Min-Max Formulation of the Balance Number in Multiobjective Global Optimization

Matthias Ehrgott
Fachbereich Mathematik
Universität Kaiserslautern
67663 Kaiserslautern
Germany
e-mail: ehrgott@mathematik.uni-kl.de
fax: (49) 631 29081

Efim A. Galperin
Département de mathématiques
Université du Québec à Montréal
Montréal (Québec) H3C 3P8
Canada
e-mail: galperin.efim@uqam.ca

Abstract

The notion of the balance number introduced in [3, page 139] through a certain set contraction procedure for nonscalarized multiobjective global optimization is represented via a min-max operation on the data of the problem. This representation yields a different computational procedure for the calculation of the balance number and allows us to generalize the approach for problems with countably many performance criteria.

1 Introduction

Consider a robust bounded closed set $X \subset I\!R^n$ and the multiobjective optimization problem $\min(f, X)$, $f : X \to I\!R^m$:

$$\min_{x \in X} f_i(x), \quad i = 1, \ldots, m. \tag{1}$$

For each i the corresponding single objective subproblem of (1) has a global optimal solution over a compact set X represented by the partial global minimum value

$$c_i^0 := \min_{x \in X} f_i(x) \tag{2}$$

and the corresponding set of all global minimizers:

$$X_i^0 := \left\{ x \in X : f_i(x) = c_i^0 \right\}. \tag{3}$$

If there is a nonempty intersection

$$X^0 := \bigcap_{i=1}^{m} X_i^0 \neq \emptyset, \tag{4}$$

then the multicriteria optimization problem (MCO) of (1) is called balanced, otherwise unbalanced [3, Chapter 8]. If the problem is unbalanced, we can relax the minimization requirements (2) – (3) and look for the uniform η-suboptimal solutions

$$X_i^0(\eta) := \left\{ x \in X : f_i(x) - c_i^0 \leq \eta, \quad \eta > 0 \right\}. \tag{5}$$

With increasing η, the intersection of $X_i^0(\eta)$ eventually becomes nonempty, and the minimal value of η for which it is nonempty is called the balance number η_0. Thus by definition [3, page 139],

$$\eta_0 = \min \left\{ \eta : X^0(\eta) = \bigcap_{i=1}^{m} X_i^0(\eta) \neq \emptyset \right\}. \tag{6}$$

Methods to compute η_0 and simultaneously determine the intersection

$$X^0(\eta) = \bigcap_{i=1}^{m} X_i^0(\eta), \quad \eta = \eta_0 \tag{7}$$

are proposed in [3, Chapter 8]. The number η_0 represents the minimal equal deviation from global minimum values for all objective functions yielding a nonempty set (7) of uniform η_0-suboptimal solutions for the MCO problem (1).

2 Min-Max Formulation for η_0

Introduce the function

$$\theta(x) := \max_{1 \leq i \leq m} \left[f_i(x) - c_i^0 \right]. \tag{8}$$

Theorem 1

$$\eta_0 = \min_{x \in X} \theta(x) = \min_{x \in X} \max_{1 \le i \le m} \left[f_i(x) - c_i^0 \right]. \tag{9}$$

Proof:
Note that, due to (2), for $x \in X$ all $f_i(x) \ge c_i^0, \quad i = 1, \ldots, m$. By definition (6), we have

$$\begin{aligned} \eta_0 &= \min \left\{ \eta : X^0(\eta) \neq \emptyset \right\} \\ &= \min \left\{ \eta : \exists\, x \in X \text{ such that } f_i(x) - c_i^0 \le \eta, \quad i = 1, \ldots, m \right\} \\ &= \min \left\{ \eta : \exists\, x \in X \text{ such that } \max_{1 \le i \le m} [f_i(x) - c_i^0] \le \eta \right\}. \end{aligned} \tag{10}$$

