# Finite Dominating Sets for Rectilinear Center Problems with Polyhedral Barriers

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**Abstract**

In planar location problems with barriers one considers regions which are forbidden for the siting of new facilities as well as for trespassing. These problems are important since they reflect various real-world situations. The resulting mathematical models have a non-convex objective function and are therefore difficult to tackle using standard methods of location theory even in the case of simple barrier shapes and distance functions. For the case of center objectives with barrier distances obtained from the rectilinear or Manhattan metric it is shown that the problem can be solved by identifying a finite dominating set (FDS) the cardinality of which is bounded by a polynomial in the size of the problem input. The resulting genuinely polynomial algorithm can be combined with bound computations which are derived from solving closely connected restricted location and network location problems. It is shown that the results can be extended to barrier center problems with respect to arbitrary block norms having four fundamental directions.

# 1 Introduction

In real-world location problems one often encounters situations in which regions are neither allowed for siting new facilities nor for trespassing. In accordance with most of the literature quoted below we call such regions *barriers*. Examples of barriers include lakes or nature parks when the location of industrial facilities is considered, obstacles in a production environment, or high risk areas in the transportation and storing of chemicals.
In spite of this practical importance, there is only a relatively small amount of literature on location problems with barriers. (Katz and Cooper, 1981) considered median (total cost) location problems using Euclidean distance and a forbidden region consisting of one circle. (Klamroth, 1996) considered the median problem where distance is derived from a norm and with a barrier consisting of a line with passages. (Aneja and Parlar, 1994) and (Butt and Cavalier, 1996) developed heuristics for the median problem with $l_p$ distance and barriers that are closed polyhedra. (Larson and Sadiq, 1983), and (Batta *et al.*, 1989) obtained discretization results for median problems with $l_1$-distance and arbitrarily shaped barriers by transforming these problems into equivalent network location problems. Their results were generalized by (Hamacher and Klamroth, 1997) for arbitrary block norms although it is not possible to transform these problems to the analogous network location problems. Location problems in which regions are excluded from siting new facilities, but trespassing is allowed are called *restricted location problems*. They have lately drawn some attention and have been successfully tackled for median and center problems, for instance, in (Hamacher, 1995), (Nickel, 1995), (Hamacher and Nickel, 1995), and (Hamacher and Schöbel, 1997).

This paper considers the weighted center problem with barriers, for which - to the best of our knowledge - no previous results exist. In the next section we will formally introduce the problem and derive lower and upper bounds on the objective value by investigating the interrelation between center barrier problems on one hand and network location and restricted location problems on the other hand. A discretization result is developed in Sections 3 and 4 for the special case that distances are measured by the Manhattan metric ($l_1$-metric) and that the barriers are pairwise disjoint convex polyhedra. It is shown that it is sufficient to consider a finite number of candidates, a finite dominating set (FDS), to find an optimal location. The resulting polynomial time algorithm using this FDS is given in Section 4.

Section 5 shows that the results of Sections 3 and 4 are more general than one might initially think: Any problem with block norms having exactly four fundamental directions can be tackled in the same way. The paper is concluded by a final section in which the results of the paper are summarized and directions for future research are outlined.

## 2 Formal definition and bounds for center problems with barriers

In this section we first give a formal definition of center problems with barriers. Then we show that by considering the restricted location problem as a relaxation we get lower and upper bounds. Further upper bounds are obtained by investigating a network location problem closely related to the input of the center problem with barriers.

Let $\{B_1, \ldots, B_N\}$ be a set of closed, convex and pairwise disjoint sets in the plane, $\mathbb{R}^2$. Each set $B_i$, $i = 1, \ldots, N$ is called a barrier. Let $\mathcal{B} = \bigcup_{i=1}^N B_i$. The location of new facilities in the interior of $\mathcal{B}$ and travel through $\text{int}(\mathcal{B})$ is forbidden. Thus the feasible region $F \subseteq \mathbb{R}^2$ for new facilities is given by

$$F = \mathbb{R}^2 \setminus \text{int}(\mathcal{B}).$$

The distance $d_{\mathcal{B}}(X, Y)$ between two points $X, Y \in F$ is defined as the length of a shortest path from $X$ to $Y$ that does not intersect a barrier. A finite set

$$\mathcal{E}x = \{Ex_m \in F \; : \; m \in \mathcal{M} = \{1, \ldots, M\}\}$$

of existing facilities is given in a connected subset of the feasible region $F$. A positive weight $w_m = w(Ex_m)$, $m \in \mathcal{M}$ is associated with each existing facility that represents the demand of facility $Ex_m$.
Define the function

$$f_{\mathcal{B}}(X) = \max_{m \in \mathcal{M}} w_m d_{\mathcal{B}}(X, Ex_m).$$

Then the weighted center problem with barriers is to minimize $f_{\mathcal{B}}(X)$ over all $X \in F$. In the notation of (Hamacher and Nickel, 1996), this problem has the classification $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$.

While center location problems in the plane without barriers are extensively discussed in the literature (see, e.g. the books of (Francis *et al.*, 1992), (Hamacher, 1995), (Drezner, 1995), and (Love *et al.*, 1988)) no references can be found on the corresponding barrier problems. The decisive distinction between the former and the latter problem is that the distance measure $d_{\mathcal{B}}$ for the problem considered in this paper reflects the fact that trespassing of the barriers is not allowed.

Let $d$ be a given distance function derived from a norm $\| \bullet \|_d$. Then the distance $d_{\mathcal{B}}(X, Y)$ between two points $X, Y \in F$ is defined as the length of a shortest path (with respect to the given distance function $d$) from $X$ to $Y$ that does not intersect the interior of a barrier. Formally, let $p$ be a piecewise, continuously differentiable parametrization, $p : [a, b] \to \mathbb{R}^2$, $a, b \in \mathbb{R}$, $a < b$, of a *permitted path connecting* $X$ *and* $Y$, i.e. a curve not intersecting the interior of a barrier, $p([a, b]) \cap \mathrm{int}(\mathcal{B}) = \emptyset$, with $p(a) = X$ and $p(b) = Y$. Then $d_{\mathcal{B}}$ is given by

$$d_{\mathcal{B}}(X, Y) := \inf \left\{ \int_a^b \| p'(t) \|_d \, \mathrm{d}t \, : p \text{ permitted path connecting } X \text{ and } Y \right\}.$$

Any path connecting $X$ and $Y$ with length $d_{\mathcal{B}}(X, Y)$ not intersecting the interior of $\mathcal{B}$ is called a *d-shortest permitted path connecting* $X$ *and* $Y$. Any two points $X$ and $Y$ in $F$ that satisfy $d_{\mathcal{B}}(X, Y) = d(X, Y)$ are called *d-visible.* If $d$ is the Manhattan metric, $d_{\mathcal{B}}(X, Y)$ is denoted by $l_{1,\mathcal{B}}(X, Y)$.

Note that $d_{\mathcal{B}}$ is symmetric and satisfies the triangle inequality, but is in general not positively homogeneous. Therefore the objective function $f_{\mathcal{B}}$ is non-convex. However, instead of tackling the problem with methods of non-convex optimization we will choose a different approach by investigating the structure of the problem in more detail.

Next, upper and lower bounds for the optimal objective value of the center problem with barriers, $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$, will be discussed. These bounds are analogous to bounds given for the median objective function in (Hamacher and Klamroth, 1997). Since these results can be easily transferred to other and more general objective functions we refer to their work for a more detailed discussion.

Two different approaches are suggested. The first approach is based on a relaxation of the barrier problem to a restricted location problem of the type

$1/P/\mathcal{R} = \mathcal{B}/d/\max$. Here trespassing through the barrier regions is allowed whereas the placement of a new facility within the region $\mathcal{R} = \mathcal{B}$ is prohibited. The second approach makes use of the visibility graph G of the problem to relate the barrier problem with a network location problem $1/G/\bullet/d_G/\max$ on $G$.
In both cases the non-convex optimization problem $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ is relaxed to a location problem that is easier to solve.

**Lemma 1 (see (Hamacher and Klamroth, 1997))** *Let $z^*_{\mathcal{B}}$ be the optimal objective value of the barrier problem $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ and let $X^*_{\mathcal{R}}$ be an optimal solution of the corresponding restricted problem $1/P/\mathcal{R} = \mathcal{B}/d/\max$. Then*

$$f(X^*_{\mathcal{R}}) = \max_{m\in\mathcal{M}}\{w_m\, d(Ex_m, X^*_{\mathcal{R}})\} \le z^*_{\mathcal{B}} \le \max_{m\in\mathcal{M}}\{w_m\, d_{\mathcal{B}}(Ex_m, X^*_{\mathcal{R}})\} = f_{\mathcal{B}}(X^*_{\mathcal{R}}).$$

**Corollary 1 (see (Hamacher and Klamroth, 1997))** *Let $X^*_{\mathcal{R}}$ be an optimal solution of the restricted problem $1/P/\mathcal{R} = \mathcal{B}/d/\max$ with objective value $z^*$. If $z^* \ge w_m d_{\mathcal{B}}(Ex_m, X^*_{\mathcal{R}})$ for all $m \in \mathcal{M}$, then $X^*_{\mathcal{R}} = X^*_{\mathcal{B}}$ is an optimal solution of $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$.*

For the case that distances are measured with respect to the Manhattan metric $d = l_1$ or the Chebychev metric $d = l_\infty$, the restricted problem $1/P/\mathcal{R} = \mathcal{B}/d/\max$ can be solved by an algorithm developed in (Hamacher and Nickel, 1995). If distances are measured with respect to polyhedral gauges $d = \gamma$, the optimal solution of the restricted problem can be obtained using an algorithm proposed in (Nickel, 1995).

