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Application Of The Heath-Platen Estimator In Pricing Barrier And Bond Options

Sema COŞKUN

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Abstract

In this thesis, we focus on the application of the Heath-Platen (HP) estimator in option pricing. In particular, we extend the approach of the HP estimator for pricing path dependent options under the Heston model. The theoretical background of the estimator was first introduced by Heath and Platen [32]. The HP estimator was originally interpreted as a control variate technique and an application for European vanilla options was presented in [32]. For European vanilla options, the HP estimator provided a considerable amount of variance reduction. Thus, applying the technique for path dependent options under the Heston model is the main contribution of this thesis.

The first part of the thesis deals with the implementation of the HP estimator for pricing one-sided knockout barrier options. The main difficulty for the implementation of the HP estimator is located in the determination of the first hitting time of the barrier. To test the efficiency of the HP estimator we conduct numerical tests with regard to various aspects. We provide a comparison among the crude Monte Carlo estimation, the crude control variate technique and the HP estimator for all types of barrier options. Furthermore, we present the numerical results for at the money, in the money and out of the money barrier options. As numerical results imply, the HP estimator performs superior among others for pricing one-sided knockout barrier options under the Heston model.

Another contribution of this thesis is the application of the HP estimator in pricing bond options under the Cox-Ingersoll-Ross (CIR) model and the Fong-Vasicek (FV) model. As suggested in the original paper of Heath and Platen [32], the HP estimator has a wide range of applicability for derivative pricing. Therefore, transferring the structure of the HP estimator for pricing bond options is a promising contribution. As the approximating Vasicek process does not seem to be as good as the deterministic volatility process in the Heston setting, the performance of the HP estimator in the CIR model is only relatively good. However, for the FV model the variance reduction provided by the HP estimator is again considerable.

Finally, the numerical result concerning the weak convergence rate of the HP estimator for pricing European vanilla options in the Heston model is presented. As supported by numerical analysis, the HP estimator has weak convergence of order almost 1.

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Chapter 1

Introduction

Monte Carlo (MC) methods were initially introduced in the 1940s with the aim of solving problems in mathematical physics and thereafter they were often used to solve several problems in a wide range of fields. The reason behind their popularity is that they are easy to implement and applicable for higher dimensional problems. The principles of MC methods originate from probabilistic approaches. That is, the main idea is to estimate an expected value of a random variable X, i.e. $\mathbb{E}[X]$, by an arithmetic average of randomly generated outcomes drawn from the distribution of X. MC methods draw their theoretical strength from two fundamental theorems of probability theory, namely the strong law of large numbers and the central limit theorem.

For the first time MC methods were applied to price financial derivatives in the seminal work of Boyle [7]. In fact, MC methods are suitable to tackle problems in both finance and insurance applications due to the use of probabilistic models in these areas. One typical example of financial applications is the use of MC methods to price complex structured options. The term complexity is often understood in relation to the dynamics of the underlying asset and the characteristics of the option itself. For instance, if one aims to price exotic options whose payoff may involve certain exotic features and further if there exist no closed form solution for the option price, then MC methods may appear as a convenient choice. Barrier options are popular examples of exotic options. Depending on whether the stock price process crosses a predefined barrier level during the lifetime of the option, the option either provides a payment at the terminal time or nullifies. Due to this additional barrier feature, barrier options are strongly path dependent and conditional options. Although a closed form solution under the Black-Scholes [6] model is available for certain types of barrier options, as the market dynamics gets complicated, very few choices except the use of numerical methods remain. In this regard, pricing barrier options under the stochastic volatility model of Heston [34] is a challenging problem. Contrary to the constant volatility assumption of the Black-Scholes model, Heston assumes that the volatility is stochastic and in particular modeled via a separate stochastic variance process. Thus, in the Heston model the stock price and the variance processes are driven by the following stochastic differential equations

$$dS_t = S_t r dt + S_t \sqrt{\nu_t} dW_t^1$$
$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^2$$

where the initial values are $S_0 = s > 0$, $\nu_0 = \nu > 0$ and W^1, W^2 are two Brownian motions which are correlated under the risk neutral measure $\mathbb{Q}$ such that $\mathbb{E}_{\mathbb{Q}}(dW_t^1 dW_t^2) = \rho dt$.

In this thesis our main aim is to price barrier options under the Heston model via an efficient Monte Carlo technique. For this purpose we employ the Heath-Platen (HP) estimator which was initially introduced by Heath and Platen [32] under the name of *Differential Operator Integral* (DOI). Although the HP estimator is originally interpreted as a control variate technique, it employs a rather sophisticated approach compared to the crude control variate technique. Roughly speaking, the crude technique mainly focuses on the difference between the expected values of two random variables. On the contrary, the HP estimator aims to couple the processes via their actual underlying dynamics. Therefore, by its nature the implementation of the HP estimator in any derivative pricing problem involves a certain amount of theoretical knowledge. Throughout the thesis we examine the specific properties of the Heath-Platen estimator in detail, particularly for barrier options. Moreover, we present numerical results regarding the implementation of the HP estimator in pricing all types of one-sided knockout options. Although the application of the HP estimator in pricing European vanilla options was presented in [32], the application for the barrier options has remained novel in the literature. Hence, this application constitutes one of the main contributions of this thesis.

Another part of the novelty of this thesis comes from the application of the HP estimator to bond options. Basically, we extend the idea of the HP estimator for pricing bond options. Bond options are financial derivatives whose underlying asset dynamics are determined by the short rate process. In particular, we consider zero coupon bond (ZCB) options. A ZCB is a financial derivative which has a final value of 1 at maturity T and makes no coupon payment during the lifetime of the bond. We begin our extension with pricing ZCB options under the Cox-Ingersoll-Ross (CIR) model [15]. The short rate in the CIR model is the only state variable which determines the complete structure of the yield curve. Therefore, the CIR model belongs to the class of one factor models. In the CIR model, the short rate process follows a mean reverting square root process, i.e.

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

where the initial value for the short rate is $r(0) = r > 0$ and W is a Brownian motion. Moreover, the parameters κ, θ and σ are positive constants.

Finally, we consider another extension of the application of the HP estimator in pricing ZCB options under the stochastic variance model of Fong and Vasicek (FV) [22]. In the FV model, the short rate process contains a stochastic volatility parameter which is modeled via a stochastic variance process. The FV model is then given by the following stochastic differential equations

$$dr(t) = \kappa_1(\theta_1 - r(t))dt + \sqrt{\nu(t)}dW^1(t)$$
$$d\nu(t) = \kappa_2(\theta_2 - \nu(t))dt + \sigma\sqrt{\nu(t)}dW^2(t)$$

where the initial values are $r(0) = r > 0$, $\nu(0) = \nu > 0$ and W^1, W^2 are two Brownian motions which are correlated under the risk neutral measure $\mathbb{Q}$ such that $\mathbb{E}_{\mathbb{Q}}(dW^1(t)dW^2(t)) = \rho dt$. Again, due to the complicated dynamics of the underlying asset process, pricing ZCB options under the FV model is challenging. Therefore, proposing an efficient pricing method is significant. For this purpose, we employ again the HP estimator to price ZCB options in the FV model.

The thesis is structured as follows. Chapter 2 provides a brief introduction to Monte Carlo methods. Exact simulation and discretization schemes for the solution of the stochastic differential equations are also introduced. The Euler-Maruyama (EM) scheme is of particular interest since we adopt the fully truncated EM scheme to discretize the Heston model. The control variate technique is briefly introduced, since the HP estimator is supposed to be a powerful control variate technique.

Chapter 3 briefly introduces the basics of option pricing and in particular the partial differential equations approach. In fact, this approach has strong relations with the theoretical background of the HP estimator. Furthermore, the dynamics of the Heston process are explained in detail and the methods which are used to price options under the Heston model are discussed.

In Chapter 4, a rigorous examination of the properties of the HP estimator and its theoretical background are presented. The difficulties to price barrier options under the Heston model are pointed out. Subsequently, the application of the HP estimator for pricing all types of one-sided knockout barrier options is presented in detail. To test the efficiency of the estimator, several numerical tests are performed. The numerical results for each option type are given at the end of the chapter.

Chapter 5 extends the application of the HP estimator for pricing bond options under both the CIR and the FV model. It examines the theoretical aspects of the application of the HP estimator in pricing ZCB options, in connection with conveying the idea of the HP estimator into a different context of the option pricing problem. Moreover, numerical results for both models are presented.

In Chapter 6, the weak error behavior of the HP estimator for European vanilla options under the Heston model is briefly explained. Since European vanilla options have semi-closed form solution in the Heston model, it is possible to numerically analyze the weak convergence rate of the HP estimator. Numerical tests for the weak convergence rate are provided for various parameter sets.

Finally, Chapter 7 concludes the thesis with a summary of the results obtained throughout the thesis. Furthermore, several potential problems regarding the future research topics are proposed.

Chapter 2

Monte Carlo Methods

In this chapter we give a brief introduction to Monte Carlo (MC) methods and their applications in finance, particularly in derivative pricing. The essential idea of MC methods was introduced by S. Ulam and J. von Neumann to solve the problems in mathematical physics. The following remark from S. Ulam explains briefly how he ended up with the idea of MC methods [20]:

> *The first thoughts and attempts I made to practice [the Monte Carlo method] were suggested by a question which occurred to me in* 1946 *as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield solitaire laid out with* 52 *cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than "abstract thinking" might not be to lay it out say one hundred times and simply observe and count the number of successful plays. This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operations. Later ... [in* 1946*, I] described the idea to John von Neumann and we began to plan actual calculations.*

As seen from the quote of S. Ulam, MC methods are established on a probabilistic basis. In MC methods, instead of examining the evolution of an experiment, one focuses directly on the expected outcomes of it. Roughly speaking, one samples randomly from a universe of possible outcomes and takes the arithmetic average over this sample. In principle, the same experiment is performed with different random inputs for each independent realization. If the number of these realizations is large enough, then eventually the average of these outcomes converges to the expected value of the experiment. In the next section, we explain the theoretical background in detail. Although MC methods were first invented to solve problems related to mathematical physics, thereafter they have been widely used in other fields as well. For example in 1977, Boyle [7] developed the MC simulation method to solve derivative valuation problems. This study is the first of its kind to exhibit how one can use MC methods in financial mathematics. Afterwards, MC methods were often used to solve various different problems in finance and insurance. For the detailed examination of the MC methods and models in finance, the following references are very useful [42, 27, 36].

2.1 Basics of Monte Carlo Methods

Monte Carlo methods are mainly based on a random sampling procedure to approximate an expected value $\mathbb{E}(X)$ by an arithmetic average of independent realizations of X. Consider, for example, the problem of estimating the integral of any function f over the unit interval. Namely, we aim to calculate the following value

$$I = \int_0^1 f(x)dx.$$

We can consider the problem of calculating the value of an integral over the unit interval as an expected value of a random variable, i.e.

$$I = \int_0^1 f(U)dU := \mathbb{E}(f(U)) \tag{2.1}$$

where U is a random variable. For a given random variable U, we can compute the expected value of a function of U over the unit interval as follows

$$\mathbb{E}(f(U)) = \int_0^1 f(U)f_u(U)dU \tag{2.2}$$

where $f_u(U)$ is the probability density function of the random variable. Equations (2.1) and (2.2) have to coincide, thus we get $f_u(U) = 1$. We rewrite it explicitly for the interval $[0, 1]$

$$f_u(U) = \begin{cases} 1 & \text{for } 0 \leq U \leq 1, \\ 0 & \text{otherwise .} \end{cases}$$

In fact, this is the probability density function of the uniform distribution on $[0, 1]$. Consequently, we can estimate the integral $\int_0^1 f(x)dx$ on $[0, 1]$ by considering it as an expected value of uniformly distributed random numbers. In order to compute this expected value, we use the MC estimation. Suppose that we have a mechanism to generate n independent uniformly distributed random numbers $U_1, U_2, ...$ in the unit interval $[0, 1]$, then the MC estimation of the integral in our example will be the following

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

So, the integral is estimated by using only n independent random numbers. Now, a natural question arises how one can be sure that this estimation indeed converges to the true expected value that we aimed to obtain. In this regard, we recall the following theorem, Kolmogorov's strong law of large numbers.

Theorem (Strong Law of Large Numbers [42]). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of integrable, real-valued random variables that are independent, identically distributed (i.i.d.) and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let further*

$$\mu = \mathbb{E}[X_1].$$

Then, we have for $\mathbb{P}$-almost all $\omega \in \Omega$

$$\frac{1}{n}\sum_{i=1}^{n} X_i(\omega) \to \mu \quad as \quad n \to \infty, \tag{SLLN}$$

i.e. the arithmetic mean of the realizations of X_i tends to the theoretical mean of every X_i as the number n is large enough.

Hence, this theorem ensures that our MC estimation for the integral $\int_0^1 f(x)dx$ tends to the real value of the integral, i.e.

$$\hat{I} \to I \quad \text{with probability 1 as} \quad n \to \infty.$$

Note that, the main condition is that the number of random variables has to be large enough to ensure this convergence. Basically, the more random numbers we use, the closer we get to the true value. For a given random variable X with a finite expectation $\mathbb{E}(X)$, the crude MC algorithm to compute an expected value is the following:

Algorithm 1 The Crude Monte Carlo estimation

Generate randomly n independent experiment results
$X_i(\omega)$ for $i = 1, 2, ..., n$, $n \in \mathbb{N}$
Approximate $\mathbb{E}(X)$ by the arithmetic mean,
$\frac{1}{n}\sum_{i=1}^{n} X_i(\omega)$

Note that, here the $X_i(\omega)$ originate from the same probability distribution as X. Furthermore, the MC estimation in the crude algorithm is *unbiased*.

Theorem (Unbiasedness of the crude MC estimator [42]). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of integrable real-valued random variables that are i.i.d. as X. All the random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the MC estimator*

$$\hat{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}$$

is an unbiased estimator for $\mu = \mathbb{E}(X)$, i.e. we have

$$\mathbb{E}(\hat{X}_n) = \mu \quad \forall n \in \mathbb{N}.$$

By the help of the previous theorems, it is ensured that MC methods approximate the relevant expectation correctly in the mean. Our next concern is the absolute value of the error produced for each trial. For this, we look at the variance between the MC estimation $\hat{X}_n$ and the expected value μ, i.e.

$$\mathbb{V}ar(\hat{X}_n - \mu) = \mathbb{V}ar(\hat{X}) = \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{V}ar(\hat{X}_i) = \frac{\sigma^2}{n} \tag{2.3}$$

where $\sigma^2 := \mathbb{V}ar(X)$, provided it exists. The variance is a measure of the accuracy of our crude MC estimation by the central limit theorem given as follows.

Theorem (Central Limit Theorem (i.i.d. case) [42]). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent real-valued random variables that are identically distributed and are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume also that they all have a finite variance $\sigma^2 = \mathbb{V}ar(X)$. Then the normalized and centralized sum of these random variables converges in distribution towards the standard normal distribution, i.e. we have*

$$\frac{\sum_{i=1}^{n} X_i - \mu n}{\sigma\sqrt{n}} \xrightarrow{D} \mathcal{N}(0,1) \quad as \quad n \to \infty\,. \tag{CLT}$$

In other words, this theorem asserts that the distribution of the error in our estimation is approximately normally distributed, i.e.

$$\hat{X}_n - \mu \approx \mathcal{N}(0, \sigma^2/n).$$

Since the error has approximately a normal distribution with a standard deviation σ, we can assign a confidence interval to each n given by

$$\left[\frac{1}{n}\sum_{i=1}^{n} X_i - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \frac{1}{n}\sum_{i=1}^{n} X_i + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right]. \tag{2.4}$$

Here $z_{1-\alpha/2}$ stands for the $1-\alpha/2$-quantile of the standard normal distribution. This confidence interval indeed gives us a measure for the accuracy of our crude MC estimator. In addition, if we aim to compare two different MC estimators of the same quantity, by keeping all other conditions equal, we should prefer the one with lower variance [27].

To illustrate the procedure we give the following example.

Example 1. *Monte Carlo estimation of $\int_0^1 \sqrt{x}dx$.*

As we already mentioned, over the interval $[0, 1]$ we can approximate this integral as an expected value of some uniformly distributed random numbers U_i, $i = 1, 2, ..., n$ with $n \in \mathbb{N}$. Then we have the following MC estimator for our problem

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(U_i) := \frac{1}{n}\sum_{i=1}^{n} \sqrt{U_i}$$

We generate n uniformly distributed random numbers and obtain the MC estimation by plugging these numbers into our function $f(x) = \sqrt{x}$. The following figure is useful to have a better understanding.

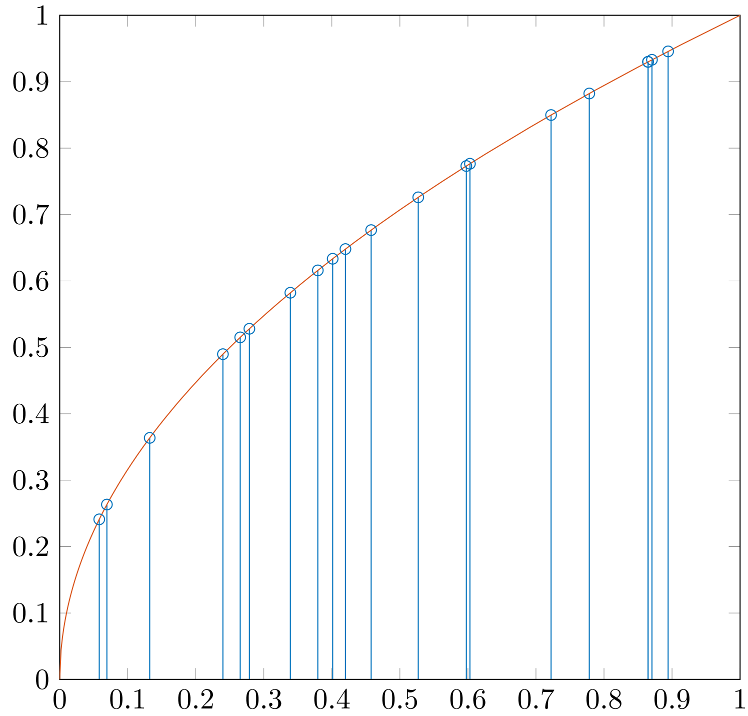

Figure 2.1: MC estimation of the integral $\int_0^1 \sqrt{x}dx$

In Figure 2.1 the red line is the graph of the function $f(x) = \sqrt{x}$. The blue dots on the graph correspond to the pair $(U_i, f(U_i))$ with uniformly distributed random numbers U_i. In fact, we estimate the integral by assigning the weight of $\frac{1}{n}$ to each random number U_i. As we increase n, we obtain more finer approximations which allow us to estimate more accurately the true value of the integral. Thus, by increasing the number of random numbers, we obtain a better estimation for the integral. The following table provides numerical results for 100, 1000, 10000 and 100000 random numbers.

n	100	1000	10000	100000
$\hat{I}$	0.6587	0.6616	0.6656	0.6672
$\hat{I}_{low}$	0.6024	0.6439	0.6600	0.6654
$\hat{I}_{up}$	0.7126	0.6797	0.6712	0.6690

Table 2.1: Crude MC estimation of the integral $\int_0^1 \sqrt{x}dx$, Exact value 2/3

Here, the lower and upper bounds are determined for the 95%-quantile of the standard normal distribution which corresponds to the value of 1.96. It can be seen from Table

2.1 that the length of the confidence interval gets smaller as n increases, i.e. the length is proportional to $1/\sqrt{n}$. This leads to the fact that one has to increase the number n of simulation trials by a factor of 100 to reduce the length of the confidence interval by a factor of 0.1. This indeed addresses the slow convergence of the crude MC methods.

2.1.1 Simulation of Stochastic Differential Equations

Stochastic differential equations (SDE)s are used to model stochastic processes that have continuous trajectories and a source of randomness contained within them. In fact, SDEs can be motivated by introducing an additional randomness to ordinary differential equations (ODE)s in that

$$dX_t = a(t, X_t)dt + \text{"}randomness\text{"}. \tag{2.5}$$

This source of randomness is commonly referred as *white noise* and is modeled as increments of Brownian motion (BM). Therefore, we begin our discussion on stochastic differential equations with the mathematical definition of Brownian motion.

Definition (Brownian Motion [38])**.** *A standard one-dimensional Brownian motion is a continuous, adapted process $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the following properties*

(i) $W_0 = 0$ $\mathbb{P}$*-a.s.,*

(ii) For $0 \leq s < t$ the increment $W_t - W_s$ is independent of $\mathcal{F}_s$,

(iii) The increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$, i.e. $W_t - W_s \sim \mathcal{N}(0, t - s)$.

If $\{W_t\}_{t\geq 0}$ is a Brownian motion and $0 = t_0 < t_1 < \ldots < t_n < \infty$, then the increments $\{W_{t_i} - W_{t_{i-1}}\}_i^n$ are independent, so that the Brownian motion has *independent increments*. Furthermore, the distribution of $W_{t_i} - W_{t_{i-1}}$ depends on t_i and t_{i-1} only through the difference $t_i - t_{i-1}$, thus the Brownian motion has *stationary increments*. A one-dimensional Brownian motion is a real valued process, i.e. for each $t \in I$ where I is the index set for time variable and for each trajectory $\omega \in \Omega$ the Brownian motion is defined as a function $W(t, \omega) : I \times \Omega \to \mathbb{R}$ such that $(t, \omega) \to W(t, \omega)$. It is also possible to define a d-dimensional Brownian motion such that

$$W_t = (W_t^1, W_t^2, \ldots, W_t^d)$$

where the components W_i are independent one-dimensional Brownian motions. For further details about Brownian motion, see [38]. Since BM may attain negative values, if one aims to model a positive valued stochastic process then the *Geometric Brownian Motion* (GBM), also known as exponential Brownian Motion, is one of the well known transformations of the BM. A one-dimensional GBM is driven by the following SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.6}$$

where W_t is the BM, $S_0 = s$ is an arbitrary constant and additionally the parameters μ and σ are constants. Here, μ is the drift coefficient which basically reflects the future tendency of the process and σ is the diffusion coefficient which determines the variation of the process. It is possible to obtain the closed form solution for Equation (2.6) by applying a logarithmic transformation, i.e. $\tilde{S}_t = \ln(S_t)$, and further by an application of the Itô formula. Thus the solution reads as

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}. \tag{2.7}$$

In addition to possessing a closed form solution, the process is popular due to its relatively simple dynamics. For instance, the process has a log-normal distribution and in particular, the expected value and the variance can be explicitly computed at any time t as follows

$$\mathbb{E}\left(S_t\right) = S_0 e^{\mu t}, \tag{2.8}$$

$$\mathbb{V}ar\left(S_t\right) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right). \tag{2.9}$$

The fundamental properties of the process are situated in the values of the drift and diffusion parameters μ and σ, respectively. In addition, if one knows the initial value S_0 of the process, then one can exactly simulate the values of S_t at any time t. The reason behind is that the Brownian increments $W_{t_{i+1}} - W_{t_i}$ are normally distributed and can be generated by the help of any standard normally distributed random numbers R_i in that

$$W_{t_{i+1}} - W_{t_i} \sim \sqrt{t_{i+1} - t_i} R_i.$$

As a result, one can exactly simulate a path of a GBM. Figure 2.2 illustrates 50 trajectories of GBM.

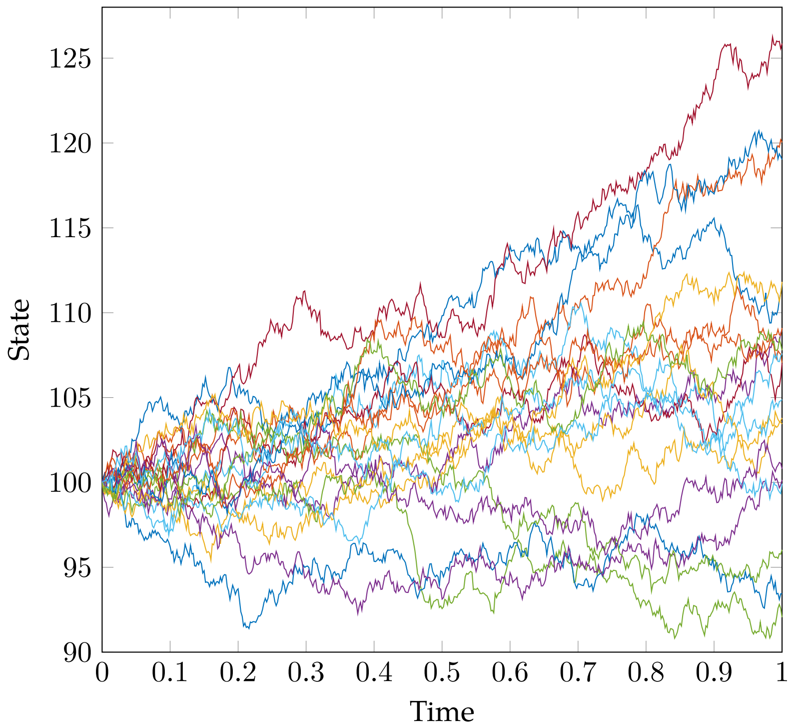

Figure 2.2: Sample paths of GBM with $\mu = 0.05$, $\sigma = 0.07$, $S_0 = 100$

For this simulation we used 1000 steps, hence the step size was equal to $\delta = 0.001$. Note that, there is no bias involved in our simulation, since at each discrete point we simulate the closed form solution given in Equation (2.7). We draw a great deal of attention to the GBM, since it is used by Black and Scholes [6] to model the stock price process. Therefore, it has a huge effect on a varying range of financial implementations, see Section 3.1 for details.

The general form of SDEs driven by Brownian motion is given by

$$\begin{aligned} dX_t &= a(t, X_t)dt + b(t, X_t)dW_t \qquad t \in [0, T] \\ X_0 &= x \end{aligned} \tag{2.10}$$

where $a(t, X_t)$ and $b(t, X_t)$ are real valued time inhomogeneous functions and $x \in \mathbb{R}$ is the initial value of the process. Here, $X_t \in \mathbb{R}$ represents the state of the process modeled at any time t. Note that, it is again possible to consider X_t as a d-dimensional process by taking a m-dimensional Brownian motion and defining the functions $a(t, X_t)$ as a $\mathbb{R}^d$-valued function and $b(t, X_t)$ as a $d \times m$ matrix valued function. However, we keep the introduction restricted for the one-dimensional case. One can equivalently write Equation (2.10) in the following integral form

$$X_t = x + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s \tag{2.11}$$

where $X_0 = x$ is an arbitrary constant. We draw attention to the fact that X_0 may also be a $\mathcal{F}(0)$ measurable random variable which is independent of the driving BM. However, for the sake of simplicity we keep it constant. The first term is a standard Lebesgue integral for each sample path ω, while the second integral is a stochastic Itô integral. The existence and uniqueness of the solutions is essentially determined by the form of the coefficient functions $a(s, x)$ and $b(s, x)$, for details see [48, 38]. We assume that there exists a unique solution X_t to the SDE given in (2.10). Unless a closed form solution for X_t is available, numerical approaches are suitable to simulate the possible outcomes of it.

2.1.2 Discretization Schemes for Stochastic Differential Equations

In the previous section we have seen how one can exactly simulate a stochastic process, in particular the exact simulation of a geometric Brownian motion. However in some cases, especially when the closed form solution of the SDE is not available, exact simulation of the process becomes impossible. Therefore, to discretize the process with a discrete scheme and approximate the original process via the simulation of this discrete scheme appears as a preferable solution. The formulation of the discrete schemes for SDEs is based on the stochastic generalization of the Taylor expansion for ordinary differential equations, for details see [40]. This generalization is obtained by the iterated application of the Itô formula and is also known as Itô-Taylor expansion. To give more detail let us consider the one-dimensional diffusion process $X = \{X_t, t_0 \leq t \leq T\}$ which is given by

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \tag{2.12}$$

on $t_0 \leq t \leq T$ with the initial value $X_{t_0} = X_0$. The solution X_t can be written in the integral form as follows

$$X_t = X_0 + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s. \tag{2.13}$$

Consider a functional $f : \mathbb{R} \to \mathbb{R}$ of the stochastic process X_t given in Equation (2.13). If f is smooth enough, i.e. $f \in C^{1,2}(\mathbb{R} \times \mathbb{R})$, then by applying the Itô formula we get

$$\begin{aligned} f(t, X_t) =& f(t_0, X_{t_0}) \\ &+ \int_{t_0}^{t} \left(\frac{\partial}{\partial t} f(s, X_s) + a(s, X_s) \frac{\partial}{\partial x} f(s, X_s) + \frac{1}{2} b^2(s, X_s) \frac{\partial^2}{\partial x^2} f(s, X_s) \right) ds \\ &+ \int_{t_0}^{t} b(s, X_s) \frac{\partial}{\partial x} f(s, X_s) dW_s. \end{aligned} \tag{2.14}$$

By defining the following differential operators

$$\mathcal{K}^0 = \frac{\partial}{\partial t} + a(s, X_s) \frac{\partial}{\partial x} + \frac{1}{2} b^2(s, X_s) \frac{\partial^2}{\partial x^2} \tag{2.15}$$

and

$$\mathcal{K}^1 = b(s, X_s) \frac{\partial}{\partial x} \tag{2.16}$$

Equation (2.14) gets the form

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{K}^0 f(s, X_s)ds + \int_{t_0}^{t} \mathcal{K}^1 f(s, X_s)dW_s \tag{2.17}$$

for $t \in [t_0, T]$. For a particular choice of $f(x) \equiv x$, where $\mathcal{K}^0 f = a$ and $\mathcal{K}^1 f = b$, we get the original Itô process X_t given in Equation (2.13), that is

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s. \tag{2.18}$$

For the choice of $f \equiv a(s, X_s)$ we get the following,

$$a(t, X_t) = a(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{K}^0 a(s, X_s)ds + \int_{t_0}^{t} \mathcal{K}^1 a(s, X_s)dW_s. \tag{2.19}$$

Again for the choice of $f \equiv b(s, X_s)$, we get

$$b(t, X_t) = b(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{K}^0 b(s, X_s)ds + \int_{t_0}^{t} \mathcal{K}^1 b(s, X_s)dW_s. \tag{2.20}$$

Finally, substitution of Equations (2.19) and (2.20) into Equation (2.18) leads to

$$\begin{aligned} X_t =& X_{t_0} \\ &+ \int_{t_0}^{t} \left(a(t_0, X_{t_0}) + \int_{t_0}^{s} \mathcal{K}^0 a(z, X_z)dz + \int_{t_0}^{s} \mathcal{K}^1 a(z, X_z)dW_z \right) ds \\ &+ \int_{t_0}^{t} \left(b(t_0, X_{t_0}) + \int_{t_0}^{s} \mathcal{K}^0 b(z, X_z)dz + \int_{t_0}^{s} \mathcal{K}^1 b(z, X_z)dW_z \right) dW_s \\ =& X_{t_0} + a(t_0, X_{t_0}) \int_{t_0}^{t} ds + b(t_0, X_{t_0}) \int_{t_0}^{t} dW_s + \mathcal{R} \end{aligned} \tag{2.21}$$

where the remainder term $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R} = &\int_{t_0}^{t}\int_{t_0}^{s} \mathcal{K}^0 a(z, X_z) dz ds + \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{K}^1 a(z, X_z) dW_z ds \\ &\int_{t_0}^{t}\int_{t_0}^{s} \mathcal{K}^0 b(z, X_z) dz dW_s + \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{K}^1 b(z, X_z) dW_z dW_s. \end{aligned} \tag{2.22}$$

Equation (2.21) together with the remainder given in Equation (2.22) corresponds to the simplest nontrivial Itô-Taylor expansion. The Euler-Maruyama (EM) approximation can be formulated by the truncation of the remainder term in Equation (2.21) and the exact values of the time and Itô integrals, i.e.

$$\int_{t_0}^{t} ds = t - t_0 \quad \text{and} \quad \int_{t_0}^{t} dW_s = W_t - W_{t_0}.$$

It is also possible to obtain the Milstein approximation by using a higher order Itô-Taylor expansion, see Remark 1.

Remark 1. *If we continue the expansion by applying the Itô formula to the function $\mathcal{K}^1 b(z, X_z)$ appearing in the remainder term given in Equation* (2.22)*, then we obtain the following*

$$\begin{aligned} X_t &= X_{t_0} \\ &+ a(t_0, X_{t_0}) \int_{t_0}^{t} ds + b(t_0, X_{t_0}) \int_{t_0}^{t} dW_s + \mathcal{K}^1 b(0, X_{t_0}) \int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s + \bar{\mathcal{R}}, \end{aligned} \tag{2.23}$$

where the remainder $\bar{\mathcal{R}}$ is given by

$$\begin{aligned}\bar{\mathcal{R}} = &\int_{t_0}^{t}\int_{t_0}^{s} \mathcal{K}^0 a(z, X_z) dz ds + \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{K}^1 a(z, X_z) dW_z ds \\ &\int_{t_0}^{t}\int_{t_0}^{s} \mathcal{K}^0 b(z, X_z) dz dW_s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} \mathcal{K}^0 \mathcal{K}^1 b(u, X_u) du dW_z dW_s \\ &\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} \mathcal{K}^1 \mathcal{K}^1 b(u, X_u) dW_u dW_z dW_s. \end{aligned}$$

One can formulate the Milstein scheme by truncating the remainder term $\bar{\mathcal{R}}$ in Equation (2.23) *and evaluating the Itô integral such that*

$$\int_{t_0}^{t}\int_{t_0}^{s} dW_z dW_s = \frac{1}{2}\left(W_t - W_{t_0}\right)^2 - \frac{1}{2}\left(t - t_0\right).$$

In this thesis, we employ the EM approximation for our implementations. Now, let us give more details about the EM approximation, see also [40]. Consider a time discretization $(\tau)_\delta$ with

$$0 = \tau_0 < \tau_1 < \dots < \tau_n < \dots < \tau_N = T$$

of a time interval $[0, T]$ which in the simplest equidistant case has the step size $\delta = \frac{T}{N}$. Then a process $Y = \{Y(t), t \geq 0\}$ which is piecewise constant exactly on the intervals

(τ_{i-1}, τ_i) is called a time discrete approximation with a step size δ. The conditions that the discrete approximation Y has to fulfill are related to its measurability and recursive structure. Thus, it is assumed that $Y(\tau_n)$ which is based on a time discretization $(\tau)_\delta$ is $\mathcal{F}_{\tau_n}$-measurable. Moreover, $Y(\tau_{n+1})$ can be expressed as a function of $Y(\tau_0), ..., Y(\tau_n)$ for $\tau_0, ..., \tau_n, \tau_{n+1}$ and also for a finite number k of $\mathcal{F}_{\tau_n}$-measurable random variables $R_{n+1,j}$ with $j = 1, ..., k$ and each $n = 0, 1, ...$ Due to this recursive structure of the approximation algorithm Kloeden and Platen [40] refers to the term *scheme*. Thus, from now on we also use the term EM scheme instead of EM approximation. Consequently, for Equation (2.12) the EM scheme has the form

$$Y_{n+1} = Y_n + a(n, Y_n)\Delta_n + b(n, Y_n)\Delta W_n \tag{2.24}$$

for $n = 0, 1, ..., N-1$ with initial value $Y_0 = X_0$ where $\Delta_n = \tau_{n+1} - \tau_n = \delta$ and $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$ for $n = 0, 1, ..., N-1$. From the previous section, we know that these Brownian increments are independent Gaussian random variables with mean

$$\mathbb{E}(\Delta W_n) = 0$$

and variance

$$\mathbb{E}((\Delta W_n)^2) = \Delta_n.$$

Therefore, they can be simulated by using a finite number k of random variables $R_{n+1,j}$ with $j = 1, ..., k$ and each $n = 0, 1, ...,$ namely $\Delta W_n \sim \sqrt{\Delta_n} R_{n+1,j}$. The EM scheme is an explicit scheme which means that the current value is always obtained by using the previous values. It is also necessary to note, that the EM scheme determines the values of the approximating process only at discrete times. Namely, we start with time $\tau_0 = 0$ and take Y_{τ_0} as an initial value. Then we obtain the next value Y_{τ_1} by adding the approximated values at the previous time τ_0 on the value Y_{τ_0}. Recursively Y_{τ_2} is obtained by adding the approximated values at time τ_1 on the value Y_{τ_1}. We continue this procedure until we reach the final value Y_{τ_N}, i.e.

$$\begin{aligned}
Y_{\tau_1} &= Y_{\tau_0} + a(\tau_0, Y_{\tau_0})\Delta_{\tau_0} + b(\tau_0, Y_{\tau_0})\Delta W_{\tau_0} \\
Y_{\tau_2} &= Y_{\tau_1} + a(\tau_1, Y_{\tau_1})\Delta_{\tau_1} + b(\tau_1, Y_{\tau_1})\Delta W_{\tau_1} \\
&\vdots \\
Y_{\tau_N} &= Y_{\tau_{N-1}} + a(\tau_{N-1}, Y_{\tau_{N-1}})\Delta_{\tau_{N-1}} + b(\tau_{N-1}, Y_{\tau_{N-1}})\Delta W_{\tau_{N-1}}.
\end{aligned}$$

Overall, the approximation is provided by the values $Y_{\tau_1}, Y_{\tau_2}, ..., Y_{\tau_N}$ which are calculated only at the discrete times $0 = \tau_0 < \tau_1 < ... < \tau_n < ... < \tau_N = T$. Therefore, the accuracy of the approximation is strongly dependent on the number of steps and relatedly the step size δ. The following figure illustrates the relation between the accuracy of an approximation and the step size.

