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A NONLINEAR RAY THEORY

Phoolan Prasad*

*Department of Mathematics
Indian Institute of Science

Bangalore - 560 012, India

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Postfach 3049

D-67653 Kaiserslautern

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Phoolan Prasad
Department of Mathematics
Indian Institute of Science
Bangalore - 560 012

Abstract

A proof of the famous Huygens' method of wavefront construction is reviewed and it is shown that the method is embedded in the geometrical optics theory for the calculation of the intensity of the wave based on high frequency approximation. It is then shown that Huygens' method can be extended in a natural way to the construction of a weakly nonlinear wavefront. This is an elegant nonlinear ray theory based on an approximation published by the author in 1975 which was inspired by the work of Gubkin. In this theory, the wave amplitude correction is incorporated in the eikonal equation itself and this leads to a system of ray equations coupled to the transport equation. The theory shows that the nonlinear rays stretch due to the wave amplitude, as in the work of Choquet-Bruhat (1969), followed by Hunter, Majda, Keller and Rosales, but in addition the wavefront rotates due to a non-uniform distribution of the amplitude on the wavefront. Thus the amplitude of the wave modifies the rays and the wavefront geometry, which in turn affects the growth and decay of the amplitude. Our theory also shows that a compression nonlinear wavefront may develop a kink but an expansion one always remains smooth. In the end, an exact solution showing the resolution of a linear caustic due to nonlinearity has been presented. The theory incorporates all features of Whitham's "geometrical shock dynamics".

1 Huygens' method of construction of a linear wavefront

This paper is a tribute to the great mathematician Christiaan Huygens (1629-1695), who proposed in 1690 that all points of a wavefront of light may be regarded as new sources of wavelets that expand in every direction at a rate depending on their velocities. At a later time, the envelope of the wavefronts emitted by these sources constitute the new wavefront. This rule of Huygens for the construction of the successive positions of a wavefront implies that the wavefront is self propagating in the sense that the geometry and the position of the wavefront at any later time depend only on its initial geometry and position (and of course, on the property of the medium) and are not influenced by the wavefronts which follow or precede it. Thus a wavefront, which may be embedded in a one parameter family of wavefronts in an infinity of ways, has its own predetermined course of motion.

Huygens discovered the method intuitively, but his method is capable of being given an exact mathematical proof, which we present briefly for the wave equation in a homogeneous medium with constant wave speed a_0:

$$u_{tt} - a_0^2 \nabla^2 u = 0. \tag{1.1}$$

If we represent a characteristic surface of (1.1) in the form $t = \psi(x)$, then the function ψ satisfies the reduced characteristic partial differential equation

$$a_0^2 \left(\psi_{x_1}^2 + \psi_{x_2}^2 + \psi_{x_3}^2 \right) = 1. \tag{1.2}$$

Let us consider for (1.1) the spherical wavefronts $t = S(x, x^*)$ originating from a point x^*, then

$$S = \frac{1}{a_0} |x - x^*| \tag{1.3}$$

containing three parameters $x^* = (x_1^*, x_2^*, x_3^*)$, is a complete integral of the equation (1.2). Let $x = x^*(\eta_1, \eta_2)$ be a parametric representation of the position of a wavefront at $t = 0$, then the method of solution of a Cauchy problem, using the complete integral (1.3), shows that the envelope of the two parameter family of the spherical wavefronts $S = S(x, x^*(\eta_1, \eta_2))$ gives a solution $\psi(x)$ of (1.2) such that $t = \psi(x)$ is the equation of the wavefront at time t. This is nothing but Huygens' method.

It is a remarkable mathematical fact that Huygens stated a method of construction of a wavefront, which is true not only for light waves but also for waves governed by an arbitrary hyperbolic system of linear partial differential equations in any number of independent variables. Hyperbolicity of the system ensures that there exist "spherical wavefronts" $t = S(x, x^*)$. The method of construction of the wavefront at any time t with the help of this complete integral S is exactly the same as that outlined for (1.1) even in this most general case [1].

A method of construction of the wavefront, equivalent to that contained in the original statement of Huygens, makes use of rays, which for light waves or sound waves in a uniform medium at rest are straight lines normal to the wavefronts. Starting from an initial position of the wavefront, the wavefront at any time t is obtained as the locus of the end points of the normals to the initial wavefront, the length the normals being equal to the distance travelled by the wave in time t. It is simple to deduce this method of wavefront construction by using the bicharacteristic equations [2] of (1.1):

$$\frac{dx}{dt} = a_0 n, \ \frac{dn}{dt} = 0 \tag{1.4}$$

where n is the unit normal to the wavefront. Equation (1.4) says that the position x on the wavefront at time t is obtained by

$$x = x^* + a_0 n t \tag{1.5}$$

where x^* is a point on the wavefront at $t = 0$.
In this paper, we shall show that Huygens' method can be extended to weakly nonlinear waves and these wavefronts are also self propagating. In the end we shall present one interesting solution of the weakly nonlinear ray equations.

2 Linear ray theory in high frequency approximation

Huygens' method of wavefront construction tells only about the location of the wavefront and not about the intensity of the wave on the wavefront. The latter problem cannot be solved exactly and hence its approximate solution, in various limiting cases, has been a subject of study by many. Huygens' method of wavefront construction is embedded in a mathematical theory of finding an approximate value of the intensity of the wave based on the assumption that either a solution in the neighbourhood of a leading wavefront is nearly discontinuous and hence, is dominated by high frequency waves or the solution is a periodic function of a rapidly changing phase function.
We first define a ray starting from a given point x^* of an initial position of a wavefront to be the spatial projection of a bicharacteristic curve (in space-time) starting from ($x = x^*$, $t = 0$). Next we define a function A, called ray tube area, along a ray as the limit (as the maximum diameter of the ray tube tends to zero) of the ratio of the cross-sectional area at any location along the ray tube to the area at a standard reference cross section.
Let us consider now the wave equation (1.1). Here, the rays given by (1.5) are orthogonal to the successive positions of a wavefront. The ray tube area is related to the mean curvature Ω of the wavefront by

$$\text{mean curvature } = \Omega = -\frac{1}{2}\text{div }(n) = -\frac{1}{2Aa_0}\frac{dA}{dt} \tag{2.1}$$

where $\frac{d}{dt}$ is the time rate of change along the ray moving with the wavefront. Let us write the equation (1.1) in terms of new variables (x'_α, ϕ) instead of (x_α, t), where

$$x'_\alpha = x_\alpha,\ i = 1,2,3 \text{ and } \phi = t - \psi(x). \tag{2.2}$$

The equation (1.1) becomes [2]

$$2\frac{d}{dt}\left(\frac{\partial u}{\partial \phi}\right) + \frac{1}{A}\frac{dA}{dt}\frac{\partial u}{\partial \phi} + \frac{a_0}{|\nabla_x \phi|}\sum_{\alpha=1}^{3}\frac{\partial^2 u}{\partial x'^2_\alpha} = 0 \tag{2.3}$$

where we have used the result $\text{div}(n) = -(\phi_{tt} - a_0^2\nabla^2\phi)\,/\,\{a_0^2|\nabla\phi|\}$. For a solution u of (1.1) for which the first order derivatives are continuous and the second order derivatives are discontinuous across $\phi = 0$, we get the transport equation

$$\frac{dw}{dt} + \frac{1}{2A}\frac{dA}{dt}w = 0, \quad w = \left[\frac{\partial^2 u}{\partial \phi^2}\right]_{\phi=0}. \tag{2.4}$$

Thus, the amplitude w of the discontinuity in the second derivative satisfies

$$w = w_0/A^{1/2}. \tag{2.5}$$

This is an exact ressult, the second derivatives (and also the higher derivatives [1]) tend to become infinite at a focus or a caustic where $A \to 0$.

