FACTORIZATION THEORY FOR STABLE, DISCRETE-TIME INNER FUNCTIONS

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Abstract

We develop a factorization theory for stable inner functions relative to the unit circle.

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1 Introduction

Inner functions play an important role in both mathematics and its applications, particularly in the area of control and filtering theory.

A classical result, proved by Beurling [1949] links inner functions with the theory of invariant subspaces. This provides another link with the theory of functional models for general, non selfadjoint operators.

Our object in this paper is to describe a factorization theory for stable, matrix valued inner functions in state space terms. Characterizations of inner functions in the right half plane or in (outside) the unit circle have been known for a long time, e.g. Genin et al. [1983].

Factorizations of inner functions are related bijectively to the set of invariant subspaces of a model operator, that is a restricted shift operator. This was in fact one of the reasons that made the study of inner functions so attractive.

From the application point of view, emphasizing numerical computability, it became of interest to relate the factorization of inner functions to their state space realizations. Such a factorization theory was initiated by Finesso and Picci [1982] and continued in Picci and Pinzoni [1994]. In this connection see also Fuhrmann [1994] where this study has been completed. The papers just quoted all focused on the continuous time case, namely the case that the function was inner in the right half plane. The central result established a bijective correspondence between factorizations of an inner function U, invariant subspaces of the generator in any minimal realization of U and the set of nonnegative definite solutions of a homogeneous Riccati equation. Our aim in this paper is to describe an analogous result in the discrete time case. Specifically, we focus on contractive, matrix valued analytic functions in the exterior of the unit circle (including the point at infinity), which have unitary (isometric) radial limits on the unit circle. Since we are interested in matrix formulas, we have to assume of course that the inner functions under study are all rational.

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2 Characterization of inner functions

We will discuss stable inner functions. Thus a rational $p \times m$ matrix valued function U is called **inner**, if it is analytic in $\{z \mid |z| > 1\}$, including at ∞, and satisfies $U^*U = I$. Here U^* is defined by $U^*(z) = U(\overline{z}^{-1})^*$. Similarly, a rational matrix valued function is called **coinner** if it is stable and satisfies $UU^* = I$. Assume U has a minimal realization of the form $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$. Necessarily A is discrete time stable, i.e. its spectrum is included in the open unit disk. We have the following slightly strengthened form of a characterization obtained in Genin et al. [1983].

Proposition 2.1 *Let*

$$U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{1}$$

be the transfer function of an asymptotically stable discrete time system. Then

1. *If there exists a solution X of the matrix equation*

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \tag{2}$$

or equivalently, of the system

$$\left\{ \begin{array}{l} X = A^*XA + C^*C \\ D^*C + B^*XA = 0 \\ D^*D + B^*XB = I \end{array} \right. \tag{3}$$

then U is inner.

2. *If U is inner and the realization (1) is reachable, then there exists a nonnegative definite solution of the system (3).*

3. *$X = X^* \geq 0$ satisfying (3) is the observability gramian of the realization. Thus $KerX$ is the unobservable subspace and $X > 0$ if and only if (A, C) is an observable pair. If the realization (1) is reachable we have for the McMillan degree $\delta(U) = n - \dim KerX$, where n is the dimension of the state space.*

Proof:

1. Let U be a rational, $p \times m$ matrix valued analytic function with all poles inside the unit disk. Then it has an expansion at ∞ of the form

$$U(z) = \sum_{\nu=0}^{\infty} \frac{U_\nu}{z^\nu}, \qquad |z| > |\lambda(A)|_{max} \ (< 1),$$

which implies

$$U^*(z) = \sum_{\nu=0}^{\infty} U_\nu^* z^\nu, \qquad |z| < [|\lambda(A)|_{max}]^{-1} \ (> 1),$$

Hence with Ω defined by

$$\Omega(z) = U^*(z)U(z) = \sum_{k=-\infty}^{\infty} \Omega_k z^k, \qquad |\lambda(A)|_{max} < |z| < [|\lambda(A)|_{max}]^{-1},$$

we have

$$\Omega_k = \left\{ \begin{array}{ll} \sum_{i=0}^{\infty} U_{k+i}^* U_i, & k \geq 0 \\ \\ \sum_{i=0}^{\infty} U_i^* U_{-k+i}, & k < 0 \end{array} \right.$$

As U has the realization (1), we have

$$U_i = \begin{cases} D & i = 0 \\ CA^{i-1}B & i > 0 \end{cases}$$

and thus we compute

$$\Omega_k = \begin{cases} \sum_{i=1}^{\infty} B^*(A^*)^{k+i-1}C^*CA^{i-1}B + B^*(A^*)^{k-1}C^*D & k > 0 \\ \\ D^*D + \sum_{i=1}^{\infty} B^*(A^*)^{i-1}C^*CA^{i-1}B & k = 0 \\ \\ \sum_{i=1}^{\infty} B^*(A^*)^{i-1}C^*CA^{-k+i-1}B + D^*CA^{-k-1}B & k < 0 \end{cases} \tag{4}$$