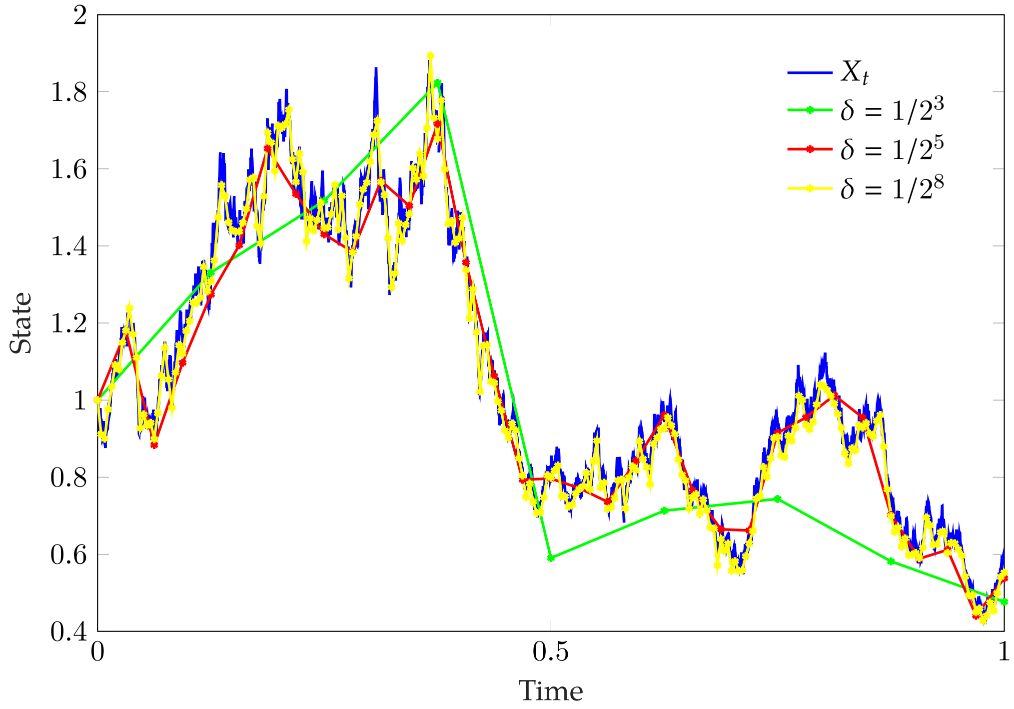

Figure 2.3: EM scheme for varying step sizes

In Figure 2.3 X_t is a geometric Brownian motion with the parameters $\mu = 0.5$ and $\sigma = 0.9$ and the initial value $X_0 = 1$. Under the necessary conditions on the drift and diffusion coefficients, it can be inferred that the EM scheme mimics the real process accurately as the step size gets finer. Therefore, we see that the step size plays a crucial role in discrete approximations. In fact, the error behavior of the underlying discretization scheme is a powerful reflector of the quality of the numerical solution. Therefore, we may briefly discuss the error behavior of the EM scheme, for details see Chapter 6. Error analysis of a discretization scheme focuses mainly on two aspects. The first concern is the convergence of the discrete scheme and the second one is the convergence rate of the discrete scheme. Although these two concepts seem compatible, the proof techniques might enormously differ from each other. In the usual examination of the weak error of a discrete scheme one has to check whether the following assumptions are satisfied, i.e. Lipschitz continuity and the linear growth conditions. We can express the conditions by considering the one-dimensional diffusion given in Equation (2.12) with time homogeneity, as follows

$$|a(x) - a(y)| + |b(x) - b(y)| \leq L\,|x - y|, \qquad x, y \in \mathbb{R} \tag{LC}$$

for some Lipschitz constant $L > 0$. Furthermore,

$$|a(x)| + |b(x)| \leq C\,(1 + |x|), \qquad x \in \mathbb{R} \tag{LG}$$

for some constant $C > 0$. Under these conditions, the definition of weak convergence of a discrete scheme is given in the following.

Definition (Weak Convergence [40]). *A general time discrete approximation Y^δ corresponding to a time discretization $(\tau)_\delta$ converges weakly to X at time T as $\delta \to 0$ with respect to a class C of test functions $g : \mathbb{R} \to \mathbb{R}$ if we have*

$$\lim_{\delta \to 0} |\mathbb{E}(g(X_T)) - \mathbb{E}(g(Y^\delta(T)))| = 0$$

for all $g \in C$. Let $C_P^\ell(\mathbb{R}, \mathbb{R})$ denotes the space of ℓ times continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$ which, together with their partial derivatives of orders up to and including order ℓ, have polynomial growth. This space is used as the class of test functions.

Since for financial applications the weak convergence is mostly of concern, here we present the related results of weak convergence of the EM scheme. Besides knowing that a discretization scheme is weakly convergent, it is advantageous to know the weak convergence rate. Moreover, the rate of weak convergence is a good criterion to compare several discrete schemes. The following definition clearly explains the concept.

Definition (Weak Convergence Rate [40]). *A time discrete approximation Y^δ converges weakly with order $\beta > 0$ to X at time T as $\delta \to 0$ if for each $g \in C_P^{2(\beta+1)}(\mathbb{R}, \mathbb{R})$ there exists a positive constant C, which does not depend on δ, and a finite $\delta_0 > 0$ such that*

$$|\mathbb{E}(g(X_T)) - \mathbb{E}(g(Y^\delta(T)))| \leq C\delta^\beta$$

for each $\delta \in (0, \delta_0)$.

It is known that under usual assumptions (see Equations (LC) and (LG) above) the EM scheme has a weak convergence of order 1. However, for many cases in financial applications these conditions are violated [39].

Another aspect concerning the error analysis is dependent on whether the diffusion process has continuous trajectories until the terminal time. There exist certain processes which are known as killed diffusions whose survival is dependent on whether the process reaches a certain boundary level. These killed diffusions also draw attention in financial applications. In many financial problems the boundaries are artificially determined. One popular example of these killing diffusions is the pricing of barrier options. This boundary condition brings an additional ingredient into the weak error analysis of the discrete scheme. Basically, the difficulty is located in the barrier checking. One possibility is to do the barrier checking only at discrete time points which is known as discrete EM scheme. In the meantime, there is a possibility that the process can cross the barrier in between two discrete time points even if the process does not cross the barrier exactly at these discrete time points. This indeed addresses the discrepancy between the discretization scheme of the process and the discretization of the survival function of the process. Now, one has to check if the process crosses the barrier between the two successive discrete points. One possible way is to apply a Brownian bridge technique which allows us to determine the distribution of the maximum values of Brownian motion conditioned on the two discrete time points, see [27] for details. If the maximum of the Brownian motion never hits the barrier then one assures that the process never hits the barrier. In other words, if one checks the barrier condition at the discrete times and also continuously in the meantime between these discrete points, then this approach is called the continuous EM scheme. It is shown by Gobet [29] that under the assumptions regarding the underlying process, domain and the test function g the weak error of the discrete Euler scheme is of order $\frac{1}{2}$ and for the continuous Euler scheme is of order 1 for the killed diffusions.

To sum up this section, once we are not able to simulate the process exactly, we have to employ a numerical scheme to discretize the process. However, this additional operation brings another source of error into our implementation. In short, if we use MC methods with a discrete scheme we have to deal with both the statistical error (i.e. MC variance) and the discretization error (i.e. bias). In order to obtain accurate and efficient results, we have to control the following mean squared error (MSE) given for any stochastic process X and a final time point T

$$
\begin{aligned}
MSE(\Delta, N) &= \mathbb{E}\Big(\mathbb{E}(X(T)) - \frac{1}{N}\sum_{i=1}^{N} X^{\Delta,i}(T)\Big)^2 \\
&= \underbrace{\Big(\mathbb{E}(X(T)) - \mathbb{E}(X^{\Delta}(T))\Big)^2}_{Bias^2} + \underbrace{\mathbb{E}\Big(\mathbb{E}(X^{\Delta}(T)) - \frac{1}{N}\sum_{i=1}^{N} X^{\Delta,i}(T)\Big)^2}_{MC\ Variance}.
\end{aligned}
$$

Here, X^{Δ} refers to the discrete time approximation of the continuous process X with a discretization step Δ. The first term in the MSE corresponds to the bias of the discretization which is dependent on the step size Δ. The second term stands for the MC variance which is dependent on the number of realizations N. To have a better control on the bias one has to choose an appropriate discretization scheme, possibly with a known convergence rate. On the other hand, to control the MC variance one has to either use very large number of paths which is practically not optimal or employ a variance reduction technique. In the following section we introduce the control variate technique which is used to reduce the MC variance.

2.2 Variance Reduction Techniques

In its crude form, the MC estimator is unbiased and thus the variance of the estimator is a measure for its accuracy. However, slow convergence of the crude MC estimator is a drawback for the numerical implementations. Therefore, reducing the variance by applying suitable techniques is the usual way of speeding up the MC method [42]. An effective use of any variance reduction technique must be substantially dependent on the specific features of the problem. Rather than selecting any technique and implementing it to solve any problem, one has to identify the technique which matches the characteristics of the problem most. Then one gains a significant efficiency. Although there exist several variance reduction techniques, in this thesis we only consider the control variate technique, for further information about the remaining techniques see [42, 27, 8].

2.2.1 Control Variates

The control variate technique is a variance reduction technique which is based on the following idea. Assume that we have a random variable X and we aim to compute the expectation of it, namely $\mathbb{E}(X)$. However, in some cases due to the complicated dynamics of this variable to explicitly compute its expectation becomes almost impossible. In addition, assume that we have another random variable Y whose $\mathbb{E}(Y)$ is explicitly known and in particular this random variable Y somehow mimics the behavior of the target variable X. Then, the random variable Y is called control variate which basically helps us to control the variation in our target variable X. By using the linearity of expectation, one has the following equation

$$\mathbb{E}(X) = \mathbb{E}(X - Y) + \mathbb{E}(Y).$$

Provided that $\mathbb{V}ar(X - Y) \ll \mathbb{V}ar(X)$, we gain an important variance reduction in our MC implementation. The variance of the simulation of $(X - Y)$ is given by

$$\mathbb{V}ar(X - Y) = \mathbb{V}ar(X) + \mathbb{V}ar(Y) - 2\mathbb{C}ov(X, Y).$$

It is obvious that the method is effective if the covariance between X and Y is large. Therefore the amount of the variance reduction of variable X is equal to

$$2\mathbb{C}ov(X, Y) - \mathbb{V}ar(Y).$$

A slight improvement can be done by defining a family of unbiased estimators via inserting a scalar parameter as a multiplier to the control variate Y such that

$$\mathbb{E}(X) = \mathbb{E}(X - \alpha Y) + \alpha\mathbb{E}(Y).$$

We can rewrite the variance of X^α as follows

$$\mathbb{V}ar(X - \alpha Y) = \mathbb{V}ar(X) + \alpha^2\mathbb{V}ar(Y) - 2\alpha\mathbb{C}ov(X, Y).$$

An optimal choice of α which minimizes the variance will be the following

$$\alpha^* = \frac{\mathbb{C}ov(X, Y)}{\mathbb{V}ar(Y)}. \tag{2.25}$$

Then the actual variance reduction reads as

$$2\alpha^*\mathbb{C}ov(X, Y) - (\alpha^*)^2\mathbb{V}ar(Y) = \frac{\mathbb{C}ov(X, Y)^2}{\mathbb{V}ar(Y)}.$$

It is possible to consider the covariance between the variables X and Y in terms of their standard deviations such that

$$\sigma_{XY} = \rho_{XY}\sigma_X\sigma_Y$$

where the ρ_{XY} denotes the correlation. Therefore, the correlation between the X and Y plays a crucial role making the control variate technique more efficient. It can be concluded that only using a control variate which has a high value of positive correlation with the target process will help making an effective use of the control variate technique. In the context of option pricing where the ultimate goal is computing the expected value of the payoff, one can think of the random variable X as the payoff of an option. If the expected value of X is not possible to compute explicitly or is too cumbersome to compute, then using another payoff Y which is positively correlated with the original payoff X is an applicable idea. Indeed, this technique is really promising. However, what makes the control variate technique difficult is to find a suitable candidate which really provides a substantial amount of variance reduction. To find a better control variate often requires some intuition and is not always based on a systematic search algorithm. See [42, 27] for different approaches to select efficient control variates. Note that, a bad choice of the control variate might even lead to more variance than the crude MC estimator.

Chapter 3

Option Pricing in the Heston Model

In this chapter we first summarize the two different approaches for option pricing, i.e. the replication principle and the partial differential equation approach. The last part of the chapter is dedicated to the dynamics of the Heston model. Furthermore, a brief discussion with regard to the methods employed for pricing options in the Heston model is presented.

3.1 Basics of Option Pricing

Ever since the contemporary form of option pricing has been introduced in the pioneering work of Black and Scholes [6], a considerable amount of attention has been put on option pricing for both academical and practical purposes. Options are financial derivatives whose holder has the right to exercise exactly at or any time earlier than a predefined time, i.e. maturity, at a predefined price, i.e. strike price. Now, some aspects mentioned in this brief explanation have to be clarified. One aspect is the exercise time based on which the options can be separated into two main classes, e.g. European options and American options. European options can only be exercised at maturity whereas American options can be exercised at any time until maturity.

Mainly, options are contracts whose price is derived from the current state of the underlying asset. Therefore, another classification is done with respect to the type of trading one share of the underlying asset. If the option ensures that the writer has the right to buy one share of the asset, then this is known as a *call option*. On the other hand, if the writer has the right to sell one share of the asset, then this is called a *put option*. Options are also classified based on the underlying assets that they are written on. For example, there exist commonly traded options on stocks, equities, bonds and goods such as oil, energy, currencies etc. In option pricing the main concern is to find a reasonable price as the premium of the option. In order to do that, several factors such as the dynamics of the underlying asset, the market structure and the type of the specific option have to be taken into account. In the next section we present two different approaches to price options.

3.1.1 Option Pricing via the Replication Principle

In the Black and Scholes (BS) market, there exist one riskless asset and one risky asset which constitute the market dynamics. The riskless asset B, e.g. the bond, whose value at time t is equal to e^{rt} in response to an investment of one unit of money at the beginning. Furthermore, it satisfies the following ordinary differential equation (ODE)

$$\frac{dB_t}{B_t} = rdt \quad t \in [0, T] \tag{3.1}$$

where r is the instantaneous interest rate. In the BS market, r is assumed to be constant. Then, the solution of the ODE can be written as follows

$$B_t = B_0 e^{rT} \tag{3.2}$$

where B_0 represents the initial amount which is invested in the riskless asset. Another component of the BS market is the risky asset, e.g. the stock. As we mentioned in Section 2.1.1 Black and Scholes modeled the stock price process as a geometric Brownian motion. Therefore, the stock price is log-normally distributed and has the following closed form solution under the physical measure $\mathbb{P}$,

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{3.3}$$

where W_t is the Brownian motion, S_0 is the initial value of the stock price and μ, σ are some positive constants. If one invests the amount of S_0 into the stock, then at time t the stock is worth S_t. Consequently, the option price is derived from the current states of these two underlying assets in the BS market. Our goal is to generate a synthetic portfolio which replicates the option price by having the same payoff at any time t. Next, we briefly clarify this idea. Assume that we have a trading strategy which is a pair of adapted processes, $\varphi_t = (\alpha_t, \beta_t)$. Here α_t and β_t represent the number of stock and bond units held at time t, respectively. Therefore, the value of a portfolio with α_t units of stock and β_t units of bond is equal to $\alpha_t S_t + \beta_t B_t$. It is assumed that the portfolio will replicate an option at time T, if its value is almost surely equal to the payoff of the option, i.e.

$$\alpha_T S_T + \beta_T B_T = h(S_T) \tag{3.4}$$

where h is the payoff function. The portfolio is required to be self-financing namely its wealth is only dependent on the stock and bond price and there is no additional cash flow. Consequently, this portfolio is known as the *replication portfolio*.

Furthermore, there exist two fundamental assumptions which have to be satisfied by the market model. The first one is the *completeness*. In a complete market each desired final wealth at time T can be exactly attained via trading according to an appropriate self-financing pair if one possesses sufficient initial capital [42]. The second assumption is about the arbitrage opportunity which refers gaining money at any time t without having no initial capital. To guarantee that Equation (3.4) always holds, it is necessary to have an arbitrage-free market. Thanks to Girsanov's measure transformation theorem, see Theorem (3.11) in [40], which ensures that it is possible to obtain an equivalent martingale measure, one can introduce a risk neutral measure $\mathbb{Q}$. Under the risk neutral measure, all the arbitrage opportunities are eliminated. With the help of the martingale representation theorem one can show that the BS market is complete and one can obtain the fair price of an option by using a unique self-financing replicating strategy. See [40] for a detailed examination of this approach. In this thesis, we mainly focus on the PDE approach which we present in detail in the following section. Moreover, we give the Black and Scholes pricing formula for a European call option.

3.1.2 Option Pricing by the Partial Differential Equation Approach

In their original paper Black and Scholes [6] derive the price of a European call option by using the PDE approach. In order to present the approach clearly we begin our discussion by considering a European call option with maturity T and strike price K. The price of this option under the risk neutral measure $\mathbb{Q}$ has the following stochastic representation

$$V(t, S_t, T, K) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}^{t,S_t}\left[h(S_T)\right] \tag{3.5}$$

where the payoff is equal to

$$h(S_T) = (S_T - K)^+.$$

The Feynman-Kac theorem ensures that under some technical conditions the option price V_t satisfies the following PDE

$$\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} - rV = 0 \tag{3.6}$$

with the terminal condition

$$V(T, s) = h(s).$$

for any $S_0 = s > 0$ and $t \in [0, T]$. Indeed, this problem is known as the Cauchy problem. The stochastic differential equation satisfied by the stock price under the risk neutral measure $\mathbb{Q}$ is given by

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t. \tag{3.7}$$

One can define the following differential operator which belongs to the stock price process given in Equation (3.7)

$$\mathcal{L}V = \frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} \tag{3.8}$$

with a smooth function V. Then one can rewrite the PDE (3.6) as follows

$$\mathcal{L}V - rV = 0 \tag{3.9}$$

with the terminal condition

$$V(T, s) = h(s).$$

For this Cauchy problem, by changing the variables and rearranging the terminal condition, one ends up with the heat equation which is a parabolic PDE. After solving the heat equation and reverting back to the original problem, one obtains an explicit solution for the function V.

A European call option under the BS model is then priced by solving Equation (3.9) with the payoff function

$$h(S_T) = (S_T - K)^+.$$

For notational purposes, we denote the price of a European call option under the BS model by $BS(t, S, T, K)$ where $S_t = S$. It can be written explicitly as follows

$$BS(t, S, T, K) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{3.10}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.11}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{3.12}$$

Here, Φ is the standard Gaussian distribution.

3.2 Dynamics of the Heston Model

To account for so-called volatility clustering in empirical stock price data, i.e. phases of very high price activity change with phases of nearly no activity, stochastic volatility models are introduced. The most popular such model is the Heston model, see [34]. In the Heston model, the stock price and the variance processes are driven by the following SDEs, respectively;

$$dS_t = S_t\mu dt + S_t\sqrt{v_t}dW_t^1 \tag{3.13}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \tag{3.14}$$

where the initial values are $S_0 = s$, $v_0 = v$ are arbitrary constants and W^1, W^2 are Brownian motions with $\mathbb{E}_{\mathbb{P}}(dW_1 dW_2) = \rho dt$. It can be seen that the volatility $\sqrt{v_t}$ itself is not modeled in the Heston model, but it is modeled via the variance process v_t. The variance process arises from the Ornstein-Uhlenbeck (OU) process which is given by

$$d\upsilon_t = -\beta\upsilon_t dt + \xi dW_t.$$

By applying the Itô formula and setting $v_t = \upsilon_t^2$, we see that v_t follows the process

$$dv_t = (\xi^2 - 2\beta v_t)dt + 2\xi\sqrt{v_t}dW_t.$$

Now, by setting $\kappa = 2\beta$, $\theta = \frac{\xi^2}{2\beta}$ and $\sigma = 2\xi$, we obtain the SDE for the variance process expressed in Equation (3.14). In particular this process is known as the Cox-Ingersoll-Ross (CIR) process which is often used to model the short term interest rate, the default intensity in credit risk or the stock volatility processes. The dynamics of the variance process are given in the following

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t$$

where $\kappa > 0$ is the mean reversion speed, $\theta > 0$ is the long-term mean reversion level and $\sigma > 0$ is the volatility of the variance. It is known that the SDE of the CIR model has a non-negative solution and this solution is pathwise unique. Additionally, under the Feller condition, i.e $2\kappa\theta > \sigma^2$, the CIR process stays strictly positive [15]. The steady state distribution of the CIR process is non-central $\chi^2(\mu; d, \gamma)$ with d degrees of freedom and non-centrality parameter γ. The stock price and the variance processes follow the SDEs given in Equations (3.13) and (3.14), respectively, under the physical measure. However, for pricing purposes we need to consider (S_t, v_t) under the risk-neutral measure $\mathbb{Q}$. By applying Girsanov's theorem to the SDEs in the Heston model, we obtain the following risk-neutral process for the stock price:

$$dS_t = S_t r dt + S_t\sqrt{v_t}d\tilde{W}_t^1$$

where

$$\tilde{W}_t^1 = \left(W_t^1 + \frac{\mu - r}{\sqrt{v_t}}t\right)$$

is a Brownian motion under $\mathbb{Q}$. The risk-neutral process for the volatility is obtained by introducing a function $\lambda(S_t, v_t, t)$ into the drift term of the SDE of the variance process, given in the following

$$dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)]dt + \sigma\sqrt{v_t}d\tilde{W}_t^2$$

where

$$\tilde{W}_t^2 = \left(W_t^2 + \frac{\lambda(S_t, v_t, t)}{\sigma\sqrt{v_t}}t\right)$$

and $\tilde{W}_t^2$ is again a Brownian motion under $\mathbb{Q}$. The function $\lambda(S_t, v_t, t)$ is called the volatility risk premium. As stated in Heston [34], the premium is assumed to be proportional to the volatility, i.e. $\lambda(S_t, v_t, t) = \lambda v_t$ where λ is constant. If we rewrite the variance process with the new value of the risk premium, we get the risk-neutral variance equation as follows

$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t}d\tilde{W}_t^2$$

where

$$\kappa^* = \kappa + \lambda \quad \text{and} \quad \theta^* = \frac{\kappa\theta}{\kappa + \lambda}.$$

Consequently, we get the following equations for the risk-neutral processes for the stock price and the variance:

$$dS_t = S_t r dt + S_t\sqrt{v_t}d\tilde{W}_t^1 \tag{3.15}$$

$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t}d\tilde{W}_t^2 \tag{3.16}$$

where $E_{\mathbb{Q}}(d\tilde{W}_t^1 d\tilde{W}_t^2) = \rho dt$ under the risk-neutral measure $\mathbb{Q}$. It is shown by Heston [34] that the stock price and the variance processes are negatively correlated which is referred as *leverage effect*. Also note that, if we take $\lambda = 0$ then we see the processes under the physical measure and the risk-neutral measure coincide. One could estimate the risk premium parameter λ by using average returns on option positions that are hedged against the risk of changes in spot asset [34]. However, in this thesis for notational simplicity we consider Equation (3.16) under the assumption of $\lambda = 0$.

In his original paper Heston [34] employs a natural approach to obtain the option price by assuming the price of an option under the stochastic volatility model will imitate the BS formula, e.g. for a call option it may have the following form

$$V_{call}(t, S, v) = S\mathbb{P}_1 - KP(t, T)\mathbb{P}_2.$$

Then, by considering the log-transform of the stock price the probability density functions will also satisfy the original pricing PDE with the following terminal condition

$$\mathbb{P}_j[\ln(s), v, T; \ln(K)] = \mathbf{1}_{\{\ln(s) \geq \ln(K)\}}$$

where s, v are the initial values for the stock and the variance processes, respectively and K is the strike price. These conditional probabilities do not admit an immediate closed form solution. Instead of treating these probabilities Heston [34] uses the relation between the characteristic functions and the probability density functions, namely he obtains the probabilities by an inverse Fourier transform of the characteristic function. The motivation behind his approach is that the characteristic functions also satisfy the same PDE with the terminal condition as follows

$$f_j(\ln(s), v, T; \varphi) = e^{i\varphi \ln(s)}.$$

More precisely, the solution of the PDE is given by

$$f_i(\ln(s), v, t; \varphi) = e^{C(T-t;\varphi) + D(T-t;\varphi)v + i\varphi \ln(s)} \tag{3.17}$$

where C and D correspond to the solutions of the ODEs reduced from the PDE

$$\begin{aligned} -\frac{1}{2}\sigma^2\varphi + \rho\sigma\varphi i D + \frac{1}{2}D^2 + u_j\varphi i - b_j D + \frac{\partial D}{\partial t} &= 0 \\ r\varphi i + \kappa\theta D + \frac{\partial C}{\partial t} &= 0 \end{aligned}$$

with $u_j = \pm\frac{1}{2}$ for $j = 1, 2$. Finally, the desired probabilities are obtained by the inverse Fourier transform

$$\mathbb{P}_j[\ln(s), v, T; \ln(K)] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \left[\frac{e^{-i\varphi \ln(K)}\, f_j(\ln(s), v, T; \varphi)}{i\varphi}\right] d\varphi. \tag{3.18}$$

It is necessary to emphasize that this approach also works for pricing bond options which in fact is considered by Heston [34]. We will point out the similarities in the following sections.

Due to analytically tractable nature of the Heston model, there exist also studies which exactly simulate the Heston model. Before giving a brief introduction about these studies, let us mention some useful facts about the Heston model. Since the two Brownian motions in the Heston model are correlated, a Cholesky decomposition of the covariance matrix leads to the following equation

$$W_t = \rho W_t^1 + \sqrt{1-\rho^2} W_t^2$$

or vice versa. This new Brownian motion W_t can be utilized either in the stock price process or in the variance process. It is also possible to consider the stock price process with a logarithmic transformation, i.e. set $\tilde{S}_t = \ln S_t$. Therefore, one can express the log-price process for the Heston model under the risk-neutral measure $\mathbb{Q}$ by

$$dS_t = (r - \frac{1}{2}v_t)dt + \sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2). \tag{3.19}$$

In addition, one can rearrange the variance process in the integral form given by

$$\int_0^t \sqrt{v_s} dW_s^2 = \frac{1}{\sigma}\left(v_t - v_0 - \kappa\theta t - \kappa \int_0^t v_s ds\right). \tag{3.20}$$

Hence, one can plug this term into the log-price process given in Equation (3.19) as

$$S_t = S_0 + \int_0^t (r - \frac{1}{2}v_s)ds + \rho \int_0^t \sqrt{v_s} dW_s^2 + \sqrt{1-\rho^2} \int_0^t \sqrt{v_s} dW_s^1. \tag{3.21}$$

Substituting Equation (3.20) into Equation (3.21) yields

$$\begin{aligned} S_t = {} & S_0 + rt + \frac{\rho}{\sigma}(v_t - v_0 - \kappa\theta t) \\ & + \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right) \int_0^t v_s ds + \sqrt{1-\rho^2} \int_0^t \sqrt{v_s} dW_s^1. \end{aligned} \tag{3.22}$$

Therefore, simulation of the pair (S_t, v_t) is then reduced to the following joint distribution

$$(v_t, \int_0^t v_s ds).$$

If one manages to find a closed form expression for this joint distribution which corresponds on the right hand side of Equation (3.20), then one can obtain the distribution of $\int_0^t \sqrt{v_s} dW_s^2$ accordingly. In fact, once the expression for the joint distribution is obtained, then one can generate a path of v_t. Moreover, Broadie and Kaya [10] state that given the path generated by v_t, one can also determine the distribution of the second Itô integral $\int_0^t \sqrt{v_s} dW_s^1$. Since v_t is independent of the Brownian motion W_t^1, the distribution of $\int_0^t \sqrt{v_s} dW_s^1$ has the following form [10]

$$\int_0^t \sqrt{v_s} dW_s^1 \sim \mathcal{N}\left(0, \int_0^t v_s ds\right).$$

Furthermore, as going to be stated in Section 5.1, the joint distribution $(v_t, \int_0^t v_s ds)$ does not admit an easy explicit representation under the CIR model. However, there exist studies in which the exact simulation of the Heston model is considered. Based on those facts given above, in their paper Broadie and Kaya [10] generate an exact sample from the distribution of S_t by conditioning on the endpoints v_0 and v_t generated by the variance process, i.e.

$$\left(\int_0^t v_s ds \mid v_0, v_t \right). \tag{3.23}$$

In fact, they sample this distribution through numerical inversion of its characteristic function. Since the direct inversion is numerically expensive, instead they use an acceptance-rejection technique. In another paper, Glasserman and Kim [28] deal with same problem with a more tractable representation of the integral of the variance process given in Equation (3.23). In particular, they represent the distribution conditioned on the endpoints (v_0, v_t) via an infinite sum of mixtures of gamma random variables. Note that, the transition density of non-central chi-square variable $\chi^2(d, \lambda)$ involves a gamma increment, i.e.

$$\begin{aligned} \mathbb{P}(\chi^2(d,\lambda) \leq k) &= \mathcal{F}_{\chi^2(d,\lambda)}(k) \\ &\equiv e^{\frac{-\lambda}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\lambda\right)^j / j!}{2^{\frac{d}{2}+j}\,\Gamma\left(\frac{d}{2}+j\right)} \int_0^k z^{\frac{d}{2}+j-1} e^{\frac{-z}{2}} dz \end{aligned} \tag{3.24}$$

for $k > 0$. Therefore, in [28] the integrated variance process decomposed into gamma variables by following a similar approach of the decomposition of Bessel bridges. The CIR process is a modified OU process, indeed a squared OU process, thus in principle conveying the Bessel bridges decomposition to the CIR process is possible. In order to suggest an applicable decomposition [28] truncate the infinite sum of gamma variables.

The main advantage of the proposed exact simulations is that both methods are unbiased, since they do not adopt a discrete scheme for the simulation. Therefore, we do not encounter the discretization error in the exact simulation of the Heston model. Although the methods are unbiased, the computational effort for both of them can be cumbersome especially if one has to simulate the process along the whole time horizon. This actually occurs in the computation of the price of path-dependent options. Hence, if one aims to price more complex structured options in the Heston model, then MC methods together with discretization schemes appear as a promising solution.

3.2.1 Discrete Schemes in the Heston Model

In this section, we introduce how to generate sample paths (S_t, v_t) by discretizing the Heston model. However, before going into the detail of discrete methods, we may briefly present the almost exact simulation of the Heston model introduced by Andersen [4]. As clarified in the previous section, the main difficulty of exact simulation emerges from the dynamics of the variance process. Since the pure exact simulation is computationally expensive, an almost exact simulation of the CIR model and an EM scheme for the subsequent simulation of the stock price are suggested in [4]. The intuitive idea is to replace the non-central chi-square distribution with a Gaussian distribution for the large values of v_t under some moment matching conditions. For small values of v_t an asymptotic expansion of the density is chosen. For the detailed examination, see [4].

In addition to all these exact or almost exact simulations of the Heston model, using an underlying discretization scheme is suitable in many MC implementations. One approach based on the idea of splitting the operators coming from the Itô-Taylor expansion, see Section 2.1.2, is given by Ninomiya and Victoir [47]. They proved that the splitting method has a weak convergence of order 2 and provided an example of pricing Asian options in the Heston model. Recently, to discretize the Heston model Altmayer and Neuenkirch [3] introduced the drift implicit Milstein scheme for the variance process and the Euler scheme for the log-price of the stock process. Their scheme has a weak convergence of order 1. In this thesis, due to its simplicity we use the EM scheme to discretize the Heston model. Now let us consider the Heston model where both the log-price and the variance processes are discretized by the EM scheme, i.e.

$$\begin{aligned} S_{n+1} &= S_n + \left(r - \frac{1}{2}v_n\right)\Delta_n + \sqrt{v_n}(\rho\Delta W_n^1 + \sqrt{1-\rho^2}\Delta W_n^2) \\ v_{n+1} &= v_n + \kappa(\theta - v_n)\Delta_n + \sigma\sqrt{v_n}\Delta W_n^2 \end{aligned} \tag{3.25}$$

where Δ_n is the time step and ΔW_n is the Brownian increment. For discretization of the Heston model a great care has to be taken about the negative values that the variance process might attain during the simulation. This is the main numerically challenging aspect of the simulation of the Heston model. As already mentioned above in continuous time, the variance process stays strictly positive as long as the Feller condition ($2\kappa\theta > \sigma^2$) is satisfied. However, the positiveness of the discretized variance process is not guaranteed. Therefore, it is necessary to examine the discretization of the CIR process in detail. Because of its wide applicability the CIR process has gained considerable attention in the finance literature. In their paper, Deelstra and Delbaen [17] extend the CIR model by allowing a stochastic mean reversion level which is constant in the usual CIR model. In particular, Alfonsi [1, 2] gives a detailed examination of the CIR process and relatedly introduces several different discrete schemes to discretize the CIR model. He also points out that the main difficulty during the discretization of the CIR process is located in 0, where the square-root is not Lipschitz continuous. He also provides a criterion to choose the appropriate discretization scheme for the CIR process, which involves checking the capacity of the scheme to support large values of σ (i.e $\sigma^2 \geq 4\kappa\theta$). If we use the CIR process to model the short rate, then we do not need the larger values of σ. However, in the Heston model the CIR process is used to model the volatility of stock prices. Thus, it is possible to observe large values for σ. When σ attains large values, then the CIR process spends much time in the neighborhood of 0 [2]. Therefore it is inevitable to take precautions against the negative values of the variance process in the Heston model in order to obtain accurate results. Lord, Koekkoek and van Dijk [45] compared several proposed solutions in a unified general framework, see Table 1 in the relevant paper. They also proposed a solution, which is known as "Fully truncated EM scheme". Full truncation refers to plugging an auxiliary function $f(t) = \max\{v_t, 0\}$ for both the drift and the diffusion coefficients of the variance process and the current state of the variance, i.e.

$$\begin{aligned} v_{n+1} &= v_n + \kappa(\theta - \max\{v_n, 0\})\Delta_n + \sigma\sqrt{\max\{v_n, 0\}}\Delta W_n \\ v_{n+1}^{+} &= \max\{v_{n+1}, 0\} \end{aligned}$$

where the discretization is done according to the EM scheme given in Equation (2.24). This auxiliary function helps to keep the values of the variance process positive, since in the case of $v_n < 0$ the right-hand side of the EM scheme reduces to $v_{n+t} = v_n + \kappa\theta\Delta_n$. Therefore, the variance process always remains positive.

