Multifacility Location Problems with Tree Structure and Finite Dominating Sets [*]

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Abstract

Multifacility location problems arise in many real world applications. Often, the facilities can only be placed in feasible regions such as development or industrial areas. In this paper we show the existence of a finite dominating set (FDS) for the planar multifacility location problem with polyhedral gauges as distance functions, and polyhedral feasible regions, if the interacting facilities form a tree. As application we show how to solve the planar 2-hub location problem in polynomial time. This approach will yield an ε-approximation for the euclidean norm case polynomial in the input data and $1/\varepsilon$.

Keywords: Multifacility Location, Finite Dominating Set, Polyhedral Gauges, Planar Hub Location

1 Introduction

Continuous facility location deals with locating one or multiple facilities in $\mathbb{R}^n$ to serve a finite set of existing demand points under consideration of minimizing a general cost function. If the underlying space is $\mathbb{R}^2$, these problems are called planar. Other types of location problems are network- or discrete location problems. In the network type, we consider an underlying graph, where a facility can be placed on a vertex or an edge. In the discrete case, there is a finite set of possible location sites to put on the facility. For an overview of location problems, see [LNSdG15]. The most common cost functions are the *Median*-objective, which minimizes the weighted distances between the new facilities and the demand points, and the *Center*-objective, minimizing the weighted maximum distance between demand points and facilities. However, only considering these objectives might fail to model some real life problems in a realistic manner. Often, the facilities can only be sited within a given region, such as developing or industrial areas.

In this paper, we consider location problems with polyhedral gauges as distance functions and polyhedral feasible regions. We derive a finite dominating set, i.e., a finite set which contains an optimal solution. We show how to construct this set based on the geometric interpretation of the

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optimality conditions for this problem.

As an application of this approach we finally consider the planar 2-hub location problem. We show how to use this set to derive a polynomial algorithm.

Regarding the outline of this paper we start with a literature review in Section 2. In Section 3 we give a formal definition of the problem and state the optimality conditions. In the remaining Sections we explain how to construct the Finite Dominating Set (Section 4) and how to apply it to the planar 2-hub location problem to get a polynomial algorithm (Section 5). We conclude the paper with a summary and an outlook to further research (Section 6).

2 Literature Review

Location problems can be divided in three main types, continuous, network and discrete location problems. Continuous location problems concern with determining possible sites (for one ore more new facilities) in a continuous space. Often, there are several fixed (or existing) facilities, also called demand points. The new facilities interact with each other and the demand points. These interactions can be the transport of goods, the reachability of service people or a communication link or any other physical links. For examples refer to [Dre95]. Depending on the interaction, these location problems are usually modeled by a center or median objective, in which either the weighted maximum distance between the facilities is minimized, or the weighted sum.

Continuous location problems have a long history, most literature summaries credit Pierre de Fermat (1601-1665) with proposing a basic form of the median problem [DH02]. In this paper we will deal with a special type of continuous location problems, namely the ones in the plane. As distance function we will consider polyhedral gauges and the demand (weights) between each pair of facilities will be positive. Using polyhedral gauges the planar location problem can be formulated as a linear programming problem [WW85]. In [DM85], it was shown that for the single facility location problem in the plane can be partitioned into polyhedral cells and that there exists an optimal solution that is a cell vertex. This concept was generalized for multifacility location problems, where the distance function is a block norm with four extreme points [Mic87]. Additionally, the existence of a solution for which all optimal location sites are contained in the metric hull which is a subset of the convex hull was shown. Other dominating set results can be found in [Pla92, CF98], where the classical Dominance Theorem from [Pla65] was generalized to skewed norms. Using subdifferential calculus and the concept of subgradients, optimality conditions were derived and a geometric interpretation was given [ILM89, LMP90]. In [ILM89] the dual was stated and a primal-dual solution approach was given. This was also done in a more general setting, called general Goal Programming problem [CF02], where optimality conditions were derived and an interior point method was stated and its polynomial complexity was proven. Another polynomial algorithm for gauge distances can be found in [Fli98].

Regarding the application part of this paper we also want to give a short review of hub location literature. The hub location problem as far as we know, was firstly introduced in [O'K86]. Since then many researchers focused on different versions of the problem, especially on the discrete version. However, to the best of our knowledge, for the continuous version of the hub location problem there was a lack of research during the past 20 years. For an extensive review on hub location literature see for example [FHAN13]. As far as we know, we are the first to apply a finite dominating set approach to the continuous hub location problem.

3 Problem Definitions and Optimality Conditions

In this section, we will give exact mathematical definitions of the problems and state optimality conditions. But first, we will start with some basic notations and definitions.

3.1 Preliminaries

In what follows, we will use the notation $[N] := \{1, \ldots, N\}$ to denote the set of indices from 1 to $N \in \mathbb{N}_{>0}$ and $aS := \{as \in \mathbb{R}^n \mid s \in S\}$ for $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}^n$. By $\times_{i=1}^N S_i$ we denote the cartesian product of some sets $S_i \subseteq \mathbb{R}^n$ and by $\mathrm{bd}(S), \mathrm{int}(S), \mathrm{cl}(S), S^c$ we denote the boundary, interior, closure and the complement of S, respectively. Finally, by $U_\varepsilon(x)$ we denote the closed ε-neighborhood of a given point x. The following definitions from convex analysis (cf. [Roc72]) are needed to understand the basic concept and the optimality conditions stated in this paper.

Definition 3.1
Let B be a subset of $\mathbb{R}^n$. Then the *gauge* of B for a given point $v \in \mathbb{R}^n$ is given by

$$\gamma_B(v) := \inf\{\mu > 0 \mid v \in \mu B\}.$$

If B is a polytope with the origin in its interior, γ_B is called a *polyhedral gauge*. The set of extreme points is denoted with $\mathrm{Ext}(B) = \{b_1, \ldots, b_R\}$.

If it is clear from context which unit ball B is considered, we will shortly write γ instead of γ_B.

Definition 3.2
The *polar set* of B is defined by

$$B^\circ := \left\{ v \in \mathbb{R}^n \;\middle|\; \sup_{b \in B} \langle b, v \rangle \leq 1 \right\}.$$

If B is polyhedral, this can be rewritten as

$$B^\circ = \left\{ v \in \mathbb{R}^n \;\middle|\; \langle b_r, v \rangle \leq 1, \forall r = 1, \ldots, R \right\}$$

with extreme points $\mathrm{Ext}(B^\circ) = \{b_1^\circ, \ldots, b_{R'}^\circ\}$. Its corresponding gauge is called *polar gauge* and will be denoted by γ_B°, or shortly γ°. If $B \subseteq \mathbb{R}^2$, the number of extreme points is equal, i.e., $R = R'$.

Definition 3.3
The *normal cone* $N_S(p)$ to a convex set S at $p \in S$ is defined by

$$N_S(p) := \{x \in \mathbb{R}^n \mid \forall q \in S : \langle x, q - p \rangle \leq 0\}.$$

3.2 Problem Definitions

The planar multifacility location problem deals with finding locations for K facilities in the plane, where $K \in \mathbb{N}_{>0}$. The locations of these (new) facilities will be denoted by $x_1, \ldots, x_K \in \mathbb{R}^2$. With $X = (x_1, \ldots, x_K) \in \mathbb{R}^{2K}$, we refer to the solution vector. As input data, there are $M \in \mathbb{N}_{>0}$ demand points (or also called existing facilities), given by the set $\mathcal{A} := \{a_m \in \mathbb{R}^2 \mid m \in [M]\}$. Between each pair

of facilities, a weight is given. The weights between the new facilities are denoted by $\tilde{w}_{kl} \in \mathbb{R}_{\geq 0}$ for $k, l \in [K]$, and the weights between new and existing facilities with $w_{km} \in \mathbb{R}_{\geq 0}$ for $k \in [K]$, $m \in [M]$. Additionally, to each weight, there will be polyhedral gauges γ_{km} and $\tilde{\gamma}_{kl}$ given by their unit balls $B_{km}, \tilde{B}_{km} \subseteq \mathbb{R}^2$, respectively. The unit balls have the extreme points $\text{Ext}(B_{km}) = \left\{ b_1^{km}, \ldots, b_{R_{km}}^{km} \right\}$, resp. $\text{Ext}\left(\tilde{B}_{kl}\right) = \left\{ \tilde{b}_1^{kl}, \ldots, \tilde{b}_{\tilde{R}_{kl}}^{kl} \right\}$ as input data. For each $k \in [K]$, let $F_k \subseteq \mathbb{R}^2$ be a closed, convex set.

Definition 3.4

The *Multifacility Location Problem* finds K locations $x_1, \ldots, x_K$ in the plane minimizing

$$
\begin{aligned}
\min \quad & \Phi(X) := \sum_{k=1}^{K} \sum_{l=1}^{K} \tilde{w}_{kl} \tilde{\gamma}_{kl}(x_k - x_l) + \sum_{k=1}^{K} \sum_{m=1}^{M} w_{km} \gamma_{km}(x_k - a_m) \\
\text{s. t.} \quad & x_k \quad \in \quad F_k \qquad \forall k \in [K].
\end{aligned}
\tag{P_F}
$$

If F_k is equal to $\mathbb{R}^2$ for all $k \in [K]$, the problem is called *unconstrained*, otherwise it is called *constrained*.

The following notations will come in handy to analyze this problem. Let $\mathcal{A}_k := \{a_m \in \mathcal{A} \mid w_{km} > 0\}$ and $\mathcal{M}_k := \{m \in [M] \mid w_{km} > 0\}$ be the set of demand points and their indices associated with facility $k \in [K]$.

The underlying directed graph, denoted by $G = (V_X \cup V_A, E_X \cup E_A)$ of a planar location problem has vertex sets $V_X = \{\tilde{1}, \ldots, \tilde{K}\}$ (where $|V_X| = K$), $V_A = \{1, \ldots, M\}$ and edge sets $E_X = \{(\tilde{k}, \tilde{l}) \mid \tilde{k}, \tilde{l} \in V_X, \tilde{w}_{kl} > 0\}$, $E_A = \{(\tilde{k}, m) \mid k \in V_X, m \in V_A, w_{km} > 0\}$. With $G_X = (V_X, E_X)$ we denote the subgraph consisting of the new facilities. In the following, we will often write $(k, l) \in E_X$ and $(k, m) \in E_A$ instead of $(\tilde{k}, \tilde{l}) \in E_X$, resp. $(\tilde{k}, m) \in E_A$.

3.3 Optimality Conditions

Since (P_F) is a convex optimization problem, it is possible to state necessary and sufficient optimality conditions. These conditions can be derived by using sub-differential calculus.