Now assume there exists an X for which the equations (3) hold. If A is stable, there exists a unique $X' = (X')^*$ satisfying the equation $X' = A^*X'A + C^*C$ and it is given by $X' = \sum_{i=0}^{\infty}(A^*)^iC^*CA^i$. Now, with $X = X'$ we compute

$$\begin{aligned} \Omega_0 &= D^*D + \textstyle\sum_{i=1}^{\infty} B^*(A^*)^{i-1}C^*CA^{i-1}B \\ &= D^*D + B^*\left(\textstyle\sum_{i=1}^{\infty}(A^*)^{i-1}C^*CA^{i-1}\right)B = D^*D + B^*XB = I. \end{aligned}$$

For $k > 0$ we compute

$$\begin{aligned} \Omega_k &= B^*(A^*)^{k-1}C^*D + \textstyle\sum_{i=1}^{\infty} B^*(A^*)^{k+i-1}C^*CA^{i-1}B \\ &= B^*(A^*)^{k-1}C^*D + B^*(A^*)^k \textstyle\sum_{i=1}^{\infty}(A^*)^{i-1}C^*CA^{i-1}B \\ &= B^*(A^*)^{k-1}C^*D + B^*(A^*)^kXB = B^*(A^*)^{k-1}(C^*D + A^*XB) = 0. \end{aligned}$$

A similar computation holds for $k < 0$. Thus we can conclude $U^*U = I$, i.e. U is inner.

2. Assume now that U is inner and the pair (A, B) is reachable. Let $X = X^*$ be the unique solution of the equation

$$X = A^*XA + C^*C.$$

This solution is given by $X = \sum_{i=0}^{\infty}(A^*)^iC^*CA^i$. Therefore $X \geq 0$ is the observability gramian of the realization and $X > 0$ if and only if the realization is observable. Now, by (4), we have for $k = 0$

$$\begin{aligned} I &= D^*D + \textstyle\sum_{i=1}^{\infty} B^*(A^*)^{i-1}C^*CA^{i-1}B \\ &= D^*D + B^*\left(\textstyle\sum_{i=1}^{\infty}(A^*)^{i-1}C^*CA^{i-1}\right)B \\ &= D^*D + B^*XB. \end{aligned}$$

For $k > 0$ we get

$$\begin{aligned} 0 &= B^*(A^*)^{k-1}C^*D + \textstyle\sum_{i=1}^{\infty} B^*(A^*)^{k+i-1}C^*CA^{i-1}B \\ &= B^*(A^*)^{k-1}\left\{C^*D + A^*\textstyle\sum_{i=1}^{\infty}(A^*)^{i-1}C^*CA^{i-1}B\right\} \\ &= B^*(A^*)^{k-1}\left\{C^*D + A^*XB\right\} = 0 \end{aligned}$$

which, by the assumption of reachability, implies $C^*D + A^*XB = 0$. For $k < 0$ an analogous computation leads to the same result. ∎

By duality considerations we get the following

Proposition 2.2 *Let*

$$U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{5}$$

be the transfer function of an asymptotically stable, discrete time system. Then:

1. *If there exists a solution* Y *of the matrix equation*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \tag{6}$$

   or equivalently, of the system

$$\left\{ \begin{array}{l} Y = AYA^* + BB^* \\ BD^* + AYC^* = 0 \\ DD^* + CYC^* = I \end{array} \right. \tag{7}$$

   then U *is coinner.*

2. *If* U *is coinner and the realization (5) is observable, then there exists a nonnegative definite solution of the system (7).*

3. $Y = Y^*$ *satisfying (7) is the reachability gramian of the realization. Thus* $Y > 0$ *if and only if* (A, B) *is a reachable pair. If the realization (5) is observable, we have for the McMillan degree* $\delta(U) = \dim ImY = rankY$.

We note that the previous results extend easily to the case of J-inner functions.

Proposition 2.3 *Let*

$$U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{8}$$

be the square transfer function of an asymptotically stable, discrete time system. Then, if $Y > 0$ *is a solution of*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \tag{9}$$

then the system

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \tag{10}$$

is solvable. Moreover $X = Y^{-1} > 0$.

Proof: By a simple computation, using the invertibility of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. ∎

We introduce now a notion of balancing that is appropriate for the class of inner functions. For an exhaustive study of balancing, see Ober [1991] and the references therein. We say that a minimal realization of an asymptotically stable, discrete-time inner function is **balanced** if the reachability and observability gramians are both equal to the identity matrix.