Chapter 4

General Properties of the Heath-Platen Estimator

The Heath-Platen (HP) estimator is initially introduced by Heath and Platen under the name of *Differential Operator Integral* (DOI) in [32]. The general framework of the HP estimator and an example of its use for pricing European vanilla options are given in [32]. Since the main concern of this thesis is the application of the HP estimator in pricing barrier and bond options, it is inevitable to explain the dynamics of the HP estimator in rigorous detail. Hence, in this chapter we present the general properties of the estimator and show what makes the HP estimator powerful and efficient as a control variate technique. Let us start by considering a general d-dimensional diffusion process $X^{s,x} = \{X_t^{s,x}, t \in [s,T]\}$ which is driven by the following SDE

$$dX_t^{s,x} = a(t, X_t^{s,x})dt + \sum_{j=1}^{m} b^j(t, X_t^{s,x})dW_t^j \tag{4.1}$$

for $t \in [s,T]$ with initial value $X_s^{s,x} = x \in \Gamma$ for $s \in [0,T]$ where $T \in (0,\infty)$ is fixed and Γ is an open connected subset of $\mathbb{R}^d$. As always we assume that the coefficient functions a, b satisfy the conditions ensuring the existence of a unique strong solution of the SDE given in Equation (4.1). We denote by $\tau : \Omega \to [s,T]$ the first exit time of $(t, X_t^{s,x})$ from $[s,T) \times \Gamma$ that is

$$\tau = \inf\{t \geq s : (t, X_t^{s,x}) \notin [s,T) \times \Gamma\}. \tag{4.2}$$

The corresponding differential operators $\mathcal{L}^0$ and $\mathcal{L}^j$ of the process $X^{s,x}$ defined on a sufficiently smooth function $f : [0,T] \times \Gamma \to \mathbb{R}$ are given by

$$\mathcal{L}^0 f(t,x) = \frac{\partial f(t,x)}{\partial t} + \sum_{i=1}^{d} a^i(t,x)\frac{\partial f(t,x)}{\partial x^i} + \frac{1}{2}\sum_{i,k=1}^{d}\sum_{j=1}^{m} b^{i,j}(t,x)b^{k,j}(t,x)\frac{\partial^2 f(t,x)}{\partial x^i \partial x^k} \tag{4.3}$$

and

$$\mathcal{L}^j f(t,x) = \sum_{i=1}^{d} b^{i,j}(t,x)\frac{\partial f(t,x)}{\partial x^i} \tag{4.4}$$

for $(t, x) \in (0, T) \times \Gamma$. Let us consider a payoff function $h : B \to \mathbb{R}$ where

$$B = ([0, T) \times \partial\Gamma) \cup (\{T\} \times (\Gamma \cup \partial\Gamma))$$

and a so-called valuation function, namely the undiscounted option price, $u : [0, T] \times \Gamma \to \mathbb{R}$ given by

$$u(t, x) = \mathbb{E}\left[h(\tau, X_\tau^{t,x})\right] \tag{4.5}$$

for $(t, x) \in [0, T] \times \Gamma$. Our ultimate goal is to find an unbiased estimator for $u(0, x)$ which corresponds to the expected payoff of the relevant option in our applications in option pricing. However, due to the complex dynamics of the underlying stochastic process $X^{s,x}$, this turns out to be a non-trivial problem. Thus, we aim to approximate this complex process via another relatively simple process which mimics the behavior of our target process. This indeed captures the essential nature of the control variate technique. Let $\bar{X}^{s,x} = \{\bar{X}_t^{s,x}, t \in [s, T]\}$ be another d-dimensional diffusion process driven by the SDE

$$d\bar{X}_t^{s,x} = \bar{a}(t, \bar{X}_t^{s,x})dt + \sum_{j=1}^{m} \bar{b}^j(t, \bar{X}_t^{s,x})dW_t^j \tag{4.6}$$

for $t \in [s, T]$ and $s \in [0, T]$. The key feature of $\bar{X}^{s,x}$ is that it mimics the behavior of the target process $X^{s,x}$, as e.g. described below in Section 4.1. Assume that $\bar{X}^{s,x}$ has a smooth valuation function $\bar{u} : [0, T] \times \Gamma \to \mathbb{R}$, namely $\bar{u} \in C^{1,2}([0, T] \times \Gamma)$. Moreover, we assume that $\bar{u}(t, x)$ approximates to the valuation function $u(t, x)$ of our target process $X^{s,x}$ under the following boundary condition

$$\bar{u}(\tau, X_\tau^{0,x}) = u(\tau, X_\tau^{0,x}) = h(\tau, X_\tau^{0,x}). \tag{4.7}$$

Since the function $\bar{u}(t, x)$ is smooth enough, an application of the differential operators given in Equations (4.3) and (4.4) is possible. Furthermore, if we apply the Itô formula to $\bar{u}(t, x)$, we get

$$\bar{u}(\tau, X_\tau^{0,x}) = \bar{u}(0, x) + \int_0^\tau \mathcal{L}^0 \bar{u}(t, X_t^{0,x})dt + \sum_{j=1}^{m} \int_0^\tau \mathcal{L}^j \bar{u}(t, X_t^{0,x})dW_t^j. \tag{4.8}$$

Here, it is assumed that $\mathcal{L}^j \bar{u}$ satisfies the appropriate integrability conditions so that the process $\int_0^\tau \mathcal{L}^j \bar{u}(t, X_t^{0,x})dW_t^j$ is martingale. Furthermore, together with the boundary condition (4.7), (4.5), (4.8) and Fubini's theorem, we obtain the following

$$\begin{aligned} u(0, x) &= \mathbb{E}(h(\tau, X_\tau^{0,x})) \\ &= \mathbb{E}(\bar{u}(\tau, X_\tau^{0,x})) \\ &= \bar{u}(0, x) + \mathbb{E}\left(\int_0^\tau \mathcal{L}^0 \bar{u}(t, X_t^{0,x})dt\right) \\ &= \bar{u}(0, x) + \int_0^T \mathbb{E}(\mathbf{1}_{\{t<\tau\}} \mathcal{L}^0 \bar{u}(t, X_t^{0,x}))dt \end{aligned} \tag{4.9}$$

for $x \in \Gamma$ and for the indicator function $\mathbf{1}_{\{t<\tau\}}$ applied to the event $\{t < \tau\}$. Consequently, the following *unbiased* approximation for $u(0, x)$ is obtained [32]

$$\bar{Z}_\tau = \bar{u}(0, x) + \int_0^\tau \mathcal{L}^0 \bar{u}(t, X_t^{0,x})dt. \tag{4.10}$$

Note that, $\bar{Z}_\tau$ is unbiased due to Equation (4.9)

$$u(0,x) = \mathbb{E}(\bar{Z}_\tau) = \bar{u}(0,x) + \mathbb{E}\left(\int_0^\tau \mathcal{L}^0 \bar{u}(t, X_t^{0,x})dt\right).$$

As a result, this approximation can be utilized in MC simulations to estimate $u(0,x)$, which can be an undiscounted option price, via the following HP estimator

$$I_{HP,N} = \frac{1}{N}\sum_{i=1}^{N} \bar{Z}_\tau^{(i)}. \tag{4.11}$$

In addition, the HP approach allows us to apply some iterative extensions to the approximation given in Equation (4.10). One can simply define the differential operator $\bar{\mathcal{L}}^0$ which belongs to the approximating process $\bar{X}^{s,x}$ by using its coefficients $\bar{a}^i$ and $\bar{b}^{i,j}$. On the other hand, if $\bar{u}$ is sufficiently smooth, then an application of Kolmogorov backward equation yields the PDE

$$\bar{\mathcal{L}}^0 \bar{u}(t,x) = 0 \tag{4.12}$$

for $(t,x) \in [0,T] \times \Gamma$ with boundary condition

$$\bar{u}(t,x) = h(t,x)$$

for $(t,x) \in B$. Therefore, with this choice of $\bar{u}$ and the PDE given in Equation (4.12) the iterative HP estimator takes the form

$$\bar{Z}_\tau = \bar{u}(0,x) + \int_0^\tau (\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t, X_t^{0,x})dt. \tag{4.13}$$

In fact, taking the difference between the operators $(\mathcal{L}^0 - \bar{\mathcal{L}}^0)$ is a simple but powerful way of coupling the two processes. In the following section, a detailed examination of the effectiveness of the iterative HP method is presented.

4.1 Application in the Heston Model

In this subsection, the application of the iterative HP estimator to the Heston model is described in a detailed framework, which is originally given in [32] for the European vanilla options. Supplementary to their results regarding the use of the HP estimator in pricing European vanilla options in the Heston model, we provide a performance comparison among the HP estimator, the crude MC estimation and the crude control variate technique for pricing European call options. Furthermore, as a novel contribution we apply the HP estimator in pricing barrier options in the Heston model. We begin our discussions with the detailed examination of the mechanism of the HP estimator in the Heston model. Since the Heston model is our target process, let us first recall the model dynamics under the risk-neutral measure $\mathbb{Q}$

$$dS_t = S_t r dt + S_t\sqrt{v_t}dW_t^1 \tag{4.14}$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2 \tag{4.15}$$

where W_1, W_2 are Brownian motions with $E^{\mathbb{Q}}(dW^1 dW^2) = \rho dt$. Moreover, the initial values are $S_0^{0,s} = s > 0$ and $v_0^{0,v} = v > 0$. In order to reduce the MC variance in the Heston model, the Black-Scholes (BS) model appears as a possible candidate. However, for a better approximation we consider the generalized BS model (GBS), i.e. a BS model with deterministic volatility

$$d\bar{S}_t = \bar{S}_t r dt + \bar{S}_t \sqrt{\bar{v}_t} dW_t^1 \tag{4.16}$$

$$d\bar{v}_t = \kappa(\theta - \bar{v}_t)dt. \tag{4.17}$$

The initial values of the processes are given such that $\bar{S}_0^{0,\bar{s}} = \bar{s} > 0$ and $\bar{v}_0^{0,\bar{v}} = \bar{v} > 0$. We can explicitly solve the equation for deterministic variance where the solution is given as follows

$$\bar{v}_t^{0,\bar{v}} = \theta + (\bar{v}_t - \theta)e^{-\kappa t}. \tag{4.18}$$

As a further concern, we are able to obtain a type of BS-formula for the call option price under the generalized BS model.

Proposition 1. *[26] Let us consider the generalized BS model given by Equations* (4.16) *and* (4.17). *Then for the GBS model, one can obtain the following type of BS-formula*

$$GBS(t, T, \bar{S}, K, \bar{\sigma}_t) = \bar{S}\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t))$$

where

$$d_1(t) = \frac{\ln\left(\frac{\bar{S}}{K}\right) + (r + \frac{1}{2}\bar{\sigma}_t^2)(T-t)}{\bar{\sigma}_t\sqrt{T-t}}$$

$$d_2(t) = d_1(t) - \bar{\sigma}_t\sqrt{T-t}$$

with the choice of

$$\bar{\sigma}_t = \sqrt{\frac{1}{T-t}\int_t^T \bar{v}_z dz}. \tag{4.19}$$

Here $\bar{v}_t$ corresponds to the deterministic variance given in Equation (4.18).

Proof. Let us assume that the function $V(t, \bar{S})$ is the call option price at time t under the generalized BS model. Then we can rewrite this function as a discounted expected value of the payoff, namely $h(\bar{S}_T) = (\bar{S}_T - K)^+$, under the risk-neutral measure $\mathbb{Q}$

$$
\begin{aligned}
V(t,\bar{S}) &= \mathbb{E}_{\mathbb{Q}}^{t,\bar{S}}\left(e^{-\int_t^T r dz} h(\bar{S}_T)\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r dz} h(\bar{S}_T) \Big| \bar{S}_t = \bar{S}\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r dz} h\left(\bar{S}_t e^{\int_t^T (r-\frac{1}{2}\bar{v}(z))dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)}\right) \Big| \bar{S}_t = \bar{S}\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r dz} h\left(\bar{S} e^{\int_t^T (r-\frac{1}{2}\bar{v}(z))dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)}\right)\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r dz} \left(\bar{S} e^{\int_t^T (r-\frac{1}{2}\bar{v}(z))dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)} - K\right)^+\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(e^{-r(T-t)} \left(\bar{S} e^{r(T-t) - \frac{1}{2}\int_t^T \bar{v}(z)dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)} - K\right)^+\right) \\
&= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left(\left(\bar{S} e^{r(T-t) - \frac{1}{2}\int_t^T \bar{v}(z)dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)} - K\right) \mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \\
&= e^{-r(T-t)} \left[\mathbb{E}_{\mathbb{Q}}\left(\bar{S} e^{r(T-t) - \frac{1}{2}\int_t^T \bar{v}(z)dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)} \mathbf{1}_{\{\bar{S}_T \geq K\}}\right) - \mathbb{E}_{\mathbb{Q}}\left(K \mathbf{1}_{\{\bar{S}_T \geq K\}}\right)\right] \\
&:= e^{-r(T-t)}(A - B) \qquad (4.20)
\end{aligned}
$$

We compute the expectations separately. We first start with the relatively easy one,

$$
\begin{aligned}
B &:= \mathbb{E}_{\mathbb{Q}}\left(K \mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \\
&= K \mathbb{E}_{\mathbb{Q}}\left(1 \mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \\
&= K \mathbb{Q}\left(\mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \\
&= K \mathbb{Q}\left(\bar{S} e^{r(T-t) - \frac{1}{2}\int_t^T \bar{v}(z)dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)} \geq K\right) \\
&= K \mathbb{Q}\left(\frac{\int_t^T \sqrt{\bar{v}(z)}dW(z)}{\sqrt{\int_t^T \bar{v}(z)dz}} \geq \frac{\ln\left(\frac{K}{\bar{S}}\right) - r(T-t) + \frac{1}{2}\int_t^T \bar{v}(z)dz}{\sqrt{\int_t^T \bar{v}(z)dz}}\right).
\end{aligned}
$$

We have to seek the distribution of the stochastic integral, $\int_t^T \sqrt{\bar{v}(z)}dW(z)$. For this purpose, we refer Theorem (2.42) in [41] which ensures that the left hand side of the inequality is normally distributed, i.e. $\tilde{z} := \dfrac{\int_t^T \sqrt{\bar{v}(z)}dW(z)}{\sqrt{\int_t^T \bar{v}(z)dz}} \sim \mathcal{N}(0,1)$. If we set $\int_t^T \bar{v}(z)dz = (T-t)\bar{\sigma}_t^2$, then we obtain the following

$$
\begin{aligned}
B &:= K \mathbb{Q}\left(\tilde{z} \geq \frac{\ln\left(\frac{K}{\bar{S}}\right) - (r + \frac{1}{2}\bar{\sigma}_t^2)(T-t)}{\bar{\sigma}_t \sqrt{T-t}}\right) \\
&= K \Phi\left(\frac{\ln\left(\frac{\bar{S}}{K}\right) + (r - \frac{1}{2}\bar{\sigma}_t^2)(T-t)}{\bar{\sigma}_t \sqrt{T-t}}\right) \\
&= K \Phi(d_2(t)) \qquad (4.21)
\end{aligned}
$$

where $d_2(t) := \dfrac{\ln\left(\frac{\bar{S}}{K}\right) + (r - \frac{1}{2}\bar{\sigma}_t^2)(T-t)}{\bar{\sigma}_t\sqrt{T-t}}$. Note that, the $d_2(t)$ function resembles the function d_2 of the BS formula with constant volatility. The first expectation can also be computed in a similar manner.

$$\begin{aligned} A &:= \mathbb{E}_{\mathbb{Q}}\left(\bar{S}e^{r(T-t)-\frac{1}{2}\int_t^T \bar{v}(z)dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)}\mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \\ &= \bar{S}e^{(r-\frac{1}{2}\bar{\sigma}_t^2)(T-t)}\mathbb{E}_{\mathbb{Q}}\left(e^{\int_t^T \sqrt{\bar{v}(z)}dW(z)}\mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \end{aligned}$$

We also know that the following equation holds for a normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\mathbb{E}\left(e^X\right) = e^{\mu + \frac{\sigma^2}{2}}.$$

Since in our case the integrand of the stochastic integral is equal to $\sqrt{\bar{v}_z}$ which is a deterministic function and belongs to the space $L^2[0, T]$, we have

$$\begin{aligned} \mathbb{E}\left(e^{\int_t^T \sqrt{\bar{v}(z)}dW(z)}\right) &= e^{\frac{1}{2}\int_t^T \bar{v}(z)dz} \\ &=: e^{\frac{1}{2}\bar{\sigma}_t^2(T-t)}. \end{aligned}$$

Thus, if we proceed with the computation of the expectation, after some rearrangement of the terms we get

$$\begin{aligned} A &:= \bar{S}e^{(r-\frac{1}{2}\bar{\sigma}_t^2)(T-t)}\mathbb{E}_{\mathbb{Q}}\left(e^{\int_t^T \sqrt{\bar{v}(z)}dW(z)}\bar{S}e^{r(T-t)-\frac{1}{2}\int_t^T \bar{v}(z)dz + \int_t^T \sqrt{\bar{v}(z)}dW(z)}\mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \\ &= \bar{S}e^{(r-\frac{1}{2}\bar{\sigma}_t^2)(T-t)}\mathbb{E}_{\mathbb{Q}}\left(e^{\left(\int_t^T \sqrt{\bar{v}(z)}dW(z)\right)^2}\mathbf{1}_{\{\bar{S}_T \geq K\}}\right) \\ &\vdots \\ &= \bar{S}e^{r(T-t)}\Phi(d_1(t)) \end{aligned} \tag{4.22}$$

If we substitute the expressions given in Equations (4.22) and (4.21) into Equation (4.20), then we obtain the following

$$V(t, \bar{S}) = e^{-r(T-t)}\left(\bar{S}e^{r(T-t)}\Phi(d_1(t)) - K\Phi(d_2(t))\right).$$

Consequently, by setting

$$\bar{\sigma}_t = \sqrt{\frac{1}{T-t}\int_t^T \bar{v}(z)dz}$$

we obtain the following type of BS formula for a call option in the generalized BS model

$$GBS(t, T, \bar{S}, K, \bar{\sigma}_t) = \bar{S}\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t))$$

where

$$\begin{aligned} d_1(t) &= \frac{\ln\left(\frac{\bar{S}}{K}\right) + (r + \frac{1}{2}\bar{\sigma}_t^2)(T-t)}{\bar{\sigma}_t\sqrt{T-t}} \\ d_2(t) &= d_1(t) - \bar{\sigma}_t\sqrt{T-t}. \end{aligned}$$

This completes the proof. □

This choice of $\bar{\sigma}_t$ also appears in the work of Fouque et. al. [25]. Coming from a PDE-based approach they call $\bar{\sigma}_t$ the *effective volatility*. We may give a brief introduction to their work while in a sense their work is complementary to ours. In their work they consider a multi-scale model for the stock price process given by

$$
\begin{aligned}
dX_t &= \mu X_t dt + f(Y_t, Z_t) X_t dW_t^{(0)} \\
dY_t &= \frac{1}{\varepsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)} && \text{(Fast-scale)} \\
dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t) dW_t^{(2)} && \text{(Slow-scale)}
\end{aligned}
$$

where all the Brownian motions are mutually correlated. Then an option price under this two-scale model is obtained approximately by an asymptotic expansion of the PDE. The asymptotic expansion method is well-known from PDE theory, where one focuses on the small perturbations of an explicitly solvable PDE. One then determines the missing/unobservable parameters by calibrating the resulting model to the market. Therefore, the main idea of the paper is to write the pricing equation as a "singular-regular perturbation" problem around the BS operator where the unobservable parameter is then the volatility. They address that the fast-scale equation and the related operator correspond to the singular perturbations, on the other hand the slow-scale equation and the related operator correspond to the regular perturbations in the domain of the pricing function. So, they deal with the option pricing problem by the following parabolic PDE

$$
\begin{aligned}
\mathcal{L}^{\varepsilon,\delta} P^{\varepsilon,\delta} &= 0 \\
P^{\varepsilon,\delta}(T, x, y, z) &= h(x)
\end{aligned}
$$

where the partial differential operator is given by

$$
\mathcal{L}^{\varepsilon,\delta} = \underbrace{\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1}_{\text{fast-scale}} + \underbrace{\mathcal{L}_2}_{\text{BS}} + \underbrace{\sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2}_{\text{slow-scale}} + \underbrace{\sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3}_{\text{mixed}} .
$$

The decomposition of the operator $\mathcal{L}^{\varepsilon,\delta}$ essentially aims to capture the real market movements via the operators of the pricing PDE. In order to provide a better representation of the market dynamics one main concern is the behavior of the implied volatility and accordingly the parameter calibration. Thus, one may have several preferences for the volatility of the BS model and relatedly the BS operator. Fouque et. al. [25] consider the BS operator with a constant volatility as it exists in the classical BS model and also with the effective volatility. The effective volatility corresponds to the invariant distribution of the Ornstein-Uhlenbeck process considered in the fast-scale equation. In particular, their main result shows that it is possible to obtain more precise results when one uses the BS operator with an effective volatility. For more details, see [25, 24, 23]. Furthermore, a detailed discussion about the implied volatility can be found in Chapter 2 in [51]. As a conclusion, this study in the multi-scale model ensures that our choice of the BS model with deterministic volatility $\bar{\sigma}_t$ will provide us a good approximation to the Heston model. We sum up the whole information about the deterministic volatility used in the generalized BS formula. The relevant value for the time dependent volatility is derived from the deterministic variance equation given in Equation (4.18) with the following procedure visualized as follows

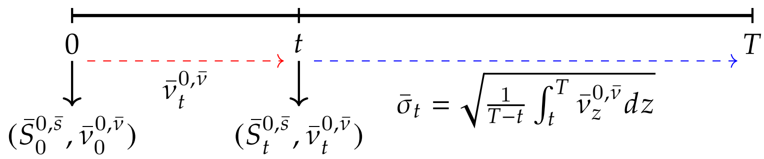

Figure 4.1: Volatility of the GBS model

Thus, we follow the deterministic variance process from the initial time 0 until any time t. Starting from time t until maturity T, we need to find the volatility, i.e $\bar{\sigma}_t$, for the GBS formula. As proven in Proposition 1 we obtain the $\bar{\sigma}_t$ value by the integration of $\bar{v}_t^{0,\bar{v}}$ over the time interval $[t, T]$, given explicitly by

$$\begin{aligned} \bar{\sigma}_t &= \sqrt{\frac{1}{T-t}\int_t^T \bar{v}_z^{t,\bar{v}} dz} \\ &= \sqrt{\frac{1}{T-t}\int_t^T [\theta + (\bar{v}_t - \theta)e^{-\kappa(z-t)}] dz} \\ &= \sqrt{\theta + (\bar{v}_t - \theta)\frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)}}. \end{aligned} \tag{4.23}$$

For the sake of notational consistency we now rewrite the GBS pricing formula in accordance with the valuation function $\bar{u}$ given for the HP estimator

$$\begin{aligned} GBS(\bar{S}_t, K, r, \bar{\sigma}_t, T-t) &:= e^{-r(T-t)}\ \bar{u}(t, \bar{S}_t, \bar{v}_t) \\ \bar{u}(t, \bar{S}_t, \bar{v}_t) &= e^{r(T-t)}\ GBS(\bar{S}_t, K, r, \bar{\sigma}_t, T-t) \end{aligned} \tag{4.24}$$

In the general framework the remaining task is to use this GBS formula as a control variate for the option price in the Heston model. However, we may emphasize that the HP estimator does not directly use the GBS as a control variate in its crude form but rather obtains a kind of expansion around the GBS pricing formula. Thus, before performing several numerical applications in detail, it is necessary to understand the theoretical background of the HP method in the Heston setting. For more general settings, see [33]. Consider that we have a function $u(t, S_t, v_t)$ which is defined on a domain $[0, T] \times \mathcal{D}$ where $\mathcal{D} = (0, \infty)^2$ and represents the expected payoff of the relevant option under the Heston model, e.g. Equation (4.5). Assume that $u(t, S_t, v_t)$ is smooth enough to allow for an application of the Itô formula which yields the following PDE

$$\frac{\partial u}{\partial t} + \frac{1}{2}v_t S_t^2 \frac{\partial^2 u}{\partial S_t^2} + \rho\sigma v_t S_t \frac{\partial^2 u}{\partial v_t \partial S_t} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 u}{\partial v_t^2} + r S_t \frac{\partial u}{\partial S_t} + \kappa(\theta - v_t)\frac{\partial u}{\partial v_t} = 0. \tag{4.25}$$

If we rewrite the PDE in compact form by the help of the differential operator $\mathcal{L}^0$ given by Equation (4.3), then we get the following

$$\mathcal{L}^0 u = 0 \tag{4.26}$$

with the terminal condition

$$u(T,s,v) = h(T,s) \tag{4.27}$$

where h is the payoff of the relevant option in the Heston setting. Indeed, this terminal value problem is known as the Kolmogorov backward equation and may turn into a Cauchy problem by introducing the constant discount factor. Under suitable technical conditions on the function h, the Feynman-Kac theorem ensures that $u(t,S_t,v_t)$ has the following stochastic representation

$$u(t,S_t,v_t) = \mathbb{E}^{t,S_t,v_t}[h(T,S_T)]. \tag{4.28}$$

On the other hand, we have another price function $\bar{u}(t,\bar{S}_t,\bar{v}_t)$ which is defined on a sub-domain of $[0,T]\times\mathcal{D}$ and also satisfies the following PDE for the generalized BS model

$$\frac{\partial\bar{u}}{\partial t} + \frac{1}{2}\bar{v}_t\bar{S}_t^2\frac{\partial^2\bar{u}}{\partial\bar{S}_t^2} + r\bar{S}_t\frac{\partial\bar{u}}{\partial\bar{S}_t} + \kappa(\theta-\bar{v}_t)\frac{\partial\bar{u}}{\partial\bar{v}_t} = 0. \tag{4.29}$$

If we again rewrite this equation in a compact form, we get

$$\bar{\mathcal{L}}^0\bar{u} = 0 \tag{4.30}$$

with the terminal condition

$$\bar{u}(T,\bar{s},\bar{v}) = h(T,\bar{s}). \tag{4.31}$$

Now imagine that we aim to cover the domain of the Heston price function $u(t,S_t,v_t)$ by using the function $\bar{u}(t,\bar{S}_t,\bar{v}_t)$ which is defined on the sub-domain, i.e. the domain of the GBS price function. To achieve this goal, we allow the function $\bar{u}(t,\bar{S}_t,\bar{v}_t)$ to move across the whole domain by replacing its components with the Heston dynamics. See the following Figure 4.2 for a better visual understanding.

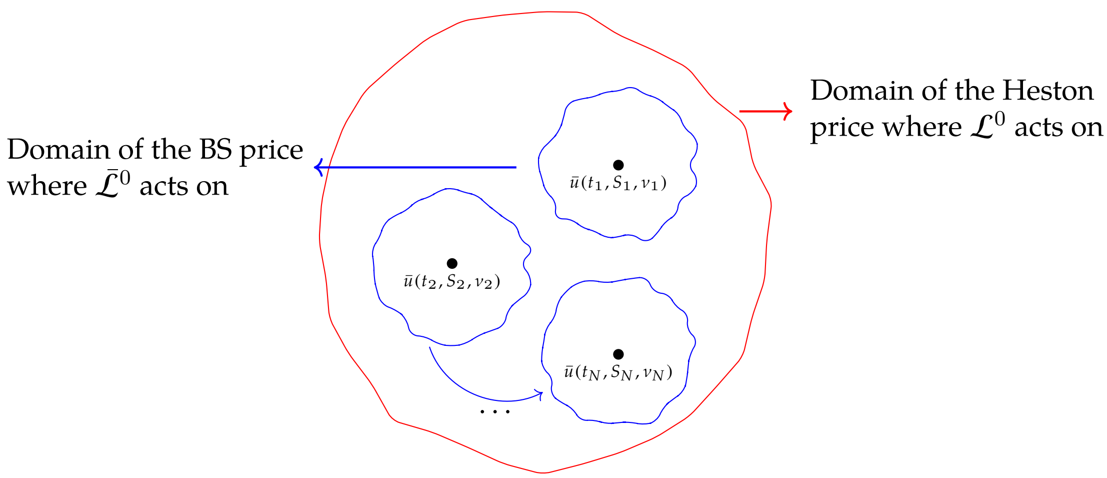

Figure 4.2: Symbolic representation of the PDE problem

However, in its usual form the components of the function $\bar{u}$ are the time, the stock price and the deterministic volatility. Therefore to be able to use the $\bar{u}$ function with the components of the Heston model, we transform the stochastic volatility of the Heston model into a deterministic one. Namely, at each time t we plug the volatility of the Heston model as an initial value of the deterministic volatility given in Equation (4.18). Thus, the function $\bar{\sigma}_t$ with the components of the Heston model is expressed by

$$\bar{\sigma}_t = \sqrt{\theta + (\nu_t - \theta)\frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)}}.$$

In fact, by doing this transformation we reduce one of the state variables and the corresponding dimension of the PDE, eventually we restrict ourselves into the sub-domain. If we have a closer look at both the Heston and the generalized BS processes which are given at the beginning of this section, we see that the price difference between these processes is generated by the stochastic part of the variance process in the Heston model. By, so to say, "removing" this stochastic part of the variance we obtain a coupled system of PDEs with a reduced number of dimensions. To sum up, we consider the price function of the generalized BS model with the stock price process of the Heston model and the transformed stochastic volatility, i.e. the final form is equal to $\bar{u}(t, S_t, \nu_t)$. Due to the movement of this $\bar{u}$ function through the whole domain it is reasonable to consider the boundary condition for this function at a first exit time τ as follows

$$\bar{u}(\tau, S_\tau, \nu_\tau) = h(\tau, S_\tau, \nu_\tau) = u(\tau, S_\tau, \nu_\tau)$$

where h is the payoff function of the relevant option. In fact, the reduction in the dimension of the PDE problem is then compensated by a Taylor expansion with respect to the removed variable. Since $\bar{u}$ is smooth enough, we can apply a Taylor expansion of the removed state variable around the GBS price. This Taylor expansion in the neighborhood of the GBS price on the sub-domain provides us with a smooth approximation for the Heston price on the whole domain. Consequently, during the simulation process for each discrete point we obtain a Taylor expansion and each final value of $\bar{Z}_\tau$ given by Equation (4.13) is an expansion around the GBS formula, namely an approximation to the option price in the Heston model. Subsequently, we calculate the mean over the $\bar{Z}_\tau$ values via the HP estimator given in Equation (4.11). In fact, the difference between the differential operators of both processes stands for the Taylor expansion of the removed state variable, i.e.

$$(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t, S_t, \nu_t) = \sigma\nu_t\Big(S_t\rho\frac{\partial^2\bar{u}(t, S_t, \nu_t)}{\partial S_t\partial\nu_t} + \frac{1}{2}\sigma\frac{\partial^2\bar{u}(t, S_t, \nu_t)}{\partial\nu_t^2}\Big). \tag{4.32}$$

Since $\bar{u}$ function corresponds to the GBS price with the deterministic volatility $\bar{\sigma}_t$, see Equations (4.24) and (4.23), we are able to compute the values for $(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t, S_t, \nu_t)$ in an explicit form. Finally, plugging all the partial derivatives into Equation (4.32) yields

$$\begin{aligned}(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t, S_t, \nu_t) = \sigma\nu_t e^{r(T-t)}\Big[&S_t\rho\frac{\partial^2 GBS(S_t, K, r, \bar{\sigma}_t, T-t)}{\partial S_t\partial\bar{\sigma}_t}\frac{\partial\bar{\sigma}_t}{\partial\nu_t} \\ &+ \frac{1}{2}\sigma\Big(\frac{\partial^2 GBS(S_t, K, r, \bar{\sigma}_t, T-t)}{\partial\bar{\sigma}_t^2}\Big(\frac{\partial\bar{\sigma}_t}{\partial\nu_t}\Big)^2 \\ &+ \frac{\partial GBS(S_t, K, r, \bar{\sigma}_t, T-t)}{\partial\bar{\sigma}_t}\frac{\partial^2\bar{\sigma}_t}{\partial\nu_t^2}\Big)\Big].\end{aligned} \tag{4.33}$$

Here, the partial derivatives $\frac{\partial GBS}{\partial \bar{\sigma}_t}$, $\frac{\partial^2 GBS}{\partial S_t \partial \bar{\sigma}_t}$ and $\frac{\partial^2 GBS}{\partial \bar{\sigma}_t^2}$ are often referred as the Greeks of the relevant option. They are known as *vega*, *vanna* and *volga*, respectively. In order to be able to obtain them in closed form, one has to check whether the option has a closed form solution in the GBS model. As a result, the HP approach aims to approximate the option price in the complex model via using the simple pricing function and having a Taylor expansion in the neighborhood of this price.

Considering the two processes, i.e. the Heston and the GBS model, the crude idea of the control variate technique would be the following

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(X - GBS) + \mathbb{E}(GBS) \\ &\approx \frac{1}{N}\sum_{i=1}^{N}(X_i - GBS_i) + \mathbb{E}(GBS)\end{aligned}$$

where X corresponds to the Heston model. The difference is basically calculated between the expectations of the two processes. In the context of option pricing, we run the simulation to obtain the expected payoffs and then take the difference over them. However, this is not the case for the HP estimator. As we already explained, initially we have the difference between the two processes via the differential operators and we run the simulation over this difference until maturity. At maturity what we obtain is indeed a Taylor expansion of the GBS price with respect to the removed variable, not the expected payoffs. Now, let us compare the crude idea of the control variate technique and the HP estimator to see how powerful the HP estimator is. We consider a European call option in the Heston setting with maturity of $T = 0.5$ years. The parameters for this numerical example are as follows: initial stock price $S_0 = 100$, strike $K = 100$, number of paths $N = 10000$, number of steps $n = 500$, interest rate $r = 0.04$, mean reversion speed $\kappa = 0.6$, volatility of variance $\sigma = 0.2$ and correlation coefficient $\rho = -0.8$. In order to avoid the negative values that the variance process might attain during the numerical simulations, we employ the fully truncated EM scheme which is introduced in [45]. By keeping the discretization bias in mind, for details see Chapter 6, we obtain the following results.

Method	$\nu(0) = \bar{\nu}(0) = \theta = 0.04$	$\nu(0) = \bar{\nu}(0) = 0.04, \theta = 0.08$
HP Estimator	6.5925	6.9530
Lower 95% bound	6.5851	6.9459
Upper 95% bound	6.5999	6.9600
Crude CV	6.6196	6.9319
Lower 95% bound	6.5620	6.8774
Upper 95% bound	6.6772	6.9864
Crude MC	6.6623	6.9746
Lower 95% bound	6.5028	6.8006
Upper 95% bound	6.8217	7.1485

Table 4.1: Comparison of the crude control variate technique, the HP estimator and the crude MC estimation, Exact values 6.5944 and 6.9544, respectively

As inferred from Table 4.1, even with $N = 10000$ simulations the HP estimator not only provides a significant variance reduction but also accurately estimates the exact option price. Moreover, the crude idea of the control variate technique also reduces the variance compared to the crude MC estimation. However, the HP estimator performs the best among the other estimations. Therefore, it is promising to examine the performance of the HP estimator for path-dependent options and possible extensions to any other option pricing problems. For this purpose, in the next section we apply the HP estimator in pricing barrier options under the Heston model.

4.2 Barrier Options in the Heston Model

In this section we consider one-sided knockout barrier options under the Heston model. Barrier options are financial derivatives whose payoff depends on whether the price of the underlying asset reaches a certain level during a certain period of time [35]. There are two main types of barrier options: knock-in and knockout options. A knock-in option is an option whose holder is entitled to receive a European option if the barrier is hit. A knockout option is an option whose holder is entitled to receive a European option if the barrier is never hit during the lifetime of the option [56]. Another way of classifying barrier options depends on whether the barrier is located below or above the initial underlying asset price. One can easily see that in total there are eight different types of one-sided barrier options. Although the structure of a barrier option can be more complicated depending on the number of the barriers and exotics features, the present study only covers all types of one-sided knockout vanilla options, i.e. down-and-out call, down-and-out put, up-and-out call and finally up-and-out put. The additional barrier feature makes the option path-dependent which is reflected by an indicator function in the payoff of the relevant option. For instance, if we consider the one-sided down-and-out call option, the payoff function is given as follows

$$h_{do}^{Call} = (S_T - K)^+ \, \mathbf{1}_{\{S_t > H,\ t \in [0,T]\}} \tag{4.34}$$

where H is the barrier level. For the remaining formulas of the one-sided barrier options, see [41, 56, 31, 50]. It can be seen from the formula that obtaining a European payoff at maturity is dependent on the barrier condition. Thus, the barrier options are conditional. Overall, barrier options are path-dependent and conditional and thus exotic options. These additional features bring some difficulties into pricing barrier options especially when the dynamics of the underlying asset is complicated. However, besides these mathematical complexities of pricing them, in practice barrier options are often traded in real life by practitioners. The reasons of their popularity are stated in [16] as follows:

(i) If an investor has a view about the future behavior of the underlying asset relatedly the market movements, then barrier options may more closely match the beliefs of the investor. By buying a barrier option, one can eliminate paying for some scenarios which might be unlikely to occur.

(ii) The premium of barrier options are generally lower than those of corresponding European options since an additional condition has to be met to receive the relevant payoff. Namely, they are cheaper than their corresponding European options.

(iii) Barrier options might match hedging needs more closely in certain situations. For instance, if the barrier is hit, then there is no need to hedge any further.

In the literature, there exist studies which focus on the issue of pricing barrier options under the Heston model. In [30] a semi-analytical solution for the discretely monitored barrier options under the Heston model is derived by using the same approach with Heston [32]. Namely, in order to obtain the required probability density function they use the n-variate characteristic functions and the inverse Fourier transform. On the other hand, in [13] the method of lines approach is developed to price both continuously and discretely monitored barrier options under the Heston model.

4.2.1 Monitoring Bias

Due to the additional barrier feature, pricing barrier options is not a trivial problem especially in the Heston model. Even in the Black-Scholes model, pricing barrier options is computationally challenging due to the monitoring bias, see Figure 2 in [19]. For example, let us consider a down-and-out call option with barrier level H, strike price K and maturity T. Although the closed form pricing formula in the BS model was first given by Merton [46], we present the pricing formula for down-and-out barrier call option given in [31] by

$$V(t,T,K,H) = \begin{cases} A - C & \text{if } H < K \\ B - D & \text{if } H > K \end{cases} \tag{4.35}$$

where

$$\begin{aligned}
A &= Se^{-q(T-t)}\Phi(x_1) - Ke^{-r(T-t)}\Phi(x_1 - \sigma\sqrt{T-t}) \\
B &= Se^{-q(T-t)}\Phi(x_2) - Ke^{-r(T-t)}\Phi(x_2 - \sigma\sqrt{T-t}) \\
C &= Se^{-q(T-t)}\Phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r(T-t)}\Phi(y_1 - \sigma\sqrt{T-t})\left(\frac{H}{S}\right)^{2\gamma} \\
D &= Se^{-q(T-t)}\Phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r(T-t)}\Phi(y_2 - \sigma\sqrt{T-t})\left(\frac{H}{S}\right)^{2\gamma}
\end{aligned}$$

with $\gamma = \frac{r-q-\frac{1}{2}\sigma^2}{\sigma^2}$ and

$$x_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}$$
$$x_2 = \frac{\ln\left(\frac{S}{H}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}$$
$$y_1 = \frac{\ln\left(\frac{H^2}{SK}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}$$
$$y_2 = \frac{\ln\left(\frac{H}{S}\right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}.$$

From the formula, one can see that the barrier level H is also involved in the cumulative distribution function. This indeed implies that the monitoring is done continuously within the pricing formula. This is known as continuously monitored barrier options. However, depending on the real market expectations it might be also desirable to specify the monitoring times, e.g. doing the barrier check at daily closing. If the barrier check is done only at some specific time points during the lifetime of the option, this is called discretely monitored barrier option. The discretization of the monitoring times is related to the simulation of the indicator function within the payoff of the option, see Equation (4.34). It is addressed in [11, 12] that there exists a mis-pricing between the continuously and discretely monitored barrier options due to the monitoring bias. Therefore, a continuity correction of the barrier level improves the accuracy of discretely monitored barrier options. Assume that for any barrier option within the BS model the monitoring is done at m different times, then let $V_m(H)$ be the price of the discretely monitored barrier option and $V(H)$ be the price of the corresponding continuously monitored barrier option. Then it is given in [11] that

$$V_m(H) = V(He^{\pm\beta\sigma\sqrt{T/m}}) + o\left(\frac{1}{\sqrt{m}}\right)$$

where + is for the case $H > S_0$, − is for the case $H < S_0$ and $\beta = -\zeta(\frac{1}{2})/\sqrt{2\pi} \approx 0.5826$ with ζ being the Riemann zeta function. Hence, it can be concluded that monitoring bias has a non-negligible effect on the accuracy of pricing the discretely monitored barrier options in the BS model. If we consider the Heston model, there exist closed form solutions only for few types of barrier options. Therefore, numerical solutions with an underlying discrete scheme are possible ways to tackle this problem. For example, MC simulation via using a discrete scheme is suitable for pricing barrier options in the Heston model. However, in this case one has to take all the errors into account, i.e. the MC variance, the bias of the underlying scheme and the monitoring bias.