Theorem 3.5 (Optimality Conditions for (P_F), [ILM89] and [LMP90])

3.5(i) Let F_k be convex sets in $\mathbb{R}^2$. The feasible point $X^* = \left(x_1^*, \ldots, x_K^*\right) \in \bigtimes_{k \in [K]} F_k$ is optimal for (P_F) if and only if there exist vectors $\tilde{u}_{kl} \in \mathbb{R}^2$ for $(k, l) \in E_X$, $u_{km} \in \mathbb{R}^2$ for $(k, m) \in E_A$ and $\bar{u}_k \in \mathbb{R}^2$ for $k \in [K]$ satisfying:
- the *ball conditions*

$$
\tilde{\gamma}_{kl}^{\circ}(\tilde{u}_{kl}) \leq 1 \qquad (k, l) \in E_X \tag{1a}
$$

$$
\gamma_{km}^{\circ}(u_{km}) \leq 1 \qquad (k, m) \in E_A \tag{1b}
$$

- the *cone conditions*

$$
x_k^* \in x_l^* + N_{\tilde{B}_{kl}^{\circ}}(\tilde{u}_{kl}) \qquad (k, l) \in E_X \tag{2a}
$$

$$
x_k^* \in a_m + N_{B_{km}^{\circ}}(u_{km}) \qquad (k, m) \in E_A \tag{2b}
$$

$$
\bar{u}_k \in N_{F_k}\left(x_k^*\right) \qquad k \in [K] \tag{2c}
$$

- the *flow conservation constraints*

$$\sum_{m:(k,m)\in E_A} w_{km}u_{km} + \sum_{l:(k,l)\in E_X} \tilde{w}_{kl}\tilde{u}_{kl} - \sum_{l:(l,k)\in E_X} \tilde{w}_{lk}\tilde{u}_{lk} + \bar{u}_k = 0, \qquad k \in [K] \qquad (3)$$

3.5(ii) If $F_k = \mathbb{R}^2$ for all $k \in [K]$, then if (X^*, U^*) is a pair satisfying the above optimality condition, then for any $X \in \mathrm{OPT}(P_F)$, the pair (X, U) must also satisfy the optimality conditions.

4 Finite Dominating Sets

In this section we will consider the problem

$$\min \sum_{(k,l)\in E_X} \tilde{w}_{kl}\tilde{\gamma}_{kl}(x_k - x_l) + \sum_{(k,m)\in E_A} w_{km}\gamma_{km}(x_k - a_m) \qquad (P_F^{Tree})$$
$$\text{s. t.} \quad x_k \ \in \ F_k \qquad \forall k \in [K],$$

where the undirected version of $G_X = (V_X, E_X)$ is a spanning tree and each F_k is polyhedral. This problem can be solved by linear programming algorithms. However, the theoretical result that there exists a finite dominating set (FDS), is shown in this section.

The cone conditions (2b) for the demand points

$$x_k \in a_m + N_{B_{km}^\circ}(u_{km}) \qquad\qquad \forall(k, m) \in E_A \qquad (2b)$$

will play an important role in this proof. Fixing k, then for any $u \in \underset{m\in\mathcal{M}_k}{\bigtimes} B_{km}^\circ$ let

$$G_k(u) := \bigcap_{m\in\mathcal{M}_k} a_m + N_{B_{km}^\circ}(u_{km})$$

be the geometric object – possibly empty – defined by those cone conditions.

Definition 4.1
Given Problem (P_F^{Tree}). The set of *intersection points* $\mathcal{I}_k(P_F^{Tree})$ for a facility $k \in [K]$ is defined as

$$\mathcal{I}_k(P_F^{Tree}) := \left\{ x \in \mathbb{R}^2 \ \middle|\ \exists u \in \underset{m\in\mathcal{M}_k}{\bigtimes} B_{km}^\circ : x \in \mathrm{Ext}(G_k(u) \cap F_k) \right\}.$$

If it is clear which problem is considered we will just write the short version $\mathcal{I}_k$.

Definition 4.2
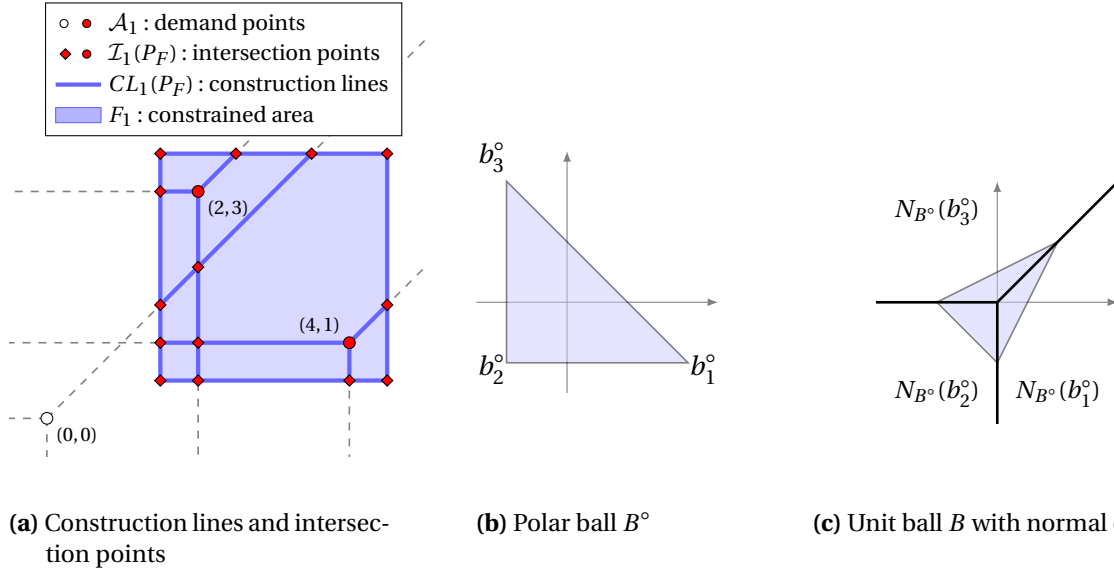Given Problem (P_F^{Tree}). The set of *construction lines* for a facility $k \in [K]$ is defined as

$$CL_k(P_F^{Tree}) := \left\{ x \in \mathbb{R}^2 \ \middle|\ \exists u \in \underset{m\in\mathcal{M}_k}{\bigtimes} B_{km}^\circ : x \in \mathrm{bd}(G_k(u) \cap F_k) \right\}.$$

Again, if it is clear which problem is considered we will just write the short version CL_k.

Example 4.3
For a single facility location problem, consider the set of demand points for facility x_1 given by $\mathcal{A}_1 = \{(0,0), (4,1), (2,3)\}$ and let

$$F_1 = \mathrm{conv}((1.5, 0.5), (4.5, 0.5), (4.5, 3.5), (1.5, 3.5))$$

(a) Construction lines and intersection points

(b) Polar ball $B°$

(c) Unit ball B with normal cones

Figure 1: Part 1a: Intersection points and Construction lines of Example 4.3. Part 1b: The polar set is given by the extreme points $\text{Ext}(B°) = \{(2,-1),(-1,-1),(-1,2)\}$. Part 1c: the normal cones are spanned by the extreme points of B.

be the constrained area as depicted in Figure 1a. We consider only a single block norm, induced by the unit ball with extreme points $\text{Ext}(B) = \{(1,1),(0,-1),(-1,0)\}$, to measure the distance between the facility and each demand point. Then the construction lines $CL_1(P_F^{Tree})$ consist of the boundary of F_1 and the fundamental directions originating at each demand point $a \in \mathcal{A}_1$ in direction of the extreme points of B intersecting with F. The intersection points are the intersections of the construction lines and the extreme points of F. Hereby, we only consider intersections which are points, i.e., have dimension zero. The set of construction lines and intersection points is shown in Figure 1a. The polar set $B° = \text{conv}((2,-1),(-1,-1),(-1,2))$ and the normal cones incduced by the extreme points of the polar set are illustrated in Figures 1b and 1c.

Definition 4.4
We say (P_F) has *minimal dimension* if for all optimal solutions $X \in \text{OPT}$ holds

$$x_k \neq x_l \qquad \forall (k,l) \in E_X.$$

To keep the analysis simple, we assume as a first step that the problem has minimal dimension. Note that any K-facility location problem where $x_k = x_l$ and $(k,l) \in E_X$ in an optimal solution $X \in \text{OPT}$ can be reformulated as a $(K-1)$-facility location problem with the same optimal objective value by contracting the edge (k,l). The new constrained area is $F_k \cap F_l$. Iteratively doing so will yield a location problem with minimal dimension. However, although the objective value will stay the same, the solution set OPT might not contain all the solutions of the original problem. We will extend the results for location problems without minimal dimension later on.

Notation 4.5
In the following we will set $\bar{w}_{lk} = \bar{w}_{kl}, \tilde{B}_{lk} = \tilde{B}_{kl}$ and $\tilde{R}_{lk} = \tilde{R}_{kl}$ for each edge $(k,l) \in E_X$. Note that $\bar{w}_{lk}, \tilde{B}_{lk}, \tilde{R}_{lk}$ are well-defined, as E_X is a tree and, hence, from $(k,l) \in E_X$ follows $(l,k) \notin E_X$ and vice versa.

Remark 4.6

Observe that for a gauge γ_B with unit ball B and a weight $w > 0$, it holds

$$w\gamma_B(-x) = \gamma_{B/w}(-x) = \gamma_{-B/w}(x) = w\gamma_{-B}(x).$$

Furthermore, the addition of any two gauges can be written as a single third gauge

$$\gamma_{B_1}(x) + \gamma_{B_2}(x) = \gamma_B(x).$$

This can be seen by using the polar set to reformulate the definition of a gauge

$$
\begin{aligned}
\gamma_{B_1}(x) + \gamma_{B_2}(x) &= \sup_{b_1 \in B_1^\circ} \langle b_1, x \rangle + \sup_{b_2 \in B_2^\circ} \langle b_2, x \rangle \\
&= \sup\{\langle b_1 + b_2, x \rangle \mid b_1 \in B_1^\circ, b_2 \in B_2^\circ\} \\
&= \sup\{\langle b, x \rangle \mid b \in \underbrace{B_1^\circ + B_2^\circ}_{=:B^\circ}\} \\
&= \gamma_B(x),
\end{aligned}
$$

where $B_1^\circ + B_2^\circ$ is the Minkowski sum. Hence, by setting $B = (B_1^\circ + B_2^\circ)^\circ$, we can write the sum of two polyhedral gauges as one. Therefore, we can assume without loss of generality that either edge $(k, l) \in E_X$ or edge $(l, k) \in E_X$. For the same reason, there is no need to consider distances $\gamma(a_m - x_k)$.

