

A Reduction Result for Planar Location Problems with Polygonal Barriers

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Abstract

In this paper we consider the problem of locating one new facility in the plane with respect to a given set of existing facilities where a set of polygonal barriers restricts traveling. This non-convex optimization problem can be reduced to a finite set of convex subproblems if the objective function is a convex function of the travel distances between the new and the existing facilities (like e.g. the Median and Center objective functions). An exact Algorithm and a heuristic solution procedure based on this reduction result are developed.

1 Introduction

In times of increasing transportation costs and just-in-time delivery schedules, good locational decisions are needed in many different fields. The location of a warehouse with respect to a given set of customers or the location of an emergency facility in an expanding neighborhood are only two examples for a wide range of applications.

The development of realistic location models is a crucial step in every locational decision process. Especially in the case of planar location models we deal with a geometric representation of the problem, and the geographical reality has to be incorporated into this representation. Restrictions of different types occur in almost every real-world location problem since there are in general regions to exclude from placement of new facilities. These regions can also often not be used for transportation which can be modeled by the introduction of barrier regions in the plane \mathbb{R}^2 . To give only some examples of possible barrier regions, consider military regions, mountain ranges, lakes, big rivers or highways, or, on a smaller scale, conveyor belts in an industrial plant.

The increasing interest in location models incorporating restrictions and barrier regions is reflected in the recent literature. Katz and Cooper 1981 [7] developed a heuristic for the

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Median problem in case that one circular barrier is given and distances are measured with the Euclidean distance function.

Likewise for the Median problem Aneja and Parlar 1994 [1] and more recently Butt and Cavalier 1996 [3] developed heuristics for the case that the barriers are closed polygons and the distance is given by the l_p -metric. In the special case of the Manhattan metric l_1 discretization results were proven by Larson and Sadiq 1983 [10] and by Batta, Ghose and Palekar 1989 [2] for arbitrarily shaped barriers. A similar discretization was derived for a more general class of distance functions in Hamacher and Klamroth 1997 [5], namely the class of block norms.

Klamroth 1996 [8] considered the Median problem for the case that the barrier is a line with a finite number of passages. A reduction of the non-convex original problem to a polynomial number of unrestricted Median problems was given for any metric derived from a norm. This approach was extended to the multiple criteria case in Klamroth and Wiecek 1998 [9].

In this paper we develop a reduction result that implies a general solution strategy for location problems with polygonal barriers. We consider objective functions that are convex functions of distances between a set of existing facilities and one new facility. This definition of the objective function includes the well known Median (Weber) and Center objective functions as well as ordered Weber objective functions.

Let a finite set of convex, closed, polygonal and pairwise disjoint barriers $\{B_1, \dots, B_N\}$ be given in \mathbb{R}^2 , representing those regions in the plane where neither trespassing nor location of new facilities is allowed. We denote the union of these barrier regions by $\mathcal{B} := \bigcup_{i=1}^N B_i$ and the sets of extreme points and facets of \mathcal{B} by $\mathcal{P}(\mathcal{B})$ and $\mathcal{F}(\mathcal{B})$, respectively, where $\mathcal{P}(\mathcal{B}) := \{P_i : i = 1, \dots, P\}$ and $\mathcal{F}(\mathcal{B}) := \{F_i : i = 1, \dots, Q\}$. The feasible region F for new locations is given by

$$F := \mathbb{R}^2 \setminus \text{int}(\mathcal{B}).$$

A finite set of existing facilities $\mathcal{E}x := \{Ex_m \in F : m \in \mathcal{M}\}$, $\mathcal{M} = \{1, \dots, M\}$ is given in a connected subset of the feasible region F .

Furthermore we assume that a distance measure d derived from a norm $\|\bullet\|_d$ is given by $d(X, Y) = \|Y - X\|_d$ for all $X, Y \in \mathbb{R}^2$. Taking the restriction due to the barrier regions into account, we can find the corresponding barrier distance function $d_{\mathcal{B}}(X, Y)$ for two points $X, Y \in F$ as the length of a shortest path between X and Y not intersecting the interior of a barrier. More formally, let P be a *permitted X-Y-path* in F , i.e. a curve connecting X and Y not intersecting the interior of a barrier. Furthermore, let p be a piecewise continuous differentiable parameterization of P , $p : [a, b] \rightarrow \mathbb{R}^2$ with $a, b \in \mathbb{R}$, $a < b$, $p(a) = X$, $p(b) = Y$ and $p([a, b]) \cap \text{int}(\mathcal{B}) = \emptyset$. Then $d_{\mathcal{B}}(X, Y)$ can be defined as

$$d_{\mathcal{B}}(X, Y) := \min \left\{ \int_a^b \|p'(t)\|_d dt : P \text{ permitted } X\text{-}Y\text{-path} \right\}. \quad (1)$$

A permitted X - Y -path with length $d_{\mathcal{B}}(X, Y)$ will be called a *d-shortest permitted X-Y-path*.

Additionally, we call two points X and Y in F *d-visible* if they satisfy $d_{\mathcal{B}}(X, Y) = d(X, Y)$, i.e. the distance between X and Y is not lengthened by the barrier regions.

Note that for $d_{\mathcal{B}}$ the triangle inequality is satisfied (provided it holds for the original distance function d , which is guaranteed by the fact that d is derived from a norm), but that $d_{\mathcal{B}}$ is in general not positively homogeneous.