Linear ray theory in the well known high frequency approximation (geometrical optics [3]) gives a value of the amplitude u of the wave whose leading term is also given in terms of the ray tube area by the same law as (2.5). This is only an approximate result; the amplitude of the wave when evaluated more accurately at points where $A \to 0$, does remain finite [4].

3 Nonlinear ray theory (NLRT)

A NLRT or kinematic wave theory for any system requires derivation of two equations, the first one being a dispersion relation in the form

$$\omega = \Omega(k, x, t, w(x, t)) \tag{3.1}$$

where ω is the frequency, k the wave number and w the amplitude of the wave. The second equation is a transport equation for the amplitude along nonlinear rays defined below. In terms of the phase function ϕ, the frequency and the wave number are given by

$$\omega = -\phi_t \text{ and } k = \nabla\phi. \tag{3.2}$$

The dispersion relation (3.1) is then equivalent to a partial differential equation

$$Q\left(\nabla\phi, \phi_t, x, t, w(x,t)\right) = 0 \tag{3.3}$$

for the function ϕ. The *nonlinear rays* are the curves $x = x(t)$ obtained from the solution of the characteristic equations or Hamilton-Jacobi equations of (3.3). We define a wave to be *hyperbolic* if the equation (3.3) is a homogeneous function of ϕ_t and the components of $\nabla\phi$. Thus, for a hyperbolic wave, the ray equations can be expressed as equations for x and the unit normal $n = \nabla\phi/|\nabla\phi|$. These equations, together with the transport equation for the amplitude along the nonlinear rays, form the basic equations of a NLRT.

We first review the derivation of the equations of NLRT for a hyperbolic system of quasilinear equations

$$Au_t + B^{(\alpha)}u_{x_\alpha} + C = 0 \tag{3.4}$$

where a repeated suffix represent sum, $u \in \mathbb{R}^n$, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^n$, and $A = A(u,x,t)$ etc. Let c be a simple root of the characteristic equation, $\varphi(x,t) =$ constant be the corresponding one parameter family of characteristic surfaces and let ℓ and r be the generalized left and right eigenvectors satisfying

$$\ell M = 0,\ Mr = 0 \text{ with } M = A\varphi_t + B^{(\alpha)}\phi_{x_\alpha} \tag{3.5}$$

Sections of the characteristic surface by $t =$ constant planes give successive positions of a wavefront. Unit normal n to the wavefront and c are given by

$$n = \nabla\varphi/|\nabla\varphi|, \quad c = -\varphi_t/|\nabla\varphi|. \tag{3.6}$$

For the waves corresponding to the simple characteristic velocity c, the eikonal equation (3.3) can be taken to be

$$Q \equiv (\ell Ar)\varphi_t + (\ell B^{(\alpha)}r)\varphi_{x_\alpha} = 0. \tag{3.7}$$

It is easy to prove the following theorem [5], see appendix.

Theorem

Along the rays of (3.4), given by

$$\frac{dx_\alpha}{dt} = \frac{\ell B^{(\alpha)}r}{\ell Ar} \equiv \chi_\alpha, \tag{3.8}$$

$$\frac{dn_\alpha}{dt} = -\frac{1}{(\ell Ar)}n_\beta \ell \left(-c\frac{\partial A}{\partial \eta_\beta^\alpha} + n_\gamma \frac{\partial B^{(\gamma)}}{\partial \eta_\beta^\alpha}\right) r \equiv \psi_\alpha, \tag{3.9}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi^{(\alpha)}\frac{\partial}{\partial x_\alpha}, \quad \frac{\partial}{\partial \eta_\beta^\alpha} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta} \tag{3.10}$$

the function u satisfies the compatibility condition

$$\ell A\frac{du}{dt} + \ell\left(B^{(\alpha)} - \chi_\alpha A\right)\frac{\partial u}{\partial x_\alpha} + \ell C = 0. \tag{3.11}$$

Note. In the second term in (3.11), the differential operator $\tilde{\partial}_j$ on u_j is given by

$$\tilde{\partial}_j = \ell_i\left(B_{ij}^{(\alpha)} - \chi_\alpha A_{ij}\right)\frac{\partial}{\partial x_\alpha} \equiv s_j^\alpha \frac{\partial}{\partial x_\alpha}, \tag{3.12}$$

where repeated suffix i implies sum from 1 to n and α implies from 1 to m. It can be easily verified that $n_\alpha s_j^\alpha = 0$ so that $\tilde{\partial}_j$ is a derivative in a direction tangential to the wavefront. In (3.9) there appear other m derivatives

$$L_\alpha = n_\beta \frac{\partial}{\partial \eta_\beta^\alpha} \tag{3.13}$$

which are components of the vector operator

$$L = \nabla - n < n, \nabla > . \tag{3.14}$$

Since $< n, L >= 0$, these are also tangential derivatives to the wavefront and hence only $m-1$ of L_α are linearly independent. We can always express $\tilde{\partial}_j$ as a linear combination of any $m-1$ components of the operator L.

As $|n| = 1$, only $m-1$ of the m equations in (3.9) are independent. Hence, relations (3.8), (3.9) and (3.11) along a ray represent $2m$ equations in $2m+n-1$ components of x, n, u. Hence, not much information from these can be obtained unless $n = 1$, a case which can be easily studied. Choquet-Bruhat [6] developed a weakly nonlinear ray theory by using high frequency approximation and a formal expansion of the perturbation $v = u - u_0$ on a basic solution $u_0(x,t)$ in terms of a small quantity ϵ which is inversely proportional to the frequency (see also [7] - [8]). A great disadvantage of this procedure is that we end up with an approximation valid in the neighbourhood of the linearized characteristic surface. Since the eikonal equation decouples from the amplitude equation, an attempt to improve the solution does not improve the position of the characteristic surfaces but only produces a stretching of the rays in the direction of the linearized rays. This is certainly not correct since the rays rotate not only by the inhomogeneity ahead but also by the nonuniform distribution of the amplitude of the perturbation along the wavefront. This rotation is important not only in the caustic region but also quite often at large distances. One may attempt to improve the solution by Lighthill's method [9] of stretching the coordinates not only in a direction along the rays but also along certain curves on the moving wavefront. Obermeier [10] used this method to get a valuable solution but even here only the coordinate in the direction of the rays has been stretched.

An entirely new approach to this problem is in approximating uniformly all the three equations (3.8), (3.9) and (3.11) for the perturbation in the neighbourhood of an *exact* nonlinear characteristic surface. In the short wave or high frequency approximation, it is possible to express the n components of the perturbation $v = u - u_0$ in terms of the unit normal of the *exact* nonlinear wavefront and an amplitude variable w [11]. Using this in (3.11), we can get a transport equation for w along the exact rays, which deviate significantly from the linear rays calculated in the base state u_0. Such a transport equation was first derived for gasdynamic equations by Gubkin [12] as early as in 1958 and for a general hyperbolic system by Prasad [5] in 1975. However, both Gubkin and Prasad missed the real strength of their transport equation since they did not use (3.8) and (3.9) to calculate the deviation of the exact nonlinear rays from the linear one (see equations (4.6) to (4.9) and in [14]). It was only a few years later that Prasad and his collaborators realized this [15] - [17].