As an immediate corollary to Proposition 2.3, we obtain

Corollary 2.1 *Let U be an asymptotically stable, discrete-time, square inner function. Then a balanced realization exists, and this realization is unique up to unitary equivalence.*

Proof: By Proposition 2.2, using the minimality of the realization, there exists a positive definite solution Y of equation (9). Applying Proposition 2.3, we have (10) solvable with $X = Y^{-1}$. Defining $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} X^{\frac{1}{2}}AX^{-\frac{1}{2}} & X^{\frac{1}{2}}B \\ CX^{-\frac{1}{2}} & D \end{pmatrix}$, this matrix is clearly unitary and hence provides a balanced realization of U.

For uniqueness assume $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a balanced, necessarily minimal, realization of U. Any other minimal realization has the form $U = \left(\begin{array}{c|c} R^{-1}AR & R^{-1}B \\ \hline CR & D \end{array}\right)$. The reachability and observability gramians for this realization are $R^{-1}R^{-*}$ and R^*R respectively. Thus the realization is balanced if and only if $R^*R = I$, i.e. R is unitary. ∎

Next we proceed to study the way an inner function is determined via its left pole structure. Of course a similar result will hold for a specified right pole structure. By specifying a left pole structure for a (stable) inner function in the exterior of the unit disk we mean specifying a nonsingular, stable polynomial matrix $D(z)$ such that, for some $N(z)$, we have $U := D^{-1}N$ is inner and the factorization is left coprime. Using the shift realization, introduced in Fuhrmann [1976], the pair (A, C) in any minimal realization of U is completely determined, up to similarity, by the polynomial matrix $D(z)$. Thus we can state our problem alternatively in the following way. Suppose we are given an observable pair (A, C), with A stable. We look for matrices B, D so that $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a minimal realization of an inner function U. Naturally we will apply the characterization of inner functions in terms of their state space representation.

Theorem 2.1 *Given an observable pair (A, C) with A asymptotically stable, there exists a square inner function U in the exterior of the unit disk, uniquely determined up to a constant right unitary factor, such that for some B and D*

$$U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \tag{11}$$

and this realization is minimal.

Proof: Let X be the observability gramian of the pair (A,C), i.e. the unique solution of the equation $A^*XA + C^*C = X$. By observability we have $X > 0$. We can rewrite the last equation as $(X^{-\frac{1}{2}}A^*X^{\frac{1}{2}})(X^{\frac{1}{2}}AX^{-\frac{1}{2}}) + (X^{-\frac{1}{2}}C^*)(CX^{-\frac{1}{2}}) = I$. Setting

$$\begin{aligned} A_1 &= X^{\frac{1}{2}}AX^{-\frac{1}{2}} \\ C_1 &= CX^{-\frac{1}{2}} \end{aligned}$$

we have $A_1^*A_1 + C_1^*C_1 = I$. Thus the matrix $\begin{pmatrix} A_1 \\ C_1 \end{pmatrix}$ is isometric, hence its columns form an orthonormal set of vectors. Let $\begin{pmatrix} B_1 \\ D_1 \end{pmatrix}$ be an arbitrary completion of $\begin{pmatrix} A_1 \\ C_1 \end{pmatrix}$ to a unitary matrix $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$. Since the orthogonal complement of a subspace of an inner product space is uniquely determined, the matrix $\begin{pmatrix} B_1 \\ D_1 \end{pmatrix}$ is uniquely determined up to a right unitary factor. By Proposition 2.1.(1), the transfer function $U = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right)$ is inner. Using the similarity $X^{\frac{1}{2}}$, we get $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, where

$$\begin{aligned} B &= X^{-\frac{1}{2}}B_1 \\ D &= D_1. \end{aligned}$$

∎

3 Factorization of inner functions

Before attempting the factorization problem, we analyze the product of two square inner functions and the solution of the related systems.

Proposition 3.1 *Let* $U_i = \left(\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right)$, $i = 1, 2$, *be minimal realizations of two square inner functions. Let*

$$\begin{aligned} U &= U_1U_2 = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \times \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right) \\ &= \left(\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right) =: \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \end{aligned} \tag{12}$$

Let, for $i = 1, 2$, X_i be the solution of the system

$$\begin{cases} X_i = A_i^* X_i A_i + C_i^* C_i \\ D_i^* C_i + B_i^* X_i A_i = 0 \\ D_i^* D_i + B_i^* X_i B_i = I \end{cases} \tag{13}$$

Then $X = \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \end{pmatrix}$ *is the solution of the system*

$$\begin{cases} X = A^* X A + C^* C \\ D^* C + B^* X A = 0 \\ D^* D + B^* X B = I \end{cases} \tag{14}$$

In particular, the realization in (12) is minimal.