The second approach to derive bounds for the problem $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ makes use of the visibility graph of the problem in order to relax the non-convex barrier problem to a network location problem.

In this case an additional assumption is needed, namely that the set of barriers is given by a set of polyhedra with extreme points $\mathcal{P}(\mathcal{B}) := \{p_i : i = 1, \ldots, g\}$. The embedded visibility graph of $\mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ is defined as $G = (V, E)$ with node set $V(G) = \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ and weights $w(v) = 0$ if $v = p \in \mathcal{P}(\mathcal{B})$ and $w(v) = w(Ex_m)$ if $v = Ex_m \in \mathcal{E}x$. Two nodes $v_i, v_j \in V(G)$ are connected by an edge if the corresponding points are $d$-visible in the feasible region $F$, i.e., $d_{\mathcal{B}}(v_i, v_j) = d(v_i, v_j)$, and in this case the length of the edge is $d(v_i, v_j)$. The embedding of this edge is represented by a $d$-shortest permitted path between the points $v_i$ and $v_j$. The length of a shortest network path between two vertices $u$ and $v$ is denoted by $d_G(u,v)$. Analogously the length of a shortest network path between a vertex $v$ and a point $X$ on an edge $e \in E(G)$ is denoted by $d_G(X,v)$. Then the network location problem $1/G/\bullet/d_G/\max$ on $G$ is defined by

$$\begin{aligned} &\min_{X\in G} f_G(X) \\ \text{where} \quad & f_G(X) = \max_{v\in V(G)} w(v)\, d_G(X,v). \end{aligned}$$

**Lemma 2 (see (Hamacher and Klamroth, 1997))** *If $X_G^*$ is an optimal solution of the network location problem $1/G/\bullet/d_G/\max$ on $G$, then a point $X_{\mathcal{B}}^*$ in the feasible region $F$ that corresponds to the point $X_G^*$ on the embedded graph, is feasible for $1/P/\mathcal{B}/d_{\mathcal{B}}/\max$ and*

$$f_{\mathcal{B}}(X_{\mathcal{B}}^*) \leq f_G(X_G^*).$$

# 3 The special case of the Manhattan metric and convex, disjoint polyhedral barriers

In this section a different network than the one used in the previous section is constructed for the special case that distances are measured by the Manhattan metric $d = l_1$ and that all barriers are closed, convex, pairwise disjoint polyhedra. Using this network we will develop a polynomial time algorithm that determines at least one optimal solution of the problem $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

## 3.1 Shortest $l_1$-paths in the presence of barriers

Let $l_{1,\mathcal{B}}(X,Y)$ denote the length of an $l_1$-*shortest permitted path* connecting $X$ and $Y$ in $F$, i.e. a shortest permitted path with respect to length $l_{1,\mathcal{B}}(X,Y)$. As special case of the $d$-visibility definition above, any two points $X$ and $Y$ in $F$ that satisfy

$$l_{1,\mathcal{B}}(X,Y) = l_1(X,Y)$$

are called $l_1$-*visible*.

The set of points $Y \in F$ that are non $l_1$-visible from a point $X \in F$ is called the *shadow of $X$ with respect to $l_1$*, i.e.

$$shadow_{l_{1,\mathcal{B}}}(X) := \{Y \in F \ : \ l_{1,\mathcal{B}}(X,Y) > l_1(X,Y)\}.$$

Note that for all $X \in F$ the set $shadow_{l_{1,\mathcal{B}}}(X)$ is bounded by parts of the boundaries of barriers or by horizontal or vertical line segments in $F$. Furthermore some $l_1$-visible points are obviously not $l_2$-visible, i.e. not visible in the usual sense of straight line visibility. On the other hand, the following result holds.

**Lemma 3** *Every point $X \in F$ that is $l_2$-visible from the origin is also $l_1$-visible from the origin. Furthermore, in this case the straight line segment connecting the origin and $X$ is an $l_1$-shortest permitted path.*

**Proof:** Let $X \in F$ be a point that is $l_2$-visible from the origin. Then the straight-line segment connecting the origin and $X$ is a permitted path $P$ from the origin to $X = (x_1, x_2)^T$ with length $l_1(P) = |x_1| + |x_2| = l_1(0, X)$.

□

Since we assume that all barriers are convex polyhedra, the relation between $l_1$-visibility and $l_2$-visibility can be used to obtain a simpler description of the barrier distance $l_{1,\mathcal{B}}$. The following lemma is a special case of a result of (Viegas and Hansen, 1985) for $l_p$-distance functions $(1 \leq p \leq \infty)$.

**Lemma 4** *Let $X, Y \in F$. Then there exists an $l_1$-shortest permitted path, $SP$, connecting $X$ and $Y$ with the following property.*

*$SP$ is a piecewise linear path with breaking points only in extreme points of barriers.* (1)

## 3.2 Constructing a cell partitioning of the feasible region

In the following a network $\mathcal{N}$ will be constructed such that $l_1$-shortest permitted paths between all existing facilities and extreme points of barriers are represented by network paths in $\mathcal{N}$, similar to the visibility graph given in Section 2. Additional edges are added resulting in a partitioning of the feasible region into cells.

The four *fundamental directions* $e^1 = (0,1)^T$, $e^2 = (1,0)^T$, $e^3 = (0,-1)^T$ and $e^4 = (-1,0)^T$ defining the unit ball of the Manhattan metric play a central role in the construction of $\mathcal{N}$ (see Figure 1).

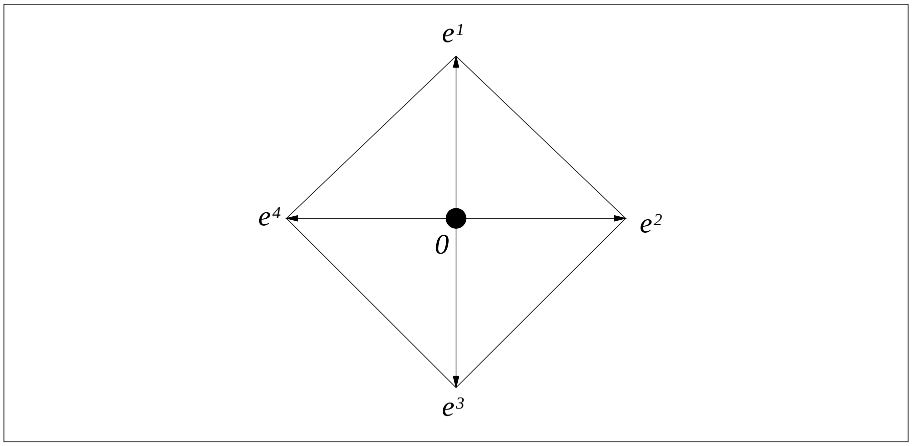


Figure 1: The unit ball of the $l_1$-norm and its four fundamental directions.

Let $\mathcal{P}(\mathcal{B})$ and $\mathcal{F}(\mathcal{B})$ denote the set of extreme points and facets, respectively, of the convex barrier polyhedra. For any $X \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$ and for any fundamental direction $e^i$, $i = 1, \ldots, 4$, define a *construction line*

$$(X + e^i)_{\mathcal{B}} := \{X + \lambda e^i \ : \ \lambda \in I\!R_+; \ (X + \mu e^i) \cap \operatorname{int}(\mathcal{B}) = \emptyset \ \forall 0 \leq \mu \leq \lambda\}$$

as the set of points in the plane which are $l_2$-visible from $X$ in the fundamental direction $e^i$. Then

$$\mathcal{G} := \left( \bigcup_{X \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})} \bigcup_{i=1}^{4} (X + e^i)_{\mathcal{B}} \right) \cup \mathcal{F}(\mathcal{B})$$

defines a grid which is a subset of $F$. All possible intersection points of construction lines, or the intersection points of construction lines and facets of a barrier in $\mathcal{G}$ define the set $V(\mathcal{G}) = V(\mathcal{N})$ of vertices of the corresponding network $\mathcal{N}$. Two vertices $v_1$, $v_2 \in V(\mathcal{N})$ are connected by an edge in $E(\mathcal{N})$ if they are adjacent on some construction line or facet in $\mathcal{G}$. The length of this edge is then given by the $l_1$-length of the corresponding line-segment.

