On balanced realizations of bounded real and positive real functions

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Abstract

Based on normalized coprime factorizations with respect to indefinite metrics and the construction of suitable characteristic functions, the Ober balanced canonical forms for the classes of bounded real and positive real are derived. This uses a matrix representation of the shift realization with respect to a basis related to sets of orthogonal polynomials.

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1 Introduction

In this paper we deal with the derivation of balanced realizations for the classes of scalar bounded real and positive real functions. This work is an extension of a method that has been initiated in [Fu 1991] and continued in [FO 1993a, Fu 1993a, HF 1994]. The central idea is to use the shift realization, introduced in [Fu 1976], and choose an appropriate basis for a matrix representation.

This has been done first in [Fu 1991] for the class of antistable transfer functions, considering the generic case where all Hankel singular values are distinct as well as the other extreme case where all the singular values coincide. In the first case, a basis made up of Hankel singular vectors led to the canonical form [Ob 1987a, Ob 1991]. On the other hand, for the case all the singular values coincide, which up to an additive constant is the case of an antistable all-pass function, the appropriate basis turned out to be a suitably normalized set of orthogonal polynomials. Again this led to a canonical form studied earlier in [1987b]. Naturally these results have their stable counterparts. We note that here balancing refers to Moore balancing, using a pair of Lyapunov equations.

The limitations on the multiplicity of the Hankel singular values has been lifted in [HF 1994] where an appropriate basis has been constructed, consisting of the union of sets of orthogonal polynomials with respect to weights determined by a special set of singular vectors.

The method, as described above, obviously fails if we consider other classes of functions. In order to apply the theory of Hankel operators to other types of balancing, we have to find a way of encoding the information carried by a transfer function in a given class into an antistable (or stable) transfer function so that an appropriate Hankel operator can be defined. This is the key idea introduced and studied in [FO 1993a]. The method is based on the use of (left and right) normalized coprime factorizations over H^∞, and the corresponding characteristic function.

Let us give briefly the details. Consider an arbitrary, strictly proper transfer function of McMillan degree n. Let $G = NM^{-1} = \overline{M}^{-1}\overline{N}$ be the (right and left) normalized coprime factorizations of G respectively. The normalization being $M^*M + N^*N = I$ and $\overline{M}\,\overline{M}^* + \overline{N}\,\overline{N}^* = I$. Coprimeness is equivalent to the solvability of the Bezout equations $\overline{V}M - \overline{U}N = I$ and $\overline{M}V - \overline{N}U = I$. Without loss of generality, via a minor modification, we can assume that we have the doubly coprime factorization

$$\begin{pmatrix} \overline{V} & -\overline{U} \\ -\overline{N} & \overline{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Given a particular solution $\begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$ of the Bezout equation, an arbitrary solution is given by

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q$$

for some $Q \in H^\infty_+$. This implies the existence of a special solution $\begin{pmatrix} U_L \\ V_L \end{pmatrix}$ for which $R_L^* = M^* U_L + N^* V_L \in H^\infty_-$ and is strictly proper. This gives us the required handle on Hankel theory. Moreover, the map from G to its characteristic function can be inverted. The inversion can be carried out via the solution of a pair of Lyapunov equations or, alternatively, via spectral factorizations.

The interesting thing in this construction is the fact that Moore, or Lyapunov, balancing for R_L^* is very closely related to LQG, or Riccati, balancing for G. Since, in the scalar case, for Lyapunov balancing we can use the shift realization with respect to a basis related to the Hankel operator with symbol R_L^*, this basis can be lifted to a basis in the natural realization space of G. With respect to the lifted basis, the shift realization is LQG balanced. The details can be found in [FO 1993a].

Now, in [FO 1993b], normalized coprime factorizations, with respect to indefinite metrics, have been derived also for the classes of bounded real and positive real functions. These factorizations are obtained via spectral factorization and they establish the connection with the Riccati equation. Moreover, the method of characteristic functions generalizes easily to the new context. A brief description of this can be found in [Fu 1994b]. Thus we have the B-characteristic and the P-characteristic at our disposal.

This leads us to the subject under study, namely the derivation of the balanced canonical forms of Ober, see [Ob 1991], for these two classes of functions, utilizing the method based on Hankel singular vectors. More precisely, we use normalized coprime factorizations to study bounded real balanced realizations. In [OMcF 1989] it was shown for general minimal systems that there is a close connection between the Lyapunov balanced realization of the normalized coprime factors and the LQG balanced realization of the transfer function itself. In the present paper, we study this problem for bounded real functions from an input–output point of view. Indeed, we examine the Hankel operator based on the normalized coprime factors, and the Hankel operator with the (adjoint) of the B–characteristic as its symbol. These operators share the same singular vectors, although with different singular values. Using the methods in [HF 1994], a Lypunov balanced realization of the B–characteristic is obtained. There is a natural lifting of the corresponding basis to the state–space for the shift realization of the original plant. With a suitable normalization, this leads to bounded real balancing. In fact, the bounded real singular values are exactly the Lyapunov singular values of the Hankel operator with the B–characteristic as its symbol. We remark that the positive real case is treated using the Cayley transform (see eg. [Bo 1974]). However, in [HF 1993] it has been indicated how the direct calculations analogous to the ones in this paper have to be done.

By and large, considering previous work described before, the results derived in this paper are expected. We find that, in spite of its highly technical nature, the paper may be of interest for three principal reasons. First, we obtain a characterization of the bounded real and positive real balanced canonical forms which is completely in input–output terms. Secondly, because the setting is special (i.e. the scalar case), all the computations can be carried out explicitly, leading to very concrete formulas. A close examination of these formulas may lead to appropriate generalizations in the multivariable case. An example of this type can be found in the derivation of optimal Nehari extensions of the normalized coprime factors, see Theorem 5.2.6.(c)-(d) in [Fu 1994a]. Finally, we find it of significance the way that indefinite metric spaces or, more specifically, Krein spaces come into the picture, as well as plus operators in these spaces. We feel that this is a direction that will see much progress in the future.

2 Preliminaries

We start with a very short introduction on Hankel operators with rational symbols. H^2_+ is the Hilbert space of all analytic functions in the open right half plane with

$$\|f\|^2 := \sup_{x>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mid f(x+iy) \mid dy < \infty.$$

The space H^2_- is defined analogously on the open left half plane. Moreover, the boundary functions exist, and hence $H^2_\pm$ can be considered as a closed subspace of $L^2(i\mathbb{R})$, the space of Lebesgue square integrable functions on the imaginary axis. It follows from Fourier–Plancherel and Paley–Wiener theorems that

$$L^2(i\mathbb{R}) = H^2_+ \oplus H^2_-.$$

Next, H^∞_+ and H^∞_- denote the spaces of bounded analytic functions on the open right and left half planes respectively. These spaces can be considered as subspaces of $L^\infty(i\mathbb{R})$, the space of Lebesgue measurable and essentially bounded functions on the imaginary axis.
As usual, given a matrix function $\phi \in L^\infty$ we define the *Hankel operator* H_ϕ by

$$H_\phi : \begin{array}{ccc} H^2_+ & \longrightarrow & H^2_- \\ f & \longrightarrow & P_-\phi f \end{array}$$

here P_- denotes the orthogonal projection of L^2 onto H^2_-. Similarly, the *involuted Hankel operator* $\hat{H}_\phi$ is defined by

$$\hat{H}_\phi : \begin{array}{ccc} H^2_- & \longrightarrow & H^2_+ \\ h & \longrightarrow & P_+\phi h \end{array}$$

Clearly we have $H^*_\phi = \hat{H}_{\phi^*}$, where $\phi^*(z) := \phi(-\bar{z})^*$.
In [Fu 1991], a detailed analysis of Hankel operators with proper rational, scalar, antistable symbol with special emphasis on the Schmidt structure was carried out. We recall that, given a bounded operator A on a Hilbert space, σ is a *singular value* of A if there exists a nonzero vector f such that

$$A^*Af = \sigma^2 f.$$

On can go over to the equivalent system

$$\begin{aligned} Af &= \sigma g \\ A^*g &= \sigma g \end{aligned}$$

by defining $g := \frac{1}{\sigma} Af$. (f, g) is called a *Schmidt pair* of A corresponding to σ. Now for the above mentioned function class it is shown in [Fu 1991] that all Schmidt pairs corresponding to a particular singular value of the associated Hankel operator can be obtained by multiplying the minimal (numerator) degree Schmidt vector, which is characterized as the solution of the so–called *fundamental polynomial equation* (as introduced in (1)), by polynomials of suitable degree. To be more specific, we present some of the results from the above mentioned paper in the following Proposition; the notation introduced is valid for the whole remaining paper.

Proposition 2.1 *Let* $\frac{r^*}{t^*} \in H^\infty_-$ *be a scalar, strictly proper transfer function, with* r *and* t *coprime polynomials and* t *monic of degree* n.

(i) We have:

(a) $\ker H_{\frac{r^*}{t^*}} = \frac{t^*}{t} H^2_+$

(b) $\{\ker H_{\frac{r^*}{t^*}}\}^\perp = X^t \; (= \{\frac{p}{t} : p \in \mathbb{R}[z], \deg p < \deg t\})$

(c) $\operatorname{im} H_{\frac{r^*}{t^*}} = X^{t^*}$

(ii) Assume that $\mu_1 > \mu_2 > \ldots > \mu_k > 0$ *are the singular values of* $H_{\frac{r^*}{t^*}}$, *where* μ_i *is of multiplicity* n_i, $i = 1, \ldots, k$ *and* $\sum_{i=1}^{k} n_i = n$.

(a) The set of all Schmidt pairs corresponding to μ_i *of* $H_{\frac{r^*}{t^*}}$ *is of the form*

$$\{\left(\frac{p_i a}{t}, \epsilon_i \frac{p_i^* \cdot a}{t^*}\right) : a \in \mathbb{R}[z], \deg a < n - \deg p_i\}.$$

Here p_i *is the unique (up to a constant factor) minimal degree solution of the* fundamental polynomial equation

$$r^* p_i = \lambda_i t p_i^* + t^* \pi_i, \tag{1}$$

where π_i *is polynomial and* $\lambda_i = \epsilon_i \mu_i$; $\epsilon_i = \pm 1$ *are the uniquely determined signs. We have* $\deg p_i = n - n_i$.

(b) There exist polynomials α_{ij} *of degree less than or equal to* $n - (n_i + n_j)$ *with the properties*

$$\alpha_{ij} = -\alpha_{ji}, \quad \alpha_{ii} = 0 \text{ for } 1 \le i, j \le k$$

such that

(1) $\lambda_i p_i^* p_j - \lambda_j p_j^* p_i = t^* \alpha_{ij}$

(2) If $i \neq j$, *then*

$$p_i p_j^* = \frac{1}{\lambda_i^2 - \lambda_j^2}\{\lambda_j t^* \alpha_{ij} + \lambda_i t \alpha_{ij}^*\} \quad \text{for } 1 \leq i,j \leq k \tag{2}$$

■

In [HF 1994], a balanced realization in canonical form as derived in [Ob 1987a] of a proper, antistable, rational transfer function is constructed as the matrix representation of the abstract shift realization introduced in [Fu 1976]. In the scalar case, given a proper rational function $g(z) = \frac{e(z)}{d(z)}$, the state–space for this realization is chosen to be X^d, and the parameters (A, B, C, D) of the realization are defined by

$$\begin{cases} A = S^d, & \\ B\xi = \pi_- \frac{e}{d} & \text{for } \xi \in \mathbb{R}, \\ Cf = (zf)(\infty) & \text{for } f \in X^d, \\ D = g(\infty); & \end{cases}$$

here π_- denotes the projection on the strictly proper part, and S^d is the *shift* on X^d defined by $S^d h := \pi_- zh$ for $h \in X^d$. The required basis of the state–space in [HF 1994] is constructed as a union of sets of polynomials orthogonal with respect to weights given by the square of the absolute values of minimal degree Schmidt vectors of the corresponding Hankel operator. [FO 1993a] obtain LQG-balanced realizations of arbitrary scalar rational functions g by examining the singular values/vectors of the Hankel operator associated to the normalized coprime factors of g.

