University of Kaiserslautern

Master's Thesis

Homogeneity and Derivations on Analytic Algebras

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Contents

Abstract

In the present master's thesis we investigate the connection between derivations and homogeneities of complete analytic algebras. From now on, denote a complete analytic algebra by R. We prove a theorem, which describes a specific set of generators for the module of derivations of R, which map the maximal ideal of R into itself. It turns out, that this set has a structure similar to a Cartan subalgebra and contains the information regarding the maximal multi-homogeneity of R. In order to prove this theorem, we extend the notion of grading by Scheja and Wiebe (see [30],[32]) to projective systems and state the connection between multi-gradings and pairwise commuting diagonalizable derivations. We prove a theorem similar to Cartan's Conjugacy Theorem in the setup of infinite-dimensional Lie algebras, which arise as projective limits of finite-dimensional Lie algebras. Using this result, we can show that the structure of the aforementioned set of generators is intrinsic to the analytic algebra R and does not depend on any choice of coordinates. Finally, we state an algorithm, which is theoretically able to compute the maximal multi-homogeneity of a complete analytic algebra.

1 Introduction

In the present thesis we investigate the relation between homogeneities and derivations of (complete) analytic algebras. Consider for example the polynomial $f := x^2+y^3 \in \mathbb{C}[[x,y]]$, then it easy to see, that f is homogeneous with respect to the weights $(3,2)$. This induces a grading on the complete analytic algebra $R := \mathbb{C}[[x,y]]/\langle f \rangle$. We say that R has a $(\mathbb{Q},+)$ grading with respect to the weight-vector $(3,2)$. An important question is, whether there are more possibilities for the grading of R, which are $\mathbb{Q}$-linear independent. In our example it turns out, that the grading induced by $(3,2)$ is the only grading for which R is graded. To investigate this topic, we use the connection between derivations of R, which map the maximal ideal of R into itself, and homogeneities of R. One of the first to investigate the connection between homogeneities and derivations was K. Saito in 1971 (see [29]). Saito proved, that a convergent power series f with an isolated hypersurface singularity at 0 is homogeneous, if it is an eigenfunction of a derivation δ of R into itself. In 1972, G. Scheja and H. Wiebe extended this idea to analytic algebras (see [30]). They stated, that homogeneities of an analytic algebra correspond to semi-simple derivations of R. In 1977 and 1980 they extend their previous results by using methods from linear algebra and projective limits (see [31] respectively [32]). One of the most important results was, that any derivation δ of a complete analytic algebra R, which maps the maximal ideal of R into itself, has a Chevalley decomposition, that is a decomposition $\delta = \delta_S + \delta_N$, where δ_S is a semi-simple derivation and δ_N is a nilpotent one. E. Kunz and W. Ruppert used the idea of derivations to show, that $f \in R$ is homogeneous, if and only if there exists a derivation δ with $\delta(f) = \lambda f$ for some constant λ (see [24]). These are the most important results connecting derivations to homogeneities. Now the question is, why do we need to investigate homogeneities of analytic algebras? We do so, for example, because we can use this information to classify isolated Gorenstein-curve singularities, as done by G.-M. Greuel, B. Martin and G. Pfister in 1985 (see [17]). Furthermore, the investigation of maximal multi-homogeneities, as in our thesis, is very useful in the classification of complete analytic algebras in general, as the dimension s of the $\mathbb{Q}$-vector space generated by the homogeneities is an invariant.

We show, that maximal multi-homogeneities arise from so called *multi-gradings*, which are gradings by K-vector spaces. The connection to derivations comes from the fact, that pairwise commuting derivations can be simultaneously diagonalized. In order to prove our results, we start Chapter 2 by stating basic results regarding projective limits, the notion of grading of rings and Lie algebras. Chapter 3 then deals with the connection between gradings of analytic algebras and derivations as in [30]. These chapters are meant to be an introduction into the basic tools and results we are going to use during the course of our thesis and are not meant to be a full treatment of the aforementioned topics. They do not contain any new results. In Chapter 4 we introduce the notion of grading to projective systems and so called Lie-Rinehart algebras to prove a general version of the Formal Structure Theorem by Granger and Schulze (see [13]), which makes it possible to state, for example, the structure of the module of $\mathfrak{m}_R$-invariant derivations $\mathrm{Der}'(R)$, where R is a complete analytic algebra. Chapter 5 is concerned with the topic of profinite Lie(-Rinerhart) algebras. We generalize Cartan's Conjugacy Theorem (see for example [6, Theorem 3.5.1]) to the setup of profinite Lie algebras. This seemingly new result makes it possible to prove, that the dimension s of the K-vector space corresponding to our maximal multi-homogeneity is uniquely determined, hence can be considered as an invariant of the complete analytic algebra R. In Chapter 6 we deal with the theory of standard bases in the setup of convergent power series rings and use methods regarding them to state an algorithm for the computation of the maximal multi-homogeneity of ideals. This information is encoded in semi-simple matrices. Our algorithm returns a basis of a Lie algebra $\mathfrak{g}$, containing the needed information, but does not necessarily compute it explicitly. The latter means, that this basis does not contain all semi-simple matrices, which are contained in $\mathfrak{g}$, but at least gives a lower bound.

2 Projective Limits, Gradings and Lie Algebras

The following chapter is a summary of basic results regarding projective limits, Lie algebras and gradings of rings. We stay close to [30] for the results about grading and [6] for the results regarding Lie algebras. We start by stating results on projective limits, then about gradings of rings and after that, we state basic results regarding Lie algebras. We omit proofs in this chapter, as long as they are not of further concern for our thesis or give any insight on the topic.

2.1 Projective Limits and Completions

In the following section we introduce the notion of projective limits and the notion of completions. This is the basic object we are going to work with in the course of this thesis.

We start with the set theoretical definition of projective limits and after that pass to a category theoretical result. The definition is taken from [27, Chapter 12].

Definition 2.1
Let J be a (partially) ordered set of indexes. Assume further, that J is directed, which means, that for any $j, j' \in J$ there exists $k \in J$ with $j \leq k$ and $j' \leq k$. Assume given a family $(G_j)_{j \in J}$ of sets (groups, rings, topological spaces, etc.) together with maps (homomorphisms)

$$f_{ij} : G_j \to G_i$$

for each pair (i, j) of indexes in J, such that $i \leq j$. This setup is called projective system*, if in addition we have*

$$f_{ik} = f_{ij} \circ f_{jk},$$

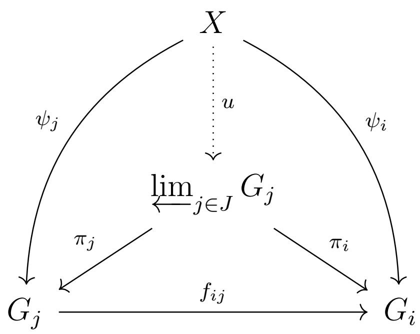
for all $i \leq j \leq k$. We use (G_j, f_{ij}) as a short notation for a projective system.
The projective limit *of such a projective system is defined as the following subset of the Cartesian product of the G_j:*

$$\varprojlim_{j \in J} G_j := \{(\sigma_j)_j \in \prod_{j \in J} G_j | f_{ij}(\sigma_j) = \sigma_i \text{ for } i \leq j\}.$$

The following result is the universal property of projective limits. Sometimes this is used as the definition of a projective limit, see for example [26, Definition 5.1.19 b)].

Theorem 2.2

Let J be a (partially) ordered and directed set of indexes and (G_j, f_{ij}) a projective system of sets (groups, rings, topological spaces, etc.) as in Definition 2.1. Assume the projective limit $\varprojlim_{j\in J} G_j$ exists, then it satisfies the following universal property:
Denote by $\pi_i : \varprojlim_{j\in J} G_j \to G_i$ the natural projections of the projective limit and let X be an arbitrary set (group, ring,topological space, etc.) with maps (respectively morphisms) $\psi_i : X \to G_i$ such that $f_{ij} \circ \psi_j = \psi_i$ for all $i \leq j$, then there exists a unique map (respectively morphism) $u : X \to \varprojlim_{j\in J} G_j$, such that the following diagram commutes:



Proof:
See [26, Example 5.1.22] for a proof in the category of sets. The proof works analogously in all our other categories like groups, rings or modules. □

Corollary 2.3

The projective limit, if it exists, is unique up to unique isomorphism.

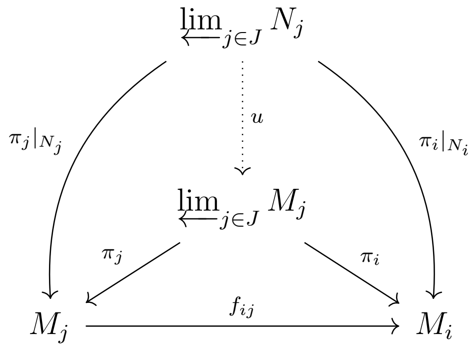
Proof:
This follows immediately from the universal property in Theorem 2.2, see for example [26, Corollary 6.1.2]. □

Next, we take a look at a setup, in which we have two projective systems with the same projective limit.

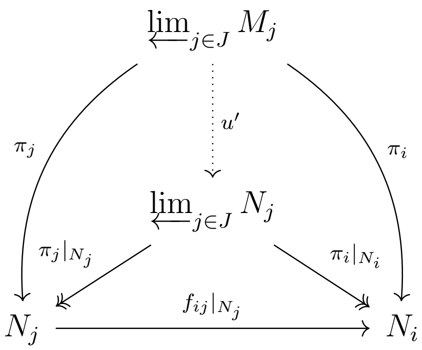
Proposition 2.4

Let J be a (partially) ordered and directed set of indexes and let $(M_j, f_{ij})_{i,j\in J}$ be a projective system of sets (groups, rings, topological spaces, etc.). Define $M := \varprojlim_{j\in J} M_j$. Denote by $\pi_j : M \to M_j$ the projections from M to the M_j and define $N_j := \pi_j(M)$. Then $(N_j, f_{ij}|_{N_j})$ is a projective system and $M \cong \varprojlim_{j\in J} N_j$.

Proof:
$(N_j, f_{ij}|_{N_j})$ is clearly a projective system, so we only have to show, that it is isomorphic to M. Consider the following commutative diagram:



Furthermore, we get another commutative diagram:



Now, as $u \circ u'$ equals the identity on $\varprojlim_{j\in J} M_j$, by the universal property of projective limits, and as $u' \circ u$ equals the identity on $\varprojlim_{j\in J} N_j$, again by the universal property, we get, that $M = \varprojlim_{j\in J} M_j \cong \varprojlim_{j\in J} N_j$. $\square$

Let us take a look at a simple example of a projective limit.

Example 2.5
Let J be a non-empty set. Then we can define a partial ordering on J by simply stating $a \leq b :\iff a = b$ for all $a, b \in J$. This implies for any projective system (G_j, f_{ij}) of sets (groups, rings, topological spaces, etc.) indexed by J as in Definition 2.1, that we have:

$$\varprojlim_{j\in J} G_j \cong \prod_{j\in J} G_j.$$

Before we go on with more advanced results, we state a useful lemma regarding the commutativity of projective limits.

Lemma 2.6
Let $(I, \leq)$ and $(J, \leq)$ be (partially) ordered and directed sets. Endow $I \times J$ with the ordering $(i,j) \leq (i',j') :\iff i \leq j$ and $i' \leq j'$. Then for any projective system $(G_{ij}, f_{(i,j),(i',j')})$ of sets (groups, rings, topological spaces, etc.), we have that:

$$\varprojlim_{i\in I}\varprojlim_{j\in J} G_{ij} \cong \varprojlim_{(i,j)\in I\times J} G_{ij} \cong \varprojlim_{j\in J}\varprojlim_{i\in I} G_{ij}.$$

Proof:
See [26, Proposition 6.2.8].

□

The last topic regarding projective limits in general we are investigating, is the behavior of projective limits as a functor. The following result states, that $\varprojlim$ is a left-exact functor.

Lemma 2.7
Let J be a (partially) ordered and directed set of indexes and (A_j, f_{ij}^A), (B_j, f_{ij}^B) and (C_j, f_{ij}^C) be projective systems of sets (groups, rings, topological spaces, etc.). If we have for all $j \in J$ an exact sequence

$$0 \to A_j \to B_j \to C_j \to 0,$$

then we have an exact sequence

$$0 \to \varprojlim_{j\in J} A_j \to \varprojlim_{j\in J} B_j \to \varprojlim_{j\in J} C_j.$$

Proof:
The proof works the same way as in [1, Lemma 1.9]. □

In general, projective limits do not exist, but these cases are not of our concern. Next, we work in a setup, where projective limits exists, namely in the setup of completions. The following results are taken from [9, Chapter 7].
Let us start with the definition.

Definition 2.8
Let R be a Noetherian ring and $R = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \dots$, where $\mathfrak{m}_i, i \in \mathbb{N}$, are ideals of R. Then we define the completion *$\hat{R}$* of *R as the projective limit $\hat{R} := \varprojlim_{i\in\mathbb{N}} R/\mathfrak{m}_i$. If we have $\mathfrak{m}_i = \mathfrak{m}^i$, then we call $\hat{R}$ the* $\mathfrak{m}$-adic completion*. Furthermore, if M is an R-module and $\mathfrak{m}_i$ as before, we define the completion $\hat{M}$ of M, as $\hat{M} := \varprojlim_{i\in\mathbb{N}} M/\mathfrak{m}_i M$.*

As the projective limit does not always exist, we need the following theorem.

Theorem 2.9

Let R be a Noetherian ring, $\mathfrak{m}$ an ideal of R and M an finitely generated R-module. Denote by $\hat{R}$ the $\mathfrak{m}$-adic completion of R, respectively by $\hat{M}$ the $\mathfrak{m}$-adic completion of M, then:

i) $\hat{R}$ *exists and is Noetherian.*

ii) $\hat{R}/\mathfrak{m}^j\hat{R} \cong R/\mathfrak{m}^j$.

iii) $\hat{M} \cong \hat{R} \otimes_R M$.

Proof:
See [9, Theorem 7.1 and 7.2]. □

Let us have a look at a standard example in the context of completions.

Example 2.10

Let R be a polynomial ring in n variables over a field K, that is, $R = K[x_1, \ldots, x_n]$. Consider the ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. Then the $\mathfrak{m}$-adic completion $\hat{R} = \varprojlim_{i \in \mathbb{N}} R/\mathfrak{m}^i \cong K[[x_1, \ldots, x_n]]$, the power series ring over K in n variables.

The last theorem we are stating, is Cohen's famous Structure Theorem. For details see [9, Theorem 7.7].

Theorem 2.11

Let R be a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field K. If R contains a field, then $R \cong K[[x_1, \ldots, x_n]]/I$ for some $n \in \mathbb{N}$ and I an ideal of $K[[x_1, \ldots, x_n]]$.

2.2 Gradings of Rings and Modules

In the following chapter we state a more general definition of the grading of a ring respectively a module. The definitions we state are taken from [30, Chapter 1]. For the classical definition of grading in the context of rings or modules, we refer the reader to [18, Chapter 2.2]. We start with the basic definition of finitely graded rings and modules:

Definition 2.12

Let $(G, +)$ be an abelian group, R a ring and M an R-module. R is a finitely graded *ring, if we have a system of group homomorphisms $\pi_g^R : R \to R$ for $g \in G$ with the property $\pi_g^R(R)\pi_h^R(R) \subseteq \pi_{g+h}^R(R)$ for all $g, h \in G$, such that R can be written as a direct sum of the subgroups $\pi_g^R(R)$. Furthermore, M is a* finitely graded *module, if R is graded with respect to a system of group homomorphisms π_g^R, $g \in G$ as before, which is compatible with group homomorphisms $\pi_g^M : M \to M$, that is $\pi_g^R(R)\pi_h^M(M) \subseteq \pi_{g+h}^M(M)$ for all $g, h \in G$, such that M can be written as a direct sum of the subgroups $\pi_g^M(M)$.*

Remark 2.13
Definition 2.12 basically extends the well known idea of grading rings in the multivariate polynomial case. Consider for example the polynomial ring $R := \mathbb{Q}[x_1, \ldots, x_n]$*. Using multi-indices* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$*, we can write any* $f \in R$ *as* $f = \sum_{|\alpha|=0}^{|\alpha|=m} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$*, where* m *is the total degree of* f*. To keep notation short, we write* $f = \sum_\alpha f_\alpha$*, where* f_α *denotes the homogeneous degree* $|\alpha|$ *part of* f*. For more details on the grading of multivariate polynomial rings see [18]. Now* R *can be written as* $R = \bigoplus_{|\alpha|\geq 0} \mathbb{Q}x_1^{\alpha_1} \cdots x_n^{\alpha_n}$*. If we consider the group* $(G, +) := (\mathbb{Z}, +)$ *and the group homomorphisms*

$$\pi_g : R \to R$$

$$f \mapsto \begin{cases} 0, & \text{if } g < 0 \\ f_\alpha, & \text{with } |\alpha| = g \end{cases}$$

We directly get the desired properties of $(\pi_g)_{g \in G}$ *as in Definition 2.12.*

The next interesting aspect is the general, not necessarily finite, grading of rings and modules. We start with the definition of Zariski rings (see for example [2, Chapter 10, Exercise 6]), as this is the setup in which we are able to define general gradings.

Definition 2.14
Let R *be a ring. We say* R *is a* Zariski ring*, if* R *is a commutative unitary Noetherian topological ring, whose topology is defined by an ideal* $\mathfrak{m}$ *contained in the Jacobson ideal of* R*.*

Now we can define general gradings.

Definition 2.15
Let $(G, +)$ *be an abelian group,* R *a Zariski ring and* M *a finitely generated* R*-module.* R *is a* graded *ring, if we have a system of group homomorphisms* $\pi_g^R : R \to R$ *for* $g \in G$*, which induce group homomorphisms* $\overline{\pi_g^R} : R/\mathfrak{m}^n \to R/\mathfrak{m}^n$ *that define a finite grading on* $R/\mathfrak{m}^n$ *for all* $n \in \mathbb{N}$*.* M *is a* graded *module, if* R *is graded with respect to a system of group homomorphisms* π_g^R*,* $g \in G$ *as before, which is compatible with group homomorphisms* $\pi_g^M : M \to M$ *which induce group homomorphism* $\overline{\pi_g^M} : M/\mathfrak{m}^n M \to M/\mathfrak{m}^n M$ *that define a finite grading on* $M/\mathfrak{m}^n M$ *as an* $R/\mathfrak{m}^n$*-module for all* $n \in \mathbb{N}$*.*

Remark 2.16
The grading in the sense of Definition 2.15, is basically a grading of $\mathfrak{m}$*-adic completions, as we reduce the grading of a ring* R*, to gradings on all* $R/\mathfrak{m}^k$*. The same holds also for modules. We extend this idea to the grading of projective limits in Chapter 4.*

Example 2.17
Let us consider the ring $R := \mathbb{Q}[[x_1, \ldots, x_n]]$*,* $\mathfrak{m} := \langle x_1, \ldots, x_n \rangle$ *and* $(G, +) := (\mathbb{Z}, +)$*. Define* π_g *as in Remark 2.13, just extended to power series. We get that the* π_g *induce a finite grading on* $R/\mathfrak{m}^k$ *for all* $k \in \mathbb{N}$*, as* $R/\mathfrak{m}^k = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1, \ldots, x_n \rangle^k$ *by Theorem 2.9 ii), hence* R *is graded in the sense of Definition 2.15.*

Due to the fact, that we have a topology on our rings and modules, we can define a notion of convergence, which is the same notion of convergence, as in the context of completions.

Definition 2.18
Let M be a graded R-module. The sum $\sum_{g \in G} m_g$, $m_g \in M$, converges *to $m \in M$, if and only if for any $n \in \mathbb{N}$ there exists a finite $E_0 \subseteq G$, such that for all $E \subseteq G$ with $E_0 \subseteq E$ we have: $m - \sum_{g \in E} m_g \in \mathfrak{m}^n M$. Then we write $m = \sum_{g \in G} m_g$.*

The following statements generalize basic results of graded modules, as stated for example in [18].

Theorem 2.19
Let M be a graded R-module with system of group homomorphisms $(\pi_g^M)_{g \in G}$. Every $m \in M$ can be written as $m = \sum_{g \in G} \pi_g^M(m)$. If $m = \sum_{g \in G} m_g$ with $m_g \in \pi_g^M(M)$, then we already have $m_g = \pi_g^M(m)$ for all $g \in G$. m_g is called the g-th homogeneous component *of m.*

Proof:
See [30, (1.1)]. □

Proposition 2.20
For all $g, h \in G$ we have: $\pi_g^2 = \pi_g$, $\pi_g \circ \pi_h = 0$, if $g \neq h$, and $\pi_g^R(R)\pi_h^M(M) \subseteq \pi_{g+h}^M(M)$.

Proof:
See [30, (1.2)]. □

The next natural step is to take a look at submodules of graded modules.

Definition 2.21
Let M be a graded R-module and N a subgroup of M. N is called homogeneous, *if $\pi_g^M(N) \subseteq N$ for all $g \in G$.*

The following three theorems characterize homogeneous submodules, resulting quotient modules and their grading.

Theorem 2.22
Let M be a graded R-module and N a submodule of M. N is homogeneous if and only if N can be generated by homogeneous elements.

Proof:
See [30, (1.3)]. □

Theorem 2.23
Let M be a graded R-module with system of group homomorphisms $(\pi_g^M)_{g \in G}$ and N a homogeneous submodule of M. Then the group homomorphisms $\pi_g^M|_N : N \to N$, $g \in G$, induce a grading of N as an R-module.

Proof:
See [30, (1.4)]. □

Theorem 2.24
Let M be a graded R-module with system of group homomorphisms $(\pi_g^M)_{g \in G}$ and N a homogeneous submodule of M. Then the group homomorphisms $\overline{\pi_g^M} : M/N \to M/N$, $g \in G$, induce a grading of M/N as an R-module.

Proof:
See [30, (1.5)]. □

2.3 Basic Results on Lie Algebras

In this section we present the basic results regarding Lie algebras, which we are going to use in the underlying thesis. We stay close to [6], but we use the notation from [33].

Remark 2.25
All vector spaces in this chapter are finite-dimensional, although the definition of a Lie algebra naturally extends to the infinite-dimensional case. The latter is not of further concern at the moment.

2.3.1 Basic Definitions and Constructions regarding Lie Algebras

Let us start with the definition of an algebra.

Definition 2.26
An algebra *is vector space $\mathfrak{g}$ over a field K together with a bilinear map*

$$[\cdot,\cdot] \ : \ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

Remark 2.27
The brackets used in Definition 2.26 are the so called Lie brackets*.*

Now we can define Lie algebras.

Definition 2.28
An algebra $\mathfrak{g}$ over a field K is said to be a Lie algebra*, if its multiplication has the following properties:*

a) $[x, x] = 0$ *for all* $x \in \mathfrak{g}$,

b) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *for all* $x, y, z \in \mathfrak{g}$.

A Lie algebra is called finite-dimensional*, if it is finite dimensional as a Kvector space.*
A subspace $\mathfrak{h}$ of $\mathfrak{g}$ satisfying the previous properties is called Lie subalgebra*.*

Let us take a look at a typical example of a Lie algebra, which is our standard example for a Lie algebra. A Lie subalgebra of the latter is in the focus of our computations in Chapter 6.

Example 2.29
Let V be an n-dimensional vector space over the field K. Denote by $\mathrm{End}(V)$ the set of all linear maps from V to V. We can turn $\mathrm{End}(V)$ into a Lie algebra using the following definition of the Lie brackets:

$$[a, b] := ab - ba$$

for all $a, b \in \mathrm{End}(V)$. It is easy to see, that the properties of Definition 2.28 are satisfied. We denote this Lie algebra by $\mathfrak{gl}(K, n)$.

The first natural structure arising in algebra, are quotient algebras.

Definition 2.30
Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ an ideal of $\mathfrak{g}$. Then the algebra $\mathfrak{g}/\mathfrak{i}$ is called the quotient algebra *of $\mathfrak{g}$ and $\mathfrak{i}$.*

Remark 2.31
The induced operations of quotient algebras are well defined (see [6, Proposition 1.15]) and $\mathfrak{g}/\mathfrak{i}$ is also a Lie algebra.

The next structures regarding Lie algebras we are talking about, are the centralizer and the normalizer.

Definition 2.32
Let $\mathfrak{g}$ be a Lie algebra and $S \subset \mathfrak{g}$. Then the set

$$\mathrm{C}(S) := \{x \in \mathfrak{g} |\ [x, s] = 0 \textit{ for all } s \in S\}$$

is called centralizer of S*. Furthermore, if $S = \mathfrak{g}$, we call $\mathrm{C}(\mathfrak{g})$ the* centre *of $\mathfrak{g}$.*

Definition 2.33
Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a subspace of $\mathfrak{g}$. Then the set

$$\mathrm{N}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} |\ [x, h] \in \mathfrak{h} \textit{ for all } h \in \mathfrak{h}\}$$

is called normalizer *of $\mathfrak{h}$ in $\mathfrak{g}$.*

Remark 2.34
We write $\mathrm{N}(\mathfrak{h})$ instead of $\mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$, if it is obvious, in which Lie algebra we are working.

Remark 2.35
Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subspace of $\mathfrak{g}$. It can be shown, that $\mathrm{C}(\mathfrak{h})$ and $\mathrm{N}(\mathfrak{h})$ are subalgebras of $\mathfrak{g}$. Furthermore, if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ is an ideal of the Lie algebra $\mathrm{N}(\mathfrak{h})$.

2.3.2 Morphisms of Lie Algebras

The next objects, which are typically investigated when dealing with a new algebraic structure, are morphisms. In the following section we present basic results regarding morphisms between Lie algebras.

Definition 2.36
Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over the field K. A K-linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ satisfying $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$ is called a morphism of Lie algebras*. If ϕ is a bijection, we call ϕ an* isomorphism *and we say that $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic. The latter is denoted by $\mathfrak{g} \cong \mathfrak{h}$.*

Proposition 2.37
Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over the field K and $\phi : \mathfrak{g} \to \mathfrak{h}$ a Lie algebra morphism. Then $\phi(\mathfrak{g})$ and $\phi^{-1}(\mathfrak{h})$ are subalgebras of $\mathfrak{h}$ resp. $\mathfrak{g}$.

To see how to work with morphisms of Lie algebras, we prove the statement about the preimage, as the one for the image works using the same idea.

Proof:
Set $\mathfrak{g}' := \phi^{-1}(\mathfrak{h})$. Then $\mathfrak{g}'$ is a K vector space, as ϕ is a linear map, thus we only need to show, that the operation of the Lie brackets is closed. Take $x, y \in \mathfrak{g}'$, then $\phi([x,y]) = [\phi(x), \phi(y)]$, hence $[x,y]$ is the preimage of an element of $\mathfrak{h}$ and we have shown, that $\phi^{-1}(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$. □

The following example is a morphism of Lie algebras, which is also used for the representation of Lie algebras (see for example [6, Chapter 1.12]).

Example 2.38
Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. Then the map $[x, \cdot] : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x,y]$ is a morphism of Lie algebras, which can easily be seen. It is the so called adjoint map*, which is denoted by ad_x.*

Remark 2.39
Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. We denote the kernel of ϕ by $\ker(\phi)$ and the image of ϕ by $\mathrm{im}(\phi)$. $\ker(\phi)$ is an ideal of $\mathfrak{g}$ and $\mathrm{im}(\phi)$ is a subalgebra of $\mathfrak{h}$.

The following results are the isomorphism theorems for Lie algebras.

Theorem 2.40
Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. Then

$$\mathfrak{g}/\ker(\phi) \cong \mathrm{im}(\phi).$$

Proof:
See [6, Lemma 1.8.1]. □

Theorem 2.41
Let $\mathfrak{g}$ be a Lie algebra with ideals $\mathfrak{i}$ and $\mathfrak{j}$. Then the following statements hold:

i) If $\mathfrak{i} \subset \mathfrak{j}$, then the quotient Lie algebra $\mathfrak{j}/\mathfrak{i}$ is an ideal of the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$ and we have $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}$.

ii) We have $(\mathfrak{i}+\mathfrak{j})/\mathfrak{j} \cong \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$.

Proof:
See [6, Proposition 1.8.2]. □

The next topic we need to talk about are automorphisms of Lie algebras. We start with their definition.

Definition 2.42
Let $\mathfrak{g}$ be a Lie algebra. An automorphism *of $\mathfrak{g}$ is an isomorphism of $\mathfrak{g}$ onto itself. The set of all automorphisms of $\mathfrak{g}$ is denoted by $\mathrm{Aut}(\mathfrak{g})$.*

Proposition 2.43
Let $\mathfrak{g}$ be a Lie algebra. $\mathrm{Aut}(\mathfrak{g})$ is a group, the so called automorphism group *of $\mathfrak{g}$.*

Proof:
See the discussion in [6] prior to Example 1.11.1. □

The next type of morphisms we are taking a look at are the so called inner automorphism, which are playing an important role in Chapter 5.

Lemma 2.44
Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. If ad_x is nilpotent, that is, there exists some $n \in \mathbb{N}$ such that $\mathrm{ad}_x^n = 0$, then $\exp(\mathrm{ad}_x) \in \mathrm{Aut}(\mathfrak{g})$, where $\exp(\mathrm{ad}_x) := \sum_{i=0}^{n-1} \frac{ad_x^i}{i!}$.

Proof:
For a proof see [6, Lemma 1.11.2]. □

Definition 2.45
Let $\mathfrak{g}$ be a Lie algebra. The automorphisms of the type described in Lemma 2.44 are called inner automorphisms*. The set of all inner automorphisms is denoted by $\mathrm{Inn}(\mathfrak{g})$.*

Proposition 2.46
Let $\mathfrak{g}$ be a Lie algebra. $\mathrm{Inn}(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Proof:
See the discussion in [6] after the proof of Lemma 1.11.2. □

We finish this section with an important result regarding the inner automorphisms.

Proposition 2.47
Let K be an algebraically closed field, V a finite-dimensional vector space and $\mathfrak{g} := \mathfrak{gl}(K, n)$. If $x \in \mathfrak{g}$ is diagonalizable, then $\phi(x)$ is diagonalizable for all $\phi \in \mathrm{Inn}(\mathfrak{g})$.

Proof:
A simple computation shows, that if $y \in \mathfrak{g}$ is a nilpotent endomorphism, we get

$$\exp([y, x]) = \exp(y) x \exp(-y)$$

for any $x \in \mathfrak{g}$. Using this, we get that every element of $\mathrm{Inn}(\mathfrak{g})$ operates by conjugation on the elements of $\mathfrak{g}$. Due to the fact, that being diagonalizable is invariant under conjugation (see [25, Chapter XIV, §3]), we get that $\phi(x)$ is diagonalizable, if $x \in \mathfrak{g}$ is so, for any $\phi \in \mathrm{Inn}(\mathfrak{g})$. □

2.3.3 Nilpotent Lie Algebras and Cartan Subalgebras

An important class of Lie algebras are so called nilpotent Lie algebras. In this subsection we define them and state important results regarding them. The results regarding the finite-dimensional case are playing an important role in the infinite-dimensional case, which we are treating in Chapter 5.

Before we can define nilpotent Lie algebras, we need the following result regarding ideals.

Lemma 2.48
Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$, $\mathfrak{j}$ ideals of $\mathfrak{g}$. Then $[\mathfrak{i}, \mathfrak{j}]$ is an ideal of $\mathfrak{g}$.

Proof:
See [6, Lemma 1.7.1]. □

The next definition is the basis for the definition of nilpotent Lie algebras.

Definition 2.49
Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{g}^{(1)} := \mathfrak{g}$ and $\mathfrak{g}^{(i)} := [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$ for $i \in \mathbb{N}_{\geq 2}$. Then the sequence

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \ldots \supset \mathfrak{g}^{(i)} \supset \ldots$$

is called lower central series *of* $\mathfrak{g}$.

Now we can define nilpotent Lie algebras.

Definition 2.50
Let $\mathfrak{g}$ be a Lie algebra. If there exists an integer k, such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is called nilpotent.

An important result concerning nilpotent finite-dimensional Lie algebras, is the following theorem.

Theorem 2.51 (Engel's Theorem)
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if ad_x is nilpotent for all $x \in \mathfrak{g}$.

Proof:
See [6, Theorem 2.1.5]. □

Now we can define a special type of nilpotent subalgebras, namely Cartan subalgebras.

Remark 2.52
In the following we restrict ourselves to the case of Lie algebras over algebraically closed fields of characteristic 0.

Definition 2.53
Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ is called Cartan subalgebra*, if the following is satisfied:*

i) *$\mathfrak{h}$ is nilpotent.*

ii) $\mathrm{N}(\mathfrak{h}) = \mathfrak{h}$.

Proposition 2.54
Let $\mathfrak{g}$ be a Lie algebra. Then there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Proof:
This follows from [6, Corollary 3.2.8], as we are in the case of characteristic zero and our field has infinite elements. □

The following theorem shows, that Cartan algebras of a finite-dimensional Lie algebra form a single conjugacy class. A similar result holds in suitable cases for infinite dimensional Lie algebras, as we will see in Chapter 5.

Theorem 2.55
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two Cartan subalgebras of $\mathfrak{g}$. Then there exists a $\sigma \in \mathrm{Inn}(\mathfrak{g})$, such that $\mathfrak{h} = \sigma(\mathfrak{h}')$.

Proof:
See [6, Theorem 3.5.1]. □

After stating some theory about Lie Algebras, we state an example.

Example 2.56
Let K be a field and consider the Lie algebra $\mathfrak{gl}(K, n)$. How does a Cartan subalgebra of $\mathfrak{gl}(K, n)$ look like? We claim, that the set of diagonal matrices is a Cartan subalgebra of $\mathfrak{gl}(K, n)$. Denote this set by $\mathfrak{h}$. It is easy to see, that $\mathfrak{h}$ is nilpotent, as diagonal matrices commute with each other. To verify the normalizer property, we need some notation. Denote by E_{ij} the canonical basis of the vector space $\mathfrak{gl}(K, n)$, then it can be easily verified, that $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$, where δ_{ij} is the Kronecker delta. Using this relation, it can easily be seen, that no non-diagonal basis vector satisfies the normalizer property , as all *diagonal matrices are contained in $\mathfrak{h}$.*

Before we finish this section on Lie algebras, we state two final results regarding the behavior of Cartan subalgebras under surjective Lie algebra morphisms.

Theorem 2.57
Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over the field K and $\phi : \mathfrak{g} \to \mathfrak{g}'$ a surjective Lie algebra morphism. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\phi(\mathfrak{h})$ is Cartan subalgebra of $\mathfrak{g}'$.

Theorem 2.58
Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over the field K and $\phi : \mathfrak{g} \to \mathfrak{g}'$ a surjective Lie algebra morphism. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}'$. Then $\phi^{-1}(\mathfrak{h}')$ is Cartan subalgebra of $\phi^{-1}(\mathfrak{g}')$ and also one of $\mathfrak{g}$.