Using the distance measure $d_{\mathcal{B}}$ as defined in (1), we consider the following general location problem:

$$\begin{aligned} \min \quad & f_{\mathcal{B}}(X) = f(d_{\mathcal{B}}(X, Ex_1), \dots, d_{\mathcal{B}}(X, Ex_M)) \\ \text{s.t.} \quad & X \in F, \end{aligned} \tag{2}$$

where f is any convex function with respect to the barrier distance $d_{\mathcal{B}}$ between the new facility X and the existing facilities in $\mathcal{E}x$. Well known examples are the Median objective function $f_{\mathcal{B}}(X) = \sum_{m \in \mathcal{M}} w_m d_{\mathcal{B}}(X, Ex_m)$ and the Center objective function $g_{\mathcal{B}}(x) = \max_{m \in \mathcal{M}} w_m d_{\mathcal{B}}(X, Ex_m)$ where the positive weights w_m represent the demand of the facility Ex_m , $m \in \mathcal{M}$.

Note that the barrier distance function $d_{\mathcal{B}}$ is in general not convex and that therefore $f_{\mathcal{B}}$ is also in general not convex.

To simplify further notation we will use the classification (*Pos1/Pos2/Pos3/Pos4/Pos5*) of location problems as introduced in Hamacher 1995 [4] or Hamacher and Nickel 1996 [6]. Following their notation, problem (2) is classified as (*1/P/B/d_B/f convex*), where *Pos1* gives the number of new facilities sought (1 for a single-facility problem), *Pos2* denotes the type of location problem (*P* for planar location problems), *Pos3* contains special assumptions (*B* for barrier regions), *Pos4* contains the information about the distance function ($d_{\mathcal{B}}$ in case of barrier distances) and *Pos5* indicates the objective function, which in this case is a convex function of the distances between the new and the existing facilities (*f convex*).

In the following section some basic properties and concepts related to shortest paths in the presence of polygonal barriers will be derived. In Section 3 a reduction result is developed that interrelates location problems with polygonal barriers with a set of unrestricted location problems. The algorithmic consequences of this result are discussed in Section 4, and the paper is concluded with Section 5.

2 Shortest paths in the presence of barriers

In this section we focus on shortest permitted paths in the presence of barriers where the set of barriers \mathcal{B} consists of pairwise disjoint convex polygonal sets.

Let d be a given distance function derived from a norm $\|\bullet\|_d$. The set of points $Y \in F$ that are not d -visible from a point $X \in F$ is called the *shadow of X with respect to d* , i.e.

$$\text{shadow}_d(X) := \{Y \in F : d_{\mathcal{B}}(X, Y) > d(X, Y)\}.$$

In Figure 1 two examples are given for the Euclidean metric l_2 and the Manhattan metric l_1 , respectively.

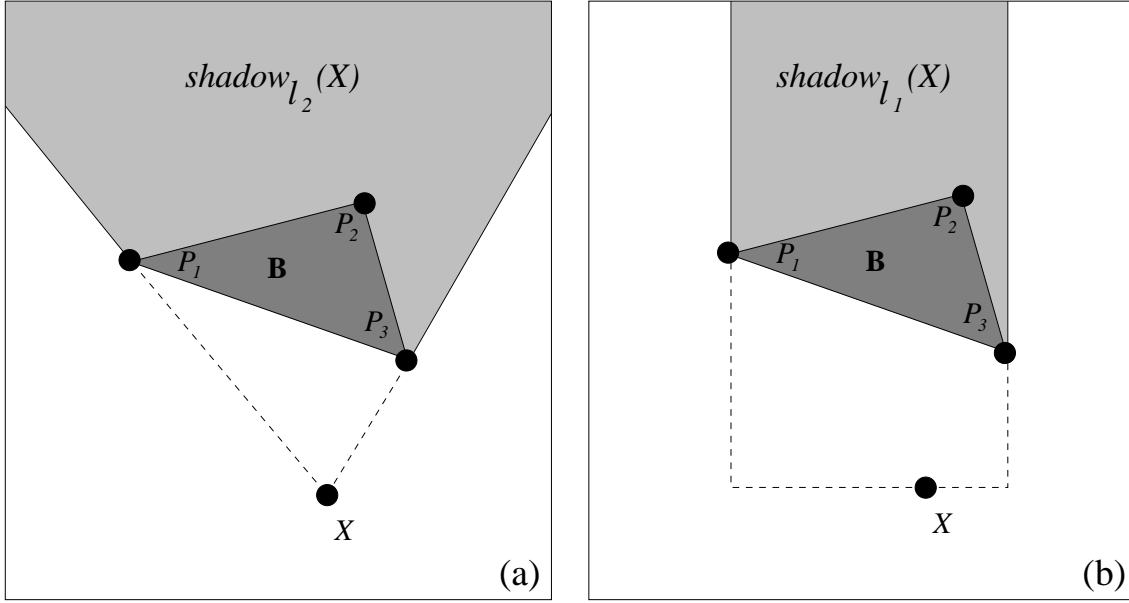


Figure 1: Part (a) depicts $shadow_{l_2}(X)$ whereas part (b) shows $shadow_{l_1}(X)$.