4.3 The HP Estimator for Pricing Barrier Options

We aim to price one-sided knockout barrier options under the Heston model via MC simulation with a variance reduction provided by the HP estimator. For this purpose, we consider the Heston model under the risk-neutral measure $\mathbb{Q}$. The pricing equation for down-and-out call option has the form

$$V(t,T,S_t,K,H) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}^{t,S_t}\left[(S_T-K)^+ \mathbf{1}_{\{S_t>H,\ t\in[0,T]\}}\right]. \tag{4.36}$$

Again, by using the Feynman-Kac theorem, the pricing function given by Equation (4.36) satisfies the following PDE

$$\mathcal{L}V = rV \tag{4.37}$$

with terminal and boundary conditions given respectively by

$$V(T,s) = h(T,s) \tag{4.38}$$

$$V(t,H) = 0 \tag{4.39}$$

where h is the payoff function and $\mathcal{L}$ is the differential operator of the Heston model. We note that the PDE problem has an additional boundary condition besides the usual terminal condition which always exists in pricing any option. In fact, this additional condition has a precise effect on the HP estimator. For instance, in pricing European vanilla options we have only a terminal condition in the PDE problem. Therefore, we simulate the HP estimator until maturity and the arithmetic mean taken over the final values corresponds to the undiscounted option price. Thus, what we only consider in pricing European vanilla options via the HP estimator is the expected value of $\bar{Z}_T$. However, for pricing barrier options the PDE problem contains an additional boundary condition and this is also reflected in the simulation of the HP estimator. Now let us recall Equation (4.9),

$$u(0,x) = \bar{u}(0,x) + \int_0^T \mathbb{E}(\mathbf{1}_{\{t<\tau\}}\mathcal{L}^0\bar{u}(t,X_t^{0,x}))dt.$$

Here, $\mathbf{1}_{\{t<\tau\}}$ denotes the indicator function applied to the event $\{t<\tau\}$ where τ is given in Equation (4.2). This indicator function plays a slightly different role than the one given in the payoff function in Equation (4.34). Let us explain the difference in detail, since this is crucial for pricing the barrier options via the HP estimator. In the crude MC simulation, it is not necessary to determine the first exit time while it is enough to check if the barrier is hit or not. In case of a hit we plug the value 0 for the payoff of the relevant path and for the paths which never hit barrier we take the value $(S_T-K)^+$ as an expected final payoff. Then the expected payoff of a down-and-out call may be either 0 or $(S_T-K)^+$, i.e.

$$h(T,H,K,S) = \begin{cases} (S_T-K)^+ & \text{if } S_t > H \text{ for all } t\in[0,T] \\ 0 & \text{if } S_\tau \le H \text{ for some } \tau\in[0,T]. \end{cases}$$

The role of the indicator function in the crude MC estimation is to determine if the path provides a final payment or not. However, this is not the case for the HP estimator. Since the HP approximation is based on the idea of having a Taylor expansion around the GBS price, we proceed with the expansion until the first exit time. In fact, the indicator function within the HP approximation implies that one has to perform the Taylor expansion until the first exit time. This is the main difference between the crude MC estimation and the HP estimation with respect to pricing barrier options. Due to this property of the indicator function, it is inevitable for the HP estimator to exactly determine the first exit/hitting time. Subsequently, we simulate the HP estimator until the first exit time and then take this value as a final expected value for the estimator. If the stock price never hits the barrier during the lifetime of the option, then we proceed with the simulation of the HP estimator until maturity. Hence, we have to precisely determine the value of the HP estimator at the first exit time. We can write the expression of $\bar{Z}$ for the barrier option as follows

$$\bar{Z}(T,H,K,S) = \begin{cases} \bar{Z}_T & \text{if } S_t > H \text{ for all } t \in [0,T] \\ \bar{Z}_\tau & \text{if } S_\tau \leq H \text{ for some } \tau \in [0,T]. \end{cases} \tag{4.40}$$

As a result, we see that in the approximation model there exist some paths which hit the barrier during the lifetime of the option but still provide us a positive payment at the end, even though these paths have zero payoff in the crude estimation. This is a significant point for the numerical simulations. In the following, we sum up the procedure for the application of the HP estimator to price one-sided knockout barrier options.

Remark 2. *Let us consider a down-and-out call option with the payoff function given in Equation* (4.34) *for some $K \geq 0$. Assume that the underlying stock price hits the barrier for the first time at τ given in Equation* (4.2). *Then we have the following valuation function for the Heston model*

$$u(0,s,\nu) = \mathbb{E}_{\mathbb{Q}}\left[(S_T - K)^+ \mathbf{1}_{\{S_t > H,\ t\in[0,T]\}}\right]. \tag{4.41}$$

Here, we approximate this function with a valuation function of the generalized BS model

$$\begin{aligned} GBS(S_t,K,r,\bar{\sigma}_t,T-t) &:= e^{-r(T-t)}\,\bar{u}(t,S_t,\nu_t) \\ \bar{u}(t,S_t,\nu_t) &= e^{r(T-t)}\,GBS(S_t,K,r,\bar{\sigma}_t,T-t) \end{aligned} \tag{4.42}$$

where $GBS(S_t,K,r,\bar{\sigma}_t,T)$ denotes the price of the down-and-out call option with strike K and maturity T, see Equation (4.35). *Further we have the deterministic volatility given by*

$$\bar{\sigma}_t = \sqrt{\theta + (\nu_t - \theta)\frac{1 - e^{-\kappa(T-t)}}{\kappa(T-t)}}. \tag{4.43}$$

The price of the barrier option in the Heston model is obtained by the following unbiased estimation

$$\bar{Z}_\tau = \bar{u}(0,s,\nu) + \int_0^T \left(\mathbf{1}_{\{t<\tau\}}(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t,S_t,\nu_t)\right)dt.$$

with the relevant Greeks values. Finally, we have to calculate the values with the HP estimator

$$I_{HP,N} = e^{-rT}\frac{1}{N}\left(\sum_{i=1}^{M}\bar{Z}_\tau^{(i)} + \sum_{i=M+1}^{N}\bar{Z}_T^{(i)}\right). \tag{4.44}$$

Here, M is the number of paths which hit the barrier at any time $\tau \in [0,T]$.

To determine the first exit time is one challenge for pricing barrier options in the Heston model. Nevertheless, what is even more important is the derivation of the Greeks values for the relevant barrier options. The explicit formulas of the Greeks for all types of one-sided knockout barrier options are presented in the next section.

4.3.1 Derivation of the Greeks

In this section, we present the derivation of the Greeks for the one-sided knockout options under the BS model. For this, we first present the pricing formulas and then derive the relevant Greeks. A detailed derivation can be found in Appendix B. First, we give the following equations which will be used in all the pricing formulas and relatedly the Greeks.

$$x_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad x_3 = x_1 - \sigma\sqrt{\tau} \tag{4.45}$$

$$x_2 = \frac{\ln\left(\frac{S}{H}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad x_4 = x_2 - \sigma\sqrt{\tau} \tag{4.46}$$

$$y_1 = \frac{\ln\left(\frac{H^2}{SK}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad y_3 = y_1 - \sigma\sqrt{\tau} \tag{4.47}$$

$$y_2 = \frac{\ln\left(\frac{H}{S}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad y_4 = y_2 - \sigma\sqrt{\tau} \tag{4.48}$$

$$\gamma = \frac{r - q - \frac{1}{2}\sigma^2}{\sigma^2} \quad \text{and} \quad \frac{\partial \gamma}{\partial \sigma} = \frac{-2(r-q)}{\sigma^3} \quad \text{and} \quad \frac{\partial^2 \gamma}{\partial \sigma^2} = \frac{6(r-q)}{\sigma^4} \tag{4.49}$$

Here, the equations for x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 stand for the variables plugged into the distribution functions and $\tau = T - t$ denotes time to maturity. Now, we start with *down-and-out call* option pricing formula

$$V_{call}^{do}(t, T, K, H) = \begin{cases} A - C & \text{if } H < K \\ B - D & \text{if } H > K \end{cases} \tag{4.50}$$

where

$$\begin{aligned}
A &= Se^{-q\tau}\Phi(x_1) - Ke^{-r\tau}\Phi(x_3) \\
B &= Se^{-q\tau}\Phi(x_2) - Ke^{-r\tau}\Phi(x_4) \\
C &= Se^{-q\tau}\Phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r\tau}\Phi(y_3)\left(\frac{H}{S}\right)^{2\gamma} \\
D &= Se^{-q\tau}\Phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r\tau}\Phi(y_4)\left(\frac{H}{S}\right)^{2\gamma}
\end{aligned}$$

where $\tau = T - t$ denotes the time to maturity. Since the formula has two parts conditioned on the barrier level against the strike price, we derive the Greeks also in two parts. First, we consider the case $K > H$ where the pricing formula is $A - C$. The derivatives of function A are given by

$$\begin{aligned}
\frac{\partial A}{\partial \sigma} &= Se^{-q\tau}\sqrt{\tau}\phi(x_1) \\
\frac{\partial^2 A}{\partial S \partial \sigma} &= -e^{-q\tau}\phi(x_1)\frac{x_3}{\sigma} \\
\frac{\partial^2 A}{\partial \sigma^2} &= Se^{-q\tau}\sqrt{\tau}\phi(x_1)\frac{x_1 x_3}{\sigma}
\end{aligned}$$

and the derivatives for C are equal to

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma} &= Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} + 2\ln\left(\frac{H}{S}\right)\frac{\partial \gamma}{\partial \sigma}C \\
\frac{\partial^2 C}{\partial S \partial \sigma} &= e^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left(\frac{y_3}{\sigma\sqrt{\tau}} - 2\gamma\right) + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-C}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial C}{\partial S}\right] \\
\frac{\partial C}{\partial S} &= -e^{-q\tau}\Phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} - \left(2\gamma\frac{C}{S}\right) \\
\frac{\partial^2 C}{\partial \sigma^2} &= Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left[\frac{y_1 y_3}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right] + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}C + \frac{\partial \gamma}{\partial \sigma}\frac{\partial C}{\partial \sigma}\right].
\end{aligned}
$$

Finally, we obtain the Greeks for the first part

$$
\frac{\partial V^{do}_{call}}{\partial \sigma} = \frac{\partial A}{\partial \sigma} - \frac{\partial C}{\partial \sigma} \tag{4.51}
$$

$$
\frac{\partial^2 V^{do}_{call}}{\partial S \partial \sigma} = \frac{\partial^2 A}{\partial S \partial \sigma} - \frac{\partial^2 C}{\partial S \partial \sigma} \tag{4.52}
$$

$$
\frac{\partial^2 V^{do}_{call}}{\partial \sigma^2} = \frac{\partial^2 A}{\partial \sigma^2} - \frac{\partial^2 C}{\partial \sigma^2}. \tag{4.53}
$$

Now, we deal with the derivatives of the second case $K < H$ where the pricing formula is $B - D$. The derivatives of B read as

$$
\begin{aligned}
\frac{\partial B}{\partial \sigma} &= Se^{-q\tau}\phi(x_2)\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) \\
\frac{\partial^2 B}{\partial S \partial \sigma} &= e^{-q\tau}\phi(x_2)\left[\frac{-x_4}{\sigma\sqrt{\tau}}\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) + \frac{1}{\sigma^2\sqrt{\tau}}\left(\frac{K}{H} - 1\right)\right] \\
\frac{\partial^2 B}{\partial \sigma^2} &= \frac{\partial B}{\partial \sigma}\frac{x_2 x_4}{\sigma} + Se^{-q\tau}\phi(x_2)\left[\frac{x_2 + x_4}{\sigma^2}\left(1 - \frac{K}{H}\right)\right]
\end{aligned}
$$

and the derivatives for D equal to

$$
\begin{aligned}
\frac{\partial D}{\partial \sigma} &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right) + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)D \\
\frac{\partial^2 D}{\partial S \partial \sigma} &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_4}{S\sigma\sqrt{\tau}} - \frac{2\gamma}{S}\right) - \left(\frac{K - H}{\sigma^2 S\sqrt{\tau}}\right)\right] \\
&\quad + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-D}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial D}{\partial S}\right] \\
\frac{\partial D}{\partial S} &= e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\left[(-2\gamma - 1)\Phi(y_2) - \frac{\phi(y_2)}{\sigma\sqrt{\tau}}\right] - \frac{K}{S}\left(\frac{H}{S}\right)^{2\gamma} e^{-r\tau}\left[(-2\gamma)\Phi(y_4) - \frac{\phi(y_2)}{\sigma\sqrt{\tau}}\right] \\
\frac{\partial^2 D}{\partial \sigma^2} &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_2 y_4}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right) + (H - K)\frac{y_2 + y_4}{\sigma^2}\right] \\
&\quad + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}D + \frac{\partial \gamma}{\partial \sigma}\frac{\partial D}{\partial \sigma}\right].
\end{aligned}
$$

Finally, we obtain the Greeks for $B - D$ by

$$\frac{\partial V_{call}^{do}}{\partial \sigma} = \frac{\partial B}{\partial \sigma} - \frac{\partial D}{\partial \sigma} \tag{4.54}$$

$$\frac{\partial^2 V_{call}^{do}}{\partial S \partial \sigma} = \frac{\partial^2 B}{\partial S \partial \sigma} - \frac{\partial^2 D}{\partial S \partial \sigma} \tag{4.55}$$

$$\frac{\partial^2 V_{call}^{do}}{\partial \sigma^2} = \frac{\partial^2 B}{\partial \sigma^2} - \frac{\partial^2 D}{\partial \sigma^2}. \tag{4.56}$$

Consequently, the required Greeks for a down-and-out call option under the BS model are derived.

We continue with the derivation of the Greeks of a *down-and-out put* option. The pricing formula is given as

$$V_{put}^{do}(t, T, K, H) = \begin{cases} A - B + C - D & \text{if } H < K \\ 0 & \text{if } H > K \end{cases} \tag{4.57}$$

where

$$\begin{aligned}
A &= -Se^{-q\tau}\Phi(-x_1) + Ke^{-r\tau}\Phi(-x_3) \\
B &= -Se^{-q\tau}\Phi(-x_2) + Ke^{-r\tau}\Phi(-x_4) \\
C &= -Se^{-q\tau}\Phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} + Ke^{-r\tau}\Phi(y_3)\left(\frac{H}{S}\right)^{2\gamma} \\
D &= -Se^{-q\tau}\Phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+2} + Ke^{-r\tau}\Phi(y_4)\left(\frac{H}{S}\right)^{2\gamma}.
\end{aligned}$$

Note that, if the strike price K is located below the barrier level, then we get zero value at the end, unless a rebate payment is defined. Throughout the thesis we consider barrier options without a rebate, thus we represent the pricing function with zero value. Although the derivation of the Greeks is similar with the down-and-out call option, due the negative sign involved in the distribution functions the results for the Greeks of down-and-out put are affected. We point out that in the BS model, these *vega*, *vanna* and *volga* values are the same for European call and put options. However, this is not the case for barrier options. Only the derivatives of the function A are the same for all types of barrier options, since it indeed corresponds to the classical BS pricing formula. Therefore, we do not repeat the derivatives for the function A. We present the results for functions B, C and D for a down-and-out put option under the BS model. The derivatives of the function B read as

$$\begin{aligned}
\frac{\partial B}{\partial \sigma} &= Se^{-q\tau}\phi(x_2)\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) \\
\frac{\partial^2 B}{\partial S \partial \sigma} &= e^{-q\tau}\phi(x_2)\left[\frac{-x_4}{\sigma\sqrt{\tau}}\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) + \frac{1}{\sigma^2\sqrt{\tau}}\left(\frac{K}{H} - 1\right)\right] \\
\frac{\partial^2 B}{\partial \sigma^2} &= \frac{\partial B}{\partial \sigma}\left(\frac{x_2 x_4}{\sigma}\right) + Se^{-q\tau}\phi(x_2)\left[\frac{x_2 + x_4}{\sigma^2}\left(1 - \frac{K}{H}\right)\right]
\end{aligned}$$

and also the derivatives of the function C

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= -Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} + 2\ln\left(\frac{H}{S}\right)\frac{\partial \gamma}{\partial \sigma}C \\
\frac{\partial^2 C}{\partial S \partial \sigma} &= -e^{-q\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left(\frac{y_3}{\sigma\sqrt{\tau}} - 2\gamma\right) + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-C}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial C}{\partial S}\right] \\
\frac{\partial C}{\partial S} &= e^{-q\tau}\sqrt{\tau}\Phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} - \left(2\gamma\frac{C}{S}\right) \\
\frac{\partial^2 C}{\partial \sigma^2} &= -Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left[\frac{y_1 y_3}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right] + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}C + \frac{\partial \gamma}{\partial \sigma}\frac{\partial C}{\partial \sigma}\right]
\end{aligned}$$

and finally, the derivatives for D

$$\begin{aligned}
\frac{\partial D}{\partial \sigma} &= -e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right) + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)D \\
\frac{\partial^2 D}{\partial S \partial \sigma} &= -e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_4}{S\sigma\sqrt{\tau}} - \frac{2\gamma}{S}\right) - \left(\frac{K-H}{\sigma^2 S\sqrt{\tau}}\right)\right] \\
&\quad + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-D}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial D}{\partial S}\right] \\
\frac{\partial D}{\partial S} &= -e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\left[(-2\gamma-1)\Phi(y_2) - \frac{\phi(y_2)}{\sigma\sqrt{\tau}}\right] + \frac{K}{S}\left(\frac{H}{S}\right)^{2\gamma}e^{-r\tau}\left[\frac{-\phi(y_4)}{\sigma\sqrt{\tau}} - 2\gamma\Phi(y_4)\right] \\
\frac{\partial^2 D}{\partial \sigma^2} &= -e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_2 y_4}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right) + (H-K)\frac{y_2+y_4}{\sigma^2}\right] \\
&\quad + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}D + \frac{\partial \gamma}{\partial \sigma}\frac{\partial D}{\partial \sigma}\right].
\end{aligned}$$

Eventually, the Greeks for the case $K > H$ are given by

$$\frac{\partial V_{put}^{do}}{\partial \sigma} = \frac{\partial A}{\partial \sigma} - \frac{\partial B}{\partial \sigma} + \frac{\partial C}{\partial \sigma} - \frac{\partial D}{\partial \sigma} \tag{4.58}$$

$$\frac{\partial^2 V_{put}^{do}}{\partial S \partial \sigma} = \frac{\partial^2 A}{\partial S \partial \sigma} - \frac{\partial^2 B}{\partial S \partial \sigma} + \frac{\partial^2 C}{\partial S \partial \sigma} - \frac{\partial^2 D}{\partial S \partial \sigma} \tag{4.59}$$

$$\frac{\partial^2 V_{put}^{do}}{\partial \sigma^2} = \frac{\partial^2 A}{\partial \sigma^2} - \frac{\partial^2 B}{\partial \sigma^2} + \frac{\partial^2 C}{\partial \sigma^2} - \frac{\partial^2 D}{\partial \sigma^2}. \tag{4.60}$$

Subsequently, we derive the Greeks of an *up-and-out call* option where the pricing formula is given by

$$V_{call}^{uo}(t,T,K,H) = \begin{cases} A - B + C - D & \text{if } H < K \\ 0 & \text{if } H > K \end{cases} \tag{4.61}$$

where

$$
\begin{aligned}
A &= Se^{-q\tau}\Phi(x_1) - Ke^{-r\tau}\Phi(x_3) \\
B &= Se^{-q\tau}\Phi(x_2) - Ke^{-r\tau}\Phi(x_4) \\
C &= Se^{-q\tau}\Phi(-y_1)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r\tau}\Phi(-y_3)\left(\frac{H}{S}\right)^{2\gamma} \\
D &= Se^{-q\tau}\Phi(-y_2)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r\tau}\Phi(-y_4)\left(\frac{H}{S}\right)^{2\gamma}.
\end{aligned}
$$

Since the values for A and B are the same with the down-and-out call option, there is no need to repeat them here. However, because of the negative values inside the cumulative distribution function, we have different values for C and D. The derivatives of them are given, respectively, by

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma} &= -Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} + 2\ln\left(\frac{H}{S}\right)\frac{\partial \gamma}{\partial \sigma}C \\
\frac{\partial^2 C}{\partial S \partial \sigma} &= -e^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left(\frac{y_3}{\sigma\sqrt{\tau}} - 2\gamma\right) + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-C}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial C}{\partial S}\right] \\
\frac{\partial C}{\partial S} &= -e^{-q\tau}\left(\frac{H}{S}\right)^{2(\gamma+1)}\Phi(y_1) - \left(2\gamma\frac{C}{S}\right) \\
\frac{\partial^2 C}{\partial \sigma^2} &= -Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left[\frac{y_1 y_3}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right] + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}C + \frac{\partial \gamma}{\partial \sigma}\frac{\partial C}{\partial \sigma}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial D}{\partial \sigma} &= -e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right) + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)D \\
\frac{\partial^2 D}{\partial S \partial \sigma} &= -e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_4}{S\sigma\sqrt{\tau}} - \frac{2\gamma}{S}\right) - \left(\frac{K-H}{\sigma^2 S\sqrt{\tau}}\right)\right] \\
&\quad + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-D}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial D}{\partial S}\right] \\
\frac{\partial D}{\partial S} &= e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\left[(-2\gamma-1)\Phi(-y_2) + \frac{\phi(y_2)}{\sigma\sqrt{\tau}}\right] - \frac{K}{S}\left(\frac{H}{S}\right)^{2\gamma}e^{-r\tau}\left[\frac{\phi(y_4)}{\sigma\sqrt{\tau}} - 2\gamma\Phi(-y_4)\right] \\
\frac{\partial^2 D}{\partial \sigma^2} &= -e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_2 y_4}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right) - (K-H)\frac{y_2+y_4}{\sigma^2}\right] \\
&\quad + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}D + \frac{\partial \gamma}{\partial \sigma}\frac{\partial D}{\partial \sigma}\right].
\end{aligned}
$$

Consequently, the Greeks can be written as follows

$$
\frac{\partial V_{call}^{uo}}{\partial \sigma} = \frac{\partial A}{\partial \sigma} - \frac{\partial B}{\partial \sigma} + \frac{\partial C}{\partial \sigma} - \frac{\partial D}{\partial \sigma} \tag{4.62}
$$

$$
\frac{\partial^2 V_{call}^{uo}}{\partial S \partial \sigma} = \frac{\partial^2 A}{\partial S \partial \sigma} - \frac{\partial^2 B}{\partial S \partial \sigma} + \frac{\partial^2 C}{\partial S \partial \sigma} - \frac{\partial^2 D}{\partial S \partial \sigma} \tag{4.63}
$$

$$
\frac{\partial^2 V_{call}^{uo}}{\partial \sigma^2} = \frac{\partial^2 A}{\partial \sigma^2} - \frac{\partial^2 B}{\partial \sigma^2} + \frac{\partial^2 C}{\partial \sigma^2} - \frac{\partial^2 D}{\partial \sigma^2}. \tag{4.64}
$$

Lastly, we consider an *up-and-out put* option. We again have a pricing function given in two parts, i.e.

$$V_{put}^{uo}(t,T,K,H) = \begin{cases} B - D & \text{if } H < K \\ A - C & \text{if } H > K \end{cases} \tag{4.65}$$

where

$$\begin{aligned}
A &= -Se^{-q\tau}\Phi(-x_1) + Ke^{-r\tau}\Phi(-x_3) \\
B &= -Se^{-q\tau}\Phi(-x_2) + Ke^{-r\tau}\Phi(-x_4) \\
C &= -Se^{-q\tau}\Phi(-y_1)\left(\frac{H}{S}\right)^{2\gamma+2} + Ke^{-r\tau}\Phi(-y_3)\left(\frac{H}{S}\right)^{2\gamma} \\
D &= -Se^{-q\tau}\Phi(-y_2)\left(\frac{H}{S}\right)^{2\gamma+2} + Ke^{-r\tau}\Phi(-y_4)\left(\frac{H}{S}\right)^{2\gamma}.
\end{aligned}$$

Now, for the case $K > H$ we have the following derivatives with the pricing formula $B-D$. Since the derivatives of the function B is the same as down-and-out put, we do not repeat them here. Therefore, the derivatives of the function D reads as

$$\begin{aligned}
\frac{\partial D}{\partial \sigma} &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right) + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)D \\
\frac{\partial^2 D}{\partial S\partial \sigma} &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_4}{S\sigma\sqrt{\tau}} - \frac{2\gamma}{S}\right) - \left(\frac{K-H}{\sigma^2 S\sqrt{\tau}}\right)\right] \\
&\quad + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-D}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial D}{\partial S}\right] \\
\frac{\partial D}{\partial S} &= -e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\left[(-2\gamma-1)\Phi(-y_2) + \frac{\phi(y_2)}{\sigma\sqrt{\tau}}\right] + \frac{K}{S}e^{-r\tau}\left(\frac{H}{S}\right)^{2\gamma}\left[\frac{\phi(y_4)}{\sigma\sqrt{\tau}} - 2\gamma\Phi(-y_4)\right] \\
\frac{\partial^2 D}{\partial \sigma^2} &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_2 y_4}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right) - \frac{y_2+y_4}{\sigma^2}(K-H)\right] \\
&\quad + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}D + \frac{\partial \gamma}{\partial \sigma}\frac{\partial D}{\partial \sigma}\right].
\end{aligned}$$

Moreover, the Greeks for the first part of the pricing function are given in the following

$$\frac{\partial V_{put}^{uo}}{\partial \sigma} = \frac{\partial B}{\partial \sigma} - \frac{\partial D}{\partial \sigma} \tag{4.66}$$

$$\frac{\partial^2 V_{put}^{uo}}{\partial S\partial \sigma} = \frac{\partial^2 B}{\partial S\partial \sigma} - \frac{\partial^2 D}{\partial S\partial \sigma} \tag{4.67}$$

$$\frac{\partial^2 V_{put}^{uo}}{\partial \sigma^2} = \frac{\partial^2 B}{\partial \sigma^2} - \frac{\partial^2 D}{\partial \sigma^2}. \tag{4.68}$$

Now, the remaining task is to derive the Greeks for the second case $K < H$ where the pricing formula is equal to $A - C$. The only required derivatives are due to function C

which are given by

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} + 2\ln\left(\frac{H}{S}\right)\frac{\partial \gamma}{\partial \sigma}C \\
\frac{\partial^2 C}{\partial S \partial \sigma} &= e^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left(\frac{y_3}{\sigma\sqrt{\tau}} - 2\gamma\right) + 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-C}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial C}{\partial S}\right] \\
\frac{\partial C}{\partial S} &= e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\Phi(-y_1) - \left(2\gamma\frac{C}{S}\right) \\
\frac{\partial^2 C}{\partial \sigma^2} &= Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left[\frac{y_1 y_3}{\sigma} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\right] + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}C + \frac{\partial \gamma}{\partial \sigma}\frac{\partial C}{\partial \sigma}\right].
\end{aligned}$$

And finally, we obtain the Greeks with the following equations

$$\frac{\partial V_{put}^{uo}}{\partial \sigma} = \frac{\partial A}{\partial \sigma} - \frac{\partial C}{\partial \sigma} \tag{4.69}$$

$$\frac{\partial^2 V_{put}^{uo}}{\partial S \partial \sigma} = \frac{\partial^2 A}{\partial S \partial \sigma} - \frac{\partial^2 C}{\partial S \partial \sigma} \tag{4.70}$$

$$\frac{\partial^2 V_{put}^{uo}}{\partial \sigma^2} = \frac{\partial^2 A}{\partial \sigma^2} - \frac{\partial^2 C}{\partial \sigma^2}. \tag{4.71}$$

As a result, these Greeks allow us to obtain the Taylor expansion around the BS price and ultimately an approximation to the Heston price. In the following sections we present several numerical results regarding the implementation of the HP algorithm for pricing one-sided knockout options in the Heston model.

4.3.2 Numerical Results for Down-and-Out Options

In this section we present the numerical results regarding the application of the HP estimator for pricing down-and-out barrier options in the Heston model. For the numerical tests we utilize the parameter sets given in Table 4.2.

Set 1:	$T = 0.5$, $S_0 = 100$, $r = 0.04$, $\kappa = 0.6$, $v_0 = 0.04$, $\theta = 0.04$, $\sigma = 0.2$, $\rho = -0.8$
Set 2:	$T = 1$, $S_0 = 100$, $r = 0.0319$, $\kappa = 6.21$, $v_0 = 0.010201$, $\theta = 0.019$, $\sigma = 0.61$, $\rho = -0.7$
Set 3:	$T = 5$, $S_0 = 100$, $r = 0.05$, $\kappa = 2$, $v_0 = 0.09$, $\theta = 0.09$, $\sigma = 1$, $\rho = -0.7$

Table 4.2: Parameters sets for the numerical tests

The first parameter set is taken from [42] and the latter parameter sets 2 and 3 are taken from [10]. The diversity of the parameter sets is dependent mostly on the variety of maturity of the options and the volatility of variance parameter σ. Especially, the parameter set 3 is a long running option with a high level of volatility of variance.

Pricing Down-and-Out Call Option

We present the numerical results for discretely monitored down-and-out call option in the Heston model. In order to perform the numerical tests systematically, we start with the extreme cases. For example, if we set the volatility of variance very small (e.g. $\sigma = 0.0001$) and set the speed of mean reversion very large (e.g. $\kappa = 20$), then we expect that the Heston price will be almost the same with the BS price. In the following we see the results for the parameter set 1, for barrier level $H = 90$, for varying number of paths and the step size $\delta = 0.004$.

	N		
Method	1000	10000	100000
HP estimator	14.9158	14.9158	14.9158
Lower 95% bound	14.9158	14.9158	14.9158
Upper 95% bound	14.9158	14.9158	14.9158
Crude MC	15.1731	15.0084	14.6189
Lower 95% bound	12.0657	13.9838	14.2995
Upper 95% bound	18.2805	16.0331	14.9382

Table 4.3: Down-and-Out call option, Exact discrete BS price 14.9158

It can be seen from Table 4.3 that the HP estimator provides a significantly better estimation of the exact value in a very fast and accurate way even for very few number of paths. However, the crude MC can only approximate to the exact value as the number of paths gets larger. In fact, the result for the HP estimator is not surprising while the main principle of the estimator is based on the idea to obtain the price of an option in the Heston model via a Taylor expansion in the neighborhood of the BS price. For the Taylor expansion, we only utilize the partial derivatives with respect to the diffusion part of the variance process, i.e. $\sigma\sqrt{\nu_t}dW_t^2$. Therefore, if we set σ really small, then basically the partial derivatives have no big effect as a correction to the BS price. In another words, we calculate the Heston price by taking the BS price as a base price and then we adjust it by the help of the Greeks of the relevant option. This feature of the HP estimator is advantageous for pricing barrier options in the Heston model. More precisely, if we take the discretely monitored BS price as an initial value for the HP estimator, then the HP result gives us the discretely monitored Heston price. The same applies for the continuously monitored case, i.e. if we take the continuously monitored BS price as an initial value for the HP estimator, then the HP result is the continuously monitored Heston price. This can not be achieved by the crude MC estimation, since it only gives us the discretely monitored price. Hence, we are only able to compare the crude MC and the HP estimator for the discretely monitored barrier options. We begin our numerical test with the comparison among the crude MC estimation, the crude control variate technique and the HP estimator. For this test $N = 10000$ paths and the steps size $\delta = 0.004$ are used.

Method	Set 1	Set 2	Set 3
HP Estimator	6.0615	6.3446	15.0579
Lower 95% bound	6.0542	6.3315	15.0445
Upper 95% bound	6.0688	6.3576	15.0712
Crude CV	6.0840	6.3601	16.5246
Lower 95% bound	6.0232	6.2394	15.6967
Upper 95% bound	6.1448	6.4808	17.3525
Crude MC	6.1719	6.3365	15.6938
Lower 95% bound	6.0061	6.1871	14.7240
Upper 95% bound	6.3377	6.4858	16.6636

Table 4.4: Comparison of the HP estimator, the crude control variate technique and the crude MC estimation, Barrier level $H = 90$

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	6.5937	(6.5861, 6.6014)	6.5827	(6.4206, 6.7448)
$H = 1$ *	Set 2	6.8068	(6.7926, 6.8210)	6.7066	(6.5571, 6.8561)
	Set 3	34.8340	(34.7514, 34.9165)	34.5534	(33.4315, 35.6752)
	Set 1	6.0648	(6.0576, 6.0720)	6.0950	(5.9321, 6.2579)
$H = 90$	Set 2	6.3414	(6.3284, 6.3543)	6.3588	(6.2066, 6.5110)
	Set 3	15.0678	(15.0554, 15.0802)	15.9896	(15.0256, 16.9536)
	Set 1	5.9052	(5.8984, 5.9120)	5.9910	(5.8276, 6.1543)
$H = 91$	Set 2	6.1898	(6.1776, 6.2020)	6.2786	(6.1266, 6.4305)
	Set 3	13.9385	(13.9269, 13.9501)	14.6127	(13.6907, 15.5346)
	Set 1	5.6910	(5.6849, 5.6971)	5.7121	(5.5502, 5.8741)
$H = 92$	Set 2	5.9762	(5.9649, 5.9874)	5.9886	(5.8371, 6.1400)
	Set 3	12.7866	(12.7768, 12.7963)	13.4373	(12.5367, 14.3378)
	Set 1	5.4248	(5.4195, 5.4302)	5.5005	(5.3377, 5.6633)
$H = 93$	Set 2	5.7276	(5.7176, 5.7375)	5.8260	(5.6728, 5.9791)
	Set 3	11.5759	(11.5671, 11.5847)	12.3687	(11.4917, 13.2456)
	Set 1	5.0880	(5.0834, 5.0926)	5.0627	(4.9042, 5.2212)
$H = 94$	Set 2	5.4004	(5.3917, 5.4090)	5.5270	(5.3740, 5.6801)
	Set 3	10.3166	(10.3082, 10.3251)	11.0393	(10.2066, 11.8719)
	Set 1	4.6761	(4.6724, 4.6799)	4.7307	(4.5714, 4.8899)
$H = 95$	Set 2	4.9746	(4.9675, 4.9816)	4.9609	(4.8132, 5.1086)
	Set 3	9.0195	(9.0108, 9.0281)	9.6808	(8.8902, 10.4715)
	Set 1	2.8158	(2.8147, 2.8168)	2.7626	(2.6285, 2.8966)
$H = 98$	Set 2	2.9453	(2.9434, 2.9473)	3.0163	(2.8859, 3.1466)
	Set 3	4.8298	(4.8213, 4.8382)	4.9919	(4.4121, 5.5716)

* In fact, the case $H = 1$ delivers the results of corresponding European vanilla options, where the exact values are 6.5944, 6.8061 and 34.8348 for Set 1, Set 2 and Set 3, respectively.

Table 4.5: Down-and-Out call price - At the money option

Table 4.4 shows that although a variance reduction compared to the crude MC estimation is provided by the crude CV technique, the HP outperforms among these estimations. Therefore, we aim to analyze the performance of the HP estimator in detail. In the following we present the results of discretely monitored down-and-out call option prices for all parameter sets given in Table 4.2 and for $N = 10000$ number of paths and for $\delta = 0.004$ step size which roughly corresponds to 250 trading days per year for a one year maturity option. Moreover, we do the barrier check on these discrete time points, which also corresponds to doing the barrier checking at each daily closing. We first consider at-the-money options, i.e. $S_0 = K$ where K is strike price.

From Table 4.5, we see that the HP estimator provides a distinctly visible variance reduction compared to the crude MC estimation. Moreover, during the numerical tests we have noticed that the performance of the HP estimator is quite robust. In particular, for the parameter Set 3, since the volatility of variance is high ($\sigma = 1$) the crude MC result with $N = 10000$ paths fluctuates with a high frequency, see Figure 4.3.