Proof:
For a proof of the previous two results see [6, Lemma 3.6.2 and Lemma 3.6.3]. □

2.3.4 The Root Space Decomposition

The final topic regarding Lie algebras, we are taking a look at, is the so called root space decomposition. This is basically the decomposition of our Lie algebra into direct sums, which have some properties regarding a fixed Cartan subalgebra of our Lie algebra. To keep the computations and definitions as simple as possible, we are going to work in the context of algebraically closed fields of characteristic 0. Before we start with the root space decomposition, we need the so called primary decomposition.

Definition 2.59
Let V be a finite-dimensional vector space over a field K of dimension $n \in \mathbb{N}$ and consider a Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(K, n)$. A decomposition

$$V = V_1 \oplus \ldots \oplus V_s$$

of V into $\mathfrak{h}$-submodules V_i is said to be primary, *if the minimum polynomial of the restriction of x to V_i is a power of an irreducible polynomial for all $x \in \mathfrak{h}$ and $1 \leq i \leq s$. The subspaces V_i are called* primary components.

In general, a primary decomposition does not exist, but in a suitable setup, it does.

Proposition 2.60
Suppose that $\mathfrak{h}$ is nilpotent. Then V has a primary decomposition with respect to $\mathfrak{h}$.

Proof:
See [6, Corollary 3.1.8]. □

The next result regarding general primary decompositions is a uniqueness statement. First of all we need a precise definition for the circumstances, in which we get the uniqueness result.

Definition 2.61
A primary decomposition of V relative to $\mathfrak{h}$ is called collected, *if for any two primary components V_i and V_j, $i \neq j$, there is an $x \in \mathfrak{h}$, such that the minimum polynomials of the restrictions of x to V_i and V_j are powers of different irreducible polynomials.*

Theorem 2.62
Let $\mathfrak{h}$ be nilpotent. Then V has a unique collected primary decomposition relative to $\mathfrak{h}$.

Proof:
See [6, Theorem 3.1.10]. □

Now we can define the root space decomposition. We use the adjoint map, to map any Lie algebra $\mathfrak{h}$ to $\mathfrak{gl}(K, n)$, such that we can compute primary decompositions in the general setup, where our Lie algebras is not necessarily a subalgebra of $\mathfrak{gl}(K, n)$.

Definition 2.63
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. The collected primary decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$$

is called roots space decomposition.

Where does the name arise from? Consider any $h \in \mathfrak{h}$. Then the minimum polynomial of the restriction of ad_h to a primary component $\mathfrak{g}_i$ is a power of an irreducible polynomial. As our field is algebraically closed, this polynomial is of the form $x - \alpha_i(h)$, where $\alpha_i(h)$ is a scalar depending on i and h. By fixing the primary component $\mathfrak{g}_i$, we get a function $\alpha_i : \mathfrak{h} \to K$. This function is called a *root*. The corresponding primary component is called a *root space*. In the further course of this thesis, we index the root space by the corresponding root. Define

$$\mathfrak{g}_{\alpha_i} := \mathfrak{g}_i = \{g \in \mathfrak{g} | \text{ for all } h \in \mathfrak{h} \text{ there is a } k > 0 \text{ such that } (\mathrm{ad}_h - \alpha_i(h))^k(g) = 0\}.$$

Then we write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \ldots \oplus \mathfrak{g}_{\alpha_s}.$$

3 Derivations and Gradings of Analytic Algebras

In the following chapter, we state the definition of an analytic algebra in the context of our thesis, as well as results regarding it. After that, we state results regarding the module of derivations of analytic algebras. For the latter, we stay close to [32, Chapter 1]. Results regarding analytic algebras are taken from [16, Chapter 1] and [14]. To keep notation short and if it is obvious from the context, we write $\underline{x}$ for $(x_1, \ldots, x_n)$ and $K\langle\langle \underline{x} \rangle\rangle$ for $K\langle\langle x_1, \ldots, x_n \rangle\rangle$.

Remark 3.1
From now on, we work only over complete real valuation fields of characteristic 0. For details on valuation fields, see [27, Chapter 23]. We also assume, that the reader is familiar with the notion of convergent power series rings. For a treatment of the latter, see [14, Chapter 1 and Chapter 3].

3.1 Analytic Algebras

This section is dedicated to analytic algebras, as we are only concerned with rings being analytic algebras in the further course of the underlying thesis. We start with basic definitions. After that, we introduce basic results regarding analytic algebras. We omit proofs, as long as they are not of further concern for our thesis.

Definition 3.2
Let R be an algebra over the field K. R is called analytic algebra*, if it is the quotient ring of a convergent power series ring, that is, $R = K\langle\langle x_1, \ldots, x_n \rangle\rangle / I$ for some ideal I of the convergent power series ring $K\langle\langle x_1, \ldots, x_n \rangle\rangle$.*

Remark 3.3
From now on, all algebras R over a field K, are analytic algebras, if not stated otherwise. As all analytic algebras R are local rings (see for example [16, Chapter 1]), they have a unique maximal ideal, which we are denoting by $\mathfrak{m}_R$.

Definition 3.4
Let R be an analytic algebra, such that $R = \varprojlim_{k \in \mathbb{N}} R/\mathfrak{m}_R^k$, then R is called complete analytic algebra.

Lemma 3.5
Let R be a complete analytic algebra, then $R \cong K[[x_1, \ldots, x_n]]/I$ for some ideal I of $K[[x_1, \ldots, x_n]]$ and $n \in \mathbb{N}$.

Proof:
As R is a complete local Noetherian ring containing a field K, we can apply Theorem 2.11 and get the result immediately. □

To get a feeling for analytic algebras, we state two examples of analytic algebras, which show, that the valuation of the field plays an important role.

Example 3.6
i) Let $\mathbb{C}$ be the complex numbers endowed with the valuation induced by the absolute norm. Then the convergent power series ring $R := \mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$ is a proper subset of the formal power series ring $\mathbb{C}[[x_1, \ldots, x_n]]$ (see [16, Excercise 1.1.3]), where $n \in \mathbb{N}$. Clearly R is an analytic algebra over the complex numbers and its completion $\hat{R}$ equals $\mathbb{C}[[x_1, \ldots, x_n]]$.

ii) Let $\mathbb{C}$ be the complex numbers endowed with the trivial valuation. Then the convergent power series ring $R := \mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$ is equal to the formal power series ring $\mathbb{C}[[x_1, \ldots, x_n]]$ (see [16, Remark 1.1.1]), where $n \in \mathbb{N}$.

In the following we are listing important results regarding analytic algebras.

Theorem 3.7
Let R be an analytic algebra. Then the following hold:

i) R is Noetherian, that is, every ideal of R is finitely generated.

ii) Let $R := K\langle\langle x_1, \ldots, x_n\rangle\rangle$. Then R is a factorial ring.

Proof:
See [16, Theorem 1.15 and Theorem 1.16]. □

Our next theorem is the famous Implicit Function Theorem.

Theorem 3.8 (Implicit Function Theorem)
Let K be a field and let $f_i \in R = K\langle\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle\rangle$, $i = 1, \ldots, m$, satisfy $f_i(0, \ldots, 0) = 0$ and

$$\det \begin{pmatrix} \partial_{y_1} f_1(0, \ldots, 0) & \ldots & \partial_{y_m} f_1(0, \ldots, 0) \\ \vdots & & \vdots \\ \partial_{y_1} f_m(0, \ldots, 0) & \ldots & \partial_{y_m} f_m(0, \ldots, 0) \end{pmatrix} \neq 0.$$

Then $R/\langle f_1, \ldots, f_m\rangle \cong K\langle\langle x_1, \ldots, x_n\rangle\rangle$*, and there exists unique power series* $Y_1, \ldots, Y_m \in \mathfrak{m}_{K\langle\langle \underline{x}\rangle\rangle}$ *solving the implicit system of equations*

$$f_1(\underline{x}, \underline{y}) = \ldots = f_m(\underline{x}, \underline{y}) = 0$$

in $\underline{y}$*, that is, satisfying*

$$f_i(\underline{x}, Y_1(\underline{x}), \ldots, Y_m(\underline{x}) = 0,\ i = 1, \ldots, m.$$

Moreover, $\langle f_1, \ldots, f_m\rangle = \langle y_1 - Y_1, \ldots, y_m - Y_m\rangle$*.*

Proof:
See [16, Theorem 1.18]. □

The following theorem is the famous Inverse Function Theorem.

Theorem 3.9 (Inverse Function Theorem)
Let $\phi : R \to K\langle\langle x_1, \ldots, x_n\rangle\rangle$ *be a morphism of analytic algebras over the field* K*, and denote by* $\mathfrak{m}_R$ *the maximal ideal of* R*. Then the following are equivalent:*

i) ϕ *is an isomorphism.*

ii) $\dot{\phi} : \mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_{K\langle\langle \underline{x}\rangle\rangle}/\mathfrak{m}^2_{K\langle\langle \underline{x}\rangle\rangle}$ *is an isomorphism.*

Proof:
See [16, Theorem 1.21]. □

Remark 3.10
The Inverse Function Theorem for analytic algebras states basically, that we can check if a morphism is an isomorphism, by passing to the morphism induced on the K*-vector space* $\mathfrak{m}_R/\mathfrak{m}_R^2$*.*

Our next lemma is a useful result regarding the lifting of morphisms.

Lemma 3.11 (Lifting Lemma)
Let ϕ *be a morphism of analytic algebras over a field* K*, that is,*

$$\phi : R = K\langle\langle x_1, \ldots, x_n\rangle\rangle/I \to S = K\langle\langle y_1, \ldots, y_m\rangle\rangle/J.$$

Then ϕ *has a lifting* $\tilde{\phi} : K\langle\langle \underline{x}\rangle\rangle \to K\langle\langle \underline{y}\rangle\rangle$*, which can be chosen as an isomorphism in the case that* ϕ *is an isomorphism and* $n = m$*, respectively as an epimorphism in the case that* ϕ *is an epimorphism and* $n \geq m$*.*

Proof:
See [16, Lemma 1.23]. □

3.2 Derivations of Analytic Algebras

This section is dedicated to derivations and their properties, which we state in the context of analytic algebras. For a more detailed treatment of derivations, we refer the reader for example to [23], as we are only presenting results, which are relevant for the underlying thesis.

Let us start with the basic definition of this section, namely the definition of a derivation, which is a modification of [16, Definition 1.105], as we restrict our setup to maps between R-algebras.

Definition 3.12
Let R be an algebra over a field K and S an R-algebra. A derivation *δ is a K-linear map $\delta : R \to S$ satisfying*

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in R$. This property is the so called Leibniz rule. *The set $\mathrm{Der}(R, S)$ denotes the set of all derivations $\delta : R \to S$.*

Example 3.13
We have already seen an example for a derivation, namely the adjoint map of an element of a Lie Algebra. Let R be a Lie algebra and $x \in R$, then ad_x is a derivation, as we can use property $b)$ of Definition 2.28:

$$\begin{aligned}
\mathrm{ad}_x([y,z]) &= [x,[y,z]] = -[z,[x,y]] - [y,[z,x]] \\
&= [[x,y],z] + [y,[x,z]] \\
&= [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]
\end{aligned}$$

for all $y, z \in R$.

Remark 3.14
Let R be an algebra over the field K. By $\mathrm{Der}(R)$ we denote the set of all derivations of R into itself. Then $\mathrm{Der}(R)$ is a vector space over K and it is also a Lie algebra, if we define the multiplication as follows:

$$[\delta, \sigma](xy) := (\delta \circ \sigma - \sigma \circ \delta)(xy),$$

with $\delta, \sigma \in \mathrm{Der}(R)$, $x, y \in R$. A simple computation yields $[\delta,\sigma](xy) = [\delta,\sigma](x)y + x[\delta,\sigma](y)$, hence the multiplication is closed. The other properties of a Lie algebra can also be verified by simple computations.

Proposition 3.15
Let R be an algebra over the field K. Then $\mathrm{Der}(R)$ is an R-module.

Proof:
This result follows from the fact, that for any $f \in R$, we have that $f \cdot \mathrm{Der}(R) \subset \mathrm{Der}(R)$. Furthermore, we get that for any $\delta, \epsilon \in \mathrm{Der}(R)$, $\delta + \epsilon \in \mathrm{Der}(R)$. □

Next we state, how to write the elements of $\mathrm{Der}(R)$ explicitly.

Theorem 3.16

Let R be an analytic algebra over a field K, with $R = K\langle\langle x_1, \ldots, x_n\rangle\rangle/I$ for some $n \in \mathbb{N}$ and I an ideal of $K\langle\langle x_1, \ldots, x_n\rangle\rangle$. Then every $\delta \in \mathrm{Der}(R)$ is of type

$$\delta = \sum_{i=1}^{n} a_i \partial_{x_i},$$

where ∂_{x_i} denotes the partial derivation with respect to x_i and $a_i \in K\langle\langle x_1, \ldots, x_n\rangle\rangle/I$.

Before we state the proof, we need the following lemma.

Lemma 3.17

Let $P := K\langle\langle X_1, \ldots, X_n\rangle\rangle$, I a proper ideal of P, $R := P/I$, $\phi : P \to R$ the natural projection and $x_i := \phi(X_i)$. If δ is a derivation on R, then there exists a derivation α on P with $\phi \circ \alpha = \delta \circ \phi$, such that $\alpha(X_i)$ equals any fixed and prescribed value from the residue class of $\phi^{-1}(\delta(x_i))$, with $1 \leq i \leq n$. Moreover, if $\delta(x_i) = \lambda_i x_i$, $\lambda_i \in K$, then we can choose α such that $\alpha(X_i) = \lambda_i X_i$.

Proof:
See [30, (2.1)]. □

Now we can prove Theorem 3.16.

Proof:
We sketch the proof, as its details are technical and do not give us any more insight on the topic.
We first consider the case $I = 0$. Let $f \in R$, then we can write $f = \sum_{\alpha \in \mathbb{N}^n}^{\infty} a_\alpha \underline{x}^\alpha$. Denote by f_k the truncation of f up to degree k, that is,

$$f_k := \sum_{\substack{\alpha \in \mathbb{N}^n, \\ \alpha_1 + \ldots + \alpha_n \leq k}} a_\alpha \underline{x}^\alpha.$$

Consider any $\delta \in \mathrm{Der}(R)$, then for any monomial x_i^k we have $\delta(x_i^k) = \delta(x_i) k x_i^{k-1} = \delta(x_i)\partial_{x_i} x_i^k$ for all $k \in \mathbb{N}, k \geq 1$. We get the previous result using induction and the Leibniz rule. As $\delta(x_i) \in R$, it follows, that $\delta(\mathfrak{m}_R^k) \subset \mathfrak{m}_R^{k-1}$. As $R \subseteq K[[x_1, \ldots, x_n]]$, we can consider the elements $g_k := \sum_{i=1}^{n} \delta(x_i)\partial_{x_i}(f_k) \in K[[x_1, \ldots, x_n]]$. Due to the fact, that we are dealing with polynomials, $\delta(f_k) = g_k$ for all $k \in \mathbb{N}$. Furthermore, it follows that $\delta(f) - g_k \in \mathfrak{m}_R^k$. If we denote the limit of the g_k by $g \in K[[x_1, \ldots, x_n]]$, we have that $\delta(f) - g \in \mathfrak{m}_R^k K[[x_1, \ldots, x_n]]$ for all $k \in \mathbb{N}$, hence, by Krull's intersection theorem (see for example [16, Theorem B.4.2]), $\delta(f) = g \in K[[x_1, \ldots, x_n]]$. Using, that

$\delta(f) \in R$, we get that $\delta(f) = g \in R$ and, as this holds for any $f \in R$, we have that any $\delta \in \mathrm{Der}(R)$ can be written as $\delta = \sum_{i=1}^{n} \delta(x_i)\partial_{x_i}$. By [14, Satz 1.3], we get $\partial_{x_i} \in \mathrm{Der}(R)$ for $i = 1, \ldots, n$, hence $\delta := \sum_{i=1}^{n} a_i \partial_{x_i} \in \mathrm{Der}(R)$, with $a_i \in R$.
The proof for the case $R := K\langle\langle x_1, \ldots, x_n \rangle\rangle / I$ for some ideal I of $K\langle\langle x_1, \ldots, x_n \rangle\rangle$ follows immediately from Lemma 3.17. □

Remark 3.18
Let $R = K\langle\langle x_1, \ldots, x_n \rangle\rangle$ for some $n \in \mathbb{N}$. Consider the standard grading on R as introduced in Example 2.17 and denote R_i the component of degree i. Then every derivation $\delta \in \mathrm{Der}(R)$ can be written as

$$\delta = \sum_{i=0}^{\infty} \sum_{j=1}^{n} a_{ij} \partial_{x_j},$$

where $a_{ij} \in R_i$. By δ_0 we denote summand $\sum_{j=1}^{n} a_{1j} \partial_{x_j}$ and we call it the linear part *of δ. Denote $(x_1, \ldots, x_n)$ by $\underline{x}$ and $(\partial_{x_1}, \ldots, \partial_{x_n})$ by $\underline{\partial}$. Then there exists a matrix $A \in K^{n \times n}$, such that $\delta_0 = \underline{x} A \underline{\partial}^T$. We call A the* representation matrix *of δ_0*

In the context of analytic algebras we can prove, that $\mathrm{Der}(R)$ is a Noetherian module, which implies, that $\mathrm{Der}(R)$ is a finitely generated module. For details on Noetherian modules see for example [18, p.126 ff.].

Corollary 3.19
Let R be an analytic algebra. Then $\mathrm{Der}(R)$ is a Noetherian R-module.

Proof:
We have to show, that $\mathrm{Der}(R)$ is finitely generated and R is a Noetherian ring. By Theorem 3.7, we have that R is a Noetherian ring. By Theorem 3.16, we have that $\mathrm{Der}(R)$ is finitely generated by the partial derivatives ∂_{x_i}, $i = 1, \ldots, n$, if $x_1, \ldots, x_n$ is a minimal generating system for $\mathfrak{m}_R$, hence $\mathrm{Der}(R)$ is a Noetherian module. □

Before we can state results, we introduce a subset of $\mathrm{Der}(R)$, which is important in the further course of our thesis.

Definition 3.20
*Let R be an analytic algebra, I an ideal of R and $\delta \in \mathrm{Der}(R)$. I is called δ-*invariant*, if $\delta(I) \subseteq I$. By $\mathrm{Der}'(R)$ we denote the set of derivations for which $\mathfrak{m}_R$ is invariant.*

The following two results state, that $\mathrm{Der}'(R)$ is a finitely generated R-module and that it is complete, if R is complete.

Proposition 3.21
Let R be an analytic algebra. Then $\mathrm{Der}'(R)$ is a finitely generated R-module.

Proof:
Let $\delta \in \operatorname{Der}'(R)$. We have for any $f \in R$, that $f\delta(\mathfrak{m}_R) \subseteq \mathfrak{m}_R$, as δ is $\mathfrak{m}_R$-invariant and $\mathfrak{m}_R$ is an ideal. Let $\epsilon \in \operatorname{Der}'(R)$, then $\delta + \epsilon \in \operatorname{Der}'(R)$, as $(\delta + \epsilon)(\mathfrak{m}_R) = \delta(\mathfrak{m}_R) + \epsilon(\mathfrak{m}_R)$, hence $\operatorname{Der}'(R)$ is an R-module. As $\operatorname{Der}'(R)$ is a submodule of $\operatorname{Der}(R)$ and as $\operatorname{Der}(R)$ is a Noetherian R-module by Corollary 3.19, we get that $\operatorname{Der}'(R)$ is finitely generated. □

In the following, we are presenting three ways of obtaining $\operatorname{Der}'(R)$ as a projective limit. The first one we are presenting seems more appealing, but turns out not to be very useful. We still state it, as it is helpful for the reader to understand *why* this approach is not the right one, at least in our context.

Proposition 3.22
Let R be a complete analytic algebra. Then

$$\operatorname{Der}'(R) = \varprojlim_{k \in \mathbb{N}} \operatorname{Der}'(R)/\mathfrak{m}_R^k \operatorname{Der}'(R).$$

Proof:
Using the previous proposition, we have that $\operatorname{Der}'(R)$ is a finitely generated R-module. By Theorem 2.9, we can write

$$\varprojlim_{k \in \mathbb{N}} \operatorname{Der}'(R)/\mathfrak{m}_R^k \operatorname{Der}'(R) = \operatorname{Der}'(R) \otimes_R \hat{R}.$$

As $\hat{R} = R$, we get that $\operatorname{Der}'(R) = \varprojlim_{k \in \mathfrak{n}} \operatorname{Der}'(R)/\mathfrak{m}_R^k \operatorname{Der}'(R)$. □

Scheja and Wiebe in [32] work with $\operatorname{Der}'(R)$ and its projections to $\operatorname{Der}(R/\mathfrak{m}_R^k)$, for some $k \in \mathbb{N}$. We follow this approach, with the difference, that we are also taking the module structure of $\operatorname{Der}'(R)$ into account for our most important result in Chapter 4, whereas Scheja and Wiebe are considering $\operatorname{Der}'(R)$ only as a Lie algebra. The notion of so called *Lie-Rinehart algebras*, which we state in Chapter 4, combines both points of views.

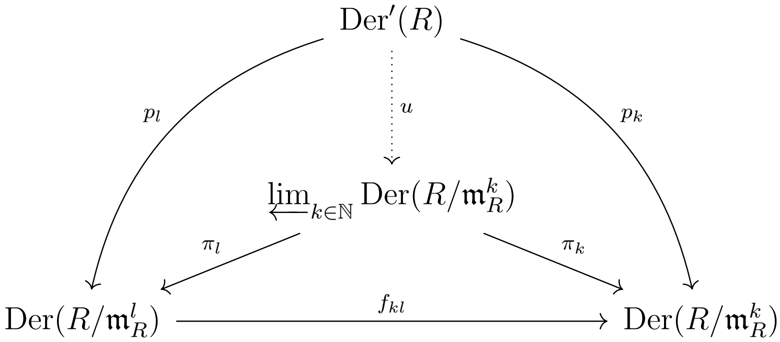
Next we show, that $\operatorname{Der}'(R) = \varprojlim_{k \in \mathbb{N}} \operatorname{Der}(R/\mathfrak{m}_R^k)$ in the case, where R is a complete analytic algebra.

Proposition 3.23
Let R be a complete analytic algebra over a field K. Then

$$\operatorname{Der}'(R) = \varprojlim_{k \in \mathbb{N}} \operatorname{Der}(R/\mathfrak{m}_R^k),$$

where the projections $f_{kl} : \operatorname{Der}(R/\mathfrak{m}_R^l) \to \operatorname{Der}(R/\mathfrak{m}_R^k)$ for $l \geq k$ are induced by the projections $R/\mathfrak{m}_R^l \twoheadrightarrow R/\mathfrak{m}_R^k$.

Proof:
We have projections $p_k : \mathrm{Der}'(R) \to \mathrm{Der}(R/\mathfrak{m}_R^k)$, such that the following diagram commutes:



u denotes the unique morphism of Lie algebras from $\mathrm{Der}'(R)$ to $\varprojlim_{k\in\mathbb{N}} \mathrm{Der}(R/\mathfrak{m}_R^k)$, we get by the universal property of projective limits. Our claim is, that u is an isomorphism. Let us start with injectivity. Consider any $\delta \in \mathrm{Der}'(R)$, with $u(\delta) = 0$. The latter means, that the projection δ_k of δ in $\mathrm{Der}(R/\mathfrak{m}_R^k)$ is the trivial derivation. From this it follows, that for all $x \in R$, we have that $\delta_k(\bar{x}) = 0$ in $R/\mathfrak{m}_R^k$, which translates to $\delta(x) \in \bigcap_{k\in\mathbb{N}} \mathfrak{m}_R^k$ for all $x \in R$. Using Krull's intersection theorem, we get that $\delta(x) = 0$ for all $x \in R$, hence δ can only be the trivial derivation and u is injective.
Now we can prove surjectivity. Consider any $\delta \in \varprojlim_{k\in\mathbb{N}} \mathrm{Der}(R/\mathfrak{m}_R^k)$, then we know, that we can consider δ as a sequence of elements $\delta_k \in \mathrm{Der}(R/\mathfrak{m}_R^k)$, as we work with a projective limit. We are going to construct a $\delta' \in \mathrm{Der}'(R)$, such that $u(\delta') = \delta$. We do this, by defining δ' for all $x \in R$, which also can be considered as a sequence of elements $x_k \in R/\mathfrak{m}_R^k$. Using, that f_{kl} is induced by $g_{kl} : R/\mathfrak{m}_R^l \to R/\mathfrak{m}_R^k$, we get the compatibility of $\delta_k(x_k)$ with the latter projection, that is, $g_{kl}(\delta_l(x_l)) = \delta_k(x_k)$ for all $l \geq k$. Thus, we can define for any $x \in R$ an element $y_x \in R$, which is the limit of the sequence $(\delta_k(x_k))_{k\in\mathbb{N}}$ and we set $\delta'(x) := y_x$. δ' is clearly a derivation, as for any $a, b \in R$, we have that $\delta'(ab)$ is the limit of the sequence $(\delta_k(a_k b_k))_{k\in\mathbb{N}}$ and as the δ_k are derivations, we get, using the same argument regarding limits as before, $\delta'(ab) = \delta'(a)b + a\delta'(b)$. By construction, we have that $u(\delta') = \delta$. □

Next, we state a third way of obtaining $\mathrm{Der}'(R)$, which is closely related to the previous one.

Corollary 3.24
Let R be a complete analytic algebra over a field K. Denote by $\mathfrak{g}_k$ the image of $\mathrm{Der}'(R)$ in $\mathrm{Der}(R/\mathfrak{m}_R^k)$. Then

$$\mathrm{Der}'(R) = \varprojlim_{k\in\mathbb{N}} \mathfrak{g}_k,$$

where the projections $f_{kl} : \mathfrak{g}_l \to \mathfrak{g}_k$ for $l \geq k$ are induced by the projections $R/\mathfrak{m}_R^l \twoheadrightarrow R/\mathfrak{m}_R^k$.

Proof:
The result follows immediately from Proposition 2.4. □

Before we go on, we want to sketch, why the first method of obtaining $\operatorname{Der}'(R)$ as a limit is not very useful in our context. At first, we have that the $\mathfrak{g}_k \subseteq \operatorname{End}(R/\mathfrak{m}_R^k)$, which means, that the elements of $\mathfrak{g}_k$ can be considered as endomorphisms of a finite-dimensional vector space. It is obvious, that any derivation of $\operatorname{Der}'(R)/\mathfrak{m}_R^k \operatorname{Der}'(R)$ maps to a corresponding derivation of $\operatorname{Der}(R/\mathfrak{m}_R^k)$. The problem with this map is, that it is not injective. Consider the derivation $\delta := 3y^2\partial_x - 2x\partial_y$ of $R := K[[x,y]]/\langle x^2 + y^3\rangle$. Clearly, $\delta \in \operatorname{Der}'(R)$, but $\delta \notin \mathfrak{m}_R \operatorname{Der}'(R)$, as $2x\partial_y \notin \mathfrak{m}_R \operatorname{Der}'(R)$, but $3y^2\partial_x \in \mathfrak{m}_R \operatorname{Der}'(R)$. Hence δ is mapped to a non-zero derivation $\bar{\delta}$ in $\operatorname{Der}'(R)/\mathfrak{m}_R \operatorname{Der}'(R)$. Now $\bar{\delta}$ operates on $R/\mathfrak{m}_R$ as the zero derivation, thus the natural map

$$\operatorname{Der}'(R)/\mathfrak{m}_R \operatorname{Der}'(R) \to \operatorname{Der}(R/\mathfrak{m}_R)$$

is not injective. As our goal is to transfer properties like semi-simplicity and nilpotency from linear algebra on finite-dimensional vector spaces to our limit, this excludes the first approach, as we cannot state an injective morphism from $\operatorname{Der}'(R)/\mathfrak{m}_R^k \operatorname{Der}'(R)$ to $\operatorname{End}(R/\mathfrak{m}_R^k)$. Due to this fact, we are from now on always considering $\operatorname{Der}'(R)$ as the projective limit of the $\operatorname{Der}(R/\mathfrak{m}_R^k)$, respectively the $\mathfrak{g}_k$.

Using, that $\mathfrak{g}_k \subseteq \operatorname{End}(R/\mathfrak{m}_R^k)$ for all $k \in \mathbb{N}$, we can define semi-simple and nilpotent derivations.

Definition 3.25
Let R be an analytic algebra and $\delta \in \operatorname{Der}'(R)$. We call δ semi-simple*, if the linear operator induced by δ in $\mathfrak{g}_k$ is semi-simple on $R/\mathfrak{m}_R^k$ for all $k \in \mathbb{N}$. δ is called* nilpotent*, if the linear operator induced by δ in $\mathfrak{g}_k$ is nilpotent on $R/\mathfrak{m}_R^k$ for all $k \in \mathbb{N}$. δ is called* diagonalizable*, if $\mathfrak{m}_R$ has a system of generators containing only eigenvectors of δ.*

Lemma 3.26
Let R be an analytic algebra over a field K and $\delta \in \operatorname{Der}'(R)$. Then δ is nilpotent if and only if the K-linear operator induced by δ on $\mathfrak{m}_R/\mathfrak{m}_R^2$ is nilpotent.

Proof:
Assume δ is nilpotent, then it induces a nilpotent K-linear operator on $\mathfrak{m}_R/\mathfrak{m}_R^2$ by definition. Now assume δ induces a nilpotent K-linear operator on $\mathfrak{m}_R/\mathfrak{m}_R^2$. This means, there exists some $n \in \mathbb{N}$, such that $\delta^n(\mathfrak{m}_R) \subseteq \mathfrak{m}_R^2$. Assume, that we have an n, such that $\delta^n(\mathfrak{m}_R^{k-1}) \subseteq \delta(\mathfrak{m}_R^k)$, for some $k \in \mathbb{N}$. Our result for $k+1$ follows by a application of the Leibniz rule:

$$\delta^n(\mathfrak{m}_R^k) = \delta^n(\mathfrak{m}_R^{k-1}\mathfrak{m}_R) = \underbrace{\delta^n(\mathfrak{m}_R^{k-1})\mathfrak{m}_R}_{\subseteq \mathfrak{m}_R^{k+1}} + \underbrace{\mathfrak{m}_R^{k-1}\delta^n(\mathfrak{m}_R)}_{\subseteq \mathfrak{m}_R^{k+1}} \subseteq \mathfrak{m}_R^{k+1}.$$

Thus, δ induces a nilpotent K-linear operator on $\mathfrak{m}_R/\mathfrak{m}_R^k$ for all $k \in \mathbb{N}$. As $\delta(K) = 0$ and $R = K \oplus \mathfrak{m}_R$, we get that it induces a nilpotent operator on $R/\mathfrak{m}_R^k$ for all $k \in \mathbb{N}$. Finally, δ is nilpotent, as we can always take $m := n \cdot k$ and get that $\delta^m(R) \subseteq \mathfrak{m}_R^k$. $\square$

Remark 3.27
If we work over an algebraically closed field, semi-simple derivations are diagonalizable.

Definition 3.28
Let R be an analytic algebra and $\delta \in \mathrm{Der}'(R)$. We say that δ has a Chevalley decomposition, *if δ can be written as $\delta = \delta_S + \delta_N$ with $[\delta_S, \delta_N] = 0$, where δ_S is a semi-simple derivation, δ_N is a nilpotent derivation and $\delta_S, \delta_N \in \mathrm{Der}'(R)$.*

Obviously the Chevalley decomposition from Definition 3.28 is analogous to the Jordan decomposition known from linear algebra (see for example [25, Chapter XIV, Theorem 2.4]). Before we go on with results regarding the Chevalley decomposition, we show, that endomorphisms of finite-dimensional vector spaces, which are also derivations, have the property, that their semi-simple and nilpotent part are also derivations.

Proposition 3.29
Let R be a K-algebra as well as a finite-dimensional K-vector space, where K is an algebraically closed field. Then for any $\delta \in \mathrm{Der}(R) \subseteq \mathrm{End}_K(R)$, we get, that $\delta_S, \delta_N \in \mathrm{Der}(R)$, where δ_S and δ_N arise from the Chevalley decomposition of δ as an endomorphism.

Proof:
Consider $\delta_S \in \mathrm{End}_K(R)$, which arises from the Chevalley decomposition as an endomorphism of a derivation $\delta \in \mathrm{Der}(R)$. We decompose R into eigenspaces R_λ, where $\lambda \in K$ is an eigenvalue of δ_S. By definition of semi-simplicity, we get that there exists a $n \in \mathbb{N}$, such that $(\delta - \lambda\,\mathrm{id}_R)^n(x) = 0$, for any $x \in R_\lambda$. If we take n large enough, we can get for $x \in R_\lambda$ and $y \in R_\mu$:

$$(\delta - (\lambda + \mu)\,\mathrm{id}_R)^n(xy) = \sum_{i=0}^{n} \binom{n}{i} (\delta - \lambda\,\mathrm{id}_R)^{n-i}(x)(\delta - \mu\,\mathrm{id}_R)^i(y) = 0,$$

hence $R_\lambda R_\mu \subseteq R_{\lambda+\mu}$. Now it suffices to show, that δ_S acts as a derivation on elements of the eigenspaces. By the previous result, we get that $\delta_S(xy) = (\lambda + \mu)xy$, for $x \in R_\lambda$ and $y \in R_\mu$. This is the same result as for $\delta_S(x)y + x\delta_S(y) = \lambda xy + \mu xy$, hence δ_S is a derivation. Using $\delta - \delta_S = \delta_N$, we conclude that $\delta_N \in \mathrm{Der}(R)$. $\square$

Remark 3.30
Proposition 3.29 basically states, that if we have a derivation, which operates on a finite-dimensional vector space, we can compute its Chevalley decomposition by computing its Chevalley decomposition as an endomorphism.

As in the linear algebra case, we cannot expect the Chevalley decomposition to exist without any restrictions to the analytic algebra. The following three theorems are the most important results regarding derivations, which we are going to use. We state the proofs for all three results, as they cannot be found explicitly in [32] and as they show, how to transfer properties from finite-dimensional linear algebra to projective limits.

Theorem 3.31
Let R be an analytic algebra and $\delta \in \mathrm{Der}'(R)$ admitting a Chevalley decomposition $\delta = \delta_S + \delta_N$. Then the Chevalley decomposition of δ is unique, that is, if $\delta = \delta_S + \delta_N = \delta'_S + \delta'_N$ with $[\delta_S, \delta_N] = [\delta'_S, \delta'_N] = 0$, then $\delta_S = \delta'_S$ and $\delta_N = \delta'_N$.