We briefly explain a derivation equivalent to that of Prasad [13] in the next paragraph. Before we do this, let us comment that this approximation is valid not only for short waves [13] but also for high frequency periodic waves. A periodic wave may have a mean field v_m [7]. But we emphasize that the mean field part v_m of the perturbation can also not be calculated by solving a linear equation as done in the theory presented in [7]. The reason is the same i.e. there may be a significant variation of v_m along the

wavefront leading to a large rotation of the rays due to this mean field. Therefore, we assume here that $v_m = 0$ whenever our perturbation is periodic.
The perturbation $v = u - u_0$ satisfies

$$Av_t + B^{(\alpha)}v_{x_\alpha} + F = 0 \tag{3.15}$$

where

$$F = (A - A_0)u_{0t} + (B^{(\alpha)} - B_0^{(\alpha)})u_{0x_\alpha} + C - C_0, \tag{3.16}$$

and a subscript 0 on any quantity represents its value in the base state i.e. $A_0 = A(x, t, u_0)$. All terms in (3.16) are of the same order ϵ.
We now express high frequency approximation by assuming that v depends on x, t and a fast variable $\theta = \frac{\varphi}{\epsilon}$. Substituting $v = v(x, t, \theta)$ in (3.15) we get

$$\frac{1}{\epsilon}\left(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}\right)v_\theta + \left[Av_t + B^{(\alpha)}v_{x_\alpha} + F\right] = 0. \tag{3.17}$$

Unlike the theory of Choquet-Bruhat, we do not expand (3.17) in powers of ϵ but use it only to calculate the approximate form of the solution, which we shall use to get the approximate transport equation along the *exact* rays. This approach is not more formal than that of Choquet-Bruhat and her followers. We believe that instead of proving the convergence of an expansion to the exact solution (which also has not been achieved for $m > 1$ so far), it is better to derive an equation for the error and study it. Equating the most dominant term on the left hand side of (3.17) to zero, we get

$$\left(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}\right)v_\theta = 0 \tag{3.18}$$

which gives $v_\theta = \tilde{w}(x, t, \theta)r(x, t, u, n) = 0(\epsilon)$, where r depends on θ through $u = u_0 + v$ but it varies slowly with θ since $v = 0(\epsilon)$. Therefore, while integrating with respect to θ, if we treat r constant, we commit an error of order ϵ^2, which is consistent with the derivation of (3.18). Thus,

$$v = w(x, t, \theta)r, \quad w = \int_0^\theta \tilde{w}(x, t, \theta)d\theta. \tag{3.19}$$

Now we go back to the original equation (3.15) and write it in the form

$$\left(A\varphi_t + B^{(\alpha)}\varphi_{x_\alpha}\right)v_\varphi + B^{(\alpha)}\frac{\partial v}{\partial x'_\alpha} + F = 0 \tag{3.20}$$

where

$$\frac{\partial}{\partial x'_\alpha} = \frac{\partial}{\partial x_\alpha} + \frac{n_\alpha}{c}\frac{\partial}{\partial t} \tag{3.21}$$

represents a derivative in a direction tangential to the wavefront. Multiplying (3.20) by ℓ, using (3.5) and substituting (3.19), we get

$$\ell A r \frac{dw}{dt} + \left(\ell B^{(\alpha)} \frac{\partial r}{\partial x'_\alpha}\right) w + \ell F = 0 \tag{3.22}$$

where we have used

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi^{(\alpha)} \frac{\partial}{\partial x_\alpha} = \chi_\alpha \frac{\partial}{\partial x'_\alpha}.$$

The equation (3.22) is correct up to order ϵ. It contains derivative $\frac{d}{dt}$ along the *exact* unperturbed ray and tangential derivatives $\frac{\partial}{\partial x'_\alpha}$ along the *exact* characteristic surface. r also contains n, the unit normal of the *exact* wavefront. We approximate the quantities in (3.22) keeping the operators $\frac{d}{dt}$, $\frac{\partial}{\partial x'_\alpha}$ and n intact. This gives the transport equation along the *exact* ray for the amplitude w correct up to order ϵ

$$(\ell_0 A r_0) \frac{dw}{dt} + \left(\ell_0 B_0^{(\alpha)} \frac{\partial r_0}{\partial x'_\alpha}\right) w + \{\ell_0 (\nabla_u F)_0 r_0\} w = 0 \tag{3.23}$$

where ℓ_0 and r_0 depend on the unit normal n of the *exact* wavefront. The full set of equations of our NLRT is obtained by substituting (3.19) (or (3.25) below) on the right hand side of (3.8) and (3.9) and retaining terms up to first degree in w and its derivatives $L_\alpha w$ i.e. terms of order ϵ. This means that the operator $\frac{d}{dt}$ in (3.10) is approximated by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left[\chi_0^{(\alpha)} + \left\{\left(\nabla_u \chi^{(\alpha)}\right)_0 r_0\right\} w\right] \frac{\partial}{\partial x_\alpha}. \tag{3.24}$$

An expression equivalent to (3.19) up to terms of order ϵ is

$$v = w r_0. \tag{3.25}$$

This result can be used to give a quick derivation of the equations of the NLRT from the equations (3.8), (3.9) and (3.11). First we must express the operator $\tilde{\partial}_j$ defined by (3.12) in terms of the operators L_α defined by (3.13). Then we substitute (3.25) in these equations and retain terms up to order ϵ leaving the opertors $\frac{d}{dt}$ and L_α intact. We shall use this simple method for gasdynamic equations below.

In the rest of this paper, we take the propagation of waves in a polytropic gas without a dissipative mechanism. The governing equations are hyperbolic and the equation (3.3), for a forward facing wave, is the characteristic partial differential equation

$$Q \equiv \phi_t + < q, \nabla\phi > + a|\nabla\phi| = 0 \tag{3.26}$$

where q is the particle velocity and the sound speed a is given in terms of the pressure p and density ρ as $a = (\gamma p/\rho)^{1/2}$. Here γ is a constant appearing in the polytropic relation p/ρ^γ = a function of the specific entropy.