Note that for some choices of d a point that is d -visible may not be l_2 -visible, i.e. not visible in the usual sense of straight line visible. On the other hand every pair of l_2 -visible points is also d -visible if d is a distance function derived from a norm. This result is a generalization of an earlier result in [5] where it was proven for symmetric polyhedral gauges.

Lemma 1 *Let d be a distance function derived from a norm. Then*

$$shadow_d(X) \subseteq shadow_{l_2}(X), \quad X \in F.$$

Furthermore if $X, Y \in F$ are l_2 -visible, $X \neq Y$, then the straight line segment connecting X and Y is a d -shortest permitted path with length $d(X, Y)$.

Proof: Wlog let $X = 0$ be the origin and let $Y \in F$ be a point that is l_2 -visible from X . Then the straight-line segment connecting X and Y is a permitted path P given by the parameterization $p : [0, 1] \rightarrow \mathbb{R}^2$, $p(t) = t \cdot Y$, $t \in [0, 1]$. Using (1), the length of P can be calculated as

$$\begin{aligned} d_{\mathcal{B}}(0, Y) &\leq \int_0^1 \|p'(t)\|_d dt = \int_0^1 \left\| \frac{d}{dt}(tY) \right\|_d dt = \int_0^1 \|Y\|_d dt = \|Y\|_d \\ &= d(0, Y). \end{aligned}$$

□

In the case that distances are measured by an l_p -metric, $1 \leq p \leq \infty$ Viegas and Hansen 1985 [11] showed that for any two points $X, Y \in F$, $X \neq Y$ there always exists an l_p -shortest permitted path connecting X and Y that is a piecewise linear path with breaking points only in extreme points of a barrier. This property was generalized for polyhedral gauges in [5]. The following result shows that it also holds for any other distance function d that is derived from a norm.

Lemma 2 *Let d be a distance function derived from a norm and let $X, Y \in F$. Then there exists a d -shortest permitted path SP connecting X and Y with the following property.*

Property 1: SP is a piecewise linear path with breaking points only in extreme points of barriers.

Proof: Let $X, Y \in F$ and let SP be any d -shortest permitted path connecting X and Y in F that does not satisfy Property 1. Note that, since the set of barriers and correspondingly the set of extreme points $\mathcal{P}(\mathcal{B})$ of barriers is finite, SP can be partitioned by a finite set of points so that two consecutive points on SP are l_2 -visible. Lemma 1 therefore implies that the straight line segment connecting two consecutive points on SP is a d -shortest permitted path connecting these two points. We can therefore construct a piecewise linear path SP' with a finite set of breaking points that has the same length as SP . A d -shortest permitted path SP'' with Property 1 can be constructed from SP' similar to the construction given in [11] for l_p -distances:

Let $[T_{i-1}, T_i]$ and $[T_i, T_{i+1}]$ be two consecutive straight line segments of SP' . First assume that T_{i-1} and T_{i+1} are l_2 -visible. Then the two segments $[T_{i-1}, T_i]$ and $[T_i, T_{i+1}]$ can be replaced by one straight line segment $[T_{i-1}, T_{i+1}]$ without increasing the length of SP' . Otherwise, using again Lemma 1, the breaking point T_i can be moved along $[T_{i-1}, T_i]$ or along $[T_i, T_{i+1}]$ towards T_{i-1} or T_{i+1} , respectively, without increasing the length of SP' , until one of these line segments becomes tangent to a barrier.

While iterating both operations every extreme point of a barrier which lies on SP' is interpreted as a breaking point T_i even if the straight line segment $[T_{i-1}, T_{i+1}]$ is part of SP' . Thus the iteration of both operations yields a path SP'' with the desired property after a finite number of steps since every breaking point of SP' which is not yet an extreme point of a barrier can be moved towards X , Y , or an extreme point of a barrier.

□

An immediate consequence of Lemma 2 is that the barrier distance $d_{\mathcal{B}}(X, Y)$, $X, Y \in F$ can be calculated with respect to a so-called *intermediate point* $I_{X,Y} \neq Y$, i.e. a breaking point on a d -shortest permitted X - Y -path with Property 1 so that $I_{X,Y}$ is d -visible from Y . (Note that in case that X and Y are d -visible the intermediate point $I_{X,Y}$ equals X .)

Corollary 1 *Let d be a distance function derived from a norm and let $X, Y \in F$. Furthermore let SP be a d -shortest permitted X - Y -path with Property 1 and let the point $I_{X,Y} \neq Y$ be a breaking point on SP that is d -visible from Y . Then*

$$d_{\mathcal{B}}(X, Y) = d_{\mathcal{B}}(X, I_{X,Y}) + d(I_{X,Y}, Y).$$

Note that the intermediate points $I_{X,Y}$ are not necessarily unique. Furthermore, as a result of Lemma 2, an intermediate point $I_{X,Y}$ can always be chosen such that it is not only d -visible from Y , but also l_2 -visible from Y .

A visibility graph as proposed in Butt and Cavalier 1996 [3] can be used to determine distances between the existing facilities and all those points that are candidates for intermediate points on a d -shortest permitted path between an existing facility and a point $X \in F$. The node set of this visibility graph G is given by $V(G) := \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$. Two nodes $v_i, v_j \in V(G)$ are connected by an edge of length $d(v_i, v_j)$ if the corresponding points in the plane are d -visible and have distance $d(v_i, v_j)$. In Figure 2 an example is given for the case that distances are measured by the Manhattan metric l_1 .