Definition 4.7

Two facilities (locations) x_k and x_l are called *adjacent* if $(k, l) \in E_X$ or $(l, k) \in E_X$. For short, we write k and l are adjacent.

Definition 4.8 (Recursive construction of intersection points)

Let a location problem (P_F^{Tree}) with minimal dimension be given. We say $X = (x_1, \ldots, x_K)$ is a *recursive intersection point* if and only if $x_k \in \mathcal{I}_k^{K-1}(P_F^{Tree})$ for all $k \in [K]$, where we define the initial set $\mathcal{I}_k^0(P_F^{Tree}) := \mathcal{I}_k(P_F^{Tree})$ and for $i = 1, 2, \ldots, K-1$ recursively:

$x_k \in \mathcal{I}_k^i(P_F^{Tree}) :\Leftrightarrow$ One of the following conditions hold

4.8(i) $x_k \in \mathcal{I}_k^{i-1}(P_F^{Tree})$.

4.8(ii) There exists an adjacent l to k with $x_l \in \mathcal{I}_l^{i-1}(P_F^{Tree})$ and an $r \in [\tilde{R}_{kl}]$, such that

$$
x_k \in \begin{cases}
\text{Ext}\left((x_l + \mathbb{R}_{>0}\tilde{b}_r^{kl}) \cap CL_k(P_F^{Tree})\right) & \text{if } (k, l) \in E_X, \\
\text{Ext}\left((x_l - \mathbb{R}_{>0}\tilde{b}_r^{lk}) \cap CL_k(P_F^{Tree})\right) & \text{if } (l, k) \in E_X.
\end{cases}
$$

4.8(iii) There exists two adjacent l_1, l_2 to k with $x_{l_1} \in \mathcal{I}_{l_1}^{i-1}(P_F^{Tree})$ and $x_{l_2} \in \mathcal{I}_{l_2}^{i-1}(P_F^{Tree})$, such that for suitable $r_1 \in [\tilde{R}_{kl_1}]$ and $r_2 \in [\tilde{R}_{kl_2}]$, the location x_k is the unique intersection point of the two rays E_1, E_2 defined by

$$
E_z = \begin{cases}
x_{l_z} + \mathbb{R}_{\geq 0}\tilde{b}_{r_z}^{kl_z} & \text{if } (k, l_z) \in E_X, \\
x_{l_z} - \mathbb{R}_{\geq 0}\tilde{b}_{r_z}^{l_z k} & \text{if } (l_z, k) \in E_X
\end{cases}
$$

for $z = 1, 2$. In particular

$$\{x_k\} = E_1 \cap E_2 \cap F_k.$$

The set of all recursive intersection points is denoted by $\mathcal{I}_{rec}(P_F^{Tree})$, or shortly, $\mathcal{I}_{rec}$ if it is clear which location problem is considered. Also the sets $\mathcal{I}_k^i(P_F^{Tree})$ will be abbreviated by $\mathcal{I}_k^i$.

Example 4.9

Let the gauge γ for all pair of facilities be given by the unit ball

$$B = \text{conv}\left((2,0),(0,2),(-3,-4)\right).$$

Consider a 5-facility location problem with the following demand points and weights:

$$
\begin{array}{lllllll}
a_1 = (2,5), & a_2 = (8,3), & a_3 = (12,0), & a_4 = (4,-3), & a_5 = (6,-1), & a_6 = (10,5), & a_7 = (14,5), \\
w_{11} = 1, & w_{32} = 1, & w_{33} = 1, & w_{44} = 1, & w_{45} = 1, & w_{56} = 1, & w_{57} = 1,
\end{array}
$$

with weights between the new facilities

$$\bar{w}_{21} = 1, \ \bar{w}_{32} = 1, \ \bar{w}_{35} = 0.5, \ \bar{w}_{43} = 1,$$

Let the following feasible regions be given:

$$
\begin{aligned}
F_1 &= \text{conv}\left(\{(5,6),(5,4),(6,4)\}\right), \\
F_2 &= \text{conv}\left(\{(0,0),(2,0),(2,2),(0,2)\}\right), \\
F_3 &= \text{conv}\left(\{(10,0),(10,3),(7,3),(7,0)\}\right), \\
F_4 &= \text{conv}\left(\{(6,-4.5),(6.5,-4.5),(7.25,-3.75),(6,-2.5)\}\right), \\
F_5 &= \text{conv}\left(\{(10.5,4),(10.5,7),(13,5)\}\right)
\end{aligned}
$$

This problem can be solved by linear programming algorithms which would return a solution like

$$x_1 = (5,5), \ x_2 = (2,1), \ x_3 = (8.625,1.0), \ x_4 = (6.0,-2.5), \ x_5 = (11.625,5).$$

Figure 2 illustrates the set of intersection points with the optimal solution of this problem.

Note that $x_k \in \mathcal{I}_k^0$ for $k = 1,4$. Location x_2 is not an intersection point, but lies on the construction line of $CL_2(P_F^{Tree})$ as it lies on the boundary of its feasible region. Therefore, by setting $l = 1$, we have $x_1 \in \mathcal{I}_1(P_F^{Tree}) = \mathcal{I}_1^0$ and

$$x_2 \in \text{Ext}\left((x_1 + \mathbb{R}_{>0}(-3,-4)) \cap CL_2(P_F^{Tree})\right),$$

thus, case 4.8(ii) is satisfied for $k = 2$, making $x_2 \in \mathcal{I}_2^1$. The facility x_3 is not an intersection point nor lies on a construction line of $CL_3(P_F^{Tree})$, therefore, we have to find l_1 and l_2 such that 4.8(iii) applies. Setting $l_1 = 2$ and $l_2 = 4$, we have that $x_{l_i} \in \mathcal{I}_{l_i}^1$ (for $i = 1,2$) and

$$x_3 \in \text{Ext}\left((x_2 + \mathbb{R}_{>0}(2,0)) \cap (x_4 - \mathbb{R}_{>0}(-3,-4))\right),$$

which makes $x_3 \in \mathcal{I}_3^2$. The only point that is not considered yet is x_5. However, we have that
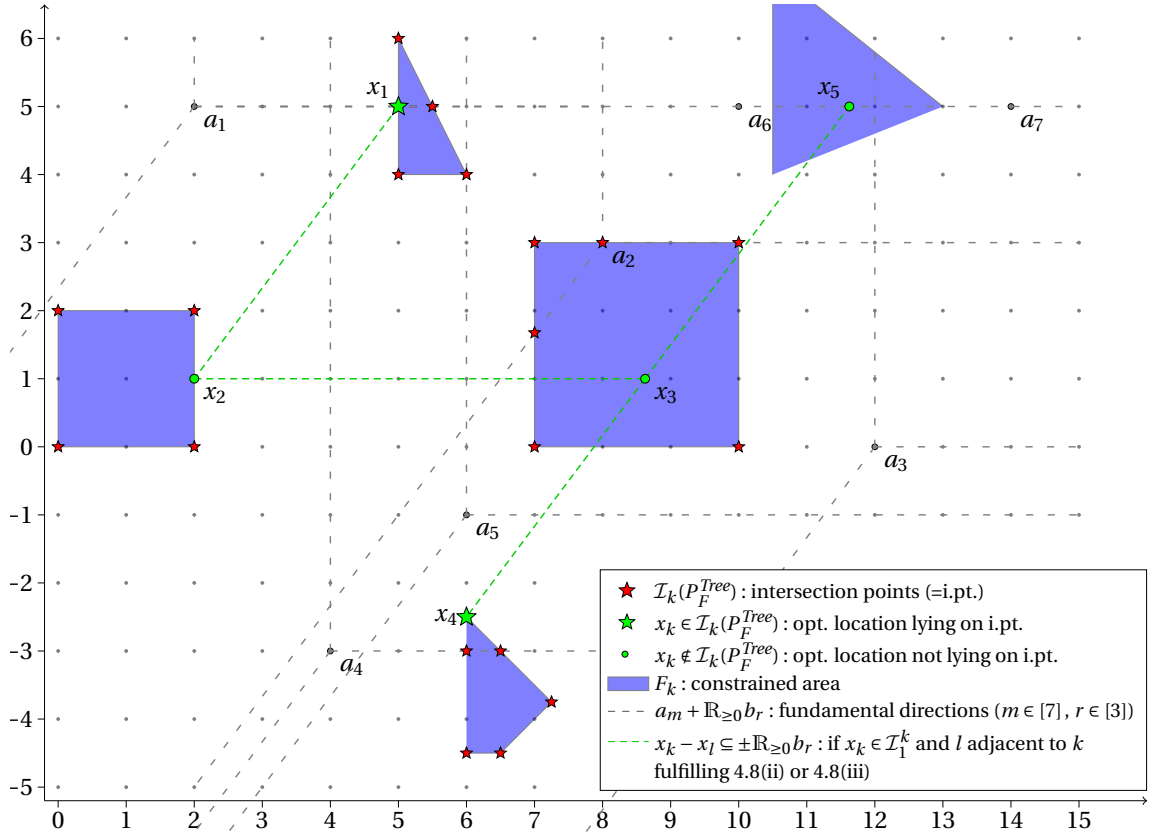
$$x_5 \in \text{Ext}\left((x_3 - \mathbb{R}_{>0}(-3,-4)) \cap CL_5(P_F^{Tree})\right),$$

making $x_5 \in \mathcal{I}_5^3$ and, consequently, $x_k \in \mathcal{I}_k^{K-1}$ for all $k \in [K]$.

Theorem 4.10

Given (P_F^{Tree}) with minimal dimension and polyhedral F_k, then there exists an optimal solution $X \in \text{OPT}(P_F^{Tree})$ with $X \in \mathcal{I}_{rec}(P_F^{Tree})$.

Figure 2: Illustration of Example 4.9 and optimal solution X. The dashed lines show fundamental directions $a + \mathbb{R}_{\geq 0} b_r$ for $a \in \mathcal{A}$ and $b_r \in \text{Ext}(B)$, resp. $x_k \pm \mathbb{R}_{\geq 0} b_r$ as described in the example.