Proof: We check

1.

$$\begin{pmatrix} A_2^* & C_2^* B_1^* \\ 0 & A_1^* \end{pmatrix} \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ B_1 C_2 & A_1 \end{pmatrix} + \begin{pmatrix} C_2^* D_1^* \\ C_1^* \end{pmatrix} \begin{pmatrix} D_1 C_2 & C_1 \end{pmatrix}$$

$$= \begin{pmatrix} A_2^* X_2 A_2 + C_2^* B_1^* X_1 B_1 C_2 + C_2^* D_1^* D_1 C_2 & C_2^* B_1^* X_1 A_1 + C_2^* D_1^* C_1 \\ A_1^* X_1 B_1 C_2 + C_1^* D_1 C_2 & A_1^* X_1 A_1 + C_1^* C_1 \end{pmatrix}$$

$$= \begin{pmatrix} A_2^* X_2 A_2 + C_2^* [B_1^* X_1 B_1 + D_1^* D_1] C_2 & C_2^* [B_1^* X_1 A_1 + D_1^* C_1] \\ [A_1^* X_1 B_1 + C_1^* D_1] C_2 & A_1^* X_1 A_1 + C_1^* C_1 \end{pmatrix}$$

$$= \begin{pmatrix} A_2^* X_2 A_2 + C_2^* C_2 & 0 \\ 0 & A_1^* X_1 A_1 + C_1^* C_1 \end{pmatrix} = \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \end{pmatrix}$$

2.

$$D_2^* D_1^* \begin{pmatrix} D_1 C_2 & C_1 \end{pmatrix} + \begin{pmatrix} B_2^* & D_2^* B_1^* \end{pmatrix} \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ B_1 C_2 & A_1 \end{pmatrix}$$

$$= D_2^* [D_1^* D_1 + B_1^* X_1 B_1] C_2 + D_2^* [D_1^* C_1 + B_1^* X_1 A_1] + B_2^* X_2 A_2$$

$$= D_2^* C_2 + B_2^* X_2 A_2 = 0.$$

3.

$$D_2^* D_1^* D_1 D_2 + \begin{pmatrix} B_2^* & D_2^* B_1^* \end{pmatrix} \begin{pmatrix} X_2 & 0 \\ 0 & X_1 \end{pmatrix} \begin{pmatrix} B_2 \\ B_1 D_2 \end{pmatrix}$$
$$D_2^* [D_1^* D_1 + B_1^* X_1 B_1] D_2 + B_2^* X_2 B_2$$
$$D_2^* D_2 + B_2^* X_2 B_2 = I.$$

The realization in (12) is minimal as $X = diag\{X_2, X_1\} > 0$ is the observability gramian and, by the dual result of Proposition 2.3, $Y = X^{-1}$ is the reachability gramian. Of course this follows also from the fact that, for two square inner functions U_1, U_2, we have $\delta(U_1U_2) = \delta(U_1) + \delta(U_2)$, as there cannot be any pole-zero cancellations. ∎

To see how the left factor U_1 can be recovered, it is natural to assume that the left factor in $U = U_1U_2$ is associated with the nonnegative definite matrix $Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix}$, where $Y_1 > 0$ is the positive definite solution of

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} Y_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1^* & C_1^* \\ B_1^* & D_1^* \end{pmatrix} = \begin{pmatrix} Y_1 & 0 \\ 0 & I \end{pmatrix} \tag{15}$$

We consider now the equation

$$\begin{pmatrix} A & B_0 \\ C & D_0 \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B_0^* & D_0^* \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \tag{16}$$

which we expand into

$$\begin{pmatrix} A_2 & 0 & B_0'' \\ B_1C_2 & A_1 & B_0' \\ D_1C_2 & C_1 & D_0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_2^* & C_2^*B_1^* & C_2^*D_1^* \\ 0 & A_1^* & C_1^* \\ (B_0'')^* & (B_0')^* & D_0^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & I \end{pmatrix} \tag{17}$$

From this we obtain the equality

$$\begin{pmatrix} B_0''(B_0'')^* & B_0''(B_0')^* & B_0''D_0^* \\ B_0'(B_0'')^* & A_1Y_1A_1^* + B_0'(B_0')^* & A_1Y_1C_1^* + B_0'D_0^* \\ D_0(B_0'')^* & C_1Y_1A_1^* + D_0(B_0')^* & C_1Y_1C_1^* + D_0D_0^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & I \end{pmatrix}$$

This leads to $B_0'' = 0$, and

$$\begin{pmatrix} A_1 & B_0' \\ C_1 & D_0 \end{pmatrix} \begin{pmatrix} Y_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1^* & C_1^* \\ (B_0')^* & D_0^* \end{pmatrix} = \begin{pmatrix} Y_1 & 0 \\ 0 & I \end{pmatrix}$$

We compare this equality with (15) to obtain

$$\begin{pmatrix} B_0' \\ D_0 \end{pmatrix} \begin{pmatrix} (B_0')^* & D_0^* \end{pmatrix} = \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} \begin{pmatrix} B_1^* & D_1^* \end{pmatrix}$$

This clearly implies the existence of a unitary matrix V such that $\begin{pmatrix} B_0' \\ D_0 \end{pmatrix} = \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} V$, since the matrix $\begin{pmatrix} B_1 \\ D_1 \end{pmatrix}$ is full column rank.