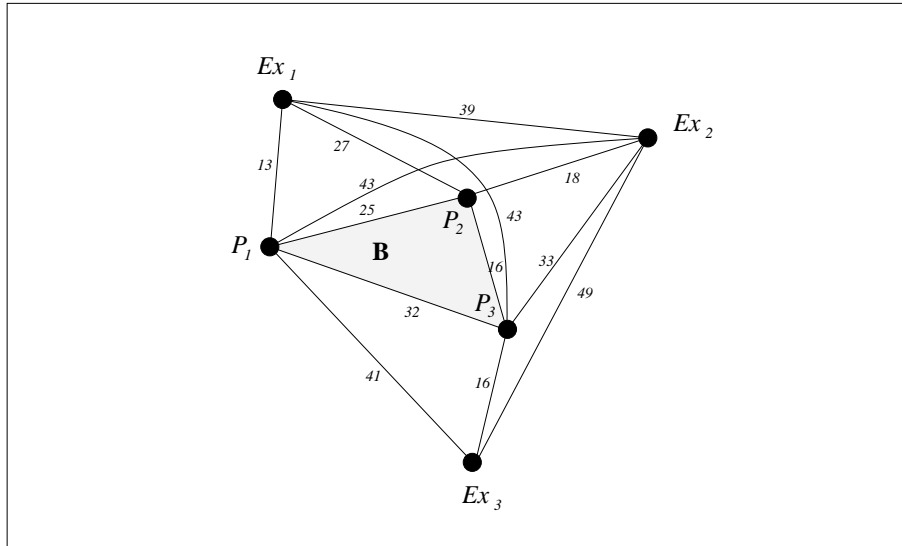


Figure 2: The visibility graph for an example problem where distances are measured with respect to l_1 .

The barrier distance $d_{\mathcal{B}}(Ex_m, X)$ between an existing facility $Ex_m \in \mathcal{E}x$ and a point $X \in F$ can now be calculated as

$$d_{\mathcal{B}}(Ex_m, X) = d_G(Ex_m, I_{Ex_m, X}) + d(I_{Ex_m, X}, X), \quad (3)$$

where $d_G(Ex_m, I_{Ex_m, X})$ denotes the length of a shortest path between Ex_m and the intermediate point $I_{Ex_m, X}$ in the visibility graph G .

Another consequence of Lemmas 1 and 2 is that the boundary of $shadow_d(X)$,

$$\begin{aligned} \partial(shadow_d(X)) &:= \{Y \in F : N_{\varepsilon}(Y) \cap shadow_d(Y) \neq \emptyset \\ &\quad \text{and } N_{\varepsilon}(Y) \not\subseteq shadow_d(Y) \ \forall \varepsilon > 0\}, \end{aligned}$$

where $N_{\varepsilon}(Y) := \{Z \in \mathbb{R}^2 : d(Z, Y) < \varepsilon\}$, is piecewise linear for any distance function d that is derived from a norm. Therefore $shadow_d(X)$ has a simple analytic representation for all $X \in F$.

Obviously those parts of $\partial(\text{shadow}_d(X))$ that are part of the boundary of a barrier region are piecewise linear. For all other parts of $\partial(\text{shadow}_d(X))$, consider a point Y on $\partial(\text{shadow}_d(X))$ and let $I_{X,Y}$ be an intermediate point on a d -shortest permitted X - Y -path with Property 1. Note that in this case Y is d -visible from X . If all the points Z on the line-segment starting at $I_{X,Y}$, passing through Y and ending as soon as it intersects the interior of a barrier are d -visible from X and thus not in $\text{shadow}_d(X)$, then $\partial(\text{shadow}_d(X))$ is piecewise linear. To simplify further discussion assume wlog that $X = 0$ is the origin.

Lemma 3 *Let d be a distance function derived from a norm. Furthermore let $Y \in F$ be a point that is d -visible from the origin and let $I := I_{0,Y}$ be an intermediate point on a d -shortest permitted 0 - Y -path with Property 1. Let $Z = I + \lambda(Y - I)$, $\lambda \geq 0$ be a point in F such that Z is l_2 -visible from I .*

Then Z is d -visible from the origin.

Proof: First assume that Y is l_2 -visible from the origin. Then $I = 0$ and thus Z is also l_2 -visible and d -visible from the origin.

Now consider the case that Y is not l_2 -visible from the origin and thus $I \neq 0$. Then $d(0, I) + d(I, Y) = d(0, Y)$ since I is a point on a d -shortest permitted 0 - Y -path.

Assume that there exist $\lambda, \mu \in [0, 1]$ such that $d(0, Z) < \lambda d(0, I) + \mu d(I, Y)$ where $Z = \lambda I + \mu(Y - I)$. Using the triangle inequality we obtain

$$\begin{aligned} d(0, Y) &= d(0, \lambda I + (1 - \lambda)I + \mu(Y - I) + (1 - \mu)(Y - I)) \\ &\leq d(0, Z) + (1 - \lambda)d(0, I) + (1 - \mu)d(I, Y) \\ &< \lambda d(0, I) + \mu d(I, Y) + (1 - \lambda)d(0, I) + (1 - \mu)d(I, Y) \\ &= d(0, I) + d(I, Y), \end{aligned}$$

contradicting the assumption that I is a point on a d -shortest 0 - Y -path.