Proof. Let $X^* = (x_1^*, \ldots, x_K^*), U^* = (u^*, \tilde{u}^*, \bar{u}^*)$ be a pair satisfying the optimality conditions in Theorem 3.5. In the following proof, U^* stays fixed. For $J_k \in \mathbb{N}_{>0}$, let $F_k = \{x \in \mathbb{R}^2 \mid \langle \Omega_i^k x \rangle \leq \omega_i^k, i \in [J_k]\}$ be the constrained areas for given suitable vectors $\Omega_i^k \in \mathbb{R}^2$ and real numbers $\omega_i^k \in \mathbb{R}$. Define

$$J_k^* := \left\{ i \in [J_k] \mid \Omega_i^k x_k^* = \omega_i^k \right\},$$

$$F_k^* := \left\{ x \in F_k \mid \Omega_i^k x = \omega_i^k, i \in J_k^* \right\},$$

where F_k^* is the smallest face containing x_k^* and let

$$G_k^* := \left(\bigcap_{m \in \mathcal{M}_k} a_m + N_{B_{km}^\circ}\left(u_{km}^*\right) \right) \cap F_k^*.$$

Note that G_k^* might be unbouded. The idea is to translate x_k^* within G_k^* for all $k \in [K]$ such that they lie in $\mathcal{I}_k^{K-1}$ and are still optimal. To do the proof, we first have to consider bounded subsets of all G_k^*.

Observe that minimizing the function

$$\Phi_F(X) := \sum_{k=1}^{K} \sum_{l=1}^{K} \tilde{w}_{kl} \tilde{\gamma}_{kl}(x_k - x_l) + \sum_{k=1}^{K} \sum_{m=1}^{M} w_{km} \gamma_{km}(x_k - a_m) + \sum_{k=1}^{K} \mathcal{X}_{F_k}(x_k),$$

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where

$$\mathcal{X}_F(x) := \begin{cases} 0 & \text{if } x \in F, \\ \infty & \text{if } x \notin F, \end{cases}$$

is equivalent to (P_F^{Tree}).

Let

$$\mathrm{OPT}_k(P_F^{Tree}) := \left\{ x \in \mathbb{R}^2 \mid \exists X = (x_1, \dots, x_K) \in \mathrm{OPT}(P_F^{Tree}) : x_k = x \right\}.$$

Since all the level sets for $z \in \mathbb{R}_{\geq 0}$

$$L_{\leq}(z) := \left\{ X \in \mathbb{R}^{2K} \mid \Phi_F(X) \leq z \right\}$$

are compact, the k-th projection of $X = (x_1, \dots, x_K) \in \mathrm{OPT}(P_F^{Tree})$ defined by

$$P_k((x_1, \dots, x_K)) = x_k$$

is a projection of the compact set $L_{\leq}(z^*)$ for an optimal objective value z^*, and, is again a compact set, that is $\mathrm{OPT}_k(P_F^{Tree})$.

Hence, there exists a bounding box $Q = [a^1, b^1] \times [a^2, b^2] \subset \mathbb{R}^2$, s.t. $\mathrm{OPT}_k(P_F^{Tree}) \subset \mathrm{int}(Q)$ for all $k \in [K]$. By definition of $\mathrm{OPT}_k(P_F^{Tree})$ it holds that

$$x \in \mathrm{bd}(Q) \implies x \notin \mathrm{OPT}_k(P_F^{Tree}) \forall k \in [K]. \tag{4}$$

Let

$$G_k^Q := \underbrace{\left(G_k(u^*) \cap F_k^* \right)}_{=G_k^*} \cap Q.$$

We will now redefine the set of recursive intersection points for geometric objects. Therefore, define for a family of polygons $G := (G_k)_{k \in [K]}$ analogue to Definition 4.8 the initial set of recursive intersectionpoints as

$$\mathcal{I}_k^0(G_k) := \mathrm{Ext}(G_k)$$

and recursively for $i = 1, 2, \dots, K-1$:

$x_k \in \mathcal{I}_k^i(G_k) :\Leftrightarrow$ One of the following conditions holds

  5(i) $x_k \in \mathcal{I}_k^{i-1}(G_k)$

  5(ii) If $x_k \in \mathrm{bd}(G_k)$, there exists an adjacent l to k with $x_l \in \mathcal{I}_l^{i-1}(G_l)$ and $u \in \tilde{B}_{kl}$ such that
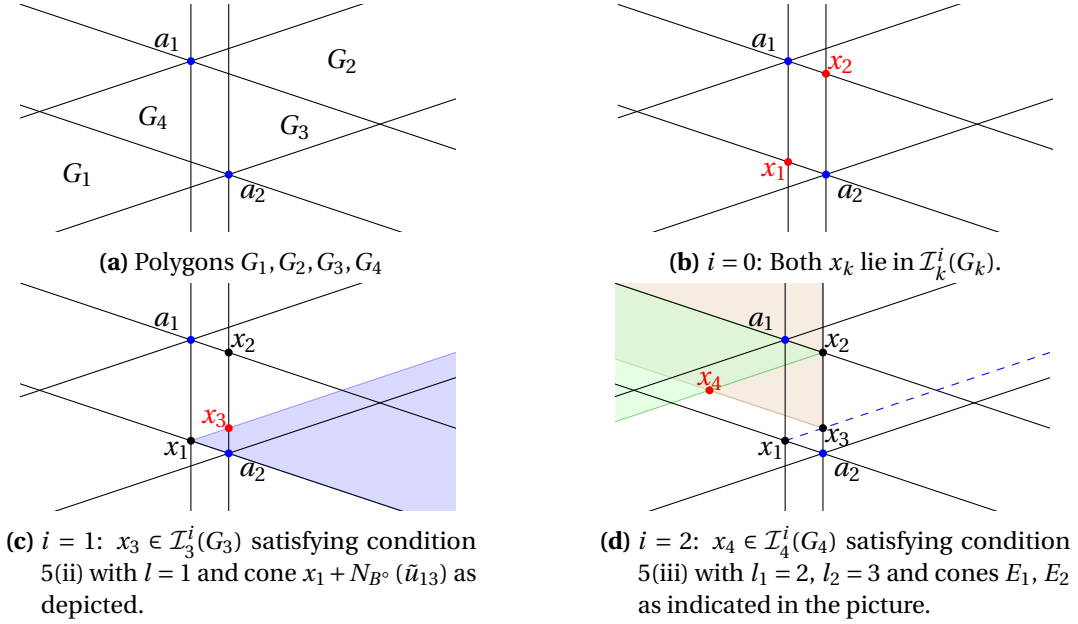 $$x_k \in \begin{cases} \mathrm{Ext}\left((x_l + N_{\tilde{B}_{kl}}(u)) \cap G_k \right) & \text{if } (k, l) \in E_X, \\ \mathrm{Ext}\left((x_l - N_{\tilde{B}_{lk}}(u)) \cap G_k \right) & \text{if } (l, k) \in E_X \end{cases}$$

  5(iii) If $x_k \in \mathrm{int}(G_k)$, there exists two adjacent l_1, l_2 to k with $x_{l_1} \in \mathcal{I}_{l_1}^{i-1}(G_{l_1})$ and $x_{l_2} \in \mathcal{I}_{l_2}^{i-1}(G_{l_2})$, such that for $u_1 \in \tilde{B}_{kl_1}$ and $u_2 \in \tilde{B}_{kl_2}$, the location x_k is an extreme point of the intersection of the two cones E_1, E_2 defined by
  $$E_z = \begin{cases} x_{l_z} + N_{\tilde{B}_{kl_z}}(u_z) & \text{if } (k, l_z) \in E_X, \\ x_{l_z} - N_{\tilde{B}_{kl_z}}(u_z) & \text{if } (l_z, k) \in E_X \end{cases}$$
  for $z = 1, 2$. In particular
  $$x_k \in \mathrm{Ext}(E_1 \cap E_2).$$

(a) Polygons G_1, G_2, G_3, G_4

(b) $i = 0$: Both x_k lie in $\mathcal{I}_k^i(G_k)$.

(c) $i = 1$: $x_3 \in \mathcal{I}_3^i(G_3)$ satisfying condition 5(ii) with $l = 1$ and cone $x_1 + N_{B^\circ}(\tilde{u}_{13})$ as depicted.

(d) $i = 2$: $x_4 \in \mathcal{I}_4^i(G_4)$ satisfying condition 5(iii) with $l_1 = 2$, $l_2 = 3$ and cones E_1, E_2 as indicated in the picture.

Figure 3: Illustration of the recursive construction of an intersection point $X = (x_1, x_2, x_3, x_4) \in \mathcal{I}_{rec}(G)$ with $G = \times_{k \in [4]} G_k$ as shown in part (a). Parts (b),(c),(d) show the recursion steps.

For illustration of the concept see Figure 3.

The set of all $X = (x_1, \ldots, x_K)$ with $x_k \in \mathcal{I}_k^{K-1}(G_k)$ for all $k \in [K]$ will be denoted by $\mathcal{I}_{rec}(G)$. Note that for $G_k = G_k^*$ the above definition is equivalent to Definition 4.8. To see that, recall that all intersection points $\mathcal{I}_k(P_F^{Tree})$ can be written as extreme points of some polyhedron

$$\underbrace{\left(\bigcap_{m \in \mathcal{M}_k} a_m + N_{B_{km}^\circ}(u_{km}) \right)}_{=G_k(u)} \cap \tilde{F}_k \tag{6}$$

for a suitable $u = (u_{km})_{m \in \mathcal{M}_k}$ and a face $\tilde{F}_k$ of F_k. The same holds for the points on the construction lines, being on the boundary of such a polyhedron. Hence, if we show that $X \in \mathcal{I}_{rec}(G^*) := \mathcal{I}_{rec}((G_k^*)_{k \in [K]})$ we are done with the proof.

The remaining proof is done in the following way:

Step 1: Given any bounded polyhedral subsets $G_k \subseteq G_k^*$ with $x_k^* \in G_k$ for all $k \in [K]$, we show by induction that we can construct a vector $X \in G := \times_{k=1}^K G_k$, such that $X \in \mathcal{I}_{rec}(G)$ and the cone conditions for the new facilities
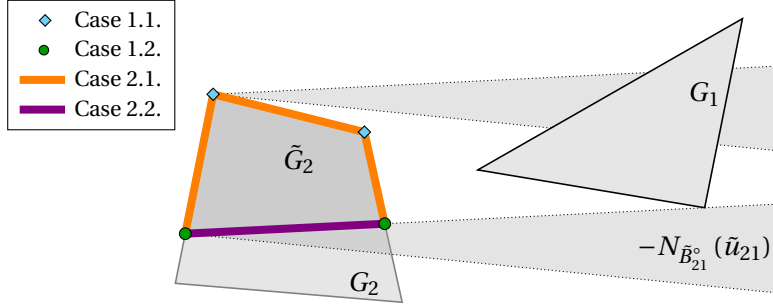
$$x_k \in x_l + N_{\tilde{B}_{kl}^\circ}(\tilde{u}_{kl}^*) \qquad \forall (k, l) \in E_X \tag{2a}$$

are still satisfied.