Now consider the realization

$$\left(\begin{array}{cc|c} A_2 & 0 & 0 \\ B_1C_2 & A_1 & B_1V \\ \hline D_1C_2 & C_1 & D_1V \end{array}\right) = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) V = U_1(z)V.$$

The unitary matrix V is clearly not significant as in any factorization $U = U_1U_2$ the left inner factor is only determined up to a right constant unitary factor. ∎

Observe that

$$\left(\begin{array}{c|c} A & B_0 \\ \hline C & D_0 \end{array}\right) = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \times V;$$

this indicates that left square inner factors of a square inner function $U = \left(\begin{array}{c|c} A & B_\mu \\ \hline C & D_\mu \end{array}\right)$ may be related to solutions of the system (16). This theme is picked up by the next theorem.

Theorem 3.1 *Let*

$$U = \left(\begin{array}{c|c} A & B_\mu \\ \hline C & D_\mu \end{array}\right) \tag{18}$$

be a minimal realization of a square inner function with $\delta(U) = n$. *Then there exists a bijective correspondence between*

1. *Left square inner factors of* U.
2. *Invariant subspaces of* A.
3. *Solutions* Y, *for suitable* B *and square* D, *of the matrix equation*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix}, \tag{19}$$

or equivalently, of the system

$$\left\{\begin{array}{l} Y = AYA^* + BB^* \\ BD^* + AYC^* = 0 \\ DD^* + CYC^* = I. \end{array}\right. \tag{20}$$

Proof: (1) ⇒ (2). Let U_1 be a left factor of U, i.e. $U = U_1U_2$ for some U_2. Assuming the notation of Proposition 3.1, then the minimal realization (12) exhibits an invariant subspace of A of dimension equal to the McMillan degree of U_1.

(2) ⇒ (1). Given a minimal realization of U as in (18) and an invariant subspace of A, then by similarity there exists an invariant subspace of the generator in the shift realization of U. This immediately implies the existence of a corresponding factorization.

(1) ⇒ (3). Starting from a left factor U_1, a solution of equation (19) can be constructed by letting $Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix}$, where $Y_1 > 0$ is the positive definite solution of

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} Y_1 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A_1^* & C_1^* \\ B_1^* & D_1^* \end{pmatrix} = \begin{pmatrix} Y_1 & 0 \\ 0 & I \end{pmatrix}, \tag{21}$$

and

$$\begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ B_1 \\ D_1 \end{pmatrix}. \tag{22}$$

Then it is easily checked that

$$\begin{pmatrix} A_2 & 0 & 0 \\ B_1C_2 & A_1 & B_1 \\ D_1C_2 & C_1 & D_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_2^* & C_2^*B_1^* & C_2^*D_1^* \\ 0 & A_1^* & C_1^* \\ 0 & B_1^* & D_1^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & I \end{pmatrix} \tag{23}$$

(3) $\Rightarrow$ (2), (1). Assume Y, B, D_1 solve the system (20), i.e.

$$\begin{pmatrix} A & B \\ C & D_1 \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D_1^* \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix}, \tag{24}$$

with $U_1 = \left(\begin{array}{c|c} A & B \\ \hline C & D_1 \end{array}\right)$ a coinner function which is square and hence necessarily inner.

Clearly, $Y = AYA^* + BB^*$ implies $Y = \sum_{i=0}^{\infty} A^i BB^*(A^*)^i \geq 0$. Let us assume, without loss of generality, that we work in a basis where the solution has the form $\begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix}$, with $Y_1 > 0$. Putting the other matrices in compatible block form, we write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_2' \\ B_1 \end{pmatrix}, C = \begin{pmatrix} C_2' & C_1 \end{pmatrix}.$$

The equation (24) can now be expanded into

$$\begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix} + \begin{pmatrix} B_2' \\ B_1 \end{pmatrix} \begin{pmatrix} (B_2')^* & B_1^* \end{pmatrix} \\ = \begin{pmatrix} A_{12}Y_1A_{12}^* & A_{12}Y_1A_{22}^* \\ A_{22}Y_1A_{12}^* & A_{22}Y_1A_{22}^* \end{pmatrix} + \begin{pmatrix} B_2'(B_2')^* & B_2'B_1^* \\ B_1(B_2')^* & B_1B_1^* \end{pmatrix} \\ \begin{pmatrix} B_2' \\ B_1 \end{pmatrix} D_1^* + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} (C_2')^* \\ C_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ D_1D_1^* + \begin{pmatrix} C_2' & C_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} (C_2')^* \\ C_1^* \end{pmatrix} = I. \end{cases} \tag{25}$$

From the first of these equations we obtain

$$\begin{pmatrix} 0 & 0 \\ 0 & Y_1 \end{pmatrix} = \begin{pmatrix} A_{12}Y_1A_{12}^* + B_2'(B_2')^* & A_{12}Y_1A_{22}^* + B_2'B_1^* \\ A_{22}Y_1A_{12}^* + B_1(B_2')^* & A_{22}Y_1A_{22}^* + B_1B_1^* \end{pmatrix}$$