Thus $d(0, Z) = \lambda d(0, I) + \mu d(I, Y)$ for all $\lambda, \mu \in [0, 1]$, which, using $\lambda = 1$, proves the result for all $\mu \in [0, 1]$.

The remaining case is that $\lambda = 1$ but $\mu > 1$, i.e. that $Z = I + \mu(Y - I)$, $\mu > 1$. Assume that there exists $\mu > 1$ such that $d(0, Z) < d(0, I) + \mu d(I, Y)$. It follows that

$$\begin{aligned} \mu d(0, \frac{1}{\mu}I + (Y - I)) &< d(0, I) + \mu d(I, Y) \\ \Leftrightarrow d(0, \frac{1}{\mu}I + (Y - I)) &< d(0, \frac{1}{\mu}I) + d(I, Y) \end{aligned}$$

which completes the proof using the inequalities derived for the previously discussed case. \square

Lemma 3 implies that the boundary of $\text{shadow}_d(X)$ is piecewise linear for all points $X \in F$. Note that $\text{shadow}_d(X)$ is nevertheless not necessarily convex as can be seen in Figure 1.

Corollary 2 *Let d be a distance function derived from a norm and let $X \in F$ be a feasible point. Then $\partial(\text{shadow}_d(X))$ is piecewise linear.*

3 Reducing the non-convex barrier problem to a set of convex location problems

For the Median problem with barriers and the Euclidean distance function Butt and Cavalier [3] proposed a partitioning of the feasible region into a finite set of subregions $R_k \subseteq F$ such that the shortest barrier distance from every point $X \in R_k$ to all of the existing facilities in $Ex_m \in \mathcal{E}x$ can be calculated with respect to the same intermediate points $I_{Ex_m, X}$, $m \in \mathcal{M}$. Using this partitioning an optimal solution to the original problem can be found by solving a finite set of convex subproblems on each of the subregions R_k . Since this approach is not efficient in practice due to the nonlinearity of the boundaries of the regions R_k (their determination is difficult especially as the number of barrier regions and existing facilities increases) a heuristic method is suggested in [3]. Starting with some initial solution $X \in F$, the procedure iteratively solves unrestricted Median problems with respect to the intermediate points corresponding to the current solution.

A different partitioning of the feasible region is suggested in this paper which has two major advantages. First it uses a smaller number of subregions and second the boundaries of the subregions are all piecewise linear. A major drawback though is that the objective function is not necessarily convex on each of the subregions. We will prove a reduction result that nevertheless implies an exact solution procedure using this grid tessellation of the feasible region.

Consider the grid \mathcal{G}_d in the plane that is defined by the boundaries of the shadows of all existing facilities and all extreme points of the barrier regions, i.e.

$$\mathcal{G}_d := \left(\bigcup_{X \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})} \partial(\text{shadow}_d(X)) \right) \cup \mathcal{F}(\mathcal{B}). \quad (4)$$

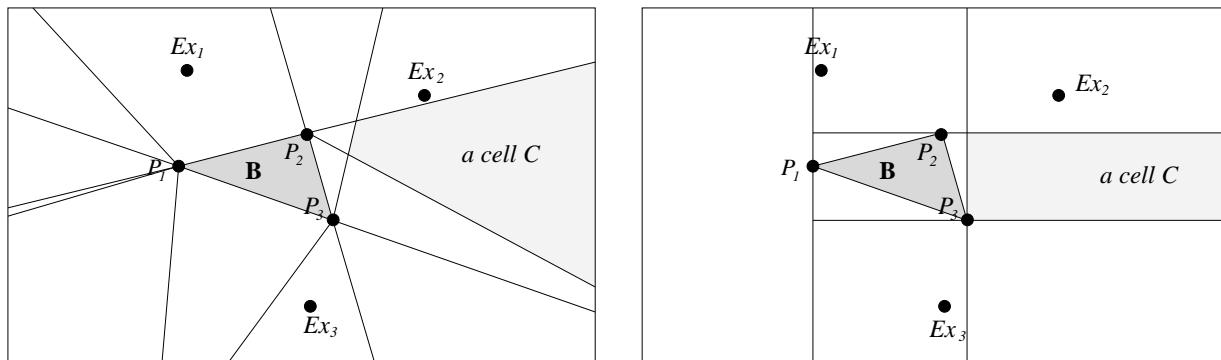


Figure 3: The grids \mathcal{G}_{l_2} and \mathcal{G}_{l_1} , respectively, for the example problem.

Since the barriers are convex polygons and since the boundary of $\text{shadow}_d(X)$ is piecewise linear for all $X \in F$ (Corollary 2), the grid \mathcal{G}_d consists of a finite set of line segments in F .

The set of *cells of \mathcal{G}_d* , i.e. the set of smallest (not necessarily convex or closed) polyhedra not intersected by a line segment in \mathcal{G}_d , is denoted by $\mathcal{C}(\mathcal{G}_d)$.

Similar to the representation of the barrier distance with respect to intermediate points in Corollary 1, the objective function $f_{\mathcal{B}}(X)$ can be rewritten for every point X in a cell $C \in \mathcal{C}(\mathcal{G}_d)$.