Step 2: As (X^*, U^*) satisfied the optimality conditions beforehand, U^* still satisfies the ball constraints (1) and the flow conservation (3) constraints. Then, for $X \in \times_{k=1}^K G_k \subseteq \times_{k=1}^K G_k^*$ obtained in the previous step, also satisfies the cone conditions (2b) and (2c)

$$x_k \in a_m + N_{B_{km}^\circ}(u_{km}^*) \qquad (k, m) \in E_A \tag{2b}$$

$$\bar{u}_k^* \in N_{F_k}(x_k^*) \qquad k \in [K]. \tag{2c}$$

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Figure 4: Choice if $\tilde{G}_2$ and case distinction of induction step.

The last cone condition (2a) is obtained from the induction hypotheses, i.e., (X, U^*) is a pair satisfying all optimality conditions in Theorem 3.5, making X optimal.

Step 3: By choosing the subsets $G^Q := \left(G_k^Q\right)_{k \in [K]}$ we can construct by the first step an $X \in \mathcal{I}_{rec}(G^Q)$ that is still optimal. However, by (4) no x_k can lie on the boundary of Q, making $X \in \mathcal{I}_{rec}(G^*)$.

It remains to show Step 1.

Induction Base: $K = 1$

Let a polyhedral $G_1 \subseteq G_1^*$ be given, such that $x_1^* \in G_1$. Then, for only one facility, we can just choose $x_1 \in \text{Ext}(G_1)$ and are done.

Induction Hypotheses: For an arbitrary, but fixed K, given a solution pair (X^*, U^*) and bounded sets $G_k \subseteq G_k^*$ with $x_k^* \in G_k$ for all $k \in [K]$, we can construct a vector $X \in \times_{k=1}^{K} G_k$, such that $X \in \mathcal{I}_{rec}(G)$ and the cone conditions

$$x_k \in x_l + N_{\tilde{B}_{kl}^\circ}\left(\tilde{u}_{kl}^*\right) \qquad\qquad \forall (k, l) \in E_X \qquad (7)$$

are still satisfied.

Induction Step: $K \mapsto K + 1$

As before, let (X^*, U^*) be an optimal pair satisfying the optimality conditions stated in Theorem 3.5 and assume that $x_k^* \neq x_l^*$ for all $k, l \in [K]$, since (P_F^{Tree}) is minimal. Without loss of generality let $(2, 1) \in E_X$ and let 1 be a leaf of G_X. Define (see Figure 4)

$$\tilde{G}_k := \begin{cases} \left\{ x \in G_2 \,\middle|\, \left(x - N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right) \cap G_1 \neq \emptyset \right\}, & \text{if } k = 2, \\ G_k, & \text{if } k \in [K+1] \setminus \{2\}. \end{cases}$$

Note, that all the points in $G_2 \setminus \tilde{G}_2$ are exactly the points that cannot satisfy the cone conditions (7). Also $x_2^* \in \tilde{G}_2$, therefore, we can use the induction hypotheses on facilities $2, \dots, K+1$ and edge set $E_X \setminus \{(2, 1)\}$. Hence, we can construct $\hat{X} := (x_2, \dots, x_{K+1}) \in \times_{k=2}^{K+1} \tilde{G}_k$ such that for each x_k one point of 5(i) - 5(iii) holds for facilities $2, \dots, K+1$, i.e.,

$$\hat{X} \in \mathcal{I}_{rec}\left(\left(\tilde{G}_k\right)_{k \in [K+1] \setminus \{1\}}\right).$$

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Choose

$$x_1 \in \underbrace{\begin{cases} \left(x_2 - N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right) \cap \mathrm{Ext}\left(G_1\right), & \text{if } \left(x_2 - N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right) \cap \mathrm{Ext}\left(G_1\right) \neq \emptyset, \\ \mathrm{Ext}\left(\left(x_2 - N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right) \cap G_1\right), & \text{else.} \end{cases}}_{\neq \emptyset \text{ by construction of } \tilde{G}_2}$$

By the choice of x_1 and the induction hypothesis, the cone condition (7) are satisfied. Also observe that by definition of $\tilde{G}_k = G_k$ for $k = 3, \dots, K+1$, it especially holds

$$\mathrm{Ext}\left(\tilde{G}_k\right) = \mathrm{Ext}\left(G_k\right) \quad \wedge \quad \mathrm{bd}\left(\tilde{G}_k\right) = \mathrm{bd}\left(G_k\right) \qquad \forall k = 3, \dots, K+1, \tag{8a}$$

implying, together with the fact that x_1 is only interacting with x_2, that for $j := \min\{i \mid x_2 \in \mathcal{I}_2^i(\tilde{G}_2)\}$, we also have

$$\left(x_k \in \mathcal{I}_k^i(\tilde{G}_k) \implies x_k \in \mathcal{I}_k^i(G_k)\right) \qquad \forall k = 3, \dots, K+1, i \le j. \tag{8b}$$

This means we are done with the induction step by showing

$$\begin{aligned} &x_2 \in \mathcal{I}_2^j(\tilde{G}_2) \text{ considering edge set } E_X \setminus \{(2,1)\} \\ &\implies x_2 \in \mathcal{I}_2^{j+1}(G_2) \text{ considering edge set } E_X. \end{aligned} \tag{9a}$$

and

$$x_1 \in \mathcal{I}_1^{j+1}(G_1). \tag{9b}$$

From (9) and the induction hypothesis immediately follows for the remaining locations

$$\left(x_k \in \mathcal{I}_k^i(\tilde{G}_k) \implies x_k \in \mathcal{I}_k^{i+1}(G_k)\right) \qquad \forall k = 3, \dots, K+1 \text{ with } i > j.$$

As (P_F^{Tree}) has minimal dimension, we have $x_1 \neq x_2$ and there are the following cases (see Figure 4 for illustration):

Case 1. $x_2 \in \mathrm{Ext}\left(\tilde{G}_2\right)$: Since $x_1 \neq x_2$ we must have $x_2 \in \mathrm{bd}\left(G_2\right)$. Therefore, we can only have two subcases:

Case 1.1. $x_2 \in \mathrm{Ext}\left(G_2\right)$: As $\mathrm{Ext}\left(G_2\right) = \mathcal{I}_2^0(G_2)$ and x_1 is a leaf, we already have by induction hypotheses that $x_k \in \mathcal{I}_k^{K-1}(G_k)$ for all $k \in [K+1] \setminus \{1\}$. Furthermore, either

$$x_1 \in \mathrm{Ext}\left(G_1\right) \text{ or } x_1 \in \mathrm{Ext}\left(G_1 \cap \left(x_2 - N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right)\right)$$

showing that $x_1 \in \mathcal{I}_1^0(G_1)$ or $x_1 \in \mathcal{I}_1^1(G_1)$. Hence, $X = (x_1, \dots, x_{K+1}) \in \mathcal{I}_{rec}(G)$ for this case.

Case 1.2. $x_2 \in \mathrm{bd}\left(G_2\right) \setminus \mathrm{Ext}\left(G_2\right)$: In this case we have not necessarily $x_k \in \mathcal{I}_k^{K-1}(G_k)$ for all $k = 2, \dots, K$ as $x_2 \in \mathcal{I}_k^0(\tilde{G}_2)$ but not $x_2 \in \mathcal{I}_k^0(G_2)$. However, this case can only appear if $x_1 \in \left(x_2 - N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right) \cap \mathrm{Ext}\left(G_1\right)$ by definition of $\tilde{G}_2$. In particular,

$$x_2 \in \mathrm{Ext}\left(G_2 \cap \left(x_1 + N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right)\right).$$

Therefore, (9) holds since $x_1 \in \mathrm{Ext}\left(G_1\right) = \mathcal{I}_1^0(G_1)$ and $x_2 \in \mathcal{I}_2^1(G_2)$. Thus, $x_k \in \mathcal{I}_k^K(G_k)$, $\forall k \in [K+1]$ which means $X = (x_1, \dots, x_{K+1}) \in \mathcal{I}_{rec}(G)$.

Case 2. $x_2 \in \mathrm{bd}\left(\tilde{G}_2\right) \setminus \mathrm{Ext}\left(\tilde{G}_2\right)$: By induction hypotheses $x_2 \in \mathcal{I}_2^j(\tilde{G}_2)$, thus, there exists an adjacent $l_1 > 2$ with $x_{l_1} \in \mathcal{I}_{l_1}^{j-1}(\tilde{G}_{l_1})$ such that 5(ii) is fulfilled, i.e.,

$$
x_2 \in \begin{cases} \mathrm{Ext}\left(\left(x_{l_1} + N_{\tilde{B}_{2l_1}^\circ}\left(\tilde{u}_{2l_1}^*\right)\right) \cap \tilde{G}_2\right) & \text{if } (2, l_1) \in E_X, \\ \mathrm{Ext}\left(\left(x_{l_1} - N_{\tilde{B}_{l_1 2}^\circ}\left(\tilde{u}_{l_1 2}^*\right)\right) \cap \tilde{G}_2\right) & \text{if } (l_1, 2) \in E_X \end{cases} \tag{10}
$$

for x_2. Moreover, by (8), we must have $x_{l_1} \in \mathcal{I}_{l_1}^{j-1}(G_{l_1})$. By construction of $\tilde{G}_2$, this case has two sub-cases:

Case 2.1. $x_2 \in \mathrm{bd}\left(G_2\right)$: Then we already have that $x_2 \in \mathcal{I}_2^j(G_2)$. By the same argumentation as above $x_1 \in \mathcal{I}_1^{j+1}(G_1)$, which means $X \in \mathcal{I}_{rec}(G)$.

Case 2.2. $x_2 \in \mathrm{int}\left(G_2\right)$: By the argumentation of Case 1, this is only possible if

$$
\left(x_2 - N_{\tilde{B}_{21}^\circ}\left(\tilde{u}_{21}^*\right)\right) \cap \mathrm{Ext}\left(G_1\right) \neq \emptyset,
$$

thus, $x_1 \in \mathrm{Ext}\left(G_1\right) = \mathcal{I}_1^0(G_1)$. Therefore, as (10) is satisfied for a $l_1 \geq 3$, we can choose $l_2 = 1$, and have that

$$
x_2 \in \mathrm{Ext}\left(E_1 \cap E_2\right),
$$

where

$$
E_z = \begin{cases} x_{l_z} + N_{\tilde{B}_{2l_z}^\circ}\left(\tilde{u}_{2l_z}^*\right) & \text{if } (2, l_z) \in E_X, \\ x_{l_z} - N_{\tilde{B}_{l_z 2}^\circ}\left(\tilde{u}_{l_z 2}^*\right) & \text{if } (l_z, 2) \in E_X \end{cases} \tag{11}
$$

for $z = 1, 2$. Thus, $x_2 \in \mathcal{I}_2^j(G_2)$ by 5(iii).