The bicharacteristic equations or the ray equations are

$$\frac{dx}{dt} = q + na \tag{3.27}$$

$$\frac{dn}{dt} = -La - \sum_{\beta=1}^{3} n_\beta L q_\beta. \tag{3.28}$$

For a high frequency nonlinear wave of small amplitude running into a medium at rest ($q = 0$, $p = p_0$, $\rho = \rho_0$) we get from (3.25)

$$q = nw, \quad p - p_0 = \rho_0 a_0 w, \quad \rho - \rho_0 = (\rho_0/a_0)w. \tag{3.29}$$

From the equations of motion of a polytropic gas, we can derive a compatibility condition on the characteristic surface satisfying (3.26) in the form

$$a\frac{d\rho}{dt} + \rho < n, \frac{dq}{dt} > + \rho a < L, q >= 0 \tag{3.30}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + < q + an, \nabla > \tag{3.31}$$

is the time rate of change along a bicharacteristic curve given by (3.27) and (3.28). Substituting (3.29) in (3.27), (3.28), (3.30) and (3.31) and retaining terms only up to order ϵ, we get the equations of the NLRT:

$$\frac{dx}{dt} = \left(a_0 + \frac{\gamma+1}{2}w\right) n, \tag{3.32}$$

$$\frac{dn}{dt} = -\frac{\gamma+1}{2} Lw \tag{3.33}$$

and

$$\frac{dw}{dt} = \Omega a_0 w = -\frac{1}{2A}\frac{dA}{dt}w, \text{ from (2.1)}, \tag{3.34}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(a_0 + \frac{\gamma+1}{2}w\right) < n, \nabla > \tag{3.35}$$

and

$$\Omega = -\frac{1}{2} < \nabla, n >= -\frac{1}{2}\left(\frac{\partial n_1}{\partial \eta_3^1} + \frac{\partial n_2}{\partial \eta_3^2}\right). \tag{3.36}$$

We note that Ω is the mean curvature of the wavefront F_t.
Consider a plane wavefront perpendicular to the x_1-axis. Then $n = (1, 0, 0)$, $\Omega = 0$ and the system (3.32)-(3.34) is equivalent to the single equation

$$\frac{\partial w}{\partial t} + \left(a_0 + \frac{\gamma+1}{2}w\right)\frac{\partial w}{\partial x} = 0. \tag{3.37}$$

With a proper scaling of dependent variables, this equation can be reduced to the famous Burgers' equation $u_t + uu_\xi = 0$ in a moving frame $\xi = x - a_0 t$. Thus the system of equations (3.32)-(3.34) is an extension of the Burgers' equation to multi-dimensions for the propagation of a weakly nonlinear wave.
Since $|n| = 1$, only two of the three equations (3.33) are independent. Therefore, the equations (3.32)-(3.34) form a system of 6 coupled equations for the determination of successive positions x of a nonlinear wavefront, the unit normal n and the wavefront intensity w. In the linear theory, w drops out of the (3.32) and (3.33) so that the ray equations decouple from the amplitude equation (3.34). In this case the rays and the successive positions of the wavefront can be constructed without any reference to the amplitude of the wave. This corresponds to the statement of Huygens' wavefront construction. In our weakly nonlinear theory, the amplitude is related to the curvature of the wavefront (or the ray tube area) by the same equation (compare equations (2.4) and (3.34)) but the nonlinear rays stretch due to the presence of w in (3.32) and the wavefront rotates due to a non-uniform distribution of the amplitude on the wavefront (represented by Lw in (3.33)). Thus the amplitude of the wave modifies the rays and the wavefront geometry which in turn affects the growth and decay of the amplitude. Further, we note that only the tangential derivatives, on a wavefront F_t at a time t, of w and n_α appear on the right hand side of the equations of NLRT. Therefore, given the initial position F_0 of the wavefront and the distribution of the amplitude on it, all quantities on the right hand side of the equations (3.32) to (3.34) can be completely determined at $t = 0$ as in the case of a non-characteristic Cauchy problem. Hence, the evolution of the wavefront and the distribution of the amplitude on it at later times can be determined from these equations. This implies that, in the short wave approximation, the nonlinear wavefront is self-progapating. The result is true not only for a compressible medium but for any continuum medium governed by the hyperbolic system (3.4). Huygens' method of wavefront construction has now been very elegantly extended to the construction of a nonlinear wavefront in the short wave limit - in this extension the amplitude also affects the position of the wavefront. Finally we mention that this theory can be easily extended to waves of arbitrary amplitude in short wave limit [11]. However, in this case the equations cannot be put in the elegant form of (3.32)-(3.34).

4 Nonlinear ray equations in two dimensions

In two space dimensions, the components n_1, n_2 of the unit normal can be expressed in terms of θ, the angle which the normal to the wavefront makes with the x_1-axis: $n_1 = \cos\theta$, $n_2 = \sin\theta$. The equations (3.32) to (3.34) reduce to

$$\frac{dx_1}{dt} = \left(a_0 + \frac{\gamma+1}{2}w\right)\cos\theta \tag{4.1}$$

$$\frac{dx_2}{dt} = \left(a_0 + \frac{\gamma+1}{2}w\right)\sin\theta \tag{4.2}$$

$$\frac{d\theta}{dt} = -\frac{\gamma+1}{2}\frac{\partial w}{\partial \lambda} \tag{4.3}$$

and

$$\frac{dw}{dt} = -\frac{1}{2}a_0 w \frac{\partial \theta}{\partial \lambda} \tag{4.4}$$

where

$$\frac{\partial}{\partial \lambda} = -\frac{1}{\sin\theta}L_1 \equiv \cos\theta\frac{\partial}{\partial x_2} - \sin\theta\frac{\partial}{\partial x_1}. \tag{4.5}$$

Setting $s = a_0 t$ and noting that the rate of change $\frac{1}{a_0}\frac{d}{dt}$ along a ray is actually a partial derivative in the characteristic surface ϕ =constant, we rewrite (4.3) and (4.4) in the form

$$\frac{\partial \theta}{\partial s} + \frac{\gamma+1}{2}\frac{\partial \bar{w}}{\partial \lambda} = 0, \qquad \frac{\partial \bar{w}}{\partial s} + \frac{1}{2}\bar{w}\frac{\partial \theta}{\partial \lambda} = 0 \tag{4.6}$$

where $\bar{w} = w/a_0$ is the nondimensional amplitude. (4.6) is a pair of equations for $(\theta, \bar{w})$ on a characteristic surface F. s is a well defined variable but λ is not, $\frac{\partial}{\partial\lambda}$ is only a symbol for an operator defined by (4.5). λ can be defined only locally. However, following Prasad, Ravindran, Sangeeta and Morton [18], it is possible to define a variable ξ such that $gd\xi$ is an element of length along the wavefront and $\frac{\partial}{\partial\lambda} = \frac{1}{g}\frac{\partial}{\partial\xi}$ (see also [19]). The metric g is given in terms of the Mach number $M = 1 + \frac{\gamma+1}{2}w$ of the wave by $g = (M-1)^{-2}e^{-2(M-1)}$. Since g never tends to zero, even in the caustic region or across a discontinuity of M and θ on the wavefront, it follows that the ray tube area A has a minimum positive value > 0 except in isolated cases of geometrically symmetric foci. Hence from (3.34) it follows that w always remains finite in this theory, which is only the first nonlinear approximation in the high frequency limit. This result is of great importance since the first linear approximation in high frequency limit gives infinite amplitude at a caustic.