Proof:
Denote by $\overline{\delta_S}$ the image of δ_S to $\mathrm{End}(R/\mathfrak{m}_R^k)$ and by $\bar{\delta}_S$ the semi-simple part of the image of δ in $\mathrm{End}(R/\mathfrak{m}_R^k)$. The analogous notation is used for the nilpotent parts. We show, that $\overline{\delta_S} = \bar{\delta}_S$ respectively $\overline{\delta_N} = \bar{\delta}_N$, as this implies that $\overline{\delta'_S} = \overline{\delta_S}$ respectively $\overline{\delta'_N} = \overline{\delta_N}$ in $\mathrm{End}(R/\mathfrak{m}_R^k)$ for all $k \in \mathbb{N}$. Note that $\overline{\delta_S}, \overline{\delta_N}, \bar{\delta}_S, \bar{\delta}_N \in \mathrm{Der}(R/\mathfrak{m}_R^k)$, due to Definition 3.25 respectively Proposition 3.29. We have that the Chevalley decomposition is unique in $\mathrm{End}(R/\mathfrak{m}_R^k)$. Now $\bar{\delta}_S + \bar{\delta}_N$ and $\overline{\delta_S} + \overline{\delta_N}$ are Chevalley decompositions of $\bar{\delta}$, hence $\overline{\delta_S} = \bar{\delta}_S$ and $\overline{\delta_N} = \bar{\delta}_N$ in $\mathrm{End}(R/\mathfrak{m}_R^k)$ for all $k \in \mathbb{N}$.
Using, that we are dealing with projective limits, due to Proposition 3.23, the corresponding sequence of δ_S respectively δ_N is uniquely determined, thus also δ_S and δ_N are uniquely determined.

□

Theorem 3.32
Let R be an analytic algebra and $\delta \in \mathrm{Der}'(R)$ admitting a Chevalley decomposition $\delta = \delta_S + \delta_N$. Furthermore let I be an ideal of R and let I be δ-invariant, then I is also δ_S and δ_N-invariant.

Proof:
We are going to use the same idea as in the proof of Theorem 3.31. We show the result only for δ_S, as the result for δ_N follows analogously. Using that $\delta \in \mathrm{Der}'(R)$, we get $\delta(I + \mathfrak{m}_R^k) \subseteq I + \mathfrak{m}_R^k$ for all $k \in \mathbb{N}$. Passing to $\mathfrak{g}_k$, we get that $\bar{\delta}(\bar{I}) \subseteq \bar{I}$ in all $\mathfrak{g}_k$. Using, that the elements of $\mathfrak{g}_k$ operate on finite-dimensional vector spaces and that the semi-simple part of a Chevalley decomposition can be written as a polynomial (see for example [25, Chapter XIV, Exercise 14]) in $\bar{\delta}$, say $\overline{\delta_S} = p_k(\bar{\delta})$ for all $k \in \mathbb{N}$, where p_k is a polynomial, we get that $\delta_S(I + \mathfrak{m}_R^k) \subseteq I + \mathfrak{m}_R^k$ for all $k \in \mathbb{N}$. Using Krull's intersection Theorem, we get that $\delta_S(I) = \delta_S(\bigcap_{k=1}^{\infty}(I + \mathfrak{m}_R^k)) \subseteq \bigcap_{k=1}^{\infty}(I + \mathfrak{m}_R^k) = I$. □

Theorem 3.33
Let R be a complete analytic algebra. Then every $\delta \in \mathrm{Der}'(R)$ admits a Chevalley decomposition.

Proof:
For the proof of this theorem, we first of all need to state, how to decompose any $\delta \in \mathrm{Der}'(R)$. In every $\mathrm{End}(R/\mathfrak{m}_R^k)$, $\bar{\delta}$ decomposes into $(\bar{\delta})_{S,k}$ and $(\bar{\delta})_{N,k}$. By Proposition 3.29, we have that $(\bar{\delta})_{S,k}$ and $(\bar{\delta})_{N,k}$ are derivations, hence they are elements of $\mathrm{Der}(R/\mathfrak{m}_R^k)$, for all $k \in \mathbb{N}$. Using Definition 3.25, we get that the $(\bar{\delta})_{N,k}, (\bar{\delta})_{S,k} \in \mathrm{Der}(R/\mathfrak{m}_R^k)$ form a sequence of nilpotent respectively semi-simple operators for all $k \in \mathbb{N}$, and are uniquely determined, as they arise from the Chevalley decomposition of $\bar{\delta}$ in $\mathrm{End}(R/\mathfrak{m}_R^k)$. It is also obvious, that $(\bar{\delta})_{N,l}$ respectively $(\bar{\delta})_{S,l} \in \mathrm{Der}(R/\mathfrak{m}_R^l)$ project onto $(\bar{\delta})_{N,k}$ respectively $(\bar{\delta})_{S,k} \in \mathrm{Der}(R/\mathfrak{m}_R^k)$ for $l \geq k$, as the respective Chevalley decomposition of $\bar{\delta} = (\bar{\delta})_{S,l} + (\bar{\delta})_{N,l} \in \mathrm{Der}(R/\mathfrak{m}_R^l)$ is unique and as the images of $(\bar{\delta_S})_{S,l}$ and $(\bar{\delta_S})_{S,l}$ in $\mathrm{Der}(R/\mathfrak{m}_R^k)$ induce a Chevalley decomposition of $\bar{\delta}$ in $\mathrm{Der}(R/\mathfrak{m}_R^k)$. Due to this, we can define the element $\delta_N := ((\bar{\delta})_{N,k})_{k\in\mathbb{N}}$ and $\delta_S := ((\bar{\delta})_{S,k})_{k\in\mathbb{N}}$, as by Corollary 3.24 $\mathrm{Der}'(R) = \varprojlim_{k\in\mathbb{N}} \mathrm{Der}(R/\mathfrak{m}_R^k)$. We get that $\delta = \delta_S + \delta_N$ is a Chevalley decomposition, as $[\delta_S, \delta_N] = 0$ follows by the result on all $\mathrm{Der}(R/\mathfrak{m}_R^k)$, using Proposition 3.23. Now we have shown, that we can decompose any $\delta \in \mathrm{Der}'(R)$ as $\delta = \delta_S + \delta_N$, where $[\delta_S, \delta_N] = 0$, $\delta_S \in \mathrm{Der}'(R)$ is a semi-simple derivation and $\delta_N \in \mathrm{Der}'(R)$ is a nilpotent derivation. □

Remark 3.34
Example 3.6 i) concerns a setup, where we cannot *apply Theorem 3.33. Example 3.6 ii) states, that if we have a field K of characteristic 0 and if $R := K[[x_1, \ldots, x_n]]$, for some $n \in \mathbb{N}$, then every $\delta \in \mathrm{Der}'(R)$ admits a Chevalley decomposition $\delta = \delta_S + \delta_N$.*

Let us take a look at an example for the Chevalley decomposition.

Example 3.35
Let $K = \mathbb{C}$ and $R := K[[x, y]]$. Consider the derivation $\delta := (x + y)\partial_x + y\partial_y$. Then $\delta_S = x\partial_x + y\partial_y$ is the semi-simple part of δ and $\delta_N = y\partial_x$ is the nilpotent part of δ. The first statement follows, as $\delta_S(x) = x$ and $\delta_S(y) = y$. The second statement follows from the fact, that $\delta_N^2 = 0$.
Now consider $\delta := (x + y + xy)\partial_x + y\partial_y$. We want to show, that the semi-simple part of the linear part of our derivation is not necessarily the semi-simple part of our derivation. Assume, that $\delta_S = x\partial_x + y\partial_y$, then $\delta_N = (y + xy)\partial_x$. Using the same argument as before, δ_S is semi-simple, but $[\delta_S, \delta_N] = xy\partial_x \neq 0$, hence δ_S cannot be the semi-simple part of δ. This example shows, that it is a non-trivial task to compute the semi-simple part of a derivation. For details on the theoretical computation of the Chevalley decomposition see [29].

Before we finish this section, we state a final result, which follows from the proof of Theorem 3.32.

Proposition 3.36
Let R be a complete analytic algebra and $\delta, \epsilon \in \mathrm{Der}'(R)$. If $[\epsilon, \delta] = 0$, then we have $[\epsilon, \delta_S] = 0$ and $[\epsilon, \delta_N] = 0$.

Proof:
Denote by $\bar{\delta}$ and $\bar{\epsilon}$ the images of δ and ϵ to $\mathrm{Der}(R/\mathfrak{m}_R^k)$, for any $k \in \mathbb{N}$. As in the proof of Theorem 3.32, we can write $\overline{\delta_S}$ as a polynomial in $\bar{\delta}$. Due to the fact, that $[\bar{\epsilon}, \bar{\delta}] = 0$, we get that $\bar{\epsilon}$ commutes with any polynomial expression in $\bar{\delta}$, hence with $\overline{\delta_S}$. The analogous result follows for $\overline{\delta_N}$. The result follows, as δ_S and δ_N can be considered as sequences of the $\overline{\delta_S}$ respectively $\overline{\delta_N}$ by Proposition 3.23. □

3.3 Gradings and Derivations

In this section we state results from [30, Chapter 2 and 3] regarding derivations and the notion of grading from Chapter 2.2.

The first two theorems are very important, as they state, that every grading of an analytic algebra arises from a derivation and vice versa.

Theorem 3.37
Let R be an analytic algebra over a field K and $\delta \in \mathrm{Der}'(R)$, such that $\mathfrak{m}_R$ has a system of generators containing only eigenvectors of δ. Then there exits a unique $(K, +)$ grading π_g of R, $g \in K$, such that each $\pi_g^R(R)$ contains only g-eigenvectors of δ.

Proof:
See [30, (2.2)]. □

Theorem 3.38
Let R be an analytic algebra over a field K and let π_g^R, $g \in K$, be a $(K, +)$ grading of R. Then there exists a unique diagonalizable derivation $\delta \in \mathrm{Der}'(R)$, such that each $\pi_g(R)$ contains only g-eigenvectors of δ.

Proof:
See [30, (2.3)]. □

Remark 3.39
By Theorem 3.37 and 3.38 the diagonalizable derivations are in one-to-one correspondence with the $(K, +)$ gradings of analytic algebras.

The next theorems are crucial in an application of the Formal Structure Theorem, which we are going to state in Chapter 4.

Theorem 3.40
Let R be an analytic algebra over a field K, which is $(K, +)$ graded. Furthermore, let I be an ideal of R and $\delta \in \mathrm{Der}'(R)$ be the derivation corresponding to the grading. Then I is homogeneous, if and only if I is δ-invariant.

Proof:
See [30, (2.4)]. □

Theorem 3.41
Let R be an analytic algebra over a field K, I be an ideal of R and $\delta \in \mathrm{Der}'(R)$. If I is δ-invariant, then every associated prime ideal P of I is δ-invariant.

Proof:
See [30, (2.5)]. □

The next theorem in this section is a surprising result, which states, that we can write every diagonalizable derivation as a finite sum of diagonalizable derivations with rational eigenvalues.

Theorem 3.42
Let R be an analytic algebra over a field K and let $\delta \in \mathrm{Der}'(R)$ be diagonalizable. Then there exist diagonalizable $\delta_j \in \mathrm{Der}'(R)\backslash\{0\}$ and $a_j \in K$, $j = 1, \ldots, s$ for some $s \in \mathbb{N}$, such that $\delta = \sum_{j=1}^{s} a_j\delta_j$, every δ_j has the same eigenvectors as δ and the δ_j have only rational eigenvalues.

Proof:
See [30, (3.2)]. □

The last lemma in this section characterizes diagonalizable and nilpotent derivations by their linear part, using Remark 3.18 and Lemma 3.26.

Lemma 3.43
Let R be an analytic algebra over a field K and $\delta \in \mathrm{Der}'(R)$. Then δ is diagonalizable if and only if there exists a set of coordinates, such that $\delta = \delta_0$ and the representation matrix is diagonalizable. δ is nilpotent if and only if δ_0 is nilpotent.

Proof:
We start with the statement regarding diagonizability. First assume δ is diagonalizable, then there exists a set of coordinates, say $x_1, \ldots, x_n$, for some $n \in \mathbb{N}$, such that $R = K\langle\langle x_1, \ldots, x_n\rangle\rangle/I$ for some ideal I of $K\langle\langle x_1, \ldots, x_n\rangle\rangle$ and with the property that there exist $\lambda_i \in K$, such that $\delta(x_i) = \lambda_i x_i$. By the proof of Theorem 3.16 we get that $\delta = \sum_{i=1}^{n} \lambda_i x_i \partial_{x_i}$, hence $\delta = \delta_0$ and the representation matrix is obviously diagonalizable. Now if $\delta = \delta_0$ and the representation matrix is diagonalizable, there exists a linear coordinate change, such that δ is of type $\sum_{i=1}^{n} \lambda_i x_i \partial_{x_i}$ for a set of coordinates $x_1, \ldots, x_n$, some $\lambda_i \in K$ and some $n \in \mathbb{N}$. Then δ is obviously diagonalizable. The statement for nilpotency follows immediately from Lemma 3.26. □

4 The Formal Structure Theorem for Complete Analytic Algebras

In the following chapter, we extend the abstract definition of grading from Chapter 2.2 to projective systems. Furthermore, we introduce a special type of Lie algebras, namely so called Lie-Rinehart algebras, which combine the structure of a module with the one of a Lie algebra. We use previous ideas to generalize the Formal Structure Theorem from [13] to Lie-Rinehart subalgebras of $\mathrm{Der}'(R)$, where R is a complete analytic algebra over an algebraically closed field K of characteristic 0.

4.1 Grading of Projective Systems

In this section we extend the notion of grading from Chapter 2.2 to the setup of projective limits. For simplicity, we only consider the case, where our indexes are natural numbers.

Remark 4.1
All rings in the following are assumed to be Noetherian and all modules are assumed to be finitely generated. By Proposition 2.4, we can assume that all projections from a projective limit to its component are surjective, hence all f_{ij} are surjective, using, that (M_i, f_{ij}) is a projective system over any indexed set.

First of all, we start with the grading of rings.

Definition 4.2
Let $(G, +)$ be an abelian group and (R_k, f_{kl}^R) a projective system of rings, with $k, l \in \mathbb{N}$ and $k \leq l$. Define $R := \varprojlim_{k\in\mathbb{N}} R_k$ and denote the projections $R \to R_k$ by p_k^R for all $k \in \mathbb{N}$. We write p_k, if the ring we are working with is clear. We say (R_k, f_{kl}) is graded with respect to G, if there are group homomorphisms $\pi_g^{R_k} : (R_k, +) \to (R_k, +)$ for all $g \in G$, $k \in \mathbb{N}$, such that the group homomorphisms $\pi_g^{R_k}$ induce a finite grading on the R_k in the sense of Definition 2.12 for all $k \in \mathbb{N}$ and such that the following diagram commutes:

$$\begin{array}{ccc} R_k & \xrightarrow{\pi_g^{R_k}} & R_k \\ {\scriptstyle f_{kl}^R}\uparrow & & \uparrow{\scriptstyle f_{kl}^R} \\ R_l & \xrightarrow{\pi_g^{R_l}} & R_l \end{array}$$

The commutativity means, that the f_{kl}^R have to be compatible with gradings on R_k and R_l for all $l \geq k$ and $g \in G$, that is, $f_{kl}^R(R_{l,g}) \subseteq R_{k,g}$, where $R_{k,g}$ is the image of R_k under the group homomorphism $\pi_g^{R_k}$ on R_k. We denote the limit of the $\pi_g^{R_k}$ by π_g^R for all $g \in G$.

Remark 4.3
Consider the case, where R is a complete analytic algebra, then we can set $R_k := R/\mathfrak{m}_R^k$ and Definition 4.2 generalizes Definition 2.15 to the setup of projective systems.

Now let us extend the notion of grading to projective systems of modules.

Definition 4.4
Let $(G,+)$ be an abelian group, (R_k, f_{kl}^R) a projective system of rings and (M_k, f_{kl}^M) a projective system of modules, where the M_k are R_k-modules, with $k,l \in \mathbb{N}$ and $k \leq l$. Define $R := \varprojlim_{k\in\mathbb{N}} R_k$, $M := \varprojlim_{k\in\mathbb{N}} M_k$ and denote the projections $R \to R_k$ by p_k^R for all $k \in \mathbb{N}$ and the projections $M \to M_k$ respectively by p_k^M. We say (M_k, f_{kl}^M) is graded with respect to G, if there are group homomorphisms $\pi_g^{R_k} : (R_k,+) \to (R_k,+)$ and $\pi_g^{M_k} : (M_k,+) \to (M_k,+)$ for all $g \in G$, $k \in \mathbb{N}$, such that the group homomorphisms $\pi_g^{R_k}$ as well as the group homomorphisms $\pi_g^{M_k}$ induce a finite grading on the M_k as R_k-modules in the sense of Definition 2.12 for all $k \in \mathbb{N}$. Furthermore, the following diagrams have to commute:

$$\begin{array}{ccc} M_k & \xrightarrow{\pi_g^{M_k}} & M_k \\ {\scriptstyle f_{kl}^M}\uparrow & & \uparrow{\scriptstyle f_{kl}^M} \\ M_l & \xrightarrow{\pi_g^{M_l}} & M_l \end{array}$$

$$\begin{array}{ccc} R_k & \xrightarrow{\pi_g^{R_k}} & R_k \\ {\scriptstyle f_{kl}}\uparrow & & \uparrow{\scriptstyle f_{kl}} \\ R_l & \xrightarrow{\pi_g^{R_l}} & R_l \end{array}$$

The commutativity means, that the f_{kl}^M have to be compatible with the gradings on M_k and M_l for all $l \geq k$ and $g \in G$, that is, that $f_{kl}^M(M_{l,g}) \subseteq M_{k,g}$, where $M_{k,g}$ is the image of M_k under the group homomorphism $\pi_g^{M_k}$ induced on M_k. As in the setup of rings, we denote the limit of the $\pi_g^{M_k}$ by π_g^M for all $g \in G$.

Remark 4.5
As before, in the case of a complete analytic algebra R and an R-module M, Definition 4.4 extends Definition 2.15 to the setup of projective systems. It is also important to note, that Definition 4.2 and 4.4 need, that the R_k and M_k admit a finite grading.

The following theorem extends the property of graded modules, that every element can be written as a sum of graded elements.

Theorem 4.6
Let $(G, +)$ be an abelian group, (R_k, f_{kl}^R) a projective system of rings and (M_k, f_{kl}^M) a projective system of modules, where the M_k are R_k-modules, with $k, l \in \mathbb{N}$ and $k \leq l$. Furthermore, define $R := \varprojlim_{k\in\mathbb{N}} R_k$, $M := \varprojlim_{k\in\mathbb{N}} M_k$ and denote the projections $R \to R_k$ by p_k^R for all $k \in \mathbb{N}$ and the projections $M \to M_k$ respectively by p_k^M. Assume, that (M_k, f_{kl}^M) is a graded projective system in the sense of Definition 4.4, where the respective systems of group homomorphisms are denoted by $(\pi_g^R)_{g\in G}$ and $(\pi_g^M)_{g\in G}$. Then every $m \in M$ can be written as

$$m = \sum_{g\in G} \pi_g^M(m).$$

In particular, if $m = \sum_{g\in G} m_g$ with $m_g \in \pi_g^M(M)$ is another representation of m, then we have that $m_g = \pi_g^M(m)$.

Proof:
By assumption, we can write any M_k as $M_k = \bigoplus_{g\in G} M_{k,g}$ and

$$p_k^M(m) = \sum_{g\in G} \pi_g^{M_k}(p_k^M(m)) = \sum_{g\in G} p_k^M(\pi_g^M(m)),$$

using that π_g^M is the limit of the $\pi_g^{M_k}$ and thus has to commute with p_k^M. Define $M_g := \varprojlim_{k\in\mathbb{N}} M_{k,g}$ and we get by construction $\pi_g^M(M) = M_g$ for all $g \in G$. Using this, we get the following group homomorphism

$$u : M \to \prod_{g\in G} M_g,\ m \mapsto (\pi_g^M(m))_{g\in G}.$$

Next, we show that u is injective, because this already results in our claim, that we can write any $m \in M$ as $m = \sum_{g\in G} \pi_g^M(m)$. Let $m \in M$ with $u(m) = 0$, then $\pi_g^M(m) = 0$ for all $g \in G$, hence $p_k^M(m) = \sum_{g\in G} p_k^M(\pi_g^M(m)) = 0$ for all $k \in \mathbb{N}$. Using that M is a projective limit, we immediately get $m = 0$ and u is injective.
Now assume, that $m = \sum_{g\in G} m_g$, with $m_g \in \pi_g^M(M)$, then $p_k^M(m_g) = p_k^M(\pi_g^M(m))$, as the M_k are decomposed as direct sums. Knowing, that the representation of $p_k^M(m)$ is unique in all M_k, we get $p_k^M(m) = \pi_g^M(m)$ for all $k \in \mathbb{N}$. Using the fact, that we are dealing with projective limits, we already have that $m_g = \pi_g^M(m)$. □

Proposition 4.7
Let $(G, +)$ be an abelian group, (R_k, f_{kl}^R) a projective system of rings and (M_k, f_{kl}^M) a projective system of modules, where the M_k are R_k-modules, with $k, l \in \mathbb{N}$ and $k \leq l$. Furthermore, define $R := \varprojlim_{k\in\mathbb{N}} R_k$, $M := \varprojlim_{k\in\mathbb{N}} M_k$ and denote the projections $R \to R_k$ by p_k^R for all $k \in \mathbb{N}$ and the projections $M \to M_k$ respectively by p_k^M. Assume, (M_k, f_{kl}^M) is a graded projective system in the sense of Definition 4.4, where the respective systems of group homomorphisms are denoted by $(\pi_g^R)_{g\in G}$ and $(\pi_g^M)_{g\in G}$ for any $g \in G$. Then $\pi_g^R(R)\pi_h^M(M) \subseteq M_{g+h}$ for all $g, h \in G$.

Proof:
The result holds on the M_k as R_k-modules, by assumption. This means, that for all $g \in G$ and $h \in H$ the following holds:

$$\pi_g^{R_k}(R_k)\pi_h^{M_k}(M_k) \subseteq M_{k,g+h}.$$

For $l \geq k$, we have that

$$f_{kl}^M(\pi_g^{R_l}(R_l)\pi_h^{M_l}(M_l)) = f_{kl}^R \circ \pi_g^{R_l}(R_l) f_{kl}^M \circ \pi_h^{M_l}(M_l) = \pi_g^{R_k}(R_k)\pi_h^{M_k}(M_k),$$

hence we get our result by passing to the limit and using, that $\varprojlim_{k\in\mathbb{N}} M_{k,g+h} = M_{g+h}$.
□

Before we finish this section, we extend the abstract definition of grading to Lie algebras, as we need this notion from now on.

Definition 4.8
Let $(G, +)$ be an abelian group and $\mathfrak{g}$ a Lie algebra over a field K. We call the Lie algebra finitely graded, *if there is a system of group homomorphisms $(\pi_g^{\mathfrak{g}})_{g\in G}$, with $\pi_g^{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$, such that $\mathfrak{g} = \bigoplus_{g\in G} \pi_g^{\mathfrak{g}}(\mathfrak{g})$ and $[\pi_g^{\mathfrak{g}}(\mathfrak{g}), \pi_h^{\mathfrak{g}}(\mathfrak{g})] \subseteq \pi_{g+h}^{\mathfrak{g}}(\mathfrak{g})$.*

Now we can extend the notion of grading to the case of projective systems of Lie algebras.

Definition 4.9
Let $(\mathfrak{g}_k, f_{kl}^{\mathfrak{g}})$ be projective system of Lie algebras over a field K, with $k, l \in \mathbb{N}$ and $k \leq l$. Define $\mathfrak{g} := \varprojlim_{k\in\mathbb{N}} \mathfrak{g}_k$ and denote the projections $\mathfrak{g} \to \mathfrak{g}_k$ by $p_k^{\mathfrak{g}}$ for all $k \in \mathbb{N}$. We say $(\mathfrak{g}_k, f_{kl}^{\mathfrak{g}})$ is graded with respect to G, if there are group homomorphisms $\pi_g^{\mathfrak{g}_k} : (\mathfrak{g}_k, +) \to (\mathfrak{g}_k, +)$ for all $g \in G$, $k \in \mathbb{N}$, such that the group homomorphisms $\pi_g^{\mathfrak{g}_k}$ induce a finite grading on the $\mathfrak{g}_k$ in the sense of Definition 4.8 for all $k \in \mathbb{N}$ and such that the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{g}_k & \xrightarrow{\pi_g^{\mathfrak{g}_k}} & \mathfrak{g}_k \\
{\scriptstyle f_{kl}^{\mathfrak{g}}}\uparrow & & \uparrow{\scriptstyle f_{kl}^{\mathfrak{g}}} \\
\mathfrak{g}_l & \xrightarrow{\pi_g^{\mathfrak{g}_l}} & \mathfrak{g}_l
\end{array}$$

The commutativity means, that the $f^{\mathfrak{g}}_{kl}$ have to be compatible with the gradings on $\mathfrak{g}_k$ and $\mathfrak{g}_l$ for all $l \geq k$ and $g \in G$, this means, that $f^{\mathfrak{g}}_{kl}(\mathfrak{g}_{l,g}) \subseteq \mathfrak{g}_{k,g}$, where $\mathfrak{g}_{k,g}$ is the image of $\mathfrak{g}_k$ under the group homomorphism $\pi^{\mathfrak{g}}_g$ induced on $\mathfrak{g}_k$. As in the setup of rings and modules, we denote the limit of the $\pi^{\mathfrak{g}_k}_g$ by $\pi^{\mathfrak{g}}_g$ for all $g \in G$.

The following result is the analogous result to Theorem 4.6 for Lie algebras.

Theorem 4.10
Let $(\mathfrak{g}_k, f^{\mathfrak{g}}_{kl})$ be projective system of Lie algebras over a field K, with $k, l \in \mathbb{N}$ and $k \leq l$. Furthermore, define $\mathfrak{g} := \varprojlim_{k\in\mathbb{N}} \mathfrak{g}_k$ and denote the projections $\mathfrak{g} \to \mathfrak{g}_k$ by $p^{\mathfrak{g}}_k$ for all $k \in \mathbb{N}$. Assume, that $(\mathfrak{g}_k, f^{\mathfrak{g}}_{kl})$ is a graded projective system of Lie algebras in the sense of Definition 4.9, where the respective system of group homomorphisms is denoted by $(\pi^{\mathfrak{g}}_g)_{g\in G}$. Then every $m \in \mathfrak{g}$ can be written as

$$m = \sum_{g\in G} \pi^{\mathfrak{g}}_g(m).$$

In particular, if $m = \sum_{g\in G} m_g$ with $m_g \in \pi^{\mathfrak{g}}_g(\mathfrak{g})$ is another representation of m, then we have that $m_g = \pi^{\mathfrak{g}}_g(m)$.

Proof:
The proof is the same as for Theorem 4.6. □

Remark 4.11
It is possible to show, that if we have two graded projective systems (R_k, f^Rkl) and (R'_k, f'^R_{kl}), which have the same limit, say R, and induce the same system of group homomorphisms $(\pi^R_g)_{g\in G}$, then gradings of the projective systems are compatible. By the latter we mean, that we get a commutative diagram as follows:

$$\begin{array}{ccc} R'_l & \xrightarrow{\pi_g^{R'_l}} & R'_l \\ {\scriptstyle\psi_{kl}}\uparrow & & {\scriptstyle\psi_{kl}}\uparrow \\ R_k & \xrightarrow{\pi_g^{R_k}} & R_k \end{array}$$

We omit a proof for the existence of the ψ_{kl}, as we do not need this result for the further course of our thesis.

Before we go on to the next section, we take a look at substructures of the previous objects. Scheja and Wiebe did not define gradings on the $\mathfrak{m}$-adic completion of a ring, but on the quotient rings $R/\mathfrak{m}^k$. This allows us to grade rings like analytic algebras, which are not necessarily complete. We are now using this idea to define gradings of projective systems of subrings, submodules or Lie subalgebras of projective systems of the respective type, as this gives a more general notion of grading. Using this, we can grade for example convergent power series rings, which are contained in a formal power series ring. We are using the notation from Definition 4.2, 4.4 or 4.9.

Definition 4.12
Let $(S_k, f_{kl}^R|_{S_k})$ be a projective system of subrings (submodules, Lie subalgebras) of a projective system of rings (modules, Lie algebras) (R_k, f_{kl}^R), which is graded as in Definition 4.2 (4.4, 4.9). We define $\pi_g^{S_k} := \pi_g^{R_k}|_{S_k}$ for all $g \in G$, $k \in \mathbb{N}$. Then $(S_k, f_{kl}^R|_{S_k})$ is a graded projective system of subrings (submodules, Lie subalgebras) if and only if $\pi_g^{S_k}$ induces a grading of S_k as a subring (submodule, Lie subalgebra) for all $k \in \mathbb{N}$.

Remark 4.13
From now on, we call a ring (module, Lie algebra), which is the projective limit of a graded projective system, a graded ring (module, Lie algebra). We do so, as the grading of a projective system induces a system of group homomorphisms, which satisfy all properties postulated by Scheja and Wiebe in the setup, where R is a Zariski ring.

Using the notation from Definition 4.4, we get the following result.

Lemma 4.14
Let (N_k, f_{kl}^N) be a projective system of submodules of the R_k-modules M_k, where the latter is graded in the sense of Definition 4.4 and $f_{kl}^N := f_{kl}^M|_{N_l}$. Assume, that $N = \varprojlim_{k\in\mathbb{N}} N_k \subseteq M = \varprojlim_{k\in\mathbb{N}} M_k$ and that M is a Noetherian module. Then N is a graded submodule in the sense of Definition 4.12, if and only if N can be generated by homogeneous elements.

Proof:
First, assume N is graded. Then the N_k are finitely graded submodules of M_k. As the f_{kl}^N are surjective and compatible with our grading, we can lift any homogeneous set of generators of N_k to N_l, for $l \geq k$, and extend it to a set of homogeneous generators of N_l. This means, that we can lift any set of homogeneous generators of N_k, say $\bar{I}_k$, to a set of homogeneous elements of N, which we denote by I_k. Starting with $k = 1$, we can build a sequence of submodules generated by homogeneous elements of N, namely

$$\langle I_1 \rangle \subset \langle I_2 \rangle \subset \langle I_3 \rangle \subset \dots$$

As M is a Noetherian module, the previous chain has to become stationary for some $k \in \mathbb{N}$. This means, that the images of the elements of I_k to N_l generate N_l for all $l \in \mathbb{N}$. So N is generated by finitely many homogeneous elements. Now assume, N can be generated by homogeneous elements. Then it is easy to see, that all N_k are generated by the projection of those, and the result follows from the result in the finitely graded case. □

4.2 Grading of Lie-Rinehart Algebras

In the following section, we introduce the notion of a Lie-Rinehart algebra, which combines the structure of a module with the structure of a Lie algebra. We also define grading of Lie-Rinehart algebras.
Let us start with the definition of a Lie-Rinehart algebra. The definition is taken from [22] and is slightly modified to fit in our context.

Definition 4.15
Let R be an algebra over a field K. Furthermore, let $\mathfrak{g}$ be a Lie algebra over the field K. We call the pair $(R, \mathfrak{g}, \rho)$ a Lie-Rinehart algebra, *if the following conditions are satisfied:*

i) $\mathfrak{g}$ *is an R-module.*

ii) $\mathfrak{g}$ *acts on the left of R by derivations, that is, there exists a morphism of Lie algebras $\rho : \mathfrak{g} \to \mathrm{Der}(R)$. Define $\alpha(f) := \rho(\alpha)(f)$ for all $\alpha \in \mathfrak{g}$ and $f \in R$.*

iii) $[\alpha, f\beta] = \alpha(f)\beta + f[\alpha, \beta]$ *for all $f \in R$, $\alpha, \beta \in \mathfrak{g}$.*

iv) $(f\alpha)(g) = f(\alpha(g))$ *for all $f, g \in R$, $\alpha \in \mathfrak{g}$.*

Remark 4.16
Condition iii) in the previous definition implies, that the Lie algebra morphism ρ is also R linear.

The next topic we need to talk about, is morphisms of Lie-Rinehart algebras. The following definition is taken from [21, Chapter 1].

Definition 4.17
Let $(R, \mathfrak{g}, \rho)$ and $(S, \mathfrak{h}, \sigma)$ be Lie-Rinehart algebras, where R, S are algebras over a field K. Then (ϕ, ψ) is a morphism of Lie-Rinehart algebras, *if:*

i) $\phi : R \to S$ *is a morphism of K-algebras,*

ii) $\psi : \mathfrak{g} \to \mathfrak{h}$ *is a morphism of Lie algebras, which in the same time is a morphism of R-modules, where R acts on S by ϕ and*

iii) for all $f \in R, \alpha \in \mathfrak{g}$ it holds, that

$$\phi \circ \alpha(f) = \psi(\alpha)(\phi(f)).$$

Our standard example for a Lie-Rinehart algebra is the module of derivations of an analytic algebra.

Example 4.18
Let R be an analytic algebra and $\mathfrak{g} = \mathrm{Der}(R)$. Then $\mathfrak{g}$ is a Lie-Rinehart algebra, as all properties are basic properties of the module of derivations.

Let us now define a notion of grading for a special type of Lie-Rinehart algebras.

Definition 4.19

Let $(G,+)$ be an abelian group, R be an algebra over a field K and $(R,\mathfrak{g},\rho)$ a Lie-Rinehart algebra, with $\mathfrak{g}\subset \mathrm{Der}(R)$ and $\rho:\mathfrak{g}\hookrightarrow \mathrm{Der}'(R)$. We say $(R,\mathfrak{g},\rho)$ is finitely graded, *if the following conditions hold:*

i) R is finitely graded in the sense of Definition 2.12

ii) $\mathfrak{g}$ is finitely graded as an R-module in the sense of Definition 2.12

iii) The group homomorphisms π_g, $g\in G$, arising from Definition 2.12, need to satisfy $[\pi_g(\mathfrak{g}),\pi_h(\mathfrak{g})]\subseteq \pi_{g+h}(\mathfrak{g})$ for all $g,h\in G$.

Next, we take a look at the grading of projective systems of Lie-Rinehart algebras. We restrict ourselves to the case, where R is a complete analytic algebra. We denote the natural projection $R/\mathfrak{m}_R^l \twoheadrightarrow R/\mathfrak{m}_R^k$ by f_{kl}^R for $l\geq k$.