3 Bounded real functions

In this section we recall some factorization results for bounded real functions from [Fu 1993b]. A rational $p \times m$ transfer function G is called *bounded real* if it is analytic in the right half plane and satisfies

$$I - G(i\omega)^* G(i\omega) > 0$$

for all $\omega \in \mathbf{R} \cup \{\pm\infty\}$. Let

$$J_B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

A representation $G = NM^{-1}$ with N, M stable proper rational transfer functions such that M^{-1} is proper and N, M are *right coprime*, i.e. there exist $\bar{U}$, $\bar{V} \in H_+^\infty$ such that $\bar{V}M - \bar{U}N = I$, is called a *right coprime factorization*. Such a right coprime factorization $G = NM^{-1}$ is called a J_B*-NRCF* of G if

$$\begin{pmatrix} M^* & N^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = I. \tag{3}$$

Similarly as above, a left coprime factorization $G = \tilde{M}^{-1}\tilde{N}$ is defined and is called a J_B*-NLCF* of G if

$$\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} = I.$$

Observe that we have

$$\begin{pmatrix} M^* & N^* \\ \tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} M & \tilde{N}^* \\ N & \tilde{M}^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{4}$$

Now in the solution set of the Bezout equation

$$\tilde{M}V - \tilde{N}U = I \tag{5}$$

there is a uniquely determined $(U_B, V_B) \in H_+^\infty$ such that

$$R_B^* := M^* U_B - N^* V_B$$

is in H_-^∞ and strictly proper. R_B is called the *bounded real characteristic* (*B-characteristic*) of G. Equivalently, R_B^* can be constructed by calculating the unique solution $(\bar{U}_B, \bar{V}_B) \in H_+^\infty$ to the Bezout equation

$$\bar{V}M - \bar{U}N = I \tag{6}$$

which is such that

$$\bar{U}_B \tilde{M}^* - \bar{V}_B \tilde{N}^*$$

is in H_-^∞ and strictly proper. As R_B^* is rational and in H_-^∞, it has coprime *Douglas–Shapiro–Shields* factorizations

$$R_B^* = \Phi_K S_K^* = S_L^* \Phi_L \tag{7}$$

with S_K and S_L inner functions in H_+^∞ (cf. e.g. [Fu 1981]). Hence

$$\ker H_{R_B^*} = S_K H_+^2$$

and

$$\operatorname{im} H_{R_B^*} = \{S_L H_-^2\}^\perp = H_-^2 \ominus S_L^* H_-^2.$$

Moreover, S_K is the *minimal right inner function* of $\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}$, i.e. there holds $S_K \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} \in H_+^\infty$ and S_K is a right divisor of any inner function T for which $T \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} \in H_+^\infty$. Similarly, S_L is the minimal left inner function of $\begin{pmatrix} M^* & N^* \end{pmatrix}$. Thus it is easily seen that

$$\ker H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} = S_K H_+^2. \tag{8}$$

Given the $p \times m$ bounded real function G with J_B–normalized coprime factorizations as above, we define

$$\begin{pmatrix} K_1 & K_2 \end{pmatrix} := S_L \begin{pmatrix} N^* & M^* \end{pmatrix} \tag{9}$$

and set

$$\Omega_L := \begin{pmatrix} \tilde{M} & \tilde{N} \\ K_1 & K_2 \end{pmatrix}. \tag{10}$$

Next, we consider the H^2_- space of $\mathbb{C}^{p+m}$–valued functions. On this space we define a new, indefinite inner product by letting

$$[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}] = (J_B \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}) = (f_1, g_1) - (f_2, g_2)\,. \qquad (11)$$

We denote this space by $H^{2,J}_-$. Observe that $H^{2,J}_-$ is a *Krein space*; we refer to [Bo 1974] for an introduction into the general theory of these spaces.
We proceed by defining the map

$$P_{\{\Omega_L^* H_-^{2,J}\}^{[\perp]}} h = \Omega_L^* J_B P_+ \Omega_L J_B h \qquad (12)$$

for $h \in H^{2,J}_-$; here $[\perp]$ denotes the orthogonal complement in the Krein space metric. It can be shown that this map is a projection in $H^{2,J}_-$ with

$$\operatorname{im} P_{\{\Omega_L^* H_-^{2,J}\}^{[\perp]}} = \{\Omega_L^* H_-^{2,J}\}^{[\perp]}. \qquad (13)$$

Given a linear transformation T in a Krein space, we define its adjoint with respect to the indefinite metric as the unique transformation $T^{\#}$ satisfying, for all x, y,

$$[Tx, y] = [x, T^{\#} y].$$

Next, we study the maps

$$Z_L : \begin{array}{ccc} \{S_L^* H_-^2\}^{\perp} & \longrightarrow & \{\Omega_L^* H_-^{2,J}\}^{[\perp]} \\ h & \longrightarrow & P_{\{\Omega_L^* H_-^{2,J}\}^{[\perp]}} \begin{pmatrix} \bar{U}_B^* \\ \bar{V}_B^* \end{pmatrix} h \end{array} \qquad (14)$$

and

$$Y_L : \begin{array}{ccc} \{\Omega_L^* H_-^{2,J}\}^{[\perp]} & \longrightarrow & \{S_L^* H_-^2\}^{\perp} \\ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} & \longrightarrow & -P_{\{S_L^* H_+^2\}^{\perp}} \begin{pmatrix} N^* & M^* \end{pmatrix} J_B \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \end{array} \qquad (15)$$

The indefinite adjoint of Y_L is given by

$$Y_L^{\#} h = -P_{\{\Omega_L^* H_-^{2,J}\}^{[\perp]}} P_- \begin{pmatrix} N \\ M \end{pmatrix} h \qquad (16)$$

for $h \in \{S_L^* H_-^2\}^{\perp}$. The reason for examining these functions in detail is that they allow us to derive important results concerning the Hankel operators associated to the J_B–coprime factors. To be more specific, it has been shown that

$$\hat{H}_{R_B} = \hat{H}_{\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} J_B} Z_L, \qquad (17)$$

$$H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} = Y_L^{\#} H_{R_B^*} \qquad (18)$$

and

$$\ker \hat{H}_{\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} J_B} = \Omega_L^* H_-^{2,J}. \tag{19}$$

Finally, it can be shown that the McMillan degree of R_B^* is equal to the McMillan degree of G. We like to mention that by investigating dual operators for the minimal right inner function S_K, in [Fu 1993b] there is obtained a key commutative diagram relating the different Hankel operators.
The previous analysis can be used to prove the following relation, which has implications as far as balancing is concerned and hence is central for the present paper.

Theorem 3.1 *Let G be a rational, proper bounded real function, and let R_B be its $B-$characteristic. Let $1 > \mu_1 \geq \ldots \geq \mu_n > 0$ be the singular values of $H_{R_B^*}$, and let $\{f_i, h_i\}$ the corresponding Schmidt pairs. Additionally, set*

$$k_i := (1 - \mu_i^2)^{-\frac{1}{2}} Z_L h_i \in \{\Omega_L^* H_-^{2,J}\}^{[\perp]}. \tag{20}$$

Then

$$H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} f_i = -\mu_i (1 - \mu_i^2)^{-\frac{1}{2}} k_i. \tag{21}$$

■

With the help of this result it is possible to investigate the singular values and singular vectors of the operator $H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}}$.

Corollary 3.1 *Under the same assumptions as in Theorem 3.1 there holds*

$$H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} f_i = -\mu_i (1 - \mu_i^2)^{-\frac{1}{2}} k_i \tag{22}$$

and

$$H^*_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} J_B} k_i = \mu_i (1 - \mu_i^2)^{-\frac{1}{2}} f_i \tag{23}$$

for $i = 1, \ldots, n$, i.e. $\{f_i, k_i\}$ are the Schmidt pairs (up to a sign) of the operator $H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}}$ corresponding to the singular values

$$\sigma_i = \mu_i (1 - \mu_i^2)^{-\frac{1}{2}}. \tag{24}$$

Proof:
Relation (22) is equal to (21). Furthermore, by definition of k_i we get

$$H^*_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} J_B} k_i = \hat{H}_{\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} J_B} k_i = (1 - \mu_i^2)^{-\frac{1}{2}} \hat{H}_{\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} J_B} Z_L h_i.$$

By (17) this can be rewritten as

$$H^*_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} J_B} k_i = (1-\mu_i^2)^{-\frac{1}{2}} \hat{H}_{R_B} h_i = (1-\mu_i^2)^{-\frac{1}{2}} H^*_{R_B^*} h_i = \mu_i (1-\mu_i^2)^{-\frac{1}{2}} f_i.$$

■

Observe that in the indefinite metric we have for $y \in H_-^{2,J}$, $x \in H_+^2$ that

$$\begin{aligned} [H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} x \,,\, y] &= (J_B H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} x \,,\, y) \\ &= (H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} x \,,\, J_B y) \\ &= (x \,,\, H^*_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} J_B y) \\ &= (x \,,\, \hat{H}_{\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} J_B} y), \end{aligned} \tag{25}$$

i.e. there holds

$$H^{\#}_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} = \hat{H}_{\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} J_B}.$$

4 Analysis of Hankel singular vectors

Our aim is the examination of the Hankel operator $H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}}$ corresponding to a J_B-NLCF, $g = \tilde{M}^{-1}\tilde{N}$, of the bounded real function $g = \frac{e}{d}$, where e and d are coprime polynomials, with d monic of degree n. To this end we will heavily use the relationship between $H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}}$ and $H_{R_B^*}$ as established in Theorem 3.1. Consider the polynomial spectral factorization

$$tt^* = dd^* - ee^* \tag{26}$$

respectively

$$\left(\frac{d}{t}\right)\left(\frac{d}{t}\right)^* - \left(\frac{e}{t}\right)\left(\frac{e}{t}\right)^* = 1, \tag{27}$$

where t is stable and monic. In terms of J_B-NCF this corresponds to

$$\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} d/t \\ e/t \end{pmatrix} \tag{28}$$

and

$$\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} = \begin{pmatrix} \frac{d}{t} & \frac{e}{t} \end{pmatrix}. \tag{29}$$

The associated B-characteristic is of the form $R_B = \frac{r}{t}$, $r \wedge t = 1$, as can be easily seen from the definition of R_B^*. Indeed, let $U_B = \frac{n_1}{d_1}$ and $V_B = \frac{n_2}{d_2}$ be polynomial coprime factorizations; then we obtain

$$R_B^* = M^* U_B - N^* V_B = \frac{d^* n_1 d_2 - e^* n_2 d_1}{t^* d_1 d_2} \in H_-^\infty.$$

However, U_B and V_B are H_+^∞–functions, and hence d_1 and d_2 are stable polynomials. This means that $d_1 d_2 \mid (d^* n_1 d_2 - e^* n_2 d_1)$, and consequently $R_B^* = \frac{r^*}{t^*}$ for $r^* := \frac{d^* n_1 d_2 - e^* n_2 d_1}{d_1 d_2}$. Finally, r and t are coprime, since the McMillan degree of R_B is equal to the McMillan degree of g, which is $\deg d = \deg t$. By (19) we have

$$\ker \hat{H}_{\left(\tilde{M} \quad \tilde{N} \right) J_B} = \Omega_L^* H_-^{2,J} = \begin{pmatrix} \tilde{M}^* & K_1^* \\ \tilde{N}^* & K_2^* \end{pmatrix} H_-^{2,J} \supseteq \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} H_-^2. \tag{30}$$