The grid defined by $\mathcal{G}$ partitions the feasible region $F$ into a finite set of cells denoted by $\mathcal{C}(\mathcal{G})$, i.e. the set of smallest 2-dimensional convex polyhedra with extreme points in $V(\mathcal{G})$ (see Figure 2). The extreme points of a cell are called *corner points* of the cell $C$. Note that each cell is bounded by construction lines or facets of the barriers.

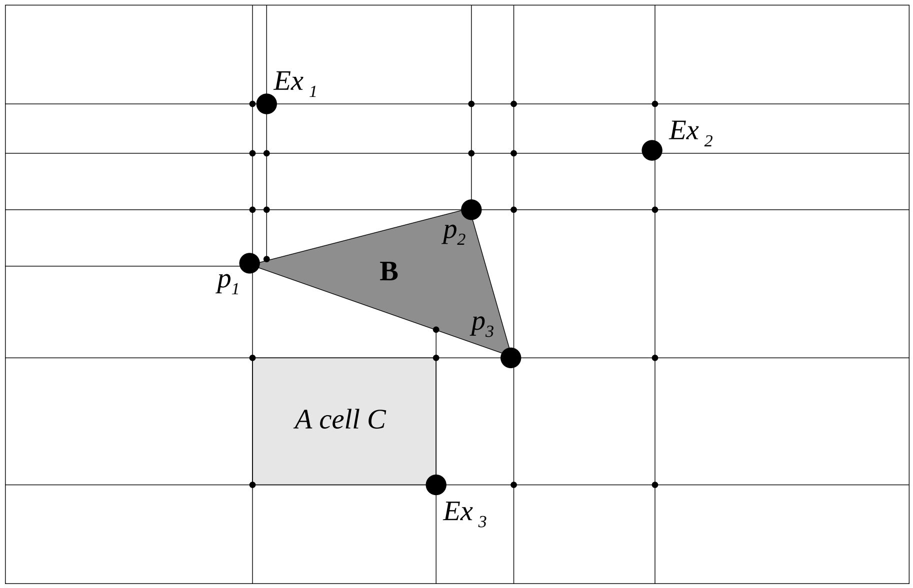


Figure 2: The network $\mathcal{N}$ for an example problem with three existing facilities and one triangular barrier.

(Larson and Sadiq, 1983) defined a similar network omitting some of the construction lines introduced above, namely those construction lines for which an extreme point of the barrier polyhedra is an end point of the construction line. Even though the properties of $l_1$-shortest permitted paths with respect to the smaller network need some further discussion, Larson and Sadiq showed that in case of the median objective function the problem can be transformed into a network location problem. Note that an analogous result cannot be proven for the center objective function even in the unrestricted case as can be seen in Figure 3.

## 3.3 Grid vertices on $l_1$-shortest permitted paths

The partitioning of $F$ into cells $C(\mathcal{G})$ will be used in this section to derive further properties of $l_1$-shortest permitted paths to the existing facilities. Since an extended network is used compared to that defined in (Larson and

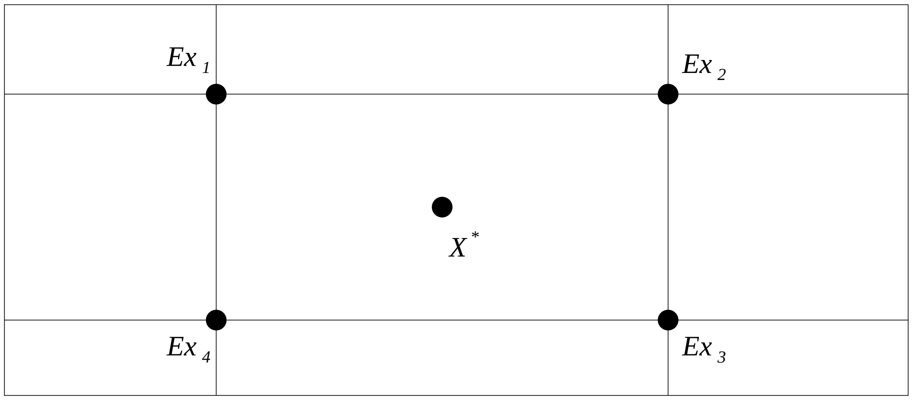


Figure 3: An example with four existing facilities with equal weights ($w_i = 1$, $i = 1, \ldots, 4$). The unique optimal solution $X^*$ of $1/P/\bullet/l_1/\max$ lies in the interior of a cell of the corresponding grid $\mathcal{G}$.

Sadiq, 1983), the following two results which can also be found in (Larson and Sadiq, 1983) can now be proven in an easier, straight forward way.

**Lemma 5** *Let $v \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$ be an existing facility or an extreme point of a barrier, and let $C$ be a cell in $C(\mathcal{G})$.*
*If $v$ is $l_1$-visible from some point in $\text{int}(C)$, then $v$ is $l_1$-visible from all points in $C$.*

**Proof:** Let $v \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$. Then the shadow of $v$ with respect to $l_1$ is bounded by facets of the barriers and by construction lines rooted at extreme points of the barriers. Thus the result follows directly from the construction of the grid $\mathcal{G}$.

□

**Lemma 6** *Let $Ex_m \in \mathcal{E}x$ be an existing facility and let $C$ be a cell in $C(\mathcal{G})$ with $X \in C$. Then there exists an $l_1$-shortest permitted path connecting $Ex_m$ and $X$ that passes through a corner point of $C$.*

**Proof:** Let $Ex_m \in \mathcal{E}x$ and let $X \in C$. Since the case that $X$ is a corner point of $C$ is trivial, assume that $X$ lies either in the interior of $C$ or on a facet of $C$. Furthermore let $P(X, Ex_m)$ be an $l_1$-shortest permitted path connecting $Ex_m$ and $X$ and satisfying Property (1) of Lemma 4. Then there exists an intermediate point $I_m \in (\mathcal{E}x \cup \mathcal{P}(\mathcal{B}))$ on $P(Ex_m, X)$ that is $l_1$-visible from

$X$. Using Lemma 5 we can conclude that $I_m$ is $l_1$-visible from all points in $C$.
Let $x_1 := \min\{x : (x,y)^T \in C\}$, $x_2 := \max\{x : (x,y)^T \in C\}$, $y_1 := \min\{y : (x,y)^T \in C\}$, and $y_2 := \max\{y : (x,y)^T \in C\}$ and let $I_m = (a,b)^T$. Then $a \notin (x_1, x_2)$ and $b \notin (y_1, y_2)$ since otherwise there would exist a construction line intersecting $\text{int}(C)$ .
Using the fact that $I_m$ is $l_1$-visible from every corner point of $C$ and that the corner points of $C$ are $l_1$-visible from every point in $C$, we can construct an $l_1$-shortest permitted path connecting $X$ and $I_m$ that passes through a corner point of $C$ (see Figure 4).

□

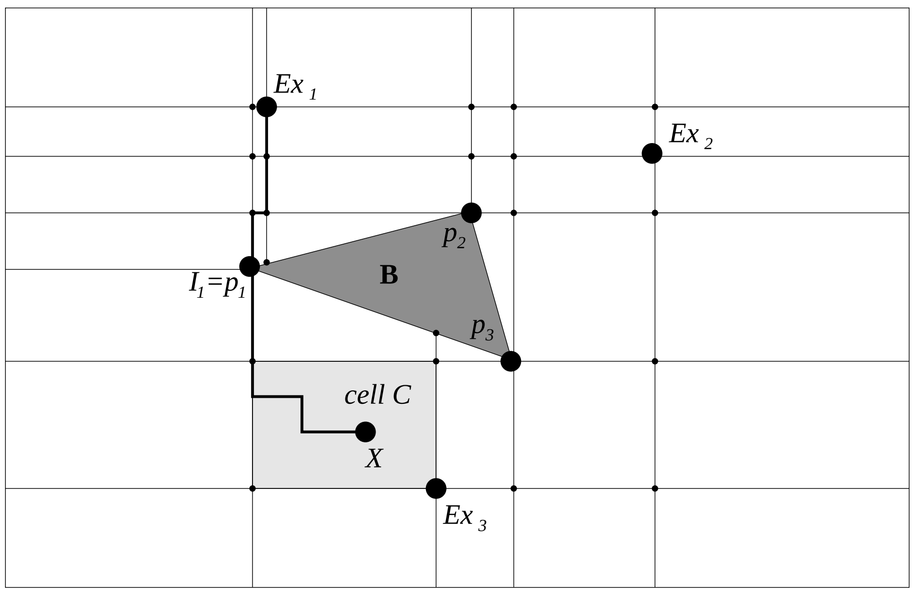


Figure 4: An example of an $l_1$-shortest permitted path connecting $Ex_1$ and $X$ that passes through a corner point of the cell $C$.