Case 3. $x_2 \in \mathrm{int}\left(\tilde{G}_2\right)$: By induction hypotheses and (8), there exist two different adjacent $l_1 > 2$ and $l_2 > 2$ with $x_{l_1} \in \mathcal{I}_{l_1}^{j-1}(G_{l_1})$ and $x_{l_2} \in \mathcal{I}_{l_2}^{j-1}(G_{l_2})$ such that 5(iii) is satisfied, i.e.,

$$
x_2 \in \mathrm{Ext}\left(E_1 \cap E_2\right),
$$

where

$$
E_z = \begin{cases} x_{l_z} + N_{\tilde{B}_{2l_z}^\circ}\left(\tilde{u}_{2l_z}^*\right) & \text{if } (2, l_z) \in E_X, \\ x_{l_z} - N_{\tilde{B}_{l_z 2}^\circ}\left(\tilde{u}_{l_z 2}^*\right) & \text{if } (l_z, 2) \in E_X \end{cases}
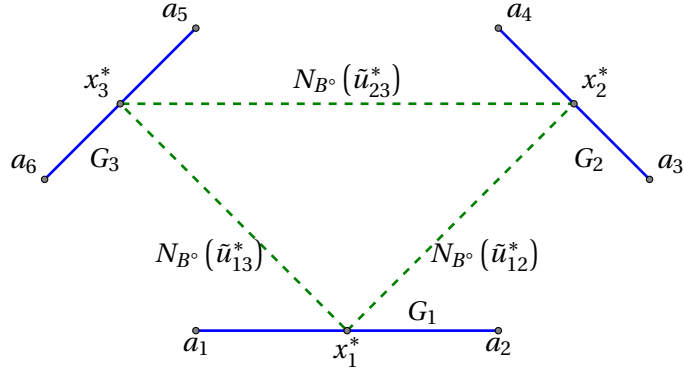$$

for $z = 1, 2$. Therefore, $x_2 \in \mathcal{I}_2^j(G_2)$ and by the choice of $x_1 \in \mathcal{I}_1^{j+1}(G_1)$.

In all cases we have $X = (x_1, \dots, x_{K+1}) \in \mathcal{I}_{rec}(G)$ and (X, U^*) satisfies the optimality conditions.

$\square$

Note that for general convex constraints, this proof does not work out as x_k^* might be any point on the boundary of F_k and we cannot translate x_k along the boundary such that the cone conditions

$$
\bar{u}_k^* \in N_{F_k}(x_k) \qquad\qquad \forall k \in [K]
$$

are still satisfied.

Figure 5: Counterexample for FDS for location problem without tree structure given in Example 4.11: Optimal solution with sets $G_k := \bigcap_{m \in \mathcal{M}_k} a_m + N_{B^\circ}(u_{km})$ plotted as solid lines. The dashed lines show the direction of the normal cones $N_{B^\circ}(\tilde{u}_{kl})$.

Example 4.11

In this example, we will show that the finite dominating set in Definition 4.8 only works for problems, where G_X is a tree. Consider the 3-facility problem with demand points

$$a_1 = (-2,0), \ a_2 = (2,0), \ a_3 = (4,2), \ a_4 = (2,4), \ a_5 = (-2,4), \ a_6 = (-4,2)$$

as depicted in Figure 5.

Let the weights between each facility be given by

$$w_{11} = w_{12} = 10,$$
$$w_{23} = w_{24} = 10,$$
$$w_{35} = w_{36} = 10,$$
$$\tilde{w}_{12} = \tilde{w}_{23} = \tilde{w}_{13} = 1.$$

All feasible regions will be $F_k = \mathbb{R}^2$ for any $k \in [K]$. There will be a single gauge between the facilities, which is given by the unit ball

$$B = \text{conv}\left((1,1),(2,0),(1,-1),(-1,-1),(-2,0),(-1,1)\right).$$

The polar set of B is given by

$$B^\circ = \text{conv}\left((0,1),(0.5,0.5),(0.5,-0.5),(0,-1),(-0.5,-0.5),(-0.5,0.5)\right)$$

Then an optimal solution pair (X^*, U^*) of Theorem 3.5 is given by

$$x_1^* = (0,0), \ x_2^* = (3,3), \ x_3^* = (-3,3)$$

and

$$
\begin{aligned}
u_{11}^* &= (0.5,-0.35), & u_{12}^* &= (-0.5,0.5), \\
u_{23}^* &= (-0.075,0.925), & u_{24}^* &= (0,-1), \\
u_{35}^* &= (-0.425,-0.575), & u_{36}^* &= (0.5,0.5),
\end{aligned}
$$

$$\tilde{u}_{12}^* = (-0.25,-0.75), \ \tilde{u}_{13}^* = (0.25,-0.75), \ \tilde{u}_{23}^* = (0.5,0).$$

By Theorem 3.5 all other optimal solutions $X \in \text{OPT}(P_F)$ fulfill the cone conditions (2), i.e.,

$$
\begin{aligned}
x_k &\in x_l + N_{B^\circ}\left(\tilde{u}^*_{kl}\right) && (k, l) \in E_X, \\
x_k &\in a_m + N_{B^\circ}\left(u^*_{km}\right) && (k, m) \in E_A.
\end{aligned}
\tag{12}
$$

As the normal cones are spanned by the extreme points of B, any $x \in N_{B^\circ}(u)$ can be written as

$$
x = \begin{cases}
\lambda_1 b_{r_1} + \lambda_2 b_{r_2} & \text{if } u \in \text{Ext}(B^\circ) \\
\lambda_1 b_{r_1} & \text{if } u \in \text{bd}(B^\circ) \setminus \text{Ext}(B^\circ) \\
0 & \text{else}
\end{cases}
$$

for suitable $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ and adjacent extreme points $b_{r_1}, b_{r_2} \in \text{Ext}(B)$. Therefore, as b_{r_1}, b_{r_2} are already defined by U^*, plugging in the above values, the cone conditions (12) form a linear system of equations:

$$
\begin{aligned}
x_1 &= a_1 + \lambda_{11}(2, 0) \\
x_1 &= a_2 + \lambda_{12}(-2, 0) + \lambda'_{12}(-1, 1) \\
x_2 &= a_3 + \lambda_{13}(-1, 1) \\
x_2 &= a_4 + \lambda_{24}(1, -1) + \lambda'_{24}(-1, 1) \\
x_3 &= a_5 + \lambda_{35}(-1, -1) \\
x_3 &= a_6 + \lambda_{36}(1, 1) + \lambda'_{36}(2, 0) \\
x_1 &= x_2 + \tilde{\lambda}_{12}(-1, -1) \\
x_1 &= x_3 + \tilde{\lambda}_{13}(1, -1) \\
x_2 &= x_3 + \tilde{\lambda}_{23}(2, 0).
\end{aligned}
$$

This system has 18 variables and 18 equalities. It is easy to verify that the matrix of the equivalent matrix equation has full rank. Therefore, there exists a unique solution which is $x_1^* = (0, 0)$, $x_2^* = (3, 3)$, $x_3^* = (-3, 3)$.

Location Problems without minimal Dimension

In the previous theorem we have described a finite dominating set for a problem (P_F) with tree structure and minimal dimension. However, it is possible that adjacent facilities coincide in an optimal solution and we do not know which facilities will coincide before solving the problem. Assume we have a spanning forest $T = \{T_i = (V_i, E_i) \mid i \in [|T|]\}$ of G_X, such that for all $k, l \in V_i$ we have $x_k = x_l$, i.e., all locations in a tree T_i coincide. Then we can define a $|T|$-facility location problem with $\mathcal{A}'_i := \bigcup_{k \in V_i} \mathcal{A}_k$ and $F'_i := \bigcap_{k \in V_i} F_k$ for $i \in [|T|]$. The edge set between the new facilities is given by

$$
E'_X := \left\{(i, j) \in [|T|] \times [|T|] \mid i \neq j, \exists k \in V_i, l \in V_j : (k, l) \in E_X\right\}.
$$

Note that if G_X is a tree, the cardinality $|\{(k, l) \mid k \in V_i, l \in V_j\}| = 1$ for $i \neq j$. The interaction between new and existing facilities is given by

$$
\begin{aligned}
\mathcal{M}_i &:= \{m \in [M] \mid \exists k \in V_i : (k, m) \in E_A\}, \\
E'_A &:= \{(i, m) \mid i \in [|T|], m \in \mathcal{M}_i\}.
\end{aligned}
$$

By Remark 4.6 the unit balls for $(i, j) \in E'_X$ are given by

$$\tilde{B}'_{ij} = \left(\sum_{k \in V_i} \sum_{\substack{l \in V_j: \\ (k,l) \in E_X}} \left(1/\bar{w}_{kl} \cdot \tilde{B}_{kl} \right)^\circ \right)^\circ$$

and for $(i, m) \in E'_A$

$$B'_{im} = \left(\sum_{\substack{k \in V_i: \\ (k,m) \in E_A}} \left(1/w_{km} \cdot B_{km} \right)^\circ \right)^\circ .$$

Then the corresponding location problem to a spanning forest T can be written as

$$\begin{aligned} \min_{X'} \quad & \Phi'(X') := \sum_{(i,j) \in E'_X} \tilde{\gamma}'_{ij}(x'_i - x'_j) + \sum_{(i,m) \in E'_A} \gamma'_{im}(x'_i - a_m) \\ \text{s. t.} \quad & x'_i \ \in \ F'_i, \qquad \forall i \in [|T|] . \end{aligned} \qquad (P_F(T))$$

Definition 4.12
A spanning forest T of G_X is called optimal for (P_F), if $(P_F(T))$ has the same optimal objective value as (P_F^{Tree}).

As consequence we get the following Corollary.

Corollary 4.13
Given problem (P_F^{Tree}). Then there exists an optimal spanning forest

$$T = \{ T_i = (V_i, E_i) \mid i \in [|T|] \}$$

of G_X, such that there exists an optimal solution $X \in \mathrm{OPT}(P_F(T))$, such that X is an recursive intersection point, i.e., $X \in \mathcal{I}_{rec}(P_F(T))$.