Definition 4.20

Let $(G,+)$ be an abelian group, R a complete analytic algebra with projective system (R_k, f_{kl}^R), where $R_k := R/\mathfrak{m}_R^k$, and $(\mathfrak{g}_k, f_{kl}^{\mathfrak{g}})$ a projective system of Lie-algebras, where $(R_k,\mathfrak{g}_k,\rho_k)$ are also Lie-Rinehart algebras, with $\rho_k:\mathfrak{g}_k\hookrightarrow \mathrm{Der}'(R_k)$, $k,l\in\mathbb{N}$ and $k\leq l$. Define $\mathfrak{g}:=\varprojlim_{k\in\mathbb{N}}\mathfrak{g}_k$ and denote the projections $R\to R_k$ by p_k^R for all $k\in\mathbb{N}$ and the projections $\mathfrak{g}\to\mathfrak{g}_k$ respectively by $p_k^{\mathfrak{g}}$. We say $(\mathfrak{g}_k, f_{kl}^{\mathfrak{g}})$ is graded with respect to G, if the following hold:

i) for all $g\in G$, $k\in\mathbb{N}$, there are group homomorphisms $\pi_g^{R_k}:(R_k,+)\to(R_k,+)$ grading R in the sense of Definition 2.12,

ii) for all $g\in G$, $k\in\mathbb{N}$, there are group homomorphisms $\pi_g^{\mathfrak{g}_k}:(\mathfrak{g}_k,+)\to(\mathfrak{g}_k,+)$ grading $(R_k,\mathfrak{g}_k,\rho_k)$ in the sense of Definition 4.19

iii) and the following diagrams have to commute:

$$\begin{array}{ccc} \mathfrak{g}_k & \xrightarrow{\pi_g^{\mathfrak{g}_k}} & \mathfrak{g}_k \\ {\scriptstyle f_{kl}^{\mathfrak{g}}}\uparrow & & \uparrow{\scriptstyle f_{kl}^{\mathfrak{g}}} \\ \mathfrak{g}_l & \xrightarrow{\pi_g^{\mathfrak{g}_l}} & \mathfrak{g}_l \end{array}$$

$$\begin{array}{ccc} R_k & \xrightarrow{\pi_g^{R_k}} & R_k \\ {\scriptstyle f_{kl}^{R}}\uparrow & & \uparrow{\scriptstyle f_{kl}^{R}} \\ R_l & \xrightarrow{\pi_g^{R_l}} & R_l \end{array}$$

Write ρ for the limit of the ρ_k, then $(R,\mathfrak{g},\rho)$ is called a graded Lie-Rinehart algebra. *As in the setup of rings, modules and Lie algebras, we denote by $\pi_g^{\mathfrak{g}}$ the limit of the $\pi_g^{\mathfrak{g}_k}$ for all $g\in G$.*

Our definition of a graded Lie-Rinehart algebra allows us to use our results regarding graded modules. We can also switch the perspective from which we are looking at our Lie-Rinehart algebra, as it is useful to consider it sometimes as a module, sometimes as a Lie algebra. Before we go on with examples and the most important theorem of this section, we have the following remark regarding the usual notion of grading of finite Lie-algebras.

Remark 4.21
The usual grading of a finite Lie algebra $\mathfrak{g}$ over a field K is a special case of Definition 4.19. If we let $\mathfrak{g}$ operate trivially on K, this is, $\alpha(f) = 0$ for all $f \in K$ and $\alpha \in \mathfrak{g}$, we can satisfy all conditions from Definition 4.15, hence $(K, \mathfrak{g}, \rho)$ is a Lie-Rinehart algebra, with $\rho : \mathfrak{g} \to \mathrm{Der}(K)$ being the trivial morphism. Now we can simply take $R = K$ and grade it trivially. Then condition i) in Definition 4.19 is superfluous and conditions ii) and iii) state basically, that our Lie algebra can be written as a direct sum of graded components, which are compatible with the Lie brackets, which is the usual definition of a graded Lie algebra.

The following theorem shows, that gradings of analytic algebras induce gradings of the corresponding Lie-Rinehart algebra. For simplicity, we assume that our field is algebraically closed.

Theorem 4.22
Let R be a complete analytic algebra over an algebraically closed field K and let $\mathfrak{g} := \mathrm{Der}'(R)$. Denote the projections $\mathrm{Der}'(R) \to \mathrm{Der}(R/\mathfrak{m}_R^k)$ by p_k, with $\mathfrak{g}_k := p_k(\mathrm{Der}'(R))$ for $k \in \mathbb{N}$. Assume, that R is $(K, +)$ graded, where the grading is induced by $\delta \in \mathrm{Der}'(R)$. Then δ induces a grading on $(R, \mathfrak{g}, \rho)$ in the sense of Definition 4.20. Every homogeneous $\epsilon \in \mathfrak{g}$ satisfies $\mathrm{ad}_\delta(\epsilon) = \lambda\epsilon$, for some $\lambda \in K$.

Proof:
In the following proof, we use, that if $\delta \in \mathrm{Der}'(R)$ is semi-simple, also $\mathrm{ad}_{\bar{\delta}}$ is semi-simple on the finite-dimensional Lie algebras $\mathfrak{g}_k$. Next we show, that this property on the finite-dimensional Lie algebras induces our grading on $\mathfrak{g}$. The first property of Definition 4.20 is satisfied automatically, as we assume, that R is graded. To show the second property, we use that $\mathfrak{g}_k = \bigoplus_{\lambda \in K} \mathfrak{g}_{k,\lambda}$, where $\mathfrak{g}_{k,\lambda}$ denotes the eigenspace with respect to the eigenvalue λ. Define $\pi_\lambda^{\mathfrak{g}_k} : (\mathfrak{g}_k, +) \to (\mathfrak{g}_k, +)$ as the projection to $\mathfrak{g}_{k,\lambda}$, for any $\lambda \in K$. Next we show, that the $\mathfrak{g}_k$ are finitely graded as R_k-modules. Consider any $k \in \mathbb{N}$, and $\lambda, \mu \in K$, then we have for any homogeneous elements $f_\mu \in R_k$ and $\tau_\lambda \in \mathfrak{g}_{k,\lambda}$:

$$\mathrm{ad}_{\bar{\delta}}(f_\mu \tau_\lambda) = \mu f_\mu \tau_\lambda + \lambda f_\mu \tau_\lambda = (\mu + \lambda) f_\mu \tau_\lambda \in \mathfrak{g}_{k,\mu+\lambda},$$

hence $\mathfrak{g}_k$ is a graded R_k-module. The last thing we need to show for the second property of Definition 4.20, is the finite grading as a Lie algebra, that is $[\mathfrak{g}_{k,\lambda}, \mathfrak{g}_{k,\mu}] \subset \mathfrak{g}_{k,\lambda+\mu}$. Consider any $\tau_\mu \in \mathfrak{g}_{k,\mu}$ and $\tau_\lambda \in \mathfrak{g}_{k,\lambda}$, then

$$\mathrm{ad}_{\bar{\delta}}([\tau_\mu, \tau_\lambda]) = -[\tau_\mu, [\tau_\lambda, \bar{\delta}]] - [\tau_\lambda, [\bar{\delta}, \tau_\mu]] = \lambda[\tau_\mu, \tau_\lambda] - \mu[\tau_\lambda, \tau_\mu] = (\mu + \lambda)[\tau_\mu, \tau_\lambda],$$

hence $[\mathfrak{g}_{k,\mu}, \mathfrak{g}_{k,\lambda}] \subseteq \mathfrak{g}_{k,\mu+\lambda}$.
The third property of Definition 4.20 has to be shown only for the $\pi_\lambda^{\mathfrak{g}_k}$, as the respective property for the $\pi_\lambda^{R_k}$ hold trivially. Consider any $\epsilon \in \mathfrak{g}_{l,\lambda}$, then $[f_{kl}^{\mathfrak{g}}(\bar{\delta}), f_{kl}^{\mathfrak{g}}(\epsilon)] = f_{kl}^{\mathfrak{g}}([\bar{\delta}, \epsilon]) = f_{kl}^{\mathfrak{g}}(\lambda\epsilon) = \lambda f_{kl}^{\mathfrak{g}}(\epsilon)$. As every element of $\mathfrak{g}_l$ can be written as a sum of homogeneous elements, we get that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}_k & \xrightarrow{\pi_\lambda^{\mathfrak{g}_k}} & \mathfrak{g}_k \\ {\scriptstyle f_{kl}^{\mathfrak{g}}}\uparrow & & \uparrow{\scriptstyle f_{kl}^{\mathfrak{g}}} \\ \mathfrak{g}_l & \xrightarrow{\pi_\lambda^{\mathfrak{g}_l}} & \mathfrak{g}_l \end{array}$$

Let $\epsilon \in \mathfrak{g}$ be homogeneous. Then $\mathrm{ad}_\delta(\epsilon) = \lambda\epsilon$ follows by the previous computation, as ϵ is a limit of homogeneous elements of the $\mathfrak{g}_k$.

□

The next corollary is analogous to Theorem 3.40.

Corollary 4.23
Let R be a graded complete analytic algebra over an algebraically closed field K with grading induced by a diagonalizable derivation $\delta \in \mathrm{Der}'(R)$ and let $\mathfrak{g} \subseteq \mathrm{Der}'(R)$ be a Lie-Rinehart subalgebra. Assume, that $\mathfrak{g} = \varprojlim_{k\in\mathbb{N}} p_k(\mathfrak{g})$. If $[\delta, \mathfrak{g}] \subseteq \mathfrak{g}$, then $\mathfrak{g}$ is a graded Lie-Rinehart subalgebra of $\mathrm{Der}'(R)$ with respect to δ.

Proof:
By Theorem 4.22 we have a grading on $\mathfrak{h} := \mathrm{Der}'(R)$ induced by δ. Let $\mathfrak{h}_k := \mathrm{Der}(R/\mathfrak{m}_R^k)$ and denote the respective grading by $\pi_\lambda^{\mathfrak{h}_k}$ for any $k \in \mathbb{N}$ and $\lambda \in K$. As $[\delta, \mathfrak{g}] \subset \mathfrak{g}$, we can write $\mathfrak{g}_k = \bigoplus_{\lambda\in K}(\mathfrak{g}_k \cap \mathfrak{h}_{k,\lambda})$. This means, that $\pi_\lambda^{\mathfrak{g}_k} := \pi_\lambda^{\mathfrak{h}_k}|_{\mathfrak{g}_k}$ is a group homomorphism of $\mathfrak{g}_k$ into itself. It satisfies all assumptions of Definition 4.20, using the exact same computations as in the proof of Theorem 4.22, hence $\mathfrak{g}$ is a graded Lie-Rinehart subalgebra of $\mathrm{Der}'(R)$.

□

4.3 A General Formal Structure Theorem

In this section, we generalize the Formal Structure Theorem from [13]. Before we state our version of the aforementioned theorem, we need a few preparing results regarding derivations of analytic algebras. To formulate our statements properly, we need some terminology. We start with so called multi-gradings, that is, a grading of an algebra, module or Lie-Rinehart algebra, by Cartesian products of groups. We show, that we can reconstruct gradings by each factor of the Cartesian product and that we can induce a grading by a Cartesian product from two given gradings, if a certain property is satisfied. The following lemmas state our results. We start with a finitely graded ring and prove the results for this setup, as all other results follow by (almost) the same computations.

Lemma 4.24
Let $(G,+)$ and $(H,+)$ be abelian groups and R a ring. R is finitely graded by $(G\times H,+)$, say by $\Psi^R_{(g,h)}, (g,h)\in G\times H$, if and only if there exist commuting group homomorphisms π^R_g and ψ^R_h, $g\in G, h\in H$, finitely grading R with

$$R=\bigoplus_{g\in G}\bigoplus_{h\in H}(\pi^R_g(R)\cap\psi^R_h(R)).$$

Furthermore, we have $\Psi^R_{(g,h)}=\psi^R_h\circ\pi^R_g$ for all $(g,h)\in G\times H$.

Proof:
First assume, that $\Psi^R_{(g,h)}$ finitely grades R. Then

$$R=\bigoplus_{(g,h)\in G\times H}\Psi_{(g,h)}(R),$$

and we can write any $m\in R$ as $m=\sum_{(g,h)\in G\times H}m_{(g,h)}$, by Theorem 2.19. Now we define for any $m\in R$ and for all $g\in G, h\in H$, $\pi^R_g(m):=\sum_{h\in H}m_{(g,h)}$ and $\psi_h(m):=\sum_{g\in G}m_{(g,h)}$. Both are clearly group homomorphisms form $(R,+)$ into itself. We have $\pi^R_g(R)\pi^R_{g'}(R)\subseteq\pi^R_{g+g'}(R)$, as this property is inherited from the $\Psi_{(g,h)}$. The same holds for the ψ^R_h. Using Proposition 2.20 and commutativity of ψ^R_h and π^R_g, we get $\pi^R_g\circ\psi^R_h\circ\pi^R_g(R)=\psi^R_h\circ(\pi^R_g)^2(R)=\psi^R_h\circ\pi^R_g(R)$. As the analogous result holds for ψ^R_h, we can see that

$$\psi^R_h\circ\pi^R_g(R)=\pi^R_g(R)\cap\psi^R_h(R),$$

as the decomposition of any m into homogeneous components is unique.
Now consider the π^R_g and ψ^R_h as given. Define $\Psi^R_{(g,h)}:=\psi^R_h\circ\pi^R_g$. By construction $\Psi^R_{(g,h)}$ is group homomorphism of $(R,+)$ into itself. We also get by construction, that $\Psi^R_{(g,h)}(R)=\pi^R_g(R)\cap\psi^R_h(R)$, hence we can decompose R by assumption as $R=\bigoplus_{(g,h)\in G\times H}\Psi_{(g,h)}(R)$. Finally, we need to show that for any $(g,h),(g',h')\in G\times H$, we have that $\Psi^R_{(g,h)}(R)\Psi^R_{(g,h)}(R)\subseteq\Psi^R_{(g+g',h+h')}(R)$, but this follows immediately from the corresponding property of the $\pi^R_g,\pi^R_{g'}$ and $\psi^R_h,\psi^R_{h'}$. □

Corollary 4.25
Let $(G_1,+),\ldots,(G_k,+)$ be abelian groups, R a ring. R is finitely graded by $(G_1\times\ldots\times G_k,+)$ with group homomorphism $\Psi_{(g_1,\ldots,g_k)},(g_1,\ldots,g_k)\in G_1\times\ldots\times G_k$, if and only if there exist pairwise commuting group homomorphisms $\pi_{g_1},\ldots,\pi_{g_k}$, $g_i\in G_i$, finitely grading R as in Lemma 4.24. Furthermore, $\Psi_{(g_1,\ldots,g_k)}=\pi_{g_k}\circ\ldots\circ\pi_{g_1}$ for all $(g_1,\ldots,g_k)\in G_1\times\ldots\times G_k$.

Proof:
The proof follows by induction from Lemma 4.24. □

The next lemmas and corollaries are the analogous results to the previous two.

Lemma 4.26
Let $(G, +), (H, +)$ be abelian groups, R a Zariski ring. R is graded by $(G \times H, +)$, say by $\Psi_{(g,h)}, (g, h) \in G \times H$, if and only there exist commuting group homomorphisms π_g, ψ_h, $g \in G, h \in H$ grading R, where $R/\mathfrak{m}_R^k$ can be written as $R/\mathfrak{m}_R^k = \bigoplus_{g\in G} \bigoplus_{h\in H}(\pi_g(R/\mathfrak{m}_R^k) \cap \psi_h(R/\mathfrak{m}_R^k))$, for all $k \in \mathbb{N}$. Furthermore, we have $\Psi_{(g,h)} = \psi_h \circ \pi_g$ for all $(g, h) \in G \times H$.

Proof:
As the notion of grading of R depends only on finite gradings, we can define the grading group homomorphisms in both directions and the remaining steps, which are to prove, follow as in the proof of Lemma 4.24, as the notion of grading depends on the notion of finite grading on the $R/\mathfrak{m}_R^k$ for all $k \in \mathbb{N}$.
Assume, the $\Psi^R_{(g,h)}$ are given. Then we can write by Theorem 2.19 any $m \in R$ as

$$m = \sum_{(g,h)\in G\times H} \Psi^R_{(g,h)}(m).$$

Now we define for any $m \in R$ and for all $g \in G, h \in H$, $\pi_g^R(m) := \sum_{h\in H} m_{(g,h)}$ and $\psi_h(m) := \sum_{g\in G} m_{(g,h)}$. The remaining steps of this direction of the proof are as in the proof of Lemma 4.24.
Now consider the π_g^R and ψ_h^R as given. Define $\Psi^R_{(g,h)} := \psi_h^R \circ \pi_g^R$. From here, again, the remaining steps of the proof are identical to the ones in the proof of Lemma 4.24. □

Corollary 4.27
Let $(G_1, +), \ldots, (G_k, +)$ be abelian groups, R a Zariski ring. R is graded by $(G_1 \times \ldots \times G_k, +)$ with group homomorphism $\Psi_{(g_1,\ldots,g_k)}, (g_1, \ldots, g_k) \in G_1 \times \ldots \times G_k$, if and only if there exist pairwise commuting group homomorphisms $\pi_{g_1}, \ldots, \pi_{g_k}$, $g_i \in G_i$, grading R, which induce finite gradings on $R/\mathfrak{m}_R^k$ for all $k \in \mathbb{N}$, as in Lemma 4.26. Furthermore, $\Psi_{(g_1,\ldots,g_k)} = \pi_{g_k} \circ \ldots \circ \pi_{g_1}$ for all $(g_1, \ldots, g_k) \in G_1 \times \ldots \times G_k$.

Proof:
The result follows by induction from Lemma 4.26. □

As we did not really need the fact, that R is a ring in the proof of Lemma 4.26, we can state the following two lemmas and corollaries for modules and Lie-Rinehart algebras. We omit the proofs, as it uses exactly the same idea as the proof of Lemma 4.26.

Lemma 4.28
Let $(G,+),(H,+)$ be abelian groups, R a graded Zariski ring, M an R-module. M is graded by $(G\times H,+)$, say by $\Psi^M_{(g,h)}, (g,h)\in G\times H$, if and only there exist commuting group homomorphisms π^M_g,ψ^M_h, $(g,h)\in G\times H$ grading M and π^R_g,ψ^R_h the corresponding gradings of R, where $M/\mathfrak{m}^k_R M$ can be written as $M/\mathfrak{m}^k_R M=\bigoplus_{g\in G}\bigoplus_{h\in H}(\pi^M_g(M/\mathfrak{m}^k_R M)\cap\psi^M_h(M/\mathfrak{m}^k_R M))$ and $R/\mathfrak{m}^k_R$ can be written as $R/\mathfrak{m}^k_R=\bigoplus_{g\in G}\bigoplus_{h\in H}(\pi^R_g(R/\mathfrak{m}^k_R)\cap\psi^R_h(R/\mathfrak{m}^k_R))$, for all $k\in\mathbb{N}$. Furthermore, $\Psi^M_{(g,h)}=\psi^M_h\circ\pi^M_g$ and $\Psi^R_{(g,h)}=\psi^R_h\circ\pi^R_g$ for all $(g,h)\in G\times H$, where the latter is the corresponding grading of R.

Corollary 4.29
Let $(G_1,+),\ldots,(G_k,+)$ be abelian groups, R a graded Zariski ring, M an R-module. M is graded by $(G_1\times\ldots\times G_k,+)$ with group homomorphism $\Psi^M_{(g_1,\ldots,g_k)}, (g_1,\ldots,g_k)\in G_1\times\ldots\times G_k$, if and only if there exist pairwise commuting group homomorphisms $\pi^M_{g_1},\ldots,\pi^M_{g_k}$, $g_i\in G_i$ grading M and $\psi^R_{g_1},\ldots,\psi^R_{g_k}$ the corresponding gradings of R, where the gradings induce finite gradings on $M/\mathfrak{m}^k_R M$ for all $k\in\mathbb{N}$, as in Lemma 4.28. Furthermore, $\Psi^M_{(g_1,\ldots,g_k)}=\pi^M_{g_k}\circ\ldots\circ\pi^M_{g_1}$ and $\Psi^R_{(g_1,\ldots,g_k)}=\psi^R_{g_k}\circ\ldots\circ\psi^R_{g_1}$ for all $(g_1,\ldots,g_k)\in G_1\times\ldots\times G_k$, where the latter is the corresponding grading of R.

As the previous results also extend naturally to the setup of projective limits, we state the result in this setup only for Lie-Rinehart algebras, as the other results look similar. We keep the notation from Definition 4.20.

Lemma 4.30
Let $(G,+),(H,+)$ be abelian groups, R a graded complete analytic algebra and $(R,\mathfrak{g},\rho)$ a Lie-Rinehart algebra as in Definition 4.20. Keeping the notation and conditions of Definition 4.20, we say $\mathfrak{g}$ is graded by $(G\times H,+)$, say by $\Psi^{\mathfrak{g}}_{(g,h)}, (g,h)\in G\times H$, if and only there exist commuting group homomorphisms $\pi^{\mathfrak{g}_k}_g,\psi^{\mathfrak{g}_k}_h$, $(g,h)\in G\times H$ grading $\mathfrak{g}_k$ and $\pi^{R_k}_g,\psi^{R_k}_h$ the corresponding gradings of R_k, where $\mathfrak{g}_k$ can be written as $\mathfrak{g}_k=\bigoplus_{g\in G}\bigoplus_{h\in H}(\pi_g(\mathfrak{g}_k)\cap\psi_h(\mathfrak{g}_k))$ and R_k can be written as $R_k=\bigoplus_{g\in G}\bigoplus_{h\in H}(\pi^R_g(R_k)\cap\psi^R_h(R_k))$, for all $k\in\mathbb{N}$. Furthermore, $\Psi^{\mathfrak{g}}_{(g,h)}=\psi^{\mathfrak{g}}_h\circ\pi^{\mathfrak{g}}_g$ and $\Psi^R_{(g,h)}=\psi^R_h\circ\pi^R_g$ for all $(g,h)\in G\times H$, where the latter is the corresponding grading of R.

Proof:
We only sketch the following proof, as its details are similar to Lemma 4.24 and Lemma 4.26.
Given $\pi^{\mathfrak{g}_k}_g$ and $\psi^{\mathfrak{g}_k}_h$, we can set $\Psi^{\mathfrak{g}_k}_{(g,h)}:=\psi^{\mathfrak{g}_k}_h\circ\pi^{\mathfrak{g}_k}_g$ and get immediately that the following diagram commutes

$$\begin{array}{ccccc}
\mathfrak{g}_k & \xrightarrow{\pi^{\mathfrak{g}_k}_g} & \mathfrak{g}_k & \xrightarrow{\psi^{\mathfrak{g}_k}_h} & \mathfrak{g}_k \\
{\scriptstyle f^{\mathfrak{g}}_{kl}}\uparrow & & \uparrow{\scriptstyle f^{\mathfrak{g}}_{kl}} & & \uparrow{\scriptstyle f^{\mathfrak{g}}_{kl}} \\
\mathfrak{g}_l & \xrightarrow{\pi^{\mathfrak{g}_l}_g} & \mathfrak{g}_l & \xrightarrow{\psi^{\mathfrak{g}_l}_h} & \mathfrak{g}_l,
\end{array}$$

hence this diagram commutes

$$\begin{array}{ccc} \mathfrak{g}_k & \xrightarrow{\Psi^{\mathfrak{g}_k}_{(g,h)}} & \mathfrak{g}_k \\ f^{\mathfrak{g}}_{kl}\uparrow & & \uparrow f^{\mathfrak{g}}_{kl} \\ \mathfrak{g}_l & \xrightarrow{\Psi^{\mathfrak{g}_l}_{(g,h)}} & \mathfrak{g}_l. \end{array}$$

The result for the gradings of our ring follow by the exact same argument, thus the $\Psi^{\mathfrak{g}_k}_{(g,h)}$ induce a grading on the $\mathfrak{g}_k$ as modules. Property (iii) from Definition 4.19, follows immediately, as the $\pi^{\mathfrak{g}_k}_g$ and $\psi^{\mathfrak{g}_k}_h$ are gradings of Lie algebras.
Assume, we have $\Psi^{\mathfrak{g}_k}_{(g,h)}$, then we can define

$$\pi^{\mathfrak{g}_k}_g : (\mathfrak{g}_k, +) \to (\mathfrak{g}_k, +),\ x \mapsto \sum_{h\in H} \Psi^{\mathfrak{g}_k}_{(g,h)}(x),$$

for all $g \in G$, $k \in \mathbb{N}$. The grading of the $\mathfrak{g}_k$ induced by the $\pi^{\mathfrak{g}_k}_g$ is inherited from $\Psi^{\mathfrak{g}_k}_{(g,h)}$, hence nothing needs to be shown, except the compatibility with the $f^{\mathfrak{g}}_{kl}$. Let $x_l \in \mathfrak{g}_l$ and $x_k := f^{\mathfrak{g}}_{kl}(x_l)$, then

$$\pi^{\mathfrak{g}_k}_g \circ f^{\mathfrak{g}}_{kl}(x_l) = \sum_{h\in H} \Psi^{\mathfrak{g}_k}_{(g,h)}(x_k) = \sum_{h\in H} f^{\mathfrak{g}}_{kl} \circ \Psi^{\mathfrak{g}_l}_{(g,h)}(x_l) = f^{\mathfrak{g}}_{kl} \circ \pi^{\mathfrak{g}_l}_g(x_l),$$

hence the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}_k & \xrightarrow{\pi^{\mathfrak{g}_k}_{g}} & \mathfrak{g}_k \\ f^{\mathfrak{g}}_{kl}\uparrow & & \uparrow f^{\mathfrak{g}}_{kl} \\ \mathfrak{g}_l & \xrightarrow{\pi^{\mathfrak{g}_l}_{g}} & \mathfrak{g}_l \end{array}$$

The analogous construction applies to the $\psi^{\mathfrak{g}_k}_h$, $\pi^{R_k}_g$ and $\psi^{R_k}_h$. This finishes our sketch of the proof, as the remaining computations are similar to the ones in Lemma 4.24 and Lemma 4.26. □

Corollary 4.31
Let $(G_1, +), \ldots, (G_j, +)$ be abelian groups, R a graded complete analytic algebra and $(R, \mathfrak{g}, \rho)$ a Lie-Rinehart algebra as in Definition 4.20. Keeping the notation and conditions of Definition 4.20, we say $\mathfrak{g}$ is graded by $(G_1\times\ldots\times G_j, +)$ with group homomorphism $\Psi^{\mathfrak{g}_k}_{(g_1,\ldots,g_j)}$, $(g_1, \ldots, g_j) \in G_1\times\ldots\times G_j$, if and only if there exist pairwise commuting group homomorphisms $\pi^{\mathfrak{g}_k}_{g_1}, \ldots, \pi^{\mathfrak{g}_k}_{g_j}$, $g_i \in G_i$ grading $\mathfrak{g}_k$ and $\pi^{R_k}_{g_1}, \ldots, \pi^{R_k}_{g_j}$ the corresponding gradings of R, where the gradings induce finite gradings on $\mathfrak{g}_k$ respectively R_k for all $k \in \mathbb{N}$, as in Lemma 4.30. Furthermore, $\Psi^{\mathfrak{g}_k}_{(g_1,\ldots,g_j)} = \pi^{\mathfrak{g}_k}_{g_j} \circ\ldots\circ \pi^{\mathfrak{g}_k}_{g_1}$ and $\Psi^{R_k}_{(g_1,\ldots,g_j)} = \pi^{R_k}_{g_j} \circ\ldots\circ \pi^{R_k}_{g_1}$ for all $(g_1, \ldots, g_j) \in G_1 \times\ldots\times G_j$, where the latter is the corresponding grading of R.

Now we can define multi-graded rings, modules and Lie-Rinehart algebras.

Definition 4.32
Let R be a Zariski ring graded in the sense of Corollary 4.27, M an R-module graded on the sense of Corollary 4.29 or $(R, \mathfrak{g}, \rho)$ be a Lie-Rinehart algebra graded in the sense of Corollary 4.31. Then R, M or $(R, \mathfrak{g}, \rho)$ is called multi-graded *with respect to $(G_1 \times \ldots \times G_k, +)$.*

Next we state terminology, which we need throughout this chapter.

Definition 4.33
Let R be an analytic algebra over a field K and $\delta \in \mathrm{Der}'(R)$ diagonalizable. We call an element $f \in R$ δ homogeneous of degree λ *or* quasi-homogeneous, *if $\delta(f) = \lambda \cdot f$ for some $\lambda \in K$. If we have a set of diagonal and commuting derivations, say $\delta_1, \ldots, \delta_s$, for some $s \in \mathbb{N}$, we call f* $\underline{\lambda}$-multihomogeneous, *if $\delta_j(f) = \lambda_j \cdot f$ for some $\lambda_j \in K$ and all $j = 1, \ldots, s$, with $\underline{\lambda} := (\lambda_1, \ldots, \lambda_s)$.*

Remark 4.34
From now on, we assume that our fields are algebraically closed. We need this assumption to assure that all semi-simple derivations are in fact diagonalizable.

Theorem 4.35
Let R be an analytic algebra over a field K and $\delta_1, \ldots, \delta_s \in \mathrm{Der}'(R)$ diagonalizable and commuting derivations, then $\delta_1, \ldots, \delta_s$ induce a $(K^s, +)$ multi-grading on R.

Proof:
We do the proof for the case $s = 2$, as the rest follows by induction. By Theorem 3.37, we get that δ_1 and δ_2 induce a $(K, +)$ grading on R. As the derivations commute, also their linear operators induced on $R/\mathfrak{m}_R^k$ commute for all $k \in \mathbb{N}$. The latter means, that we can write $R/\mathfrak{m}_R^k$ as a direct sum of eigenspaces of common eigenvectors of δ_1 and δ_2. As these are precisely the graded components of the $R/\mathfrak{m}_R^k$ with respect to the gradings induced by δ_1 and δ_2, we are in the setup of Lemma 4.26 and we get a $(K^2, +)$ grading on R applying the latter. □

Remark 4.36
Later on we will see, that all $(K^s, +)$ gradings are induced by a set of s diagonalizable and commuting derivations.

The next lemma we proof, states, that if we have derivations which equal their linear part, we can compute their Lie bracket by computing the Lie bracket of the representation matrices.

Lemma 4.37
Let R be an analytic algebra over a field K and $\delta, \epsilon \in \mathrm{Der}'(R)$. Assume $\mathfrak{m}_R$ has a minimal set of generators $x_1, \ldots, x_n$ for some $n \in \mathbb{N}$, $\delta = \sum_{i=1}^n \lambda_i x_i \partial_{x_i}$ and $\epsilon = \epsilon_0$. Then $[\delta, \epsilon] = \underline{x}[A, B]\underline{\partial}^T$, where $A, B \in K^{n \times n}$ are the representation matrices of the linear parts of δ respectively ϵ.

Proof:
See [13, Lemma 2.2]. □

The next lemma gives a nice criterion, when a given derivation is nilpotent. We use the grading introduced in Theorem 4.22, Lemma 3.43 and 4.37.

Lemma 4.38
Let R be an analytic algebra over a field K and $\delta \in Der'(R)$ diagonalizable. Furthermore, let $\epsilon \in \mathrm{Der}'(R)$, then $[\delta, \epsilon] = \lambda \cdot \epsilon$ for $\lambda \in K^*$ implies that ϵ is nilpotent.

Proof:
Assume with out loss of generality, that $\mathfrak{m}_R$ has a minimal set of generators $x_1, \ldots, x_n$ for some $n \in \mathbb{N}$, $\delta = \sum_{i=1}^{n} \lambda_i x_i \partial_{x_i}$ with diagonal representation matrix $B \in K^{n \times n}$ and $\epsilon = \epsilon_0$ with representation matrix $A \in K^{n \times n}$. By Lemma 4.37 and 3.43, we can restrict ourselves to the respective results regarding the matrices A and B. The following result from linear algebra then gives our desired result:
Let $A, B \in K^{n \times n}$ for some algebraically closed field K of characteristic 0. Then $[A, B]$ is nilpotent, if $[A, [A, B]] = 0$.
In our case we have $[A, B] = -\lambda A$, hence $[A, [A, B]] = 0$ and $-\lambda A$ is nilpotent. As $\lambda \neq 0$, we get that A is nilpotent, hence ϵ. □

Remark 4.39
The result in the proof of the previous lemma is a typical exercise regarding the connection between matrices and Lie algebras. It can be proven using, that $[A, B]^{k+1} = [A, B]^k \cdot (AB - BA) = A[A, B]B - [A, B]BA$, which has trace 0 for all $k \geq 1$. Thus we get that, over an algebraically closed field, the matrix $[A, B]$ is nilpotent.

Remark 4.40
Lemma 4.38 states, that if we have a homogeneous derivation with weight $\neq 0$, then this derivation is already nilpotent.

It is clear, that nilpotent derivations stay nilpotent under arbitrary coordinate changes. The next lemma states, when diagonal derivations keep their diagonal form.

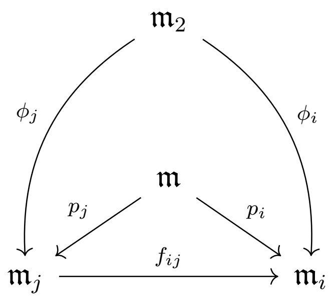
Lemma 4.41
Let R be a complete analytic algebra over a field K, let $x_1, \ldots, x_n$ be a set of coordinates for R, where $n \in \mathbb{N}$ and $\delta \in \mathrm{Der}'(R)$, with $\delta = \sum_{i=1}^{n} \lambda_i x_i \partial_{x_i}$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in K^n$. Then δ is invariant under $\underline{\lambda}$ homogeneous coordinate changes, that is, coordinate changes of type $x_i \mapsto x_i + h_i$ for some $h_i \in R$ with $\delta(h_i) = \lambda \cdot h_i$ for some $\lambda \in K$.

Proof:
See [13, Lemma 2.7]. □

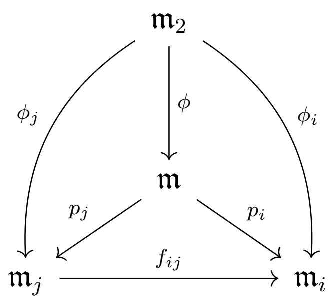
The next theorem generalizes a well known result from linear algebra, namely that we can find a linear coordinate change, such that a finite set of commuting diagonalizable matrices is simultaneously in diagonal form.

Theorem 4.42
Let R be a complete analytic algebra over a field K and $\delta_1, \ldots, \delta_s \in \mathrm{Der}'(R)$ diagonalizable and commuting. Then there exists a coordinate change, such that all δ_i are diagonal.