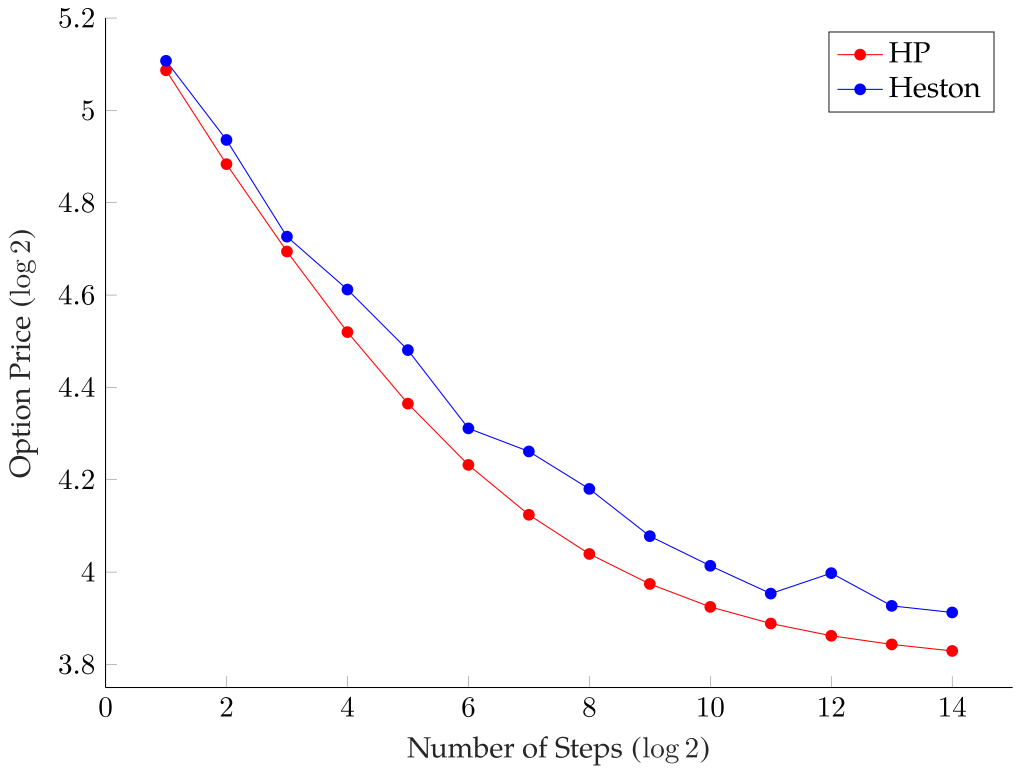

Figure 4.3: Down-and-Out call price - the HP estimator and the crude MC estimation in the Heston model for parameter set 3

Figure 4.3 illustrates that the option price is calculated for varying step sizes. As previously explained, increasing the number of steps will provide us with a better approximation of the real process. On the other hand, by increasing the number of steps one will find more violations of the barrier condition. Thus, the option prices have to be a decreasing function of the number of steps, i.e. barrier checks. Therefore, we expect that the option price will decrease as we increase the number of steps. However, the HP estimator shows a very smooth decay behavior unlike the crude MC estimation. We know that in order to have robust results in the crude MC estimation one has to have a large number of paths to keep the MC variance low. Otherwise, the resulting error in the MC simulation will be dominated by the MC variance. In fact, the interplay between the bias and the MC variance in the MSE has a substantial effect on the accuracy of the MC results. Thus, if one aims to have a control on the MC variance in the crude MC estimation, then one has to either increase the number of paths as much as possible or employ a variance reduction technique. This control on the MC variance leads to the situation that the statistical error in the MSE is in a sense eliminated and the only source of error is due to the bias. The detailed examination of this error behavior is given in the error analysis in Section 6.2. As a result, this typical behavior of the error in the MC simulations can conveniently explain the reason of the fluctuations in the crude MC estimation displayed in Figure 4.3. Our next aim is to analyze the behavior of the HP estimator for down-and-out call option regarding the all parameter sets. Figure 4.4 demonstrates the behavior of the HP estimator for the parameter Set 1. For practical purposes, only 50 simulated outcomes are presented.

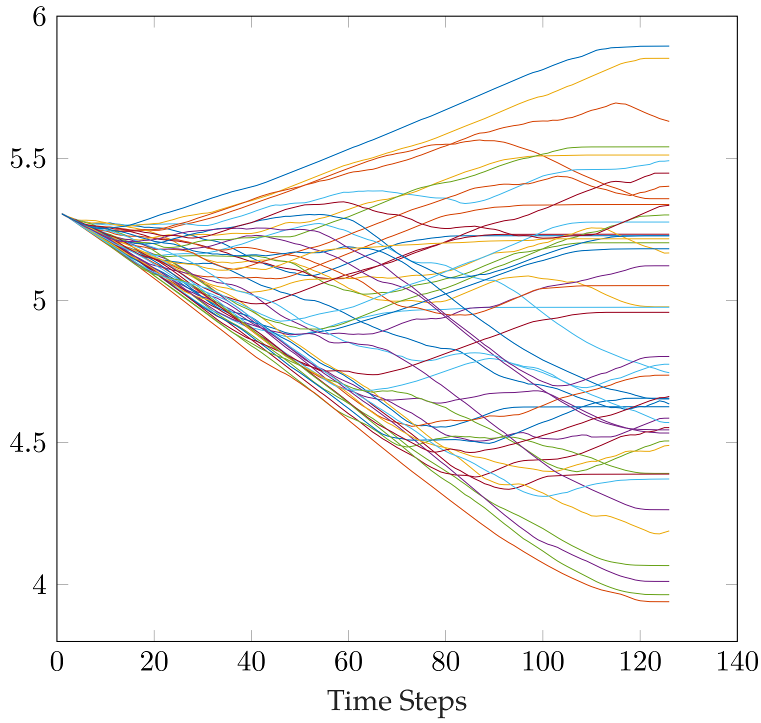

Figure 4.4: Simulated outcomes of the HP estimator for parameter set 1

Figure 4.4 demonstrates that for parameter set 1 the paths of the HP estimator are quite smooth. However, we note that these are the simulated outcomes without any barrier check, namely the barrier level is always involved in the computation of the Greeks until maturity regardless of being hit or not. Therefore, during the simulation it is also involved in the behavior of the HP estimator via the Greeks of the relevant option. Also note that, if the stock price hits the barrier or even gets closer to the barrier level, the behavior of the Greeks becomes sensitive. Hence, this sensitivity also effects the behavior of the HP estimator. To check the effect of the Greeks on the HP estimator we also give the simulated outcomes of the latter parameter sets. Figure 4.5 illustrates the behavior of the HP estimator regarding the parameter set 2.

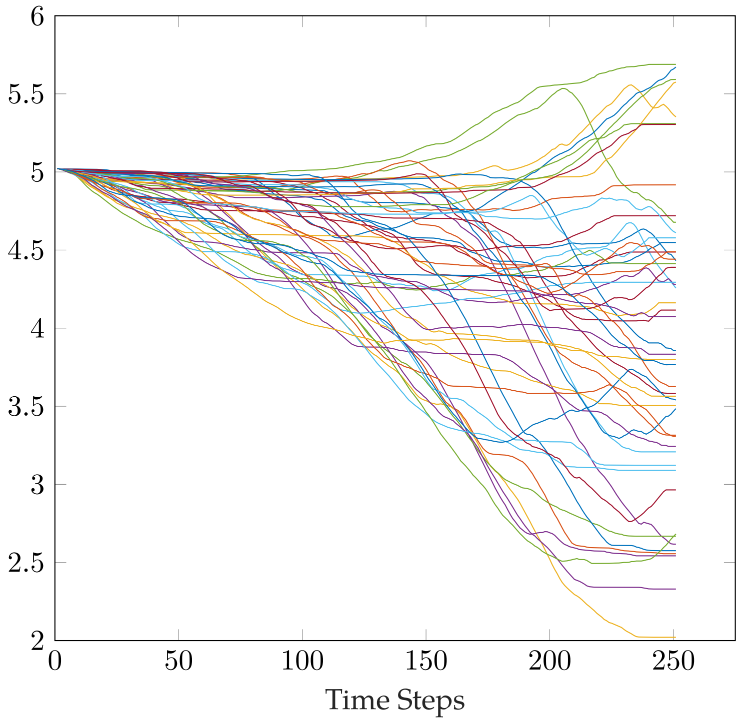

Figure 4.5: Simulated outcomes of the HP estimator for parameter set 2

In Figure 4.5 the effect of the Greeks is slightly visible for the case of parameter set 2. The smoothness of the paths of the HP estimator is decreased. Subsequently, we obtain Figure 4.6 by using the parameter set 3, which contains the most challenging parameters.

As expected, the change in parameters effects the behavior of the HP estimator. For the parameter set 3 long-run maturity and relatedly increased number of steps and also the high value of volatility of variance $\sigma = 1$ have a visible effect on the behavior of the HP estimator. For instance, there are some unexpected small jumps in Figure 4.6. However, we point out that the paths with jumps may not be valid for the final consideration of the expected values. As we mentioned before, for the paths which hit the barrier at any time $\tau < T$ we consider the value of the HP estimator specifically at this first hitting time. As defined in Equation (4.40) we only consider the $\bar{Z}$ values either at the first hitting time τ or at the final time T. As an overall evaluation, one can deduce that for all parameter sets the behavior of the HP estimator is quite smooth and also the variance of the paths of the HP estimator is remarkable small.

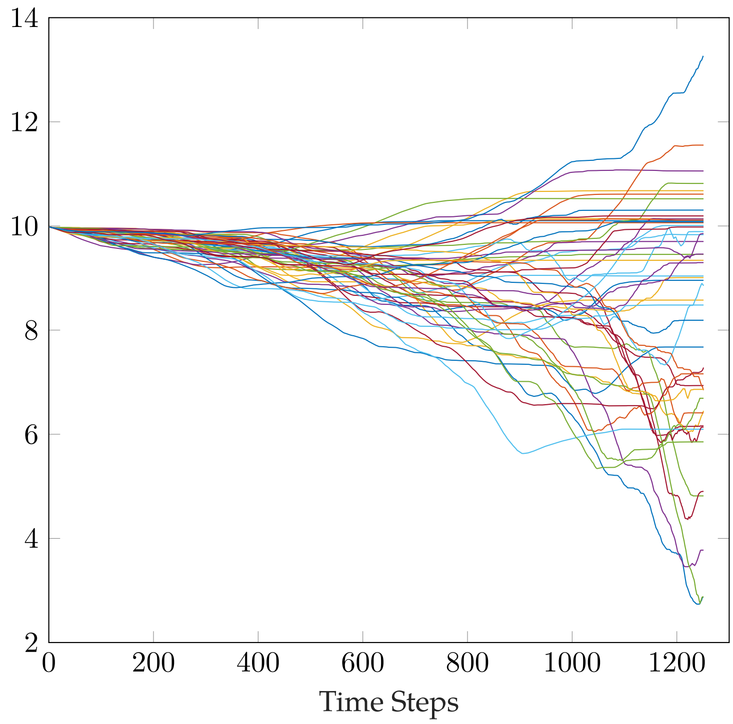

Figure 4.6: Simulated outcomes of the HP estimator for parameter set 3

		Heath-Platen		Crude MC	
		CI Length	N	CI Length	N
	Set 1	0.0198	5500	0.0195	2800000
$H = 90$	Set 2	0.0199	17000	0.0195	700000
	Set 3	0.0198	15200	−−	−−
	Set 1	0.0194	1500	0.0198	2550000
$H = 95$	Set 2	0.0195	5000	0.0195	650000
	Set 3	0.0195	7000	−−	−−

Table 4.6: Down-and-Out call option - Required number of paths

In Table 4.6 we present the comparison results between the crude MC estimation and the HP estimation with respect to the required number of paths to achieve the targeted length of the confidence interval. We know that the confidence interval is dependent on the variance of the MC estimation. Thus, it is an indicator for the amount of variance reduction. It is necessary to point out that we discretize the Heston model only once by using a fully truncated Euler scheme, see the HP algorithm for European options in Appendix A. Afterwards, we use these discrete values for both the crude MC estimation and the HP estimator. Thus, the bias regarding the discretization of the Heston model is the same for both the HP and the crude estimations. With this, we are able to compare the two estimations by assuming the target level of confidence interval length as 0.02. In the comparison, the step size for all parameter sets are set to $\delta = 0.004$.

The results given in Table 4.6 show that the required number of paths to achieve the targeted confidence interval length is dramatically lower for the HP estimator than for the crude MC. In particular, for parameter set 3 we could not manage to determine the required number of paths for the crude MC estimation due to certain memory problems. This indeed shows that how effectively the HP estimator can be utilized by using significantly less number of paths. This feature of the HP estimator makes it particularly suitable for MC implementations for path-dependent options. Especially, in the Heston model for pricing the path-dependent options via a MC simulation, the time and energy consumption of the algorithm might reach a level which is unattainable by the standard devices. On the other hand, the HP estimator performs superior, so that one can accelerate the procedure by reducing not only the number of paths but also the MC variance dramatically. In fact, these advantages initiated the further examination of the HP estimator for path-dependent options and also some other extensions of it.
In the following we present the results for in the money and out of the money down-and-out call options, respectively. For the case of in the money options (i.e. $S_0 > K$) we consider the initial stock price $S_0 = 100$ and strike $K = 90$. Note that in the BS model down-and-out barrier options have a piecewise pricing formula conditioned on the location of the barrier with respect to the strike price, see Equation (4.35). In our numerical test, if we set $K = 90$, then we should use the corresponding pricing formula for $K < H$. Therefore it is necessary to examine the case of in the money options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	10.7489	(10.7466, 10.7512)	10.6663	(10.4251, 10.9074)
$H = 92$	Set 2	11.6216	(11.6166, 11.6266)	11.6056	(11.3748, 11.8363)
	Set 3	14.3136	(14.2907, 14.3365)	14.9295	(13.9567, 15.9024)
	Set 1	8.3899	(8.3847, 8.3951)	8.2736	(8.0342, 8.5130)
$H = 95$	Set 2	9.1612	(9.1513, 9.1712)	9.3040	(9.0681, 9.5398)
	Set 3	10.0578	(10.0337, 10.0818)	10.6945	(9.8472, 11.5419)
	Set 1	4.7526	(4.7454, 4.7598)	4.7229	(4.5169, 4.9289)
$H = 98$	Set 2	5.1041	(5.0911, 5.1170)	5.2223	(5.0168, 5.4278)
	Set 3	5.3773	(5.3549, 5.3997)	5.6560	(5.0156, 6.2965)

Table 4.7: Down-and-Out call price - In the money option

Again, the results given in Table 4.7 show that the HP estimator performs well for in the money options. We also consider an out of the money call option (i.e $S_0 < K$) in the Heston model for the three parameter sets with initial stock price $S_0 = 100$ and strike $K = 110$.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
$H = 92$	Set 1	2.0608	(2.0538, 2.0679)	2.0543	(1.9636, 2.1450)
	Set 2	1.8725	(1.8605, 1.8845)	1.8944	(1.8127, 1.9761)
	Set 3	11.5992	(11.5782, 11.6203)	11.9519	(11.1162, 12.7877)
$H = 95$	Set 1	1.7750	(1.7678, 1.7823)	1.8048	(1.7169, 1.8926)
	Set 2	1.6456	(1.6334, 1.6577)	1.6604	(1.5835, 1.7373)
	Set 3	8.1965	(8.1818, 8.2112)	8.6967	(7.9641, 9.4292)
$H = 98$	Set 1	1.1523	(1.1459, 1.1587)	1.1774	(1.1042, 1.2505)
	Set 2	1.0644	(1.0539, 1.0750)	1.0937	(1.0289, 1.1586)
	Set 3	4.4081	(4.4000, 4.4161)	4.6685	(4.1191, 5.2179)

Table 4.8: Down-and-Out call price - Out of the money option

Table 4.8 shows that the HP estimator provides an easily noticeable variance reduction for the out of the money options as well.

Pricing Down-and-Out Put Option

We now present the numerical results for discretely monitored down-and-out put options in the Heston model. We again use the parameter sets given in Table 4.2. In order to avoid the option prices which are very close to zero, we set the interest rate $r = 0$ for all parameter sets. Throughout the whole numerical analyses we again set the number of paths $N = 10000$ and the step size $\delta = 0.004$. We begin our analyses with the comparison among the HP estimator, the crude MC estimation and the crude CV technique.

Method	Set 1	Set 2	Set 3
HP Estimator	0.4226	0.3631	0.0054
Lower 95% bound	0.4167	0.3494	0.0037
Upper 95% bound	0.4285	0.3768	0.0071
Crude CV	0.4155	0.3688	0.0063
Lower 95% bound	0.3897	0.3328	0.0015
Upper 95% bound	0.4412	0.4047	0.0111
Crude MC	0.4202	0.3561	0.0049
Lower 95% bound	0.3915	0.3308	0.0016
Upper 95% bound	0.4488	0.3814	0.0081

Table 4.9: Comparison of the HP estimator, the crude control variate technique and the crude MC estimation, Barrier level $H = 90$

Table 4.9 shows the comparison results of the HP estimator versus the crude CV technique and the crude MC estimations. In all the cases, the HP estimator provides with a significant amount of variance reduction. An interesting observation is that the crude CV technique did not reduce the variance for the parameter sets 2 and 3. Here, we point out that an improvement for the crude CV technique via finding an optimal multiplier for the GBS formula may be necessary. For details, see Section 2.2.1. We continue our numerical tests by considering at the money down-and-out put options, i.e. $S_0 = K$.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	1.2735	(1.2614, 1.2856)	1.2492	(1.1920, 1.3064)
$H = 85$	Set 2	1.0764	(1.0500, 1.1028)	1.0863	(1.0352, 1.1374)
	Set 3	0.0244	(0.0192, 0.0295)	0.0278	(0.0184, 0.0372)
	Set 1	0.4215	(0.4156, 0.4273)	0.4048	(0.3772, 0.4323)
$H = 90$	Set 2	0.3692	(0.3552, 0.3830)	0.3636	(0.3381, 0.3890)
	Set 3	0.0050	(0.0036, 0.0064)	0.0069	(0.0031, 0.0105)
	Set 1	0.0464	(0.0453, 0.0475)	0.0456	(0.0386, 0.0526)
$H = 95$	Set 2	0.0389	(0.0360, 0.0417)	0.0395	(0.0331, 0.0458)
	Set 3*	0.0022	(0.0012, 0.0032)	0.0039	(0.0011, 0.0066)

* Due to maturity $T = 5$ of the parameter set 3, the result was almost 0. Therefore, only for this case we set the barrier level $H = 92$.

Table 4.10: Down-and-Out put price - At the money option

Table 4.10 shows that the performance of the HP estimator is remarkable. By setting the strike $K = 105$, we analyze the case for in the money options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	2.7336	(2.7140, 2.7533)	2.7479	(2.6565, 2.8392)
$H = 85$	Set 2	2.5688	(2.5301, 2.6076)	2.5389	(2.4552, 2.6225)
	Set 3	0.0504	(0.0401, 0.0606)	0.0517	(0.0372, 0.0661)
	Set 1	1.2405	(1.2273, 1.2537)	1.2043	(1.1477, 1.2608)
$H = 90$	Set 2	1.2532	(1.2252, 1.2812)	1.2440	(1.1893, 1.2987)
	Set 3	0.0158	(0.0113, 0.0202)	0.0180	(0.0105, 0.0255)
	Set 1	0.2912	(0.2858, 0.2966)	0.2924	(0.2685, 0.3162)
$H = 95$	Set 2	0.3025	(0.2903, 0.3145)	0.3033	(0.2798, 0.3267)
	Set 3	0.0020	(0.0010, 0.0029)	0.0023	(0.0001, 0.0044)

Table 4.11: Down-and-Out put price - In the money option

The results given in Table 4.11 indicate that the HP estimator provides us with a significant variance reduction for in the money down-and-out put options as well. Finally, we consider out of the money down-and-out put options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	1.0689	(1.0586, 1.0792)	1.0470	(0.9944, 1.0996)
$H = 80$	Set 2	0.8118	(0.7885, 0.8349)	0.7975	(0.7528, 0.8422)
	Set 3	0.0346	(0.0284, 0.0407)	0.0329	(0.0228, 0.0429)
	Set 1	0.4252	(0.4198, 0.4305)	0.4118	(0.3841, 0.4394)
$H = 85$	Set 2	0.3186	(0.3058, 0.3313)	0.3193	(0.2955, 0.3430)
	Set 3	0.0092	(0.0074, 0.0109)	0.0085	(0.0042, 0.0127)
	Set 1	0.0683	(0.0671, 0.0694)	0.0689	(0.0602, 0.0775)
$H = 90$	Set 2	0.0479	(0.0449, 0.0509)	0.0467	(0.0399, 0.0534)
	Set 3*	0.0026	(0.0018, 0.0033)	0.0019	(0.0001, 0.0037)

* Due to parameters of Set 3 in this case the result was almost 0. Therefore, only for this case we set the strike $K = 98$.

Table 4.12: Down-and-Out put price - Out of the money option

To evaluate the performance of the HP estimator for out of the money options, we look at Table 4.12. As the results imply, the HP estimator provides a noticeable variance reduction.

4.3.3 Numerical Results for Up-and-Out Options

In this section, the numerical results of the application of the HP estimator for pricing up-and-out barrier options are presented. Before going into the detail, we point out that the payoff of the up-and-out options has slightly different characteristics. For instance, the payoff of an up-and-out call option reaches its highest value shortly before the stock price hits the barrier and once the stock hits the barrier the payoff drops immediately to zero. Therefore, the behavior of the payoff function is indeed fairly irregular. To be able to partially eliminate this irregularity we set the drift term of the stock price to 0. For the numerical analyses, we again use the same parameter sets given in Table 4.2.

Pricing Up-and-Out Call Option

We present the numerical results of up-and-out call option in the Heston model. We again start our discussion with the comparison of the HP estimator with the crude MC estimation and the crude CV technique.

Method	Set 1	Set 2	Set 3
HP Estimator	0.5466	0.9895	0.0311
Lower 95% bound	0.5400	0.9703	0.0264
Upper 95% bound	0.5532	1.0087	0.0357
Crude CV	0.5583	0.9930	0.0307
Lower 95% bound	0.5342	0.9478	0.0212
Upper 95% bound	0.5823	1.0382	0.0401
Crude MC	0.5579	0.9837	0.0318
Lower 95% bound	0.5252	0.9406	0.0234
Upper 95% bound	0.5905	1.0267	0.0401

Table 4.13: Comparison of the HP estimator, the crude control variate technique and the crude MC estimation, Barrier level $H = 110$

Again, we see from Table 4.13 the HP estimator performs remarkably good for pricing up-and-out call options. We also notice that the crude CV technique does not provide a significant variance reduction. To improve the performance of the crude CV technique searching for an optimal multiplier may be a helpful choice, see Section 2.2.1. In the following we consider at the money up-and-out call option with varying barrier levels.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	0.0538	(0.0526, 0.0549)	0.0594	(0.0513, 0.0673)
$H = 105$	Set 2	0.1166	(0.1112, 0.1219)	0.1190	(0.1076, 0.1302)
	Set 3	0.0025	(0.0016, 0.0034)	0.0024	(0.0006, 0.0040)
	Set 1	0.5507	(0.5439, 0.5573)	0.5725	(0.5391, 0.6059)
$H = 110$	Set 2	0.9992	(0.9799, 1.0184)	1.0116	(0.9678, 1.0554)
	Set 3	0.0260	(0.0212, 0.0307)	0.0281	(0.0203, 0.0358)
	Set 1	1.8013	(1.7870, 1.8157)	1.8267	(1.7583, 1.8950)
$H = 115$	Set 2	2.6540	(2.6218, 2.6862)	2.6443	(2.5655, 2.7230)
	Set 3	0.0965	(0.0853, 0.1076)	0.1009	(0.0826, 0.1191)

Table 4.14: Up-and-Out call price - At the money option

As inferred from Table 4.14, the variance reduction provided by the HP estimator is significant. We continue our numerical tests by setting the strike price $K = 90$ to consider in the money up-and-out call options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	0.8835	(0.8733, 0.8935)	0.9020	(0.8505, 0.9533)
$H = 105$	Set 2	1.5408	(1.5124, 1.5691)	1.5474	(1.4794, 1.6154)
	Set 3	0.0442	(0.0369, 0.0514)	0.0431	(0.0309, 0.0552)
	Set 1	3.0410	(3.0202, 3.0617)	3.1030	(3.0023, 3.2038)
$H = 110$	Set 2	4.6278	(4.5835, 4.6721)	4.6056	(4.4857, 4.7254)
	Set 3	0.1473	(0.1320, 0.1624)	0.1529	(0.1277, 0.1781)
	Set 1	6.0817	(6.0548, 6.1085)	6.0036	(5.8598, 6.1474)
$H = 115$	Set 2	8.1832	(8.1333, 8.2330)	8.2008	(8.0480, 8.3536)
	Set 3	0.3321	(0.3079, 0.3561)	0.3400	(0.2978, 0.3821)

Table 4.15: Up-and-Out call price - In the money option

Table 4.15 indicates that the performance of the HP estimator is again noteworthy. As a final consideration, we set the strike price $K = 105$. Thus, we consider out of the money up-and-out call options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	0.0909	(0.0891, 0.0926)	0.0917	(0.0820, 0.2118)
$H = 110$	Set 2	0.1932	(0.1865, 0.1999)	0.1976	(0.1832, 1.6154)
	Set 3	0.0050	(0.0036, 0.0063)	0.0051	(0.0026, 0.0075)
	Set 1	0.6484	(0.6407, 0.6560)	0.6730	(0.6369, 0.7089)
$H = 115$	Set 2	0.9875	(0.9685, 1.0063)	0.9873	(0.9450, 1.0295)
	Set 3	0.0327	(0.0277, 0.0376)	0.0362	(0.0274, 0.0448)
	Set 1	1.6354	(1.6209, 1.6499)	1.6212	(1.5570, 1.6854)
$H = 120$	Set 2	1.8798	(1.8514, 1.9082)	1.8773	(1.8101, 1.9445)
	Set 3	0.1179	(0.1061, 0.1297)	0.1184	(0.0992, 0.1375)

Table 4.16: Up-and-Out call price - Out of the money option

If we look at Table 4.16, then we see that the HP estimator provides a considerable amount of variance reduction for out of the money options as well.

Pricing Up-and-Out Put Option

As a final consideration, we present the results of the numerical tests with regard to the application of the HP estimator for pricing up-an-out put options. For numerical tests, we use the parameter sets given in Table 4.2. We begin with the comparison of the three approaches, i.e. the HP estimator, the crude MC estimation and the crude CV technique.

Method	Set 1	Set 2	Set 3
HP Estimator	5.2187	4.6842	9.1222
Lower 95% bound	5.2121	4.6728	9.1104
Upper 95% bound	5.2253	4.6956	9.1341
Crude CV	5.2254	4.6824	9.1044
Lower 95% bound	5.1282	4.5796	8.7292
Upper 95% bound	5.2059	4.7852	9.4796
Crude MC	5.2163	4.7939	9.0745
Lower 95% bound	5.0500	4.6283	8.6398
Upper 95% bound	5.3825	4.9596	9.5092

Table 4.17: Comparison of the HP estimator, the crude control variate technique and the crude MC estimation, Barrier level $H = 110$

As seen from Table 4.17, both the HP estimator and the crude CV technique provide variance reduction compared to the crude MC estimation. Note that, for this type of options, the crude CV technique also provides a good variance reduction compared to the crude MC estimation. However, the amount of the variance reduction provided by the HP estimator is substantially bigger than the crude CV technique. Next, we consider at the money up-and-out put options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	Set 1	3.9986	$(3.9953, 4.0019)$	4.0111	$(3.8526, 4.1696)$
$H = 105$	Set 2	3.6623	$(3.6559, 3.6688)$	3.6995	$(3.5467, 3.8522)$
	Set 3	5.5059	$(5.5011, 5.5108)$	5.4447	$(5.0940, 5.7954)$
	Set 1	5.2132	$(5.2065, 5.2199)$	5.2328	$(5.0653, 5.4003)$
$H = 110$	Set 2	4.6941	$(4.6827, 4.7056)$	4.7215	$(4.5576, 4.8853)$
	Set 3	9.1237	$(9.1121, 9.1353)$	9.2258	$(8.7894, 9.6621)$
	Set 1	5.4773	$(5.4696, 5.4850)$	5.5293	$(5.3604, 5.6981)$
$H = 115$	Set 2	4.8926	$(4.8795, 4.9057)$	4.7964	$(4.6348, 4.9579)$
	Set 3	12.0712	$(12.0522, 12.0903)$	11.9522	$(11.4693, 12.4351)$

Table 4.18: Up-and-Out put price - At the money option

From Table 4.18 we see that the performance of the HP estimator is again quite remarkable. Especially, the option prices are big enough to better reflect the effectiveness of the HP estimator. We continue our numerical tests by setting $K = 105$ to obtain the results for in the money up-and-out put options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
$H = 110$	Set 1	7.6035	(7.5991, 7.6079)	7.7375	(7.5339, 7.9412)
	Set 2	7.1489	(7.1412, 7.1566)	7.1923	(6.9974, 7.3872)
	Set 3	10.0587	(10.0525, 10.0650)	9.9818	(9.5133, 10.4503)
$H = 115$	Set 1	8.1671	(8.1602, 8.1740)	8.0788	(7.8795, 8.2781)
	Set 2	7.5396	(7.5289, 7.5502)	7.5275	(7.3308, 7.7242)
	Set 3	13.3249	(13.3108, 13.3390)	13.5990	(13.0696, 14.1284)
$H = 120$	Set 1	8.2367	(8.2297, 8.2437)	8.2283	(8.0288, 8.4278)
	Set 2	7.5773	(7.5661, 7.5884)	7.6348	(7.4392, 7.8305)
	Set 3	15.9728	(15.9512, 15.9945)	16.0579	(15.5040, 16.6118)

Table 4.19: Up-and-Out call price - In the money option

As seen from Table 4.19, the HP estimator performs very good for pricing in the money up-and-out put options. Subsequently, we set $K = 90$ in order to analyze out of the money up-and-out put options.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
$H = 105$	Set 1	1.6718	(1.6668, 1.6768)	1.6508	(1.5554, 1.7462)
	Set 2	1.4476	(1.4386, 1.4566)	1.4368	(1.3430, 1.5307)
	Set 3	4.4612	(4.4492, 4.4731)	4.5150	(4.2134, 4.8166)
$H = 110$	Set 1	2.0016	(1.9962, 2.0069)	2.0251	(1.9236, 2.1266)
	Set 2	1.7720	(1.7615, 1.7825)	1.7631	(1.6610, 1.8653)
	Set 3	7.3314	(7.3113, 7.3515)	7.2371	(6.8690, 7.6052)
$H = 115$	Set 1	2.0490	(2.0436, 2.0544)	2.0562	(1.9563, 2.1562)
	Set 2	1.8305	(1.8196, 1.8415)	1.7936	(1.6919, 1.8952)
	Set 3	9.6433	(9.6165, 9.6701)	9.4949	(9.0846, 9.9052)

Table 4.20: Up-and-Out call price - Out of the money option

It can easily be seen from the Table 4.20 the HP estimator dramatically reduces the variance of out of the money up-and-out put options.

Finally, we focus on the behavior of the up-and-out put options with varying number of steps.

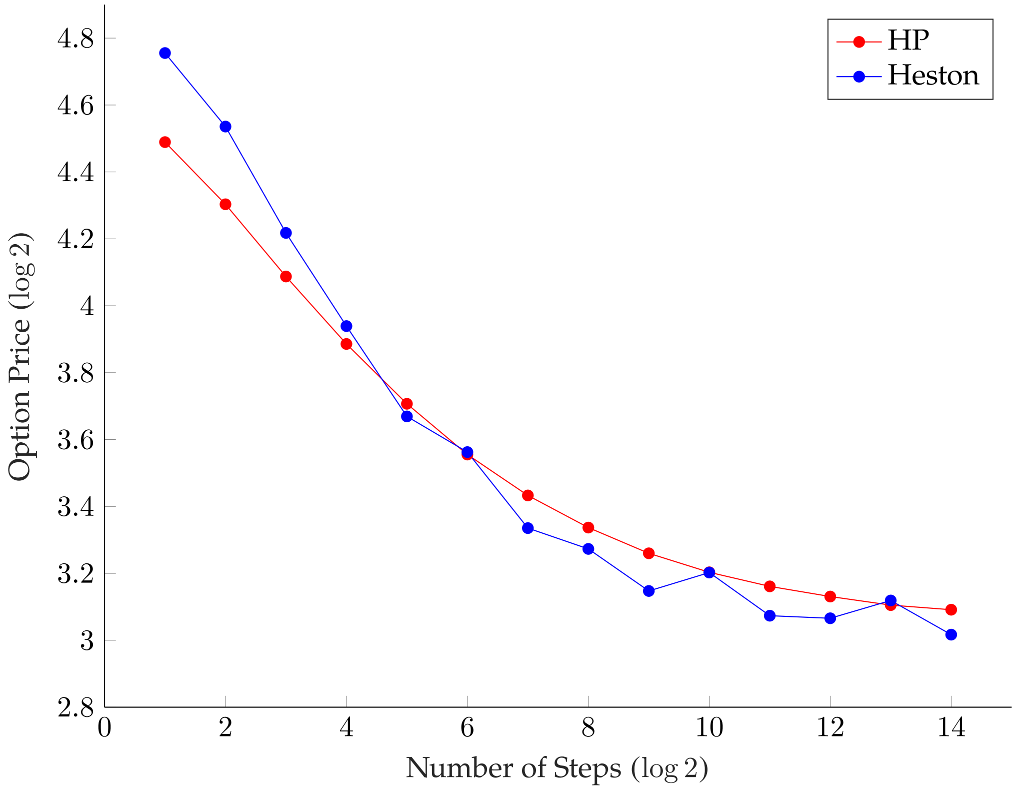

Figure 4.7: Up-and-out put price - the HP estimator and the crude MC estimation in the Heston model for parameter set 3

Figure 4.7 illustrates that the behavior of the HP estimator is remarkable smooth. Indeed, this gives us the idea to employ an extrapolation method to speed up the HP estimator. We can estimate the option price curve of the HP estimator as a function of the number of steps, and then extrapolate for large values of N.

Chapter 5

Application of the HP Estimator to Bond Options

In this chapter, we explain the application of the HP estimator in pricing bond options, in detail. As a further contribution to the literature, we extend the concept of the HP estimator for pricing bond options. Bond options are financial derivatives whose underlying asset dynamics are determined by the short rate. In one factor models it is assumed that the only state variable is the short rate. So, the term structure is completely determined by the spot interest rate. Now, the question is how to model the short rate process. In general, the short rate r_t is assumed to follow a stochastic process. In the literature many different approaches have been proposed to model r_t with an underlying SDE, however in this thesis we only consider the one factor models CIR, Vasicek and furthermore the stochastic variance model of Fong-Vasicek. It is obvious that the theory of short rate and relatedly fixed income securities covers already a wide range of topics which goes beyond the scope of this thesis. Thus, for the sake of simplicity we will keep the thesis restricted only to the models mentioned above. For more details about the theory and the models, we refer to [9, 42].

5.1 Bond Options in the CIR Model

In this section, we give a brief introduction to pricing zero coupon bond (ZCB) options under the Cox-Ingersoll-Ross (CIR) model. A ZCB, also known as pure discount bond, is a financial contract which pays its holder one unit of money at maturity T and makes no coupon payment during the lifetime of the bond. The price of a ZCB at any time t with maturity T is calculated under the risk neutral measure $\mathbb{Q}$ as follows

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left(e^{\left(-\int_t^T r(s)ds\right)} P(T,T)\right) \tag{5.1}$$

where the terminal value is $P(T,T) = 1$. We see that the discount factor is now a stochastic process driven by the short rate dynamics. Cox-Ingersoll-Ross [15] model the short interest rate as a square root diffusion process given by the following SDE

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t) \tag{5.2}$$

where κ, θ, σ are positive constants and $r(0) = r > 0$. Since the CIR process has a mean reversion property, in Equation (5.2) θ stands for the long-term mean reversion level, κ is for the speed to reach the mean reversion level and σ is the volatility of the short rate. We have already mentioned that the CIR process is considered as a variance process by Heston in the stochastic volatility model [34]. Some additional features regarding the short rate modeling will be given here. The process is favorable because the short rate never drops below zero. However, the likelihood that the process hits zero is often quite significant. At the same time, the process is strongly reflecting at the origin. So, once the process hits zero subsequently it becomes positive. There is a criterion introduced by Feller which states that the continuous CIR process remains strictly positive as long as the following condition is satisfied [15]

$$2\kappa\theta > \sigma^2. \tag{5.3}$$

Moreover, due to the mean reverting property the process has an invariant distribution in the limit. The steady state distribution of the interest rate is non-central chi-square $\chi^2(\mu; d, \lambda)$ with d degrees of freedom and non-centrality parameter λ [15]. It is possible to calculate the first two moments of the distribution of the future interest rates conditioned on the current value $r(t)$ as follows

$$\mathbb{E}\left(r(T) \to r(t)\right) = r(t)e^{-\kappa(T-t)} + \theta(1 - e^{-\kappa(T-t)}) \tag{5.4}$$

$$\mathbb{V}ar\left(r(T) \to r(t)\right) = r(t)\left(\frac{\sigma^2}{\kappa}\right)\left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\right) + \theta\left(\frac{\sigma^2}{2\kappa}\right)\left(1 - e^{-\kappa(T-t)}\right)^2. \tag{5.5}$$

We can also interpret the long term behavior of the short rate process by the help of these two moments. For instance, if κ approaches infinity then due to the mean reverting dynamics of the process the mean goes to θ and the variance goes to 0. If otherwise κ approaches zero, the mean goes to the current value of the interest rate $r(t)$ and the variance goes to $r(t)\sigma^2(T-t)$. Furthermore, as t tends to infinity, we have the following steady state mean and variance respectively,

$$\lim_{t\to\infty} \mathbb{E}\left(r(t)\right) = \theta \tag{5.6}$$

$$\lim_{t\to\infty} \mathbb{V}ar\left(r(t)\right) = \frac{\theta\sigma^2}{2\kappa}. \tag{5.7}$$

In addition, the steady state density function of the process approaches a gamma distribution

$$f\left[r(t)\right] = \frac{\omega^{\upsilon}}{\Gamma(\upsilon)} r^{\upsilon-1} e^{-\omega r} \tag{5.8}$$

where $\omega = \frac{2\kappa}{\sigma^2}$ and $\upsilon = \frac{2\kappa\theta}{\sigma^2}$ [15]. Despite the fact that the SDE for the short rate process does not have an explicit solution, one can obtain a closed form solution for the price of a zero coupon bond under the CIR model. Indeed, the price of a T-maturity zero coupon bond at time t has the form [15, 42]

$$P(t,T) = e^{-B(t,T)r(t)+A(t,T)} \tag{5.9}$$

with

$$B(t,T) = \frac{2\left[e^{h(T-t)} - 1\right]}{2h + (\kappa + h)\left[e^{h(T-t)} - 1\right]} \tag{5.10}$$

$$A(t,T) = \ln\left(\left[\frac{2he^{\frac{(T-t)(\kappa+h)}{2}}}{2h + (\kappa + h)\left[e^{h(T-t)} - 1\right]}\right]^{\frac{2\kappa\theta}{\sigma^2}}\right) \tag{5.11}$$

$$h = \sqrt{\kappa^2 + 2\sigma^2}. \tag{5.12}$$

Since we are able to obtain a closed form solution for the ZCB price under the CIR model, our next concern is to write an option on this ZCB as the underlying asset. As we already mentioned, the price of an option is equal to the discounted expected payoff of the underlying asset. Hence, the ZCB options are written on the price of the ZCB. Note that the price of ZCB is also derived from another stochastic process, i.e. the short rate. This is indeed one of the main differences compared to the stock options in the BS model where the stock price process itself determines the dynamics of the underlying asset. If we consider a European call option with maturity T and strike price K written on a S-maturity ZCB, then the price function takes the form

$$V(t,r) = \mathbb{E}_{\mathbb{Q}}\left[e^{\left(-\int_t^T r(s)ds\right)} h(T, r(T))\right] \tag{5.13}$$

with the payoff function

$$h(T, r(T)) = (P(T,S) - K)^+. \tag{5.14}$$

To be able to obtain an analytical solution, one has to be able to derive a closed form expression for the joint distribution of

$$\left(r(t), \int_0^t r(s)ds\right).$$

Unfortunately, the CIR model does not admit an easy explicit representation for this joint distribution. However, the steady state moments allow us to derive the pricing formula of a European call option under the CIR model which is given by [15, 42]

$$C(t,T,S,K) = P(t,S)\chi^2(a_1; d, \lambda_1) - KP(t,T)\chi^2(a_2; d, \lambda_2) \tag{5.15}$$

with

$$d = \frac{4\kappa\theta}{\sigma^2} \tag{5.16}$$

$$a_1 = 2\bar{r}(\xi + \psi + B(T,S)) \qquad a_2 = a_1 - 2\bar{r}B(T,S) \tag{5.17}$$

$$\bar{r} = \frac{\ln\left(\frac{A(T,S)}{K}\right)}{B(T,S)} \quad \psi = \frac{\kappa + h}{\sigma^2} \quad \xi = \frac{2h}{\sigma^2\left(e^{h(T-t)} - 1\right)} \tag{5.18}$$

$$\lambda_1 = \frac{2\xi^2 r(t) e^{h(T-t)}}{\xi + \psi + B(T,S)} \quad \lambda_2 = \frac{2\xi^2 r(t) e^{h(T-t)}}{\xi + \psi} \tag{5.19}$$

where the functions A and B are given in Equations (5.11) and (5.10), respectively.