Hence $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \{\ker \hat{H}_{\left(\tilde{M} \quad \tilde{N} \right) J_B}\}^{[\perp]}$ implies that $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} [\perp] \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} g$ for all $g \in H_-^2$, which means $\begin{pmatrix} f_1 \\ -f_2 \end{pmatrix} \perp \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} g$ or $(\tilde{M} f_1 - \tilde{N} f_2) \perp g$ for all $g \in H_-^2$. The last expression shows that $(\tilde{M} f_1 - \tilde{N} f_2) \in H_+^2$. Thus

$$\begin{aligned} \hat{H}_{\left(\tilde{M} \quad \tilde{N} \right) J_B} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= P_+ \left(\tilde{M} \quad \tilde{N} \right) J_B \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= P_+ \left(\tilde{M} \quad -\tilde{N} \right) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= P_+ (\tilde{M} f_1 - \tilde{N} f_2) = (\tilde{M} f_1 - \tilde{N} f_2) \end{aligned}$$

for all $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \{\ker \hat{H}_{\left(\tilde{M} \quad \tilde{N} \right) J_B}\}^{[\perp]}$, i.e. the restriction of the operator $\hat{H}_{\left(\tilde{M} \quad \tilde{N} \right) J_B}$ to $\{\ker \hat{H}_{\left(\tilde{M} \quad \tilde{N} \right) J_B}\}^{[\perp]}$ acts by multiplication by $\left(\tilde{M} \quad \tilde{N} \right) J_B$. Hence the equations (22) and (23) in view of Proposition 2.1, (i) now read

$$H_{\begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix}} \frac{q_i}{t} = P_- \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} \frac{q_i}{t} = -\sigma_i \begin{pmatrix} \hat{q}_1^{(i)} / t^* \\ \hat{q}_2^{(i)} / t^* \end{pmatrix} \tag{31a}$$

$$\hat{H}_{\left(\tilde{M} \quad \tilde{N} \right) J_B} \begin{pmatrix} \hat{q}_1^{(i)} / t^* \\ \hat{q}_2^{(i)} / t^* \end{pmatrix} = \left(\tilde{M} \quad \tilde{N} \right) J_B \begin{pmatrix} \hat{q}_1^{(i)} / t^* \\ \hat{q}_2^{(i)} / t^* \end{pmatrix} = \sigma_i \frac{q_i}{t} \tag{31b}$$

By partial fraction decomposition there exist polynomials $\rho_1^{(i)}, \rho_2^{(i)}$ of degree at most $n-1$ such that

$$\begin{pmatrix} d^*/t^* \\ e^*/t^* \end{pmatrix} \frac{q_i}{t} = -\sigma_i \begin{pmatrix} q_1^{(i)} / t^* \\ q_2^{(i)} / t^* \end{pmatrix} + \begin{pmatrix} \rho_1^{(i)} / t \\ \rho_2^{(i)} / t \end{pmatrix} \tag{32a}$$

$$\begin{pmatrix} \frac{d}{t} & -\frac{e}{t} \end{pmatrix} \begin{pmatrix} \hat{q}_1^{(i)}/t^* \\ \hat{q}_2^{(i)}/t^* \end{pmatrix} = \sigma_i \frac{q_i}{t} \tag{32b}$$

respectively

$$\begin{pmatrix} d^* \\ e^* \end{pmatrix} q_i = -\sigma_i t \begin{pmatrix} \hat{q}_1^{(i)} \\ \hat{q}_2^{(i)} \end{pmatrix} + t^* \begin{pmatrix} \rho_1^{(i)} \\ \rho_2^{(i)} \end{pmatrix} \tag{33a}$$

$$d\hat{q}_1^{(i)} - e\hat{q}_2^{(i)} = \sigma_i t^* q_i \tag{33b}$$

In the sequel we will now further examine this relation; we fix the following additional notation: If we regard the solution pair $\{\frac{p_i}{t}, \begin{pmatrix} \hat{p}_1^{(i)}/t^* \\ \hat{p}_2^{(i)}/t^* \end{pmatrix}\}$ of (32), then the remainder polynomials are denoted by $\begin{pmatrix} \pi_1^{(i)} \\ \pi_2^{(i)} \end{pmatrix}$.

First note that

$$q_i q_i^* = \hat{q}_2^{(i)}(\hat{q}_2^{(i)})^* - \hat{q}_1^{(i)}(\hat{q}_1^{(i)})^* \tag{34}$$

for any solution of (32), which can be seen as follows. Taking the adjoints on both sides in equation (33a) yields

$$q_i^* \begin{pmatrix} d & e \end{pmatrix} = -\sigma_i t^* \begin{pmatrix} (\hat{q}_1^{(i)})^* & (\hat{q}_2^{(i)})^* \end{pmatrix} + t \begin{pmatrix} (\rho_1^{(i)})^* & (\rho_2^{(i)})^* \end{pmatrix}. \tag{35}$$

Multiplying this equation on the right by $\begin{pmatrix} \hat{q}_1^{(i)} \\ -\hat{q}_2^{(i)} \end{pmatrix}$, multiplying (33b) by q_i^* and substracting results in

$$0 = \sigma_i t^* \{q_i q_i^* + (\hat{q}_1^{(i)})^* \hat{q}_1^{(i)} - (\hat{q}_2^{(i)})^* \hat{q}_2^{(i)}\} - t\{(\rho_1^{(i)})^* \hat{q}_1^{(i)} - (\rho_2^{(i)})^* \hat{q}_2^{(i)}\}. \tag{36}$$

Therefore, as t and t^* are coprime, we have $t \mid (q_i q_i^* + (\hat{q}_1^{(i)})^* \hat{q}_1^{(i)} - (\hat{q}_2^{(i)})^* \hat{q}_2^{(i)})$, and by symmetry also $t^* \mid (q_i q_i^* + (\hat{q}_1^{(i)})^* \hat{q}_1^{(i)} - (\hat{q}_2^{(i)})^* \hat{q}_2^{(i)})$. Hence we conclude by a degree argument that $q_i q_i^* + (\hat{q}_1^{(i)})^* \hat{q}_1^{(i)} - (\hat{q}_2^{(i)})^* \hat{q}_2^{(i)} = 0$, i.e. (34) holds true. Moreover, we have the following characterization of all Schmidt pairs that correspond to a particular singular value.

Lemma 4.1 *Let $\frac{p_j}{t}$ a minimal degree Schmidt vector of $H_{\frac{r^*}{t^*}}$ for the singular value μ_j, $j \in \{1, \ldots, k\}$ as introduced in Proposition 2.1, and let $\{\frac{p_j}{t}, \begin{pmatrix} \hat{p}_1^{(j)}/t^* \\ \hat{p}_2^{(j)}/t^* \end{pmatrix}\}$ be a corresponding solution of (32) for $\sigma_j = \mu_j(1-\mu_j^2)^{-\frac{1}{2}}$ (see (24)). Then all solutions $\{\frac{q_j}{t}, \begin{pmatrix} \hat{q}_1^{(j)}/t^* \\ \hat{q}_2^{(j)}/t^* \end{pmatrix}\}$ of (32) with respect to σ_j are of the form*

$$q_j = p_j \cdot a$$

$$\begin{pmatrix} \hat{q}_1^{(j)} \\ \hat{q}_2^{(j)} \end{pmatrix} = \begin{pmatrix} \hat{p}_1^{(j)} \\ \hat{p}_2^{(j)} \end{pmatrix} a \tag{37}$$

with $a \in \mathbb{R}[z]$ such that $\deg a < \deg t - \deg p_j$.

Proof:
By Proposition 2.1, (ii) (a) we have $q_j = p_j a$ for some polynomial a of degree $< n - \deg p_j$. Substract equation (33a) for p_j multiplied by a from equation (33a) for q_j; because of $q_j = p_j \cdot a$ this results in

$$0 = -\sigma_j t \begin{pmatrix} \hat{q}_1^{(j)} - a\hat{p}_1^{(j)} \\ \hat{q}_2^{(j)} - a\hat{p}_2^{(j)} \end{pmatrix} + t^* \begin{pmatrix} \rho_1^{(j)} - a\pi_1^{(j)} \\ \rho_2^{(j)} - a\pi_2^{(j)} \end{pmatrix}.$$

Since t and t^* are coprime and $\deg\left(\hat{q}_i^{(j)} - a\hat{p}_i^{(j)}\right) < n$, $i = 1, 2$ it follows that $\hat{q}_i^{(j)} = a\hat{p}_i^{(j)}$ and $\rho_i^{(j)} = a\pi_i^{(j)}$, $i = 1, 2$. ∎

Schmidt vectors corresponding to different singular values are orthogonal, as is shown in the next Lemma.

Lemma 4.2 *Let* $\{\frac{q_k}{t}, \begin{pmatrix} \hat{q}_1^{(k)}/t^* \\ \hat{q}_2^{(k)}/t^* \end{pmatrix}\}$ *resp.* $\{\frac{q_j}{t}, \begin{pmatrix} \hat{q}_1^{(j)}/t^* \\ \hat{q}_2^{(j)}/t^* \end{pmatrix}\}$ *be solutions of (32) corresponding to* σ_k *resp.* σ_j*, and let* $\sigma_k \neq \sigma_j$*. Then* $(\frac{q_k}{t}, \frac{q_j}{t}) = 0$ *and*

$$[\begin{pmatrix} \hat{q}_1^{(k)}/t^* \\ \hat{q}_2^{(k)}/t^* \end{pmatrix}, \begin{pmatrix} \hat{q}_1^{(j)}/t^* \\ \hat{q}_2^{(j)}/t^* \end{pmatrix}] = 0.$$

Proof:
The first statement is clear, since by Theorem 3.1 $\frac{q_k}{t}$ and $\frac{q_j}{t}$ are Schmidt vectors of $H_{\frac{r^*}{t^*}}$ corresponding to $\mu_k = \sigma_k(1+\sigma_k^2)^{-\frac{1}{2}}$ respectively $\mu_j = \sigma_j(1+\sigma_j^2)^{-\frac{1}{2}}$ (see (24)) and $\mu_k \neq \mu_j$. Multiplying

$$\begin{pmatrix} d^* \\ e^* \end{pmatrix} q_j = -\sigma_j t \begin{pmatrix} \hat{q}_1^{(j)} \\ \hat{q}_2^{(j)} \end{pmatrix} + t^* \begin{pmatrix} \rho_1^{(j)} \\ \rho_2^{(j)} \end{pmatrix}$$

by $\begin{pmatrix} (\hat{q}_1^{(k)})^* & -(\hat{q}_2^{(k)})^* \end{pmatrix}$ results in

$$\sigma_k t q_k^* q_j = -\sigma_j t\{(\hat{q}_1^{(k)})^*\hat{q}_1^{(j)} - (\hat{q}_2^{(k)})^*\hat{q}_2^{(j)}\} + t^*\{(\hat{q}_1^{(k)})^*\rho_1^{(j)} - (\hat{q}_2^{(k)})^*\rho_2^{(j)}\}. \tag{38}$$

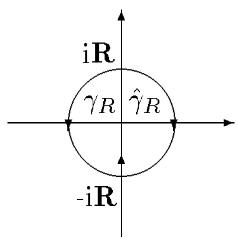
Since $t \wedge t^* = 1$, t divides $(\hat{q}_1^{(k)})^*\rho_1^{(j)} - (\hat{q}_2^{(k)})^*\rho_2^{(j)}$. Thus there exist polynomials a_{jk} such that

$$(\hat{q}_1^{(k)})^*\rho_1^{(j)} - (\hat{q}_2^{(k)})^*\rho_2^{(j)} = a_{jk}t.$$

We divide (38) by t^2t^* to get

$$\sigma_k \frac{q_k^* q_j}{tt^*} = -\sigma_j \frac{(\hat{q}_1^{(k)})^*\hat{q}_1^{(j)} - (\hat{q}_2^{(k)})^*\hat{q}_2^{(j)}}{tt^*} + \frac{a_{jk}}{t}.$$

Integrating this expression over the semicircular contour $\hat{\gamma}_R$



and taking the limit as $R \to \infty$ leads because of the stability of t to

$$0 = \sigma_k \int_{-\infty}^{\infty} \frac{q_k^* q_j}{tt^*} d\tau = -\sigma_j \int_{-\infty}^{\infty} \frac{(\hat{q}_1^{(k)})^* \hat{q}_1^{(j)} - (\hat{q}_2^{(k)})^* \hat{q}_2^{(j)}}{tt^*} d\tau =$$

$$= -\sigma_j \left(\begin{pmatrix} \hat{q}_1^{(k)}/t^* \\ -\hat{q}_2^{(k)}/t^* \end{pmatrix}, \begin{pmatrix} \hat{q}_1^{(j)}/t^* \\ \hat{q}_2^{(j)}/t^* \end{pmatrix} \right) = -\sigma_j \left[\begin{pmatrix} \hat{q}_1^{(k)}/t^* \\ \hat{q}_2^{(k)}/t^* \end{pmatrix}, \begin{pmatrix} \hat{q}_1^{(j)}/t^* \\ \hat{q}_2^{(j)}/t^* \end{pmatrix} \right].$$

■

Before we proceed with the analysis of singular vectors, we remind the reader of the following facts from AAK theory; an elemtary proof can be found in [Fu 1993a].