When the nonlinear wave is a compression wave, $\bar{w} > 0$. In this case the pair of equations (4.6) form a hyperbolic system with real characteristic roots

$$c_1, c_2 = \pm\sqrt{\frac{\gamma+1}{4}\bar{w}}. \tag{4.7}$$

Thus, changes in θ and w propagate on the nonlinear wavefront with finite velocities c_1, c_2. This is analogous to the nonlinear waves in one-dimensional motion of the original gas. When the nonlinear wave is an expansion wave $\bar{w} < 0$, the system (4.6) is elliptic. Therefore, there is no possibility of the appearance of a discontinuity in θ and $\bar{w}$ or a kink on the wavefront. The functions $\bar{w}$ and θ along the wavefront will be connected infinitely smoothly. Thus, nonlinear compression and expansion wavefronts behave in entirely different ways to small disturbances introduced on it at any time. The fact that the system (4.6) is elliptic for $\bar{w} < 0$ no way contradicts the initial assumption that we are considering the propagation of a wavefront. The propagation of the wavefront

is represented by the operator $\frac{d}{dt}$ defined by (3.35). In addition, we get genuinely nonlinear waves moving on the wavefront when $\bar{w} > 0$. For an expansion wave ($\bar{w} < 0$), the original wavefront still gets modified by the distribution of the amplitude on it but it always remains smooth.
For a compression wave, ($\bar{w} > 0$), the system (4.6) has two Riemann invariants

$$R = \theta + 2[(\gamma+1)\bar{w}]^{1/2}, \qquad S = \theta - 2[(\gamma+1)\bar{w}]^{1/2}. \tag{4.8}$$

For a simple wave, in which one Riemann invariant, say S, remains constant on the nonlinear wavefront, the amplitude $\bar{w}$ (and also θ) satisfy the partial differential equation (for details see [19])

$$\begin{aligned} &\tfrac{\partial \bar{w}}{\partial t'} + \left\{\left(1+\tfrac{\gamma+1}{2}\bar{w}\right)\cos\theta - \left(\tfrac{\gamma+1}{4}\bar{w}\right)^{1/2}\sin\theta\right\}\tfrac{\partial \bar{w}}{\partial \bar{x}} \\ &+\left\{\left(1+\tfrac{\gamma+1}{2}\bar{w}\right)\sin\theta + \left(\tfrac{\gamma+1}{4}\bar{w}\right)^{1/2}\cos\theta\right\}\tfrac{\partial \bar{w}}{\partial \bar{y}} = 0 \end{aligned} \tag{4.9}$$

where non-dimensional variables $t', \bar{x}, \bar{y}$ are suitably defined. We call a characteristic curve of this simple wave in $(\bar{x}, \bar{y}, t)$ space, a "characteristic curve on the wavefront" or briefly a CCWF. $\bar{w}$ and θ remain constant on a CCWF, say equal to $\bar{w}_0$ and θ_0 at a point on the initial wavefront. The simple wave solution of an initial value problem

$$\left.\begin{array}{c} \text{initial wave front } \bar{x} = \bar{x}_0(\eta),\ \bar{y} = \bar{y}_0(\eta) \\ \text{with } \bar{w}(t'=0) = \bar{w}_0(\eta),\ \theta(t'=0) = \theta_0(\eta) \\ \text{satisfying } \bar{w}_0 = \frac{1}{4(\gamma+1)}(\theta_0 - S_0)^2 \end{array}\right\} \tag{4.10}$$

where S_0 is independent of η, is given by

$$\bar{w}(\bar{x},\bar{y}) = \bar{w}_0\left(\bar{x}_0(\eta), \bar{y}_0(\eta)\right), \quad \theta(\bar{x},\bar{y}) = \theta\left(\bar{x}_0(\eta), \bar{y}_0(\eta)\right) \tag{4.11}$$

along CCWF

$$\bar{x} = \bar{x}_0 + \left\{\left(1+\frac{\gamma+1}{2}\bar{w}_0\right)\cos\theta_0 - \left(\frac{\gamma+1}{4}\bar{w}_0\right)^{1/2}\sin\theta_0\right\}t', \tag{4.12}$$

$$\bar{y} = \bar{y}_0 + \left\{\left(1+\frac{\gamma+1}{2}\bar{w}_0\right)\sin\theta_0 + \left(\frac{\gamma+1}{4}\bar{w}_0\right)^{1/2}\cos\theta_0\right\}t'. \tag{4.13}$$

At any time t', equations (4.12) and (4.13) give a point $(\bar{x}, \bar{y})$ on the wavefront in terms of the parameter η. The values of $\bar{w}$ and θ at this point is given by (4.11). This gives the complete history of a nonlinear wavefront. Given a curved pulse satisfying short wave approximation, we can cover it by a one-parameter family of wavefronts. The history of each of these wavefronts can be studied by the procedure discussed here. Thus, we can study the nonlinear evolution of the pulse as long as a shock does not appear in it. Equations (4.11)-(4.13) give a rare solution for such a complex nonlinear problem: an exact solution of the approximate equations of NLRT. Similarly, simple wave solution can be obtained when the Riemann invariant R is constant. We can also construct a composite simple wave solution in which the values $\bar{w}_0$, $\bar{\theta}_0$ on the initial wavefront is so prescribed that it gives rise to straight line CCWF with S=constant from one part of the initial position of the wavefront and R =constant from another part. One such solution is shown in the figure below.

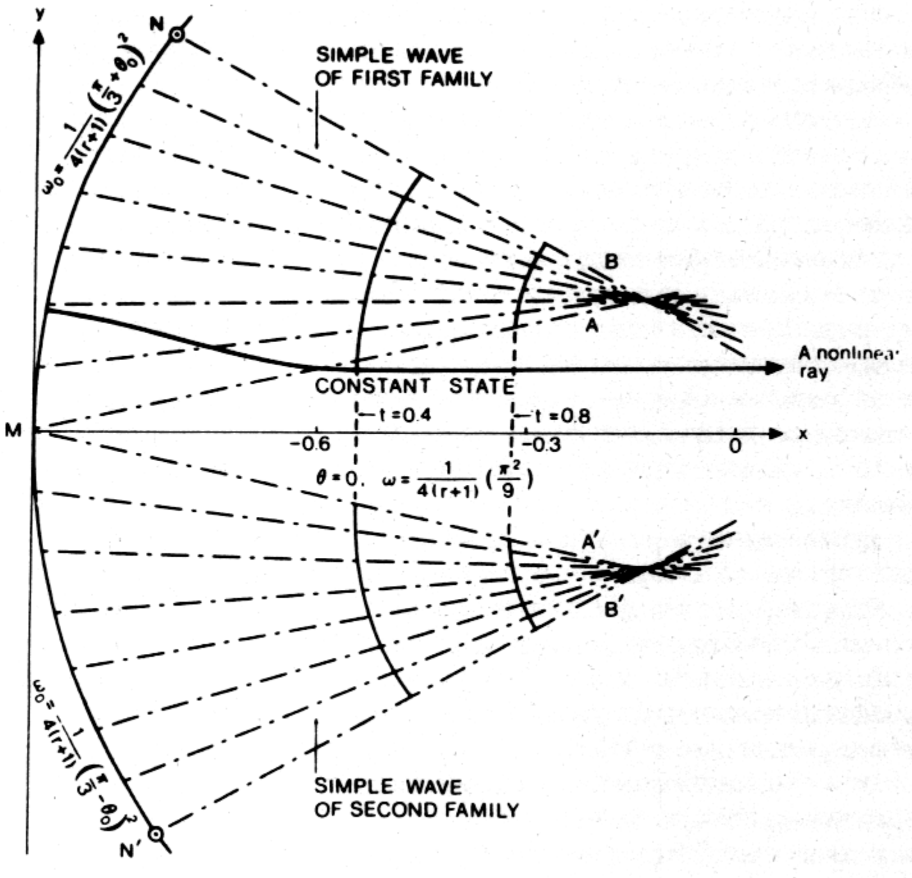

Fig. 6.1 Composite simple wave solution of the nonlinear ray equations with initial data $w_0 = \frac{1}{4(\gamma+1)}(\frac{\pi}{3} - |\theta_0|)^2$ prescribed on a circular arc. Simple wave characteristics are shown as $-\cdot-\cdot-\cdot$.