Proof:
We write $\mathfrak{m}$ for $\mathfrak{m}_R$ to keep the notation short and we consider the $\mathfrak{m}/\mathfrak{m}^k$ from now on as K-vector spaces. It is a well known fact from linear algebra, that a given set of diagonalizable and commuting matrices has common basis of eigenvectors (see for example [25, Chapter XIV, Exercise 13]). We use this result and the theory of projective limits of K-vector spaces, to show that our derivations $\delta_1, \ldots, \delta_s$ have a common basis of eigenvectors. We start by considering the spaces $\mathfrak{m}_k := \mathfrak{m}/\mathfrak{m}^k$. We have projections $p_k : \mathfrak{m} \twoheadrightarrow \mathfrak{m}_k$, $\pi_k : \mathfrak{m}_{k+1} \twoheadrightarrow \mathfrak{m}_k$ and $f_{ij} : \mathfrak{m}_j \twoheadrightarrow \mathfrak{m}_i$ for $j \geq i$. As the derivations commute in $\mathrm{Der}'(R)$, they also commute on all $\mathrm{Der}(R/\mathfrak{m}^k)$, hence we get a common basis of eigenvectors for the δ_i on all $\mathfrak{m}_k$. We write δ_i^k for the linear operator on $\mathfrak{m}_k$ induced by δ_i. Assume $\mathfrak{m}_k = \bigoplus_{j=1}^{n_k} E_j^k$ where E_j^k is an eigenspace of all derivations δ_i^k. We can lift any basis of $\mathfrak{m}_2$ to a basis of $\mathfrak{m}_3$, hence we get an injection $\sigma_3 : \mathfrak{m}_2 \hookrightarrow \mathfrak{m}_3$, with $\pi_2 \circ \sigma_3 = \mathrm{id}_{\mathfrak{m}_2}$. Inductively, we get injections $\sigma_{k+1} : \mathfrak{m}_k \hookrightarrow \mathfrak{m}_{k+1}$ with $\pi_k \circ \sigma_{k+1} = \mathrm{id}_{\mathfrak{m}_k}$. Using this construction, we get injections $\phi_k : \mathfrak{m}_2 \hookrightarrow \mathfrak{m}_k$, such that the following diagram commutes for $j \geq i$:



As $\mathfrak{m} = \varprojlim_{k \in \mathbb{N}} \mathfrak{m}_k$, we get by the universal property of projective limits a K-linear map $\phi : \mathfrak{m}_2 \to \mathfrak{m}$, such that the following diagram commutes for $j \geq i$:



ϕ is injective, as $p_2 \circ \phi = \mathrm{id}_{\mathfrak{m}_2}$.
Now consider any $x_j \in E_j^k$, then we have that $\delta_i^k(x_j) = \lambda_i x_j$ for all i. The latter holds in particular for $\phi_k(x_j)$, where $x_j \in E_j^2$. Using $\delta_i^k \circ p_k = p_k \circ \delta$, we get

$$\lambda_i \phi_k(x_j) = \delta_i^k(\phi_k(x_j)) = \delta_i^k \circ p_k(\phi(x_j)) = p_k \circ \delta(\phi(x_j)).$$

Thus we get for all $x_j \in E_j^2$:

$$\delta_i(\phi(x_j)) = \lambda_i \phi(x_j),$$

where we consider any element of $\mathfrak{m}$ as a sequence of elements of elements of $\mathfrak{m}_k$ on which the δ_i operate component wise. Applying Nakayama's Lemma, any basis of $\mathfrak{m}_2$ lifts to a minimal set of generators of $\mathfrak{m}$ as an ideal. By the application of ϕ to $\mathfrak{m}_2$, we get in our case, that $\mathfrak{m}$ has a set of generators, which are eigenvectors of all δ_i, so we have that all derivations are simultaneously diagonalizable. □

We can use the idea of the previous proof to prove the following theorem:

Theorem 4.43
Let R be a $(K^s, +)$ multi-graded analytic algebra over a field K. Then there exist diagonalizable and commuting $\delta_1, \ldots, \delta_s \in \mathrm{Der}'(R)$, such that the $(K^s, +)$ multi-grading is induced by them.

Proof:
We do the case $s = 2$, as the rest follows by induction. By Theorem 3.38, there exist diagonalizable derivations $\delta_1, \delta_2 \in \mathrm{Der}'(R)$ each inducing a $(K, +)$ grading of R, where these gradings correspond to the first and second component of our $(K^2, +)$ multi-grading. We now need to show, that they have a common eigenbasis. By Lemma 4.26, we know that δ_1 and δ_2 have a common eigenbasis on $R/\mathfrak{m}_R^k$ for all $k \in \mathbb{N}$, as the graded components of our ring R are precisely the common eigenvectors of δ_1 and δ_2. Checking the proof of Theorem 4.42, we see that this suffices to get a set of generators of $\mathfrak{m}_R$ consisting of common eigenvectors of δ_1 and δ_2, hence they can be simultaneously diagonalized and thus are commuting. □

Now we can state our more general version of the Formal Structure Theorem from [13]. It states, that we can extend a given Lie-Rinehart subalgebra of $\mathrm{Der}'(R)$, where R is a complete analytic algebra, to a larger Lie-Rinehart subalgebra of $\mathrm{Der}'(R)$, which has a concrete known structure. This structure can be used to compute possible gradings of the resulting Lie-Rinehart algebra.

Theorem 4.44 (Formal Structure Theorem)
Let R be a complete analytic algebra and $\mathfrak{g}$ be a Lie-Rinehart subalgebra of $\mathrm{Der}'(R)$. Assume, that $\mathfrak{g} = \varprojlim_{k\in\mathbb{N}} p_k(\mathfrak{g})$ and for any $\delta \in \mathfrak{g}$, we have that $\delta_S, \delta_N \in \mathfrak{g}$. Then there exist $\delta_1, \dots, \delta_s, \nu_1, \dots, \nu_r \in \mathfrak{g}$ with a uniquely determined $s \in \mathbb{N}$, such that

- *i) $\delta_1, \dots, \delta_s, \nu_1, \dots, \nu_r$ is a minimal set of generators of $\mathfrak{g}$ as an R-module,*
- *ii) if $\sigma \in \mathfrak{g}$ with $[\delta_i, \sigma] = 0$ for all i, then $\sigma_S \in \langle \delta_1, \dots, \delta_s \rangle_K$,*
- *iii) δ_i is diagonal with eigenvalues in $\mathbb{Q}$,*
- *iv) ν_i is nilpotent, and*
- *v) $[\delta_i, \nu_j] \in \mathbb{Q} \cdot \nu_j$*

Proof:
We are going to mimic the proof of [13, Theorem 5.4]. Statement iii) follows using Theorem 3.42 and 4.42. Assume, we already have $\delta_1, \dots, \delta_s \in \mathfrak{g}$ diagonalizable with s being maximal. As the δ_i induce a multi-grading of $\mathfrak{g}$, we can take any homogeneous derivation $\sigma \in \mathfrak{g}$, with multi-degree $\underline{\lambda} = (\lambda_1, \dots, \lambda_s) \in \mathbb{Q}^s$. If one of the λ_j is not equal to zero, Lemma 4.38 already states, that σ is nilpotent. So let us assume all λ_j are equal to zero. By Theorem 3.33, we get that σ has a Chevalley decomposition $\sigma = \sigma_S + \sigma_N$, with $\sigma_S, \sigma_N \in \mathfrak{g}$. As σ has multi-degree $\underline{0}$, also σ_S and σ_N have multi-degree $\underline{0}$, due to Proposition 3.36. Due to the maximality of s, we already get that $\sigma_S \in \langle \delta_1, \dots, \delta_s \rangle_K$. So we can assume $\sigma = \sigma_N$. This proves i), ii), iv) and v). We postpone the proof of the uniqueness of s to Chapter 5. □

Remark 4.45
Before we go on with a special case, in which Theorem 4.44 holds, we state a more general setup. Consider a sequence of K-vector spaces of R, say $(V_i)_{i\in\mathbb{N}}$, such that

$$V_0 \supseteq V_1 \supseteq V_2 \supset \dots,$$

and such that $RV_i \subseteq V_j$ for $j \geq i$ Define $\mathfrak{g} := \{\delta \in \mathrm{Der}'(R) | \ \delta(V_i) \subseteq V_j \text{ for } j \geq i\}$. It is easy to see, that $\mathfrak{g}$ is a Lie-Rinehart algebra. As in the proof of Theorem 3.32, we can show, that, if $\delta \in \mathfrak{g}$, then also $\delta_S, \delta_N \in \mathfrak{g}$, as the defining property is kept under taking powers as morphisms of vector spaces. In the following we consider the setup, where I is an ideal of R and we have $V_i := I$ for all $i \in \mathbb{N}$.

The Formal Structure Theorem has a nice application in the computation of homogeneities of ideals of complete analytic algebras. As all complete analytic algebras over a field K are of type $R = K[[x_1, \dots, x_n]]/I$ for some $n \in \mathbb{N}$ and an ideal I of $K[[x_1, \dots, x_n]]$, we can consider a special set of derivations, namely so called *logarithmic derivations*.

Definition 4.46
Let R be an analytic algebra over a field K and let I be an ideal of R. We call the R-module

$$\mathrm{Der}_I(R) := \mathrm{Der}_I := \{\delta \in \mathrm{Der}(R) | \delta(I) \subseteq I\}$$

the module of logarithmic derivations.

Remark 4.47
It is obvious, that the module of logarithmic derivations is a submodule of $\mathrm{Der}(R)$*. Furthermore, it is a Lie-Rinehart subalgebra of* $\mathrm{Der}(R)$*.*

Now we use Lemma 3.17 to show, that all derivations of $K[[x_1, \dots, x_n]]/I$ arise from I-invariant derivations of $K[[x_1, \dots, x_n]]$.

Corollary 4.48
Consider the setup of Lemma 3.17. Then the derivation α is I-invariant.

Proof:
We clearly have $\delta(0) = 0$, so $\phi \circ \alpha(I) = \delta \circ \phi(I) = 0$ and we get that $\alpha(I) \subseteq I$. □

We know, that the information regarding a $(K^s, +)$ grading of an ideal I can be given by stating diagonalizable derivations of $\mathrm{Der}'(R)$, with $\delta(I) \subseteq I$. This motivates the following definition.

Definition 4.49
Let R be a complete analytic algebra and I an ideal of R. Define

$$\mathrm{Der}'_I(R) := \mathrm{Der}_I(R) \cap \mathrm{Der}'(R).$$

We call $\mathrm{Der}'_I(R)$ the module of complete logarithmic derivations.

Remark 4.50
The term complete *in the previous definition arises from the fact, that the module turns out to be complete.*

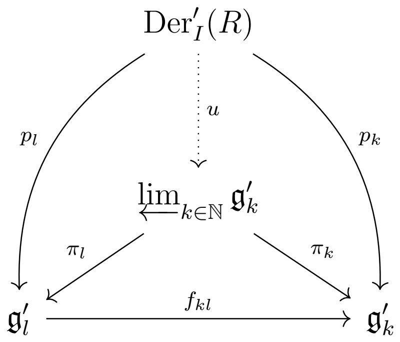
It easy to see, that $(R, \mathrm{Der}'_I(R), \rho)$ is a Lie-Rinehart algebra, with $\rho : \mathrm{Der}'_I(R) \hookrightarrow \mathrm{Der}(R)$. Next we show, that it satisfies the conditions of Theorem 4.44.

Remark 4.51
From now on, we write $\mathfrak{g}'_k$ for the images of the projections of $\mathrm{Der}'_I(R)$ *to* $\mathrm{Der}(R/\mathfrak{m}_R^k)$ *for all* $k \in \mathbb{N}$.

Corollary 4.52
Let R be a complete analytic algebra and I an ideal of R. Then $\mathrm{Der}'_I(R)$ satisfies the conditions of Theorem 4.44 and there exist δ_i and ν_j as in Theorem 4.44, such that:

$$\mathrm{Der}'_I(R) = \langle \delta_1, \ldots, \delta_s, \nu_1, \ldots, \nu_r \rangle_R.$$

Proof:
We want to apply Theorem 4.44 to $\mathrm{Der}'_I(R)$, hence we need to show, that for any $\delta \in \mathrm{Der}'_I(R)$, we have that $\delta_S, \delta_N \in \mathrm{Der}'_I(R)$ and that $\mathrm{Der}'_I(R) \cong \varprojlim_{k\in\mathbb{N}} \mathfrak{g}'_k$. The first statement follows from the fact, that any $\delta \in \mathrm{Der}'_I(R)$ has a Chevalley decomposition $\delta = \delta_S + \delta_N$ and, using that $\delta(I) \subseteq I$, we have that $\delta_S(I) \subseteq I$ and $\delta_N(I) \subseteq I$ by Theorem 3.32. To show $\mathrm{Der}'_I(R) \cong \varprojlim_{k\in\mathbb{N}} \mathfrak{g}'_k$, we consider the following commutative diagram, we get due to the definition of the $\mathfrak{g}'_k$:



The injectivity of u follows by same proof as for Proposition 3.23, so we only need to show surjectivity. We can consider every element δ of $\varprojlim_{k\in\mathbb{N}} \mathfrak{g}'_k$ as a sequence of elements $(\delta_k)_{k\in\mathbb{N}}$ with $\delta_k \in \mathfrak{g}'_k \subseteq \mathfrak{g}_k$. As $\delta_k(\bar{I}) \subseteq \bar{I}$ holds for all δ_k, with $\bar{I}$ being the projection of I to $R/\mathfrak{m}_R^k$, we get $\delta(I + \mathfrak{m}_R^k) \subseteq I + \mathfrak{m}_R^k$ for all $k \in \mathbb{N}$. Using Krull's Intersection Theorem, we get that $\delta(I) \subseteq I$ and $\delta \in \mathrm{Der}'_I(R)$, hence we can find for any $\delta \in \varprojlim_{k\in\mathbb{N}} \mathfrak{g}'_k$ a $\delta' \in \mathrm{Der}'_I(R)$ with $u(\delta') = \delta$.
Now we can apply Theorem 4.44 to $\mathrm{Der}'_I(R)$ and our statement follows immediately.
□

Remark 4.53
Due to Corollary 4.48, we get that every derivation of an complete analytic algebra $R = K[[x_1, \ldots, x_n]]/I$ arises from a derivation of $K[[x_1, \ldots, x_n]]$, which is I-invariant. Hence analyzing the derivations of R can be reduced to analyzing $\mathrm{Der}'_I(K[[x_1, \ldots, x_n]])$, as these obviously induce derivations on R. By Theorem 3.40 we get that every diagonalizable derivation of $\mathrm{Der}'_I(K[[x_1, \ldots, x_n]])$ corresponds 1:1 to a grading of I respectively a grading of R, by Lemma 3.17. We are going to use this approach to compute the possible gradings of analytic algebras in Chapter 6.

5 Profinite Lie(-Rinehart) Algebras

In this chapter we take a closer look at so called profinite Lie algebras. The idea is to investigate Lie algebras, that arise as projective limits of finite-dimensional Lie algebras. The work of Hofmann and Morris in [20] in the context of topological Lie algebras serves as a template for our work. We modify their definitions in a sense, that our definitions are compatible with the ones in [20], if we endow our Lie algebras with the discrete topology. In this chapter, we are only stating and proving basic results regarding profinite Lie algebras, as our goal is to prove an analogous statement to Theorem 2.55 for profinite Lie algebras. We include profinite Lie-Rinehart algebras in the beginning, as their construction is analogous to the construction of profinite Lie algebras.

Remark 5.1
In the following chapter, R always denotes a complete analytic algebra over a field K of characteristic 0 and $\mathfrak{g}$ always denotes a Lie algebra over the field K. We denote a Lie-Rinehart algebra by $(R, \mathfrak{g}, \rho)$. We restrict ourselves to the natural numbers as a set of indexes, as the other cases extend naturally and we only work with this setup in the following chapter.

5.1 Basic Definitions and Results

Let us start with the basic definition of the following chapter.

Definition 5.2
Let $(\mathfrak{g}_i, f_{ij})$ be projective system of finite-dimensional Lie algebras over K, with $i, j \in \mathbb{N}$ and $i \leq j$. Then we call $\mathfrak{g} := \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$ a profinite Lie algebra*.*

Remark 5.3
From now on, we assume that the projections $p_i : \mathfrak{g} \to \mathfrak{g}_i$ are surjective for all $i \in \mathbb{N}$, hence also all f_{ij} are surjective for all $j \geq i$, as $f_{ij} \circ p_j = p_i$ and the p_i are surjective. We can do so due to Proposition 2.4.

Now we can define profinite Lie-Rinehart algebras in the setup of complete analytic algebras.

Definition 5.4
Let (R_i, f_{ij}) *be a projective system of complete analytic algebras, with* $R_i := R/\mathfrak{m}_R^i$*. Furthermore, let* $(\mathfrak{g}_i, f_{ij}^{\mathfrak{g}})$ *be a projective system of Lie-Rinehart algebras, with* $\mathfrak{g}_i \subseteq \mathrm{Der}(R_i)$ *and* $\rho_i : \mathfrak{g}_i \hookrightarrow \mathrm{Der}(R_i)$*. As the* $(R_i, \mathfrak{g}_i, \rho_i)$ *are finite-dimensional Lie-Rinehart algebras, we have a projective system* $(\mathrm{Der}(R_i), f_{ij}^{\mathrm{Der}'(R)})$ *with* $\varprojlim_{i\in\mathbb{N}} \mathrm{Der}(R_i) \hookrightarrow \mathrm{Der}(R)$*. Write* $\mathfrak{g} := \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$ *and denote by* ρ *the limit of the* ρ_i*, then we call* $(R, \mathfrak{g}, \rho)$ *a* profinite Lie-Rinehart algebra.

Remark 5.5
We stated the previous result only in the setup of complete analytic algebras, as we have that the $\mathrm{Der}(R/I^i)$ *have* $\mathrm{Der}'(R)$ *as their limit due to Proposition 3.23. A detailed analysis of the proof of the aforementioned proposition yields, that we can replace the analytic algebra by any ring* R *which arises as an* I*-adic completion. Let* $\mathrm{Der}'(R)$ *denote the set of derivations, which are* I*-invariant. Then we get* $\mathrm{Der}'(R) = \varprojlim_{i\in\mathbb{N}} \mathrm{Der}(R/I^i)$*, as the proof of Proposition 3.23 only needs* $\bigcap_{i\in\mathbb{N}} I^i = 0$*.*
To guarantee, that $\mathrm{Der}(R/I^i)$ *is finite-dimensional, we need to assume, that* R/I *is finite-dimensional as a* K*-vector space.*

Let us see an immediate result from the definition.

Lemma 5.6
Keep the notation from Definition 5.2. Let $(R, \mathfrak{g}, \rho)$ *be a profinite Lie-Rinehart algebra. Set* $\mathfrak{h}_i := \mathrm{Der}(R_i)$*. Then there exists a morphism of Lie-Rinehart algebras* (id_R, ψ)*, with*

$$\psi : \mathfrak{g} \hookrightarrow \varprojlim_{i\in\mathbb{N}} \mathfrak{h}_i = \mathrm{Der}'(R).$$

Proof:
Consider the following exact sequence:

$$0 \longrightarrow \mathfrak{g}_i \xrightarrow{\rho_i} \mathfrak{h}_i$$

Using, that $\varprojlim$ is a left-exact functor, we get an injection $\rho : \mathfrak{g} \hookrightarrow \mathrm{Der}'(R)$. □

An important example for our work is the following:

Example 5.7
Consider the case where $R = \mathbb{C}[[x_1, \ldots, x_n]]$*,* $\mathfrak{g} = \mathrm{Der}'(R)$ *and* $\rho : \mathrm{Der}'(R) \hookrightarrow \mathrm{Der}(R)$*. Define the* $\mathfrak{g}_i$ *as the images of* $\mathfrak{g}$ *in* $\mathrm{Der}(R/\mathfrak{m}_R^i)$*. It is clear, that* $(R, \mathfrak{g}, \rho)$ *is a Lie-Rinehart algebra by Example 4.18. Define* $R_i := R/\mathfrak{m}_R^i$ *and* $\rho_i : \mathfrak{g}_i \hookrightarrow \mathrm{Der}(R_i)$ *for all* $i \in \mathbb{N}$ *and we get immediately, that the* $(R_i, \mathfrak{g}_i, \rho_i)$ *are finite-dimensional Lie-Rinehart algebras. We also have* $\varprojlim_{i\in\mathbb{N}} \mathrm{Der}(R_i) = \mathrm{Der}'(R) \hookrightarrow \mathrm{Der}(R)$ *and* $f_{ij}^{\mathfrak{g}} = f_{ij}^{\mathrm{Der}'(R)}$*, thus* $(R, \mathfrak{g}, \rho)$ *is a profinte Lie-Rinehart algebra.*

From Example 5.7 we get the following result.

Corollary 5.8
Let K be a field with characteristic 0 and $R = K[[x_1, \ldots, x_n]]/I$ for some ideal I of $K[[x_1, \ldots, x_n]]$. Then $(R, \mathrm{Der}'(R), \rho)$ is a profinite Lie-Rinehart algebra, with $\rho : \mathrm{Der}'(R) \hookrightarrow \mathrm{Der}(R)$.

Proof:
The computation from Example 5.7 works in the same way for this statement. □

Our next definition extends the notion of nilpotent and solvable Lie algebras to the profinite case.

Definition 5.9
Let $(R, \mathfrak{g}, \rho)$ be a profinite Lie-Rinehart algebra with projective systems (R_i, f_{ij}^R) and $(\mathfrak{g}_i, f_{ij}^{\mathfrak{g}})$, such that $R = \varprojlim_{i\in\mathbb{N}} R_i$ and $\mathfrak{g} = \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$. Then $(R, \mathfrak{g}, \rho)$ is called pronilpotent *(resp.* prosolvable*), if $\mathfrak{g}_i$ is nilpotent (resp. solvable) as a Lie algebra for all i. We call an element $x \in \mathfrak{g}$ nilpotent, if and only if $p_i^{\mathfrak{g}}(x) \in \mathfrak{g}_i$ is nilpotent for all i.*

Now we can state Engel's Theorem for profinite Lie-Rinehart algebras.

Theorem 5.10
Let $(R, \mathfrak{g}, \rho)$ be a profinite Lie-Rinehart algebra with projective systems (R_i, f_{ij}^R) and $(\mathfrak{g}_i, f_{ij}^{\mathfrak{g}})$, such that $R = \varprojlim_{i\in\mathbb{N}} R_i$ and $\mathfrak{g} = \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$. Then $(R, \mathfrak{g}, \rho)$ is pronilpotent if and only if $\mathrm{ad}_{p_i^{\mathfrak{g}}(x)}$ is nilpotent for all $x \in \mathfrak{g}$ and $i \in \mathbb{N}$.

Proof:
$(R, \mathfrak{g}, \rho)$ being pronilpotent is equivalent to $\mathfrak{g}_i$ being nilpotent for all i. Applying Engel's Theorem (see Theorem 2.51), $\mathfrak{g}_i$ being nilpotent is quivalent to $p_i^{\mathfrak{g}}(x)$ being nilpotent, which is equivalent to $\mathrm{ad}_{p_i^{\mathfrak{g}}(x)}$ being nilpotent for all $x \in \mathfrak{g}$, as $p_i^{\mathfrak{g}}$ is assumed to be surjective. Combining the results we see, that $(R, \mathfrak{g}, \rho)$ is pronilpotent if and only $\mathrm{ad}_{p_i^{\mathfrak{g}}(x)}$ is nilpotent for all $x \in \mathfrak{g}$ and $i \in \mathbb{N}$. □

Our next definitions are normalizers and centralizers of profinite Lie-Rinehart algebras.

Definition 5.11
Let $(R, \mathfrak{g}, \rho)$ be a profinite Lie-Rinehart algebra and $\mathfrak{h} \subset \mathfrak{g}$. Then

$$\mathrm{N}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} |\ [x, h] \in \mathfrak{h} \text{ for all } h \in \mathfrak{h}\}$$

is called the normalizer *of $\mathfrak{h}$ in $\mathfrak{g}$.*

Definition 5.12
Let $(R, \mathfrak{g}, \rho)$ be a profinite Lie-Rinehart algebra and $\mathfrak{h} \subset \mathfrak{g}$. Then

$$\mathrm{C}(\mathfrak{h}) := \{x \in \mathfrak{g} |\ [x, h] = 0 \text{ for all } h \in \mathfrak{h}\}$$

is called the centralizer *of $\mathfrak{h}$ in $\mathfrak{g}$.*

Remark 5.13
Let $\mathfrak{g}$ be a profinite Lie algebra and $\mathfrak{h}$ a subspace of $\mathfrak{g}$. It can be shown that $\mathrm{C}(\mathfrak{h})$ and $\mathrm{N}(\mathfrak{h})$ are subalgebras of $\mathfrak{g}$. If furthermore $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ is an ideal of the Lie algebra $\mathrm{N}(\mathfrak{h})$.

We omit an example at this point, as we are going to work with pronilpotent Lie(-Rinehart) algebras in the following section.

5.2 Pro-Cartan Subalgebras

In this section, we are going to extend the notion of a Cartan subalgebra to the profinite Lie(-Rinehart) algebra case. We show, that so called pro-Cartan subalgebras exist and that they are all conjugated in the sense of Theorem 2.55. We restrict our definitions to profinite Lie algebras, as the results only need the Lie algebra structure and not any module structure.

Let us first define pro-Cartan subalgebras.

Definition 5.14
Let $\mathfrak{g}$ be a profinite Lie algebra with projective system $(\mathfrak{g}_i, f_{ij}^{\mathfrak{g}})$, such that $\mathfrak{g} = \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$ and $\mathfrak{h}$ a profinite Lie subalgebra of $\mathfrak{g}$. We say $\mathfrak{h}$ is a pro-Cartan subalgebra, *if the following are satisfied:*

i) $\mathrm{N}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$,

ii) $\mathfrak{h}$ *is pronilpotent*

In Theorem 2.55, we used the group of inner automorphisms of a given finite dimensional Lie algebra. With the next definition, we extend this notion to profinite Lie algebras.

Definition 5.15
Let $\mathfrak{g}$ be a profinite Lie algebra with projective system $(\mathfrak{g}_i, f_{ij}^{\mathfrak{g}})$, such that $\mathfrak{g} = \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$. We call $\mathrm{Inn}_p(\mathfrak{g}) := \varprojlim_{i\in\mathbb{N}} \mathrm{Inn}(\mathfrak{g}_i)$ the group of projective inner automorphisms. *Furthermore, $\mathrm{Inn}(\mathfrak{g})$ denotes the group generated by the set $\{\exp(\mathrm{ad}_x) |\ x \text{ is nilpotent}\}$, where $\exp(\mathrm{ad}_x(y)) := (\exp(p_i^{\mathfrak{g}}([x, y])))_{i\in\mathbb{N}}$ for any $y \in \mathfrak{g}$. $\mathrm{Inn}(\mathfrak{g})$ is called* the group of inner automorphisms.

We called $\mathrm{Inn}(\mathfrak{g})$ the group of inner automorphisms, now we need to prove, that $\exp(\mathrm{ad}_x)$ is well-defined for any nilpotent $x \in \mathfrak{g}$, hence maps $\mathfrak{g}$ into itself and that it is an automorphism. The definition, at the moment, only guarantees, that $\exp(\mathrm{ad}_x)$ maps $\mathfrak{g}$ into the product $\prod_{i\in\mathbb{N}} \mathfrak{g}_i$. We split the proof regarding the properties of $\mathrm{Inn}(\mathfrak{g})$ into two parts. The first part shows, that the $\exp(\mathrm{ad}_x)$ are Lie algebra morphisms of $\mathfrak{g}$ into itself. The second part shows, that $\mathrm{Inn}(\mathfrak{g})$ injects into $\mathrm{Inn}_p(\mathfrak{g})$ and that the latter is a subgroup of $\mathrm{Aut}(\mathfrak{g})$, hence also $\mathrm{Inn}(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})$.

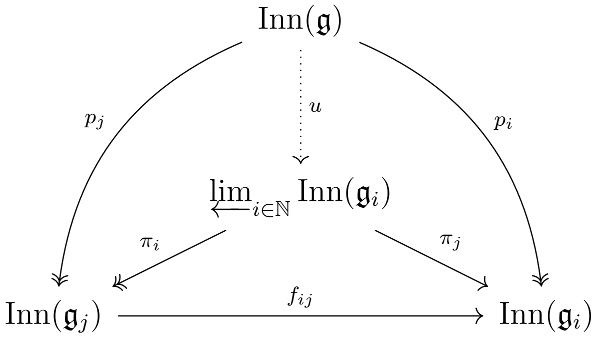
Proposition 5.16
Consider the setup of Definition 5.15. Let $x \in \mathfrak{g}$ be nilpotent, then $\exp(\mathrm{ad}_x)$ is a Lie algebra morphism of $\mathfrak{g}$ into itself.

Proof:
Let $y \in \mathfrak{g}$ be arbitrary. If we can show, that for any $j \geq i$, we have $f_{ij}^{\mathfrak{g}}(\exp(p_j^{\mathfrak{g}}([x, y]))) = \exp(p_i^{\mathfrak{g}}([x, y]))$, we haven proven, that $\exp(\mathrm{ad}_x)$ maps $\mathfrak{g}$ into itself. But this is easy too see, as $f_{ij}^{\mathfrak{g}}(\mathrm{ad}^l_{p_j^{\mathfrak{g}}(x)}(p_j^{\mathfrak{g}}(y))) = \mathrm{ad}^l_{p_i^{\mathfrak{g}}(x)}(p_i^{\mathfrak{g}}(y))$ for any $l \in \mathbb{N}$ and as the $\exp(p_i^{\mathfrak{g}}([x, y]))$ are finite sums, where the summands are powers of $\mathrm{ad}_{p_i^{\mathfrak{g}}(x)}$. Now for any $y \in \mathfrak{g}$, $\exp(\mathrm{ad}_x(y))$ is a sequence of elements compatible with the $f_{ij}^{\mathfrak{g}}$, hence it lies in $\mathfrak{g}$. To see, that $\exp(\mathrm{ad}_x)$ is a Lie algebra morphism of $\mathfrak{g}$ into itself, we only need to take a look at its behavior on every component of a sequence of elements. As we know, that $\exp(\mathrm{ad}_{p_i^{\mathfrak{g}}(x)})$ is a Lie algebra morphism of $\mathfrak{g}_i$ into itself for all $i \in \mathbb{N}$, we have that $\exp(\mathrm{ad}_x)$ is a Lie algebra morphism of $\mathfrak{g}$ into itself. □

Our next lemma shows us, that $\mathrm{Inn}(\mathfrak{g})$ is isomorphic to a subgroup of $\mathrm{Inn}_p(\mathfrak{g})$ and that the latter is a subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Lemma 5.17
Let $\mathfrak{g}$ be a profinite Lie algebra with projective system $(\mathfrak{g}_i, f_{ij}^{\mathfrak{g}})$, such that $\mathfrak{g} = \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$. Then $\mathrm{Inn}(\mathfrak{g}) \hookrightarrow \mathrm{Inn}_p(\mathfrak{g})$. In particular, all $\phi \in \mathrm{Inn}_p(\mathfrak{g})$ are automorphisms of $\mathfrak{g}$.

Proof:
First of all, we get projections $p_i : \mathrm{Inn}(\mathfrak{g}) \twoheadrightarrow \mathrm{Inn}_p(\mathfrak{g}_i)$ induced by $\exp(\mathrm{ad}_x) \mapsto \exp(\mathrm{ad}_{p_i^{\mathfrak{g}}(x)})$, where $p_i^{\mathfrak{g}}$ denotes the projection $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_i$. The p_i are surjective for all $i \in \mathbb{N}$, as the $p_i^{\mathfrak{g}}$ are surjective for all $i \in \mathbb{N}$ by Remark 5.1. The p_i commute with the group homomorphisms $f_{ij} : \mathrm{Inn}(\mathfrak{g}_j) \to \mathrm{Inn}(\mathfrak{g}_i)$, which are induced by $\exp(\mathrm{ad}_{p_j^{\mathfrak{g}}(x)}) \mapsto \exp(\mathrm{ad}_{f_{ij}^{\mathfrak{g}} \circ p_j^{\mathfrak{g}}(x)}) = \exp(\mathrm{ad}_{p_i^{\mathfrak{g}}(x)})$. By the universal property of projective limits, we get the following diagram:



where f_{ij} and π_i are as in Definition 2.1 for $i \leq j$. Due to the uniqueness of u and the proof of the universal property, we get that

$$u : \mathrm{Inn}(\mathfrak{g}) \to \varprojlim_{i\in\mathbb{N}} \mathrm{Inn}(\mathfrak{g}_i), \exp(\mathrm{ad}_x) \mapsto (\exp(\mathrm{ad}_{p_i^{\mathfrak{g}}(x)}))_{i\in\mathbb{N}}.$$

Now we can show, that u is injective. Consider any $\phi \in \mathrm{Inn}(\mathfrak{g})$, such that $u(\phi)$ is the identity. Then, using the commutativity the previous diagram, we get that $p_i(\phi) = \mathrm{id}_{\mathfrak{g}_i}$ for all $i \in \mathbb{N}$. Thus $p_i^{\mathfrak{g}}(\phi)(x) = p_i(\phi)(p_i^{\mathfrak{g}}(x)) = p_i^{\mathfrak{g}}(x)$, for any $x \in \mathfrak{g}$ and $i \in \mathbb{N}$. As $\mathfrak{g} = \varprojlim_{i\in\mathbb{N}} \mathfrak{g}_i$, we get that $\phi(x) = x$ for all $x \in \mathfrak{g}$, hence $\phi = \mathrm{id}_{\mathfrak{g}}$ and u is injective.
Finally, we can show the last statement. Let $\phi \in \mathrm{Inn}_p(\mathfrak{g})$, then it is a sequence $(\phi_i)_{i\in\mathbb{N}}$, with $\phi_i \in \mathrm{Inn}(\mathfrak{g}_i)$ for all $i \in \mathbb{N}$. Due to the construction of the f_{ij}, we get the following commutative diagram for all $j \geq i$:

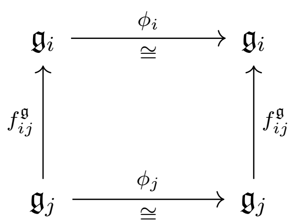


Figure 5.1: Commutative diagram regarding the ϕ_i.

First, we show that ϕ is injective. Consider the following exact sequence:

$$0 \longrightarrow 0 \longrightarrow \mathfrak{g}_i \xrightarrow{\phi_i} \mathfrak{g}_i \longrightarrow 0.$$

Due to the commutativity of the diagram in Figure 4.1, we can apply $\varprojlim$ as a left-exact functor and we get the following exact sequence:

$$0 \longrightarrow 0 \longrightarrow \mathfrak{g} \xrightarrow{\phi} \mathfrak{g}.$$

By exactness of the previous sequence, we have that ϕ is injective. Next, we have to show, that for any $y \in \mathfrak{g}$, there exists an $x \in \mathfrak{g}$, such that $\phi(x) = y$. As $\mathfrak{g}$ is a limit, we consider y as the sequence $(y_i)_{i\in\mathbb{N}}$ and show, that there is a sequence $(x_i)_{i\in\mathbb{N}}$ with $\phi_i(x_i) = y_i$ for all i. We already know, that the ϕ_i are isomorphisms, hence we have a unique $x_i \in \mathfrak{g}$, such that $\phi_i(x_i) = y_i$. If we can show, that $f^{\mathfrak{g}}_{ij}(x_j) = x_i$ for all $j \geq i$, we are done. We have

$$\phi_i(f^{\mathfrak{g}}_{ij}(x_j)) = f^{\mathfrak{g}}_{ij}(\phi_j(x_j)) = f^{\mathfrak{g}}_{ij}(y_j) = y_i,$$

hence $f^{\mathfrak{g}}_{ij}(x_j) = x_i$. □

Our next goal is to prove the existence and conjugacy of pro-Cartan subalgebras of profinite Lie algebras. Before we state the proof we need a few more preparing definitions and results.

Definition 5.18
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field K. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Furthermore, let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \ldots \oplus \mathfrak{g}_{\alpha_s}$ be the root space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$. We denote by $\mathrm{E}_{\mathfrak{g}}(\mathfrak{h})$ the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by elements of type $\exp(\mathrm{ad}_x)$, where x is contained in $\mathfrak{g}_{\alpha_i}$ for some $1 \leq i \leq s$.