Corollary 3 *Let $C \in \mathcal{C}(\mathcal{G}_d)$ be a cell and let $X \in C$ be a feasible solution of (1/P/B/ $d_{\mathcal{B}}$ /f convex). Then*

$$f_{\mathcal{B}}(X) = f_X(X) + g_X, \quad (5)$$

where

$$f_X(Y) := f(d(Y, I_1), \dots, d(Y, I_m)), \quad Y \in \mathbb{R}^2, \quad (6)$$

$$\begin{aligned} g_X &:= f(d_{\mathcal{B}}(I_1, Ex_1), \dots, d_{\mathcal{B}}(I_m, Ex_m)) \\ &= f(d_G(I_1, Ex_1), \dots, d_G(I_m, Ex_m)) \end{aligned} \quad (7)$$

and $I_m := I_{Ex_m, X} \neq X$ ($m \in \mathcal{M}$) is an intermediate point on a d -shortest permitted X - Ex_m -path with Property 1 that is d -visible from X .

Note that $f_X(Y)$ is convex in \mathbb{R}^2 since it does not depend on the non-convex distance function $d_{\mathcal{B}}$. Furthermore g_X is a constant not depending on the choice of Y . Therefore $f_X(Y) + g_X$ is a convex function of Y in \mathbb{R}^2 .

The reformulation of the objective function $f_{\mathcal{B}}$ given in Corollary 3 will be used in the following to interrelate the non-convex problem (1/P/B/ $d_{\mathcal{B}}$ /f convex) to a finite set of corresponding convex problems of type (1/P/•/ d /f convex).

Theorem 1 *Let $C \in \mathcal{C}(\mathcal{G}_d)$ be a cell and let $X_{\mathcal{B}}^* \in \text{int}(C)$ be an optimal solution of (1/P/B/ $d_{\mathcal{B}}$ /f convex). Then $X_{\mathcal{B}}^*$ is an optimal solution of the corresponding convex problem*

$$\begin{aligned} \min \quad & f_{X_{\mathcal{B}}^*}(Y) + g_{X_{\mathcal{B}}^*} \\ \text{s.t.} \quad & Y \in \mathbb{R}^2, \end{aligned} \quad (8)$$

where $f_{X_{\mathcal{B}}^*}(Y)$ and $g_{X_{\mathcal{B}}^*}$ are defined with respect to (6) and (7), respectively.

Proof: Let $X_{\mathcal{B}}^* \in \text{int}(C)$ and let $f_{X_{\mathcal{B}}^*}(Y)$ and $g_{X_{\mathcal{B}}^*}$ be defined with respect to (6) and (7). Furthermore let $I_m \neq X_{\mathcal{B}}^*$, $m \in \mathcal{M}$ be the corresponding intermediate points on d -shortest permitted $X_{\mathcal{B}}^*$ - Ex_m -paths with Property 1 that are d -visible from $X_{\mathcal{B}}^*$. Since $\text{int}(C)$ is not intersected by the boundary of the shadow of any candidate for an intermediate point (i.e. an existing facility or an extreme point of a barrier), the intermediate points I_m , $m \in \mathcal{M}$ are d -visible for all points $Y \in C$. Thus the inequality

$$f_{\mathcal{B}}(Y) = f_Y(Y) + g_Y \leq f_{X_{\mathcal{B}}^*}(Y) + g_{X_{\mathcal{B}}^*} \quad (9)$$

holds for all $Y \in C$. Assume that there exists a point $Y^* \in C$ such that

$$f_{X_B^*}(Y^*) + g_{X_B^*} < f_{X_B^*}(X_B^*) + g_{X_B^*}.$$

Using (9), we can calculate that

$$\begin{aligned} f_B(Y^*) &= f_{Y^*}(Y^*) + g_{Y^*} \leq f_{X_B^*}(Y^*) + g_{X_B^*} \\ &< f_{X_B^*}(X_B^*) + g_{X_B^*} = f_B(X_B^*), \end{aligned}$$

contradicting the optimality of X_B^* .

Using the fact that $f_{X_B^*}(Y) + g_{X_B^*}$ is a convex function of Y in \mathbb{R}^2 and that $X_B^* \in \text{int}(C)$, we can conclude that X_B^* minimizes $f_{X_B^*}(Y) + g_{X_B^*}$ in \mathbb{R}^2 . □

Theorem 1 implies that any problem of type $(1/P/\mathcal{B}/d_B/f \text{ convex})$ can be reduced to a finite set of convex subproblems within each cell in $\mathcal{C}(\mathcal{G}_d)$ even though the original objective function $f_B(X)$ is in general non-convex within the cells.

Note that Theorem 1 can be generalized to the case that the objective function $f_B(X)$ is a non-convex function of the barrier distances. Nevertheless in this case the resulting subproblems are also non-convex and the problem difficulty is not reduced as in the convex case.