Theorem 4.1 (Adamjan, Arov and Krein) *Let $\Phi = \frac{e}{d} \in H_-^\infty$ be a scalar, strictly proper transfer function, with e and d coprime polynomials, and d monic of degree n. Assume that $\mu_1 \geq \ldots \geq \mu_{k-1} > \mu_k = \ldots = \mu_{k+\nu-1} > \mu_{k+\nu} \geq \ldots \geq \mu_n > 0$ are the singular values of H_Φ.*

(i) Let p_k be the minimum degree solution of the fundamental polynomial equation (compare (1)) corresponding to μ_k. Then p_k has exactly $k-1$ antistable zeroes.

(ii) There holds

$$\mu_k = \inf \{ \|\Phi - \psi\|_\infty \ : \ \psi \in H_{[k-1]}^\infty \},$$

where $H_{[k-1]}^\infty$ denotes the set of functions such that the McMillan degree of the unstable part is at most k. Moreover, the infimum is attained for a uniquely determined function $\psi_k \in H_{[k-1]}^\infty$.

■

Theorem 4.2 *Let $g = \frac{e}{d}$ be bounded real with J_B-normalized coprime factors $\tilde{M} = \frac{d}{t}$, $\tilde{N} = \frac{e}{t}$ as before. Furthermore, let $\{(\frac{p_i}{t}, \epsilon_i \frac{p_i^*}{t^*})\}_{i=1}^k$ the minimal degree Schmidt pairs of $H_{R_B^*}$ as introduced in Proposition 2.1. Then for $i = 1, \ldots, k$:*

$$(i) \qquad d\pi_1^{(i)} - e\pi_2^{(i)} = (1+\sigma_i^2)tp_i \tag{39}$$

$$(ii) \qquad (\pi_1^{(i)})^*\hat{p}_1^{(i)} - (\pi_2^{(i)})^*\hat{p}_2^{(i)} = 0 \tag{40}$$

$$(iii) \qquad \left(\frac{1}{(1+\sigma_i^2)^{1/2}}\begin{pmatrix}\pi_1^{(i)}/p_i \\ \pi_2^{(i)}/p_i\end{pmatrix}\right)^* J_B \left(\frac{1}{(1+\sigma_i^2)^{1/2}}\begin{pmatrix}\pi_1^{(i)}/p_i \\ \pi_2^{(i)}/p_i\end{pmatrix}\right) = 1$$

$$(iv) \qquad d^*\pi_2^{(i)} - e^*\pi_1^{(i)} = \epsilon_i\sigma_i\sqrt{1+\sigma_i^2}\, tp_i^* \tag{41}$$

Proof:

(i) Multiplying (33a) from the left by $\begin{pmatrix} d & -e \end{pmatrix}$ we obtain due to (26) that

$$tt^*p_i = -\sigma_i t(d\hat{p}_1^{(i)} - e\hat{p}_2^{(i)}) + t^*(d\pi_1^{(i)} - e\pi_2^{(i)}).$$

Using (33b) gives (39).

(ii) Take the adjoint of (33a) and multiply it on the right by $\begin{pmatrix}\hat{p}_1^{(i)} \\ \hat{p}_2^{(i)}\end{pmatrix}$, multiply (33b) by p^* and substract; this yields

$$0 = \sigma_i t^*\{p_ip_i^* + \hat{p}_1^{(i)}(\hat{p}_1^{(i)})^* - \hat{p}_2^{(i)}(\hat{p}_2^{(i)})^*\} - t\{(\pi_1^{(i)})^*\hat{p}_1^{(i)} - (\pi_2^{(i)})^*\hat{p}_2^{(i)}\}.$$

Hence (40) follows from (34).

(iii) Multiply (33a) on the left by $\begin{pmatrix} (\pi_1^{(i)})^* & -(\pi_2^{(i)})^* \end{pmatrix}$. Then because of (40) there holds

$$\{d^*(\pi_1^{(i)})^* - e^*(\pi_2^{(i)})^*\}p_i = t^*\{(\pi_1^{(i)})^*\pi_1^{(i)} - (\pi_2^{(i)})^*\pi_2^{(i)}\}.$$

In view of (39) this reduces to

$$(1+\sigma_i^2)p_i^*p_i = (\pi_1^{(i)})^*\pi_1^{(i)} - (\pi_2^{(i)})^*\pi_2^{(i)};$$

this is statement (iii).

(iv) For $i = 1$ equation (39) says that

$$\begin{pmatrix} V_1 \\ U_1 \end{pmatrix} := \frac{1}{1+\sigma_1^2}\begin{pmatrix}\pi_1^{(1)}/p_1 \\ \pi_2^{(1)}/p_1\end{pmatrix} \in H_+^\infty$$

is a solution to the H_+^∞-Bezout equation $\tilde{M}V - \tilde{N}U = I$; observe that by Theorem 4.1, (i) p_1 is stable. The Kucera-Youla parametrization (see [Ku 1979], [YBJ 1976]) states that all solutions to this equation are of the form

$$\begin{pmatrix} V \\ U \end{pmatrix} = \begin{pmatrix} V_1 \\ U_1 \end{pmatrix} + \begin{pmatrix} N \\ M \end{pmatrix} h,$$

where $h \in H_+^\infty$. Hence in view of (26)

$$M^*U - N^*V = \frac{1}{1+\sigma_1^2}\{\frac{d^*\pi_2^{(1)} - e^*\pi_1^{(1)}}{t^*p_1}\} + h. \tag{42}$$

To get the bounded real characteristic we choose $h \in H_+^\infty$ so that

$$\frac{r^*}{t^*} = \frac{1}{1+\sigma_1^2}\{\frac{d^*\pi_2^{(1)} - e^*\pi_1^{(1)}}{t^*p_1}\} + h. \tag{43}$$

First, multiply equation (33a) for p_1 on the left by $\begin{pmatrix} -e^* & d^* \end{pmatrix}$ to obtain

$$0 = -\sigma_1 t\{d^*\hat{p}_2^{(1)} - e^*\hat{p}_1^{(1)}\} + t^*\{d^*\pi_2^{(1)} - e^*\pi_1^{(1)}\}. \tag{44}$$

Thus, because of the coprimeness of t and t^*, there exists a polynomial x_1 such that

$$d^*\pi_2^{(1)} - e^*\pi_1^{(1)} = x_1 t.$$

If we substitute this into equation (43), we get that

$$\frac{r^*p_1}{t^*t} = \frac{1}{1+\sigma_1^2}\frac{x_1}{t^*} + h\frac{p_1}{t}.$$

Applying the projection P_- to this equality and recalling that $\{\frac{p_1}{t}, \epsilon_1\frac{p_1^*}{t^*}\}$ is a Schmidt pair of $H_{R_B^*}$ corresponding to the singular value μ_1 we get

$$H_{\frac{r^*}{t^*}}\frac{p_1}{t} = \frac{1}{1+\sigma_1^2}\frac{x_1}{t^*} = \epsilon_1\mu_1\frac{p_1^*}{t^*}.$$

This implies the equality

$$x_1 = \epsilon_1\mu_1(1+\sigma_1^2)p_1^*. \tag{45}$$

Substituting this relation in (43) yields

$$\frac{r^*}{t^*} = \epsilon_1\mu_1\frac{p_1^*t}{t^*p_1} + h.$$

Comparison with the equation

$$\frac{r^*}{t^*} = \epsilon_1\mu_1\frac{p_1^*t}{t^*p_1} + \frac{\pi_1}{p_1}$$

leads to $h = \frac{\pi_1}{p_1}$. Now by (24) there holds

$$\sigma_1 = \mu_1(1-\mu_1^2)^{-\frac{1}{2}}.$$

Solving this for μ_1 results in

$$\mu_1 = \sigma_1(1+\sigma_1^2)^{-\frac{1}{2}}.$$

Hence (45) reads

$$x_1 = \epsilon_1 \sigma_1 \sqrt{1+\sigma_1^2}\, p_1^*,$$

and (41) follows for the case $i = 1$.

For $i > 1$ equation (39) gives the solution

$$\begin{pmatrix} V_i \\ U_i \end{pmatrix} := \frac{1}{1+\sigma_i^2} \begin{pmatrix} \pi_1^{(i)}/p_i \\ \pi_2^{(i)}/p_i \end{pmatrix}$$

of the Bezout equation $\tilde{M}V - \tilde{N}U = I$. Observe that again by Theorem 4.1, (i) $\begin{pmatrix} V_i \\ U_i \end{pmatrix} \in H^\infty_{[n^i]}$, i.e. its unstable part has at most McMillan degree $n^i := \sum_{j<i} n_j$. All solutions $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ of the Bezout equation in $H^\infty_{[n^i]}$ are of the form

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} V_1 \\ U_1 \end{pmatrix} - \begin{pmatrix} N \\ M \end{pmatrix} q$$

with $q \in H^\infty_{[n^i]}$. Out of this set of solutions we now determine a unique solution which is optimal in a certain sense. For that purpose we interpret $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ as a mapping from L^2 to the Krein space $L^2 \oplus L^2$. Following [Fr 1987] we call a mapping $T : L^2 \oplus L^2 \to L^2 \oplus L^2$ J_B-*unitary* if $T^* J_B T = J_B$; observe that $[Tx, Ty] = [x, y]$ holds for these operators. Now, starting from equation (4) one can calculate that $\begin{pmatrix} \tilde{M} & -\tilde{N} \\ -N^* & M^* \end{pmatrix}$ is J_B-unitary. Moreover,

$$\begin{aligned}
\begin{pmatrix} \tilde{M} & -\tilde{N} \\ -N^* & M^* \end{pmatrix} \begin{pmatrix} X_i \\ Y_i \end{pmatrix} &= \begin{pmatrix} \tilde{M} & -\tilde{N} \\ -N^* & M^* \end{pmatrix} \left\{ \begin{pmatrix} V_1 \\ U_1 \end{pmatrix} - \begin{pmatrix} N \\ M \end{pmatrix} q \right\} \\
&= \begin{pmatrix} \tilde{M}V_1 - \tilde{N}U_1 \\ M^*U_1 - N^*V_1 \end{pmatrix} - \begin{pmatrix} \tilde{M}N - \tilde{N}M \\ M^*M - N^*N \end{pmatrix} q \\
&= \begin{pmatrix} 1 \\ M^*U_1 - N^*V_1 - q \end{pmatrix}.
\end{aligned}$$

Hence for all $x \in L^2$ and $q \in H^\infty_{[n^i]}$ we get

$$\begin{aligned}
[\begin{pmatrix} X_i \\ Y_i \end{pmatrix} x, \begin{pmatrix} X_i \\ Y_i \end{pmatrix} x] &= [\begin{pmatrix} x \\ (M^*U_1 - N^*V_1 - q)x \end{pmatrix}, \begin{pmatrix} x \\ (M^*U_1 - N^*V_1 - q)x \end{pmatrix}] \\
&= (x, x) - \| (M^*U_1 - N^*V_1 - q)x \|_2^2 \qquad (46) \\
&\geq \left(1 - \| M^*U_1 - N^*V_1 - q \|_\infty^2\right) \cdot (x, x).
\end{aligned}$$