Using Lemma 6 we can always find $l_1$-shortest permitted paths with the following property: Every cell that is intersected by this path is entered through one corner point and left through a different corner point.
Thus extending $\mathcal{N}$ by all those edges of length $l_1(v_1, v_2)$ that connect two corner points $v_1$ and $v_2$ of the same cell $C$ that are not yet connected by a network path of length $d_{\mathcal{N}'}(v_1, v_2) = l_1(v_1, v_2)$ leads to a network $\mathcal{N}'$ such that:

**Corollary 2** *The length of an $l_1$-shortest permitted path between a corner point of a cell and an existing facility is equal to the length of a shortest network path connecting the corresponding vertices in $\mathcal{N}'$.*

Corollary 2 can easily be extended to points on lines or line-segments of the grid $\mathcal{G}$ corresponding to points on edges of $\mathcal{N}$. Thus in the case that an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ exists on $\mathcal{G}$, this solution is also an optimal solution of the corresponding network location problem $1/\mathcal{N}'/\bullet/d_{\mathcal{N}'}/\max$ on $\mathcal{N}'$. Moreover, the network $\mathcal{N}'$ can be used similar to the visibility graph to derive an improved upper bound for the optimal objective value of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ (see Lemma 2):

**Corollary 3** *Let $\mathcal{N}'$ be the extension of $\mathcal{N}$ as defined above. If $X^*_{\mathcal{N}'}$ is an optimal solution of the network location problem $1/\mathcal{N}'/\bullet/d_{\mathcal{N}'}/\max$ on $\mathcal{N}'$, then the point in the plane corresponding to the point $X^*_{\mathcal{N}'}$ in the embedding of $\mathcal{N}'$ is feasible for $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ and*

$$f_{\mathcal{B}}(X^*_{\mathcal{B}}) \leq f_{\mathcal{N}'}(X^*_{\mathcal{N}'}).$$

In general, contrary to the median case (see (Larson and Sadiq, 1983)), this bound is not sharp (see Figure 3), such that additional arguments are necessary.

## 3.4 A dominating set for an optimal solution

Corollary 2 enables us to calculate barrier distances between corner points of $\mathcal{G}$ and existing facilities in an efficient way by evaluating network distances in $\mathcal{N}'$. We can now draw our attention to the properties of an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

The following result has already been developed in the previous section. Since it will be of importance in the following, we reformulate it here as a theorem.

**Theorem 1** *If $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ has an optimal solution on $\mathcal{G}$, this solution is an optimal solution of the network location problem $1/\mathcal{N}'/\bullet/d_{\mathcal{N}'}/\max$ on $\mathcal{N}'$.*

Thus in the case that $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ has an optimal solution on $\mathcal{G}$, this solution can be found by solving a network location problem on the network $\mathcal{N}'$.

In the following we will concentrate on the case that there exists no optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ on the grid $\mathcal{G}$, i.e. $\mathcal{X}^*_{\mathcal{B}} \cap \mathcal{G} = \emptyset$ (see Figure 3 for an example). Then all optimal solutions of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ are located in the interior of cells of $C(\mathcal{G})$. Note that since the objective function is non-convex, the set of optimal solutions may also be non-convex and there may exist optimal solutions in the interior of more than one cell in $C(\mathcal{G})$.

For two points $Y_1, Y_2 \in F$ with positive weights $w_1, w_2 \in I\!R_+$ let the *weighted bisector of* $Y_1$ *and* $Y_2$ be defined as

$$b(w_1 Y_1, w_2 Y_2) := \{X \in F \,:\, w_1 l_{1,\mathcal{B}}(X, Y_1) = w_2 l_{1,\mathcal{B}}(X, Y_2)\}.$$

To simplify a further discussion of intermediate points on $l_1$-shortest permitted paths we additionally define for the constants $d_1$ and $d_2$ the *weighted bisector of* $Y_1, d_1$ *and* $Y_2, d_2$ as

$$\begin{aligned} &b(w_1(Y_1, d_1), w_2(Y_2, d_2)) := \\ &\quad \{X \in F \,:\, w_1(l_{1,\mathcal{B}}(X, Y_1) + d_1) = w_2(l_{1,\mathcal{B}}(X, Y_2) + d_2)\}. \end{aligned}$$

A well known result for center problems, that also applies to center problems with barriers, is that every optimal solution has to be located on the weighted bisector of two existing facilities. (Otherwise the objective value can be improved by moving the new location towards the existing facility at maximum weighted distance.) Note that therefore an optimal solution can always be found as a point on the farthest-point Voronoi diagram (see e.g. (Okabe *et al.*, 1992)) with respect to the existing facilities taking into account the barrier regions (see e.g. (Shamos and Hoey, 1975) for the unrestricted case).
Since the construction of weighted bisectors as well as the construction of the corresponding Voronoi diagram is difficult in the presence of barriers, this result is strengthened in the following theorem yielding a solution strategy to solve $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ without constructing all the weighted bisectors or the corresponding Voronoi diagram.

**Theorem 2** *Let $\mathcal{X}^*_{\mathcal{B}}$ be the set of optimal solutions of* $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ *such that $\mathcal{X}^*_{\mathcal{B}} \cap \mathcal{G} = \emptyset$ and let $z^*$ be the optimal objective value.*
*Then there exists at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ that has the weighted distance $z^*$ from at least three different existing facilities in $\mathcal{E}x$.*

**Proof:** Let $X^*_{\mathcal{B}}$ be an optimal solution and let $C$ be a cell such that $X^*_{\mathcal{B}} \in C$. Let $Ex_i$ and $Ex_j$ be two existing facilities with $z^* = w_i l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_i)$

$= w_j l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_j)$. Furthermore let $C_i$ (and $C_j$, respectively) be a corner point of $C$ such that there exists an $l_1$-shortest permitted path connecting $Ex_i$ and $X^*_{\mathcal{B}}$ ($Ex_j$ and $X^*_{\mathcal{B}}$, respectively) passing through $C_i$ ($C_j$, respectively), see Lemma 6.

Now assume that there exists no existing facility $Ex_m \in \mathcal{E}x$ other than $Ex_i$ and $Ex_j$ such that $w_m l_{1,\mathcal{B}}(X^*_{\mathcal{B}}, Ex_m) = z^*$. Then $C_i \neq C_j$ since otherwise the objective value $z^*$ could be improved by moving $X^*$ towards $C_i$ in $C$.

Defining $d_i := l_{1,\mathcal{B}}(C_i, Ex_i)$ and $d_j := l_{1,\mathcal{B}}(C_j, Ex_j)$ we get that

$$w_i(l_1(X^*_{\mathcal{B}}, C_i) + d_i) = w_j(l_1(X^*_{\mathcal{B}}, C_j) + d_j) = z^*$$

and thus $X^*_{\mathcal{B}} \in (b(w_i(C_i, d_i), w_j(C_j, d_j)) \cap C)$.

Due to the optimality of $X^*_{\mathcal{B}}$ and to the fact that the weighted distance from $X^*_{\mathcal{B}}$ to all other existing facilities is less than $z^*$, $X^*_{\mathcal{B}}$ has to be located on an $l_1$-shortest permitted path connecting $C_i$ and $C_j$ in $C$. Since there also exists an $l_1$-shortest permitted path connecting $C_i$ and $C_j$ on the network $\mathcal{N}'$, there exists a point $X_{\mathcal{N}'} \in \mathcal{N}'$ (not necessarily a node, i.e. $X_{\mathcal{N}'}$ may lie in the interior of an edge) different from $X^*_{\mathcal{B}}$ on this path (and in the cell $C$) such that

$$w_i(l_1(X_{\mathcal{N}'}, C_i) + d_i) = w_j(l_1(X_{\mathcal{N}'}, C_j) + d_j) = z^*.$$

Thus $X_{\mathcal{N}'} \neq X^*_{\mathcal{B}}$ is also a point on the weighted bisector of $C_i, d_i$ and $C_j, d_j$, i.e. $X_{\mathcal{N}'} \in (b(w_i(C_i, d_i), w_j(C_j, d_j)) \cap C)$. Since $C$ is convex, all points on the line-segment

$$\overline{X^*_{\mathcal{B}}, X_{\mathcal{N}'}} := \{X \in C \,:\, X = \lambda X^*_{\mathcal{B}} + (1-\lambda) X_{\mathcal{N}'},\ \lambda \in [0,1]\}$$

connecting $X^*_{\mathcal{B}}$ and $X_{\mathcal{N}'}$ lie in $C$. Furthermore for $m \in \{i, j\}$ all points $X \in \overline{X^*_{\mathcal{B}}, X_{\mathcal{N}'}}$ satisfy

$$\begin{aligned} w_m(l_{1,\mathcal{B}}(X, C_m)) &= w_m l_1(X, C_m) \\ &= w_m l_1(\lambda X^*_{\mathcal{B}} + (1-\lambda) X_{\mathcal{N}'}\,,\, C_m) \\ &= w_m(\lambda l_1(X^*_{\mathcal{B}}, C_m) + (1-\lambda) l_1(X_{\mathcal{N}'}, C_m)) \\ &= z^* - w_m d_m. \end{aligned}$$

Thus $X^*_{\mathcal{B}} \in \overline{X^*_{\mathcal{B}}, X_{\mathcal{N}'}}$ can be moved along the line-segment $\overline{X^*_{\mathcal{B}}, X_{\mathcal{N}'}}$ (which is part of the bisector $b(w_i(C_i, d_i), w_j(C_j, d_j))$) in the cell $C$ without increasing the weighted distance to $Ex_i$ and $Ex_j$, until it either reaches the boundary of $C$, i.e. the point $X_{\mathcal{N}'}$, a case that is excluded due to the assumption $\mathcal{X}^*_{\mathcal{B}} \cap \mathcal{G} = \emptyset$, or until the weighted distance to any other existing facility $Ex_m \in \mathcal{E}x$ equals $z^*$.