Now $A_{12}Y_1A_{12}^* + B_2'(B_2')^* = 0$ implies $A_{12} = 0$ and $B_2' = 0$. We condense notation by putting $A_1 = A_{22}, A_2 = A_{11}$. In particular $Y_1 = A_1Y_1A_1^* + B_1B_1^*$ implies the reachability of the pair (A_1, B_1). The observability of the pair (A_1, C_1) follows from that of (A, C). For, suppose $\xi \in \cap_{i \geq 0} Ker C_1 A_1^i$; then, $\begin{pmatrix} 0 \\ \xi \end{pmatrix} \in \cap_{i \geq 0} Ker C A^i$ which implies $\xi = 0$. Since $A = \begin{pmatrix} A_2 & 0 \\ A_{21} & A_1 \end{pmatrix}$, we have shown the existence of an A-invariant subspace of dimension equal to the size of A_1.

Now we can use the information obtained so far in the realization, to get

$$U_1 = \left(\begin{array}{cc|c} A_2 & 0 & 0 \\ A_{21} & A_1 & B_1 \\ \hline C_2' & C_1 & D_1 \end{array}\right) = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right).$$

Moreover, note that we already know that $Y_1 = A_1 Y_1 A_1^* + B_1 B_1^*$ holds. From the second equation in (25) we get $B_1 D_1^* + A_1 Y_1 C_1^* = 0$, whereas the last equation in (25) implies $D_1 D_1^* + C_1 Y_1 C_1^* = I$.

To conclude the proof, we will show that U_1 is a left inner factor of U. We write now $B_\mu = \begin{pmatrix} B_2 \\ B_1' \end{pmatrix}$ in block form, compatible with that of A. Thus

$$U = \left(\begin{array}{c|c} A & B_\mu \\ \hline C & D_\mu \end{array}\right) = \left(\begin{array}{cc|c} A_2 & 0 & B_2 \\ A_{21} & A_1 & B_1' \\ \hline C_2' & C_1 & D_\mu \end{array}\right) \tag{26}$$

Since U is square and inner, and (26) is a minimal realization, there exist positive definite solutions X and Y' to the matrix equations

$$\begin{pmatrix} A & B_\mu \\ C & D_\mu \end{pmatrix} \begin{pmatrix} Y' & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B_\mu^* & D_\mu^* \end{pmatrix} = \begin{pmatrix} Y' & 0 \\ 0 & I \end{pmatrix} \tag{27}$$

and

$$\begin{pmatrix} A^* & C^* \\ B_\mu^* & D_\mu^* \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_\mu \\ C & D_\mu \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}, \tag{28}$$

moreover $X = (Y')^{-1}$.

From the reachability of the pair $\left(\begin{pmatrix} A_2 & 0 \\ A_{21} & A_1 \end{pmatrix}, \begin{pmatrix} B_2 \\ B_1' \end{pmatrix} \right)$ follows immediately the reachability of the pair (A_2, B_2). Thus, there exist, by the dual form of Theorem 2.1, matrices C_2, D_2 for which $U_2 = \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$ is square inner and the realization is minimal. The matrices C_2, D_2 are uniquely determined up to a common left unitary factor. Let $Y_2 > 0$ be the reachability gramian of the realization of U_2, i.e. the solution of $A_2 Y_2 A_2^* + B_2 B_2^* = Y_2$. We note that from

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} Y_2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_2^* & C_2^* \\ B_2^* & D_2^* \end{pmatrix} = \begin{pmatrix} Y_2 & 0 \\ 0 & I \end{pmatrix} \tag{29}$$

we get

$$\begin{pmatrix} A_2Y_2 & B_2 \end{pmatrix} \begin{pmatrix} C_2^* \\ D_2^* \end{pmatrix} = 0$$

Since U_2 is square $p \times p$ with $\delta(U_2) = n_2$ and $\begin{pmatrix} A_2Y_2 & B_2 \end{pmatrix} \in \mathbf{C}^{n_2\times(n_2+p)}$ has full row rank and $\begin{pmatrix} C_2^* \\ D_2^* \end{pmatrix} \in \mathbf{C}^{(n_2+p)\times n_2}$ has full column rank, it follows that the last matrix is a basis matrix for $Ker \begin{pmatrix} A_2Y_2 & B_2 \end{pmatrix}$.

Next we expand equation (27) into

$$\begin{pmatrix} A_2 & 0 & B_2 \\ A_{21} & A_1 & B_1' \\ C_2' & C_1 & D_\mu \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} & 0 \\ Y_{12}^* & Y_{22} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_2^* & A_{21}^* & (C_2')^* \\ 0 & A_1^* & C_1^* \\ B_2^* & (B_1')^* & D_\mu^* \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} & 0 \\ Y_{12}^* & Y_{22} & 0 \\ 0 & 0 & I \end{pmatrix}$$

Computing the (1,1) entry, we get $A_2Y_{11}A_2^* + B_2B_2^* = Y_{11}$, which implies $Y_{11} = Y_2$.