Thus for $0 \leq \| M^*U_1 - N^*V_1 - q \|_\infty^2 < 1$ the mapping $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ is a *plus-operator* (see [Bo 1974]), i.e. for all $y \in \text{im} \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ there holds $[y, y] \geq 0$. Furthermore, (46) also gives

$$\begin{aligned} \inf_{(x,x)=1} [\begin{pmatrix} X_i \\ Y_i \end{pmatrix} x, \begin{pmatrix} X_i \\ Y_i \end{pmatrix} x] &= 1 - \sup_{(x,x)=1} \| (M^*U_1 - N^*V_1 - q)x \|_2^2 \\ &= 1 - \| M^*U_1 - N^*V_1 - q \|_\infty^2 . \end{aligned} \tag{47}$$

For plus-operators the term on the left hand side has a clear geometric interpretation; now we look for $q \in H^\infty_{[n^i]}$ which makes the operator $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ "most positive", i.e. which maximizes the term on the left hand side of expression (47). Hence we have to determine $\bar{q} \in H^\infty_{[n^i]}$ such that

$$\| M^*U_1 - N^*V_1 - \bar{q} \|_\infty^2 = \inf_{q \in H^\infty_{[n^i]}} \| M^*U_1 - N^*V_1 - q \|_\infty^2 .$$

The result of the case $i = 1$ gives (using (24)) that

$$M^*U_1 - N^*V_1 = \epsilon_1 \mu_1 \frac{tp_1^*}{t^*p_1}.$$

Moreover, the fundamental polynomial equation (1) shows that

$$\frac{r^*}{t^*} = \epsilon_1 \mu_1 \frac{tp_1^*}{t^*p_1} + \frac{\pi_1}{p_1} = \epsilon_i \mu_i \frac{tp_i^*}{t^*p_i} + \frac{\pi_i}{p_i}; \tag{48}$$

hence

$$M^*U_1 - N^*V_1 - q = \epsilon_i \mu_i \frac{tp_i^*}{t^*p_i} + \frac{\pi_i}{p_i} - \frac{\pi_1}{p_1} - q \tag{49}$$

and

$$\begin{aligned} \inf_{q \in H^\infty_{[n^i]}} \| M^*U_1 - N^*V_1 - q \|_\infty &= \inf_{q \in H^\infty_{[n^i]}} \| \frac{r^*}{t^*} - \frac{\pi_1}{p_1} - q \|_\infty \\ &= \inf_{q' \in H^\infty_{[n^i]}} \| \frac{r^*}{t^*} - q' \|_\infty . \end{aligned}$$

But now Theorem 4.1, (ii) tells us that there exists a uniquely determined $\bar{q} \in H^\infty_{[n^i]}$ for which the infimum is attained, and that

$$\| M^*U_1 - N^*V_1 - \bar{q} \|_\infty = \mu_i. \tag{50}$$

Moreover, in view of (49) and the uniqueness of the minimizing $\bar{q}$ we can conclude that

$$\bar{q} = \frac{\pi_i}{p_i} - \frac{\pi_1}{p_1}.$$

Sumarizing, this means that all solutions $\begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ in $H^\infty_{[n^i]}$ of $\tilde{M}V - \tilde{N}U = I$ satisfy

$$\inf_{(x,x)=1} [\begin{pmatrix} X_i \\ Y_i \end{pmatrix} x, \begin{pmatrix} X_i \\ Y_i \end{pmatrix} x] \leq 1 - \mu_i^2, \tag{51}$$

and that there is exactly one solution, namely the one corresponding to $q = \bar{q}$, for which equality holds.

On the other hand, $\begin{pmatrix} V_i \\ U_i \end{pmatrix} \in H^\infty_{[n^i]}$ is a solution of the Bezout equation, and by (iii) and (24)

$$\inf_{(x,x)=1} [\begin{pmatrix} V_i \\ U_i \end{pmatrix} x, \begin{pmatrix} V_i \\ U_i \end{pmatrix} x] = \frac{1}{1+\sigma_i^2} = 1 - \mu_i^2.$$

Thus $\begin{pmatrix} V_i \\ U_i \end{pmatrix} = \begin{pmatrix} V_1 \\ U_1 \end{pmatrix} - \begin{pmatrix} N \\ M \end{pmatrix} \bar{q}$ by the uniqueness of the optimal solution, and hence

$$\begin{pmatrix} V_i \\ U_i \end{pmatrix} := \frac{1}{1+\sigma_i^2} \begin{pmatrix} \pi_1^{(i)}/p_i \\ \pi_2^{(i)}/p_i \end{pmatrix} = \frac{1}{1+\sigma_1^2} \begin{pmatrix} \pi_1^{(1)}/p_1 \\ \pi_2^{(1)}/p_1 \end{pmatrix} - \begin{pmatrix} e/t \\ d/t \end{pmatrix} (\frac{\pi_i}{p_i} - \frac{\pi_1}{p_1}).$$

Now this implies with (41) for the case $i = 1$ and (48) that

$$\begin{aligned}
M^*U_i - N^*V_i &= \frac{1}{1+\sigma_i^2} \left(\frac{d^*\pi_2^{(i)} - e^*\pi_1^{(i)}}{t^*p_i} \right) \\
&= \frac{1}{1+\sigma_1^2} \left(\frac{d^*\pi_2^{(1)} - e^*\pi_1^{(1)}}{t^*p_1} \right) - \left(\frac{d^*d - e^*e}{t^*t} \right) (\frac{\pi_i}{p_i} - \frac{\pi_1}{p_1}) \\
&= \frac{1}{1+\sigma_1^2} \left(\frac{d^*\pi_2^{(1)} - e^*\pi_1^{(1)}}{t^*p_1} \right) - (\frac{\pi_i}{p_i} - \frac{\pi_1}{p_1}) \\
&= \epsilon_1\sigma_1(1+\sigma_1^2)^{-\frac{1}{2}} \frac{tp_1^*}{t^*p_1} - \frac{\pi_i}{p_i} + \frac{\pi_1}{p_1} \\
&= \epsilon_1\mu_1 \frac{tp_1^*}{t^*p_1} - \frac{\pi_i}{p_i} + \frac{\pi_1}{p_1} \\
&= \epsilon_i\mu_i \frac{tp_i^*}{t^*p_i}.
\end{aligned}$$

From this relation it follows that

$$\begin{aligned}
d^*\pi_2^{(i)} - e^*\pi_1^{(i)} &= \epsilon_i\mu_i(1+\sigma_i^2)tp_i^* \\
&= \epsilon_i \frac{\sigma_i}{\sqrt{1+\sigma_i^2}} (1+\sigma_i^2)tp_i^*,
\end{aligned}$$

which is equation (41) for general i. ∎

Corollary 4.1 *Let* $\{\frac{p_i}{t}, \begin{pmatrix} \hat{p}_1^{(i)}/t^* \\ \hat{p}_2^{(i)}/t^* \end{pmatrix}\}$ *be a minimal degree solution of (32) corresponding to* σ_i. *Then the following relations hold true:*

$$\hat{p}_1^{(i)} = \frac{1}{t}\{\sigma_i d^* p_i + \epsilon_i (1+\sigma_i^2)^{1/2} e p_i^*\}$$

$$\hat{p}_2^{(i)} = \frac{1}{t}\{\sigma_i e^* p_i + \epsilon_i (1+\sigma_i^2)^{1/2} d p_i^*\}$$

Proof:
Multiplying (33a) from the left by $\begin{pmatrix} -e^* & d^* \end{pmatrix}$ results in

$$0 = -\sigma_i t\{d^*\hat{p}_2^{(i)} - e^*\hat{p}_1^{(i)}\} + t^*\{d^*\pi_2^{(i)} - e^*\pi_1^{(i)}\}.$$

Now Theorem 4.2, (iv) gives

$$d^*\hat{p}_2^{(i)} - e^*\hat{p}_1^{(i)} = \epsilon_i(1+\sigma_i^2)^{1/2} t^* p_i^*. \tag{52}$$

Together with (33b) we can write this as

$$\begin{pmatrix} d & -e \\ -e^* & d^* \end{pmatrix} \begin{pmatrix} \hat{p}_1^{(i)} \\ \hat{p}_2^{(i)} \end{pmatrix} = \begin{pmatrix} \sigma_i t^* p_i \\ \epsilon_i(1+\sigma_i^2)^{1/2} t^* p_i \end{pmatrix}. \tag{53}$$

Multiplication from the left by $\begin{pmatrix} d^* & e \\ e^* & d \end{pmatrix}$ proves the result, because

$$\begin{pmatrix} d & -e \\ -e^* & d^* \end{pmatrix}\begin{pmatrix} d^* & e \\ e^* & d \end{pmatrix} = \begin{pmatrix} d^* & e \\ e^* & d \end{pmatrix}\begin{pmatrix} d & -e \\ -e^* & d^* \end{pmatrix} = tt^*\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

■

5 Bounded real balancing

We follow the approach taken in [HF 1994] for the derivation of a Lyapunov balanced realization of an asymptotically (anti)stable transfer function g. First we prove a similar result for the J_B-normalized coprime factors of a bounded real transfer function g.

Definition 5.1 *([Mo 1981]) A minimal, asymptotically stable system* (A, B, C, D) *is called* Lyapunov balanced *if there exists a diagonal matrix*

$$\Sigma = \mathrm{diag}(\kappa_1, \ldots, \kappa_n),\ \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n > 0$$

such that

$$\begin{aligned} A\Sigma + \Sigma A^* &= -BB^* \\ A^*\Sigma + \Sigma A &= -C^*C \end{aligned} \tag{54}$$

The matrix Σ *is called* Lyapunov gramian *of the system, and* $\kappa_1, \kappa_2, \ldots, \kappa_n$ *are called the* Lyapunov singular values *of* (A, B, C, D). □

We start with a summary of relationships between coefficients of the different polynomials involved. Recall that by $q_{j,i}$ we denote the i-th coefficient of the polynomial q_j, i.e. $q_j = \sum_{k=0}^{n} q_{j,k} z^k$.

Proposition 5.1 *(i) Let* $\frac{r^*}{t^*}$ *be scalar, strictly proper and antistable with* $r \wedge t = 1$, t *monic. Assume the notation of Proposition 2.1. Then:*

$$\alpha_{ij,n-(n_i+n_j)} = p_{i,n-n_i} p_{j,n-n_j} \frac{\lambda_i^2 - \lambda_j^2}{(-1)^{n_i}\lambda_i + (-1)^{n_j}\lambda_j} \tag{55}$$

(ii) Assume the notation of Proposition 2.1. Let $\{p_i, \begin{pmatrix} \hat{p}_1^{(i)} \\ \hat{p}_2^{(i)} \end{pmatrix}\}$ *be a solution pair to (33) corresponding to* σ_i*, and let* $g = \frac{e}{d}$ *strictly proper. Then:*

$$\hat{p}_{1,n-n_i}^{(i)} = (-1)^n \sigma_i p_{i,n-n_i} \tag{56}$$

$$\hat{p}_{2,n-n_i}^{(i)} = (-1)^{n-n_i} \epsilon_i \sqrt{1+\sigma_i^2} p_{i,n-n_i} \tag{57}$$

$$\pi_{1,n-n_i}^{(i)} = (1+\sigma_i^2) p_{i,n-n_i} \tag{58}$$

$$\pi_{2,n-n_i}^{(i)} = (-1)^{n_i} \epsilon_i \sigma_i \sqrt{1+\sigma_i^2} p_{i,n-n_i} \tag{59}$$

Proof:

(i) Equating highest degree coefficients in (2) yields

$$p_{i,n-n_i}(-1)^{n-n_j} p_{j,n-n_j} = (-1)^n \frac{\lambda_j + (-1)^{n_i+n_j}\lambda_i}{\lambda_i^2 - \lambda_j^2} \alpha_{ij,n-(n_i+n_j)},$$

which is (55).

