section we show that a sufficient condition for a convex operator to be continuous is its *K*-boundedness.

Definition 3. Assume that X and Y are normed spaces, and that the space Y is partially ordered by means of a normal cone K. The operator $F:X\to Y$ is called K-bounded from above iff for an arbitrary bounded subset $U\subset X$ there exists an element $z=z(U)\in Y$ with a finite norm which satisfies

$$z - F(x) \in K, \ \forall\, x\in U.$$

Theorem 3. Suppose that X is a normed space, Y is a Banach space which is partially ordered by means of a normal cone K, and $F:X\to Y$ is convex and K-bounded from above. Then the operator F is continuous in X.

Proof. Assume that x is an arbitrary point in X, and U is some bounded neighbourhood of x, i.e., an open ball with its center in x and a finite radius. Without loss of generality we can suppose that x=0 and $F(x)$=0$\in Y$. In this case U will be a convex and bounded neighbourhood of the point $0\in X$, and therefore, for $\varepsilon\in(0,1)$, $\varepsilon U\subset U$ holds. As the operator F is K-bounded from above, there exists an element $z=z(U)\in Y$ with a finite norm such that it satisfies the following inclusions:

$$\varepsilon z-\varepsilon F(\frac{v}{\varepsilon})\in K,\ \varepsilon\, z-\varepsilon F(\frac{-v}{\varepsilon})\in K,\ \forall\varepsilon\in(0,1),\ v\in U. \tag{26}$$

As F is a convex operator we have

$$F(v)-(1-\varepsilon)F(0)-\varepsilon F(\frac{v}{\varepsilon})=F(v)-\varepsilon F(\frac{v}{\varepsilon})\in -K,$$

and thus we get

$$F(v)-\varepsilon\, z\in -K+(-\,\varepsilon z+\varepsilon F(\frac{v}{\varepsilon})). \tag{27}$$

By taking into account inclusions (26) and convexity of the cone K, from (27) we get the inclusion

$$\varepsilon\, z-F(v)\in K,\ \forall\varepsilon\in(0,1),\ v\in\varepsilon\, U. \tag{28}$$

Note that

$$F(0)=F(\frac{v}{1+\varepsilon}v+\frac{\varepsilon}{1+\varepsilon}(\frac{-v}{\varepsilon})). \tag{29}$$

As $-v/\varepsilon$, $-v/(1+\varepsilon)\in U$, taking into account the convexity of F for the second time, (29) implies

$$F(0)-\frac{1}{1+\varepsilon}F(v)-\frac{\varepsilon}{1+\varepsilon}F(-\frac{v}{\varepsilon})\in -K\Rightarrow F(v)-(1+\varepsilon)F(0)+\varepsilon F(-\frac{v}{\varepsilon})\in K$$

$$\Rightarrow F(v)+\varepsilon F(-\frac{v}{\varepsilon})\in K\Rightarrow F(v)+\varepsilon r\in K+\varepsilon\, z-\varepsilon F(-\frac{v}{\varepsilon}).$$

Using relations (26) for the second time, we can easily show the validity of the inclusion

$$F(v)+\varepsilon\, z\in K,\ \forall\varepsilon\in(0,1),\ v\in\varepsilon\, U. \tag{30}$$

From (28) and (30) the validity of the following inequality follows:

$$-\varepsilon\, z \le F(v) \le \varepsilon\, z,\ \forall\varepsilon\ \in (0,1),\, v \in \varepsilon\, U. \tag{31}$$

As *K* is a normal cone in the Banach space *Y*, and the element $z \in Y$ has a finite norm, from normality of *K* (i.e. semimonotony of the norm, see, for instance, [17], [21]), from inequality (31) we can conclude that there exists a positive number c>0, such that the following inequality

$$\|F(v) - F(0)\|_Y \le \varepsilon\, c,\ \forall\varepsilon \in (0,1),\, v \in \varepsilon\, U,$$

holds.

Note that with $\varepsilon \to 0$ we get $\|v\|_X \to 0.$ Then from the latter inequality we get

$$\lim_{v\to 0} F(v) = F(0).$$

This exactly means that the operator *F* is continuous in *X*. q.e.d.

Remark. An analogous result for convex functionals can be found in [20].

According to Theorem 3 the continuity condition of the operator *F* in Theorem 2 can be changed to the condition of its *K*-boundness from above in *X*. Also note that $F'(x_o, h)$ (from Theorem 2) is referred to as the first variation of the operator *F* in the point $x_o \in X$ and, in general, the operator $F(x_o, \bullet): X \to Y$ is nonlinear.

6. Conclusions

In this paper we have investigated several important properties of convex operators in infinite-dimensional spaces. We have given a condition for the existence of directional derivatives and obtained an analythical representation of the direction decrease cone $K(F,x_o)$ of a continuous convex operator F.In this way, results of Miliutin and Dubovtskiy, developed for singlecriterion optimization problems, have been generalized for a vector valued case. The results obtained in this paper are steps towards solution methods for convex vector optimization problems, and thus may be of interest for researchers in various fields, where vector optimization problems are often encountered.

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