Note that, one can also price the coupon bond options by using the approach of Jamshidian [37]. The idea is to consider the option on a coupon bond as a portfolio of options. This approach is only valid when the bond prices are monotone functions of the interest rate. By following this approach, the closed form solution of European options on coupon bonds under the CIR model is derived by Longstaff [44]. However in this thesis, we aim to show the applicability of the HP estimator for the interest rate options, therefore we only consider the ZCB options.

5.2 The HP Estimator for Pricing ZCB Options in the CIR Model

In this section, we present the application of the HP estimator in pricing ZCB options under the CIR model. Since the closed form solution of the ZCB option under the CIR model is relatively complicated, we come up with an idea to transfer the main properties of the HP estimator into bond option pricing within the CIR setting. Hence, we consider the CIR process as our target process and we aim to approximate this process with another short interest rate process which has relatively simple dynamics. At this point, the short rate model of Vasicek [55] appears as a possible candidate. As supposed by Vasicek, the short interest rate is driven by the following SDE

$$d\tilde{r}(t) = \kappa(\theta - \tilde{r}(t))dt + \sigma dW(t) \tag{5.20}$$

with real, positive constants κ, θ, σ. In the Vasicek model the short interest rate follows again a mean-reverting process similar to the CIR model. On the contrary, the Vasicek process is normally distributed. The normal distribution of the short rate has computational advantages such as the simplification of the closed form solutions of both the price of the ZCB and the relevant option. Unlike the CIR model, it is possible to derive the joint distribution in closed form for

$$\left(\tilde{r}(t), \int_0^t \tilde{r}(s)ds\right). \tag{5.21}$$

Due to its simple dynamics, it is possible to obtain closed form solutions of the bond option prices in the Vasicek model. They are given in the following theorem.

Theorem 1. *[42] In the Vasicek model given by Equation* (5.20) *we have*

(a) T-zero coupon bond prices of the form

$$P(t,T) = e^{-B(t,T)\tilde{r}(t)+A(t,T)} \tag{5.22}$$

with A and B given by

$$B(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \tag{5.23}$$

$$A(t,T) = \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)(B(t,T) - T + t) - \frac{\sigma^2}{4\kappa}B^2(t,T). \tag{5.24}$$

(b) European bond call and put option prices of the form

$$Call(t,T,S,K) = P(t,S)\Phi(d_1(t)) - KP(t,T)\Phi(d_2(t)), \tag{5.25}$$

$$Put(t,T,S,K) = KP(t,T)\Phi(-d_2(t)) - P(t,S)\Phi(-d_1(t)) \tag{5.26}$$

with

$$d_{1/2}(t) = \frac{1}{\bar{\sigma}(t)} \ln\left(\frac{P(t,S)}{P(t,T)K}\right) \pm \frac{1}{2}\bar{\sigma}(t), \quad \bar{\sigma}(t) = \sigma\sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}} B(T,S) \tag{5.27}$$

where K denotes the strike and T the maturity of the options, and $S \geq T$ is the maturity of the underlying ZCB.

As seen from Theorem 1, the Vasicek pricing formula resembles the well known BS option pricing formula and has a similar interpretation. Thus, it is smooth enough to allow an application of the differential operator $\mathcal{L}^0$. So, we are able to obtain a Taylor expansion in the neighborhood of the Vasicek option pricing formula.

To sum up, the price difference between the CIR model and the Vasicek model is due to the square root function in the diffusion coefficient of the CIR model. Therefore we adjust the price difference generated by this square root function of the CIR model by using the partial derivatives of the Vasicek price. Note that, we are able to define the iterative HP estimator while the Vasicek pricing formula $\tilde{V}(T, \tilde{r}(t))$ is smooth enough to allow the application of the backward Kolmogorov equation, see [37], i.e.

$$\tilde{\mathcal{L}}^0 \tilde{V} = 0 \tag{5.28}$$

with the terminal condition

$$\tilde{V}(T, \tilde{r}) = h(T, \tilde{r}).$$

The following Remark 3 summarizes the implementation of the iterative HP estimator for pricing ZCB options in the CIR model.

Remark 3. *Consider a European ZCB call option under the CIR model, then the price is expressed by*

$$V(t, r(t)) = \mathbb{E}_{\mathbb{Q}}\left[e^{\left(-\int_t^T r(s)ds\right)}(P(T,S)-K)^+\right] \tag{5.29}$$

with the payoff $h(T, r(T)) = (P(T,S)-K)^+$ for some $K \geq 0$ where T denotes the maturity of the option whereas the underlying bond has a maturity of $S \geq T$. Moreover, $P(t,T)$ is the price of the T-maturity ZCB at time t. Then, we approximate this function by using the Vasicek price $\tilde{V}(t, \tilde{r}(t))$ which is explicitly given by Equations (5.25) *and* (5.26) *in Theorem 1. Therefore, we have the following iterative HP estimator for pricing ZCB option in the CIR model*

$$\mathbb{E}\left(\tilde{Z}_T\right) = \tilde{V}(0, r) + \mathbb{E}\left(\int_0^T (\mathcal{L}^0 - \tilde{\mathcal{L}}^0)\tilde{V}(t, r(t))dt\right). \tag{5.30}$$

where $r(t)$ is the short rate of the CIR model and the initial value is $r(0) = r$.

In the next section, we derive the required sensitivities of the Vasicek price to obtain an approximation to the CIR price.

5.2.1 Derivation of the Sensitivities

In order to price the ZCB call option under the CIR model, we use the Vasicek model as an approximation. There, the derivatives of the Vasicek model act like a correction term of the option price. Consequently, we have to compute the following derivatives

$$(\mathcal{L}^0 - \tilde{\mathcal{L}}^0)Call(t,T,S,K) = \frac{\sigma^2}{2}(r-1)\frac{\partial^2 Call(t,T,S,K)}{\partial r^2} \tag{5.31}$$

Thus, we have to compute the second order derivative of the call price with respect to the short rate. Since the Vasicek pricing formula contains the Gaussian distribution function, we are able to compute the derivatives explicitly. We derive the first order derivate which is given in the following

$$\begin{aligned}
\frac{\partial Call(t,T,S,K)}{\partial r} &= \left(\frac{\partial P(t,S)}{\partial r}\Phi(d_1) + P(t,S)\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial r}\right) \\
&\quad - K\left(\frac{\partial P(t,T)}{\partial r}\Phi(d_2) + P(t,T)\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial r}\right) \\
&= -B(t,S)P(t,S)\Phi(d_1) + P(t,S)\phi(d_1)\frac{\partial d_1}{\partial r} \\
&\quad + KB(t,T)P(t,T)\Phi(d_2) - KP(t,T)\phi(d_2)\frac{\partial d_2}{\partial r} \\
&= -B(t,S)P(t,S)\Phi(d_1) + KB(t,T)P(t,T)\Phi(d_2).
\end{aligned} \tag{5.32}$$

Observe that, here the variable of the call price formula is the ZCB price $P(t,T)$, therefore we have to take the derivatives of this function into account as well. If we think of the BS formula deriving the Greeks is straightforward, i.e. the ingredients are the stock price, the normal distribution function and the parameters. Whereas, in the bond option pricing formula, we have to consider the derivatives of the ZCB price function which is derived from the short rate process. This reflects the difference between the BS and the Vasicek option pricing formulas with regard to the derivation of sensitivities. Furthermore, we are able to simplify the expression obtained in Equation (5.32) by the help of the following equations

$$\frac{\partial d_1}{\partial r} = \frac{1}{\bar{\sigma}(t)}(B(t,T) - B(t,S)) \tag{5.33}$$

and also

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r}$$

additionally,

$$\phi(d_2) = \frac{P(t,S)}{KP(t,T)}\phi(d_1).$$

This results in the second order derivatives as follows

$$
\begin{aligned}
\frac{\partial^2 Call(t,T,S,K)}{\partial r^2} =& \frac{\partial}{\partial r}\left(\frac{\partial Call(t,T,S,K)}{\partial r}\right) \\
=& - B(t,S)\left(\frac{\partial P(t,S)}{\partial r}\Phi(d_1) + P(t,S)\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial r}\right) \\
&+ KB(t,T)\left(\frac{\partial P(t,T)}{\partial r}\Phi(d_2) + P(t,T)\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial r}\right) \\
=& B^2(t,S)P(t,S)\Phi(d_1) - KB^2(t,T)P(t,T)\Phi(d_2) \\
&+ \frac{1}{\bar{\sigma}(t)}P(t,S)\phi(d_1)(B(t,T) - B(t,S))^2. \qquad (5.34)
\end{aligned}
$$

In the following section, we present the numerical results of the HP estimator utilized to price ZCB call options under the CIR model.

5.2.2 Numerical Results in the CIR Model

In this section, the implementation of the HP estimator for pricing a ZCB call option under the CIR model is given. For pricing ZCB put options the same approach can be used in a similar manner. In order to discretize the CIR model we employ the drift implicit Milstein scheme given as

$$
r_{t+1} = \kappa(\theta - r_{t+1})dt + \sigma\sqrt{r_t}(W_{t+1} - W_t) + \frac{\sigma^2}{4}((W_{t+1} - W_t)^2 - dt). \qquad (5.35)
$$

For practical purposes one can rewrite this scheme in the following explicit form

$$
r_{t+1} = \frac{1}{1+\kappa dt}\left(\kappa\theta dt + \sigma\sqrt{r_t}(W_{t+1} - W_t) + \frac{\sigma^2}{4}((W_{t+1} - W_t)^2 - dt)\right). \qquad (5.36)
$$

This scheme is introduced in [3] to discretize the CIR process within the Heston model. In [1], several discretization schemes have been proposed and numerically tested for the CIR model by considering different parameters. Moreover, it is proved that the schemes are weakly convergent with an order of 1 under the assumptions regarding the related test functions, e.g. payoffs, and the Feller condition. Subsequently, Alfonsi [2] introduced second and third order discrete schemes for the CIR model by weakening the restrictions on the Feller index. For various discretization schemes of the CIR model and results on the strong convergence, see [18] and references therein. For the numerical tests, we use the following parameters $r(0) = 0.03$, strike $K = 0.1$, maturity of the option $T = 5$ and maturity of the ZCB $S = 6$. After initially taking the values $\kappa = 0.05$, $\theta = 0.03$ and $\sigma = 0.02$, we let these parameters vary to see their effect on the numerical results. Furthermore, we used $N = 5000$ number of paths and the step size $\delta = 0.004$.

		Heath-Platen		Crude MC	
		Value	Confidence Interval	Value	Confidence Interval
	0.05	0.74909	(0.74908, 0.74909)	0.74950	(0.74893, 0.75004)
κ	2	0.74921	(0.74921, 0.74921)	0.74920	(0.74909, 0.74925)
	10	0.74920	(0.74920, 0.74920)	0.74920	(0.74918, 0.74921)
	0.01	0.76185	(0.76184, 0.76185)	0.76167	(0.76112, 0.76222)
θ	0.06	0.73035	(0.73035, 0.73036)	0.73082	(0.73025, 0.73138)
	0.1	0.70609	(0.70609, 0.70610)	0.70652	(0.70595, 0.70709)
	0.001	0.74920	(0.74920, 0.74920)	0.74920	(0.74917, 0.74922)
σ	0.05	0.74846	(0.74840, 0.74852)	0.75041	(0.74900, 0.75181)
	0.075	0.74735	(0.74715, 0.74755)	0.75296	(0.75096, 0.75496)

Table 5.1: ZCB call option price in the CIR model for varying parameters

In fact, the results given in Table 5.1 are not so promising with regard to varying volatility σ. Therefore, we investigate the reason behind this behavior of the HP estimator under the CIR model. For this purpose, we present the simulated outcomes of the HP estimator. We use the following the parameters: $r(0) = 0.08$, $\kappa = 0.05$, $\theta = 0.09$, $\sigma = 0.1$, $K = 0.2$, $S = 5$ and $T = 1$. The following figure illustrates 50 simulated outcomes of the HP estimator.

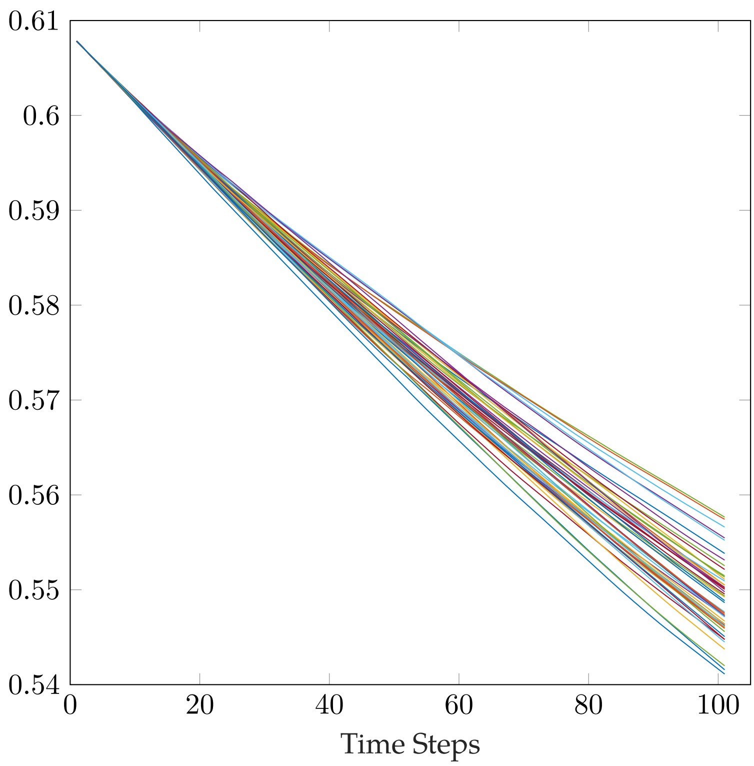

Figure 5.1: Simulated outcomes for the HP estimator in the CIR model

In fact, the behavior of the HP estimator is fairly interesting in the CIR model. Recall that the original approach of the HP estimator aims to approximate the price under a complex model via another model which has simpler dynamics. For instance, the GBS model with a deterministic volatility is a good candidate to approximate the Heston model. Here the approximation is provided by the Taylor expansion in the neighborhood of the GBS price. Indeed, this Taylor expansion corresponds to the partial derivatives of the GBS formula with respect to the volatility of variance. Since we transform the stochastic variance process into a deterministic function, the dimension in the PDE problem of the Heston model is then reduced. Basically, by reducing the dimension we restrict ourselves to a sub-domain and the Taylor expansion on this sub-domain provides us a smooth approximation for the whole domain. However, this is not the case for the CIR model. If we look at the stochastic processes of the CIR and the Vasicek models, we see that there is no reduction in the dimension of the PDE problem. Therefore, the approximation is not smooth. Considering an approximation to the CIR model by a deterministic interest rate would be a better choice for realization of the dynamics of the HP estimator under the CIR model. However, for such a choice the integrability assumptions of the approximation function would be violated. Nevertheless, the idea of implementing the HP estimator is not completely useless, since in some cases it gives a good approximation. The only slight drawback of the application of the HP estimator for the CIR model is that the smooth approximation is not always guaranteed. With this motivation, we come up with the idea to consider a two dimensional model for pricing bond options. The stochastic variance model of Fong and Vasicek [22] is such an example for these sort of processes. In the next section we focus on the application of the HP estimator under the Fong-Vasicek model.

5.3 Bond Options in the Fong-Vasicek Model

A further extension of the Vasicek model is introduced by Fong and Vasicek [22] by allowing the volatility to be stochastic in the classical model of Vasicek. This stochastic volatility is driven by a variance process which is modeled as the CIR process. In the Fong-Vasicek (FV) model the term structure of the yield is effected by both the short rate and the stochastic volatility as the state variables. The nature of this approach resembles the two factor models, though not completely the same. In two factor models the term structure is determined by two separate short rate processes which may be correlated, whereas in the FV model there exists only one short rate process whose volatility is stochastic. Let us give more details on the process and its properties. The FV model assumes that the short rate process is driven by the following SDEs under the physical measure $\mathbb{P}$

$$dr(t) = (\kappa_1(\theta_1 - r(t)) + \lambda\nu(t))dt + \sqrt{\nu(t)}d\hat{W}^1(t) \tag{5.37}$$

$$d\nu(t) = (\kappa_2\theta_2 - (\kappa_2 + \sigma\eta)\nu(t))dt + \sigma\sqrt{\nu(t)}d\hat{W}^2(t) \tag{5.38}$$

where λ and η are the risk premiums associated with interest rate and volatility risks, respectively. Moreover, the two Brownian motions $\hat{W}^1(t)$ and $\hat{W}^2(t)$ are correlated with a coefficient ρ. It is stated in [22] that an increasing level of the interest rates are typically accompanied by an increase in their volatility and vice versa. Thus, it is deduced that the correlation coefficient ρ between the interest rate and its volatility is typically positive. For pricing purposes, we rewrite the equations under the risk-neutral measure $\mathbb{Q}$ and we get the following

$$dr(t) = \kappa_1(\theta_1 - r(t))dt + \sqrt{v(t)}dW^1(t) \tag{5.39}$$

$$dv(t) = \kappa_2(\theta_2 - v(t))dt + \sigma\sqrt{v(t)}dW^2(t) \tag{5.40}$$

where the parameters $\kappa_1, \theta_1, \kappa_2, \theta_2$ and σ are positive constants and the initial values are $r(0) = r$ and $v(0) = v$. Moreover, we have

$$\mathbb{E}_{\mathbb{Q}}\left[dW^1(t)dW^2(t)\right] = \rho dt.$$

We begin our discussion by showing how to price a ZCB under the FV model and extend the case for ZCB options. Under the risk neutral measure $\mathbb{Q}$ the pricing PDE of the ZCB reads as

$$\frac{\partial P}{\partial t} + \frac{1}{2}v\frac{\partial^2 P}{\partial r^2} + \rho\sigma v\frac{\partial^2 P}{\partial v \partial r} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P}{\partial v^2} + \kappa_1(\theta_1 - r)\frac{\partial P}{\partial r} + \kappa_2(\theta_2 - v)\frac{\partial P}{\partial v} - rP = 0 \tag{5.41}$$

subject to the terminal condition $P(T, r, v, T) = 1$. Although Fong and Vasicek [22] did not provide an explicit solution for the ZCB price in their original paper, they have presented the ZCB price in its generic form. Given the current values $r(t)$ and $v(t)$, the ZCB price with maturity T is expressed as follows

$$P(t, r(t), v(t), T) = G(T-t)e^{-r(t)D(T-t)-v(t)F(T-t)}. \tag{5.42}$$

We see that the ZCB price contains an additional term coming from the stochastic variance. Here, the functions $D(T-t), F(T-t)$ and $G(T-t)$ are functions of only the time variable. Moreover, they satisfy the following ODEs which are reduced from the pricing PDE given in Equation (5.41)

$$D'(T-t) = -\kappa_1 D(T-t) + 1, \quad D(0) = 0 \tag{5.43}$$

$$F'(T-t) = \frac{-D^2(T-t)}{2} - (\kappa_2 + \rho\sigma D(T-t))F(T-t) - \frac{\sigma^2}{2}F(T-t), \quad F(0) = 0 \tag{5.44}$$

$$G'(T-t) = -G(T-t)(\kappa_1\theta_1 D(T-t) + \kappa_2\theta_2 F(T-t)), \quad G(0) = 1 \tag{5.45}$$

where D', F' and G' correspond the first order derivative of the functions D, F and G with respect to time variable, respectively. An explicit solution for the function D is available and exactly the same as the function B in the Vasicek model given by Equation (5.23). The solution for the function G can be obtained by integration once we obtain the solution for the function F. Hence, the difficulty to solve the ODEs is mostly due to the function F whose solution includes confluent hypergeometric functions. This leads one to use complex algebra which is practically not really favorable. To overcome this complexity, Selby and Strickland [52] provided a Frobenius series expansion to replace the confluent hypergeometric functions. Their method is both accurate and fast in the computational sense. Here, one has to keep in mind that it is always possible to solve the ODEs by employing some numerical methods. However, this might also bring some error to the implementation. Tahani and Li [53] follow a similar approach as Heston [34] to price the discount bond options under the FV model. They consider the cross-moment generating function of the joint distribution of the short rate, its volatility and its time integral to obtain the price of an interest rate derivative and subsequently they extend the method of Selby and Strickland [52] for the generalized ZCB price under the FV model. Moreover, a Monte Carlo implementation to price the interest rate derivatives under the FV model is proposed by Clewlow and Strickland [14]. Their approach involves the usage of hedge ratios as a control variate technique for the derivative price. Namely, they generate independent random samples for the option price and at the same time independent hedge ratios which are negatively correlated with the option price. Consequently, the variance of the payoff of the hedged option position is smaller than that of the unhedged payoff. In fact, their idea has certain similarities with the HP estimator. For instance, in the Heston model one can think of the HP estimator as the hedged price with respect to first and second order volatility sensitivities. However, in [14] they consider only the so-called delta and vega as the hedge ratios of the Vasicek in its crude form, i.e. with constant volatility. With this motivation, we apply the HP estimator to price ZCB options under the FV model.

5.4 The HP Estimator for Pricing ZCB options in the Fong-Vasicek Model

In this section, we present the application of the HP estimator in pricing bond options under the FV model. Since we aim to price ZCB call option under the FV model, let us recall the FV model under the risk neutral measure $\mathbb{Q}$

$$dr(t) = \kappa_1(\theta_1 - r(t))dt + \sqrt{v(t)}dW^1(t)$$
$$dv(t) = \kappa_2(\theta_2 - v(t))dt + \sigma\sqrt{v(t)}dW^2(t)$$

where the initial values are $r(0), v(0)$ are real umbers and $W^1(t), W^2(t)$ are Brownian motions with $\mathbb{E}^{\mathbb{Q}}(dW^1(t)dW^2(t)) = \rho dt$. To closely capture the dynamics of the FV model, one possible candidate is the Vasicek model which is given in Equation (5.20). In its usual form, the Vasicek model assumes that the volatility is constant. However, for a better approximation we propose to use the Vasicek model with a deterministic volatility. By doing this, we both reduce the complexity of the iterative HP estimator and obtain a better approximation. Now, let us consider the Vasicek model with deterministic volatility given under the risk neutral measure $\mathbb{Q}$

$$d\hat{r}(t) = \kappa_1(\theta_1 - \hat{r}(t))dt + \sqrt{\hat{v}(t)}dW^1(t) \tag{5.46}$$
$$d\hat{v}(t) = \kappa_2(\theta_2 - \hat{v}(t))dt \tag{5.47}$$

where the coefficient $\kappa_1, \theta_1, \kappa_2$ and θ_2 are positive constants and the initial values $\hat{r}(0)$ and $\hat{v}(0)$ are real numbers. We call this model the generalized Vasicek model. As we already explained in the Section 4.1, the deterministic variance equation can be solved explicitly by

$$\hat{v}_t^{0,\hat{v}} = \theta_2 + (\hat{v}_t - \theta_2)e^{-\kappa_2 t}.$$

In order to obtain the required value for the deterministic volatility to use within the generalized Vasicek pricing formula, the procedure given in Proposition 1 applies similarly. Finally, we consider the deterministic volatility given as follows

$$\hat{\sigma}_t = \sqrt{\theta_2 + (\hat{\nu}_t - \theta_2)\frac{1 - e^{-\kappa_2(T-t)}}{\kappa_2(T-t)}}. \tag{5.48}$$

We may follow the same steps with the HP estimator applied in the Heston model to obtain the price of a ZCB option under the FV model. Here, our target process is the FV model and our aim is to obtain a Taylor expansion in the neighborhood of the generalized Vasicek model. Finally, this approximation will provide us with a variance reduction in our MC implementation. Hence, we begin our discussion by considering the following valuation function under the FV model

$$v(t, r(t), \nu(t)) = \mathbb{E}_{\mathbb{Q}}^{(t,r(t),\nu(t))}\left[e^{\left(-\int_t^T r(s)ds\right)} h(T, r(T), \nu(T))\right] \tag{5.49}$$

where h is the payoff function of the option. For instance, for a ZCB call option it is actually equal to

$$h(T, r(T), \nu(T)) = [P(T, r(T), \nu(T), S) - K]^+ \tag{5.50}$$

where $P(T, r(T), \nu(T), S)$ is the price of the underlying S-maturity ZCB at time T and K is the strike price. Furthermore, the valuation function can also be obtained as the solution of the PDE

$$\frac{\partial v}{\partial t} + \frac{1}{2}\nu\frac{\partial^2 v}{\partial r^2} + \rho\sigma\nu\frac{\partial^2 v}{\partial \nu \partial r} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 v}{\partial \nu^2} + \kappa_1(\theta_1 - r)\frac{\partial v}{\partial r} + \kappa_2(\theta_2 - \nu)\frac{\partial v}{\partial \nu} = 0. \tag{5.51}$$

namely

$$\mathcal{L}^0 v = 0 \tag{5.52}$$

with the terminal condition

$$v(T, r(0), \nu(0)) = h(T, r(0), \nu(0))$$

On the other hand, if we consider the valuation function for the generalized Vasicek model, we have the following

$$\hat{v}(t, \hat{r}(t), \hat{\nu}(t)) = \mathbb{E}_{\mathbb{Q}}^{(t,\hat{r}(t),\hat{\nu}(t))}\left[e^{\left(-\int_t^T \hat{r}(s)ds\right)} h(T, \hat{r}(T), \hat{\nu}(T))\right].$$

Again, this valuation function satisfies the following PDE

$$\frac{\partial \hat{v}}{\partial t} + \frac{1}{2}\hat{\nu}\frac{\partial^2 \hat{v}}{\partial \hat{r}^2} + \kappa_1(\theta_1 - \hat{r})\frac{\partial \hat{v}}{\partial \hat{r}} + \kappa_2(\theta_2 - \hat{\nu})\frac{\partial \hat{v}}{\partial \hat{\nu}} = 0 \tag{5.53}$$

which can be written in short

$$\hat{\mathcal{L}}^0 \hat{v} = 0 \tag{5.54}$$

with the terminal condition

$$\hat{v}(T, \hat{r}(0), \hat{\nu}(0)) = h(T, \hat{r}(0), \hat{\nu}(0)).$$

Now, we are able to define the following unbiased HP estimator to obtain the ZCB option price under the FV model as

$$\mathbb{E}\left(\hat{Z}_T\right) = \hat{v}(0, r(0), v(0)) + \mathbb{E}\left(\int_0^T \left(\mathcal{L}^0 - \hat{\mathcal{L}}^0\right) \hat{v}(t, r(t), v(t))dt\right). \tag{5.55}$$

Note that $\hat{v}$ is the valuation function under the generalized Vasicek model, thus it is smooth enough to allow a Taylor expansion. Again the idea is to approximate a complicated price function by another pricing function. Here the key point is that the approximation process has relatively simple dynamics and particularly a closed form solution for the option price. The iterative HP estimator leads to a reduction of the complexity of the original pricing PDE problem in the FV model. Namely, the difference between the differential operators of the two processes provides us a Taylor expansion of the removed term, e.g. the diffusion of the variance process. Eventually, the Taylor expansion in the neighborhood of the Vasicek price on the sub-domain provides us with a smooth approximation for the FV price on the whole domain. Consequently, during the simulation process for each discrete point we obtain a Taylor expansion and each final value of $\hat{Z}_T$ given by Equation (5.55) is an option price for the FV model. Subsequently, we calculate the mean over the $\hat{Z}_T$ values via the HP estimator (4.13). The following expression represents the required derivatives for the application of the HP estimator in the FV model

$$(\mathcal{L}^0 - \hat{\mathcal{L}}^0)\hat{v}(t, r(t), v(t)) = \sigma v(t)\left(\rho \frac{\partial^2 \hat{v}(t, r(t), v(t))}{\partial r \partial v} + \frac{1}{2}\sigma \frac{\partial^2 \hat{v}(t, r(t), v(t))}{\partial v^2}\right). \tag{5.56}$$

Here, the $\hat{v}$ function corresponds to the option price in the generalized Vasicek model. Since the generalized Vasicek model has a closed form solution for pricing bond options, we are able to derive the required partial derivatives explicitly. In particular, for the ZCB call options we can write the pricing formula for the generalized Vasicek model as follows

$$Vas(t, T, S, K, \hat{\sigma}_t) = P(t, S)\Phi(d_1(t)) - KP(t, T)\Phi(d_2(t)) \tag{5.57}$$

where the functions $d_1(t)$ and $d_2(t)$ are given by

$$d_{1/2}(t) = \frac{1}{\bar{\sigma}(t)} \ln\left(\frac{P(t, S)}{P(t, T)K}\right) \pm \frac{1}{2}\bar{\sigma}(t), \quad \bar{\sigma}(t) = \hat{\sigma}_t \sqrt{\frac{1 - e^{-2\kappa_1(T-t)}}{2\kappa_1}} B(T, S)$$

and further the ZCB price $P(t, T)$ as

$$P(t, T) = e^{-B(t,T)\hat{r}(t) + A(t,T)}$$

with A and B given by

$$B(t, T) = \frac{1}{\kappa_1}\left(1 - e^{-\kappa_1(T-t)}\right)$$

$$A(t, T) = \left(\theta_1 - \frac{\hat{\sigma}_t^2}{2\kappa_1^2}\right)(B(t, T) - T + t) - \frac{\hat{\sigma}_t^2}{4\kappa_1} B^2(t, T).$$

Note that all the functions except the function B involve the deterministic volatility $\hat{\sigma}_t$. Now, if we set

$$\hat{v}(t, r(t), v(t)) = Vas(t, T, S, K, \hat{\sigma}_t)$$

then we are able to compute the partial derivatives regarding the diffusion part of the variance process in the FV model. Let us write the difference explicitly again

$$\begin{aligned}(\mathcal{L}^0 - \hat{\mathcal{L}}^0)Vas(t, T, S, K, \hat{\sigma}_t) = \sigma v(t)\Bigg[&\rho \frac{\partial^2 Vas(t, T, S, K, \hat{\sigma}_t)}{\partial r \partial \hat{\sigma}_t}\frac{\partial \hat{\sigma}_t}{\partial v} \\ &+ \frac{1}{2}\sigma\Bigg(\frac{\partial^2 Vas(t, T, S, K, \hat{\sigma}_t)}{\partial \hat{\sigma}_t^2}\left(\frac{\partial \hat{\sigma}_t}{\partial v}\right)^2 \\ &+ \frac{\partial Vas(t, T, S, K, \hat{\sigma}_t)}{\partial \hat{\sigma}_t}\frac{\partial^2 \hat{\sigma}_t}{\partial v^2}\Bigg)\Bigg]. \end{aligned} \tag{5.58}$$

Finally, the derivatives can be computed explicitly. In the following section we present the derivation of the sensitivities of the ZCB call option.