(ii) We use equations (33), i.e.

$$\begin{pmatrix} d^* \\ e^* \end{pmatrix} p_i = -\sigma_i t \begin{pmatrix} \hat{p}_1^{(i)} \\ \hat{p}_2^{(i)} \end{pmatrix} + t^* \begin{pmatrix} \pi_1^{(i)} \\ \pi_2^{(i)} \end{pmatrix} \tag{60}$$

and

$$d\hat{p}_1^{(i)} - e\hat{p}_2^{(i)} = \sigma_i t^* p_i. \tag{61}$$

Equating the highest degree coefficients in (61) results in (56) because of the strict properness of g.

From the first coordinate of (60) we get

$$(-1)^n p_{i,n-n_i} = -\sigma_i \hat{p}^{(i)}_{1,n-n_i} + (-1)^n \pi^{(i)}_{1,n-n_i}, \tag{62}$$

whereas the second coordinate yields

$$0 = -\sigma_i \hat{p}^{(i)}_{2,n-n_i} + (-1)^n \pi^{(i)}_{2,n-n_i}. \tag{63}$$

Finally, (41) gives relation (59):

$$(-1)^n \pi^{(i)}_{2,n-n_i} = \epsilon_i \sigma_i \sqrt{1+\sigma_i^2}(-1)^{n-n_i} p_{i,n-n_i} \tag{64}$$

Plugging in (56) and (59) in (62) and (63) gives the result. ∎

We now derive a canonical form for J_B-normalized coprime factors, which is Lyapunov balanced up to a sign. For the sake of simplicity we restrict ourselves to the strictly proper case.

Theorem 5.1 *Let $g = \frac{e}{d}$ be a strictly proper bounded real transfer function with $e \wedge d = 1$, d monic of degree n. Let $\frac{e}{t}$, $\frac{d}{t}$, t monic be the J_B-normalized coprime factors of g, and let $\sigma_1 > \sigma_2 > \ldots > \sigma_k > 0$ the singular values of $H_{\begin{pmatrix} d^*/t^* \\ e^*/t^* \end{pmatrix}}$, where σ_i is of multiplicity n_i, $i = 1, \ldots, k$, $\sum_{j=1}^{k} n_j = n$. Then there exists an orthogonal basis*

$$\mathcal{B} := \{\frac{p_1^{(1)}}{t}, \ldots, \frac{p_1^{(n_1)}}{t}, \frac{p_2^{(1)}}{t}, \ldots, \frac{p_2^{(n_2)}}{t}, \ldots, \frac{p_k^{(1)}}{t}, \ldots, \frac{p_k^{(n_k)}}{t}\}$$

of X^t such that

$$\| \frac{p_i^{(n_i)}}{t} \|^2 = \sigma_i\,, \quad i = 1, \ldots, k \tag{65}$$

with the following properties: the matrix representation (A, B, C, D) of the shift realization of the function $\begin{pmatrix} \frac{d}{t} & -\frac{e}{t} \end{pmatrix}$ with respect to the basis $\mathcal{B}$ satisfies the (modified) Lyapunov equations

$$A\Sigma + \Sigma A^* = B J_B B^* \tag{66}$$

$$A^*\Sigma + \Sigma A = -C^* C \tag{67}$$

with $\Sigma = diag(\underbrace{\sigma_1, \ldots, \sigma_1}_{n_1}, \underbrace{\sigma_2, \ldots, \sigma_2}_{n_2}, \ldots, \underbrace{\sigma_k, \ldots, \sigma_k}_{n_k})$ and is given by (A, B, C, D) with

$$A = (A_{ij}), \quad i, j = 1, \ldots, k, \quad A_{ij} \in \mathbb{R}^{n_i \times n_j}, \quad i, j = 1, \ldots, k$$

$$A_{ii} = \begin{pmatrix} 0 & \alpha_1^i & 0 & \cdots & \cdots & 0 \\ -\alpha_1^i & 0 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \alpha_{n_i-2}^i & 0 \\ \vdots & & \ddots & \ddots & 0 & \alpha_{n_i-1}^i \\ 0 & \cdots & \cdots & 0 & -\alpha_{n_i-1}^i & a_{ii} \end{pmatrix} \tag{68}$$

$\alpha_j^i > 0, \quad j = 1, \ldots, n_i - 1, \quad i = 1, \ldots, k$

$$A_{ij} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ \vdots & & & \vdots & 0 \\ 0 & \cdots & \cdots & 0 & a_{ij} \end{pmatrix}, \quad i, j = 1, \ldots, k \tag{69}$$

$$B = \begin{pmatrix} B' & B'' \end{pmatrix}$$

$$B' = (\underbrace{0, \ldots, 0, b_1'}_{n_1}, \underbrace{0, \ldots, 0, b_2'}_{n_2}, \ldots, \underbrace{0, \ldots, 0, b_k'}_{n_k})^T$$

$$B'' = (\underbrace{0, \ldots, 0, b_1''}_{n_1}, \underbrace{0, \ldots, 0, b_2''}_{n_2}, \ldots, \underbrace{0, \ldots, 0, b_k''}_{n_k})^T$$

$$\begin{aligned} b_i' &= \sigma_i p_{i,n-1}^{(n_i)} \\ b_i'' &= (-1)^{n_i-1}\epsilon_i(1+\sigma_i^2)^{1/2} p_{i,n-1}^{(n_i)} \end{aligned} \tag{70}$$

with $\epsilon_i = \pm 1, \ i = 1, \ldots, k,$

$$C = (\underbrace{0, \ldots, 0, c_1}_{n_1}, \underbrace{0, \ldots, 0, c_2}_{n_2}, \ldots, \underbrace{0, \ldots, 0, c_k}_{n_k})$$

$$c_i = p_{i,n-1}^{(n_i)} \quad , i = 1, \ldots k \tag{71}$$

$$D = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and

$$a_{ji} = -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \{ \frac{(-1)^{n_j-1}\epsilon_j\sqrt{1+\sigma_i^2}}{(-1)^{n_i-1}\epsilon_i\sigma_i\sqrt{1+\sigma_j^2} + (-1)^{n_j-1}\epsilon_j\sigma_j\sqrt{1+\sigma_i^2}} \} \tag{72}$$

for $i, j = 1, \ldots, k.$

Proof:
Let $R_B := \frac{r}{t}$ be the B-characteristic associated to g, and assume the notation of Proposition 2.1. In particular, the singular values of $H_{\frac{r^*}{t^*}}$ are calculated with the help of formula (24) to be

$$\mu_i = \frac{\sigma_i}{\sqrt{1+\sigma_i^2}} \quad , \quad i = 1, \ldots, k. \tag{73}$$

Now for the spectral subspace corresponding to each singular value μ_i we take the basis $\mathcal{B}_i := \{\frac{p_i}{t}, \frac{zp_i}{t}, \ldots, \frac{z^{n_i-1}p_i}{t}\}$. Application of the Gram-Schmidt orthonormalization procedure yields bases $\mathcal{B}_i' = \{\frac{\tilde{q}_i^{(1)}}{t}, \ldots, \frac{\tilde{q}_i^{(n_i)}}{t}\}$ with

$$\tilde{q}_i^{(\ell)} = p_i a_i^{(\ell)} \quad , \quad \ell = 1, \ldots, n_i. \tag{74}$$

It is immediate from the Gram-Schmidt procedure that the $a_i^{(\ell)}$ are polynomials of degree $\ell - 1$. Moreover,

$$\frac{\tilde{q}_i^{(\ell)}}{t} \perp \mathrm{span}_{\mathbb{R}}\{\frac{p_i}{t}, \ldots, \frac{z^{\ell-2}p_i}{t}\}$$

for $\ell = 2, \ldots, n_i$. In view of these properties one can derive the recursion formulas

$$\begin{aligned} \tilde{q}_i^{(2)} &= \gamma_1^i z \tilde{q}_i^{(1)}, \\ \tilde{q}_i^{(\ell+1)} &= \gamma_\ell^i z \tilde{q}_i^{(\ell)} - \beta_{\ell-1}^i \tilde{q}_i^{(\ell-1)}, \quad \ell = 2, \ldots, n_i - 1 \end{aligned} \tag{75}$$

for $i = 1, \ldots, k$ using standard Hilbert space methods. Furthermore, it can be proved that $\gamma_\ell^i > 0$ for $\ell = 1, \ldots, n_i - 1$ and $\beta_{\ell-1}^i < 0$ for $\ell = 2, \ldots, n_i$. Now from (75) it is clear that the polynomials $a_i^{(\ell)}$ contain only even(odd) powers of z depending on whether ℓ is odd(even). Moreover, this recursion formula also shows that the matrix representation of the shift S^t in the basis $\mathcal{B}_i'$ is of the form (68). Finally, by performing a similar analysis for the last basis vectors $\frac{\tilde{q}_i^{(n_i)}}{t}$ and multiplying the elements of $\mathcal{B}_i'$ by suitable constants results in bases $\mathcal{B}_i'' = \{\frac{q_i^{(1)}}{t}, \ldots, \frac{q_i^{(n_i)}}{t}\}$ with

$$\| \frac{q_i^{(n_i)}}{t} \|^2 = \mu_i \tag{76}$$

such that the matrix representation of the shift with respect to the basis $\mathcal{B}' := \bigcup_{i=1}^k \mathcal{B}_i''$ is of the form (68) and (69) with

$$a_{ji} = -q_{i,n-1}^{(n_i)} q_{j,n-1}^{(n_j)} \frac{s_i s_j}{s_i s_j \mu_j + \mu_i},$$

where $s_\ell := (-1)^{n_\ell - 1}\epsilon_\ell$, $\ell = 1, \ldots, k$; for the details we refer to [HF 1994].

Now we construct the basis $\mathcal{B}$ in a completely analogous way, only using normalization (65) instead of (76). Then

$$a_{ji} = (\frac{p_j^{(n_j)}}{t}, \frac{p_j^{(n_j)}}{t})^{-1}(S^t\frac{p_i^{(n_i)}}{t}, \frac{p_j^{(n_j)}}{t}) = \frac{1}{\sigma_j}(S^t\frac{p_i^{(n_i)}}{t}, \frac{p_j^{(n_j)}}{t}) = \frac{1}{\sigma_j}\sqrt{\frac{\sigma_i\sigma_j}{\mu_i\mu_j}}(S^t\frac{q_i^{(n_i)}}{t}, \frac{q_j^{(n_j)}}{t}) =$$

$$= -q_{i,n-1}^{(n_i)}q_{j,n-1}^{(n_j)}\frac{\mu_j}{\sigma_j}\sqrt{\frac{\sigma_i\sigma_j}{\mu_i\mu_j}}\frac{s_is_j}{s_is_j\mu_j+\mu_i} = -p_{i,n-1}^{(n_i)}p_{j,n-1}^{(n_j)}\frac{\mu_j}{\sigma_j}\frac{s_is_j}{s_is_j\mu_j+\mu_i}.$$

By application of (73) one obtains expression (72).
The constant term is given by

$$D = \begin{pmatrix} \frac{d}{t} & -\frac{e}{t} \end{pmatrix}(\infty) = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Moreover, we have

$$C\frac{p_i^{(k)}}{t} = \left(\frac{p_i^{(k)}}{t}\right)_{-1} = \begin{cases} 0 & ,k=1,\ldots,n_i-1 \\ p_{i,n-1}^{(n_i)} & ,k=n_i \end{cases}$$

We now compute the matrix representation of the input matrix B. Note that $\frac{e}{t} \in X^t$ and $\frac{d-t}{t} \in X^t$. Hence there exists a representation

$$\left(\begin{pmatrix} \frac{d}{t} & -\frac{e}{t} \end{pmatrix} - \begin{pmatrix} 1 & 0 \end{pmatrix}\right) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\begin{pmatrix} b'_{ij} & b''_{ij} \end{pmatrix}\frac{p_i^{(j)}}{t}, \tag{77}$$

and the orthogonality of $\mathcal{B}$ gives

$$b''_{ij} = -\frac{(\frac{e}{t}, \frac{p_i^{(j)}}{t})}{(\frac{p_i^{(j)}}{t}, \frac{p_i^{(j)}}{t})} \tag{78}$$

respectively

$$b'_{ij} = \frac{(\frac{d-t}{t}, \frac{p_i^{(j)}}{t})}{(\frac{p_i^{(j)}}{t}, \frac{p_i^{(j)}}{t})} \tag{79}$$

for $j=1,\ldots,n_i$, $i=1,\ldots,k$. For the calculation of the above scalar products we first remark that by (32a)

$$-\frac{ep_i^*}{tt^*} = \sigma_i\frac{(\hat{p}_2^{(i)})^*}{t} - \frac{(\pi_2^{(i)})^*}{t^*}.$$