Now a similarity T in the state space transforms the triple (A, B, C) and a solution of $Y = AYA^* + BB^*$ into the triple (TAT^{-1}, TB, CT^{-1}) and TYT^*. For our purpose, we choose the similarity $\begin{pmatrix} I & 0 \\ -Y_{12}^*Y_2^{-1} & I \end{pmatrix}$. Then simple computations lead to

$$\begin{pmatrix} I & 0 \\ -Y_{12}^*Y_2^{-1} & I \end{pmatrix} \begin{pmatrix} Y_2 & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix} \begin{pmatrix} I & -Y_2^{-1}Y_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} Y_2 & 0 \\ 0 & Y_{22} - Y_{12}^*Y_2^{-1}Y_{12} \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ -Y_{12}^*Y_2^{-1} & I \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ A_{21} & A_1 \end{pmatrix} \begin{pmatrix} I & 0 \\ Y_{12}^*Y_2^{-1} & I \end{pmatrix} = \begin{pmatrix} A_2 & 0 \\ A_{21}' & A_1 \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ -Y_{12}^*Y_2^{-1} & I \end{pmatrix} \begin{pmatrix} B_2 \\ B_1' \end{pmatrix} = \begin{pmatrix} B_2 \\ B_1' - Y_{12}^*Y_2^{-1}B_2 \end{pmatrix}$$

$$\begin{pmatrix} C_2' & C_1 \end{pmatrix} \begin{pmatrix} I & 0 \\ Y_{12}^*Y_2^{-1} & I \end{pmatrix} = \begin{pmatrix} C_2' + C_1Y_{12}^*Y_2^{-1} & C_1 \end{pmatrix}.$$

Moreover, defining $Y_1' = Y_{22} - Y_{12}^*Y_2^{-1}Y_{12}$, we must have, by Sylvester's law of inertia, that $Y_1' > 0$. Thus, redefining A_{21}, B_1' and C_2', we may assume without loss of generality that $\begin{pmatrix} A_2 & 0 & B_2 \\ A_{21} & A_1 & B_1' \\ C_2' & C_1 & D_\mu \end{pmatrix}$ is already in the new basis and $Y' = \begin{pmatrix} Y_2 & 0 \\ 0 & Y_1' \end{pmatrix}$.

Since

$$\begin{pmatrix} A_2^* & A_{21}^* \\ 0 & A_1^* \end{pmatrix} \begin{pmatrix} Y_2^{-1} & 0 \\ 0 & (Y_1')^{-1} \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ A_{21} & A_1 \end{pmatrix} + \begin{pmatrix} (C_2')^* \\ C_1^* \end{pmatrix} \begin{pmatrix} C_2' & C_1 \end{pmatrix} = \begin{pmatrix} Y_2^{-1} & 0 \\ 0 & (Y_1')^{-1} \end{pmatrix}$$

it follows that $A_1^*(Y_1')^{-1}A_1 + C_1^*C_1 = (Y_1')^{-1}$. As U_1 is square inner, we have also $A_1Y_1'A_1^* + B_1B_1^* = Y_1'$. Therefore $Y_1' = Y_1$.

Going back to equation (27), it can be expanded now into

$$\begin{pmatrix} A_2 & 0 & B_2 \\ A_{21} & A_1 & B_1' \\ C_2' & C_1 & D_\mu \end{pmatrix} \begin{pmatrix} Y_2 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_2^* & A_{21}^* & (C_2')^* \\ 0 & A_1^* & C_1^* \\ B_2^* & (B_1')^* & D_\mu^* \end{pmatrix} = \begin{pmatrix} Y_2 & 0 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & I \end{pmatrix}$$

Computing the (1,3) term, we get

$$A_2Y_2(C_2')^* + B_2D_\mu^* = \begin{pmatrix} A_2Y_2 & B_2 \end{pmatrix} \begin{pmatrix} (C_2')^* \\ D_\mu^* \end{pmatrix} = 0.$$

Since $\begin{pmatrix} C_2^* \\ D_2^* \end{pmatrix}$ is a basis matrix for $Ker \begin{pmatrix} A_2Y_2 & B_2 \end{pmatrix}$, we conclude that there exists a matrix P for which $\begin{pmatrix} (C_2')^* \\ D_\mu^* \end{pmatrix} = \begin{pmatrix} C_2^* \\ D_2^* \end{pmatrix} P^*$. Equivalently $\begin{pmatrix} C_2' & D_\mu \end{pmatrix} = \begin{pmatrix} PC_2 & PD_2 \end{pmatrix}$.