□

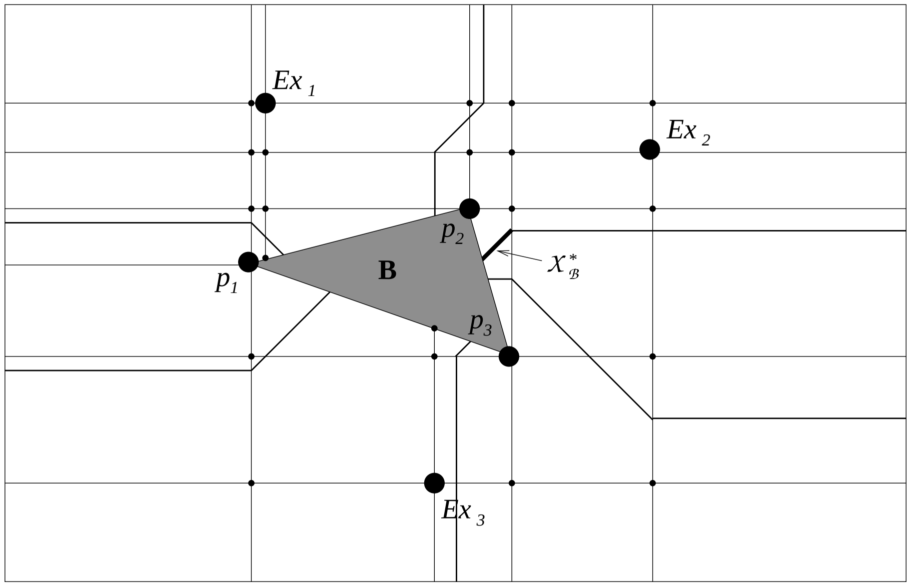


Figure 5: The bisectors $b(Ex_i, Ex_j)$ for the example problem introduced in Figure 2. Note that there exist optimal solutions on $\mathcal{N}'$, but not in an intersection of bisectors.

Theorem 2 enables us to construct a dominating set, $DS \subseteq F$, containing at least one optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$. This dominating set consists of an optimal solution of $1/\mathcal{N}'/\bullet/d_{\mathcal{N}'}/\max$ on $\mathcal{N}'$ and additionally consists of points in the interior of cells that are intersection points of two weighted bisectors determined by choosing pairs of existing facilities from a set of three existing facilities.
Before we develop an algorithm to solve $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$, we will reduce this dominating set by further exploiting the special structure of the problem.

Consider an arbitrary cell $C \in C(\mathcal{G})$ and let the *corner distance* between a point $X \in F \setminus C$ and the cell $C$ be defined as

$$l_{corn}(X, C) := \min\{l_{1,\mathcal{B}}(X, C_i) \, : \, C_i \text{ is a corner point of } C\}.$$

Then we can identify that existing facility $Ex^C_{\max} \in \mathcal{E}x$ with weight $w^C_{\max}$ that maximizes the distance to $C$, i.e.

$$w^C_{\max}\, l_{corn}(Ex^C_{\max}, C) = \max\{w_m\, l_{corn}(Ex_m, C) \,:\, Ex_m \in \mathcal{E}x\}.$$

Furthermore let

$$|C| := \max\{l_1(C_i, C_j) \,:\, C_i \text{ and } C_j \text{ are corner points of } C\}$$

denote the maximal distance between two corner points of a cell.

**Lemma 7** *Let $\mathcal{X}^*_{\mathcal{B}}$ be the set of optimal solutions of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ such that $\mathcal{X}^*_{\mathcal{B}} \cap \mathcal{G} = \emptyset$ and let $z^*$ be the optimal objective value.*
*Then there exists at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ in a cell $C \in C(\mathcal{G})$ that lies at the intersection of at least two bisectors determined by three existing facilities $Ex_i$, $Ex_j$, $Ex_k$ such that*

$$w_p\, l_{corn}(Ex_p, C) + w_p\, |C| \geq w^C_{\max}\, l_{corn}(Ex^C_{\max}, C), \qquad p = i, j, k.$$

**Proof:** The first part of this result follows directly from Theorem 2. Namely there exists at least one optimal solution $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ that has distance $z^*$ from at least three pairwise different existing facilities. Wlog denote these existing facilities by $Ex_1$, $Ex_2$, $Ex_3$. Then $X^*_{\mathcal{B}}$ lies at the intersection of $b(w_1 Ex_1, w_2 Ex_2)$ and $b(w_2 Ex_2, w_3 Ex_3)$ (and thus also in $b(w_1 Ex_1, w_3 Ex_3) \cap b(w_1 Ex_1, w_2 Ex_2)$ and in $b(w_1 Ex_1, w_3 Ex_3) \cap b(w_2 Ex_2, w_3 Ex_3)$). Now let $X^*_{\mathcal{B}} \in C$ and assume that wlog $w_1 l_{corn}(Ex_1, X^*_{\mathcal{B}}) + w_1 |C| < w^C_{\max} l_{corn}(Ex^C_{\max}, C)$. Then

$$\begin{aligned} z^* &= w_1 l_{1,\mathcal{B}}(Ex_1, X^*_{\mathcal{B}}) \leq w_1 l_{corn}(Ex_1, C) + w_1 |C| \\ &< w^C_{\max} l_{corn}(Ex^C_{\max}, C) \leq w^C_{\max} l_{1,\mathcal{B}}(Ex^C_{\max}, X^*_{\mathcal{B}}), \end{aligned}$$

contradicting the optimality of $X^*_{\mathcal{B}}$.

□

Thus with respect to each cell it is sufficient to consider only those existing facilities that satisfy the distance requirement given in Lemma 7. Especially in applications with a high number of existing facilities this result leads to a significant reduction of intersection points of weighted bisectors that have to be evaluated.

An additional reduction of the dominating set is possible since it can be shown that the set of optimal solutions $\mathcal{X}^*_{\mathcal{B}}$ is contained in the smallest rectangle $R$ with sides parallel to the coordinate axes containing all existing facilities such that the boundary of $R$, $\partial(R)$, does not intersect the interior of a barrier.

**Theorem 3** *Let $R$ be the smallest rectangle with sides parallel to the coordinate axes such that $\mathcal{E}x \subseteq R$ and $\partial(R) \cap \mathrm{int}(\mathcal{B}) = \emptyset$.*
*Then the set of optimal solutions of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ lies in $R$, i.e.*

$$\mathcal{X}^*_{\mathcal{B}} \subseteq R.$$

**Proof:** Assume that $X^*_{\mathcal{B}} \in \mathcal{X}^*_{\mathcal{B}}$ is an optimal solution that is not located in $R$, i.e. $X^*_{\mathcal{B}} \in F \setminus R$. Wlog we can assume that there exists no barrier in $I\!R^2 \setminus R$ since this assumption does not increase the objective value of any point $X \in F$. Let $R = [x_1, x_2] \times [y_1, y_2]$ and let $X^*_{\mathcal{B}} = (a, b)^T$. Since $X^*_{\mathcal{B}} \notin R$ we get that $a \notin [x_1, x_2]$ or $b \notin [y_1, y_2]$; wlog let $a < x_1$.
Let $P_m$ be an $l_1$-shortest permitted path from $X^*_{\mathcal{B}}$ to $Ex_m$ with Property (1) of Lemma 4 and let $I_m \in (\mathcal{P}(\mathcal{B}) \cup \{Ex_m\})$ be an intermediate point on $P_m$ that is $l_2$-visible from $X^*_{\mathcal{B}}$, $m \in \mathcal{M}$. Then the straight-line segment connecting $X^*_{\mathcal{B}}$ and $Ex_m$, $m \in \mathcal{M}$, intersects $\partial(R)$ in a point $(a_m, b_m)^T$ with $a < a_m$. Thus moving $X^*_{\mathcal{B}}$ towards the boundary of $R$ by increasing $a$ to $a + \epsilon$ with a small $\epsilon > 0$ decreases the distance between $X^*_{\mathcal{B}}$ and each of the intermediate points $I_m$ and thus also between $X^*_{\mathcal{B}}$ and $Ex_m$, contradicting the optimality of $X^*_{\mathcal{B}}$.