Hence in view of (74) there holds

$$-\frac{e(p_i^{(j)})^*}{tt*} = \sigma_i\frac{(\hat{p}_2^{(i)})^*(a_i^{(j)})^*}{t} - \frac{(\pi_2^{(i)})^*(a_i^{(j)})^*}{t^*}.$$

Integrating over the semicircular contour γ_R as introduced on page 14 we obtain

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e(p_i^{(j)})^*}{tt^*}d\tau = -\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_R}\frac{e(p_i^{(j)})^*}{tt^*}dz =$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_R}\sigma_i\frac{(\hat{p}_2^{(i)})^*(a_i^{(j)})^*}{t}dz = \begin{cases} 0 & ,j=1,\ldots,n_i-1 \\ (-1)^{n-1}\sigma_i\hat{p}_{2,n-n_i}^{(i)}a_{i,n_i-1}^{(n_i)} & ,j=n_i \end{cases}$$

But by (57)

$$\hat{p}_{2,n-n_i}^{(i)} = (-1)^{n-n_i}\epsilon_i\sqrt{1+\sigma_i^2}p_{i,n-n_i},$$

and thus

$$(-\frac{e}{t},\frac{p_i^{(j)}}{t}) = \begin{cases} 0 & ,j=1,\ldots,n_i-1 \\ (-1)^{n_i-1}\epsilon_i\sigma_i\sqrt{1+\sigma_i^2}p_{i,n-1}^{(n_i)} & ,j=n_i \end{cases}$$

again by (74); in view of the normalization (65) this is the second coordinate of (70). Similarly, the first coordinate is calculated. Observe that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(d-t)(p_i^{(j)})^*}{tt^*}d\tau = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\gamma_R}\frac{d(p_i^{(j)})^*}{tt^*}dz.$$

However, the first equation in (32a) in connection with (56) gives the desired result for b_i' exactly in the same way as above.
Finally we check the validity of equations (66) and (67): obviously the only interesting matrix entries are those in positions (n_i,n_j), $i,j\in\{1,\ldots,k\}$; there we have

$$(A\Sigma+\Sigma A^*-BJ_BB^*)_{n_i,n_j} = a_{ij}\sigma_j+\sigma_ia_{ji}-b_i'b_j'+b_i''b_j''$$

$$= -p_{i,n-1}^{(n_i)}p_{j,n-1}^{(n_j)}\left\{\frac{s_i\sigma_j\sqrt{1+\sigma_i^2}+s_j\sigma_i\sqrt{1+\sigma_j^2}}{s_i\sigma_i\sqrt{1+\sigma_j^2}+s_j\sigma_j\sqrt{1+\sigma_i^2}}-\sigma_i\sigma_j+s_is_j\sqrt{1+\sigma_i^2}\sqrt{1+\sigma_j^2}\right\}$$

$$= -p_{i,n-1}^{(n_i)}p_{j,n-1}^{(n_j)}\left\{\frac{\sigma_i\sigma_j(\sigma_j^2-\sigma_i^2)+s_is_j\sqrt{1+\sigma_i^2}\sqrt{1+\sigma_j^2}(\sigma_i^2-\sigma_j^2)}{\sigma_i^2-\sigma_j^2}\right.$$

$$\left.-\sigma_i\sigma_j+s_is_j\sqrt{1+\sigma_i^2}\sqrt{1+\sigma_j^2}\right\} = 0$$

and

$$(A^*\Sigma+\Sigma A+C^*C)_{n_i,n_j} = a_{ji}\sigma_j+\sigma_ia_{ij}+c_ic_j = 0.$$

■

Following [OJ 1988] we define bounded real balancing. We state here the definition for non-strictly proper systems (see [Ob 1991]).

Definition 5.2 *A bounded real system* (A, B, C, D) *is called* bounded real balanced *if*

$$P_{min} = P_{max}^{-1} = \text{diag}(\tau_1, \ldots, \tau_n) =: \Sigma > 0,\ 1 > \tau_1 \geq \tau_2 \geq \ldots \geq \tau_n > 0$$

where P_{min} *resp.* P_{max} *are the minimal resp. maximal solution to the bounded real Riccati equation (BRRE)*

$$A^T P + PA + C^T C + (PB + C^T D)S^{-1}(PB + C^T D)^T = 0$$

with $S := I - D^T D > 0$. Σ *is called the* bounded real gramian *of* (A, B, C, D)*; its diagonal entries are called* bounded real singular values. □

We now derive a bounded real balanced realization of the transfer function g. Again this realization will be shown to be the matrix representation of the shift realization with respect to a basis that is constructed from the Schmidt vectors of $H_{R_B^*}$.

Theorem 5.2 *Let* $g = \frac{e}{d}$ *be a strictly proper bounded real transfer function with* $e \wedge d = 1$, d *monic of degree* n*, and let* $\frac{e}{t}, \frac{d}{t}$*,* t *monic be the* J_B*-normalized coprime factors of* g*. Let* R_B *denote the* B*-characteristic of* g*. Assume that the singular values of* $H_{R_B^*}$ *are* $1 > \mu_1 > \mu_2 > \ldots > \mu_k > 0$*, where* μ_i *is of multiplicity* n_i*,* $i = 1, \ldots, k$*,* $\sum_{j=1}^{k} n_j = n$*. Then there exists an orthogonal basis* $\mathcal{B} := \{\frac{p_1^{(1)}}{t}, \ldots, \frac{p_1^{(n_1)}}{t}, \frac{p_2^{(1)}}{t}, \ldots, \frac{p_2^{(n_2)}}{t}, \ldots, \frac{p_k^{(1)}}{t}, \ldots, \frac{p_k^{(n_k)}}{t}\}$ *of* X^t *with the following properties:*

(i) $\bar{\mathcal{B}} := \{\frac{p_1^{(1)}}{d}, \ldots, \frac{p_1^{(n_1)}}{d}, \frac{p_2^{(1)}}{d}, \ldots, \frac{p_2^{(n_2)}}{d}, \ldots, \frac{p_k^{(1)}}{d}, \ldots, \frac{p_k^{(n_k)}}{d}\}$ *is a basis of* X^d.

(ii) If we normalize the basis $\mathcal{B}$ *so that*

$$\|\frac{p_i^{(n_i)}}{t}\|^2 = \sigma_i\sqrt{1 + \sigma_i^2} \tag{80}$$

(where σ_i *is calculated from* μ_i *according to formula (24)), then the matrix representation of the shift realization of* g *with respect to the basis* $\bar{\mathcal{B}}$ *is in bounded real balanced canonical form with bounded real gramian* $\Sigma_B = diag(\mu_1 I_{n_1}, \ldots, \mu_k I_{n_k})$*. More specifically we have:*

$$A = (A_{ij}),\ A_{ij} \in \mathbb{R}^{n_i \times n_j},\ i, j = 1, \ldots, k \tag{81}$$

A_{ii} *as in (68),* A_{ij} *as in (69),* $i, j = 1, \ldots, k$

$$b = (\underbrace{0, \ldots, 0, b_1}_{n_1}, \underbrace{0, \ldots, 0, b_2}_{n_2}, \ldots, \underbrace{0, \ldots, 0, b_k}_{n_k})^T$$

$$b_i = -s_i p_{i,n-1}^{(n_i)}, \quad i = 1, \ldots, k \tag{82}$$

$$c = (\underbrace{0,\ldots,0,c_1}_{n_1},\underbrace{0,\ldots,0,c_2}_{n_2},\ldots,\underbrace{0,\ldots,0,c_k}_{n_k},$$

$$c_i = p^{(n_i)}_{i,n-1}, \quad i=1,\ldots,k \tag{83}$$

$$d = 0$$

and

$$a_{ji} = -p^{(n_i)}_{i,n-1}p^{(n_j)}_{j,n-1}\left\{\frac{1+s_is_j\mu_i\mu_j}{s_is_j\mu_i+\mu_j}\right\} \tag{84}$$

for $i,j=1,\ldots,k$.

Proof:

(i) The statement is obvious, since the multiplication map by $\frac{t}{d}$ is an invertible map of X^t onto X^d.

(ii) Assume again the notation of Proposition 2.1. We turn X^d into a Hilbert space via the inner product

$$(\frac{p}{d},\frac{q}{d})_{[t]} := (\frac{p}{t},\frac{q}{t})_{H^2_+}.$$

Obviously orthogonality in X^t carries over to X^d equipped with $(\cdot\,,\cdot)_{[t]}$. Moreover, for deg $p < n-1$ we have

$$(S^d\frac{p}{d},\frac{q}{d})_{[t]} = (\frac{zp}{d},\frac{q}{d})_{[t]} = (S^t\frac{p}{t},\frac{q}{t}).$$

We take as the basis $\mathcal{B}$ from the statement of the Theorem the same basis as was constructed in Theorem 5.1, only replacing the normalization condition (65) by (80). Then in view of the previous discussion the entries in the A-matrix in columns different from columns $n_1, n_1+n_2,\ldots,n_1+\ldots+n_k$ are not changed. Indeed, we will show below that only the elements a_{ij} change, i.e. that A is of the form (81). Observe that for $k=1,\ldots,n_j$, $i,j=1,\ldots,k$ there holds

$$(S^d\frac{p_i^{(n_i)}}{d},\frac{p_j^{(k)}}{d})_{[t]} = (\frac{zp_i^{(n_i)}-p^{(n_i)}_{i,n-1}d}{d},\frac{p_j^{(k)}}{d})_{[t]} =$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\left\{\int_{\gamma_R}\frac{zp_i^{(n_i)}(p_j^{(k)})^*}{tt^*}dz - p^{(n_i)}_{i,n-1}\int_{\gamma_R}\frac{d(p_j^{(k)})^*}{tt^*}dz\right\}, \tag{85}$$

where γ_R is the previously defined semicircular contour. Now (32a) in connection with (74) gives

$$\frac{d(p_j^{(k)})^*}{tt^*} = -\sigma_j\frac{(\hat{p}_1^{(j)})^*(a_j^{(k)})^*}{t} + \frac{(\pi_1^{(j)})^*(a_j^{(k)})^*}{t^*}, \tag{86}$$

and hence for $k = 1, \ldots, n_j - 1$ we have because of $deg(\hat{p}_1^{(j)} a_j^{(k)})^* < n$ that the second term in (85) is zero; thus we conclude that

$$(S^d \frac{p_i^{(n_i)}}{d}, \frac{p_j^{(k)}}{d})_{[t]} = (S^t \frac{p_i^{(n_i)}}{t}, \frac{p_j^{(k)}}{t})$$

for $k = 1, \ldots, n_j - 1$, $i, j = 1, \ldots, k$.