In some applications it may be beneficial to consider the grid \mathcal{G}_{l_2} instead of the grid \mathcal{G}_d for a given distance function d , especially in the case that the construction of \mathcal{G}_{l_2} is simpler than that of \mathcal{G}_d . This is possible for any distance function d derived from a norm since Lemmas 1 and 2 imply the following reformulation of Theorem 1:

Corollary 4 *Let d be a distance function derived from a norm. Furthermore, let $C \in \mathcal{C}(\mathcal{G}_{l_2})$ be a cell in the grid \mathcal{G}_{l_2} and let $X_B^* \in \text{int}(C)$ be an optimal solution of $(1/P/\mathcal{B}/d_B/f \text{ convex})$. Then X_B^* is an optimal solution of the corresponding problem*

$$\begin{aligned} \min \quad & f_{X_B^*}(Y) + g_{X_B^*} \\ \text{s.t.} \quad & Y \in \mathbb{R}^2, \end{aligned} \tag{10}$$

where $f_{X_B^*}(Y)$ and $g_{X_B^*}$ are defined with respect to (6) and (7), respectively, and the intermediate points I_m ($m \in \mathcal{M}$) are chosen such that they are l_2 -visible from X_B^* .

An important consequence of Theorem 1 is that many of the general properties of unrestricted location problems can be transferred to the restricted case $(1/P/\mathcal{B}/d_B/f \text{ convex})$ if $X_B^* \notin \mathcal{G}_d$.

As an example consider location problems for which the set of optimal solutions lies within the convex hull of the existing facilities in the unrestricted case. Defining the *iterative convex hull* R_B of the existing facilities and the barrier regions as the smallest convex subset of F such that $\partial R_B \cap \text{int}(\mathcal{B}) = \emptyset$ (see [5] for the construction of R_B), the following analogous result can be proven in the restricted case:

Theorem 2 *Let $X_{\mathcal{B}}^* \notin \mathcal{G}_d$ be an optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex). Then*

$$X_{\mathcal{B}}^* \in R_{\mathcal{B}}$$

if, for the corresponding unrestricted problem $(1/P/\bullet/d/f$ convex), the set of optimal solutions is contained in the convex hull of the existing facilities.

Proof: Let $X_{\mathcal{B}}^*$ be an optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) such that $X_{\mathcal{B}}^* \in \text{int}(C)$ for some cell $C \in \mathcal{C}(\mathcal{G}_d)$.

Suppose that $X_{\mathcal{B}}^* \notin R_{\mathcal{B}}$. Wlog we assume that there exists no barrier in $\mathbb{R}^2 \setminus R_{\mathcal{B}}$ since this assumption does not increase the objective value of any point $X \in F$. Applying Theorem 1 we can follow that $X_{\mathcal{B}}^*$ is an optimal solution of problem (8) with respect to some intermediate points $I_m \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$, $m \in \mathcal{M}$. This problem is an unrestricted location problem of type $(1/P/\bullet/d/f$ convex) and thus $X_{\mathcal{B}}^* \in \text{conv}\{I_m : m \in \mathcal{M}\} \cap F$. Since $R_{\mathcal{B}}$ is the convex hull of all existing facilities and all barrier sets intersected with the feasible region F , we can conclude that

$$\text{conv}\{I_m : m \in \mathcal{M}\} \cap F \subseteq \text{conv}(\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap F \subseteq R_{\mathcal{B}}.$$

□

Other consequences of Theorem 1 are e.g. the discretization results developed for the Median problem with Manhattan- or gauge-distances (see [2, 5, 10]).

4 Algorithmic consequences

Reducing a problem of type $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) to a set of convex subproblems with respect to Theorem 1 (or Corollary 4), two different cases may occur. An optimal solution $X_{\mathcal{B}}^*$ of a problem of type $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) may either be located on the grid \mathcal{G}_d or in the interior of a cell $C \in \mathcal{C}(\mathcal{G}_d)$. In the first case $X_{\mathcal{B}}^*$ can be easily found by applying a line search procedure on the line segments of \mathcal{G}_d . In the latter case $X_{\mathcal{B}}^*$ is the optimal solution of a corresponding unrestricted problem (8).

Thus a two step algorithm can be suggested to solve problems of type $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex). In a first step, a line search procedure is applied on each line segment of the grid \mathcal{G}_d . In a second step, a local minimum is sought in the interior of a cell in $F \setminus \mathcal{G}_d$ by solving convex subproblems (8) for all feasible reformulations $f_{\mathcal{B}}(Y) = f_X(Y) + g_X$ of the objective function. For each solution Y^* of one of these subproblems feasibility has to be tested, i.e. it has to be verified whether $f_{\mathcal{B}}(Y^*) = f_X(Y^*) + g_X$.

Algorithm 1**Input:** *Location problem $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$.*Step 1: *Construct the grid \mathcal{G}_d .*Step 2: *Find a local minimum of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$ on \mathcal{G}_d .*Step 3: *For all feasible reformulations of the objective function, i.e. for all feasible assignments of intermediate points to existing facilities, do:*(a) *Find an optimal solution Y^* of the corresponding unrestricted problem $\min f_X(Y) + g_X, Y \in \mathbb{R}^2$.*(b) *If $f_{\mathcal{B}}(Y^*) = f_X(Y^*) + g_X$, the solution Y^* is a candidate for an optimal solution.*Step 4: *Determine the set of global minima from the candidate set found in Steps 2 and 3.***Output:** *Set of optimal solutions of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$.*Note that the grid \mathcal{G}_d can also be replaced by the grid \mathcal{G}_{l_2} (see Corollary 4).