Starting from a smooth initial geometry of the wavefront and a smooth distribution of $\bar{w}$ on it, it can be shown that initial values with $\frac{\partial w}{\partial \lambda} < 0$ lead to a break down of the solution at a finite time t_c since $\frac{\partial w}{\partial \lambda}$ tends to infinity as $t \to t_c - 0$. This conclusion was originally derived by using quite complex analysis [15] but can be given a very simple derivation using ray coordinate system [19]. It also appears that only singularities, which appear in the solution of the equations of NLRT, are shock type of discontinuities in $\bar{w}$ and θ representing a kink on the wavefront, which we call *wavefront shock*. The functions $\bar{w}$

and θ and their derivatives suffer only jump discontinuities across such a discontinuity [18].
Ramanathan [15] has worked out numerically many interesting solutions (not necessarily simple wave solutions) of the equations of NLRT. One such solution shows the initially converging nonlinear rays to start diverging before the arete of a caustic and then converging again after the arete (see figure 6.3 in [19]).

5 Conclusion

Hunter, Keller, Majda and Rosales have made a very significant contribution to the development of various aspects of a weakly nonlinear geometrical optics in a series of papers starting from 1983 (see Rosales [7]). Their work, an extension of a simple and formal version of a theory of Choquet-Bruhat [6], is based on approximating a system of hyperbolic conservation laws in the neighbourhood of the characteristic surface (in space-time) of the linearized equations. This means that the eikonal equation is independent of the wave amplitude. The transport equation for the amplitude then decouples from the eikonal equation and the only nonlinear effect on the rays is seen as an elongation of the linear rays in the direction of the rays. They miss the equally important effect of rotation of the rays due to nonlinearity. They justify their theory mainly by comparing solutions of some problems having spherical and cylindrical symmetry with well known solutions. However, the rotation of the rays is absent in these problems and 'naturally' they find good agreement. Our NLRT is based on approximating a hyperbolic system in a neighbourhood of the *exact* nonlinear characteristic surface in space-time and then retaining only the first order nonlinear terms in the bicharacteristic equations (3.8) and (3.9). In this way we are able to incorporate the wave amplitude correction in the eikonal equation itself (see [7] remark (iii), page 297). This allows our weakly nonlinear ray theory to take into account of an additional nonlinear effect namely the deviation of the rays (from linear rays) arising out of rotation due to nonlinearity. It is interesting that the two nonlinear effects, elongation of the rays (produced by the wave amplitude) and the deviation of the rays (produced by the gradient of the wave amplitude along the wavefront) can be of same order of magnitude. In fact, it is both these effects which are responsible for the resolution of the caustic in the figure. The figure also shows that a pair of singularities in the wavefront geometry appear at a finite time. To study these singularities (in the form of kinks in the wavefront across which θ and $\bar{w}$ become discontinuous) we need conservation form of the equations (4.6). Two such sets of physically realistic conservation forms have recently been derived by Prasad, Ravindran, Sangeeta and Morton [18], see also [19]. Numerical results obtained so far, show beautiful pictures of kinks propagating on the nonlinear wavefront. An important conclusion drawn from our nonlinear ray theory is that the amplitude of the wave even in first nonlinear approximation becomes finite in contrast to the result of the linear theory.
A nonlinear wavefront in an expansion wave behaves in an entirely different way. It always remains smooth, at least in the case fo two space dimensions.

Apart from the derivation of the equations (3.32) to (3.34), there are two more convincing arguments to show that their solutions represent the true physical phenomenon. A theorem proved in [20] states that if we take two nonlinear wavefronts one in a state just ahead of a shock front and another in a state just behind such that both instantaneously concide with the shock front and have same amplitude distribution, then for a weak shock the shock ray velocity and shock ray rotation are mean of those for the two nonlinear wavefronts. This theorem is true only for nonlinear wavefronts evolving according to (3.32) and (3.33). When the state ahead is the uniform state at rest, the shock ray velocity and rotation according to this theorem are given by

$$\chi_{sh} = (a_0 + \frac{\gamma+1}{4}w)n \tag{5.1}$$

and

$$\psi_{sh} = -\frac{\gamma+1}{4}Lw. \tag{5.2}$$

These are exactly the results for a weak shock when we retain terms up to first power in the shock strength [11], [21]. Moreover, it is simple to deduce the transport equation for the shock strength along the shock ray from (3.34) using (5.1) and (5.2). On the other hand there exists an infinite system of compatibility conditions along shock rays. These are exact results without any approximation and are valid for a shock of arbitrary strength. When we take the weak shock limit of the first compatibility condition (see equation (5.76) in [20]), we get exactly the same as the transport equation for the shock strength obtained from (3.34), (5.1) and (5.2). The second convincing argument is the validity of the physical principle of Fermat, which states that the time taken by a wave moving along a ray is stationary. Taking the ray velocity to be $\left(a_0 + \frac{\gamma+1}{2}w\right)n$, the Fermat's principle implies that the normal to the wavefront n must vary according to (3.33). Thus, ours is the only NLRT for which the Fermat's principle is valid and from which the transport equation for shock strength can be correctly determined.

It is also simple to see that our NLRT incorporates all features of Whitham's geometrical shock dynamics [3] derived by intuitive arguments. However, it has been shown [22], [23] that the geometrical shock dynamic is not correct for the propagation of a shock front. Instead, Whitham's theory for a weak shock is actually the NLRT provided we replace the shock ray velocity by the nonlinear ray velocity and the rate of rotation of the shock ray by that of the nonlinear ray [23]. For a weak shock, Whitham's equations have $\frac{\gamma+1}{4}$ instead of $\frac{\gamma+1}{2}$ in equations (3.32) to (3.34). Srinivasan's work [11] on the propagation of a nonlinear wavefront (in short wave limit) with an arbitrary amplitude of the wave shows that Whitham's theory, though correct in principle for a nonlinear wavefront, would require careful modification.

So far we have discussed a theory of propagation of only a single wavefront. Given a multidimensional pulse satisfying the short wave or high frequency assumption, we can define a one parameter family of wavefronts and use the nonlinear ray theory to each of these wavefronts to find the history of the pulse. The actual computational procedure is really simple as a parameter (identifying different wavefronts) would appear in the

equations (3.32) to (3.34). As in the case of the Burgers' equation, a curved shock wave may appear in the pulse. The successive positions of this shock can be obtained by the new theory of shock dynamics [24], [25].
Huygens' method has found many applications in physics. We do not know whether the nonlinear ray theory will also have success in explaining a few important nonlinear phenomena in physics.

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References

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II, Interscience, New York (1962).

[2] P. Prasad and R. Ravindran, *Partial Differential Equations*, John Wiley, New York and Wiley Eastern, New Delhi (1985).

[3] G.B. Whitham, *Linear and Nonlinear Waves*, John Wiley and Sons, New York (1974).

[4] D. Ludwig, "Uniform asymptotic expansions at a caustic", *Comm. Pure and Appl. Math. 19*, 215-250 (1966).

[5] P. Prasad, *On the Lemma on Bicharacteristic*, Appendix in A Nonlinear Ray Theory, Berichte der Arbeitsgruppe Technomathematik Nr. ... (1993).

[6] Y. Choquet-Bruhat, "Ondes asymptotiques et approchées pour des systèmes d'équations aux derivées partielles nonlinéaires", *J. Math. Pures et Appl. 48*, 117-158 (1969).