5.4.1 Derivation of the Sensitivities

We begin the derivation with the first order partial derivative $\dfrac{\partial Vas(t, T, S, K, \hat{\sigma}_t)}{\partial \hat{\sigma}_t}$ which indeed corresponds to the derivative of the call price with respect to the volatility parameter. First, we obtain

$$\begin{aligned}\frac{\partial Vas(t, T, S, K, \hat{\sigma}_t)}{\partial \hat{\sigma}_t} &= \left[\frac{\partial P(t, S)}{\partial \hat{\sigma}_t}\Phi(d_1) + P(t, S)\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial \hat{\sigma}_t}\right] \\ &\quad - K\left[\frac{\partial P(t, T)}{\partial \hat{\sigma}_t}\Phi(d_2) + P(t, T)\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \hat{\sigma}_t}\right] \\ &= P(t, S)\Phi(d_1)\frac{\partial A_s}{\partial \hat{\sigma}_t} + P(t, S)\phi(d_1)\frac{\partial d_1}{\partial \hat{\sigma}_t} \\ &\quad - K\left[P(t, T)\Phi(d_2)\frac{\partial A_t}{\partial \hat{\sigma}_t} + P(t, T)\phi(d_2)\frac{\partial d_2}{\partial \hat{\sigma}_t}\right]. \end{aligned} \tag{5.59}$$

The following equations help us to simplify the above equation. First of all, we can rewrite the probability density function of the second variable in terms of the first one, or vice versa

$$\phi(d_2) = \frac{P(t, S)}{KP(t, T)}\phi(d_1). \tag{5.60}$$

Moreover, we have that

$$\frac{\partial d_1}{\partial \hat{\sigma}_t} = \frac{-d_2}{\hat{\sigma}_t} + \frac{1}{\bar{\sigma}_t}\left(\frac{\partial A_s}{\partial \hat{\sigma}_t} - \frac{\partial A_t}{\partial \hat{\sigma}_t}\right). \tag{5.61}$$

Also, we have the following equation

$$\frac{\partial d_2}{\partial \hat{\sigma}_t} = \frac{\partial d_1}{\partial \hat{\sigma}_t} - \frac{\partial \bar{\sigma}_t}{\partial \hat{\sigma}_t}. \tag{5.62}$$

Substituting Equations (5.60) and (5.62) into Equation (5.59) we get the following

$$\begin{aligned}\frac{\partial Vas(t,T,S,K,\hat{\sigma}_t)}{\partial \hat{\sigma}_t} &= P(t,S)\Phi(d_1)\frac{\partial A_s}{\partial \hat{\sigma}_t} - KP(t,T)\Phi(d_2)\frac{\partial A_t}{\partial \hat{\sigma}_t} \\ &+ P(t,S)\phi(d_1)\frac{\partial \bar{\sigma}_t}{\partial \hat{\sigma}_t}.\end{aligned} \tag{5.63}$$

We are also able to derive the partial derivatives involved in Equation (5.63) as follows

$$\frac{\partial A_s}{\partial \hat{\sigma}_t} = \frac{\hat{\sigma}_t}{\kappa_1^2}(B(t,T) - T + t) - \frac{\hat{\sigma}_t}{2\kappa_1}B^2(t,T) \tag{5.64}$$

and also

$$\frac{\partial \bar{\sigma}_t}{\partial \hat{\sigma}_t} = \sqrt{\frac{1-e^{-2\kappa_1(T-t)}}{2\kappa_1}}B(T,S). \tag{5.65}$$

The next step is to derive the second order partial derivatives. We continue with the following sensitivity $\frac{\partial^2 Vas(t,T,S,K,\hat{\sigma}_t)}{\partial r \partial \hat{\sigma}_t}$ which in the BS setting corresponds to the Greek *vanna*. The derivation reads as

$$\begin{aligned}\frac{\partial^2 Vas(t,T,S,K,\hat{\sigma}_t)}{\partial r \partial \hat{\sigma}_t} &= \frac{\partial}{\partial r}\left(\frac{\partial Vas(t,T,S,K,\hat{\sigma}_t)}{\partial \hat{\sigma}_t}\right) \\ &= \frac{\partial A_s}{\partial \hat{\sigma}_t}\left[\frac{\partial P(t,S)}{\partial r}\Phi(d_1) + P(t,S)\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial r}\right] \\ &- K\frac{\partial A_t}{\partial \hat{\sigma}_t}\left[\frac{\partial P(t,T)}{\partial r}\Phi(d_2) + P(t,T)\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial r}\right] \\ &+ \frac{\partial \bar{\sigma}_t}{\partial \hat{\sigma}_t}\left[\frac{\partial P(t,S)}{\partial r}\phi(d_1) + P(t,S)(-d_1)\frac{\partial \Phi(d_1)}{\partial d_1}\frac{\partial d_1}{\partial r}\right].\end{aligned} \tag{5.66}$$

Again, we utilize some equations to simplify the above expression, for instance,

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r} = \frac{1}{\bar{\sigma}_t}\left(B(t,T) - B(t,S)\right). \tag{5.67}$$

Furthermore, the derivative of the ZCB price with respect to the interest rate variable is as follows

$$\frac{\partial P(t,S)}{\partial r} = -B(t,S)P(t,S). \tag{5.68}$$

By plugging Equations (5.60), (5.67) and (5.68) into Equation (5.66) we obtain the following

$$\begin{aligned}\frac{\partial^2 Vas(t,T,S,K,\hat{\sigma}_t)}{\partial r \partial \hat{\sigma}_t} &= -B(t,S)P(t,S)\Phi(d_1)\frac{\partial A_s}{\partial \hat{\sigma}_t} \\ &+ KB(t,T)P(t,T)\Phi(d_2)\frac{\partial A_t}{\partial \hat{\sigma}_t} \\ &+ P(t,S)\phi(d_1)\left[\frac{\partial d_1}{\partial r}\left(\frac{\partial A_s}{\partial \hat{\sigma}_t} - \frac{\partial A_t}{\partial \hat{\sigma}_t}\right) - \frac{\partial \bar{\sigma}_t}{\partial \hat{\sigma}_t}B(t,S) - d_1\frac{\partial d_1}{\partial r}\right].\end{aligned} \tag{5.69}$$

All the partial derivatives required in computation of Equation (5.69) are obtained recently. Therefore, we now proceed to the computation of the last term involved in the HP estimator, i.e.

$$
\begin{aligned}
\frac{\partial^2 Vas(t,T,S,K,\hat{\sigma}_t)}{\partial \hat{\sigma}_t^2} &= \frac{\partial}{\partial \hat{\sigma}_t}\left(\frac{\partial Vas(t,T,S,K,\hat{\sigma}_t)}{\partial \hat{\sigma}_t}\right) \\
&= \frac{\partial P(t,S)}{\partial \hat{\sigma}_t}\frac{\partial A_s}{\partial \hat{\sigma}_t}\Phi(d_1) + P(t,S)\frac{\partial^2 A_s}{\partial \hat{\sigma}_t^2}\Phi(d_1) \\
&+ P(t,S)\frac{\partial A_s}{\partial \hat{\sigma}_t}\frac{\partial \Phi(d_1)}{\partial \hat{\sigma}_t}\frac{\partial d_1}{\partial \hat{\sigma}_t} \\
&- K\Big[\frac{\partial P(t,T)}{\partial \hat{\sigma}_t}\frac{\partial A_t}{\partial \hat{\sigma}_t}\Phi(d_2) + P(t,T)\frac{\partial^2 A_t}{\partial \hat{\sigma}_t^2}\Phi(d_2) \\
&+ P(t,T)\frac{\partial A_t}{\partial \hat{\sigma}_t}\frac{\partial \Phi(d_2)}{\partial \hat{\sigma}_t}\frac{\partial d_2}{\partial \hat{\sigma}_t}\Big] \\
&+ \frac{\partial \bar{\sigma}_t}{\partial \hat{\sigma}_t}\left[\frac{\partial P(t,S)}{\partial \hat{\sigma}_t}\phi(d_1) + P(t,S)(-d_1)\phi(d_1)\frac{\partial d_1}{\partial \hat{\sigma}_t}\right].
\end{aligned}
\tag{5.70}
$$

If we rearrange the terms to simplify the expression we get the following

$$
\begin{aligned}
\frac{\partial^2 Vas(t,T,S,K,\hat{\sigma}_t)}{\partial \hat{\sigma}_t^2} &= P(t,S)\Phi(d_1)\left[\left(\frac{\partial A_s}{\partial \hat{\sigma}_t}\right)^2 + \frac{\partial^2 A_s}{\partial \hat{\sigma}_t^2}\right] \\
&- KP(t,T)\Phi(d_2)\left[\left(\frac{\partial A_t}{\partial \hat{\sigma}_t}\right)^2 + \frac{\partial^2 A_t}{\partial \hat{\sigma}_t^2}\right] \\
&+ P(t,S)\phi(d_1)\Big[\frac{\partial d_1}{\partial \hat{\sigma}_t}\left(\frac{\partial A_s}{\partial \hat{\sigma}_t} - \frac{\partial A_t}{\partial \hat{\sigma}_t}\right) \\
&+ \frac{\partial \bar{\sigma}_t}{\partial \hat{\sigma}_t}\left(\frac{\partial A_s}{\partial \hat{\sigma}_t} + \frac{\partial A_t}{\partial \hat{\sigma}_t}\right) - d_1\frac{\partial d_1}{\partial \hat{\sigma}_t}\Big].
\end{aligned}
\tag{5.71}
$$

The only remaining unknown partial derivative is the second order derivative of the function A which reads as

$$
\frac{\partial^2 A_s}{\partial \hat{\sigma}_t^2} = \frac{-1}{\kappa_1^2}(B(t,T) - T + t) - \frac{1}{2\kappa_1}B^2(t,T). \tag{5.72}
$$

Finally, we have all the partial derivatives required for the HP estimator. Thus, we can run the numerical tests to obtain the ZCB call option price in the FV model. The ZCB put option price can be obtained similarly, therefore we restrict our study only to the ZCB call option.

5.4.2 Numerical Results in the Fong-Vasicek Model

In this section, we present the results of the numerical analyses of the application of the HP estimator in pricing ZCB options under the FV model. To implement the MC simulation, we discretize the FV model by using a fully truncated Euler scheme for the variance process and the standard Euler scheme for the short rate process. By using the full truncation, we avoid the negative values that the variance process might attain during the simulation. However, the short rate process still might attain negative values. To analyze the HP estimator for the FV model, we first provide a graphical illustration to show the degree of variance reduction with the HP estimator. For this, 50 simulated outcomes of the HP estimator are displayed. To plot the figure we use the following parameters: initial short rate $r(0) = 0.08$, initial variance $v(0) = 0.03$, mean reversion for the short rate $\theta_1 = 0.095$, mean reversion for the variance $\theta_2 = 0.03$, mean reversion speed for the short rate $\kappa_1 = 2$, mean reversion speed for the variance $\kappa_2 = 2$, volatility of variance $\sigma = 0.6$, and the correlation coefficient $\rho = 0.8$. Furthermore, the strike $K = 0.6236$, the maturity of the option $T = 1$ and the maturity of bond $S = 5$.

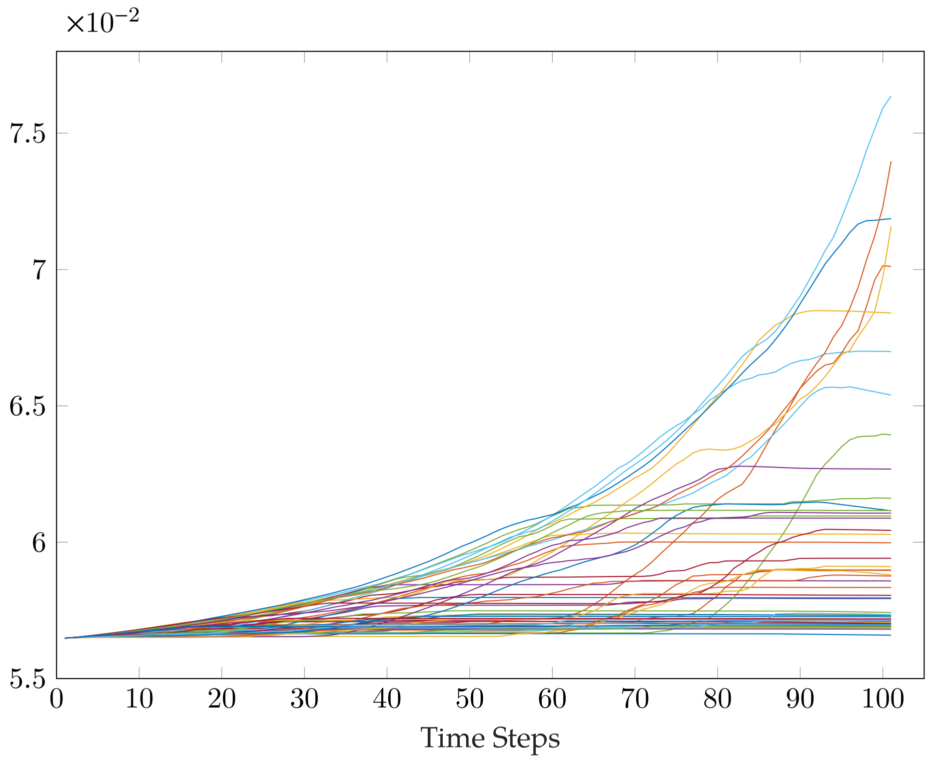

Figure 5.2: Simulated outcomes for the HP estimator in the FV model

As can be easily seen from Figure 5.2 the simulated paths of the HP estimator have a smooth behavior and further the variance of the simulated paths is substantially small. For the same parameter set we illustrate the option price and the corresponding confidence interval length in Figure 5.3.

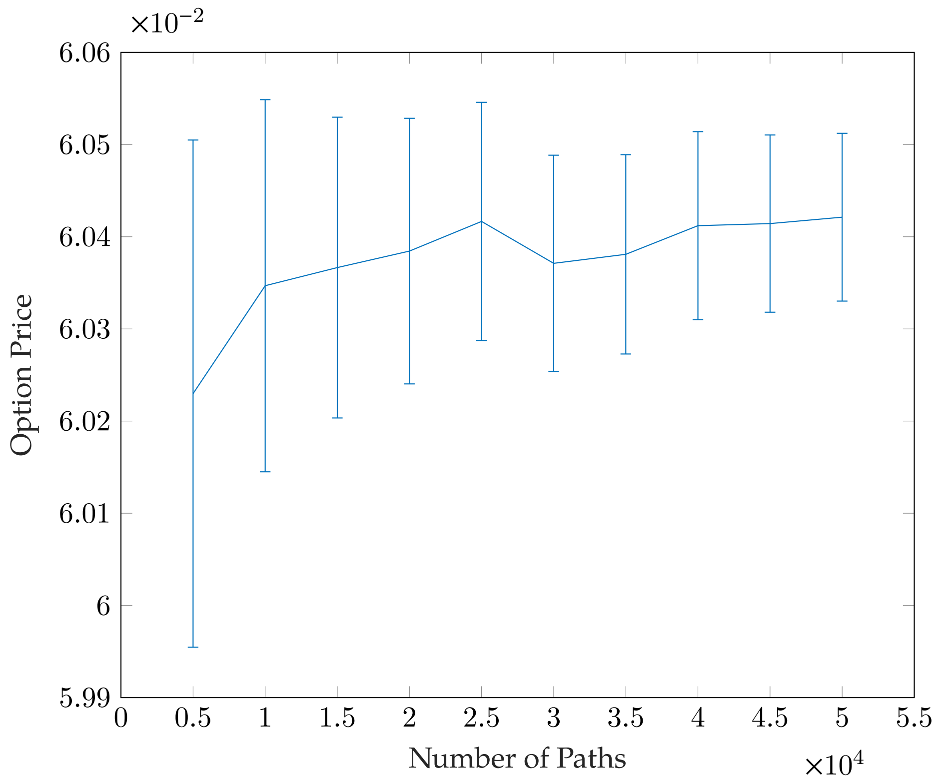

Figure 5.3: Price and corresponding error bounds as a function of number of paths

It can be seen from Figure 5.3 that the increasing number of paths reduces the variance of the HP estimator. There are two aspects regarding the results given in Figure 5.3. First, we notice that even with 5000 paths, the HP estimator produces remarkably small variance and relatedly a small confidence interval. Secondly, after reaching a certain number of paths, the length of the confidence interval does not vary significantly. Thus, after reaching that level from practical point of view there is no need to increase the number of paths for the HP estimator.

As a final consideration, we compare the computational time performance of the HP estimator regarding the discrete schemes employed to discretize the variance process. We employ the fully truncated (FT) Euler scheme and the drift implicit (DI) Milstein scheme to discretize the variance process. The discretization of the short rate process remains as Euler scheme for both cases. For comparison we use the parameter set described above with $N = 10000$ number of paths and the step size $\delta = 0.004$.

Discrete Scheme	Sensitivities	Main Function
FT Euler scheme	1.112	1.378
DI Milstein scheme	1.119	1.383

Table 5.2: Computational performance of the HP estimator

Table 5.2 suggests that there is no computationally significant difference between the FT Euler scheme and the drift implicit Milstein scheme with regard to the performance of the HP estimator. However, for both cases a big portion of the computational time is invested in the computation of sensitivities.

Chapter 6

Weak Error Analysis of the HP Estimator

In this chapter, we present the results of numerical analyses regarding the weak error behavior of the HP estimator. In principle, the error estimation is a fundamental criterion to evaluate the accuracy of numerical estimations. As we briefly explained in Section 2.1.2, the exact simulation of an SDE via the crude MC estimator is only affected by the variance of the randomly generated outcomes. Therefore, the only error involved in the exact simulation is the statistical error caused by the variance. Hence, the indicator of the accuracy of these exact simulations via the crude MC estimation is the variance. However, if one employs a discrete scheme to discretize the underlying process, then another source of error comes into the play, i.e. the discretization error. Thus, the error of the MC estimation is then expressed in terms of the statistical error and the discretization error. Up to this section, we analyzed the performance of the HP estimator with regard to the MC variance. In the next section we give more details about the discretization error of the HP estimator.

6.1 Basics of Weak Error Analysis

In numerical implementations the main concern regarding the error analysis is whether the numerical solution converges to the exact solution. This convergence which assures that the numerical approximation is reasonably accurate, is mainly dependent on the discretization error. Therefore, one has to have a control on the discretization error. To give more details, let us consider the diffusion process $X = \{X_t, t \in [0, T]\}$ satisfying the following SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t.$$

Assume that $a(t, X_t)$ and $b(t, X_t)$ satisfy the necessary conditions to ensure the existence and uniqueness of the solution X_t. To obtain the solution numerically we discretize the process X_t by employing the EM scheme with an equidistant step size $\Delta = \frac{T}{N}$ such that

$$\hat{X}_{n+1} = \hat{X}_n + a(\Delta, \hat{X}_n)\Delta + b(\Delta, \hat{X}_n)\Delta W_n$$

for $n = 0, 1, 2, 3, \cdots, N$ where N is the number of steps and $\hat{X}_0 = X_0$. Imagine that we generate a random path via using this discrete approximation $\hat{X}_n$. If one is interested in the pathwise difference of X_t and $\hat{X}_n$, then it is referred as the strong convergence. Hence, the error regarding the strong convergence is then called strong error of the approximation. The strong error of a discrete approximation $\hat{X}_n$ is given by

$$\epsilon = \mathbb{E}\left(\left|X_t - \hat{X}_n\right|\right). \tag{6.1}$$

Consequently, we say that $\hat{X}_n$ converges strongly to X_t if the following condition is satisfied [40]

$$\lim_{\Delta_n \to 0} \mathbb{E}\left(\left|X_t - \hat{X}_n\right|\right) = 0.$$

However, in financial applications we are generally interested in the moments (i.e. the expected values) of the diffusion processes. Therefore, pathwise difference is not particularly necessary. If we look at the difference between the first moments of X_t and $\hat{X}_n$, then we have to deal with the weak error. One further consideration of the weak error is to determine the difference between the expected value of some functionals of X_t and $\hat{X}_n$, i.e.

$$\varepsilon = \left|\mathbb{E}\left(f(X_T)\right) - \mathbb{E}(f(\hat{X}_T))\right| \tag{6.2}$$

where the functional f belongs to a class of test functions C. The necessary condition of the weak convergence is then expressed by

$$\lim_{\Delta_n \to 0}\left|\mathbb{E}\left(f(X_T)\right) - \mathbb{E}(f(\hat{X}_T))\right| = 0. \tag{6.3}$$

In fact, this expression implies that the global error has to approach zero as the step size approaches zero. The global error corresponds the propagation of the error starting from the initial time 0 until the terminal time T. In their seminal work, Talay and Tubaro [54] expanded the weak error of the Euler scheme in powers of the discretization step size. Let us give more details about their work, since their novel approach to analyze the weak error of the Euler scheme has certain similarities with the HP estimator. They begin their discussion by considering a time homogeneous smooth functional f of $\mathcal{F}_T$ and a function $\varphi(t, x) = \mathbb{E}^{t,x}\left(f(X_T)\right)$. This φ solves the following Kolmogorov backward equation

$$\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi = 0$$
$$\varphi(T, x) = f(x)$$

where the $\mathcal{L}$ corresponds to the differential operator. Then, the global weak error of the discrete scheme $\hat{X}_n$ takes the following form

$$Err_e(T, \Delta) = \mathbb{E}\left(\varphi(T, \hat{X}_N)\right) - \mathbb{E}\left(\varphi(0, \hat{X}_0)\right). \tag{6.4}$$

Talay and Tubaro [54] consider the global error expansion as the sum of the local errors where the local error is computed by performing a Taylor expansion around each discrete time point. They first start with the computation of the following local error

$$\mathbb{E}\left(\varphi(T,\hat{X}_N)\right) - \mathbb{E}\left(\varphi((N-1)\Delta,\hat{X}_{N-1})\right)$$

and perform the Taylor expansion for these terms. Subsequently, they proceed with the same procedure until the initial time. Eventually, the global error for the Euler scheme reads as

$$Err_e(T,\Delta) = -\Delta\int_0^T \mathbb{E}(\psi_e(s,X_s))ds + O(\Delta^2) \tag{6.5}$$

where $\psi_e(s,X_s)$ given as follows

$$\begin{aligned}\psi_e(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} b^i(t,x)b^j(t,x)\partial_{ij}\varphi(t,x)\\ &+\frac{1}{2}\sum_{i,j,k=1}^{d} b^i(t,x)a^j_k(t,x)\partial_{ijk}\varphi(t,x)\\ &+\frac{1}{8}\sum_{i,j,k,l=1}^{d} a^i_j(t,x)a^k_l(t,x)\partial_{ijkl}\varphi(t,x)+\frac{1}{2}\frac{\partial^2}{\partial t^2}\varphi(t,x)\\ &+\sum_{i=1}^{d} b^i(t,x)\frac{\partial}{\partial t}\partial_i\varphi(t,x)+\frac{1}{2}\sum_{i,j=1}^{d} a^i_j(t,x)\frac{\partial}{\partial t}\partial_{ij}\varphi(t,x).\end{aligned} \tag{6.6}$$

This expansion can be utilized to obtain a weak convergence rate of a discrete scheme. The main difficulty is located in the computation of the integral term appearing in the error expansion given in Equation (6.5). If one is able to bound all the derivatives contained in the function $\psi_e(t,x)$, then one can obtain the convergence rate of the discrete scheme. The weak convergence rate results of Bally and Talay [5] are based on the global error expansion of Talay and Tubaro [54]. To obtain the bounds for the derivatives of the function $\psi_e(t,x)$ Bally and Talay [5] employ the techniques from Malliavin calculus, further they weaken the smoothness assumption on the functional f to being only measurable and bounded. Following the approaches of [54, 5] Neuenkirch and Altmayer [3] obtained the weak convergence of order 1 for their scheme, i.e. the drift implicit Milstein scheme for the variance process and the Euler scheme for the log-stock process in the Heston model. Furthermore, for determining the bound of the derivatives appearing in the $\psi_e(t,x)$ function, they utilize the results from the work of Feehan and Pop [21].

6.2 Weak Convergence Rate of the HP Estimator

Our main concern is to obtain the weak convergence rate of the HP estimator for European call options in the Heston model. Now, let us recall the unbiased HP approximation of the European call option price in the Heston model,

$$\mathbb{E}\left(\bar{Z}_T\right) = \bar{u}(0,s,\nu) + \mathbb{E}\left(\int_0^T (\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t,S_t,\nu_t)dt\right) \tag{6.7}$$

where $S_0 = s$, $\nu_0 = \nu$ and $\bar{u}$ corresponds to GBS price given by

$$\bar{u}(t,S_t,\nu_t) = e^{r(T-t)}GBS(S_t,K,r,\bar{\sigma}_t,T-t). \tag{6.8}$$

By using Fubini's theorem we can rewrite the HP estimator as follows

$$\mathbb{E}\left(\bar{Z}_T\right) = \bar{u}(0,s,\nu) + \int_0^T \mathbb{E}\left[(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t,S_t,\nu_t)\right]dt. \tag{6.9}$$

In particular, we draw attention to the similarity between the global error expansion of Talay and Tubaro given in Equation (6.5) and the expression for the HP estimator given in Equation (6.9). The integrand appearing in the first term of the global error expansion is the expected value of the function $\psi_e(t,x)$ which is consisted of some partial derivatives of the function $\varphi(t,x) = \mathbb{E}^{t,x}\left(f(X_T)\right)$. This indeed resembles the second term in the HP estimator where the integrand is equal to the expected value of the expression $(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t,S_t,\nu_t)$ which corresponds to the partial derivatives of the function $\bar{u}(t,S_t,\nu_t)$. In the numerical implementation we simulate the discretization of the integrand appearing in the HP estimator by discretizing the stock and the variance processes of the Heston model. Thus, the discretized HP estimator $\bar{\bar{Z}}_n$ takes the following form

$$\bar{\bar{Z}}_{n+1} = \bar{\bar{Z}}_n + \Delta\left[(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}\left(\Delta,\Delta S_n,\Delta\nu_n\right)\right] \tag{6.10}$$

for $n = 0,1,2,\cdots,N$ and the equidistant step size $\Delta = \frac{T}{N}$. Note that the discrete time increment which is standing out of the brackets corresponds to the discretization of the integral. As already pointed out, in application of the HP estimator in the Heston model we consider $\bar{u}(t,S_t,\nu_t)$ as the undiscounted GBS price, see Equation (6.8), then we compute the Heston price as a Taylor expansion in the neighborhood of the GBS price. Therefore, at each discrete time we obtain the Taylor expansion by using the Greeks of the GBS price. Consequently, we deduce that the discrete HP estimator basically simulates the slightly modified version of the global error expansion. The slight modification is due to the iterative HP estimator. Namely, we do not take all the partial derivatives involved in the function $\psi_e(t,x)$ into account but only the partial derivatives generated by the difference of the differential operators.

In general, to compute the weak convergence rate we have to determine the bounds of partial derivatives involved in the function $\psi_e(t,x)$ appearing in the global error expansion. However, as we already mentioned, by using the iterative HP estimator we eliminate certain derivatives. Therefore, the resulting error for the HP estimator is only due to the remaining terms. For the Heston application, the difference corresponds to the following derivatives

$$(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t,S_t,\nu_t) = \sigma\nu_t\left(S_t\rho\frac{\partial^2\bar{u}(t,S_t,\nu_t)}{\partial S_t\partial\nu_t} + \frac{1}{2}\sigma\frac{\partial^2\bar{u}(t,S_t,\nu_t)}{\partial\nu_t^2}\right).$$

Thus, we have to find a bound for these derivatives to control the local error and eventually the associated global error. The weak error of the HP estimator is then given by

$$\varepsilon_{HP} = \left|\mathbb{E}(\bar{Z}_T) - \mathbb{E}(\bar{\bar{Z}}_N)\right|. \tag{6.11}$$

If we rewrite this difference explicitly then we obtain

$$\begin{aligned}
\varepsilon_{HP} &= \Big|\bar{u}(0,s,v) + \int_0^T \mathbb{E}\left[(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t,S_t,v_t)\right] dt \\
&\quad - \sum_{n=0}^{N} \left(\bar{\bar{Z}}_n + \Delta\left[(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(\Delta,\Delta S_n,\Delta v_n)\right]\right)\Big| \\
&= \Big|\int_0^T \mathbb{E}\left[(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t,S_t,v_t)\right] dt - \sum_{n=1}^{N} \Delta\left[(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}\left(\Delta,\Delta S_n,\Delta v_n\right)\right]\Big|.
\end{aligned} \tag{6.12}$$

Since $\bar{\bar{Z}}_0 = \bar{u}(0,s,v)$, the GBS price cancels out for the continuous and discrete parts. Hence, we have to determine the difference between the continuous and the discrete terms, which indeed is slightly different than the approach of Talay and Tubaro [54]. They expand the error by defining the φ function as the expectation of the functional of the underlying process. They perform the Taylor expansion between each discrete points and then the summation of the local errors yields the global error. However, the HP estimator already simulates an expression which is similar to this error expansion at each discrete time. Thus, to be able to examine the weak convergence rate of the HP estimator, one has to find an upper bound for the expression given in Equation (6.12). The theoretical treatment of this problem may require some improved techniques. Nevertheless, we can still examine the weak convergence rate numerically, since the European call options have semi-analytical solution in the Heston model. To evaluate the weak error of the HP estimator we consider the so-called root mean squared error (RMSE) given by

$$RMSE = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\text{Real value} - \text{Simulated value})^2}. \tag{6.13}$$

The real value is the exact price of the European call option in the Heston model and the simulated value is the result obtained by the HP estimator. Moreover, M is the number of repetitions that we perform for each different number of steps. In Figure 6.1 we provide the weak convergence rate of both the HP estimator and the crude MC estimation for all parameter sets given in Table 4.2 with $N = 200000$ number of paths. Furthermore, we discretize the HP estimator by employing fully truncated Euler scheme. To determine the weak convergence rate we have to estimate the slope of the plotted error curve. To plot the error curve we set the number of repetitions $M = 50$. In Figure 6.1 the x-axis represents the number of steps and the y-axis is the RMSE. The $\log 2$ scale is used for both x, y-axes.

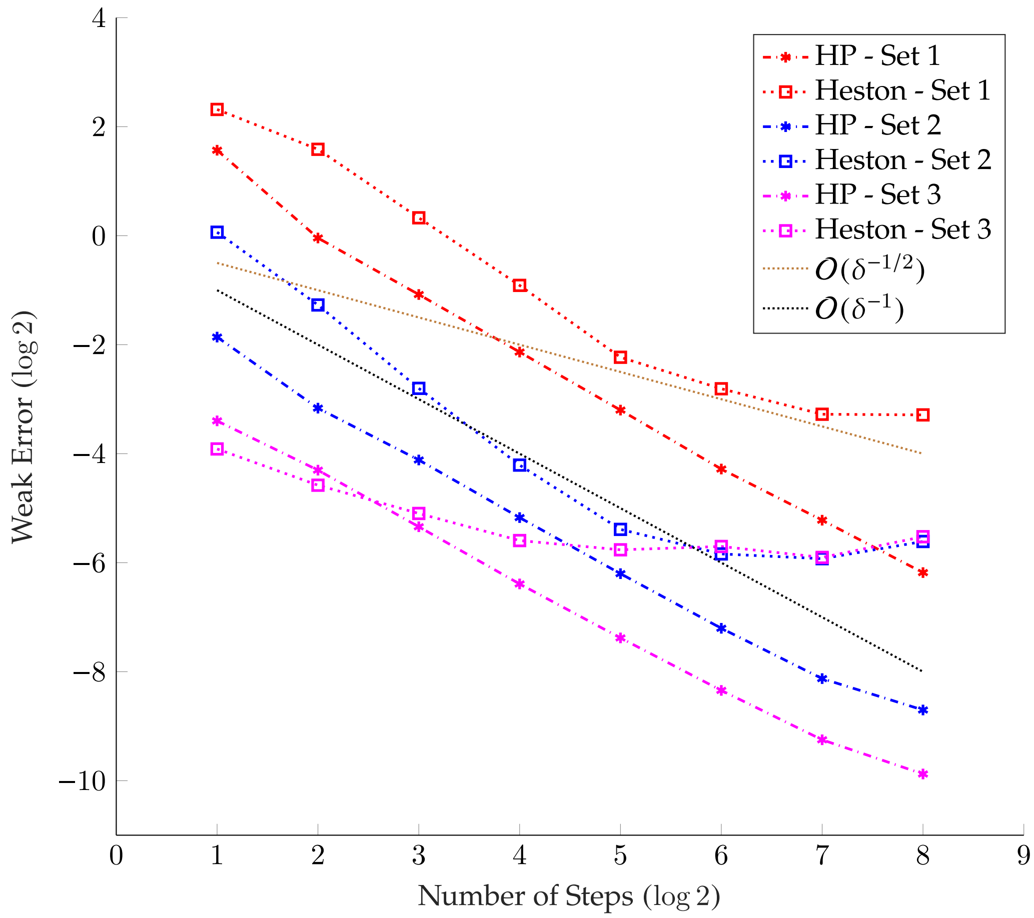

Figure 6.1: RMSE - European call option in the Heston model

We see from Figure 6.1 that the weak convergence rate of the HP estimator for pricing European call options in the Heston model is almost 1. Namely the error of the HP estimator grows only linearly in the step size. This indeed is a remarkably good achievement for the Heston model. In particular, if we look at the error behavior of the crude MC estimation, then the efficiency of the HP estimator is particularly visible.

As a further consideration, we plot the error surface of the HP estimator for varying number of paths and varying number of steps. By doing this, we can evaluate the interplay between the MC variance and the bias of the HP estimator in pricing European vanilla options in the Heston model.

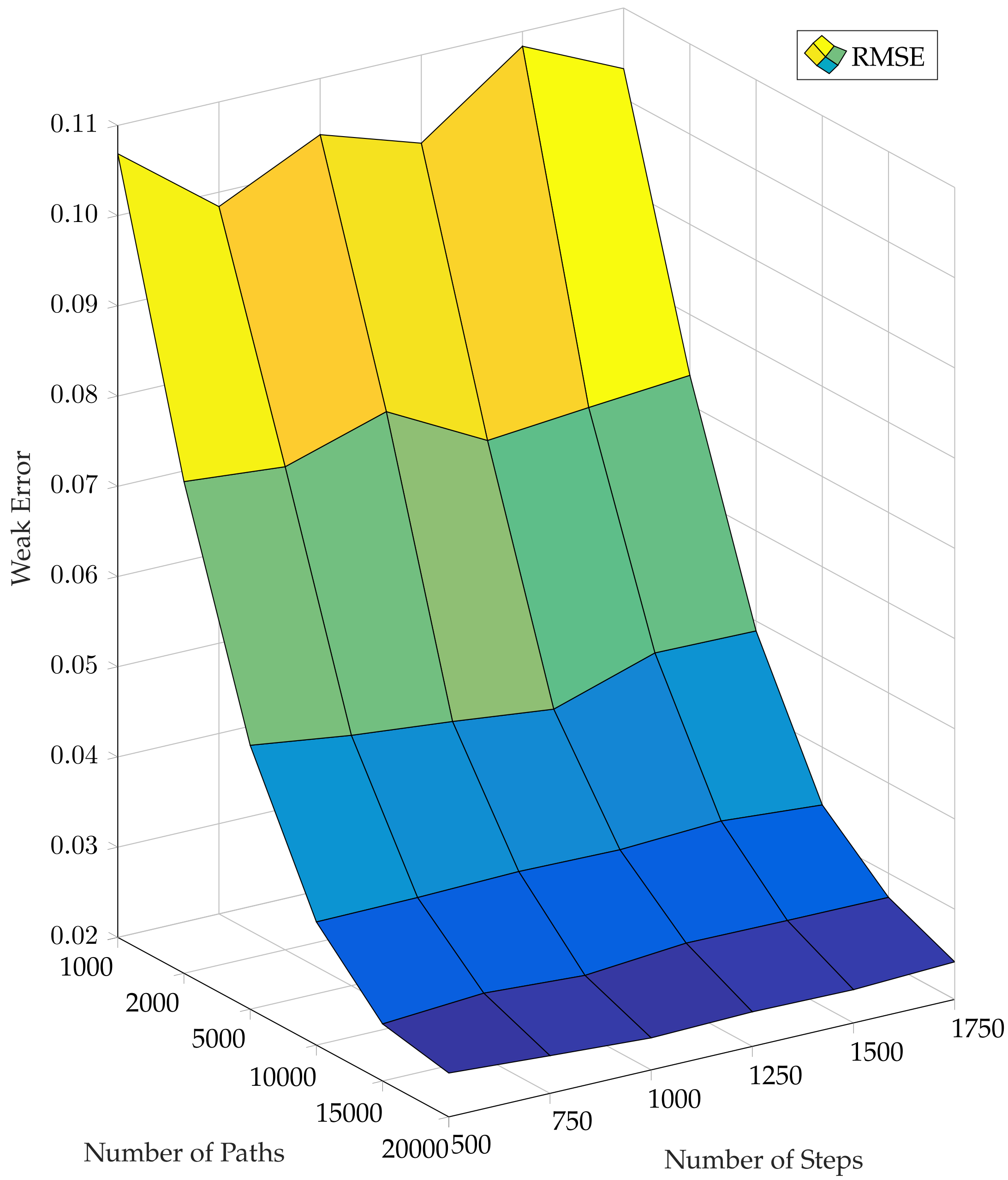

Figure 6.2: Error surface - European call option in the Heston model

Figure 6.2 shows that the error generated by the HP estimator is mainly dominated by the MC variance. Namely, the bias has no big effect on the RMSE. The fluctuations on the surface only occurs when the number of paths is really low. As we increase the number of paths then the surface becomes fairly smooth. Note that, this smoothing is achieved by the HP estimator only with $N = 20000$ paths. This is also a significant achievement of the HP estimator.

Chapter 7

Conclusion

In this chapter, we summarize the obtained results throughout the thesis and point out the challenging issues about each applications. Subsequently, we propose some potential problems regarding the future research topics.

7.1 Summary of the Main Results

In this thesis, we first focused on the application of the Heath-Platen (HP) estimator in pricing one-sided knockout barrier options in the Heston model. The HP estimator was initially introduced by Heath and Platen [32] as a control variate technique and they also applied the HP estimator for pricing European vanilla options in the Heston model. The essential idea of the HP estimator is based on the coupling of the Heston model and the generalized BS model, i.e. BS with deterministic volatility, via their differential operators. In particular, this coupling provides a dimension reduction in the PDE problem of option pricing. In fact, we restrict ourselves in the sub-domain of the GBS price, then a Taylor expansion on this sub-domain provides us with a smooth approximation to the option price in the whole domain of the Heston price. Hence, the price of an option in the Heston model is approximated by a Taylor expansion in the neighborhood of the GBS price. Thus, the HP estimator for pricing an option in the Heston model has the following form

$$\mathbb{E}\left(\bar{Z}_\tau\right) = \bar{u}(0, s, \nu) + \int_0^T \mathbb{E}\left(\mathbf{1}_{\{t<\tau\}}(\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}(t, S_t, \nu_t)\right) dt$$

where $\bar{u}$ function corresponds to the price of the option in the GBS model and the initial values are $S_0 = s$ and $\nu_0 = \nu$ for the stock and variance, respectively. Moreover, the differential operators $\mathcal{L}^0$ and $\bar{\mathcal{L}}^0$ belong to the Heston and the GBS models, respectively. Although the HP estimator is originally interpreted as a control variate technique, it employs a rather sophisticated approach compared to the crude control variate technique. We conducted numerical tests to compare the performance of the HP estimator and the crude control variate technique for a European call option in the Heston model. The performance of the HP estimator for European options was quite remarkable. Therefore, it was promising to examine the performance of the HP estimator for pricing path dependent options in the Heston model. Thus, the main contribution of this thesis is the examination of the performance of the HP estimator for pricing all types of one-sided knockout barrier options in the Heston model. There were two challenging aspects regarding the application of the HP estimator for pricing barrier options. The tedious part was related to the derivation of the required Greeks for each type of the one-sided knockout options. The second one was to determine the first hitting time τ at which the stock price hits the barrier for the first time. Since the HP estimator aims to approximate the price in the Heston model via a Taylor expansion around the GBS price, we have to proceed with the expansion until the first hitting time. Note that for the crude control variate technique which aims to compute the difference between the expected payoffs of the two processes determining the first exit time is not necessary. Therefore, for the crude control variate technique once the stock price hits the barrier we get zero payoff at maturity T. However, for the HP estimator, there exist some paths which hit the barrier during the lifetime of the option but still provide us a positive payment at the end. Thus, the HP estimator for a down-and-out call option can be written in the following form

$$\bar{Z}(T, H, K, S) = \begin{cases} \bar{Z}_T & \text{if } S_t > H \text{ for all } t \in [0, T] \\ \bar{Z}_\tau & \text{if } S_\tau \leq H \text{ for some } \tau \in [0, T]. \end{cases}$$

Another challenging aspect of the application of the HP estimator was due to the monitoring bias. In the BS model, there exists a mis-pricing between the continuously monitored and discretely monitored barrier options. This mis-pricing is also noticeable in the results obtained by the HP estimator in the Heston model. More precisely, if we take the continuously monitored BS price as an initial value of the HP estimator, then the approximated price is the continuously monitored barrier option price in the Heston model. On the other hand, if we take the discretely monitored BS price as an initial value, then the HP estimator gives us the discretely monitored barrier option price in the Heston model. Note that, this cannot be achieved by the crude Monte Carlo estimation in the Heston model, since it can only give the discretely monitored barrier price. To test the efficiency of the HP estimator in pricing one-sided knockout barrier options, we conducted several numerical tests regarding various aspects and parameters. As the detailed numerical analyses imply, the HP estimator reduces the variance dramatically and thus performs superior in pricing barrier options in the Heston model.