□

Analogous to Theorem 3 the following result obviously holds for $l_1$-shortest permitted paths between two points in $R$:

**Lemma 8** *Let $X$ and $Y$ be two points in $F \cap R$. Then every $l_1$-shortest permitted path connecting $X$ and $Y$ lies completely in $F \cap R$.*

Note that Theorem 3 implies not only that it is sufficient to consider cells in the rectangle $R$, but that, using Lemma 8, we can also reduce the network $\mathcal{N}'$ to the subnetwork $\mathcal{N}'' \subseteq \mathcal{N}'$ that results from the intersection of the embedding of $\mathcal{N}'$ with the rectangle $R$.

Summarizing the results above, we get the following dominating set for an optimal solution of $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$:

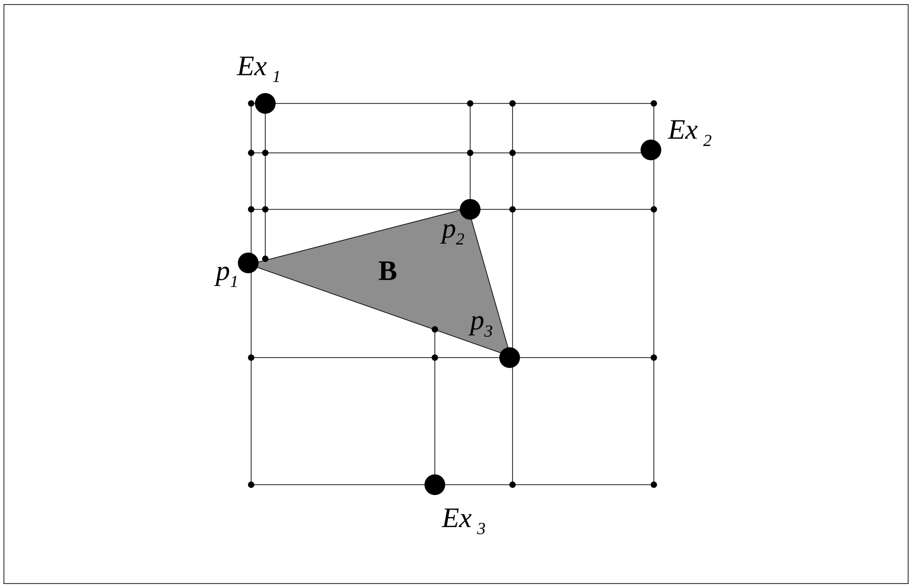


Figure 6: The rectangle $R$ and the network $\mathcal{N}''$ for the example problem introduced in Figure 2.

**Theorem 4** *The dominating set DS consisting of*

- *an optimal solution of the network location problem* $1/\mathcal{N}''/\bullet/d_{\mathcal{N}''}/\max$ *on* $\mathcal{N}''$
- *for all cells* $C \in (C(\mathcal{G}) \cap R)$*:*

  *the intersection points of the three weighted bisectors determined by three existing facilities* $Ex_i, Ex_j, Ex_k$ *satisfying*

$$w_p\, l_{corn}(Ex_p, C) + w_p\,|C| \geq w_{\max}^C\, l_{corn}(Ex_{\max}^C, C), \qquad p = i, j, k$$

*contains at least one optimal solution of* $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

Note that the dominating set $DS$ of Theorem 4 is in general not yet finite since the intersection of bisectors with respect to rectilinear distance is not necessarily a finite set. We will however see in the following section that it is sufficient to consider only a finite subset of the dominating set $DS$, thus yielding a finite dominating set $FDS$.

# 4 A finite dominating set and algorithmic consequences

Before we discuss how a finite dominating set *FDS* can be obtained using Theorem 4, we focus our attention on the computation of points in the intersection set $I = b(w_iEx_i, w_jEx_j) \cap b(w_jEx_j, w_kEx_k) \cap b(w_iEx_i, W_kEx_k) \cap C$ of the weighted bisectors of three existing facilities $(Ex_i, Ex_j, Ex_k)$ in a given cell $C$. Recall that we only have to consider triples $(Ex_i, Ex_j, Ex_k)$ of existing facilities in that subset $\mathcal{E}x(C) \subseteq \mathcal{E}x$ of existing facilities $Ex_m$ that satisfy $w_m l_{corn}(Ex_m, C) + w_m|C| \geq w^C_{\max} l_{corn}(Ex^C_{\max}, C)$.
The individual points in the set $I$ can be found in principle as the intersection points of weighted bisectors between the corresponding corner points $C_i, C_j$ and $C_k$ that lie on $l_1$-shortest permitted paths to the existing facilities $Ex_i, Ex_j, Ex_k$, adding the weighted distances $w_p l_{\mathcal{N}''}(C_p, Ex_p)$, $p = i, j, k$ from the respective corner points to the existing facilities on the network $\mathcal{N}''$ (compare Lemma 6). Since these corner points may be difficult to determine in practice, a superset of the set $I$ can be obtained by intersecting the weighted bisectors between all possible pairs of corner points and their weighted distances to the existing facilities $Ex_p$, $p = i, j, k$. This procedure is efficient since the weighted bisectors between two corner points of a cell are linear within the cell independent of the constant distances added. Thus the intersection sets $I_q$, $q = 1, \ldots, r$ of these bisectors for the $q$'th assignment of corner points $C^q_i, C^q_j, C^q_k$ to the existing facilities $Ex_i, Ex_j, Ex_k$ can be calculated in constant time. (Note that $r \leq 3^8$ since each cell can have at most 8 corner points.)
Since for each of the sets $I_q$, $q = 1, \ldots, r$ the distance $d_q(X, Ex_i) := w_p(l_1(X, C^q_i) + l_{\mathcal{N}''}(C^q_i, Ex_i))$ between a point $X \in I_q$ and the existing facility $Ex_i$ (note that $d_q(X, Ex_i) = d_q(X, Ex_j) = d_q(X, Ex_k)$ for all $X \in I_q$) may vary for different points in $I_q$, we determine that subset $\tilde{I}_q \subseteq I_q$ for which the corresponding distance $d_q$ to the three existing facilities in question is minimized and equal to the real distance $l_{1,\mathcal{B}}$, i.e.

$$\tilde{d}_q := \min_{X \in I_q} \{d_q(X, Ex_i)\}$$

and $$\tilde{I}_q := \{X \in I_q : l_{1,\mathcal{B}}(X, Ex_p) = \tilde{d}_q,\ p = i, j, k\}, \qquad q = 1, \ldots, r.$$

Let $d_C(i, j, k) := \min\{\tilde{d}_1, \ldots, \tilde{d}_r\}$ and let $I_C(i, j, k) := \{X \in \tilde{I}_q : \tilde{d}_q = d_C,\ q \in \{1, \ldots, r\}\}$. Then any representative of $I_C(i, j, k)$ can be chosen as a candidate for an optimal solution.

We can conclude that, applying this procedure, a representative of the intersection set of weighted bisectors $I$ within a cell $C$ can be found in constant time once the cell $C$ and three existing facilities $Ex_i, Ex_j, Ex_k \in \mathcal{E}x(C)$ are identified.

**Corollary 4** *It is sufficient to take as finite dominating set FDS of the center problem with barriers,* $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$*, an optimal solution of the network location problem,* $1/\mathcal{N}''/\bullet/d_{\mathcal{N}''}/\max$ *on* $\mathcal{N}''$*, and, for each cell* $C \in C(\mathcal{G}) \cap R$ *and for each triple of existing facilities* $(Ex_i, Ex_j, Ex_k)$ *in* $\mathcal{E}x(C)$*, a representative in the set* $I_C(i,j,k)$*.*

We would like to point out one more time that in case of the center problem it is not sufficient to consider intersection points of the constructed grid $\mathcal{G}$ (see Figure 3) as it was the case for the median objective function and that the problem is not equivalent to a network location problem on $\mathcal{G}$ (compare (Larson and Sadiq, 1983)). Instead, it is necessary to consider additionally points in the intersections of specific weighted bisectors between existing facilities in order to find an optimal solution of the center problem with barriers.

Based on Corollary 4 one alternative is to compare the objective values of all candidates in the finite dominating set *FDS*. In the following algorithm we avoid these lengthy computations by combining the determination of the set *FDS* with the computation of lower bounds - each derived from three existing facilities - which ultimately leads to an optimal solution of the center problem with barriers $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$.

**Algorithm for solving** $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$

**Input:** *A finite set of convex, pairwise disjoint, polyhedral barriers* $\mathcal{B}$ *and a finite set of existing facilities* $\mathcal{E}x \subseteq F$ *with positive weights.*

1. *Construct* $\mathcal{G}$*,* $R$ *and* $\mathcal{N}''$ *according to Section 3.2, Theorem 3 and remarks thereafter and calculate the distance matrix of* $\mathcal{N}''$*.*
2. *Find the set of optimal solutions* $\mathcal{X}^*_{\mathcal{N}''}$ *of* $1/\mathcal{N}''/\bullet/d_{\mathcal{N}''}/\max$ *and set* $z^*_{\mathcal{N}''} := f_{\mathcal{N}''}(\mathcal{X}^*_{\mathcal{N}''})$*.*
3. *Set* $I(i,j,k) := \emptyset$ *and* $d(i,j,k) := \infty$ *for all triples of existing facilities* $(Ex_i, Ex_j, Ex_k)$ *in* $\mathcal{E}x$*.*

   *For all cells* $C \in (C(\mathcal{G}) \cap R)$ *do*

(a) *Find the subset* $\mathcal{E}x(C) \subseteq \mathcal{E}x$ *of existing facilities* $Ex_m$ *with* $w_m l_{corn}(Ex_m, C) + w_m|C| \geq w^C_{\max} l_{corn}(Ex^C_{\max}, C)$.