Relation 10 represents the following nonlinear optimization problem

$$\begin{aligned} & \min \eta \\ \text{subject to} \quad & \max_{1 \le i \le m} \left[f_i(x) - c_i^0 \right] \le \eta, \; x \in X \end{aligned} \tag{11}$$

which, due to nonnegative brackets in (10), (11), has a solution $\min \eta = \eta_0 \ge 0$ for $x \in X^0(\eta_0)$. Thus, minimization with respect to η is implied by minimization with respect to x in (11), yielding

$$\eta_0 = \min_{x \in X} \max_{1 \le i \le m} \left[f_i(x) - c_i^0 \right].$$

□

Remarks:

1. *Clearly, the above formula admits generalization for countably many performance criteria.*

2. *The operations in (9) are not commutative, indeed:*

$$0 \le \eta_0 = \min_{x \in X} \max_{1 \le i \le m} \left[f_i(x) - c_i^0 \right] \neq \max_{1 \le i \le m} \min_{x \in X} \left[f_i(x) - c_i^0 \right] = 0,$$

   by definition of c_i^0*, see (2).*

3. *Formula (9) yielding the balance number* $\eta_0 \ge 0$ *(with* $\eta_0 = 0$*, the problem is balanced, that is, all partial minima* c_i^0 *can be attained simultaneously) does not determine the set* $X^0(\eta_0)$*, (7). However, the knowledge of* η_0 *is important as an independent measure of possible improvement, and it can facilitate computation of the set* $X^0(\eta_0)$ *by set contraction methods.*

3 Level Set Computation of $X^0(\eta_0)$

In some cases, suboptimal sets (5) rewritten as level sets

$$X_i^0(\eta) := \left\{ x \in X : f_i(x) \le c_i^0 + \eta \right\} \tag{12}$$

are easy to compute (e.g., if all $f_i(x)$ are linear functions). If η_0 is known, then the solution is immediately obtained as

$$X^0(\eta_0) = \bigcap_{i=1}^{m} X_i^0(\eta_0) = \bigcap_{i=1}^{m} \left\{ x \in X : f_i(x) \le c_i^0 + \eta_0 \right\} \neq \emptyset \tag{13}$$

$$c_i^0 \le f_i(x) \le c_i^0 + \eta_0, \quad x \in X^0(\eta_0), \quad i = 1, \dots, m \tag{14}$$

Where $c_i^0 = \min_{x \in X} f_i(x)$, yielding the minimal guaranteed deviation of η_0 for each $f_i(x)$ from its partial minimum c_i^0. This solution can be readily computed despite the fact that $X \subset I\!\!R^n$ may be nonconvex and very complicated, see Example 5.1 in [4, pages 542-544].

4 Comparison with Pareto optimality

Here we use a discrete optimization problem to illustrate the determination of the balance number as compared to the Pareto approach in multiobjective optimization. By definition, a point $x \in X$ is a Pareto solution if there does not exist $x' \in X$ such that $f_i(x') \le f_i(x), \;\; i = 1 \dots, m$, with strict inequality for at least one i.

Consider $X = \{x_1, x_2, x_3, x_4\}$ and three objective criteria f_1, f_2, f_3 evaluated as in the following matrix:

	f_1	f_2	f_3	
x_1	3	1	2	not Pareto
x_2	0	1	3	Pareto
x_3	3	1	1	Pareto
x_4	0	4	0	Pareto

Here x_1 is not Pareto because of x_3 for which $f_3(x_3) = 1 < f_3(x_1) = 2$, and x_2, x_3, x_4 are all Pareto points. Thus, the set of Pareto optimal solutions is given by $X_{Par} = \{x_2, x_3, x_4\}$, as indicated at the right of the matrix.

The vector of partial minima, $c^0 = (c_1^0, c_2^0, c_3^0) = (0, 1, 0)$. According to definitions (5) and (6) we have to look at minimal common deviations $\eta \ge f_i(x) - c_i^0$ from global optimality, or equivalently $f_i(x) \le \eta + c_i^0$ with $\eta \to \min$. We start choosing $\eta = 0$, check if intersection (7) is nonempty and increase η step by step until this is the case. Thus, both the balance number η_0 and the full set $X^0(\eta_0)$ are finally determined, as illustrated in the following matrix.