A further contribution of this thesis is the application of the HP estimator to price bond options under the Cox-Ingersoll-Ross (CIR) model [15] and the Fong-Vasicek (FV) model [22]. As suggested in the original paper of Heath and Platen [32], the HP estimator has a wide range of applicability for derivative pricing. Therefore, transferring the concept of the HP estimator for pricing bond options is a promising contribution. For this purpose, first we aimed to price zero coupon bond options under the CIR model. The Vasicek model [55] appeared as a possible candidate to consider as an approximating process. Thus, we approximated the ZCB bond option price in the CIR model by performing a Taylor expansion around the Vasicek price. If we have a look at the dynamics of the CIR and the Vasicek model, then we see that the difference is due to the square root function appearing in the drift term. We know that, in principle the HP estimator aims to reduce the dimension of the PDE problem. This principle of the HP estimator is not achieved for the CIR model, as the approximating Vasicek process does not seem to be as good as the deterministic volatility process in the Heston setting. Therefore, the performance of the HP estimator is only relatively good with regard to varying σ values. Nevertheless, the idea of implementing the HP estimator in the CIR model is not completely useless, since in some cases it provides a good approximation. The only slight drawback of the application of the HP estimator for the CIR model is that the smooth approximation is not always guaranteed. With this motivation, we come up with the idea to apply the HP estimator in pricing bond options in the FV model. For the application of the HP estimator in the FV model, we consider the Vasicek model with deterministic volatility as an approximating process. Therefore the HP estimator performs a Taylor expansion in the neighborhood of the bond option price in the generalized Vasicek model. We derived the required sensitivities to provide this Taylor expansion and then applied the HP estimator. The performance of the HP estimator in pricing bond options in the FV model is evaluated by the numerical tests. As numerical results imply, the HP estimator is applicable in the FV model and provides a considerable amount of variance reduction.

In the last part of the thesis, we numerically analyzed the weak error behavior of the HP estimator in pricing European call option in the Heston model. The numerical tests suggest that the HP estimator has a weak convergence of order almost 1 in the Heston model. This indeed is a remarkable achievement of the HP estimator.

To conclude, we summarize the contributions of this thesis as

1. The application of the HP estimator in pricing one-sided knockout barrier options in the Heston model,
2. The application of the HP estimator in pricing bond options in the CIR model,
3. The application of the HP estimator in pricing bond options in the FV model,
4. Numerical analysis of the weak convergence rate of the HP estimator for pricing European options in the Heston model.

7.2 Future Research

As we have seen that the HP estimator has a wide range of applicability for option pricing, we investigate further frameworks where the HP estimator can be applied. Therefore, we observe the similarity between the unbiased approximation $\bar{Z}_\tau$ to the option price $u(0, x)$ given in (4.10) and Dynkin's formula.

Theorem (Dynkin's Formula [48]). *Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time $\mathbb{E}^x[\tau] < \infty$, then*

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x \left[\int_0^\tau \mathcal{A} f(X_s) ds \right]. \tag{7.1}$$

where the infinitesimal generator $\mathcal{A}$ equals to

$$\mathcal{A}f(x) = \sum_i a_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(bb^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

for X_t any time homogeneous Itô diffusion process such that

$$dX_t = a(X_t)dt + b(X_t)dW_t.$$

If we have a closer look at the expression of the HP estimator given in Equation (4.10), we can easily see the similarities between Dynkin's formula and expectation of the random variable $\bar{Z}_\tau$. Indeed, by assuming that $f(x)$ in Dynkin's formula is equal to the BS price, we immediately get

$$\mathbb{E}[\bar{Z}_\tau] = \bar{u}(0,x) + \mathbb{E}\Big[\int_0^\tau \mathcal{L}^0 \bar{u}(t, X_t^{0,x})dt\Big].$$

Here, τ in the Dynkin's formula given in Equation (7.1) is formulated as a stopping time. Although in our case τ given in Equation (4.2) is defined as the first exit time, it is formulated as an optimal stopping time which indeed allows an application in pricing American options. Furthermore, in [49] an application of Dynkin's formula for the optimal stopping problem is considered. Moreover, we know that the infinitesimal generator $\mathcal{A}$ and the differential operator $\mathcal{L}$ coincide on the space C_0^2 of twice continuously differentiable functions with compact support. This resemblance between the HP approximation and Dynkin's formula gives us the insight that the HP approximation can be also utilized to price American options. However, we keep this for future research.

Another approach for pricing options can be expressed in terms of the semi-groups. The following figure will provide us with a better understanding of the relations among the different approaches to the stochastic problems.

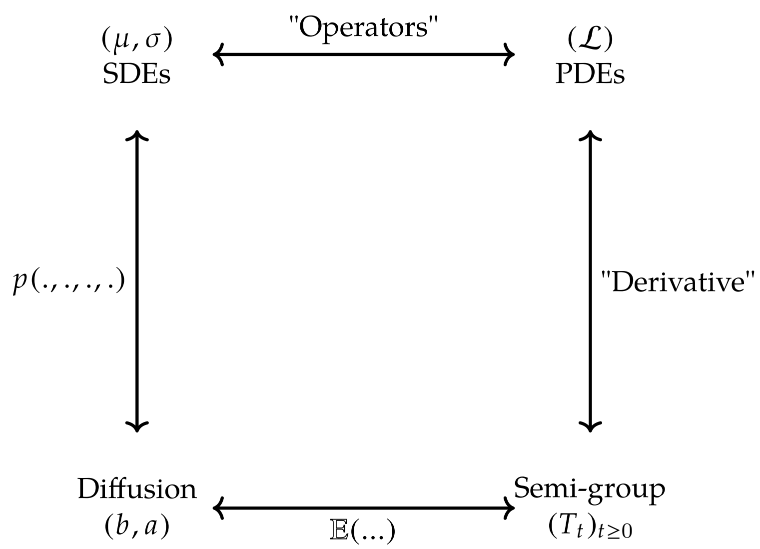

Figure 7.1: Mutual connections among the approaches to the stochastic problems

If we have a look at the relations of the approaches, we can easily infer that we can adopt the approach of semi-groups to solve the problems in SDEs. Our main concern is to price financial derivatives in an efficient way. Throughout the thesis we combined the approach of PDEs and numerical solutions of SDEs. However, there exist also studies which deal with the derivative pricing problem by means of the semi-group approach. For instance, [16, 43] consider the option pricing problem by using the pricing semi-groups. We will give a brief introduction about the approach, then we will show how the HP estimator may also be utilized in the framework of the semi-groups. Let us consider a time homogeneous regular one dimensional diffusion process $\{X_t, t \geq 0\}$ defined on a state space Γ which is some interval $I \subseteq \mathbb{R}$ with endpoints e_1 and e_2, $-\infty \leq e_1 < e_2 \leq \infty$. Here the diffusion follows the SDE

$$dX_t = a(X_t)dt + b(X_t)dW_t, \qquad X_0 = x \in \Gamma \tag{7.2}$$

where the diffusion coefficient $b(X_t)$ is strictly positive and continuous on the open interval (e_1, e_2). Moreover, the drift coefficient $a(X_t)$ is continuous on the same interval. If we consider a payoff function h, then we can write the price of an option under the risk neutral measure $\mathbb{Q}$ as follows

$$V(t,x) = e^{-r(T-t)}\mathbb{E}^x[h(X_t)] =: (\mathcal{P}_t h)(x) \tag{7.3}$$

with r stands for the interest rate. Here $(\mathcal{P}_t h)(x)$ is the pricing semigroup. Let us consider the infinitesimal generator of the pricing semigroup which is given by

$$(\mathcal{G}h)(x) = \frac{1}{2}b^2(x)h''(x) + a(x)h'(x), \qquad x \in (e_1, e_2) \tag{7.4}$$

where $\mathcal{G}$ acts on the functions on I depending on the appropriate boundary and regularity conditions. Furthermore, the option pricing problem turns into the following Cauchy problem for the evolution equation

$$\begin{aligned} V_t &= \mathcal{G}V \quad \text{for} \quad t > 0 \\ V(0,x) &= h(x). \end{aligned} \tag{7.5}$$

To solve this pricing problem under a general Markov process semi-group setting (i.e. Banach Space) the numerical methods are quite convenient. An analytical solution may exist if there exists an available Hilbert space semi-group setting where the pricing semi-group is exactly self-adjoint. One can find a unique spectral representation for pricing semi-group and subsequently a closed form solution for $V(t,x)$ in the Hilbert space. This actually is the essential property of Hilbert spaces for option pricing problems. See Figure 7.2, for a visual summary of the semi-group approach

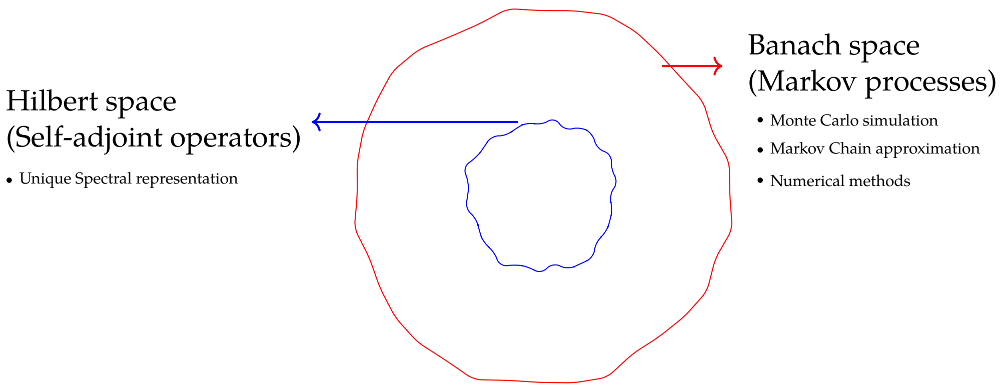

Figure 7.2: Symbolic representation of the semi-group approach

As already mentioned, the HP approach works with the underlying PDE structures and the related differential operators. The main difference between the PDE and semi-group approach emerges due to the functions where the operators act on. In the PDE case we use differential operators directly on pricing functions, e.g. the function V in Equation (7.3). On the contrary, in the semi-group approach the infinitesimal generator acts particularly on the payoff function, e.g. the function h in Equation (7.3). In fact, here the reason is obvious why the differential operator and the infinitesimal generator coincide only on the space C_0^2 of twice continuously differentiable functions. Since the payoff functions might have varying characteristics in terms of differentiability, using the infinitesimal generator and accordingly the semi-group approach is not always possible. In [43], there exists a brief discussion on L^2 and non-L^2 payoffs where L^2 refers to the Hilbert space. However, in the general perspective the idea of the HP estimator seems applicable for the approach of pricing semi-groups. For instance, we may take a L^2 payoff and utilize it as an approximation for a non-L^2 payoff. This indeed seems worth to consider as a future research.

Appendix A

THE HP ALGORITHM FOR EUROPEAN AND BARRIER OPTIONS

The HP Algorithm 2 for the European options has already been given in [42]. However, to point out the difference between the algorithms of the European and the one-sided knockout barrier options it is necessary to recall it here. In principle, we run the algorithm until maturity T and we obtain the price of a European option by the HP estimator given in the Algorithm 2. Basically we have no condition of the time variable. Note that, this algorithm also applies for the interest rate derivatives which we have considered in this thesis.

In the case of barrier options, there is an additional step involved in the HP algorithm which emerges from the barrier crossing condition. As already mentioned, the HP estimator has a different nature compared to the crude MC estimation. While in the crude simulation one focuses only on the condition whether the barrier is hit. Precisely the first exit time has no effect on the expected payoffs. However, the HP approach aims to have the Taylor expansion around the GBS price, therefore one has to continue with the expansion until the first exit time. Consequently, the first exit time has a non-negligible effect on the results obtained by the HP estimator. See the Algorithm 3.

Algorithm 2 European Call Option Price with the HP estimator [42]

Let N and $\Delta = \frac{T}{n}$ be given.

for $i = 1$ to N **do**

$S^{(i)}(0) = s, \nu^{(i)}(0) = \nu, Z^{(i)}(0) = e^{rT} GBS(s, K, r, \bar{\sigma}_0, T)$.

for $j = 1$ to n **do**

Generate two independent $\mathcal{N}(0,1)$-random numbers $Y_1^{(ij)}, Y_2^{(ij)}$.

Correlate the random numbers $W^{(ij)} = \rho Y_1^{(ij)} + \sqrt{1-\rho^2} Y_2^{(ij)}$.

Update (S, ν, Z):

$S^{(i)}(j\Delta) = S^{(i)}((j-1)\Delta) + \left(1 + r\Delta + \sqrt{\nu^{(i)}((j-1)\Delta)^+}\sqrt{\Delta} Y_1^{(ij)}\right)$

$\nu^{(i)}(j\Delta) = \nu^{(i)}((j-1)\Delta) + \kappa\left(\theta - \nu^{(i)}((j-1)^+\Delta)\right)\Delta + \sigma\sqrt{\nu^{(i)}((j-1)\Delta)^+}\sqrt{\Delta} W^{(ij)}$

$\nu_+^{(i)}(j\Delta) = \nu^{(i)}(j\Delta)^+$

$Z^{(i)}(j\Delta) = Z^{(i)}((j-1)\Delta) + (\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}\left((j-1)\Delta, S((j-1)\Delta), \nu((j-1)\Delta)\right)\Delta$

Use the relation (4.33) for the computation of the Z-update.

Estimate the call price by the HP estimator

$$I_{HP,N} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} Z^{(i)}(T)$$

Algorithm 3 Down-and-Out Call Option Price with the HP estimator

Let N and $\Delta = \frac{T}{n}$ be given.

for $i = 1$ to N **do**

$S^{(i)}(0) = s, \nu^{(i)}(0) = \nu, Z^{(i)}(0) = e^{rT} GBS(s, K, r, \bar{\sigma}_0, T)$.

for $j = 1$ to n **do**

Generate two independent $\mathcal{N}(0,1)$-random numbers $Y_1^{(ij)}, Y_2^{(ij)}$.

Correlate the random numbers $W^{(ij)} = \rho Y_1^{(ij)} + \sqrt{1-\rho^2} Y_2^{(ij)}$.

Update (S, ν, Z):

$$S^{(i)}(j\Delta) = S^{(i)}((j-1)\Delta) + \left(1 + r\Delta + \sqrt{\nu^{(i)}((j-1)\Delta)^+} \sqrt{\Delta} Y_1^{(ij)}\right)$$

$$\nu^{(i)}(j\Delta) = \nu^{(i)}((j-1)\Delta) + \kappa\left(\theta - \nu^{(i)}((j-1)^+\Delta)\right)\Delta + \sigma\sqrt{\nu^{(i)}((j-1)\Delta)^+}\sqrt{\Delta}W^{(ij)}$$

$$\nu_+^{(i)}(j\Delta) = \nu^{(i)}(j\Delta)^+$$

$$Z^{(i)}(j\Delta) = Z^{(i)}((j-1)\Delta) + (\mathcal{L}^0 - \bar{\mathcal{L}}^0)\bar{u}\left((j-1)\Delta, S((j-1)\Delta), \nu((j-1)\Delta)\right)\Delta$$

if $S^{(i)}(j\Delta) \leq H$ **then**

Store the value $Z^{(i)}(j\Delta)$ for the terminal valuation.

else

Continue to the update process until maturity.

Use the relations given in Appendix B for the computation of the Z-update.

Estimate the call price by the HP estimator

$$I_{HP,N} = e^{-rT}\frac{1}{N}\left(\sum_{i=1}^{M} \bar{Z}_\tau^{(i)} + \sum_{i=M+1}^{N} \bar{Z}_T^{(i)}\right)$$

Appendix B

DETAILED DERIVATION OF GREEKS

For the sake of simplicity, we keep our detailed derivation of the Greeks only for down-and-out call option. The other formulas can be obtained in a similar way. In order to give a coherent form of the derivation we recall the BS pricing formula, then we derive the Greeks accordingly. The BS pricing formula for the down-and-out call option reads as

$$V_{call}^{do}(t,T,K,H) = \begin{cases} A - C & \text{if } H < K \\ B - D & \text{if } H > K \end{cases} \tag{B.1}$$

with

$$\begin{aligned} A &= Se^{-q\tau}\Phi(x_1) - Ke^{-r\tau}\Phi(x_3) \\ B &= Se^{-q\tau}\Phi(x_2) - Ke^{-r\tau}\Phi(x_4) \\ C &= Se^{-q\tau}\Phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r\tau}\Phi(y_3)\left(\frac{H}{S}\right)^{2\gamma} \\ D &= Se^{-q\tau}\Phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+2} - Ke^{-r\tau}\Phi(y_4)\left(\frac{H}{S}\right)^{2\gamma} \end{aligned}$$

where $\tau = T - t$ corresponds to time to maturity and the arguments in the distribution functions are given in the following

$$\begin{aligned} x_1 &= \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad x_3 = x_1 - \sigma\sqrt{\tau} \\ x_2 &= \frac{\ln\left(\frac{S}{H}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad x_4 = x_2 - \sigma\sqrt{\tau} \\ y_1 &= \frac{\ln\left(\frac{H^2}{SK}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad y_3 = y_1 - \sigma\sqrt{\tau} \\ y_2 &= \frac{\ln\left(\frac{H}{S}\right) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad y_4 = y_2 - \sigma\sqrt{\tau} \end{aligned}$$

and also

$$\gamma = \frac{r - q - \frac{1}{2}\sigma^2}{\sigma^2} \quad \text{and} \quad \frac{\partial \gamma}{\partial \sigma} = \frac{-2(r-q)}{\sigma^3} \quad \text{and} \quad \frac{\partial^2 \gamma}{\partial \sigma^2} = \frac{6(r-q)}{\sigma^4}.$$

We start the derivation of the Greeks with the following case.

- Case: $K > H$

For this case we need the derivatives of the function A and C, respectively. If we have a closer look, we see that the function A is exactly the same as the classical BS pricing formula. However for the consistency on the derivation procedure, we derive the relevant derivatives, i.e. *vega*, *volga* and *vanna* Greeks of the function A. We start with the derivation of vega which is given by

$$\begin{aligned}\frac{\partial A}{\partial \sigma} &= Se^{-q\tau}\frac{\partial \Phi(x_1)}{\partial x_1}\frac{\partial x_1}{\partial \sigma} - Ke^{-r\tau}\frac{\partial \Phi(x_3)}{\partial x_3}\frac{\partial x_3}{\partial \sigma} \\ &= Se^{-q\tau}\phi(x_1)\left(\frac{\partial x_1}{\partial \sigma} - \frac{\partial x_3}{\partial \sigma}\right) \\ &= Se^{-q\tau}\sqrt{\tau}\phi(x_1). \end{aligned} \tag{B.2}$$

The following equation is utilized to obtain a simpler expression. By the help of the relation between the distribution functions, i.e. $x_3 = x_1 - \sigma\sqrt{\tau}$, we get the following

$$\frac{\partial x_1}{\partial \sigma} - \frac{\partial x_3}{\partial \sigma} = \sqrt{\tau}. \tag{B.3}$$

In addition, we have that

$$\frac{\partial x_1}{\partial \sigma} = \frac{-x_3}{\sigma} \quad \text{and} \quad \frac{\partial x_3}{\partial \sigma} = \frac{-x_1}{\sigma}. \tag{B.4}$$

We continue with the second order derivatives of the function A and we get

$$\begin{aligned}\frac{\partial^2 A}{\partial \sigma^2} &= \frac{\partial}{\partial \sigma}\left(\frac{\partial A}{\partial \sigma}\right) \\ &= Se^{-q\tau}\sqrt{\tau}\left(\frac{\partial \phi(x_1)}{\partial x_1}\frac{\partial x_1}{\partial \sigma}\right). \end{aligned} \tag{B.5}$$

It is necessary to present some useful equations to simplify this expression. To be able to proceed with the calculation we have to derive the second order derivative of the probability density function of the standard normal distribution. It is known that

$$\phi(x_1) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x_1^2}{2}}.$$

Thus, the aim is to calculate the derivative with respect to the variable x_1 which yields

$$\begin{aligned}\frac{\partial \phi(x_1)}{\partial x_1} &= \frac{1}{\sqrt{2\pi}}\frac{-2x_1}{2}e^{\frac{-x_1^2}{2}} \\ &= -x_1\phi(x_1). \end{aligned} \tag{B.6}$$

If we substitute Equations (B.4) and (B.6) into Equation (B.5), we get immediately the following

$$\frac{\partial^2 A}{\partial \sigma^2} = Se^{-q\tau}\sqrt{\tau}\phi(x_1)\frac{x_1 x_3}{\sigma}. \tag{B.7}$$

The last partial derivative for the function A is obtained as

$$\begin{aligned}
\frac{\partial^2 A}{\partial S \partial \sigma} &= \frac{\partial}{\partial S}\left(\frac{\partial A}{\partial \sigma}\right) \\
&= e^{-q\tau}\sqrt{\tau}\left(\phi(x_1) + S\phi(x_1)(-x_1)\frac{\partial x_1}{\partial S}\right) \\
&= e^{-q\tau}\sqrt{\tau}\phi(x_1)\left(1 - \frac{x_1}{\sigma\sqrt{\tau}}\right) \\
&= -e^{-q\tau}\sqrt{\tau}\phi(x_1)\frac{x_3}{\sigma}.
\end{aligned} \tag{B.8}$$

The following equation helped us to simplify the expression

$$\frac{\partial x_1}{\partial S} = \frac{\partial x_3}{\partial S} = \frac{1}{S\sigma\sqrt{\tau}}. \tag{B.9}$$

The next aim is to derive the partial derivatives of the function C which is obviously more complicated than the function A. We start with the first order derivative which yields

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} = Se^{-q\tau}&\left[\phi(y_1)\frac{\partial y_1}{\partial \sigma}\left(\frac{H}{S}\right)^{2\gamma+2} + \Phi(y_1)2\frac{\partial \gamma}{\partial \sigma}\left(\frac{H}{S}\right)^{2\gamma+2}\ln\left(\frac{H}{S}\right)\right] \\
- Ke^{-r\tau}&\left[\frac{S}{S}\phi(y_3)\frac{\partial y_3}{\partial \sigma}\left(\frac{H}{S}\right)^{2\gamma} + \Phi(y_3)2\frac{\partial \gamma}{\partial \sigma}\left(\frac{H}{S}\right)^{2\gamma}\ln\left(\frac{H}{S}\right)\right].
\end{aligned} \tag{B.10}$$

By the help of the following equation we are able to rearrange the terms. Moreover, we plug the $\frac{S}{S}$ as a multiplier to the first term which helps us to gather the terms involving $\phi(y_1)$ together. We have the following relation between $\phi(y_1)$ and $\phi(y_3)$

$$\phi(y_3) = \phi(y_1)\frac{H^2}{SK}e^{(r-q)\tau}. \tag{B.11}$$

Furthermore, if we arrange the terms, we see that the second part of the equation includes the function C again. Eventually, we get

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= Se^{-q\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left(\frac{\partial y_1}{\partial \sigma} - \frac{\partial y_3}{\partial \sigma}\right) + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)C \\
&= Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2} + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)C
\end{aligned} \tag{B.12}$$

Note that, the following equation also holds for the y_1 and y_3 functions

$$\frac{\partial y_1}{\partial \sigma} - \frac{\partial y_3}{\partial \sigma} = \sqrt{\tau}.$$

The next step is to calculate the second order derivatives of the function C. The partial derivative of C with respect to σ yields

$$\begin{aligned}\frac{\partial^2 C}{\partial\sigma^2} &= \frac{\partial}{\partial\sigma}\left(\frac{\partial C}{\partial\sigma}\right)\\ &= Se^{-q\tau}\sqrt{\tau}\left[\phi(y_1)(-y_1)\frac{\partial y_1}{\partial\sigma}\left(\frac{H}{S}\right)^{2\gamma+2} + 2\frac{\partial\gamma}{\partial\sigma}\ln\left(\frac{H}{S}\right)\left(\frac{H}{S}\right)^{2\gamma+2}\phi(y_1)\right]\\ &+ 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2\gamma}{\partial\sigma^2}C + \frac{\partial\gamma}{\partial\sigma}\frac{\partial C}{\partial\sigma}\right]\\ &= Se^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left[\frac{y_1 y_3}{\sigma} + 2\frac{\partial\gamma}{\partial\sigma}\ln\left(\frac{H}{S}\right)\right]\\ &+ 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2\gamma}{\partial\sigma^2}C + \frac{\partial\gamma}{\partial\sigma}\frac{\partial C}{\partial\sigma}\right].\end{aligned} \tag{B.13}$$

Note that, Equation (B.4) also holds for the functions y_1, y_3, i.e.

$$\frac{\partial y_1}{\partial\sigma} = \frac{-y_3}{\sigma} \quad\text{and}\quad \frac{\partial y_3}{\partial\sigma} = \frac{-y_1}{\sigma}.$$

Then, the last partial derivative of the function C reads as

$$\begin{aligned}\frac{\partial^2 C}{\partial S\partial\sigma} &= \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial\sigma}\right)\\ &= e^{-q\tau}\sqrt{\tau}\left[\phi(y_1)(-y_1)\frac{\partial y_1}{\partial S}\frac{H^{2\gamma+2}}{S^{2\gamma+1}} + \phi(y_1)(-2\gamma-1)\frac{H^{2\gamma+2}}{S^{2\gamma+2}}\right]\\ &+ 2\frac{\partial\gamma}{\partial\sigma}\left[\frac{-1}{S}C + \frac{\partial C}{\partial S}\ln\left(\frac{H}{S}\right)\right].\end{aligned} \tag{B.14}$$

Then by setting

$$\frac{\partial y_1}{\partial S} = \frac{\partial y_3}{\partial S} = \frac{-1}{S\sigma\sqrt{\tau}} \tag{B.15}$$

we get the following final form

$$\frac{\partial^2 C}{\partial S\partial\sigma} = e^{-q\tau}\sqrt{\tau}\phi(y_1)\left(\frac{H}{S}\right)^{2\gamma+2}\left(\frac{y_3}{\sigma\sqrt{\tau}} - 2\gamma\right) + 2\frac{\partial\gamma}{\partial\sigma}\left[\frac{-C}{S} + \frac{\partial C}{\partial S}\ln\left(\frac{H}{S}\right)\right]. \tag{B.16}$$

The remaining task is to compute the derivative of the function C with respect to the variable S which is equal to the *delta* of an option in the usual BS setting. Eventually, we obtain the following equation

$$\begin{aligned}\frac{\partial C}{\partial S} &= e^{-q\tau}\left[\phi(y_1)\frac{\partial y_1}{\partial S}\frac{H^{2\gamma+2}}{S^{2\gamma+1}} + \Phi(y_1)(-2\gamma-1)\frac{H^{2\gamma+2}}{S^{2\gamma+2}}\right]\\ &- Ke^{-r\tau}\left[\phi(y_3)\frac{\partial y_3}{\partial S}\frac{H^{2\gamma}}{S^{2\gamma}} + \Phi(y_3)(-2\gamma)\frac{H^{2\gamma}}{S^{2\gamma+1}}\right]\\ &= e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\left[\frac{-\phi(y_1)}{\sigma\sqrt{\tau}} + \Phi(y_1)(-2\gamma-1)\right]\\ &- \frac{K}{S}e^{-r\tau}\left(\frac{H}{S}\right)^{2\gamma}\left[\frac{-\phi(y_3)}{\sigma\sqrt{\tau}} + \Phi(y_3)(-2\gamma)\right].\end{aligned} \tag{B.17}$$

If we use the relation given in Equation (B.11), then the terms involving $\phi(y_1)$ cancel out. If we proceed with the derivation, we obtain

$$\begin{aligned}\frac{\partial C}{\partial S} &= -\frac{S}{S}e^{-q\tau}\Phi(y_1)(2\gamma+1)\left(\frac{H}{S}\right)^{2\gamma+2} + \frac{K}{S}e^{-r\tau}\left(\frac{H}{S}\right)^{2\gamma}\Phi(y_3)(2\gamma) \\ &= -e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\Phi(y_1) - 2\gamma\frac{C}{S}. \end{aligned} \tag{B.18}$$

By plugging the multiplier $\frac{S}{S}$ and rearranging the terms we finally obtain the simplified Equation (B.18). The first case is now completed. In the following, we consider the second case.

- Case: $K < H$

The required derivatives belong to the functions B and D, therefore we start with the function B. The first order derivative of B reads as

$$\begin{aligned}\frac{\partial B}{\partial \sigma} &= Se^{-q\tau}\phi(x_2)\frac{\partial x_2}{\partial \sigma} - Ke^{-r\tau}\phi(x_4)\frac{\partial y_3}{\partial S} \\ &= Se^{-q\tau}\phi(x_2)\left[\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right]. \end{aligned} \tag{B.19}$$

It is again possible to utilize the relation between $\phi(x_2)$ and $\phi(x_4)$ which is given by

$$\phi(x_4) = \phi(x_2)\frac{S}{H}e^{(r-q)\tau}. \tag{B.20}$$

Moreover, the derivatives of the functions x_2 and x_4 with respect to σ are equal to

$$\frac{\partial x_2}{\partial \sigma} = \frac{-x_4}{\sigma} \quad \text{and} \quad \frac{\partial x_4}{\partial \sigma} = \frac{-x_2}{\sigma}.$$

We proceed with the second order derivatives of the function B and obtain the following

$$\begin{aligned}\frac{\partial^2 B}{\partial \sigma^2} &= \frac{\partial}{\partial \sigma}\left(\frac{\partial B}{\partial \sigma}\right) \\ &= Se^{-q\tau}\left[\phi(x_2)(-x_2)\frac{\partial x_2}{\partial \sigma}\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) + \phi(x_2)\left(\frac{K}{H}\frac{(-x_4 - x_2)}{\sigma^2} - \frac{(-x_4 - x_2)}{\sigma^2}\right)\right] \\ &= Se^{-q\tau}\phi(x_2)\left[\frac{x_2 x_4}{\sigma}\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) + \left(\left(1 - \frac{K}{H}\right)\left(\frac{x_4 + x_2}{\sigma^2}\right)\right)\right]. \end{aligned} \tag{B.21}$$

We can equivalently write

$$\frac{\partial^2 B}{\partial \sigma^2} = \frac{\partial B}{\partial \sigma}\frac{x_2 x_4}{\sigma} + Se^{-q\tau}\phi(x_2)\left[\left(1 - \frac{K}{H}\right)\left(\frac{x_4 + x_2}{\sigma^2}\right)\right].$$

The last partial derivative of the function B is equal to

$$\begin{aligned}\frac{\partial^2 B}{\partial S \partial \sigma} &= \frac{\partial}{\partial S}\left(\frac{\partial B}{\partial \sigma}\right)\\ &= e^{-q\tau}\left[\phi(x_2)\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) + S\phi(x_2)(-x_2)\frac{\partial x_2}{\partial S}\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right)\right.\\ &\quad \left. + S\phi(x_2)\left(\frac{K}{H}\frac{1}{\sigma}\frac{\partial x_2}{\partial S} - \frac{1}{\sigma}\frac{\partial x_4}{\partial S}\right)\right]\\ &= e^{-q\tau}\phi(x_2)\left[\frac{-x_4}{\sigma\sqrt{\tau}}\left(\frac{K}{H}\frac{x_2}{\sigma} - \frac{x_4}{\sigma}\right) + \frac{1}{\sigma^2\sqrt{\tau}}\left(\frac{K}{H} - 1\right)\right]. \end{aligned} \tag{B.22}$$

Note that, we also have

$$\frac{\partial x_2}{\partial S} = \frac{\partial x_4}{\partial S} = \frac{1}{S\sigma\sqrt{\tau}}. \tag{B.23}$$

Finally, we deal with the derivatives of the function D and obtain the following

$$\begin{aligned}\frac{\partial D}{\partial \sigma} &= Se^{-q\tau}\left[\phi(y_2)\frac{\partial y_2}{\partial \sigma}\left(\frac{H}{S}\right)^{2\gamma+1} + \Phi(y_2)2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\left(\frac{H}{S}\right)^{2\gamma+2}\right]\\ &\quad - Ke^{-r\tau}\left[\phi(y_4)\frac{\partial y_4}{\partial \sigma}\left(\frac{H}{S}\right)^{2\gamma} + \Phi(y_4)2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\left(\frac{H}{S}\right)^{2\gamma}\right]. \end{aligned} \tag{B.24}$$

In order to reorganize the terms we use the equations

$$\phi(y_2) = \phi(y_1)\frac{H}{S}e^{(r-q)\tau} \tag{B.25}$$

and

$$\frac{\partial y_2}{\partial \sigma} = \frac{-y_4}{\sigma} \quad \text{and} \quad \frac{\partial y_4}{\partial \sigma} = \frac{-y_2}{\sigma}.$$

We also consider the expression $\left(\frac{H}{S}\right)^{2\gamma+2} = \frac{H}{S}\left(\frac{H}{S}\right)^{2\gamma+1}$ to be able to aggregate the terms with respect to $\phi(y_2)$. Hence, we obtain the following simplified form

$$\begin{aligned}\frac{\partial D}{\partial \sigma} &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(H\frac{\partial y_2}{\partial \sigma} - K\frac{\partial y_4}{\partial \sigma}\right) + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)D\\ &= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right) + 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)D. \end{aligned} \tag{B.26}$$

We consider the second order derivative of the D function and get the following

$$\begin{aligned}
\frac{\partial^2 D}{\partial \sigma^2} &= \frac{\partial}{\partial \sigma}\left(\frac{\partial D}{\partial \sigma}\right) \\
&= e^{-q\tau}\Bigg[\phi(y_2)(-y_2)\frac{\partial y_2}{\partial \sigma}\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{\partial y_2}{\partial \sigma} - H\frac{\partial y_4}{\partial \sigma}\right) \\
&+ 2\frac{\partial \gamma}{\partial \sigma}\ln\left(\frac{H}{S}\right)\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{\partial y_2}{\partial \sigma} - H\frac{\partial y_4}{\partial \sigma}\right) \\
&+ \left(\frac{H}{S}\right)^{2\gamma+1}\phi(y_2)\left(K\frac{(-y_4-y_2)}{\sigma^2} - H\frac{(-y_4-y_2)}{\sigma^2}\right)\Bigg] + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}D + \frac{\partial \gamma}{\partial \sigma}\frac{\partial D}{\partial \sigma}\right] \\
&= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\Bigg[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_2 y_4}{\sigma} + 2\ln\left(\frac{H}{S}\right)\frac{\partial \gamma}{\partial \sigma}\right) \\
&+ (H-K)\frac{y_2+y_4}{\sigma^2}\Bigg] + 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}D + \frac{\partial \gamma}{\partial \sigma}\frac{\partial D}{\partial \sigma}\right].
\end{aligned} \tag{B.27}$$

The last part of the derivations for the case $B - D$ deals with the second order derivative of the function D. Thus, we get the following

$$\begin{aligned}
\frac{\partial^2 D}{\partial S \partial \sigma} &= \frac{\partial}{\partial S}\left(\frac{\partial D}{\partial \sigma}\right) \\
&= e^{-q\tau}\Bigg[\phi(y_2)(-y_2)\frac{\partial y_2}{\partial S}\left(\frac{H}{S}\right)^{2\gamma+1}\left(K\frac{\partial y_2}{\partial \sigma} - H\frac{\partial y_4}{\partial \sigma}\right) \\
&+ \phi(y_2)(-2\gamma-1)\frac{H^{2\gamma+1}}{S^{2\gamma+2}}\left(K\frac{\partial y_2}{\partial \sigma} - H\frac{\partial y_4}{\partial \sigma}\right) + \phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left(\frac{K}{\sigma}\frac{\partial y_2}{\partial S} - \frac{H}{\sigma}\frac{\partial y_4}{\partial S}\right)\Bigg] \\
&+ 2\ln\left(\frac{H}{S}\right)\left[\frac{\partial^2 \gamma}{\partial \sigma^2}D + \frac{\partial \gamma}{\partial \sigma}\frac{\partial D}{\partial \sigma}\right] \\
&= e^{-q\tau}\phi(y_2)\left(\frac{H}{S}\right)^{2\gamma+1}\left[\left(K\frac{y_2}{\sigma} - H\frac{y_4}{\sigma}\right)\left(\frac{y_4}{S\sigma\sqrt{\tau}} - \frac{2\gamma}{S}\right) - \left(\frac{K-H}{\sigma^2 S\sqrt{\tau}}\right)\right] \\
&+ 2\frac{\partial \gamma}{\partial \sigma}\left[\frac{-D}{S} + \ln\left(\frac{H}{S}\right)\frac{\partial D}{\partial S}\right].
\end{aligned} \tag{B.28}$$

Note that, the following equation holds for the functions y_2 and y_4

$$\frac{\partial y_2}{\partial S} = \frac{\partial y_4}{\partial S} = \frac{-1}{S\sigma\sqrt{\tau}}. \tag{B.29}$$

To complete the derivation of the Greeks of the down-and-out call option it remains only one more partial derivative. We derive the term $\frac{\partial D}{\partial S}$ as

$$\begin{aligned}
\frac{\partial D}{\partial S} &= e^{-q\tau}\left[\phi(y_2)\frac{\partial y_2}{\partial S}\frac{H^{2\gamma+2}}{S^{2\gamma+1}} + \Phi(y_2)(-2\gamma-1)\frac{H^{2\gamma+2}}{S^{2\gamma+2}}\right] \\
&- Ke^{-r\tau}\left[\phi(y_4)\frac{\partial y_4}{\partial S}\frac{H^{2\gamma}}{S^{2\gamma}} + \Phi(y_4)(-2\gamma)\frac{H^{2\gamma}}{S^{2\gamma}}\right] \\
&= e^{-q\tau}\left(\frac{H}{S}\right)^{2\gamma+2}\left[(-2\gamma-1)\Phi(y_2) - \frac{\phi(y_2)}{\sigma\sqrt{\tau}}\right] \\
&- \frac{K}{S}\left(\frac{H}{S}\right)^{2\gamma} e^{-r\tau}\left[(-2\gamma)\Phi(y_4) - \frac{\phi(y_2)}{\sigma\sqrt{\tau}}\right].
\end{aligned} \tag{B.30}$$

As a result, we derived the Greeks for down-and-out call option. The remaining Greeks of the other types of one-sided knockout options can be derived in a similar way. In general, the variation in the derivations are due to the negative signs involved in the distribution functions. If one pays enough attention to the sign changes whilst the derivation, then one can obtain the remaining formulas for the Greeks in a similar manner.

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