[7] R.R. Rosales, "An introduction to weakly nonlinear geometrical optics", a chapter in *Multi-dimensional Hyperbolic Problems and Computation* (Editors J. Glimm and A.J. Majda), Springer-Verlag (1991).

[8] A.M. Anile, J.K. Hunter, P. Pantano and G. Russo, *Ray Methods for Nonlinear Waves in Fluids and Plasmas*, Longman, U.K. (1993).

[9] M.J. Lighthill, "A technique for rendering approximate solutions to physical problems uniformly valid", *Phil. Mag. (7), 44*, 1179-1201 (1949).

[10] F. Obermeier, "On propagation of weak and moderately strong, curved shock waves", *J. Fluid Mechanics 129*, 123-136 (1983).

[11] R. Srinivasan, *A Mathematical Theory on the Propagation of Multi-dimensional Shock and Nonlinear Waves*, Ph.D. Thesis, Indian Institute of Science, Bangalore (1987).

[12] K.E. Gubkin, "Propagation of discontinuities in sound waves", *PMM, J. Appl. Math. and Mech. 22*, 787-793 (1958).

[13] P. Prasad, "Approximation of perturbation equations of a quasilinear hyperbolic system in the neighbourhood of a bicharacteristic", *J. Math. Anal. and Appl. 50*, 470-482 (1975).

[14] R. Ravindran and P. Prasad, "A theory of nonlinear waves in multi-dimensions with special reference to surface-water-waves", *J. Inst. Math. and Appl. 20*, 9-20 (1977).

[15] T.M. Ramanathan, *Huygens' Method for Construction of Weakly Nonlinear Wavefronts and Shockfronts with Application to Hyperbolic Caustics*, Ph.D. Thesis, Indian Institute of Science, Bangalore (1985).

[16] R. Ravindran and P. Prasad, "Kinematics of a shockfront and resolution of a hyperbolic caustic", a chapter in *Advances in Nonlinear Waves*, Vol. III (Editor L. Debnath) Pitman Research Notes in Mathematics (1985).

[17] P. Prasad, "Extension of Huygens' construction of a wavefront to a nonlinear wavefront and a shock front", *Current Science 56*, 50-54 (1987).

[18] P. Prasad, R. Ravindran, K. Sangeeta and K.M. Morton, "Conservation form of nonlinear ray equations", communicated for publication.

[19] P. Prasad, *Propagation of a Curved Shock and Nonlinear Ray Theory*, Longman U.K. and John Wiley, New York (1993).

[20] P. Prasad, "Kinematics of a multidimensional shock of arbitrary strength in an ideal gas", *Acta Mechanica 45*, 163-176 (1982).

[21] R. Ravindran and P. Prasad, "On infinite system of compatibility conditions along a shock ray", *Quart. J. Appl. Math. and Mech. 46*, 131-140 (1993).

[22] P. Prasad, "On Shock Dynamics", *Proc. Indian Acad. Sciences: Mathematical Sciences 100*, 87-92 (1990).

[23] P. Prasad, R. Ravindran and A. Sau, "On the characteristic rule for shocks", *Applied Mathematics Letters 4*, 5-8 (1991).

[24] R. Ravindran and P. Prasad, "A new theory of shock dynamics", Part I, *Applied Mathematics Letters 3*, 77-81; Part II, same Journal *3*, 107-109 (1990).

[25] M.P. Lazarev, R. Ravindran and P. Prasad, "Shock propagation in gas dynamics: explicit form of higher order compatibility conditions", communicated for publication.

Appendix: On the Lemma on Bicharacteristics

Phoolan Prasad*
Department of Mathematics
Universität Kaiserslautern
D-67663, Kaiserslautern, Germany

Abstract

The well known lemma on bicharacteristics of Courant is extended by finding an explicit form of an equation for the unit normal of the characteristic surface. A simple form of the compatibility condition along bicharacteristics is also presented.

1 Introduction

There is an important result in Methods of Mathematical Physics, Vol. II by Courant and Hilbert, in the form of a lemma. The result, known as lemma on bicharacteristics, giving an explicit form of the direction of rays, has been found very useful in the theory of nonlinear waves (Prasad, 1993). The result is not complete in the sense that the direction of the rays is coupled with the equation for the unit normal $\mathbf{n} = \nabla_{\mathbf{x}}\phi/|\nabla_x\phi|$ of the wavefronts associated with the one parameter family of characteristic surfaces $\phi(\mathbf{x},t) =$ constant. In this note we present an explicit form of the equations for $\mathbf{n}$. We also derive a simpler form of the compatibility condition on the characteristic surface. Earlier form of the compatibility condition (derived by Prasad and Ravindron 1984, see also Prasad, 1993) is too complicated.

2 Lemmas on bicharacteristics

Consider a first order system of partial differential equations

$$A\mathbf{u}_t + B^\alpha \mathbf{u}_{x_\alpha} + C = 0 \tag{A.2.1}$$

where $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^n$. We may take (1) to be a quasilinear system, in which case $A = A(\mathbf{u}(\mathbf{x},t), x, t)$ etc. The system (A 2.1) need not hyperbolic but we assume that c is a simple real characteristic root of the

[1]Permanent adress: Department of Mathematics, Indian Institute of Science, Bangalore-560012

characteristic equation. Let $\phi(\mathbf{x},t) =$ constant be the corresponding one parameter family of characteristic surfaces and let ℓ and r be the generalised left and right eigenvectors. Then

$$\ell M = 0,\ Mr = 0,\ \text{with } M = A\xi_0 + B^{(\alpha)}\xi_\alpha, \xi_0 = \phi_t, \xi = \nabla_{\mathbf{x}}\phi \tag{A.2.2}$$

where a reapeated index α, β or γ in a term would imply sum over $1, 2, \ldots, m$. We also have

$$c = -\frac{\xi_0}{|\xi|} \quad \text{and let} \quad \mathbf{n} = \frac{\xi}{|\xi|} \tag{A.2.3}$$

Let Q denote the expression $(\ell Ar)\phi_t + \ell B^{(\alpha)} r \phi_{X_\alpha}$, then ϕ satisfies

$$Q \equiv (\ell Ar)\xi_0 + (\ell B^{(\alpha)} r)\xi_\alpha = 0 \tag{A.2.4}$$

Note that c, ℓ and r are functions of $\mathbf{x}, t, \mathbf{u}(\mathbf{x},t)$, ξ_0 and ξ.

Bicharacteristics of (A.2.1) are the characteristic curves of the first order partial differential equation (pde) $\overline{Q} \equiv \det M = 0$ for ϕ. However, for the simple characteristic field under consideration, they are also the characteristic curves of the first order pde (A.2.4) i.e. $Q = 0$ where we note that the partial derivatives ξ_0, ξ appear also in the expressions for ℓ and r. The statement of the lemma in Courant and Hilbert is

Lemma 1

$$\frac{dx_\alpha}{dt} = \frac{\ell B^{(\alpha)} r}{\ell Ar} = x_\alpha, \text{ say} \quad \alpha = 1, 2, \ldots, m \tag{A.2.5}$$

The original proof of the lemma uses Hamilton canonical equations for the function $\overline{Q}$ and some simple results in matrix theory. The results (A.2.5) can also be proved using the canonical equations for the function Q. The proof is similar (but much simpler) to the proof of the Lemma 2:

Lemma 2

$$\frac{dn_\alpha}{dt} = -\frac{1}{(\ell Ar)} n_\beta \ell(-c\frac{\partial A}{\partial \eta^\alpha_\beta} + n_\gamma \frac{\partial B^{(\gamma)}}{\partial \eta^\alpha_\beta})r \tag{A.2.6}$$

where

$$\frac{\partial}{\partial \eta^\alpha_\beta} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta} \tag{A.2.7}$$

are tangential derivatives not only with respect to the characteristic surface in space-time but also with respect to the projections on $\mathbf{x}$-space of the sections of the characteristic surface by $t =$ *constant* planes (i.e. with respect ot ***wavefronts*** W_t).