In addition, note that the objective value of $(P_F(T))$ is greater or equal than (P_F^{Tree}). Then by iterating over all spanning forests T of G_X, we will necessarily get a location problem with minimal dimension and the same objective value. In particular, for this problem the finite dominating set of Theorem 4.10 holds. We summarize this result in Algorithm 1.

Remark 4.14
Note that it is not necessary to calculate each of the unit balls $\tilde{B}'_{ij}$ and B'_{im} of $(P_F(T))$ in Algorithm 1. Since G_X is a tree $|\{(k, l) \in E_X \mid k \in V_i, l \in V_j\}| = 1$, hence, $\tilde{B}'_{ij}$ is just the polar set of the sum of one polar set, thus, $\tilde{B}'_{ij} = \tilde{B}_{kl}$ with $(k, l) \in E_X \cap (V_i \times V_j)$. Furthermore, since in the definition of the finite dominating set the demand points do not need to be different, we can introduce $|E_A|$ demand points $a_{(k,m)} = a_m$ for each $(k, m) \in E_A$, and write the objective as

$$\sum_{(i,j) \in E'_X} \tilde{w}'_{ij} \tilde{\gamma}'_{ij}(x'_i - x'_j) + \sum_{i \in [|T|]} \sum_{k \in V_i} \sum_{m \in \mathcal{M}_k} w_{km} \gamma_{km}(x'_i - a_{(k,m)}).$$

Hence, it is enough to use the original demand points and gauges to calculate the FDS for $(P_F(T))$.

Algorithm 1: Finding best solution for (P_F^{Tree})

Data: Location Problem (P_F^{Tree}) with tree structure
Result: Optimal solution of (P_F^{Tree})

1 $\mathrm{obj} = \infty$
2 **for** *All spanning forests $T = \{T_i = (V_i, E_i) \mid i \in [|T|]\}$ of G_X* **do**
3 $\quad$ Construct problem $P_F(T)$ as shown above
4 $\quad$ Calculate the set $\mathcal{I}_{rec}(P_F(T))$ in Definition 4.8
5 $\quad$ Choose $X' \in \mathrm{argmin}\{\Phi'(X) \mid X \in \mathcal{I}_{rec}(P_F(T))\}$
6 $\quad$ **if** $\Phi'(X') < \mathrm{obj}$ **then**
7 $\quad\quad$ $X^* = (x_1^*, \ldots, x_K^*)$ with $x_k^* = x_i'$ if $k \in V_i$ for $i \in [|T|]$
8 $\quad\quad$ $\mathrm{obj} = \Phi'(X')$

9 **return** X^*

Remark 4.15

Note that there are algorithms with a better running time. The number of spanning forests of G_X, where G_X is a tree, is

$$\sum_{i=0}^{K-1} \binom{K-1}{i}.$$

In addition, the set $\mathcal{I}_{rec}(P_F(T))$ is of exponential size, since there exist problem instances with $|\mathcal{I}_i^0| = |\mathcal{I}_i(P_F(T))| \in \Omega(M^2 R^2)$ for each $i \in [|T|]$, where $R := \max_{(k,m) \in E_A} R_{km}$. Consequently $\mathcal{I}_{rec}(P_F(T))$ has at least $(MR)^{2|T|}$ possibilities that $x_i \in \mathcal{I}_i^0$ for all $i \in [|T|]$. Future research will be to decrease the size of the FDS by also taking the flow conservation constraints into account. Since this is more of a theoretical result, we will not go too deeply into the running time analysis of Algorithm 1.

Example 4.16

Analogue to Example 4.9, let the gauge γ for all pair of facilities be given by the unit ball

$$B = \mathrm{conv}\left((2,0),(0,2),(-3,-4)\right).$$

Consider the 6-facility location problem with the following demand points and weights:

$$
\begin{array}{lllllll}
a_1 = (2,5), & a_2 = (8,3), & a_3 = (12,0), & a_4 = (4,-3), & a_5 = (6,-1), & a_6 = (10,5), & a_7 = (14,5), \\
w_{11} = 1, & w_{32} = 1, & w_{33} = 1, & w_{44} = 1, & w_{55} = 1, & w_{66} = 1, & w_{67} = 1,
\end{array}
$$

with weights between the new facilities

$$\tilde{w}_{21} = 1, \; \tilde{w}_{32} = 1, \; \tilde{w}_{36} = 0.5, \; \tilde{w}_{43} = 1, \; \tilde{w}_{54} = 60,$$

Let the following feasible regions be given:

$$
\begin{aligned}
F_1 &= \mathrm{conv}\left(\{(5,6),(5,4),(6,4)\}\right), \\
F_2 &= \mathrm{conv}\left(\{(0,0),(2,0),(2,2),(0,2)\}\right), \\
F_3 &= \mathrm{conv}\left(\{(10,0),(10,3),(7,3),(7,0)\}\right), \\
F_4 &= \mathrm{conv}\left(\{(5,-1.5),(8,-4.5),(5,-4.5)\}\right), \\
F_4 &= \mathrm{conv}\left(\{(6,0),(6,-5),(9,-2)\}\right), \\
F_6 &= \mathrm{conv}\left(\{(10.5,4),(10.5,7),(13,5)\}\right)
\end{aligned}
$$

Linear programming algorithms would return a solution like

$$x_1 = (5,5), \ x_2 = (2,1), \ x_3 = (8.625, 1.0), \ x_4 = x_5 = (6.0, -2.5), \ x_6 = (11.625, 5).$$

Therefore, we have a spanning forest $T = \{T_1, \ldots, T_5\}$ defined by

$$T_1 := (\{1\}, \emptyset), \quad T_2 := (\{2\}, \emptyset), \quad T_3 := (\{3\}, \emptyset), \quad T_4 := (\{4,5\}, \{(4,5)\}), \quad T_5 := (\{6\}, \emptyset).$$

As $F_4' = F_4 \cap F_5 = \text{conv}(\{(6, -4.5), (6.5, -4.5), (7.25, -3.75), (6, -2.5)\})$, the contracted location problem $(P_F(T))$ becomes the 5-facility location problem in Example 4.9 and, therefore, lies in the FDS.

5 Application to Planar Hub Location

In this section we will show how we can use the FDS result to solve hub location problems in the plane, in particular, the 2-facility hub location problem. In this problem we have given the set of demand points $\mathcal{A} = \{a_m \in \mathbb{R}^2 \mid m \in [M]\}$ and, in contrast to the planar multifacility problem (P_F), there is a weight $v_{mn} \in \mathbb{R}_{\geq 0}$ between each pair of demand points $m, n \in [M]$. Additionally, the polyhedral gauge γ will be a block norm, i.e., it has the symmetry $\gamma(x) = \gamma(-x)$ as additional property. The task is to find not only locations x_k in some polyhedral areas $F_k \subseteq \mathbb{R}^2$ for the two facilities, but also an assignment

$$z_{km} = \begin{cases} 1 & \text{if demand point } m \text{ is assigned to facility } k, \\ 0 & \text{else.} \end{cases}$$

The different demand points are interacting through the hub nodes, whereby the key feature is that a discount factor $\alpha \in (0,1)$ is taken into account for inter-hub interactions. Denoting the sum of ingoing and outgoing traffic of demand point $m \in [M]$ with

$$G_m := \sum_{n=1}^{M} v_{nm} + v_{mn},$$

the 2-hub median location problem can be written as

$$\min \quad \sum_{m=1}^{M} \sum_{k=1}^{2} G_m \gamma(a_m - x_k) z_{km}$$

$$+ \alpha \gamma(x_1 - x_2) \sum_{m=1}^{M} \sum_{n=1}^{M} v_{mn}(z_{1m} z_{2n} + z_{1n} z_{2m}) \tag{13a}$$

$$\text{s.t.} \quad \sum_{k=1}^{2} z_{km} = 1, \quad \forall m \in [M], \tag{13b}$$

$$x_k \in F_k \qquad \forall k \in [2], \tag{13c}$$

$$z_{km} \in \mathbb{B} \qquad \forall k \in [2], \ m \in [M]. \tag{13d}$$

The objective (13a) minimizes the overall ingoing and outgoing traffic going from one demand point a_m to its assigned hub x_k, plus, the interaction between each of the demand point via the hubs. Constraint (13b) assures that each demand point is assigned to exactly one hub. Constraint (13c) guarantees that x_k lies in its feasible region.

In first place this does not look like a classic location problem (P_F), but assume we have already given a feasible assignment $z = (z_{km})_{k \in [K], m \in [M]}$ between the demand points and the hubs.

With

$$w_{km} := G_m z_{km},$$

$$\bar{w} := \alpha \sum_{m=1}^{M} \sum_{n=1}^{M} v_{mn}(z_{1m} z_{2n} + z_{1n} z_{2m}),$$

we will denote the weights of the multifaciltiy location problem instance and define the location problem as

$$
\begin{aligned}
\min \quad & \Phi(X) := \bar{w}\gamma(x_1 - x_2) + \sum_{k=1}^{2} \sum_{m=1}^{M} w_{km}\gamma(x_k - a_m) \\
\text{s. t.} \quad & x_k \in F_k \qquad \forall k \in [2].
\end{aligned}
\qquad (P_F^H(z))
$$

Hence, by iterating over every possible assignment for z, we could get for each assignment a finite dominating set. However, the overall number of possibilities for the assignments is 2^M. The next part shows, that we only need to calculate a finite dominating set once. Therefore, define for $u = (u_1, \ldots, u_M) \in \bigtimes_{m \in [M]} B^\circ$

$$G(u) := \left(\bigcap_{m \in [M]} a_m + N_{B^\circ}(u_m) \right).$$

Analogue to the previous definitions, the set of intersection points for both facilities will be

$$\mathcal{I}_k = \left\{ x \in \mathbb{R}^2 \ \middle| \ \exists u \in \bigtimes_{m \in [M]} B^\circ : x \in \mathrm{Ext}\,(G(u) \cap F_k) \right\},$$

and construction lines

$$CL_k = \left\{ x \in \mathbb{R}^2 \ \middle| \ \exists u \in \bigtimes_{m \in [M]} B^\circ : x \in \mathrm{bd}\,(G(u) \cap F_k) \right\}.$$

Using these sets, the points in $\mathcal{I}_k^0 = \mathcal{I}$ and $\mathcal{I}_k^1$ will be defined as in Definition 4.8.