(b) *For all triples* $(Ex_i, Ex_j, Ex_k)$ *of existing facilities in* $\mathcal{E}x(C)$, *compute that part* $I_C(i,j,k)$ *- if it exists - of the intersection set* $I$ *of the three weighted bisectors that lies in* $C$ *and compute its minimal distance* $d_C(i,j,k)$ *to the three respective facilities (see Corollary 4).*

(c) *If* $d_C(i,j,k) < d(i,j,k)$ *set* $d(i,j,k) := d_C(i,j,k)$ *and* $I(i,j,k) := I_C(i,j,k)$.
*If* $d_C(i,j,k) = d(i,j,k)$ *set* $I(i,j,k) := I(i,j,k) \cup I_C(i,j,k)$.

4. *Set* $z := 0$ *and* $\mathcal{X} := \emptyset$.
*For all triples of existing facilities* $(Ex_i, Ex_j, Ex_k)$ *in* $\mathcal{E}x$ *do*

(a) *Determine* $\tilde{d}(i,j,k) := \max_{p,q \in \{i,j,k\}; p \neq q}\{\frac{w_p w_q}{w_p + w_q} l_{\mathcal{N}''}(Ex_p, Ex_q)\}$

(b) *If* $d(i,j,k) \leq \tilde{d}(i,j,k)$ *and if* $d(i,j,k) > z$ *set* $z := d(i,j,k)$ *and* $\mathcal{X} := I(i,j,k)$.
*Else if* $d(i,j,k) > \tilde{d}(i,j,k)$ *and if* $\tilde{d}(i,j,k) > z$ *set* $z := \tilde{d}(i,j,k)$ *and* $\mathcal{X} := \emptyset$.

5. *The optimal objective value of* $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ *can be determined as* $z^* = \min\{z^*_{\mathcal{N}''}, z\}$.
*If* $z^*_{\mathcal{N}''} > z$ *then* $\mathcal{X}_\mathcal{B} := \mathcal{X}$.
*Else if* $z^*_{\mathcal{N}''} = z$ *then* $\mathcal{X}_\mathcal{B} := \mathcal{X}^*_{\mathcal{N}''} \cup \mathcal{X}$.

**Output:** *A subset* $\mathcal{X}_\mathcal{B}$ *of the set of optimal solutions* $\mathcal{X}^*_\mathcal{B}$ *of* $1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ *and the optimal objective value* $z^*$.

In Step 1 of this algorithm the fundamental data structures are implemented and in Step 2 the corresponding center problem on the network $\mathcal{N}''$ is solved. Step 3 completes the computation of the finite dominating set *FDS* and in Steps 4 and 5 the individual candidates are compared to identify an optimal solution of the problem.

Note that the computation of the set of optimal solutions $\mathcal{X}^*_{\mathcal{N}''}$ of the center problem $1/\mathcal{N}''/\bullet/d_{\mathcal{N}''}/\max$ on the network $\mathcal{N}''$ could be incorporated into the iterative process in Step 4, avoiding its individual computation but without reducing the overall time complexity of the algorithm.

Furthermore the complete set $\mathcal{X}^*_\mathcal{B}$ could be determined if in the case that $\tilde{d}(i,j,k) \leq d(i,j,k)$ in Step 4 (b) of the algorithm the corresponding points $Y$ that satisfy $l_{1,\mathcal{B}}(Y, Ex_p) = \tilde{d}(i,j,k)$, $p = i,j,k$ would be determined and the set $\mathcal{X}$ would be updated accordingly.

The overall time complexity of the algorithm is only slightly exceeding the time complexity of the algorithm for the corresponding median problem suggested by (Larson and Sadiq, 1983). In Step 1 of the algorithm a network of the same asymptotic size of $|V(\mathcal{N}'')| \leq O((M+g)^2)$ vertices and $|E(\mathcal{N}'')| \leq O((M+g)^2)$ edges is constructed and its distance matrix has to be determined. (We use $M$ to denote the total number of existing facilities and $g$ to denote the total number of extreme points of barriers.) In Step 2 the corresponding center problem on $\mathcal{N}''$ can be solved by an available algorithm, see e.g. (Kariv and Hakimi, 1979) where an $O(|E||V|\log(|V|))$ algorithm is given for an arbitrary simple network with $|V|$ vertices and $|E|$ edges.

Different from the case of the median objective function, the additional determination of weighted bisectors is needed in the case of the center objective function in Step 3 of the algorithm.
The number of intersections of weighted bisectors between three existing facilities at a time is bounded by $O(M^3)$, where each intersection may consist of a set of points. Even if all of the $O((M+g)^2)$ cells of $\mathcal{G}$ are enumerated to identify those cells that contain a candidate from *FDS*, the overall time complexity of Step 3 of the algorithm is bounded above by $O(M^3(M+g)^2)$.

Note that in this worst case analysis the time complexity of Steps 1 and 2 of $O((M+g)^4\log(M+g))$ and that of Step 4 of $O(M^3)$ is dominated by that of Step 3 of the Algorithm if the number of extreme points of barriers $g$ is not very large (namely if $O((M+g)^2\log(M+g)) < O(M^3)$), yielding an overall time complexity of $O(M^3(M+g)^2)$. Since the additional reductions of the finite dominating set due to Lemma 7 and Theorem 3 are not reflected in this complexity analysis better results seem to be achievable in practice.

# 5 Extension to block norms with four fundamental directions

In this section we will explore possible extensions of the results developed in Section 3 to other, more general distance functions.

A *block norm* is given by a symmetric convex polyhedron $P$ in the plane $\mathbb{R}^2$ containing the origin in its interior. Then $P$ defines a norm $\gamma(\bullet)$ by

$$\gamma(X) := \min_{\lambda \in \mathbb{R}_+} \{\lambda \,:\, X \in \lambda P\},$$

see (Minkowski, 1967). Let $d^1, \ldots, d^\delta$ denote the extreme points of $P$, referred to as *fundamental directions*. To simplify further notation we assume that $d^{\delta+1} = d^1$.

If $X$ lies in the cone $C(d^i, d^{i+1})$ spanned by $d^i$ and $d^{i+1}$, then $X$ has a unique representation in terms of $d^i$ and $d^{i+1}$, i.e.

$$X = \alpha_i d^i + \alpha_{i+1} d^{i+1}$$

for two nonnegative scalars $\alpha_i, \alpha_{i+1} \in \mathbb{R}$. As was shown in (Hamacher and Klamroth, 1997; Schandl, 1998), the norm of $X = (x_1, x_2)^T \in \mathcal{C}(d^i, d^{i+1})$ can be evaluated as

$$\gamma(X) = \alpha_i + \alpha_{i+1},$$

where $\alpha_i$ and $\alpha_{i+1}$ can be calculated as

$$\alpha_i = \frac{d_1^{i+1} x_2 - d_2^{i+1} x_1}{d_2^i d_1^{i+1} - d_1^i d_2^{i+1}} \qquad \text{and} \qquad \alpha_{i+1} = \frac{d_2^i x_1 - d_1^i x_2}{d_2^i d_1^{i+1} - d_1^i d_2^{i+1}}. \tag{2}$$

Obviously, we can interpret $\gamma(X)$ as the distance $\gamma(0, X)$ between the origin and $X$ and extend the definition to define the *gauge distance*

$$\gamma(X, Y) := \gamma(0, Y - X) = \gamma(Y - X)$$

between any two points $X, Y \in \mathbb{R}^2$.

Note that the Manhattan metric $l_1$ is an example of a block norm with four fundamental directions $e^1, \ldots, e^4$ as defined in Section 3.2.

In the following we will consider block norms as defined above with exactly four fundamental directions (see Figure 7 for an example). To distinguish these more general distance functions from the Manhattan metric with the fundamental directions $e^1, \ldots, e^4$, we will denote the fundamental directions of any other block norm with four fundamental directions by the vectors $d^1, \ldots, d^4$.