On the other hand, computing the (1,2) term, we get $A_2Y_2A_{21}^* + B_2(B_1')^* = 0$. By the same reasoning as before, there exists a matrix Q such that $\begin{pmatrix} A_{21} & B_1' \end{pmatrix} = Q \begin{pmatrix} C_2 & D_2 \end{pmatrix}$. Therefore

$$U = \left(\begin{array}{cc|c} A_2 & 0 & B_2 \\ QC_2 & A_1 & QD_2 \\ \hline PC_2 & C_1 & PD_2 \end{array}\right) = \left(\begin{array}{c|c} A_1 & Q \\ \hline C_1 & P \end{array}\right) \times \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$$

Now $U_2 = \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$ is square inner and hence, necessarily, also $\left(\begin{array}{c|c} A_1 & Q \\ \hline C_1 & P \end{array}\right)$ is square inner. By the uniqueness part of Theorem 2.1, there exists a unitary matrix V such that

$$\left(\begin{array}{c|c} A_1 & Q \\ \hline C_1 & P \end{array}\right) = \left(\begin{array}{c|c} A_1 & B_1V \\ \hline C_1 & D_1V \end{array}\right) = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \times V.$$

Thus

$$U = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \times \left(\begin{array}{c|c} A_2 & B_2 \\ \hline VC_2 & VD_2 \end{array}\right)$$

and both factors are square inner.

∎

Finally we want to compare the result obtained here with the continuous time case. In Fuhrmann [1994], a bijective correspondence between factorizations of an inner function U, invariant subspaces of the generator in any minimal realization of U and the set of nonnegative definite solutions of a homogeneous Riccati equation was established. As shown in Theorem 3.1, in discrete time the Riccati equation is replaced by the set of equations (20). However, there is a close relation between these two objects.

First, we note that if the asymptotically stable square inner function is given in terms of a realization $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, then the invertibility of A implies the invertibility of D. To see this we start from the equations

$$\begin{cases} Y = AYA^* + BB^* \\ BD^* + AYC^* = 0 \\ DD^* + CYC^* = I, \end{cases}$$

solved by $Y > 0$. Assuming A is invertible, we get $-YC^* = A^{-1}BD^*$ and hence $DD^* - I = -CYC^* = CA^{-1}BD^*$, that is $(D - CA^{-1}B)D^* = I$. This shows that D^*, and hence also D, are invertible.

If we assume that the realization $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is minimal, then the invertibility of D implies that of A. To see this, we can assume, without loss of generality, that $D = (I - CYC^*)^{\frac{1}{2}}$ and hence $B = -AYC^*(I - CYC^*)^{-\frac{1}{2}}$, which implies that $Y = AYA^* + AYC^*(I - CYC^*)^{-1}CYA^* = A[Y + YC^*(I - CYC^*)^{-1}CY]A^*$. This shows that A is invertible.

This leads, in this case, to an explicit parametrization of all left square inner factors of an asymptotically stable square inner function.

Proposition 3.2 *Let the asymptotically stable square inner function* $U = \left(\begin{array}{c|c} A & B_\mu \\ \hline C & D_\mu \end{array}\right)$ *be given in terms of a minimal realization. If A is invertible, then*

$$U_Y = \left(\begin{array}{c|c} A & -AYC^*(I - CYC^*)^{-\frac{1}{2}} \\ \hline C & (I - CYC^*)^{\frac{1}{2}} \end{array}\right) \tag{30}$$

with Y any nonnegative definite solution of the Riccati equation

$$Y = AYA^* + AYC^*(I - CYC^*)^{-1}CYA^* \tag{31}$$

such that $I - CYC^* > 0$, gives a parametrization of all left square inner factors of U.

Proof: We saw that the invertibility of A implies the invertibility of D. Thus, from

$$\begin{cases} Y = AYA^* + BB^* \\ BD^* + AYC^* = 0 \\ DD^* + CYC^* = I \end{cases}, \tag{32}$$

we conclude, without loss of generality, that $I - CYC^* > 0$ and $D = (I - CYC^*)^{\frac{1}{2}}$. Plugging this into the second equation yields

$$B = -AYC^*(I - CYC^*)^{-\frac{1}{2}},$$

and finally the first equation yields the Riccati equation (31).

Conversely, suppose Y solves (31) and $I - CYC^* > 0$. Then an easy calculation shows that Y, B, D with

$$B := -AYC^*(I - CYC^*)^{-\frac{1}{2}},$$
$$D := (I - CYC^*)^{\frac{1}{2}}$$

solve (32). ∎

Observe that D invertible corresponds to the case that U^{-1} is proper. Thus the solution of the Riccati equation (31) only gives those square inner factors which are proper invertible. However, if we start with A invertible, i.e. with an inner function with proper inverse, the solution of the Riccati equation gives *all* square inner factors. This shows that the Riccati equation only gives a complete solution to the factorization problem in the case when the inner function has no poles at the origin.

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