Proposition 5.19
Consider the setup of Definition 5.18. Then $\mathrm{E}_{\mathfrak{g}}(\mathfrak{h})$ does not depend on the choice of $\mathfrak{h}$, and we can write $\mathrm{E}_{\mathfrak{g}}$ instead of $\mathrm{E}_{\mathfrak{g}}(\mathfrak{h})$.

Proof:
See [11, Proposition 1]. □

Theorem 5.20
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field K. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be be two Cartan subalgebras of $\mathfrak{g}$. Then there exists an $\sigma \in \mathrm{E}_{\mathfrak{g}}$, such that $\mathfrak{h} = \sigma(\mathfrak{h}')$.

Proof:
See [11, Theorem 2]. □

The following two results, combined with the previous theorem, are crucial for the proof of the existence and conjugacy of pro-Cartan subalgebras.

Lemma 5.21
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field K. Assume, $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{h} + \mathfrak{i}$, where $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathfrak{i}$ is an ideal of $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \ldots \oplus \mathfrak{g}_{\alpha_s}$ is a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$, then $\mathfrak{g}_{\alpha_i} \subseteq \mathfrak{i}$ for all $i = 1, \ldots, s$.

Proof:
As $\mathfrak{i}$ is an ideal, we can consider the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, which is a morphism of Lie algebras, as well as a morphism of $\mathfrak{h}$-modules. As $\mathfrak{g}/\mathfrak{i} \cong (\mathfrak{h} + \mathfrak{i})/\mathfrak{i}$, $\mathfrak{g}/\mathfrak{i}$ is nilpotent and we get that all non-trivial root spaces are 0. Using [4, Chapter 7, Proposition 9 iv)], we get that $\pi(\mathfrak{g}_{\alpha_i}) = 0$ for all $i = 1, \dots, s$, hence $\mathfrak{g}_{\alpha_i} \subseteq \mathfrak{i}$ for all $i = 1, \dots, s$. □

As an important corollary of the previous lemma, we get the following result.

Corollary 5.22
Let $\mathfrak{g}, \mathfrak{g}'$ be a finite-dimensional Lie algebras over an algebraically closed field K and $\phi : \mathfrak{g} \to \mathfrak{g}'$ be a surjective morphism of Lie algebras. Furthermore, let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}'$. Then every element of $E_{\phi^{-1}(\mathfrak{h}')}$ is generated by elements of type $\exp(\mathrm{ad}_x)$, with $x \in \ker(\phi) \cap \mathfrak{g}_{\alpha_i}$ for some $1 \leq i \leq s$, where the root space decomposition of $\phi^{-1}(\mathfrak{h}')$ equals $\mathfrak{h} \oplus \phi^{-1}(\mathfrak{h}')_{\alpha_1} \oplus \ldots \oplus \phi^{-1}(\mathfrak{h}')_{\alpha_s}$.

Proof:
By Proposition 2.37, we have that $\phi^{-1}(\mathfrak{h}')$ is a subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{h}$ a Cartan subalgebra of $\phi^{-1}(\mathfrak{h}')$, then $\phi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}'$ by Theorem 2.57. Using that $\phi(\mathfrak{h}) = \mathrm{N}_{\mathfrak{g}'}(\phi(\mathfrak{h})) = \mathrm{N}_{\mathfrak{h}'}(\phi(\mathfrak{h}))$, we see that $\phi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{h}'$. Due to the fact, that $\mathfrak{h}'$ is nilpotent, it contains only one Cartan subalgebra, namely itself, hence $\phi(\mathfrak{h}) = \mathfrak{h}'$. Now consider any $x \in \phi^{-1}(\mathfrak{h}')$, then there exists a $y \in \mathfrak{h}$, such that $\phi(y) = \phi(x)$, hence $x - y \in \ker(\phi)$. Now we have that any element x of $\phi^{-1}(\mathfrak{h}')$ can be written as $x = y + z$, with $y \in \mathfrak{h}$ and $z \in \ker(\phi)$. By Lemma 5.21 we get $\phi^{-1}(\mathfrak{h}')_{\alpha_i} \subset \ker(\phi)$ for all $i = 1, \dots, s$. Now by definition of $E_{\phi^{-1}(\mathfrak{h}')}$ and due to the fact, that by Proposition 5.19 $E_{\phi^{-1}(\mathfrak{h}')}$ does not depend on the chosen Cartan subalgebra, our claim follows. □

Now we can state the existence and conjugacy results for pro-Cartan subalgebras of profinite Lie algebras.

Theorem 5.23
Let $\mathfrak{g}$ be a profinite Lie algebra over an algebraically closed field K. Then there exists a pro-Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Proof:
As we have done before, we are going to use projective limits to prove our result. The notation is the same as in the proof of Lemma 5.17, replacing the inner automorphism groups by $\mathfrak{g}$ respectively $\mathfrak{g}_i$ for $i \in \mathbb{N}$. To keep notation short, we denote a Cartan subalgebra of $\mathfrak{g}_i$ by $\mathfrak{h}_i$, as their existence is guaranteed by Proposition 2.54. To show existence, we need to construct a sequence $(\mathfrak{h}_i)_{i \in \mathbb{N}}$, such that $\varprojlim_{i \in \mathbb{N}} \mathfrak{h}_i$ is a pro-Cartan subalgebra of $\mathfrak{g}$. Due to Theorem 2.57 and Theorem 2.58, we can use that f_{ij} is a surjection between $\mathfrak{g}_j$ and $\mathfrak{g}_i$, for $i \leq j$. Now we can lift any Cartan subalgebra $\mathfrak{h}_i$

of $\mathfrak{g}_i$ to a Cartan subalgebra $\mathfrak{h}_j$ of $\mathfrak{g}_j$, such that $\mathfrak{h}_i = f_{ij}(\mathfrak{h}_j)$. Starting with $\mathfrak{g}_1$, we can construct a sequence as mentioned before. Clearly $\mathfrak{h} = \varprojlim_{i\in\mathbb{N}} \mathfrak{h}_i$ is a pronilpotent subalgebra of $\mathfrak{g}$, so that we only need to check the normalizer property. So consider any $x \in \mathfrak{g}$, such that $[x, h] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$. Then the image of x under all projections to $\mathfrak{g}_i$, say $\bar{x}$, satisfies $[\bar{x}, \bar{h}] \in \mathfrak{h}_i$ for all $i \in \mathbb{N}$ and $\bar{h} \in \mathfrak{h}_i$. Now using that the $\mathfrak{h}_i$ are Cartan subalgebras, we get that $\bar{x} \in \mathfrak{h}_i$ for all $i \in \mathbb{N}$, hence $x \in \mathfrak{h}$ and we get $\mathfrak{h} = \mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$. □

Theorem 5.24
Let $\mathfrak{g}$ be a profinite Lie algebra over an algebraically closed field K. Then for any two pro-Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$ there exists a $\sigma \in \mathrm{Inn}_p(\mathfrak{g})$, such that $\mathfrak{h} = \sigma(\mathfrak{h}')$.

Proof:
Let $\mathfrak{h}$ and $\mathfrak{h}'$ be pro-Cartan subalgebras of $\mathfrak{g}$. Denote for all $i \in \mathbb{N}$ the projections of $\mathfrak{h}$ (respectively $\mathfrak{h}'$) to $\mathfrak{g}_i$ by $\mathfrak{h}_i$ (respectively $\mathfrak{h}'_i$). We know, that we can lift any $\exp(\mathrm{ad}_{p_i^{\mathfrak{g}}(x)}) \in \mathrm{Inn}(\mathfrak{g}_i)$ to $\exp(\mathrm{ad}_{p_j^{\mathfrak{g}}(x)}) \in \mathrm{Inn}(\mathfrak{g}_j)$ for all $j \geq i$ and $x \in \mathfrak{g}$. As $\mathfrak{g}$ projects surjectively to the $\mathfrak{g}_i$ this implies, that we can lift any $\sigma^{(i)} \in \mathrm{Inn}(\mathfrak{g}_i)$ to some $\sigma^{(j)} \in \mathrm{Inn}(\mathfrak{g}_j)$, such that the projection of $\sigma^{(j)}$ to $\mathrm{Inn}(\mathfrak{g}_i)$ equals $\sigma^{(i)}$. We show the conjugacy using this idea and an inductive argument. Assume that $\mathfrak{h}_i = \mathfrak{h}'_i$ for all $1 \leq i < k+1$ for some fixed $k \in \mathbb{N}$. Now we know, that $\mathfrak{h}_{k+1}$ and $\mathfrak{h}'_{k+1}$ are Cartan subalgebras of $f^{-1}_{k,k+1}(\mathfrak{g}_{k+1}) \subset \mathfrak{g}_{k+1}$. Applying Theorem 5.20 and Corollary 5.22 to the Lie algebra $f^{-1}_{k,k+1}(\mathfrak{g}_{k+1})$, we get the existence of an element $\sigma^{(k+1)}_{k+1}$, such that $\mathfrak{h}_{k+1} = \sigma^{(k+1)}_{k+1}(\mathfrak{h}'_{k+1})$ and $\sigma^(i)_{k+1} = \mathrm{id}_{\mathfrak{g}_i}$ for all $1 \leq i \leq k$, as $\sigma^{(k+1)}_{k+1}$ is a product of elements of type $\exp(\mathrm{ad}_x)$, with x living in a complement of $\mathfrak{h}_{k+1}$, but in the kernel of $f_{k,k+1}$. Iterating this process by lifting as explained before, we find a sequence of elements $(\sigma^{(i)}_i \circ \ldots \circ \sigma^{(i)}_1)_{i\in\mathbb{N}}$, such that the corresponding elements of $\mathrm{Inn}(\mathfrak{g}_i)$ satisfy $\mathfrak{h}_i = \sigma^{(i)}_i \circ \ldots \circ \sigma^{(i)}_1(\mathfrak{h}'_i)$. This means, that we have constructed an element $\sigma \in \mathrm{Inn}_p(\mathfrak{g})$, which satisfies $\mathfrak{h} = \sigma(\mathfrak{h}')$. □

Remark 5.25
In [20, Chapter 7] more results regarding topological profinite Lie algebras can be found, which basically generalize our previous results. Only Theorem 5.24 has no analogous result, hence our result, although it is not in a topological context, seems to be a new result, which has not been proven before.

As an example for the use of pro-Cartan subalgebras, we state the proof of the uniqueness of the dimension of the vector space s, which is generated by the pairwise commuting diagonalizable derivations in Theorem 4.44. If we can show, that the aforementioned vector space is a subspace of a pro-Cartan subalgebra, the conjugacy of pro-Cartan subalgebras gives us the uniqueness of s.

Lemma 5.26
Let R be a complete analytic algebra over an algebraically closed, complete real valuation

field K and $\mathfrak{g} \subset \mathrm{Der}'(R)$, as in Theorem 4.44. Using the notation from the aforementioned theorem, we have that

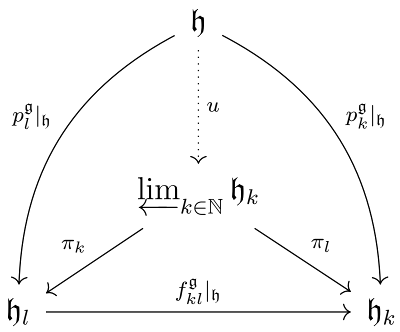
$$\mathfrak{h} := \{\tau \in \mathfrak{g} | [\tau, \delta_i] = 0,\ i = 1, \ldots, s\}$$

is a pro-Cartan subalgebra of $\mathfrak{g}$.

Proof:
From now on, we denote the projections of $\tau \in \mathfrak{g}$ to $\mathfrak{g}_k$ by $\bar{\tau}$.
First of all, we need to show that $\mathfrak{h}$ is subalgebra of $\mathfrak{g}$. It is obvious, that the sum of two elements of $\mathfrak{h}$ is again an element of $\mathfrak{h}$, as the Lie bracket is linear in each component. Now we need to show, that the commutator of any two elements of $\mathfrak{h}$ lies again in $\mathfrak{h}$. Let $\tau, \tau' \in \mathfrak{h}$, then

$$[\tau, [\tau', \delta_i]] + [\tau', [\delta_i, \tau]] + [\delta_i, [\tau, \tau']] = 0.$$

As we know from the definition of $\mathfrak{h}$, the first two summands equal 0, hence the third one also equals 0 and $[\tau, \tau'] \in \mathfrak{h}$. The previous computation also shows, that $\mathfrak{h}_k := \{\bar{\tau} \in \mathfrak{g}_k | \ [\bar{\tau}, \bar{\delta}_i] = 0\ i = 1, \ldots, s\}$ is a subalgebra of $\mathfrak{g}_k$ for all $k \in \mathbb{N}$.
Next we show, that every element of the $\mathfrak{h}_k$ is nilpotent. Using Theorem 2.51, we only need to show, that $\mathrm{ad}_{\bar{\tau}}$ is nilpotent for all $\bar{\tau}$. As we can decompose any $\bar{\tau} \in \mathfrak{h}_k$ into a semi-simple part $\bar{\tau}_S$ and a nilpotent part $\bar{\tau}_N$, we get $[\bar{\tau}_S + \bar{\tau}_N, \bar{\delta}_i] = [\bar{\tau}_N, \bar{\delta}_i] = 0$, for all $i = 1, \ldots, s$. Using, that by construction the semi-simple part of any element of $\mathfrak{h}_k$ is a linear combination of the $\bar{\delta}_i$, we only need to focus on the $\bar{\tau}_N$. As we are dealing with nilpotent derivations, they induce nilpotent linear operators on $\mathfrak{g}_k$ and we get that $\mathrm{ad}_{\bar{\tau}}$ is nilpotent. As this holds for all $\bar{\tau} \in \mathfrak{h}_k$, we get that the $\mathfrak{h}_k$ are nilpotent subalgebras, using Theorem 2.51.
Now we show the normalizer property. Clearly $\mathfrak{h}_k \subseteq \mathrm{N}_{\mathfrak{g}_k}(\mathfrak{h}_k)$. Consider any $\bar{\tau} \in \mathrm{N}_{\mathfrak{g}_k}(\mathfrak{h}_k)$. Commuting with all $\bar{\delta}_i$ means, that a derivation is of multi-degree $\underline{0}$, regarding the multi-grading induced by the $\bar{\delta}_i$. As $\bar{\tau} \in \mathrm{N}_{\mathfrak{g}_k}(\mathfrak{h}_k)$, we get that $[\bar{\delta}_j, [\bar{\tau}, \delta_i]] = 0$ for all $i, j = 1, \ldots, s$, hence $[\bar{\tau}, \bar{\delta}_i]$ is contained in the multi-degree $\underline{0}$ part of $\mathfrak{g}_k$. As all $\bar{\delta}_i$ are also contained in the multi-degree $\underline{0}$ part, we get that $\bar{\tau}$ has to be contained in there, otherwise $[\bar{\tau}, \bar{\delta}_i]$ were not contained in it, as the grading is compatible with Lie brackets. Hence, $[\bar{\tau}, \bar{\delta}_i] = 0$ for all $i = 1, \ldots, s$ and $\mathfrak{h}_k = \mathrm{N}_{\mathfrak{g}_k}(\mathfrak{h}_k)$.
So far we have shown, that all $\mathfrak{h}_k$ are Cartan subalgebras of the $\mathfrak{g}_k$. If we can show, that $\mathfrak{h} = \varprojlim_{k \in \mathbb{N}} \mathfrak{h}_k$, we get that $\mathfrak{h}$ is a pronilpotent subalgebra of $\mathfrak{g}$, which satisfies the normalizer property $\mathfrak{h} = \mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$, as all $\mathfrak{h}_k$ satisfy this property. Consider the following commutative diagram:



u is clearly injective, as for any $\tau \in \mathfrak{h}$ with $u(\tau) = 0$, we have that $\bar{\tau} = 0$ in all $\mathfrak{g}_k$, hence $\tau = 0$, as $\mathfrak{h} \subseteq \mathfrak{g}$. So we can assume, that $\mathfrak{h} \subseteq \varprojlim_{k\in\mathbb{N}} \mathfrak{h}_k$. Now consider any $\tau \in \varprojlim_{k\in\mathbb{N}} \mathfrak{h}_k$, then $[\tau, \delta_i] = 0$ for all $i = 1, \ldots, s$ as $[\bar{\tau}, \bar{\delta_i}] = 0$ in all $\mathfrak{g}_k$, hence $\tau \in \mathfrak{h}$. Finally, we have that $\mathfrak{h} = \varprojlim_{k\in\mathbb{N}} \mathfrak{h}_k$, $\mathfrak{h}$ is pronilpotent and satisfies $\mathfrak{h} = \mathrm{N}_{\mathfrak{g}}(\mathfrak{h})$, hence $\mathfrak{h}$ is a pro-Cartan subalgebra of $\mathfrak{g}$. □

Theorem 5.27
Let R be a complete analytic algebra. Consider the setup from Theorem 4.44. Then the dimension s of the vector space of pairwise commuting and diagonalizable derivations is uniquely determined.

Proof:
Using Lemma 5.26, we get that there is a pro-Cartan subalgebra $\mathfrak{h}$ containing our vector space. By Theorem 5.24, we have that all pro-Cartan subalgebras are conjugated. Being pairwise commuting and diagonalizable is kept under conjugation on finite-dimensional vector spaces (see Proposition 2.47). Due to the latter, the properties passes on to the limit. This means, that we have at least s semi-simple derivations, which are pairwise commuting and diagonalizable.
Assume we have a pro-Cartan subalgebra $\mathfrak{h}'$ containing an $s + 1^{st}$ diagonalizable derivation ϵ, then the image of ϵ in $\mathfrak{h}$ is already contained in the vector space generated by the first s diagonalizable derivations. This means, that it must have already been contained in the vector space generated by the preimages of the first s diagonalizable derivations of $\mathfrak{h}$ in $\mathfrak{h}'$, which contradicts the assumption. □

Remark 5.28
Theorem 5.27 states, that for any complete analytic algebra R, the dimension s of the $(K^s, +)$ multi-grading is uniquely determined and can be considered as an invariant of the algebra R.

6 Algorithmic Aspects of Power Series Rings

The following chapter is concerned with the theory of standard bases in the context of convergent power series rings. We present an overview of the theory of standard bases in our context and show their importance regarding basic computations in commutative algebra, as well as their usefulness in proving theoretical results. We are not going to investigate specific algorithms for the computation of standard bases, as this is a topic of its own. For an overview on the whole topic, we refer the reader for example to [34, Chapter 21] or [18, Chapter 1, 2 and 6]. Our goal is to provide a theoretical background together with a set of algorithms, such that we can compute maximal multi-homogeneities of a given ideal of an analytic algebra.

6.1 Theoretical Aspects of Standard Bases in Power Series Rings

Before we start defining standard bases, we refer the reader to [18, Chapter 1.2-1.5 and Chapter 2.5] for the basic notions as monomials, monomial orderings, leading monomials etc. We use the notation from [18]. Our results are taken from [18, Chapter 2.5, 2.8 and 6.4] and [7, Chapter 7].

Remark 6.1
From now on, K is always a complete real valuation field of characteristic 0 and $\underline{x} = (x_1, \dots, x_n)$. Throughout this section, we fix a local degree ordering $>$ on $\mathrm{Mon}(x_1, \dots, x_n)$, that is, $\underline{x}^\alpha > \underline{x}^\beta$ implies that $\text{w-deg}(\underline{x}^\alpha) \leq \text{w-deg}(\underline{x}^\beta)$ for suitable weight vector $\underline{\omega} = (\omega_1, \dots \omega_n)$ with $\omega_i > 0$. Such orderings are compatible with the $\langle \underline{x} \rangle$-adic topology, which allows us to compute standard bases in $K\langle\langle x_1, \dots, x_n \rangle\rangle$ and $K[[x_1, \dots, x_n]]$. Any non-zero $f \in K[[x_1, \dots, x_n]]$ can be written as $f = \sum_{\nu=0}^{\infty} a_\nu x^{\alpha(\nu)}, a_\nu \in K, a_0 \neq 0$ and $x^{\alpha(\nu)} > x^{\alpha(\nu)+1}$ for all ν. We denote the leading monomial by $\mathrm{LM}(f)$, the leading exponent by $\mathrm{LE}(f)$, the leading term by $\mathrm{LT}(f)$, the leading coefficient by $\mathrm{LC}(f)$ and the tail by $\mathrm{tail}(f)$. We denote the leading module of the module I by $\mathrm{L}(I)$.

To get familiar with the notation, we take a look at the following example.

Example 6.2
Consider the polynomial $f := 2x^5 + y^6 + y^4 + x \in \mathbb{C}\langle\langle x, y\rangle\rangle$ *and* $I := \langle f\rangle$*. Then the following table shows, how the leading monomial, leading coefficient and so on, depend on the choice of the monomial ordering.*

Monomial ordering	LC(f)	LM(f)	L(I)	tail(f)
lex	1	y^4	$\langle y^4\rangle$	$y^6 + x + 2x^5$
deglex	1	x	$\langle x\rangle$	$y^4 + 2x^5 + y^6$

In our example lex *denotes the* local lexicographical ordering *and* deglex *the* local degree lexicographical ordering.

Now we can define standard bases.

Definition 6.3
Let $R = K\langle\langle x_1, \ldots, x_n\rangle\rangle$ *or* $R = K[[x_1, \ldots, x_n]]$ *and* $I \subset R^N$ *an* R*-module and* $N \in \mathbb{N}$*. Then a finite set* $S \subset R^N$ *is called* standard basis *of* I *if*

$$S \subset I, \text{ and } \mathrm{L}(I) = \mathrm{L}(S).$$

That is, S *is a standard basis, if the leading monomials of the elements of* S *generate the leading module of* I*, or, in other words, if for any* $f \in I\backslash\{0\}$ *there exists a* $g \in S$ *satisfying* $\mathrm{LM}(g) | \mathrm{LM}(f)$*. If we just say that* S *is a standard basis, we mean that* S *is a standard basis of the ideal* $\langle S\rangle$ *generated by* S*.*

The next lemma guarantees us the existence of the standard basis.

Lemma 6.4
Let $I \subseteq R^N$ *be an* R*-module and* $N \in \mathbb{N}$*. Then there exists a standard basis* S *of* I*.*

Proof:
As R is Noetherian, we can assume that $\mathrm{L}(I)$ is finitely generated, that is, $\mathrm{L}(I) = \langle m_1, \ldots, m_s\rangle$ for monomials $m_i \in R^N$. As they arise from elements $g_1, \ldots, g_s \in I$, we can set $S := \{g_1, \ldots, g_s\}$ and we have $\mathrm{L}(S) = \mathrm{L}(I)$, hence S is a standard basis. □

Before we state more results regarding standard bases, we need some terminology.

Definition 6.5
Let $S \subset R^N$ *be any subset and* $N \in \mathbb{N}$*.*

i) S *is called* interreduced, *if* $0 \notin S$ *and if* $\mathrm{LM}(g) \nmid \mathrm{LM}(f)$ *for any two elements* $f \neq g$ *in* S*. An interreduced standard basis* S *is also called* minimal.

ii) $f \in R$ *is called* completely reduced with respect to S, *if no monomial in the power series expansion of* f *is contained in* $\mathrm{L}(S)$*.*

iii) S is called completely reduced, *if S is interreduced and if, for any $g \in S$, $\mathrm{LC}(g) = 1$ and $\mathrm{tail}(g)$ is completely reduced with respect to S.*

Let us take a look at an example.

Example 6.6
Let $I := \langle x^3 + y^2, y\rangle \subset \mathbb{C}\langle\langle x, y\rangle\rangle$ and consider the weight-vector $\underline{\omega} := (1, 2)$. Then $S := \{x^3 + y^2, y, y^5\}$ is a standard basis with respect to the local degree lexicographical ordering. S is not minimal, but $S' := \{x^3, y\}$, which is also a standard basis, is minimal and also completely reduced, as it contains only monomials, hence their tail is 0 and nothing has to be checked.

Theorem 6.7 (Grauert-Hironaka-Galligo Division Theorem)
Let $f, f_1, \ldots, f_m \in R^N$, for some $N \in \mathbb{N}$, then there exist $q_j \in R$ and $r \in R^N$, such that

$$f = \sum_{j=1}^{m} q_j f_j + r$$

and, for all $j = 1, \ldots, m$,

i) no monomial of r is divisible by $\mathrm{LM}(f_j)$;

ii) $\mathrm{LM}(q_j f_j) \leq \mathrm{LM}(f)$.

Proof:
This result is the famous *Grauert-Hironaka-Galligo Division Theorem*. See [15], [19] and [10]. For a compact presentation of the result, see [28, Theorem 10.1]. The module case follows, for example, from [10], by replacing the real or complex numbers with any complete real valuation field of characteristic 0. □

Definition 6.8
Using the notation from Theorem 6.7, define $S := \{f_1, \ldots, f_m\}$ and

$$\mathrm{NF}(f|S) := r.$$

In this way, we obtain a reduced normal form, *that is, a normal form, where r is completely reduced with respect to S.*

Having a reduced normal form, we get the following two corollaries. We prove the first one, to see how to actually argue with standard bases.

Corollary 6.9
Let $I \subset R^N$ be an R-module, $N \in \mathbb{N}$ and S, S' two standard bases of I. Then $\mathrm{NF}(f|S) = \mathrm{NF}(f|S')$ for all $f \in R\backslash\{0\}$.

Proof:
Let $f \in R^N\backslash\{0\}$. Define $r := \mathrm{NF}(f|S)$ and $r' := \mathrm{NF}(f|S')$. Then we have $r - r' \in I$, due to the representation of f in Theorem 6.7. Assume $r \neq r'$ and, with out loss of generality, that the leading monomial of $r - r'$ is a monomial of r. Then we have that the leading monomial of an element of S divides the leading a monomial of r, which contradicts Theorem 6.7, Property i), hence $r = r'$. □

Remark 6.10
Due to Corollary 6.9, we can represent any element f of the ring R/I, using its reduced normal form, with respect to a standard basis S of I, if the latter is actually computable.

Corollary 6.11
Let $I \subset R^N$ be an R-module, $N \in \mathbb{N}$ and S, S' two reduced standard bases of I. Then $S = S'$.

Proof:
See [7, Corollary 7.2.11], using, that we can replace $\mathbb{C}$ be any complete real valuation field of characteristic 0 or suitable fields compatible with the Grauert-Hironaka-Galligo Division Theorem. □

For actual computations in power series rings, the following theorems are important. The first theorem states, that we can reduce the case of a convergent power series ring to the formal power series ring and the second one states, that we can reduce the computation in a polynomial setup to the computation in the polynomial ring.

Theorem 6.12
Let $f_1, \ldots, f_m \in K\langle\langle x_1, \ldots, x_n\rangle\rangle \subset K[[x_1, \ldots, x_n]]$, both equipped with a compatible local degree ordering, and $I = \langle f_1, \ldots, f_m\rangle \subset K\langle\langle x_1, \ldots, x_n\rangle\rangle$. If $S := \{f_1, \ldots, f_m\}$ is a standard basis of I, then S is a standard basis of $IK[[x_1, \ldots, x_n]]$.

Proof:
Let S be a standard basis of I. Every element $f \in IK[[x_1, \ldots, x_n]]$ can be written as $\sum_{i=1}^m g_i f_i$ with $g_i \in K[[x_1, \ldots, x_n]]$. If $f \neq 0$ we can find a $c \in \mathbb{N}$, such that $\mathrm{LM}(f) \notin \langle \underline{x}\rangle^c$. So every element of $\langle \underline{x}\rangle^c$ has a smaller leading monomial than $\mathrm{LM}(f)$. Choose $g_i' \in K[x_1, \ldots, x_n]$, such that $g_i - g_i' \in \langle \underline{x}\rangle^c$. Consider $f' = \sum_{i=1}^m g_i' f_i$. Then $f' \in I$ and $f - f' \in \langle x\rangle^c$, hence $\mathrm{LM}(f) = \mathrm{LM}(f') \in \mathrm{L}(I)$. □

Theorem 6.13
Let $K[x_1, \ldots, x_n] \subset R$ be equipped with compatible local degree orderings. Let I be an ideal of $K[x_1, \ldots, x_n]$. If S is a standard basis of I, then S is a standard basis of IR.

Proof:
It is the same proof as for Theorem 6.12. □

Theorem 6.13 motivates the following definition.

Definition 6.14
Let $I \subset K[[x_1, \ldots, x_n]]$ be an ideal. We call I a polynomial ideal, *if there exists an ideal $J \subset K[x_1, \ldots, x_n]$, such that $I = JK[[x_1, \ldots, x_n]]$.*

Remark 6.15
As we cannot work with infinite sums on a computer, every actual standard basis computation is reduced to a computation on polynomial ideals.

The last theoretical aspect of standard bases we want to mention are syzygies, as these can be computed using standard bases. An algorithm for the computation follows in the upcoming section. Let us define syzygies.

Definition 6.16
A syzygy *between k elements $f_1, \ldots, f_k$ of an R-module M is a k-tuple $(g_1, \ldots, g_k) \in R^k$ satisfying*

$$\sum_{i=1}^{k} g_i f_i = 0.$$

Assume $I := \langle f_1, \ldots, f_k \rangle$, then we write $\operatorname{syz}(I) := \operatorname{syz}(f_1, \ldots, f_k)$ for the set of syzygies of I, with respect to the generators $f_1, \ldots, f_k$.

Lemma 6.17
Let R be a Noetherian ring and $f_1, \ldots, f_k$ be elements of an R-module M. Then $\operatorname{syz}(f_1, \ldots, f_k)$ is an R-module. If $f_1, \ldots, f_k$ and $g_1, \ldots, g_m$ are sets of generators for M, then

$$\operatorname{syz}(f_1, \ldots, f_k) \oplus \bigoplus_{i=1}^{m} Re_i \cong \operatorname{syz}(g_1, \ldots, g_m) \oplus \bigoplus_{i=1}^{k} Re_i.$$

If $k = m$, then $\operatorname{syz}(f_1, \ldots, f_k) \cong \operatorname{syz}(g_1, \ldots, g_k)$. Moreover, if R is a local ring and $f_1, \ldots, f_k$ and $g_1, \ldots, g_m$ are minimal sets of generators for M, then $\operatorname{syz}(M)$ is well-defined up to isomorphism.

Proof:
Let $I := \langle f_1, \ldots, f_k \rangle \subset M$. To show that $\operatorname{syz}(I)$ is an R-module, we consider the following map:

$$\psi : \bigoplus_{i=1}^{k} Re_i \to M, \; e_i \mapsto f_i,$$

where $\{e_1, \ldots, e_k\}$ denotes the canonical basis of R^k. Now it is obvious, that $\ker(\psi) = \operatorname{syz}(I)$, hence $\operatorname{syz}(I)$ is an R-module. For the proof, we are going to use *Schanuel's Lemma* (see [18, Excercise 2.5.5]). Assume we have $I = \langle f_1, \ldots, f_k \rangle = \langle g_1, \ldots, g_m \rangle$, then we get the following exact sequences:

$$0 \longrightarrow \operatorname{syz}(f_1, \ldots, f_k) \longrightarrow R^k \xrightarrow{\pi_1} I \longrightarrow 0$$

$$0 \longrightarrow \operatorname{syz}(g_1, \ldots, g_m) \longrightarrow R^m \xrightarrow{\pi_2} I \longrightarrow 0$$

where $\pi_1 : \bigoplus_{i=1}^k Re_i \to I, e_i \mapsto f_i$ and $\pi_2 : \bigoplus_{i=1}^m Re_i \to I, e_i \mapsto g_i$. Using Schanuel's Lemma, we get $\operatorname{syz}(f_1, \ldots, f_k) \oplus R^m \cong \operatorname{syz}(g_1, \ldots, g_m) \oplus R^k$.
Now assume that $k = m$, then we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{syz}(f_1, \ldots, f_k) & \longrightarrow & \operatorname{syz}(f_1, \ldots, f_k) \oplus R^k & \longrightarrow & R^k & \longrightarrow & 0 \\ & & \Big\downarrow \cong & & \Big\downarrow \cong & & \Big\downarrow \cong & & \\ 0 & \longrightarrow & \operatorname{syz}(g_1, \ldots, g_k) & \longrightarrow & \operatorname{syz}(g_1, \ldots, g_k) \oplus R^k & \longrightarrow & R^k & \longrightarrow & 0 \end{array}$$

As the second and third arrow from the top row to the bottom row are isomorphisms, we know by basic results from homological algebra, that we can choose the first one to be an isomorphism, too. If R is a local ring, we have that every minimal set of generators of a finitely generated module has the same number of elements, due to Nakayama's Lemma, hence we can always assume $k = m$ in the local case and we are done. □

As the last result of this section, we state how to compute the syzygy module using standard bases.

Lemma 6.18
Let $I = \langle f_1, \ldots, f_k \rangle \subset R^N = \bigoplus_{i=1}^N Re_i$, where $e_1, \ldots, e_N$ denotes the canonical basis of R^N. Consider the canonical embedding

$$R^N \subset R^{N+k} = \bigoplus_{i=1}^{N+k} Re_i$$

and the canonical projection $\pi : R^{N+k} \to R^k$. Let $S = \{g_1, \ldots, g_s\}$ be a standard basis of $F = \langle f_1 + e_{N+1}, \ldots, f_k + e_{N+k} \rangle$ with respect to an elimination ordering for $e_1, \ldots, e_N$. Suppose that $\{g_1, \ldots, g_l\} = S \cap \bigoplus_{i=N+1}^{N+k} Re_i$, then

$$\operatorname{syz}(I) = \langle \pi(g_1), \ldots, \pi(g_l) \rangle.$$

Proof:
See [18, Lemma 2.5.3]. □

6.2 Algorithmic Aspects of Standard Bases in Power Series Rings

This section is dedicated to results regarding the algorithmic use of standard bases. From now, R will denote either $K[x_1, \ldots, x_n]$ or $K[x_1, \ldots, x_n]_> \cong K[x_1, \ldots, x_n]_{\langle \underline{x} \rangle}$ and with a fixed local degree ordering, as we can perform computations only on polynomial input. We do not focus on actual algorithms for the computation of standard bases or normal forms, as they do not give us any insight for the algorithms we are going to need. Results on these can be found in [18, Chapter 1.6,1.7 and 2.3]. We are using SINGULAR (see [8]) for the computation of standard bases.