Note: In (A.2.6), only m linear combinations L_α of the operators $\partial/\partial\eta^\alpha_\beta$

$$L_\alpha = n_\beta \frac{\partial}{\partial \eta^\alpha_\beta} \tag{A.2.8}$$

operate on A and $B^{(\gamma)}$ and hence on all dependent variables $u_1, u_2, \ldots, u_n$. However, since $n_\alpha L_\alpha = 0$ only $m-1$ of these tangential derivatives $L_\alpha(\alpha = 1, 2, \ldots, m)$ are linearly independent.

Proof of lemma 2. From Hamilton canonical equations for (A.2.4), we get

$$\begin{aligned} \frac{d\phi_{x_\alpha}}{dt} &= -Q_{x_\alpha}/Q_{\xi_0} = -\frac{1}{\ell A r}\left[\frac{\partial}{\partial x_\alpha}\{\ell(A\xi_0 + B^{(\beta)}\xi_\beta)r\}\right] \\ &= -\frac{1}{\ell A r}\left[\ell_{x_\alpha}(Mr) + (\ell M)r_{x_\alpha} + \ell(A_{x_\alpha}\xi_0 + B^{(\beta)}_{x_\alpha}\xi_\beta)r\right] \end{aligned}$$

The first two terms in the square bracket on the right hand side vanish due to (A.2.2). Hence,

$$\frac{d\xi_\alpha}{dt} = -\frac{1}{\ell A r}\ell(A_{x_\alpha}\xi_0 + B^{(\beta)}_{x_\alpha}\xi_\beta)r \tag{A.2.9}$$

Since

$$\frac{dn_\alpha}{dt} = \frac{1}{|\xi|^3}\left\{|\xi|^2\frac{d\xi_\alpha}{dt} - \xi_\alpha(\xi_\beta\frac{d\xi_\beta}{dt})\right\}$$

we get after rearrangement of terms

$$\frac{dn_\alpha}{dt} = \frac{1}{|\xi|^3}\left\{\xi_\beta(\xi_\beta\frac{d\xi_\alpha}{dt} - \xi_\alpha\frac{d\xi_\beta}{dt})\right\} \tag{A.2.10}$$

Substituting (A.2.9) in (A.2.10) and using $n_\alpha = \xi_\alpha/|\xi|$ we finally get the result (A.2.6) stated in the lemma.

In the theory of hyperbolic equations and in general theory of hyperbolic waves, it is quite common to choose ℓ and r such that $\ell A r = 1$. However, it is better to express the result in the form (A.2.5) and (A.2.6) so that simple forms of ℓ and r can be chosen. It is particularly important to choose the right eigenvector r in a very simple form since derivatives of r frequently appear in applications (see chapter 6, Prasad, 1993 or chapter 4, Anile et all, 1993).

The result (A.2.6) can also be proved using the method of Courant and Hilbert.

3 Compatibility condition on the characteristic surface

Lemma 1 gives the direction of the bicharacteristics in space-time. The operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi_\alpha \frac{\partial}{\partial x_\alpha} \tag{A.3.1}$$

in (A.2.5) and (A.2.6) represents the time rate of change when one moves along a bicharacteristic with the bicharacteristic velocity

$$\chi = (\chi_1, \chi_2, \ldots, \chi_m), \chi_\alpha = (\ell B^{(\alpha)} r)/(\ell A r) \tag{A.3.2}$$

A linear combination of the scalar equations in (A.2.1), containing only tangential derivatives in $\phi = constant$ of the components $u_i (i = 1, 2, \ldots, n)$ of $\mathbf{u}$, is obtained by pre-multiplying by ℓ

$$\ell A \frac{\partial \mathbf{u}}{\partial t} + \ell B^{(\alpha)} \frac{\partial \mathbf{u}}{\partial x_\alpha} + \ell C = 0 \tag{A.3.3}$$

Using (A.3.1), we can write (A.3.3) in the form

$$\ell A \frac{d\mathbf{u}}{dt} + \ell (B^{(\alpha)} - \chi_\alpha A) \frac{\partial \mathbf{u}}{\partial x_\alpha} + \ell C = 0 \tag{A.3.4}$$

The derivative $\tilde{\partial}_j = s_j^\alpha \frac{\partial}{\partial x_\alpha} \equiv l_i (B_{ij}^{(\alpha)} - \chi_\alpha A_{ij}) \frac{\partial}{\partial x_\alpha}$ on u_j in the second term is a special tangential derivative with respect to the characteristic surface, it is a tangential derivative also with respect ot the *wavefronts* W_t. This follows from

$$n_\alpha s_j^\alpha = l_i A_{ij} (c - n_\alpha \chi_\alpha) = 0 \tag{A.3.5}$$

since $c = n_\alpha \chi_\alpha$.

The form (A.3.4) of the compatibility conditioin has very special feature. The derivative $\frac{d}{dt}$ in the bicharacteristic direction is the only derivative in (A.3.4) which contains $\frac{\partial}{\partial t}$. The other n tangential derivatives $\tilde{\partial}_j (j = 1, 2, \ldots, n)$ contain only spatial derivatives and can be expressed in terms of any $m - 1$ of the m tangential derivatives L_α. As we have seen, it requires only a trivial steps to get (A.3.4) from the original form (A.2.1). However, (A.3.4) is important in formulating numerical methods using bicharacteristic curves (Reddy, Tikekar and Prasad, 1982). (A.3.4) can also be used to give a simple derivation of a transport equations (see eqn. (6.27) in Prasad, 1993) for the amplitude along the exact nonlinear bicharacteristic curve.

We state all the results in this paper in a form of a

Theorem The bicharacteristic curves in a simple characteristic field of (A.2.1) are given by (A.2.5) and (A.2.6) along which the compatibility condition (A.3.4) holds.

References

[1] Anile, A.M., Hunter, J.K., Pantano, P. and Russo, G. 1993 *Ray Methods for Nonlinear Waves in Fluids and Plasmas*, Longman.

[2] Courant, R. and Hilbert, D. 1962 *Methods of Mathematical Physics*, Vol. II, Interscience.

[3] Prasad, P. 1993 *Propagation of a Curved Shock and Nonlinear Ray Theory*, Longman

[4] Prasad, P. and Ravindran, R. 1984 A canonical form of quasilinear first order hyperbolic system and its use in their numerical solution, *J. Mathematical and Physical Sciences*, **18**, 361-369.

[5] Reddy, A.S., Tikekar, V.G. and Prasad, P. 1982 Numerical solution of hyperbolic equations by method of bicharacteristics, *J. Mathematical and Physical Sciences*, **16**, 575-603.

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