Theorem 5.1
There exists an optimal solution (x_1, x_2) of (13) such that one of the following conditions holds:

1. $x_1 = x_2$: there exist $u \in \bigtimes_{m \in [M]} B^\circ$ with

$$x_1 = x_2 \in \mathrm{Ext}\,(G(u) \cap F_1 \cap F_2)$$

2. $x_1 \in \mathcal{I}_1^0$ and $x_2 \in \mathcal{I}_2^0$: both points are intersection points, thus,

$$x_1 \in \mathcal{I}_1, \ x_2 \in \mathcal{I}_2$$

3. $x_1 \in \mathcal{I}_1^0$ and $x_2 \in \mathcal{I}_2^1 \setminus \mathcal{I}_2^0$: there exist an $r \in [\tilde{R}]$ with

$$x_1 \in \mathcal{I}_1, \ x_2 \in \mathrm{Ext}\,(CL_2 \cap (x_1 + \mathbb{R}_{>0}\tilde{b}_r))$$

4. $x_1 \in \mathcal{I}_1^1 \setminus \mathcal{I}_1^0$ and $x_2 \in \mathcal{I}_2^0$: there exist an $r \in [\tilde{R}]$ with

$$x_2 \in \mathcal{I}_2, \ x_1 \in \mathrm{Ext}\left(CL_1 \cap (x_2 + \mathbb{R}_{>0}\tilde{b}_r)\right)$$

Proof. Assume $z := (z_{km})_{k \in [2], m \in [M]}$ is an assignment to problem (13). Then an optimal solution $X^* = (x_1^*, x_2^*)$ can be found using the planar location problem $(P_F^H(z))$. By Corollary 4.13 there exists a spanning forest T such that X^* also is in $\mathcal{I}_{rec}(P_F^H(T))$, where $(P_F^H(T))$ denotes in this case the contracted problem of $(P_F^H(z))$.

First assume $X^* = (x_1^*, x_2^*) \in \mathrm{OPT}(P_F^H(z))$ with $x_1^* = x_2^*$. Then the optimal spanning forest consists of the spanning tree of G_X, i.e., $T = \{(\{1,2\}, \{(1,2)\})\}$. By Corollary 4.13 and the definition of the contracted problem (i.e, $F_1' := F_1 \cap F_2$), we obtain the first case.

If $x_1^* \neq x_2^*$ for all $(x_1^*, x_2^*) \in \mathrm{OPT}(P_F^H(z))$ then

$$T = \{ (\{1\}, \{\emptyset\}), (\{2\}, \{\emptyset\}) \}.$$

We can find a $(x_1, x_2) \in \mathrm{OPT}(P_F^H(T))$ and by definition of $\mathcal{I}_{rec}(P_F^H(T))$ only the following three cases can occur.

1. $x_k \in \mathcal{I}_k^0(P_F^H(T)) = \mathcal{I}_k(P_F^H(T))$ for both $k = 1,2$:
2. $x_1 \in \mathcal{I}_1^0(P_F^H(T)) = \mathcal{I}_1(P_F^H(T))$ and there exists an $r \in [\tilde{R}]$ with
$$x_2 \in \mathrm{Ext}\left((x_1 + \mathbb{R}_{>0}\tilde{b}_r) \cap CL_2(P_F^H(T))\right)$$
3. $x_2 \in \mathcal{I}_2^0(P_F^H(T)) = \mathcal{I}_2(P_F^H(T))$ and there exists an $r \in [\tilde{R}]$ with
$$x_1 \in \mathrm{Ext}\left((x_2 + \mathbb{R}_{>0}\tilde{b}_r) \cap CL_1(P_F^H(T))\right)$$

Note that we do not have to distinguish the cases $(1,2) \in E_X$ or $(2,1) \in E_X$ as γ is symmetric. The claim for the given spanning forest T follows as

$$\mathcal{I}_k(P_F^H(T)) = \left\{ x \in \mathbb{R}^2 \ \middle| \ \exists u \in \underset{m \in \mathcal{M}_k}{\bigtimes} B^\circ : x \in \mathrm{Ext}\left(G_k(u) \cap F_k\right) \right\}$$
$$\subseteq \left\{ x \in \mathbb{R}^2 \ \middle| \ \exists u \in \underset{(k,m) \in [2] \times [M]}{\bigtimes} B^\circ : x \in \mathrm{Ext}\left(G(u) \cap F_k\right) \right\} = \mathcal{I}_k$$

and

$$CL_k(P_F^H(T)) = \left\{ x \in \mathbb{R}^2 \ \middle| \ \exists u \in \underset{m \in \mathcal{M}_k}{\bigtimes} B^\circ : x \in \mathrm{bd}\left(G_k(u) \cap F_k\right) \right\}$$
$$\subseteq \left\{ x \in \mathbb{R}^2 \ \middle| \ \exists u \in \underset{(k,m) \in [2] \times [M]}{\bigtimes} B^\circ : x \in \mathrm{bd}\left(G(u) \cap F_k\right) \right\} = CL_k. \qquad \square$$

As the FDS in Theorem 5.1 is independent of the assignment, we can iterate over all points in the FDS and then compute an optimal assignment. The assignment will be computed by using the LP of [SP97]. This is summarized in Algorithm 2.

Runnning time of Algorithm 2
The running time of the Algorithm highly depends on the number of possible locations in the FDS. Hence, in the following we analyze how many of those points exist. Let the feasible regions be given by the polyhedrons $F_k = \{x \in \mathbb{R}^2 \mid \Omega_i x \leq \omega_i, i \in [J_k]\}$, where $J_k \in \mathbb{N}_{\geq 0}$ and $\Omega_i \in \mathbb{R}^2, \omega_i \in \mathbb{R}$ for

Algorithm 2: Solving the 2-hub location problem

Data: Hub Location Problem (13)
Result: Optimal Solution (x_1^*, x_2^*), objective value obj
1   obj $= \infty$
2   Compute FDS of Theorem 5.1
3   **for** $X \in$ FDS **do**
4   Compute optimal assignment z using the LP of [SP97] with X as input
5   Compute objective value currentOpt of z and X
6   **if** currentOpt $<$ obj **then**
7   obj $=$ currentOpt
8         $(X^*, z^*) = (X, z)$

9   **return** (X^*, z^*), obj

---

all $i \in [J_k]$. Let $J := \max_{k \in [2]} J_k$ be the maximum number of facets and extreme points of the feasible regions and R be the number of extreme points of the gauge γ. We will go through the the different cases of Theorem 5.1.

1. In the first case the facilities coincide, thus, since there are R lines emerging from each of the demand points, there is a total of MR lines, which can give at most M^2R^2 intersections. Additionally, each line can intersect the feasible region at most twice, giving another $O(MR)$ intersection points. Furthermore, there are at most $2J$ extreme points of the feasible regions. In total this gives $O(M^2R^2 + J)$ possible choices for $x_1 = x_2$.
2. In the second case, by the same argumentation as before, there are $O(M^2R^2 + J)$ possibilities for each of the facilities, giving a total of $O((M^2R^2 + J)^2)$ for this case.
3. For the third case we analyze the additional number of possible locations for x_2 not considered in the second case. There are $O(M^2R^2 + J)$ possible locations for x_1 and R rays emerging from each of these locations. The possible locations for x_2 are the intersection points of these rays with the construction lines CL_2. Since there are $O(MR + J)$ lines or line segments in the construction lines CL_2, this results in $O((M^2R^2 + J) \cdot R \cdot (MR + J))$ additional locations for x_2 not considered in case 2.
4. Case 4 goes analogue to case 3 by interchanging the roles of x_1 and x_2.

Adding up all cases we get a total number of

$$O(M^2R^2 + J) + O((M^2R^2 + J)^2) + 2 \cdot O((M^2R^2 + J) \cdot R \cdot (MR + J))$$
$$= O(M^4R^4 + M^2R^3J + J^2R)$$

for the locations in the finite dominating set.

For each location in the FDS we have to determine an assignment by solving the LP of [SP97]. We denote by *poly(LP)* the running time of an arbitrary polynomial algorithm to solve the LP. We get the following result.

Theorem 5.2
Algorithm 2 solves the planar 2-hub location problem (13) in

$$O((M^4R^4 + M^2R^3J + J^2R) \cdot poly(LP))$$

time.

Corollary 5.3
The planar 2-hub location problem (13) is polynomially solvable.

Remark 5.4
It is possible to extend this approach to different gauges between each x_k and a_m, since it is possible to simplify the addition of multiple gauges to a single gauge by Remark 4.6. The number of extreme points of the resulting gauges is at most the sum of the number of extreme points of the original gauges. Setting R as the maximal number of extreme points of the new gauges, Theorem 5.2 is still valid.

Theorem 5.5
The 2-facility location problem with euclidean norm

$$\min \quad \sum_{m=1}^{M} \sum_{k=1}^{2} G_m l_2 (a_m - x_k) z_{km}$$

$$+ \alpha l_2 (x_1 - x_2) \sum_{m=1}^{M} \sum_{n=1}^{M} v_{mn}(z_{1m} z_{2n} + z_{1n} z_{2m})$$

$$\text{s.t.} \quad \sum_{k=1}^{2} z_{km} = 1, \quad \forall m \in [M],$$

$$z_{km} \in \mathbb{B} \qquad \forall k \in [2], \ m \in [M],$$

$$x_k \in F_k \qquad \forall k \in [2]$$

can be approximated with relative error $\varepsilon > 0$ in

$$O\big(\big(M^4 \,({}^1\!/\!\sqrt{\varepsilon})^4 + M^2 \,({}^1\!/\!\sqrt{\varepsilon})^3 J + J^2 \,({}^1\!/\!\sqrt{\varepsilon})\big) \cdot poly(LP)\big)$$

time.

Proof. By [CHKN00] it is possible to approximate the unit ball of the euclidean norm by a block norm with $O\,({}^1\!/\!\sqrt{\varepsilon})$ extreme points, giving an ε-approximation of the objective function. Replacing the R with ${}^1\!/\!\sqrt{\varepsilon}$ in Theorem 5.2 gives the desired result. $\qquad\square$

6 Conclusion and Future Research

In this paper we have shown the existence of a finite dominating set for the planar multifacility location problem with polyhedral feasible regions. However, this set is only valid if the underlying structure of the interacting facilities form a tree, as shown by an example where the structure is in fact a circle. We applied the FDS to the planar 2-hub location problem to get a polynomial procedure for solving the problem exactly.

For the future we try to decrease the size of the FDS by taking into account the flow conservation constraints. Furthermore, it would be interesting to see if there are other approaches to describe a FDS which can handle other interaction-structures than trees. In the example, we were able to describe the optimal solution set by a system of linear equations, which could be a promising approach to tackle this problem. Another issue arises when considering the planar K-hub location problem, with $K \geq 3$. Although, we know that for fixed K the size of the FDS is still polynomial, it is unclear how to efficiently determine an assignment in this case, since the LP of [SP97] can only handle two hubs. For general K the problem is known to be NP-hard.

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