Before we start with algorithms, we need to argue, why we can pass from $S := K\langle\langle x_1, \ldots, x_n \rangle\rangle$ or $S := K[[x_1, \ldots, x_n]]$ to the polynomial case. The fact, that we can compute the standard bases of ideals in the polynomial case follows from Theorem 6.12 and Theorem 6.13, so we only need to argue, why we can compute syzygies. First consider the case $R := K[x_1, \ldots, x_n]_{\langle \underline{x} \rangle}$ for some $n \in \mathbb{N}$, and $f_1, \ldots, f_j \in R^k$ for some $k \in \mathbb{N}$. Denote by $\phi : R \hookrightarrow S$ the natural injection of R into S, by $\mathfrak{m}_R$ the maximal ideal of R and by $\mathfrak{m}_S$ the maximal ideal of S. Using [16, Theorem B.5.1, (4)], with $M := S$ and $I := \mathfrak{m}_R$, we get that $M/I^k M \cong S/\mathfrak{m}_S^k \cong R/\mathfrak{m}_R^k$ for all $k \geq 1$, hence S is R-flat. Now let $A := K[x_1, \ldots, x_n]$, then R is A-flat by [16, Proposition B.3.3 (6)]. Using [16, Proposition B.3.3 (2)], we get that S is A-flat. For more details on the notion of flatness, see [16, Appendix B.3 and B.5]. Now consider the following exact sequence:

$$0 \longrightarrow \operatorname{syz}_R(f_1, \ldots, f_k) \longrightarrow R^k \xrightarrow{(f_1, \ldots, f_k)} \langle f_1, \ldots, f_k \rangle_R \longrightarrow 0.$$

Applying $- \otimes_R S$ yields:

$$0 \longrightarrow \operatorname{syz}_R(f_1, \ldots, f_k) \otimes_R S \longrightarrow R^k \otimes_R S \xrightarrow{(f_1, \ldots, f_k)} \langle f_1, \ldots, f_k \rangle_R \otimes_R S \longrightarrow 0.$$

Using $\langle f_1, \ldots, f_k \rangle_R \otimes_R S \cong \langle f_1 \otimes_R 1, \ldots, f_k \otimes_R 1 \rangle_S$, which holds if S is R-flat, and $R^k \otimes_R S \cong S^k$, we get that $\operatorname{syz}_R(f_1, \ldots, f_k) \otimes_R S \cong \operatorname{syz}_S(f_1 \otimes_R 1, \ldots, f_k \otimes_R 1)$.

Now we can start with the first use of standard bases, namely testing, whether a given element is contained in a given finitely generated submodule of R^N or not.

Algorithm 1 Module Membership

INPUT: $f, f_1, \dots, f_k \in R^N$ with $I = \langle f_1, \dots, f_k \rangle_R$.
OUTPUT: 1, if $f \in I$, 0 else.

```
Compute a standard basis S of I
Compute r := NF(f|S)
if r = 0 then
    return 1
else
    return 0
end if
```

Theorem 6.19
Algorithm 1 terminates and works correctly.

Proof:
As the algorithms for standard bases and normal form computation terminate, Algorithm 1 terminates. The algorithm works correctly, due to the fact, that a reduced normal form returns 0 if and only if our element f is contained in I. □

The next algorithm states how to intersect a given finitely generated submodule I of R^N with a free submodule of R^N.

Algorithm 2 Intersection with Free Submodules

INPUT: $f_1, \dots, f_k \in R^N$ with $I = \langle f_1, \dots, f_k \rangle_R$ and $s \in \mathbb{N}$.
OUTPUT: $I' = I \bigcap \bigoplus_{i=s+1}^{N} Re_i$.

Compute a standard basis S of I, with respect to the module ordering

$$x^\alpha e_i < x^\beta e_j :\Longleftrightarrow\ j < i \text{ or } (j = i \text{ and } x^\alpha < x^\beta);$$

return $S' := \{g \in S \mid \mathrm{LM}(g) \in \bigoplus_{i=s+1}^{N} Re_i\}$

Theorem 6.20
Algorithm 2 terminates and works correctly.

Proof:
As the algorithms for standard bases and normal form computation terminate, Algorithm 2 terminates. The algorithm works correctly due to [18, Lemma 2.8.2]. □

Our next algorithm is the syzygy computation algorithm.

Algorithm 3 Computation of Syzygies

INPUT: $f_1, \ldots, f_k \in R^N$.
OUTPUT: $S = \{s_1, \ldots, s_l\}$, such that $\langle S \rangle = \mathrm{syz}(f_1, \ldots, f_k) \subset R^k$.

1: Set $F := \langle f_1 + e_{N+1}, \ldots, f_k + e_{N+k} \rangle$, where $e_1, \ldots, e_{N+k}$ denotes the canonical basis of $R^{N+k} = R^N \oplus R^k$ such that $f_1, \ldots, f_k \in R^N$;
2: Compute a standard basis G of F, with respect to an elimination ordering for $e_1, \ldots, e_N$;
3: Set $G_0 := G \cap \bigoplus_{i=N+1}^{N+k} Re_i = \{g_1, \ldots, g_l\}$ with $g_i = \sum_{j=1}^{k} a_{ij} e_{N+j}, i = 1, \ldots, l$;
4: $s_i := (a_{i1}, \ldots, a_{ik}), i = 1, \ldots, l$;
5: **return** $S = \{s_1, \ldots, s_l\}$.

Theorem 6.21
Algorithm 3 terminates and works correctly.

Proof:
Algorithm 3 terminates, as all algorithms used in the steps terminate. The algorithm works correctly due to Lemma 6.18. □

The final algorithm in this section, is an algorithm for the intersection of two finitely generated submodules of R^N. Before we can state the algorithm, we need the following lemma.

Lemma 6.22
Let $f_1, \ldots, f_k, h_1, \ldots, h_s \in R^N$, $I = \langle f_1, \ldots, f_k \rangle_R$ and $I' = \langle h_1, \ldots, h_s \rangle_R$. Moreover, let $c_1, \ldots, c_{N+k+s} \in R^{2N}$ be the columns of the $2N \times (N+k+s)$-matrix

$$\left(\begin{array}{ccc|ccc|ccc} 1 & & 0 & & & & & & \\ & \ddots & & f_1 & \ldots & f_k & 0 & \ldots & 0 \\ 0 & & 1 & & & & & & \\ \hline 1 & & 0 & & & & & & \\ & \ddots & & 0 & \ldots & 0 & h_1 & \ldots & h_s \\ 0 & & 1 & & & & & & \end{array}\right).$$

Then $g \in I \cap I' \subset R^N$ if and only g appears as the first N components of some $g' \in \mathrm{syz}(c_1, \ldots, c_{N+k+s}) \subset R^{N+k+s}$.

Proof:
Consider any syzygy for the columns of the matrix in Lemma 6.22, say $\lambda_1, \ldots, \lambda_{N+k+s}$. Due to the structure of our matrix, we get for the first N rows a sum of the type

$\sum_{i=1}^{N} \lambda_i e_i + \sum_{i=N+1}^{N+k} \lambda_i f_i = 0$, hence the first N components of our relation lie in I. An analogous statement yields that they lie in I', hence in $I \cap I'$. If $g \in I \cap I'$, it is easy to see that we can construct an element of our syzygy module. □

Now we can state the algorithm to compute the intersection of two finitely generated submodules I and I' of R^N.

Algorithm 4 Intersection of Submodules

INPUT: $f_1, \ldots, f_k, h_1, \ldots, h_s \in R^N$ with $I = \langle f_1, \ldots, f_k \rangle_R$ and $I' = \langle h_1, \ldots, h_s \rangle_R$.
OUTPUT: A set $P = \{p_1, \ldots, p_l\}$, such that $\langle P \rangle = I \cap I'$.

1: Let $c_i, i = 1, \ldots, N + k + s$, be the columns of the matrix in Lemma 6.22;
2: Compute $M = \{g_1, \ldots, g_l\} = \mathrm{syz}(c_1, \ldots, c_{N+k+s})$ using Algorithm 3;
3: Define $p_i, i = 1, \ldots, l$, to be the projections of the g_i to their first N components;
4: **return** $P := \{p_1, \ldots, p_l\}$

Theorem 6.23
Algorithm 4 terminates and works correctly.

Proof:
As the algorithms used Algorithm 4 terminate, it terminates itself. The algorithm works correctly due to Lemma 6.22. □

6.3 Homogeneities of Complete Analytic Algebras

In this section we use standard bases to compute the module of logarithmic derivations $\mathrm{Der}_I(R)$, of a given ideal $I \subset K[x_1, \ldots, x_n]$. After that, we are use our results from the Formal Structure Theorem to state an algorithm for the computation of the maximal multi-homogeneities of the given ideal, respectively the resulting quotient ring $K[[x_1, \ldots, x_n]]/I$. We set $K = \mathbb{Q}$, as we expect a rational result for our multi-homogeneities, due to Theorem 3.42. At this point we cannot state any results regarding the coordinates, in which our ideal has the maximal multi-homogeneity. A formal coordinate change, consisting of power series, is possible (see [13, Theorem 5.3]), but we cannot guarantee, that it is computable. The latter means, that we do not know, if we can find a polynomial coordinate change, such that we have a coordinate system in which our ideal has the maximal multi-homogeneity.

Remark 6.24
In this section we set $R := \mathbb{Q}[x_1, \ldots, x_n]$ for some $n \in \mathbb{N}$, as long as we consider algorithms. $I \subset R$ denotes an ideal of R generated by $f_1, \ldots, f_k$ for some $k \in \mathbb{N}$.

Let us start with our first result, which is the inspiration for the idea of the computation of $\mathrm{Der}_I(R)$.

Lemma 6.25
Let $R := \mathbb{Q}[x_1, \ldots, x_n]$ and $I := \langle f_1, \ldots f_k\rangle \subset R$ an ideal. Furthermore, set

$$A := \begin{pmatrix} \partial_{x_1} f_1 & \ldots & \partial_{x_n} f_1 & f_1 & & 0 & \ldots & f_k & & 0 \\ \vdots & & \vdots & & \ddots & & & & \ddots & \\ \partial_{x_1} f_k & \ldots & \partial_{x_n} f_k & 0 & & f_1 & \ldots & 0 & & f_k \end{pmatrix}.$$

Then $\mathrm{Der}_I(R) \cong \ker(\phi) \cap R^n$, where $\phi : R^{n+n\cdot k} \to R^k$ is the module homomorphism induced by the matrix A. We consider $R^n \subset R^{n+n\cdot k}$ as the free module generated by the first n components of $R^{n+n\cdot k}$.

Proof:
Let $(g_1, \ldots, g_n) \in \ker(\phi) \cap R^n$, then $\delta := \sum_{i=1}^n g_i \partial_{x_i}$ is a derivation and using the definition of $\ker(\phi)$, we get that $\delta(f_j) \in I$ for all $j = 1, \ldots, k$, hence $\delta \in \mathrm{Der}_I(R)$. Now consider any element $\delta \in \mathrm{Der}_I(R)$, then δ can be written as $\delta = \sum_{i=1}^n g_i \partial_{x_i}$, with $g_i \in R$. As $\delta(I) \subseteq I$, we can write $\delta(f_j) = \sum_{i=1}^n g_i \partial_{x_i} f_j = \sum_{l=1}^k h_l^{(j)} f_l$, with $h_l^{(j)} \in R$. Using this information, we can construct an element of the kernel of ϕ, thus $\mathrm{Der}_I(R)$ is isomorphic to a submodule of $\ker(\phi) \cap R^n$. Combining both results, we get $\mathrm{Der}_I(R) \cong \ker(\phi) \cap R^n$. □

Now we can state our Algorithm to compute a submodule of R^n, which is isomorphic to $\mathrm{Der}_I(R)$.

Algorithm 5 Module of Logarithmic Derivations

INPUT: $f_1, \ldots, f_k \in R$ with $I = \langle f_1, \ldots, f_k\rangle$.
OUTPUT: A set $P = \{p_1, \ldots, p_l\}$, such that $\langle P\rangle_R \cong \mathrm{Der}_I(R)$.

1: Let $c_i, i = 1, \ldots, n + n \cdot k$, be the columns of the matrix in Lemma 6.25;
2: Compute $M = \{g_1, \ldots, g_l\} = \mathrm{syz}(c_1, \ldots, c_{n+n\cdot k})$ using Algorithm 3;
3: Define $p_i, i = 1, \ldots, l$, to be the projections of the g_i to their first n components;
4: **return** $P := \{p_1, \ldots, p_l\}$

Theorem 6.26
Algorithm 5 terminates and works correctly.

Proof:
As the algorithms used Algorithm 5 terminate, it terminates itself. The correctness follows from Lemma 6.25. □

For a SINGULAR implementation of Algorithm 5, see Appendix A.2, Algorithm find_der.

Before we can start the computation of multi-homogeneities, we show, that we can pass to the linear parts of our result, as we know that there exists a set of coordinates, where our semi-simple derivations, which encode the information regarding the multi-homogeneity, are simultaneously in diagonal form. As every change of coordinates in a formal power series ring results in a conjugate transformation in the linear part, hence truncating and working exclusively with the representation matrices of the derivations generating $\mathrm{Der}_I(R)$ is sufficient.

Let us turn the previous comment into more precise mathematical results. To keep our notation as simple as possible, we write morphisms of $\mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$ into itself using a vector notation. The i-th component of the vector represents the image of x_i.

Lemma 6.27
Let $R := \mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$ for some $n \in \mathbb{N}$. Then every $\phi \in \mathrm{Aut}(R)$ can be written as $\phi(\underline{x}) = A\underline{x}^T +$ higher order terms in $\underline{x}$, where $\underline{x} = (x_1, \ldots, x_n)$ and $A \in \mathbb{C}^{n\times n}$, with $\det(A) \neq 0$.

Proof:
First of all, we have that $\phi(\underline{0}) = \underline{0}$ has to be satisfied, as otherwise a non-unit is mapped to a unit. Hence, we have that we can write ϕ as $\phi(\underline{x}) = A\underline{x}^T+$ higher order terms in $\underline{x}$. Using Theorem 3.9, we have that ϕ must induce an isomorphism on $\mathfrak{m}_R/\mathfrak{m}_R^2$, thus A has to be an invertible matrix. □

Corollary 6.28
Let $R := \mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$ for some $n \in \mathbb{N}$ and $\phi \in \mathrm{Aut}(R)$. Assume $\phi(\underline{x}) = A\underline{x}^T +$ higher order terms in $\underline{x}$ for some $A \in \mathbb{C}^{n\times n}$, with $\det(A) \neq 0$. Then $\phi^{-1} = A^{-1}\underline{x}^T +$ higher order terms in $\underline{x}$.

Proof:
We know, that $\phi \circ \phi^{-1}(\underline{x}) = \underline{x}^T$. Write $\phi^{-1} = B\underline{x}^T +$ higher order terms in $\underline{x}$, with $B \in \mathbb{C}^{n\times n}$ and $\det(B) \neq 0$. As all higher order terms do not affect the linear part, we get that $AB\underline{x}^T = \underline{x}^T$, hence $B = A^{-1}$. □

Next, we investigate the affect on derivations of R. We focus on the linear part, as this is the only part, we are actually interested in.

Lemma 6.29
Let $R := \mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$ for some $n \in \mathbb{N}$ and $\phi \in \mathrm{Aut}(R)$. Furthermore, let $\underline{y}^T := \phi(\underline{x})$, with $\phi(\underline{x}) = A\underline{x}^T +$ higher order terms in $\underline{x}$ for some $A \in \mathbb{C}^{n\times n}$, with $\det(A) \neq 0$. Furthermore, let $\underline{y}B\underline{\partial_y}^T$, where $B \in \mathbb{C}^{n\times n}$, be the linear part of a given derivation $\delta \in \mathrm{Der}(R)$ in the coordinates given by $\underline{y}$. Then $\underline{x}A^TB(A^{-1})^T\underline{\partial_x}^T$ is the linear part of δ before the coordinate transformation by ϕ.

Proof:
Applying the chain rule (see [14, Chapter 4, Folgerung 1]), we get $\partial_{y_i} = \sum_{k=1}^{n} \partial_{y_i}(x_k)\partial_{x_k}$. As higher order terms do not affect our linear part during a coordinate transformation, we can directly assume $\phi(\underline{x}) = A\underline{x}^T$. This results in $\underline{\partial_y}^T = (A^{-1})^T \underline{\partial_x}^T$, as Corollary 6.28 yields $A^{-1}\underline{y}^T = \underline{x}^T$. Combining these results, we can write

$$\underline{y}B\underline{\partial_y}^T = \underline{x}A^T B(A^{-1})^T \underline{\partial_x}^T.$$

□

Remark 6.30
By Theorem 4.42, we know, that we can find a set of coordinates, in which our semi-simple derivations already equal their linear part and the latter is of diagonal form. We also know, by Lemma 6.29, that this information is still contained in the linear part of the derivations after any coordinate change, hence we can truncate our derivations and only consider the $\mathbb{C}$-vector space generated by their linear parts. Using Lemma 5.26, we can concentrate on the Cartan subalgebra of the respective vector space.

Remark 6.30 justifies the following algorithm to compute the Cartan subalgebra from Lemma 5.26.

Algorithm 6 Linear Part Cartan Subalgebra of $\mathrm{Der}_I(R) \cap \mathrm{Der}'(R)$

INPUT: $f_1, \dots, f_k \in R$ with $I = \langle f_1, \dots, f_k \rangle$.
OUTPUT: A set $C = \{A_1, \dots, A_l\}$, $A_i \in \mathbb{Q}^{n\times n}$, such that the A_i span a Cartan subalgebra of the Lie algebra generated by the representation matrices of the elements of $\mathrm{Der}_I(R)$.

1: Compute a set P', such that $\langle P' \rangle \cong \mathrm{Der}_I(R)$ using Algorithm 5;
2: Compute a set P'', such that $\langle P'' \rangle \cong \mathrm{Der}'(R)$ using Algorithm 5;
3: Compute a set P, such that $\langle P \rangle \cong \mathrm{Der}_I(R) \cap \mathrm{Der}'(R)$ using Algorithm 4;
4: Compute the set of linear parts of the $p_i \in P$. Denote it by N;
5: Compute a list $C = \{A_1, \dots A_k\}$, $A_i \in \mathbb{Q}^{n\times n}$, such that the elements of C are a basis of a Cartan subalgebra of the Lie algebra generated by the elements of N;
6: **return** C

Theorem 6.31
Algorithm 6 terminates and works correctly.

Proof:
Clearly Algortihm 5 terminates by Theorem 6.26. Truncating is also a trivial operation and terminates. The computation of a Cartan subalgebra also terminates. For an algorithm, see [6, Algorithm CartanSubAlgebraBigField]. Hence, our algorithm terminates. The correctness follows by the correctness of the used algorithms. □

In our experiments, we have seen, that in most cases the Cartan subalgebra already consisted of simultaneously diagonalizable matrices, hence we were able to compute a set of vectors, generating a maximal multi-homogeneity. The main problem using our algorithm is the fact, that we compute syzygies using standard bases, which have a double exponential worst case complexity (see for example [34, Chapter 21.7.]). Keeping the number of variables small and working with sparse polynomials or with homogeneous polynomials, we were able to compute some examples. Let us take a look at one of these examples of Algorithm 6. For further examples, see Appendix A.1.

Example 6.32
Consider the ring $R := \mathbb{C}[[X, Y, Z, W]]$ *and the ideal* $I = \langle X^4 - Y^2 + 8X^2Z - 2YZ - Z^2, 4X^2Y + Y^2 - 9X^2Z + 3YZ - XW, 6X^2Y - 3X^2Z + 2YZ - Z^2 - XW, X^3Z + \frac{4}{7}XYZ - \frac{9}{7}XZ^2 + \frac{1}{7}X^2W - \frac{2}{7}YW - \frac{2}{7}ZW, XYZ - \frac{5}{3}XZ^2 + \frac{1}{9}X^2W - \frac{1}{3}YW - \frac{4}{9}ZW, Z^3 + \frac{13}{21}X^3W + \frac{1}{3}XYW + XZW - \frac{5}{21}W^2\rangle$. *Using our implementation of Algorithm 6 (see Appendix A.2, Algorithm* LieAlg_der_homog*), we get the following basis for a Cartan subalgebra:*

$$A = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As we have only one element, we know that these are all homogeneities of our ideal. Taking a closer look at our equations for I*, we can see that we are already in a system of coordinates, where it has its maximal homogeneity.*

A problem arises, if we have more than one basis vector in the result of our computation using Algorithm 6. Theoretically we expect only rational eigenvalues for our diagonalizable matrices, due to Theorem 4.44. The following example shows, that we need to be able to handle *algebraic numbers* in our computation. As we work with rational matrices, their characteristic polynomials have rational coefficients and we get that our eigenvalues are algebraic numbers. The main problem is, that SINGULAR, at the moment, cannot handle diagonalization of matrices with non-rational eigenvalues in a way, in which it extends its base field automatically during the computation. Therefore, we have to use MAGMA (see [5],[3]) for the computation of the simultaneous diagonalization of our matrices.

Example 6.33
Consider the ring $R := \mathbb{C}[[X, Y, Z]]$ *and the ideal* $I = \langle X^7 + Y^2 + Z^2\rangle$. *Using SINGULAR we get*

$$A = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

as the results of Algorithm 6. Computing the simultaneous diagonalization using MAGMA, we get

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{7} \end{pmatrix} \text{ and } B' = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as the result, where i is the imaginary unit satisfying $i^2 = -1$. We have to keep in mind, that we actually work over a $\mathbb{C}$-vector space, hence we can multiply the second matrix with i and get as a final result:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{7} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 6.34
The reader has to be careful in the interpretation of our results. We can compute the maximal homogeneity of an ideal of a power series ring, but we cannot state, in which coordinate system our ideal has this maximal multi-homogeneity.

Example 6.35
Consider the setup from Example 6.33. Using the identity $a^2 + b^2 = (a + ib)(a - ib)$, we can write the polynomial generating our ideal as $x^7 + (y + iz)(y - iz)$. Define a coordinate change as follows $X = x, Y = y + iz$ and $Z = y - iz$, then the polynomial can be written as $X^7 + YZ$ and we can easily see, that $(\frac{2}{7}, 1, 1)$ and $(0, 1, -1)$ are homogeneities.

An open question is, whether we always have a *polynomial* coordinate change (possibly over the complex numbers), such that our ideal has its maximal multi-homogeneity in the new coordinates or not. We are still investigating this aspect, as it is not clear, if it can be proven in general or at least for some special type of ideals, as for example for isolated hypersurface singularities.

Our final example shall show another problem we have with our computations. As we have seen in the previous computations, our resulting Cartan subalgebra consisted only of diagonalizable derivations. Although we do not have any counterexamples, we assume this to be false in general. Consider the Lie algebra from Example 2.56 and denote it by $\mathfrak{g}$. We know, that a Cartan subalgebra of this special Lie algebra is the subalgebra generated by all diagonal matrices. In general we cannot expect any subalgebra of $\mathfrak{g}$ to have a similar structure for its own Cartan subalgebra. There is a special type of subalgebras where this holds, namely a subalgebra, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{i}$, with $\mathfrak{h}$ being a nilpotent subalgebra generated by diagonal matrices and $\mathfrak{i}$ being an ideal. Then $\bar{\mathfrak{h}}$ is the only Cartan subalgebra of $\mathfrak{g}/\mathfrak{i}$ and we get immediately, that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The question is, what is this having to do with our problem? Assume we have a positively graded Lie algebra $\mathfrak{g}$, that is, we can decompose $\mathfrak{g}$ as $\mathfrak{g}_0 \oplus \mathfrak{g}_{a_1} \oplus \ldots \oplus \mathfrak{g}_{a_k}$ with a_k being positive integers. Then it is easy to see, that

$\mathfrak{i} := \mathfrak{g}_{a_1} \oplus \ldots \oplus \mathfrak{g}_{a_k}$ is an ideal of $\mathfrak{g}$, as $[\mathfrak{g}_{a_i}, \mathfrak{g}_{a_j}] \subseteq \mathfrak{g}_{a_i+a_j}$. This idea can be extended to a multi-grading on our Lie algebra of linear parts. If we know, that there exists a grading, which is induced by a diagonalizable derivation and has only positive degree components, we have exactly the previous setup. Our next example, which is taken from [12, Example 1.2], shows that the previous setup cannot be expected in general.

Example 6.36
Let $R := \mathbb{C}[[x_1, \ldots, x_7]]$ *and*

$$\begin{aligned} f_1 &:= x_1x_4 + x_2x_5 + x_3^2 - x_4^5 + x_7^5 \\ f_2 &:= x_1x_5 + x_2x_6 + x_3^2 + x_6^5 + 5x_7^5. \end{aligned}$$

Now define $I := \langle f_1, f_2 \rangle$. *Then our algorithm yields the following representation matrix for our* $(\mathbb{Q}, +)$ *grading:*

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This result means, that, after clearing denominators, we only have a grading generated by the vector $(8, 8, 5, 2, 2, 2, 2)$. *Let*

$$\sigma := 8x_1\partial_{x_1} + 8x_2\partial_{x_2} + 5x_3\partial_{x_3} + 2x_4\partial_{x_4} + 2x_5\partial_{x_5} + 2x_6\partial_{x_6} + 2x_7\partial_{x_7}$$

and

$$\eta := 2x_3(x_5 - x_6)\partial_{x_1} - 2x_3(x_4 - x_5)\partial_{x_2} + (x_4x_6 - x_5^2)\partial_{x_3}.$$

Simple computations yield $\sigma, \eta \in \mathrm{Der}'_I(R)$ *and* $[\sigma, \eta] = -\eta$. *This result means, that we have a derivation* η, *which is contained in a component with negative degree.*

6.4 Prospect

Our computational results for Algorithm 6 match the theoretical results, which we expect by the Formal Structure Theorem (see Theorem 4.44). An important problem regarding this topic, which is still open, is how to handle the situation, where the resulting Cartan subalgebra contains nilpotent matrices in its vector space basis. A non-deterministic solution is to simply compute a random linear combination of our matrices. If the semi-simple part of the resulting matrix is not a semi-simple matrix, which is known to us, we add it to the list of known semi-simple matrices and compute a basis supplement. We did not implement this approach, as SINGULAR does not contain the necessary tools at the moment. A further question is, if this problem can be solved deterministically using an algorithm with polynomial complexity. A further project has to be an algorithm, that is able to state an explicit set of coordinates, in which our ideal has its maximal multi-homogeneity. This task is combined with the theoretical question, if this coordinate change can be stated constructively using polynomials, if the initial input was polynomial data. Our proofs in Chapter 4 and Chapter 5 use formal coordinate changes, but we do not know, if it can be shown, that a polynomial coordinate change suffices. A final question is, if our algorithm can be optimized in a way, such that is works faster and uses less memory (see Appendix A.1). To do so, more experimental results are needed, especially with different algorithms for the syzygy computation, as this seems to be the most expensive step in the computation.

A Appendix

A.1 Experimental Results

The following tables contain experimental results we obtained by using the implementation of Algorithm 6. The polynomials are randomly generated sparse polynomials, which were generated by using the *sparsepoly* function of SINGULAR's *random.lib*. We used the monomial ordering *ds*.
The first input were two polynomials f, g and we computed the maximal multi-homogeneities of the ideal $I = \langle f, g\rangle$, which was considered as an ideal of $\mathbb{Q}[[x, y]]$.

Polynomial f	Polynomial g	Degree	Homogeneities
$x + 2xy + x^3 + 3x^2y^2$	$5y + 3x^2 + 2x^2y + 4y^4$	4	$(1,0),(0,1)$
$4y + 4y^2 + 3xy^2 + 4xy^3 + xy^4$	$5y + 2x^2 + 4y^3 + 3x^2 + y^2 + 3y^5$	5	$(1,0),(0,1)$
$5x + 3y^2 + 5x^3 + 4xy^3 + 5x^5 + x^6 + 5x^2y^5 + 4y^8$	$5x + xy + 5xy^2 + 5x^3y + 2x^2y^3 + 3x^3y^3 + 3x^6y + 3y^8$	8	no result
$2y + y^2 + xy^2 + 4y^4 + y^5 + 4x^2y^4 + 4y^7 + 2x^2y^6 + 2x^3y^6 + y^{10}$	$y + 4xy + 5xy^2 + 4x^4 + 4x^5 + 5y^6 + 2y^7 + 3x^7y + 5x^9 + 2y^{10}$	10	no result

Table A.1: Experimental results for the computation of maximal multi-homogeneities of ideals generated by two polynomials

The second input were three polynomials f, g, h and we computed the maximal multi-homogeneities of the ideal $I = \langle f, g, h\rangle$, which was considered as an ideal of $\mathbb{Q}[[x, y, z]]$.

Polynomial f	Polynomial g	Polynomial h	Degree	Homogeneities
$4x + z^2 + 4xyz$	$4x + z^2 + 3x^3$	$x + 2y^2 + 5x^2z$	3	$(1,0,0),(0,1,1)$
$5y + z^2 + 5x^3 + xyz^2 + 3xz^4 + z^5$	$4x + 4z^2 + 3x^2z + x^4 + x^3z2 + 2z^5$	$x + 3y^2 + 2yz^2 + 5xz^3 + 5xy^2z^2 + 4xyz^3$	5	no result
$2x + 3z^2 + 3xz^2 + 2x^3y + 3x^2z^3 + 4yz^5 + 2xz^6$	$2x + 4x^2 + 3x^2z + xyz^2 + 4x^5 + xy^2z^3 + xy^6$	$x + 3z^2 + 5x^2y + 3xz^3 + 3x^4y + 2y^4z^2 + 2x^2y^3z^2$	7	no result

Table A.2: Experimental results for the computation of maximal multi-homogeneities of ideals generated by three polynomials

No result means, that our working memory, which was around 1000MB, was exceeded and SINGULAR was not able to finish the computation. This seems to happen due to coefficient explosions during the computation of the module of logarithmic derivations. We observed, that the rational coefficients of some derivations were large numbers, with more than twenty digits. The number of generators of our module was around 50, so that SINGULAR seems to deal with a large amount of data. We omit an example for the coefficient explosion, as we cannot properly include it into our thesis. The reader may do the computations for Example 6.32 using our algorithms and verify our claim.