η	$\eta + c_i^0$			$X^0(\eta)$
0	0	1	0	$\emptyset = \{x_2, x_4\} \cap \{x_1, x_2, x_3\} \cap \{x_4\}$
1	1	2	1	$\emptyset = \{x_2, x_4\} \cap \{x_1, x_2, x_3\} \cap \{x_3, x_4\}$
2	2	3	2	$\emptyset = \{x_2, x_4\} \cap \{x_1, x_2, x_3\} \cap \{x_1, x_3, x_4\}$
$\eta_0 = 3$	3	4	3	$X \neq \emptyset$
	f_1	f_2	f_3	

We see that the minimal guaranteed deviation from every partial minimum is $\eta_0 = 3$ which holds for the whole set $X = \{x_1, x_2, x_3, x_4\}$.

In contrast, the Pareto set $X_{Par} = \{x_2, x_3, x_4\}$, yielding the same guaranteed deviation $\eta = 3$ for every function f_1, f_2, f_3 vis-a-vis its partial minimum over X_{Par}, unjustly discriminates against x_1, despite the fact that at x_1 the function f_2 attains its minimum $c_2^0 = 1$, the value $f_3(x_1) = 2 > c_3^0 = 0$ by $\eta = 2 < 3$, and only $f_1(x_1) = 3 > c_1^0 = 0$ by $\eta = 3$, as for the whole Pareto set. The exclusion of x_1 is caused by the qualification of "nondomination" postulated in the definition of Pareto optimality and unrelated to the optimality represented by the partial minima $\{c_i^0\}$.

5 Example

In this section, we use Example 2.1 from [4] to demonstrate the computation of the balance number by the min-max operation, Theorem 1. At the same time we demonstrate the difference between the min-max determination of the balance number and the ordinary min-max problem

$$\min_{x \in X} \max_{1 \le i \le m} f_i(x). \tag{15}$$

We consider the problem with feasible set $X = [1, 2]$ and objective function $f = (x, 2x, -x)$. The solution of (15) is as follows:

$$\min_{x \in [1,2]} \max\{x, 2x, -x\} = 2, \tag{16}$$

at $i = 2$, $x = 1$.

Let us now consider determining the balance number. Obviously

$$c^0 = \left(\min_{x \in [1,2]} x,\ \min_{x \in [1,2]} 2x,\ \min_{x \in [1,2]} -x \right) = (1, 2, -2). \tag{17}$$

According to Theorem 1 we have to find

$$\eta_0 = \min_{x \in [1,2]} \max\{x - 1, 2x - 2, -x + 2\}, \tag{18}$$

which differs from (15), (16). We have:

$$\begin{aligned} x = 1 &\Rightarrow \max\{0, 0, 1\} = 1 \\ 1 < x < 1.5 &\Rightarrow \max\{x - 1, 2x - 2, -x + 2\} < 1 \\ x = 1.5 &\Rightarrow \max\{0.5, 1, 0.5\} = 1 \\ 1.5 < x < 2 &\Rightarrow \max\{x - 1, 2x - 2, -x + 2\} > 1 \\ x = 2 &\Rightarrow \max\{1, 2, 0\} = 2. \end{aligned}$$

In the interval $(1, 1.5)$ the minimal value of the maximum is attained at the intersection of the lines $2x - 2$ and $-x + 2$ which yields $x_0 = \frac{4}{3}$ and $\eta_0 = -\frac{4}{3} + 2 = \frac{2}{3}$ as the (unique) optimal solution. The balance number should thus be equal to $\frac{2}{3}$, which is indeed the case, since the defining inequalities read

$$\begin{array}{rcl} x-1 & \leq & \eta \\ 2x-2 & \leq & \eta \\ -x+2 & \leq & \eta \end{array}$$

for $x \in [1,2]$, and are fulfilled for $\min \eta = \frac{2}{3}$ with $x = \frac{4}{3}$, and have no solution if $\eta < \frac{2}{3}$.

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