The time complexity of Step 1 of Algorithm 1 depends on the size of the grid \mathcal{G}_d (or \mathcal{G}_{l_2} , respectively) and thus on the number of existing facilities, the number of extreme points of the barrier regions and the choice of the distance function d . In the case that distances are measured by the Euclidean distance function l_2 , the number of line segments in \mathcal{G}_{l_2} is bounded by $(|\mathcal{E}x| + |\mathcal{P}(\mathcal{B})|) \cdot |\mathcal{P}(\mathcal{B})|$.

The overall time complexity of Algorithm 1 is dominated by Step 2. If no additional information is available to reduce the possible assignments of existing facilities to intermediate points, the number of subproblems is exponential in the number of existing facilities and in the number of extreme points of the barrier regions. For specially shaped barrier regions better results are nevertheless available. As an example consider the case that the barrier is given by one line with a finite number of passages. In this case a polynomial number of subproblems is sufficient to determine an optimal solution as was shown in [8].

If no additional information is available a heuristic strategy can alternatively be applied that, in a large number of cases, finds the optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$ in a remarkably smaller number of iterations. Instead of evaluating all the theoretically possible assignments of existing facilities to intermediate points, a sample set S of (not necessarily equidistant) grid points can be constructed in $R_{\mathcal{B}}$. All the points in this sample set are used as starting points for an unrestricted location problem (8). As in Algorithm 1, the corresponding optimal solution Y^* is used as a candidate for the optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$ if Y^* is feasible, i.e. if $f_{\mathcal{B}}(Y^*) = f_X(Y^*) + g_X$.

Algorithm 2

Input: *Location problem* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$.

Step 1: *Construct the grid* \mathcal{G}_d .

Step 2: *Find a local minimum of* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$ *on* \mathcal{G}_d .

Step 3: *Define a sample set* S *of grid points in* $R_{\mathcal{B}}$.

Step 4: *For each grid point* $X \in S$ *do:*

(a) *Find an optimal solution* Y^* *of the corresponding unrestricted problem* $\min f_X(Y) + g_X, Y \in \mathbb{R}^2$.

(b) *If* $f_{\mathcal{B}}(Y^*) = f_X(Y^*) + g_X$, *the solution* Y^* *is a candidate for an optimal solution.*

Step 5: *Determine the best solution found in Steps 2 and 4.*

Output: *Approximation of the optimal solution of* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$.

An optimal solution of a problem of type $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$ can be approximated with increasing accuracy by refining the sample set S . The following result can be easily verified:

Theorem 3 *For any problem of type* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f \text{ convex})$, *Algorithm 2 yields an optimal solution if the sample set of grid points* S *is chosen sufficiently fine.*

5 Conclusions

In this paper a reduction procedure for a general class of planar location problems with polygonal barriers is developed that allows the exact solution of this type of non-convex optimization problems by solving a finite number of related convex location problems.

This result is as well of theoretical as of practical interest. It allows the transfer of theoretical results for unrestricted planar location problems to the restricted case and yields exact and heuristic Algorithms to solve planar location problems with barriers.

Future research topics include the investigation of special cases like simple barrier shapes (e.g. rectangles or circles) and specific distance functions (e.g. the Manhattan metric l_1 or the more general class of block norms). Furthermore reduction based methods should be combined with modern solution techniques to develop efficient implementations of the suggested algorithms.

References

- [1] Y.P. Aneja and M. Parlar. Algorithms for Weber facility location in the presence of forbidden regions and/or barriers to travel. *Transportation Science*, 28:70–76, 1994.

- [2] R. Batta, A. Ghose, and U.S. Palekar. Locating facilities on the Manhattan metric with arbitrarily shaped barriers and convex forbidden regions. *Transportation Science*, 23:26–36, 1989.
- [3] S.E. Butt and T.M. Cavalier. An efficient algorithm for facility location in the presence of forbidden regions. *European Journal of Operational Research*, 90:56–70, 1996.
- [4] H.W. Hamacher. *Mathematical Methods in Planar Locational Planning (in German)*. Vieweg Verlag, Braunschweig, 1995. 171 pages.
- [5] H.W. Hamacher and K. Klamroth. Planar location problems with barriers under polyhedral gauges. Technical Report No. 21, Universität Kaiserslautern, Department of Mathematics, 1997. submitted.
- [6] H.W. Hamacher and S. Nickel. Multicriteria planar location problems. *European Journal of Operational Research*, 94:66–86, 1996.
- [7] I.N. Katz and L. Cooper. Facility location in the presence of forbidden regions, I: Formulation and the case of Euclidean distance with one forbidden circle. *European Journal of Operational Research*, 6:166–173, 1981.
- [8] K. Klamroth. Planar location problems with line barriers. Technical Report in Wirtschaftsmathematik No. 13, Universität Kaiserslautern, Department of Mathematics, 1996.
- [9] K. Klamroth and M. Wiecek. A multiple objective planar location problem with a line barrier. Technical Report in Wirtschaftsmathematik No. 33, Universität Kaiserslautern, Department of Mathematics, 1998. submitted.
- [10] R.C. Larson and G. Sadiq. Facility locations with the Manhattan metric in the presence of barriers to travel. *Operations Research*, 31:652–669, 1983.
- [11] J. Viegas and P. Hansen. Finding shortest paths in the plane in the presence of barriers to travel (for any l_p -norm). *European Journal of Operational Research*, 20:373–381, 1985.