A.2 SINGULAR Library for Lie Algebras and Derivations

The following is the SINGULAR library we have written for basic computations regarding Lie algebras and for the computations in Chapter 6. For details on the algorithms regarding Lie algebras, we refer the reader to [6].

```
//////////////////////////////////////////////////////////////////////////////
version="version LieAlg.lib 4.0.2.0 May_2015 "; // $Id$
category="Non-commutative Algebra";
info="
LIBRARY:  LieAlg.lib   Compute with Lie Algebras
AUTHORS:  Raul Epure, epure@mathematik.uni-kl.de
LITERATURE:
                [Coh00] H. Cohen,   A Course In Computational Algebraic Number Theory
                [DGr00] W. deGraaf, Lie Algebras: Theory And Algorithms
                [Epe15] R.-P. Epure, Homogeneity and Derivations on Analytic Algebras, Master Thesis
/////////////////////////////////////////////////////////////////////////////

PROCEDURES:

 ////BASIC ALGORTIHMS FOR LIE ALGEBRAS////////////////////////////////////////
 LieAlg_Basis (list l, int n); // Given a list of matrices this algorithm returns a vector space basis for the given Lie Algebra
 LieAlg_dim(list B);// Returns the dimension of the given Lie Algebra with basis B
 LieAlg_coeffs(LieAlg L, list B); // Returns the coefficients of L with respect to the basis B
 LieAlg_adjointmat(LieAlg L, list B); // Returns the adjoint representation matrix of L with respect to the Lie Algebra Basis B
 LieAlg_nonnilpotentelt(list B); // Checks whether the Lie Algebra is nilpotent or returns a non-nilpotent element
 LieAlg_structureconst(list B); // Returns the structure constants for the Lie Algebra with basis B
 LieAlg_centralizer(list B, list C); // Returns the Centralizer of C in B
 LieAlg_normalizer(list B, list C); // Returns the Normalizer of C in B
 LieAlg_complement(list B, list C); // Returns the Lie Algebra Complement of C in B
 LieAlg_productspace(list B, list C); //Returns the Lie Algebra generated by [B,C]

 ////COMPUTATION OF DECOMPOSITIONS AND CARTAN SUBALGEBRAS/////////////////////
 LieAlg_fittingonecomponent(list B, list C);// Returns the fitting one component of B with respect to C (if C is nilpotent)
 LieAlg_fittingzerocomponent(list B, list C);// Returns the fitting zero component of B with respect to C (if C is nilpotent)
 LieAlg_Cartan(list B); //Computes a Cartan-subalgebra for the Lie Algebra generated by B

 ////ALGORITHMS FOR LIE ALGEBRAS OF DERIVATIONS///////////////////////////////
 find_der(ideal I); // Computes the module of I logarithmic derivations
 der_matlist (module D); // Truncates a given module of derivations and returns a generating set for the respective Lie Algebra
 LieAlg_der_homog(ideal I); // Computes the Cartan Subalgebra of the I invariant derivations.
 ";
 LIB "linalg.lib";
 LIB "matrix.lib";


/////////////////////////////////////////////////////////////////////////////
////DEFINITION OF OUR NEW STRUCTURE//////////////////////////////////////////
/////////////////////////////////////////////////////////////////////////////

 static proc mod_init()
 {
        newstruct("LieAlg","matrix Mat"); //Definition of our new structure.
        system("install","LieAlg","=",LieAlg_eq,1); //redifining "=" for "LieAlg"
        system("install","LieAlg","==",LieAlg_eqtest,2); //redifining "==" for "LieAlg"
        system("install","LieAlg","!=",LieAlg_ineqtest,2); //redifining "!=" for "LieAlg"
        system("install","LieAlg","+",LieAlg_add,2); //redifining "+" for "LieAlg"
        system("install","LieAlg","-",LieAlg_sub,2); //redifining "-" for "LieAlg"
        system("install","LieAlg","*",LieAlg_mult,2); //redifining "*" for "LieAlg
 }
//////////////////////////////////////////////////////////////////////////////
////MAIN ALGORITHMS FOR LIE ALGEBRAS ////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////

//////////////////////////////////////////////////////////////////////////////
////OPERATIONS ON LIE ALGEBRA ELEMENTS///////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////
```

```
static proc LieAlg_eq(matrix A)
"USAGE:   LieAlg_eq(A); A matrix.
@*
RETURN:   An element L of our "LieAlg" structure, with value set to A.
NOTE:     The matrix A has to be a square matrix.
EXAMPLE:  example LieAlg_eq; shows an example
"
{
          LieAlg L;
          L.Mat=A;
          return(L);
}
example
{
    ring r=0,x,dp;
    matrix A=unitmat(3);
    LieAlg L; L=A;
    L;
}


//////////////////////////////////////////////////////////////////////////////
static proc LieAlg_add(LieAlg L, LieAlg G)
"USAGE:   LieAlg_add(L,G); L LieAlg, G LieAlg.
@*
RETURN:   An element M of our "LieAlg" structure, with value set to L+G.
NOTE:     L and G need to have the same size as matrices.
EXAMPLE:  example LieAlg_add; shows an example
"
{
          LieAlg M;
          matrix A=L.Mat+G.Mat;
          M=A;
          return(M);
}
example
{
    ring r=0,x,dp;
    matrix A=unitmat(3);
    LieAlg L; LieAlg G;
    L=A;L=G;
    L+G;
}
//////////////////////////////////////////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////
static proc LieAlg_sub(LieAlg L, LieAlg G)
"USAGE:   LieAlg_sub(L,G); L LieAlg, G LieAlg.
@*
RETURN:   An element M of our "LieAlg" structure, with value set to L+G.
NOTE:     L and G need to have the same size as matrices.
EXAMPLE:  example LieAlg_add; shows an example
"
{
          LieAlg M;
          matrix A=L.Mat-G.Mat;
          M=A;
          return(M);
}
example
{
    ring r=0,x,dp;
    matrix A=unitmat(3);
    LieAlg L; LieAlg G;
    L=A;L=G;
    L-G;
}
//////////////////////////////////////////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////
static proc LieAlg_mult(L, LieAlg G)
"USAGE:   LieAlg_mult(L,G); L , G LieAlg.
@*
RETURN:   An element M of our "LieAlg" structure, with value set to L*G, where
                      * denotes the classical Lie Braket multiplication.
NOTE:     L and G need to have the same size as matrices.
EXAMPLE:  example LieAlg_add; shows an example
"
{
if(typeof(L)=="LieAlg")
{
          LieAlg M;
          M.Mat=L.Mat*G.Mat-G.Mat*L.Mat;
          return(M);
}
else
{
```

```
          LieAlg M;
          M.Mat=L*G.Mat;
          return(M);
}
}
example
{
    ring r=0,x,dp;
    matrix A=unitmat(3);
    LieAlg L; LieAlg G;
    L=A;G=A;
    L*G;
}
////////////////////////////////////////////////////////////////////////////
////////////////////////////////////////////////////////////////////////////


static proc LieAlg_eqtest(LieAlg L, LieAlg G)
"USAGE:   LieAlg_eqtest(L,G); L LieAlg, G LieAlg.
@*
RETURN:  1 if L equals G, 0 else
EXAMPLE: example LieAlg_eqtest; shows an example
"
{
          //Comparing the matrices
          if (L.Mat==G.Mat)
          {return(1);}
          else
          {return(0);}
}
example
{
    ring r=0,x,dp;
    matrix A=unitmat(3);
    LieAlg L=A; LieAlg G=A;
    L==G;
}


////////////////////////////////////////////////////////////////////////////
////////////////////////////////////////////////////////////////////////////


static proc LieAlg_ineqtest(LieAlg L, LieAlg G)
"USAGE:   LieAlg_ineqtest(L,G); L LieAlg, G LieAlg.
@*
RETURN:  0 if L equals G, 1 else
EXAMPLE: example LieAlg_ineqtest; shows an example
"
{
          if (L==G)
          {return(0);}
          else
          {return(1);}
}
example
{
    ring r=0,x,dp;
    matrix A=unitmat(3);
    LieAlg L=A; LieAlg G=A;
    L!=G;
}


////////////////////////////////////////////////////////////////////////////
////BASIC ALGORTIHMS FOR LIE ALGEBRAS///////////////////////////////////////
////////////////////////////////////////////////////////////////////////////
proc LieAlg_Basis(list l,int n)
"USAGE:    LieAlg_Basis(l,n); l list, n integer.
@*
RETURN:  A list of elements of type LieAlg, which are the basis of the Lie algebra
                generated by the input matrices.
NOTE:    The matrices contained in the list l need to have the same size.
THEORY:  This algorithm computes a basis for a Lie algebra using a simple approach:
                First we compute a vector space basis. Then we compute all pairwise products
                and add them to our list of elements. Then we compute again a vector space basis
                of the resulting space. Now are two possibilities. The first, is that our
                dimension does not increase, then we have our basis for the Lie algebra, as
                further products can be reduced to the elements already contained in our list.
                If the dimension increases, we repeat this procedure until it stops increasing.
                As we are dealing with finite dimensional Lie algebras, this process has to
                stop at some point.
EXAMPLE: example LieAlg_Basis; shows an example
"
{
```

```
        int i; int j;
        int d;
        list ll;
        LieAlg L;


        //Constructing the basis
        l=matsp_basis(l,n,n);
        if(size(l)==0)
        {
                matrix O[n][n];
                L=O;
                ll=insert(ll,L);
                return(ll);
        }


        while(size(l)>d)
        {
                d=size(l);
                for (i=1;i<=d; i++)
                {
                        for(j=i;j<=d;j++)
                        {
                                //We need to start from the 'back' of the list due to the implementation of insert
                                l=insert(l,l[size(l)+1-i]*l[size(l)+1-j]-l[size(l)+1-j]*l[size(l)+1-i]);
                        }
                }
                l=matsp_basis(l,n,n);
        }
        //Preparing LieAlg Output

        for(i=1; i<=d;i++)
        {
                L=l[i];
                ll=insert(ll,L);
        }

        return(ll);
}
example
{
    ring r=0,x,dp;
    matrix A[2][2]=1,0,0,0;
    matrix B[2][2]=0,1,1,0;
    list l=A,B;
    list j=LieAlg_Basis(l,2);
    j;
}
//////////////////////////////////////////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////
proc LieAlg_dim(list B)
"USAGE:   LieAlg_dim(B); B list.
@*
RETURN:   Returns the dimension d of the Lie algebra with Basis B.
NOTE:     We do not check, if B is a basis.
EXAMPLE: example LieAlg_dim; shows an example
"
{
        int d=size(B);
        if(d>1)
        {return(d);}

        if(d==0)
        {return(0);}

        matrix O[ncols(B[1].Mat)][ncols(B[1].Mat)];
        LieAlg OO=O;
        if(B[1]==OO)
        {return(0);}
        else
        {return(1);}

}
example
{
    ring r=0,x,dp;
    matrix A1[2][2]=1,0,0,0;
    matrix A2[2][2]=0,1,1,0;
    list l=A1,A2;
    list B=LieAlg_Basis(l,2);
    int d=LieAlg_dim(B);
    d;
}
//////////////////////////////////////////////////////////////////////////////
```

```
///////////////////////////////////////////////////////////////////////////////
proc LieAlg_coeffs(LieAlg L, list B)
"USAGE:    LieAlg_coeffs(L,B); L LieAlg, B list.
@*
RETURN:   The vector of coefficients of L with respect to the basis B.
NOTE:     The size of L as a matrix has to be compatible with the size
                  of the elements in B. We do not check, if B is a basis.
THEORY:   We compute the relations for our element L with respect to
                  the basis B by simply computing the kernel of the matrix, where
                  the first columns are the elements of B and the last column is L.
EXAMPLE:  example LieAlg_coeffs; shows an example
"
{
        //Creating auxilliary matrix for coordinate computation
        int n=nrows(L.Mat);
        matrix C[n*n][1];
        C=mat2vec(B[1].Mat);
        for (int i=2;i<=size(B);i++)
        {
                C=concat(C,mat2vec(B[i].Mat));
        }
        // Computation of our Output
        int m=size(B);
        module D;
        matrix v[m][1];
        C=concat(C,mat2vec(L.Mat));
        D=syz(C);
        v=D[1];
        if (D[1][ncols(C)]>0)
        {
                //Correcting "wrong" sign in the syzygy computation
                v=(-1)*v;
                return(v);
        }
        else
        {
                return(v);
        }


}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    LieAlg L=B[1]+5*B[2];
    LieAlg_coeffs(L,B);
}
///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////
proc LieAlg_adjointmat(LieAlg L, list B)
"USAGE:    LieAlg_adjointmat(L,B); L LieAlg, B list.
@*
RETURN:   The adjoint representation of L with respect to Basis B
NOTE:     L must be contained in the Lie Algebra generated by B. We do not check, if B is a basis.
THEORY:   Our algorithm computes the images of our basis elements under the map L*B[i] for all i.
                  After that we compute the representation matrix of this map by writing the coefficients
                  of L*B[i] as the columns of the matrix.
EXAMPLE:  example LieAlg_adjointmat; shows an example
"
{
        LieAlg G;
        int m=LieAlg_dim(B);
        if (m==0)
        {
                matrix A;
                return(A);
        }

        matrix M[m][1]=LieAlg_coeffs(L*B[1],B);

        for (int i=2; i<=m; i++)
        {
                G=L*B[i];
                M=concat(M,LieAlg_coeffs(G,B));
        }
        return(M);
}
example
{ //"EXAMPLE: Sturmfels: Algorithms in Invariant Theory 2.3.7:"; echo=2;
```

```
    ring r=0,x,dp;
    matrix A[2][2]=1,0,0,0;
    matrix AA[2][2]=0,1,1,0;
    list l=A,AA;
    list B=LieAlg_Basis(l,2);
    LieAlg L=A;
    matrix M=LieAlg_adjointmat(L,B);
    print(M);
}
////////////////////////////////////////////////////////////////////////////////
////////////////////////////////////////////////////////////////////////////////
proc LieAlg_nonnilpotentelt(list B)
"USAGE:    LieAlg_nonnilpotentelt(B); B list.
@*
RETURN:  A non-nilpotent element of the Lie Algebra with basis B or the 0 element,
             if the Lie Algebra is nilpotent.
NOTE:    Works only in characteristic zero. We do not check, if B is a basis.
THEORY:  Algorithm "NonNilpotentElement" in [DGr00].
EXAMPLE: example LieAlg_nilpotentelt; shows an example
"
{   LieAlg L;
    matrix M;


        if(LieAlg_dim(B)==0)
        {
                return(B[1]);
        }


        for(int i=1; i<=LieAlg_dim(B);i++)
        {
                M=LieAlg_adjointmat(B[i],B);
                if(nilp_test(M)==0)
                {
                        return(B[i]);
                }
        }
        for (int i=1; i<LieAlg_dim(B);i++)
        {
                for(int j=i+1;j<=LieAlg_dim(B);j++)
                {
                        matrix M=LieAlg_adjointmat(B[i]+B[j], B);
                        if(nilp_test(M)==0)
                        {
                                return(B[i]+B[j]);
                        }
                }
        }
        matrix C[nrows(B[1].Mat)][ncols(B[1].Mat)];
        L=C;
        return(L);
}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    LieAlg L=LieAlg_nonnilpotentelt(B);
    L;
}
////////////////////////////////////////////////////////////////////////////////
////////////////////////////////////////////////////////////////////////////////
proc LieAlg_structureconst(list B)
"USAGE:    LieAlg_structureconst(B); B list.
@*
RETURN:  The list of structure constants.
NOTE:    We do not check, if B is a basis.
THEORY:  We compute the structure constants, by computing all pairwise products.
EXAMPLE: example LieAlg_structureconst; shows an example
"
{
        list C;
        for(int i=size(B); i>=1;i--)
        {
                list l;
                for(int j=size(B); j>=1;j--)
                {

                        matrix v;
                        LieAlg L=B[i]*B[j];
```

```
                              v=LieAlg_coeffs(L,B);
                              l=insert(l,v);
                    }
                    C=insert(C,l);
          }
          return(C);
}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    list C=LieAlg_structureconst(B);
    print(C);
}
//////////////////////////////////////////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////
proc LieAlg_centralizer(list B, list C)
"USAGE:    LieAlg_centralizer(B,C); B list, C list.
@*
RETURN:  Returns the centralizer of C in B, where B and C are bases for Lie Algebras
               L resp. M with M being an subalgebra of L. The output is a list.
NOTE:    We do not check, if B or C are bases.
THEORY:  See Algorithm "Centralizer" in DeGraaf.
EXAMPLE: example LieAlg_centralizer; shows an example
"
{
          list S=LieAlg_structureconst(B);
          matrix M[size(B)][1]=LieAlg_coeffs(C[1],B);
          for(int i=2; i<=size(C);i++)
          {
                    M=concat(M,LieAlg_coeffs(C[i],B));
          }
          M=transpose(M); // To keep the same indices as deGraaf

          matrix L[size(B)*size(C)][size(B)];

          for (int k=1; k<=size(B);k++)
          {
                    for (int l=1;l<=size(C);l++)
                    {
                              for (int i=1; i<=size(B);i++)
                              {
                                        for (int j=1;j<=size(B);j++)
                                        {
                                                  L[(k-1)*size(C)+l,i]=L[(k-1)*size(C)+l,i]+M[l,j]*S[i][j][k,1];
                                        }
                              }
                    }
          }

          module D=syz(L);
          list BB;
          for (int i=1; i<=size(D);i++)
          {
                    matrix E[nrows(B[1].Mat)][ncols(B[1].Mat)];
                    LieAlg G=E;
                    for (int j=1;j<=size(B);j++)
                    {
                              G=G+D[i][j]*B[j];
                    }
                    BB=insert(BB,G);
          }
          return(BB);
}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    list D=LieAlg_centralizer(B,B);
    D;
}
//////////////////////////////////////////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////
proc LieAlg_normalizer(list B, list C)
"USAGE:    LieAlg_normalizer(B,C); B list, C list.
```

```
@*
RETURN:   Returns the normalizer of C in B, where B and C are bases for Lie
                algebras L resp. M, with M being an subalgebra of L.
NOTE:     We do not check, if B or C are bases.
THEORY:   See Algorithm "Normalizer" in [DGr00].
EXAMPLE:  example LieAlg_normalizer; shows an example
"
{
        list S=LieAlg_structureconst(B);
        LieAlg G;
        matrix M[size(B)][1]=LieAlg_coeffs(C[1],B);
        for(int i=2; i<=size(C);i++)
        {
                M=concat(M,LieAlg_coeffs(C[i],B));
        }
        M=transpose(M); // To keep the same indices as deGraaf

        matrix L[size(B)*size(C)][size(B)+size(C)*size(C)];

        for (int k=1; k<=size(B);k++)
        {
                for (int l=1;l<=size(C);l++)
                {
                        for (int i=1; i<=size(B);i++)
                        {
                                for (int j=1;j<=size(B);j++)
                                {
                                        L[(k-1)*size(C)+l,i]=L[(k-1)*size(C)+l,i]+M[l,j]*S[i][j][k,1];
                                }
                        }
                        for (int m=1;m<=size(C);m++)
                                {
                                        L[(k-1)*size(C)+l,size(B)+(l-1)*size(C)+m]=-M[m,k];
                                }
                }
        }

        module D=syz(L);
        list BB;
        for (int i=1; i<=size(D);i++)
        {
                matrix E[nrows(B[1].Mat)][ncols(B[1].Mat)];
                G=E;
                for (int j=1;j<=size(B);j++)
                {
                        G=G+D[i][j]*B[j];
                }
                BB=insert(BB,G);
        }
        return(BB);
}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    list D=LieAlg_centralizer(B,B);
    list C=LieAlg_normalizer(B,D);
   C;
}
///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////
proc LieAlg_intersect(list B, list C)
"USAGE:    LieAlg_intersect(B,C); B list, C list.
@*
RETURN:  Returns the intersection of the Lie Algebras generated by B and C.
NOTE:     Both Lie Algebras have to be subalgebras of the same Lie Algebra.
                We do not check, if B or C are bases.
THEORY:   Having a vector space basis of our Lie algebras, we can intersect them
                as vector spaces, and get the intersection as Lie algebras.
EXAMPLE:  example LieAlg_intersect; shows an example
"
{
        int n=ncols(B[1].Mat);
        list B1; list C1;
        LieAlg L;

        matrix A[n*n][size(B)];
        for (int i=1;i <=size(B);i++)
        {
                A[1..n*n,i]=mat2vec(B[i].Mat);
```

```
        }

        matrix AA[n*n][size(C)];
        for (int i=1;i <=size(C);i++)
        {
                AA[1..n*n,i]=mat2vec(C[i].Mat);
        }

        matrix AAA=sub_intersect(AA,A);
        list l;
        matrix E[n][n];
        for (int i=1;i<=ncols(AAA);i++)
        {
                E=AAA[1..n*n,i];
                L=E;
                l=insert(l,L);
        }

        return(l);
}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    list C=LieAlg_centralizer(B,B);
    list D=LieAlg_intersect(B,C);
    print("B:");print(B); print("C:");print(C);print("D:");print(D);
}
///////////////////////////////////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////
proc LieAlg_complement(list B, list C)
"USAGE:   LieAlg_complement(B,C); B list, C list.
@*
RETURN:  Returns the complement of the Lie Algebra generated by the basis C in the
                 Lie algebra generated by the basis B.
NOTE:    The first Lie Algebra has to contain the second. We do not check,
                 if B or C are bases.
THEORY:  Having a vector space basis of our Lie algebras, we can compute a vector
                 space supplement and get one as a Lie algebra.
EXAMPLE: example LieAlg_complement; shows an example
"
{
        int n=ncols(B[1].Mat);
        int i;

        LieAlg L;

        matrix A[n*n][size(B)];
        for (i=1;i <=size(B);i++)
        {
                A[1..n*n,i]=mat2vec(B[i].Mat);
        }

        matrix AA[n*n][size(C)];
        for (i=1;i<=size(C);i++)
        {
                AA[1..n*n,i]=mat2vec(C[i].Mat);
        }

        matrix AAA=sub_supplement(AA,A);
        list l;
        matrix E[n][n];
        for (int i=1;i<=ncols(AAA);i++)
        {
                E=AAA[1..n*n,i];
                l=insert(l,E);
        }
        l=LieAlg_Basis(l,n);
        return(l);
}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    list C=LieAlg_centralizer(B,B);
```

```
    list D=LieAlg_complement(B,C);
    print("B:");print(B); print("C:");print(C);print("D:");print(D);
}
////////////////////////////////////////////////////////////////////////////////
////////////////////////////////////////////////////////////////////////////////
proc LieAlg_productspace(list B, list C)
"USAGE:    LieAlg_productspace(B,C); B list, C list.
@*
RETURN:   Returns the Lie Algebra generated by [B,C].
NOTE:     Both Lie Algebras have to be subalgebras of a common Lie Algebra. We do
                not check, if B or C are bases.
THEORY:   See Algorithm "ProductSpace" in [DGr00].
EXAMPLE:  example LieAlg_productspace; shows an example
"
{
        list l; LieAlg G;
        if (LieAlg_dim(B)==0)
        {
                l=insert(l,B[1]);
                return(l);
        }
        if (LieAlg_dim(C)==0)
        {
                l=insert(l,C[1]);
                return(l);
        }

        int n=nrows(B[1].Mat);

        for (int i=1;i<=LieAlg_dim(B);i++)
        {
                for(int j=1;j<=LieAlg_dim(C);j++)
                {
                        G=B[i]*C[j];
                        l=insert(l,G.Mat);
                }
        }
        l=matsp_basis(l,n,n);
        list ll;
        for(int i=1;i<=size(l);i++)
        {
                        G=l[i];
                        ll=insert(ll,G);
        }
        return(ll);
}
example
{ "EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[3][3]=unitmat(3);
    matrix A2[3][3]=1,1,0,0,1,0,0,0,1;
    matrix A3[3][3]=1,0,0,0,1,1,0,0,1;
    matrix A4[3][3]=1,0,1,0,1,0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,3);
    list C=B[1];
    list D=B[3];
    list E=LieAlg_productspace(B,C);
        print(E);
}
////////////////////////////////////////////////////////////////////////////////
////COMPUTATION OF DECOMPOSITIONS AND CARTAN SUBALGEBRAS///////////////////////
////////////////////////////////////////////////////////////////////////////////
proc LieAlg_fittingonecomponent(list B, list C)
"USAGE:    LieAlg_fittingonecomponent(B,C); B list, C list.
@*
RETURN:   Returns the fitting one component of the Lie Algebra generated by B
                with respect to the nilpotent subalgebra generated by C.
NOTE:     The Lie Algebra generated by C has to be a nilpotent subalgebra of the
                one generated by B. We do not check, if B or C are bases.
THEORY:   See algorithm "FittingOneComponent" in [DGr00].
EXAMPLE:  example LieAlg_fittingonecomponent; shows an example
"
{       list l=LieAlg_productspace(C,B);
        list ll=LieAlg_productspace(C,l);
        // These are actually no Lie Algebras, but LieAlg_dim computes the vector space dimension
        while(LieAlg_dim(l)>LieAlg_dim(ll))
        {
                l=LieAlg_productspace(C,ll);
                ll=LieAlg_productspace(C,l);
        }
        return(ll);
}
example
{ "//EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
```

```
    ring r=0,x,dp;
    matrix A1[2][2]=1,0,0,0;
    matrix A2[2][2]=0,1,0,0;
    matrix A3[2][2]=0,0,1,0;
    matrix A4[2][2]=0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,2);
    list E=A1,A4;
    list C=LieAlg_Basis(E,2);
    list D=LieAlg_fittingonecomponent(B,C);
    print(D);
}
////////////////////////////////////////////////////////////////////////////////
////////////////////////////////////////////////////////////////////////////////
proc LieAlg_fittingzerocomponent(list B, list C)
"USAGE:   LieAlg_fittingzerocomponent(B,C); B list, C list.
@*
RETURN:   Returns the fitting zero component of the Lie algebra generated by B
                with respect to the nilpotent subalgebra generated by C.
NOTE:     The Lie Algebra generated by C has to be a nilpotent subalgebra of the one
                generated by B. We do not check, if B or C are bases.
THEORY:   The fitting zero component together with the fitting one component form
                our Lie algebra as a direct sum, hence computing a basis supplement of the
                fitting one component yields the fitting zero component.
EXAMPLE:  example LieAlg_fittingonecomponent; shows an example
"
{
        int n=nrows(B[1].Mat);
        list l=LieAlg_fittingonecomponent(B,C);
        l=LieAlg_complement(B,l);
        list ll;
        for (int i=1;i<=size(l);i++)
        {
                ll=insert(ll,l[i].Mat);
        }
        l=LieAlg_Basis(ll,n);

        return(l);
}
example
{ "//EXAMPLE: HEISENBERG ALGEBRA IN 3 DIMENSIONS:"; echo=2;
    ring r=0,x,dp;
    matrix A1[2][2]=1,0,0,0;
    matrix A2[2][2]=0,1,0,0;
    matrix A3[2][2]=0,0,1,0;
    matrix A4[2][2]=0,0,0,1;
    list l=A1,A2,A3,A4;
    list B=LieAlg_Basis(l,2);
    nrows(B[1].Mat);
    list E=A1,A4;
    list C=LieAlg_Basis(E,2);
    list D=LieAlg_fittingzerocomponent(B,C);
    print(D);
}
////////////////////////////////////////////////////////////////////////////////
////////////////////////////////////////////////////////////////////////////////
proc LieAlg_Cartan(list B)
"USAGE:   LieAlg_Cartan(B); B list.
@*
RETURN:   Returns the Cartan Subalgebra for the Lie Algebra generated by B.
NOTE:     The characteristic of the field has to be at least size(B)+1. We do not check,
                if B is a basis.
THEORY:   See Algorithm "CartanSubAlgebraBigField" in DeGraaf.
EXAMPLE:  example LieAlg_Cartan; shows an example
"
{
        //INITIALIZATIONS
        matrix O[nrows(B[1].Mat)][ncols(B[1].Mat)];
        int i;
        LieAlg OO=O;
        LieAlg JJ;
        list lll;


        // TEST IF OUR LIE ALGEBRA IS ALREADY NILPOTENT
        LieAlg GG=LieAlg_nonnilpotentelt(B);
        if (GG==OO)
        {
                return(B);
        }

        //MAIN COMPUTATIONS

        list C=GG;
        list ll=LieAlg_fittingzerocomponent(B,C);
```

```
        LieAlg HH=LieAlg_nonnilpotentelt(ll);
        while (HH!=OO)
        {
                i=1;
                while (i<=LieAlg_dim(B)+1)
                        {
                        JJ=GG+i*(HH-GG);
                        C=JJ;
                        lll=LieAlg_fittingzerocomponent(B,C);
                        if(LieAlg_dim(lll)<LieAlg_dim(ll))
                        {
                                GG=JJ;
                                i=LieAlg_dim(B)+4;
                                ll=lll;
                        }
                        i++;
                        }

                if(i==LieAlg_dim(B)+2)
                {return("ERROR");}

                HH=LieAlg_nonnilpotentelt(ll);
        }

        return(ll);

}
example
{ "EXAMPLE: NON-GORENSTEIN CURVE:"; echo=2;
  ring r=0,(X,Y,Z,W),ds;
  ideal I = X4-Y2+8X2Z-2YZ-Z2, 4X2Y+Y2-9X2Z+3YZ-XW, 6X2Y-3X2Z+2YZ-Z2-XW,
  X3Z+4/7XYZ-9/7XZ2+1/7X2W-2/7YW-2/7ZW, XYZ-5/3XZ2+1/9X2W-1/3YW-4/9ZW, Z3+13/21X3W+1/3XYW+XZW-5/21W2;
  module D=find_der(I);
  list B=der_matlist(D);
  B=LieAlg_Basis(B,nvars(r));
  list C=LieAlg_Cartan(B);
  print(C);
}
//////////////////////////////////////////////////////////////////////////////
//////////////////////////////////////////////////////////////////////////////


//////////////////////////////////////////////////////////////////////////////
////ALGORITHMS FOR LIE ALGEBRAS OF DERIVATIONS////////////////////////////////
//////////////////////////////////////////////////////////////////////////////


//Computing the module of logarithmic derivations
proc find_der(ideal I)

"USAGE:   find_der(I); I ideal.
RETURN:   The Module D of logarithmic derivations.
NOTE:     Does not work in qring or with mixed orderings.
THEORY:   [Epe15], Algorithm 5.
EXAMPLE:  example find_der; shows an example"
{
//Testing for the trivial case:
if(I==0)
{
        return(freemodule(nvars(basering)));
}


// Dummy variables and Initialization:
int k,i,n,m;

//generating matrix for syzygie computation:
n=nvars(basering);
m=size(I);
ideal j=jacob(I);


matrix M=matrix(j,m,n);
for (i=1;i<=m;i++)
{
        M=concat(M,diag(I[i],m));
}
module C=syz(M);
module D;

for(i=1;i<=size(C);i++)
{
        D=D+C[i][1..n];
}

```

```
//Clearing memory
kill j;
kill C;
kill M;
return(D);
}

example
{
"EXAMPLE:";
echo=2;
ring A=0,(x,y,z,w),ds;
poly f=x4w+y6+y5x+x5y;
find_der(f);

}
/////////////////////////////////////////////////////////////////////////////

//Getting matrices from the Module of derivations
proc der_matlist(module D)

"USAGE:   der_matlist (D); D module.
RETURN:   List l truncating the derivations generating D, leaving only degree 1 coefficients.
NOTE:     D has to be a module of derivations, like in the output of find_der.
EXAMPLE:  example der_matlist; shows an example"

{
// Dummy variables and Initialization:
int k,i,j,n,m;
n=nvars(basering);
D=jet(D,1);
D=compress(D);
list l;

for(i=1;i<=size(D);i++)
{
        matrix A[n][n];
        for(j=1; j<=n;j++)
        {
                poly f;
                f=D[i][j];
                for(k=1;k<=n;k++)
                {
                                        A[j,k]=diff(f,var(k));
                }
        }
        l=insert(l,A);
}
l=matsp_basis(l,n,n);
return(l);
}


example
{
"EXAMPLE:";
echo=2;
ring A=0,(x,y,z,w),ds;
ideal I=x4w+y6+y5x+x5y;
module D=find_der(I);
list P=der_matlist(D);
print(P);
}
/////////////////////////////////////////////////////////////////////////////
proc LieAlg_der_homog(ideal I)
"USAGE:   LieAlg_der_homog(I); I ideal.
RETURN:   Returns the Cartan Subalgebra of the Lie Algebra generated by the I
                  homogeneous derivations, which keep the maximal ideal invariant.
NOTE:     A local ordering like ds has to be used.
THEORY:   [Epe15], Algorithm 6.
EXAMPLE:  example LieAlg_der_homog; shows an example
"
{
  //Constructing the maximal ideal
  int i=1;
  ideal M;
  for(i;i<=nvars(basering);i++)
  {
                M=M+var(i);
  }

  //Computing the necessary modules of logarithmic derivations
  module D1=find_der(I);
  module D2=find_der(M);
  module D=intersect(D1,D2);
```

```

  //Truncating the module and computing the Cartan subalgebra
  list B=der_matlist(D);
  B=LieAlg_Basis(B,nvars(basering));
  list C=LieAlg_Cartan(B);
  return(C);
}
example
{ "EXAMPLE: CURVE WHICH IS NOT GORENSTEIN:"; echo=2;
  ring r=0,(X,Y,Z,W),ds;
  ideal I = X4-Y2+8X2Z-2YZ-Z2, 4X2Y+Y2-9X2Z+3YZ-XW, 6X2Y-3X2Z+2YZ-Z2-XW, X3Z+4/7XYZ-9/7XZ2+1/7X2W-2/7YW-2/7ZW, XYZ-5/3XZ2+1/9X2W-1/3YW-4/9ZW, Z3+13/21X3W
  list C=LieAlg_der_homog(I);
  print(C);
}
///////////////////////////////////////////////////////////////////////////////
//Computing the product of the ring variables
static proc var_prod()

"
RETURN: Product of all ring variables
EXAMPLE: example var_prod; shows an example"

{
int i;
poly f=1;
for(i=1;i<=nvars(basering);i++)
{
        f=f*var(i);
}
return(f);
}

example
{
"EXAMPLE:";
echo=2;
ring A=0,(x,y,z,w),ds;
var_prod();
}

///////////////////////////////////////////////////////////////////////////////
////LINEAR ALGEBRA ALGORITHMS///////////////////////////////////////////////////
///////////////////////////////////////////////////////////////////////////////

// Matrix to Vector

static proc mat2vec(matrix A)

"USAGE: Transforms a given matrix A into a vector v.
EXAMPLE: example mat2vec; shows an example"

{
vector v;
int i;int j;
int k=1;

for (i=1;i<=nrows(A);i++)
{
        for(j=1;j<=ncols(A);j++)
        {
                v=v+A[i,j]*gen(k);
                k++;
        }
}
return(v);
}

example
{
"EXAMPLE:";
echo=2;
ring r=0,(x),ds;
matrix A[3][3]=diag([1,2,3]);
vector v=mat2vec(A);
print(v);
}

// Matrix Vector Space Basis

static proc matsp_basis(list l, int n, int m)

"USAGE:   Given a list of n times m matrices, this procedure
                returns a vector space basis for the matrices.
THEORY:   Using Gaussian elimination, we compute a basis for
                our vector space.
```

```
                   See for example Algorithm 2.3.11 in [Coh00].
EXAMPLE: example matsp_basis; shows an example"

{
if (size(l)==0)
{
        matrix O[n][m];
        return(O);

}

int k=size(l);
int i; int j;
matrix B[n*m][k];
matrix C[n][m];
list p;

for (i=1;i<=k;i++)
{
        B=concat(B,mat2vec(l[i]));
}
B=gauss_col(B);
B=compress(B);
k=ncols(B);

for(j=1;j<=k;j++)
{
        C=B[j];
        p=insert(p,C);
}

return(p);
}

example
{
"EXAMPLE:";
echo=2;
ring r=0,(x),ds;
matrix A[3][3]=diag([1,2,3]);
matrix B[3][3]=1,1,0,0,1,1,0,0,1;
matrix C[3][3]=0,0,0,0,0,0,0,1,0;
list l=A,B,C;
print(matsp_basis(l,3,3));
}


//Testing nilpotency
static proc nilp_test(matrix A)

"USAGE:   Deciding whether a given matrix A is nilpotent or not.
RETURN:  1 if A is nilpotent, 0 else.
NOTE:    A has to be a square matrix.
THEORY:  If a matrix A is nilpotent its characteristic polynomial has
                   to be a power of a ring variable. The maximal degree of the
                   characteristic polynomial is the number of rows/columns of A, say n.
                   Using this, it suffices to test A^n, if it is the zero matrix.
EXAMPLE: example nilp_test; shows an example"

{
int n=nrows(A);
matrix O[n][n]; // Dummy  testing for 0 matrix
if (A==O)
{return(1);}

if(power(A,n)==O)
{ return(1);}
else
{ return(0);}
}

example
{
"EXAMPLE:";
echo=2;
ring r=0,(x),ds;
LIB "linalg.lib";
matrix A[3][3]=0,1,0,0,0,1,0,0,0;
nilp_test(A);
}


//Supplement of a Basis
static proc basis_supplement(matrix A)

```

```
"USAGE:   Computes the supplement of a subvectorspace V generated by the columns of the matrix A.
RETURN:   A matrix B, such that the columns of B are a basis for the supplement of V in the ambient vector space.
THEORY:   See Algorithm 2.3.6 in [Coh00].
EXAMPLE:  example basis_supplement; shows an example"

{
poly d; poly a;
int s;int t;int j;

matrix M=compress(gauss_col(A));
int n=nrows(A);
int k=ncols(M);


if(k==n)
{return(compress(0*unitmat(n)));}
if(compress(transpose(M))==0)
{return(unitmat(n));}
matrix B=unitmat(n);
for(s=1;s<=k;s++)
{
        t=s;
        while(M[t,s]==0)
        {
                t++;
        }
        d=1/M[t,s];
        B[1..n,t]=B[1..n,s];
        B[1..n,s]=M[1..n,s];
        for (j=s+1;j<=k;j++)
        {
                if(t!=s)
                {
                        a=M[s,j];
                        M[s,j]=M[t,j];
                        M[t,j]=a;
                        M[s,j]=d*M[s,j];
                        for(int i=1;i<=n;i++)
                        {
                                if(i!=t && i!=s)
                                {
                                        M[i,j]=M[i,j]-M[i,s]*M[s,j];
                                }
                        }
                }
        }
}


return(submat(B,1..n,k+1..n));
}
example
{
"EXAMPLE:";
echo=2;
ring r=0,(x),ds;
LIB "linalg.lib";
matrix A[3][3]=0,1,1,0,1,1,1,0,0;
print("Basis:");
print(A);
print("Basis Supplement:");
print(basis_supplement(A));
}

//Supplement a subsapce in another
static proc sub_supplement(matrix A, matrix B)

"USAGE:   Computes the supplement of a subvectorspace F in another subvectorspace E, which contains F.
                 They are generated by the columns of the matrix A resp. B.
RETURN:   A matrix M, such that the columns of M are a basis for the supplement of F in E.
NOTE:     The vector space generated by A has to be contained in the one generated by B.
THEORY:   See Algorithm 2.3.7 in [Coh00].
EXAMPLE:  example sub_supplement; shows an example"

{
matrix N=compress(gauss_col(A));
matrix M=compress(gauss_col(B));
matrix C=concat(M,N);
matrix X=syz(C);
X=submat(X,1..ncols(M),1..ncols(X));
matrix D=basis_supplement(X);
return(M*D);
}
example
{
```

```
"EXAMPLE:";
echo=2;
ring r=0,(x),ds;
LIB "linalg.lib";
matrix A[3][2]=gen(1),gen(1);
matrix B[3][3]=gen(1)+gen(2),gen(2),gen(2);
print(sub_supplement(A,B));
}

// Intersection of Subspaces
static proc sub_intersect(matrix A, matrix B)

"USAGE:   sub_intersect(A,B); A matrix, B matrix.
RETURN:  A matrix C, whose rows are a basis for the intersection of U and V.
THEORY:  See Algorithm 2.3.9 in [Coh00].
EXAMPLE: example sub_intersect; shows an example"

{
int i;
matrix M=concat(A,B);
matrix N=syz(M);
N=submat(N,1..ncols(A),1..ncols(N));
matrix C=A*N;
C=gauss_col(C);
return(C);
}

example
{
"EXAMPLE:";
echo=2;
ring r=0,(x),ds;
LIB "linalg.lib";
matrix A[3][2]=1,2,3,5,4,1;
matrix B[3][2]=1,2,1,2,2,1;
matrix C=sub_intersect(A,B);
print(C);
}
```

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