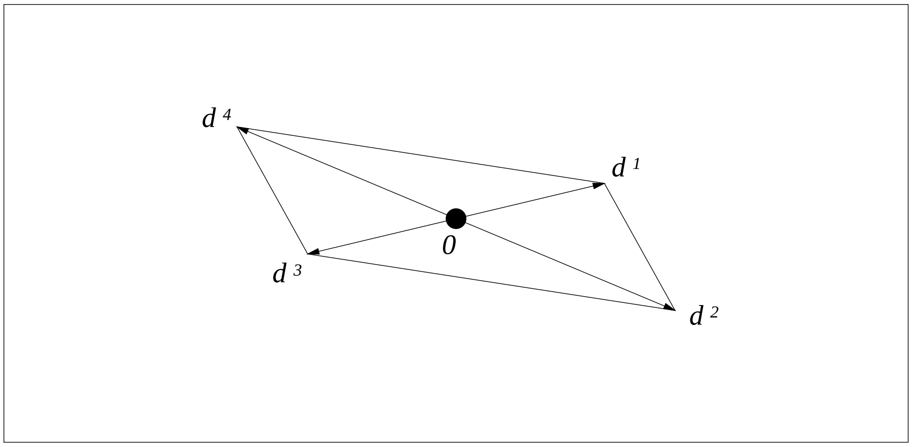


Figure 7: The unit ball of a block norm with four fundamental directions $d^1, \ldots, d^4$.

In the following let $\gamma$ be a block norm with the four fundamental directions

$$d^1 = (d^1_1, d^1_2)^T, \;\; d^2 = (d^2_1, d^2_2)^T, \;\; d^3 = (d^3_1, d^3_2)^T, \;\; d^4 = (d^4_1, d^4_2)^T$$

such that $d^3 = -d^1$ and $d^4 = -d^2$. We define a linear transformation $T$ such that $T(d^1) = Td^1 = e^1$ and $T(d^2) = Td^2 = e^2$ are satisfied. This yields

$$T \;=\; \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \;:=\; \frac{1}{d^1_2 d^2_1 - d^1_1 d^2_2} \begin{pmatrix} -d^2_2 & d^2_1 \\ d^1_2 & -d^1_1 \end{pmatrix}. \tag{3}$$

Denoting $T(X) := TX$ for all $X \in I\!R^2$, we can prove the following result relating gauge distances to the $l_1$-distance function:

**Lemma 9** *Let $\gamma$ be a block norm with four fundamental directions and let the linear transformation $T$ be defined as in (3). Then*

$$\gamma(X, Y) = l_1(T(X), T(Y))$$

*for all $X, Y \in I\!R^2$.*

**Proof:** First consider the special case that $Y = (0,0)^T$. Let $X \in \mathcal{C}(d^i, d^{i+1})$ for some $i \in \{1, \ldots, 4\}$ and let $X = \alpha_i d^i + \alpha_{i+1} d^{i+1}$ be the unique representation of $X$ in terms of $d^i$ and $d^{i+1}$. Using the fact that $\gamma$ has only four fundamental directions we can assume wlog that $d^i \in \{d^1, -d^1\}$ and $d^{i+1} \in \{d^2, -d^2\}$. Using (2) we can calculate

$$\begin{aligned} l_1(T(X), 0) &= |t_{11}x_1 + t_{12}x_2| + |t_{21}x_1 + t_{22}x_2| \\ &= \left| \frac{d_1^2 x_2 - d_2^2 x_1}{d_2^1 d_1^2 - d_1^1 d_2^2} \right| + \left| \frac{d_2^1 x_1 - d_1^1 x_2}{d_2^1 d_1^2 - d_1^1 d_2^2} \right| \\ &= |\alpha_i| + |\alpha_{i+1}| \\ &= \gamma(X, 0). \end{aligned}$$

Now consider the general case that $X, Y \in I\!R^2$. Since $T$ is a linear transformation, we immediately get

$$\begin{aligned} l_1(T(X), T(Y)) &= l_1(T(X) - T(Y), 0) = l_1(T(X - Y), 0) \\ &= \gamma(X - Y, 0) = \gamma(X, Y). \end{aligned}$$

□

This result is well known for the special case of the Chebychev-metric $l_\infty$. In this case the transformation $T$ can be defined as

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

see e.g. (Hamacher, 1995). Note that the definition of $T$ depends on the choice of the vectors $d^1$ and $d^2$ for a given gauge $\gamma$ and is therefore not unique.

In the following we will strengthen the result given in Lemma 9 so that it can be applied to center problems with barriers. For this purpose let $\mathcal{B}$ be a set of pairwise disjoint convex polyhedral sets in $I\!R^2$ as introduced in Section 3. By $T(\mathcal{B})$ and $T(F)$ we denote the image of $\mathcal{B}$ and the feasible region $F$ under the linear transformation $T$, respectively. Note that since $T$ is linear and nonsingular (the fundamental directions $d^1$ and $d^2$ of a block norm are linearly independent), the set $T(\mathcal{B})$ of barriers in the image space is again a set of closed, convex and pairwise disjoint barriers. Namely $T$ is a linear bijective mapping that defines a one-to-one correspondence between the extreme points of $\mathcal{B}$ in the original space and the extreme points of $T(\mathcal{B})$ in the image space. Thus the decision space of a problem of type

$1/P/\mathcal{B}/l_{1,\mathcal{B}}/\max$ can be transformed using the transformation $T$ resulting in a problem of type $1/P/T(\mathcal{B})/l_{1,T(\mathcal{B})}/\max$.

Using the fact that Lemmas 3 and 4 also hold in the more general case of symmetric block norms (even without the restriction on the number of fundamental directions), see (Hamacher and Klamroth, 1997), we can derive the following relation between the barrier distances $\gamma_{\mathcal{B}}$ and $l_{1,T(\mathcal{B})}$:

**Lemma 10** *Let $\gamma$ be a block norm with four fundamental directions and let the linear transformation $T$ be defined as in (3). Then*

$$\gamma_{\mathcal{B}}(X,Y) = l_{1,T(\mathcal{B})}(T(X),T(Y))$$

*for all $X, Y \in F$.*

**Proof:** Let $X, Y \in F$ be two feasible points and let $SP$ be a $\gamma$-shortest permitted path connecting $X$ and $Y$ with Property (1) of Lemma 4. Thus $SP$ is a piecewise linear path with a finite number of breaking points $I_1, \ldots, I_k \in F$. Since $T$ is linear and bijective, the image $T(SP)$ of $SP$ is feasible in $T(F)$, i.e. $T(SP) \subseteq T(F)$. Furthermore line-segments in $SP$ are transformed into line-segments in $T(SP)$ and thus $T(SP)$ is also a piecewise linear path with breaking points $T(I_i)$, $i = 1, \ldots, k$. Denoting $I_0 := X$ and $I_{k+1} := Y$ and using Lemma 9, we get

$$\begin{aligned}
\gamma_{\mathcal{B}}(X,Y) &= \sum_{i=0}^{k} \gamma(I_i, I_{i+1}) \\
&= \sum_{i=0}^{k} l_1(T(I_i), T(I_{i+1})) \\
&\geq l_{1,T(\mathcal{B})}(T(X),T(Y)).
\end{aligned}$$

Analogously let $SP'$ be an $l_1$-shortest permitted path in $T(F)$ connecting $T(X)$ and $T(Y)$ with Property (1) of Lemma 4. Let $I'_1, \ldots, I'_{k'} \in T(F)$ be the finite set of breaking points on $SP'$ and let $I'_0 := T(X)$ and $I'_{k'+1} := T(Y)$. Then

$$\begin{aligned}
l_{1,T(\mathcal{B})}(T(X),T(Y)) &= \sum_{i=0}^{k'} l_1(I'_i, I'_{i+1}) \\
&= \sum_{i=0}^{k'} \gamma(T^{-1}(I'_i), T^{-1}(I'_{i+1})) \\
&\geq \gamma_{\mathcal{B}}(X,Y),
\end{aligned}$$

completing the proof.

□

Lemma 10 enables us to derive the main result of this section.

**Theorem 5** *Let $\gamma$ be a block norm with four fundamental directions and let $T$ be a linear transformation defined as in (3). Furthermore let $\mathcal{E}x = \{Ex_m : m \in \mathcal{M}\}$ be a set of existing facilities in $F$.*
*$X^*_{\mathcal{B}}$ is an optimal solution of $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\max$ with existing facilities $\mathcal{E}x$ if and only if $T(X^*_{\mathcal{B}})$ is an optimal solution of the transformed problem $1/P/T(\mathcal{B})/l_{1,T(\mathcal{B})}/\max$ with existing facilities $T(\mathcal{E}x) := \{T(Ex_m) : m \in \mathcal{M}\}$.*

**Proof:** Follows immediately from Lemma 10 since

$$\max_{m \in \mathcal{M}} \{\gamma_{\mathcal{B}}(X, Ex_m)\} = \max_{m \in \mathcal{M}} \{l_{1,T(\mathcal{B})}(T(X), T(Ex_m))\}$$

for all $X \in F$.

□

# 6 Conclusions

In this paper a discretization result is developed for center location problems with Manhattan distance where polyhedral barriers restrict traveling in the plane. A polynomial time algorithm to solve this non-convex optimization problem is suggested which is based on this discretization result.

This paper can be seen as a continuation of earlier work on the discretization of planar location problems which has proven to be a powerful method in location theory. Future research includes the analysis of level curves for barrier problems which will be helpful, for instance, in dealing with multi-criteria location problems with barriers.

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