In the case $k = n_j$ the second term in (85) can be calculated using (86) as

$$\lim_{R\to\infty} \frac{1}{2\pi i} p_{i,n-1}^{(n_i)} \int_{\gamma_R} \frac{d(p_j^{(n_j)})^*}{tt^*} dz = -p_{i,n-1}^{(n_i)} \sigma_j (-1)^{n-1} \hat{p}_{1,n-n_j}^{(j)} a_{j,n_j-1}^{(n_j)} =$$

$$= (-1)^n \sigma_j p_{i,n-1}^{(n_i)} a_{j,n_j-1}^{(n_j)} \sigma_j (-1)^n p_{j,n-n_j} = \sigma_j^2 p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)}, \tag{87}$$

where we also have made use of relation (56) and (74). Applying the same calculations as in [HF 1994] to the first term in (85) gives

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{z p_i^{(n_i)} (p_j^{(n_j)})^*}{tt^*} dz = -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \frac{s_j \mu_j}{s_j \mu_j + s_i \mu_i}. \tag{88}$$

Summarizing, we get in view of the normalizing condition (80) and equation (73) that

$$a_{ji} = (\frac{p_j^{(n_j)}}{d}, \frac{p_j^{(n_j)}}{d})_{[t]}^{-1} (S^d \frac{p_i^{(n_i)}}{d}, \frac{p_j^{(n_j)}}{d})_{[t]} =$$

$$= -\frac{p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)}}{\sigma_j \sqrt{1+\sigma_j^2}} \left\{ \frac{s_j \mu_j}{s_j \mu_j + s_i \mu_i} + \sigma_j^2 \right\} =$$

$$= -\frac{p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)}}{\sigma_j \sqrt{1+\sigma_j^2}} \left\{ \frac{s_j \sigma_j \sqrt{1+\sigma_i^2}}{s_j \sigma_j \sqrt{1+\sigma_i^2} + s_i \sigma_i \sqrt{1+\sigma_j^2}} + \sigma_j^2 \right\}$$

$$= -\frac{p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)}}{\sigma_j \sqrt{1+\sigma_j^2}} \left\{ \frac{s_j \sigma_j \sqrt{1+\sigma_i^2}(1+\sigma_j^2) + s_i \sigma_i \sigma_j^2 \sqrt{1+\sigma_j^2}}{s_j \sigma_j \sqrt{1+\sigma_i^2} + s_i \sigma_i \sqrt{1+\sigma_j^2}} \right\}$$

$$= -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \left\{ \frac{s_j \sqrt{1+\sigma_i^2}\sqrt{1+\sigma_j^2} + s_i \sigma_i \sigma_j}{s_j \sigma_j \sqrt{1+\sigma_i^2} + s_i \sigma_i \sqrt{1+\sigma_j^2}} \right\} = -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \left\{ \frac{1 + s_i s_j \mu_i \mu_j}{\mu_j + s_i s_j \mu_i} \right\};$$

this is (84)!

The calculation of the c- and d-matrix is again obvious.

Finally, we calculate the b-matrix. Observe that $\frac{e}{d} \in X^d$. Hence

$$(\frac{e}{d}, \frac{p_j^{(k)}}{d})_{[t]} = (\frac{e}{t}, \frac{p_j^{(k)}}{t}), \quad k = 1, \ldots, n_j, \, j = 1, \ldots, k.$$

Thus we can use the calculation of the B-matrix in Theorem 5.1, and by taking into account the modified normalization condition we obtain (82).

In the end we remark that the constructed realization is in bounded real balanced canonical form (compare [Ob 1991], Corollary 5.1). ■

6 Positive Real Functions

In this chapter we examine positive real transfer functions. A proper square stable rational function G is called *positive real* if

$$G(i\omega) + G^*(i\omega) > 0$$

for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$. Let

$$J_P := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

A right coprime factorization of $G = NM^{-1}$ is called a *J_P-NRCF* of G if

$$\begin{pmatrix} M^* & N^* \end{pmatrix} J_P \begin{pmatrix} M \\ N \end{pmatrix} = I. \tag{89}$$

Analogously, a *J_P-NLCF* of $G = \tilde{M}^{-1}\tilde{N}$ is defined. Observe that there holds

$$\begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix}^* J_P \begin{pmatrix} M & -\tilde{M}^* \\ N & \tilde{N}^* \end{pmatrix} = J_B. \tag{90}$$

Moreover, proceeding as in the bounded real case we regard the unique solution $(U_P, V_P) \in H_+^\infty$ of the Bezout equation

$$\tilde{M}V - \tilde{N}U = I \tag{91}$$

such that

$$R_P^* := M^*V_P + N^*U_P \in H_-^\infty$$

and R_P^* is strictly proper. R_P is called the *positive real characteristic (P-characteristic)* of G.

Now in principle one could go through the same analysis as was done for bounded real functions in [Fu 1993b], only replacing the indefinite metric induced by J_B by the one induced by J_P, in order to derive a result analogous to Theorem 3.1. Then, an analogous reasoning as in section 4 and 5 could be applied for obtaining a realization in positive

real balanced canonical form. However, a reduction of the positive real case to the (more general) bounded real case may be done by use of the *Cayley transform* (see eg. [Bo 1974]). Let $B_n^{m,m}$ denote the set of all bounded real transfer functions of dimension $m \times m$ and of Mc Millan degree n. The corresponding positive real transfer functions are denoted by P_n^m.
The map

$$\mathcal{C}: \begin{array}{ccc} B_n^{m,m} & \longrightarrow & P_n^m \\ B(z) & \longrightarrow & (I - B(z))^{-1}(I + B(z)) \end{array}$$

is called the *Cayley transform* . $\mathcal{C}$ is a bijection with inverse

$$\mathcal{C}^{-1}: \begin{array}{ccc} P_n^m & \longrightarrow & B_n^{m,m} \\ P(z) & \longrightarrow & (P(z) - I)(P(z) + I)^{-1} \end{array}$$

The corresponding state space formulas are given by

$$\mathcal{C}(A,B,C,D) =$$

$$\Big(A + B(I-D)^{-1}C\,,\ \sqrt{2}B(I-D)^{-1}\,,\ \sqrt{2}(I-D)^{-1}C\,,\ (I-D)^{-1}(I+D)\Big) \tag{92}$$

respectively

$$\mathcal{C}^{-1}(A,B,C,D) =$$

$$\Big(A - B(I+D)^{-1}C\,,\ \sqrt{2}B(I+D)^{-1}\,,\ \sqrt{2}(I+D)^{-1}C\,,\ (D-I)(D+I)^{-1}\Big) \tag{93}$$

(see eg. [AV 1973]). The Cayley transform preserves normalization in the respective indefinite metrics, as is shown in the next Lemmas. Before proceeding with that we mention the following result from [FO 1993b].

Proposition 6.1 *(i) Let G be proper rational and positive real, and let (N, M) $((\tilde{N}, \tilde{M}))$ be a J_P-NRF (J_P-NLF) of G. Then this factorization is right (left) coprime and unique up to right (left) multiplication by a constant unitary matrix.*

(ii) Analogous for J_B-NF of bounded real functions. ■

Lemma 6.1 *(i) Let (N, M) be a J_P-NRCF of $P \in P_n^m$. Then $(\frac{1}{\sqrt{2}}(N-M), \frac{1}{\sqrt{2}}(N+M))$ is a J_B-NRCF of $\mathcal{C}^{-1}P$.*

(ii) Let $(\tilde{N}, \tilde{M})$ be a J_B-NLCF of $B \in B_n^{m,m}$. Then $(\frac{1}{\sqrt{2}}(\tilde{M}+\tilde{N}), \frac{1}{\sqrt{2}}(\tilde{M}-\tilde{N}))$ is a J_P-NLCF of $\mathcal{C}B$.

(iii) Let $(\tilde{N}, \tilde{M})$ be a J_P-NLCF of $P \in P_n^m$. Then $(\frac{1}{\sqrt{2}}(\tilde{N}-\tilde{M}), \frac{1}{\sqrt{2}}(\tilde{N}+\tilde{M}))$ is a J_B-NLCF of $\mathcal{C}^{-1}P$.

Proof:

(i) Calculating $\mathcal{C}^{-1}P$ and using the J_P-normalization of (N, M) gives the J_B-normalization of $(\frac{1}{\sqrt{2}}(N-M), \frac{1}{\sqrt{2}}(N+M))$, which proves the statement using Proposition 6.1, (ii).

(ii) Analogous to (i).

(iii) Assume $\mathcal{C}^{-1}P$ has a J_B-NLCF (L, K), i.e. $\mathcal{C}^{-1}P = K^{-1}L$. Then by (ii) $(\frac{1}{\sqrt{2}}(K + L), \frac{1}{\sqrt{2}}(K - L))$ is a J_P-NLCF of P. Hence by Proposition 6.1, (i) there exists a constant unitary matrix A such that

$$\left(A\tilde{N}\,,\,A\tilde{M}\right) = (\frac{1}{\sqrt{2}}(K+L)\,,\,\frac{1}{\sqrt{2}}(K-L))$$

or

$$K = \frac{1}{\sqrt{2}}A(\tilde{N}+\tilde{M}), \quad L = \frac{1}{\sqrt{2}}A(\tilde{N}-\tilde{M}).$$

Thus $\mathcal{C}^{-1}P = (\tilde{N}+\tilde{M})^{-1}(\tilde{N}-\tilde{M})$, and multiplication on the left by A^{-1} gives the result. ■

The following can be verified by direct calculation.

Lemma 6.2 *(i) Let $\tilde{V}M - \tilde{U}N = I$. Then:*

$$\frac{1}{2}(\tilde{V}-\tilde{U})(N+M) - \frac{1}{2}(\tilde{V}+\tilde{U})(N-M) = I$$

(ii) Let $\tilde{M}V - \tilde{N}U = I$. Then:

$$\frac{1}{2}(\tilde{N}+\tilde{M})(V-U) - \frac{1}{2}(\tilde{N}-\tilde{M})(V+U) = I$$

■

We have now made all the preparations for the statement and the proof of the main theorem of this section.

Theorem 6.1 *Let P be a proper rational positive real transfer function with J_P-NRCF (N, M) and P-characteristic R_P. Furthermore, let $\mathcal{C}^{-1}P$ be the associated bounded real function; denote its B-characteristic by R_B. Then $R_P = R_B$.*

Proof:
Let $(U_P, V_P) \in H_+^\infty$ denote the unique solution to the Bezout equation $\tilde{M}V - \tilde{N}U = I$ such that $R_P^* := M^*V_P + N^*U_P \in H_-^\infty$ and R_P^* is strictly proper. Then by Lemma 6.2, (ii) the Bezout equation

$$\frac{1}{2}(\tilde{N}+\tilde{M})(V_P-U_P) - \frac{1}{2}(\tilde{N}-\tilde{M})(V_P+U_P) = I \tag{94}$$

holds. Observe that $(\frac{1}{\sqrt{2}}(V_P+U_P), \frac{1}{\sqrt{2}}(V_P-U_P)) \in H_+^\infty$ and

$$\frac{1}{\sqrt{2}}(N+M)^*\frac{1}{\sqrt{2}}(V_P+U_P) - \frac{1}{\sqrt{2}}(N-M)^*\frac{1}{\sqrt{2}}(V_P-U_P) = R_P^*. \tag{95}$$

This gives the result, because by Lemma 6.1, (iii) $(\frac{1}{\sqrt{2}}(\tilde{N}-\tilde{M}), \frac{1}{\sqrt{2}}(\tilde{N}+\tilde{M}))$ is a J_B-NLCF of $\mathcal{C}^{-1}P$, and hence the right hand side of (95) is equal to R_B^*. ■

Finally we turn to positive real balancing (see [DP 1984]).

Definition 6.1 *A system* $(A, B, C, D) \in P_n^m$ *is called* positive real balanced *if*

$$P_{min} = P_{max}^{-1} = diag(\nu_1, \ldots, \nu_n) =: \Sigma > 0, 1 > \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n > 0$$

where P_{min} *resp.* P_{max} *are the minimal resp. maximal solution to the positive real Riccati equation (PRRE)*

$$A^T P + PA + (C - B^T P)^T (D + D^T)^{-1} (C - B^T P) = 0$$

Σ *is called the* positive real gramian *of the system; its diagonal entries are called* positive real singular values . □

Now we can derive the balanced canonical form for positive real functions as obtained in [Ob 1991] in our context. Since we will heavily use the results from the bounded real case, where we considered only strictly proper transfer functions, we only regard here functions with feedthrough term equal to 1.

Theorem 6.2 *Let* $g = \frac{e}{d}$ *be a positive real transfer function with* $e \wedge d = 1$*,* d *monic of degree* n*, and* $g(\infty) = 1$*; moreover, let* $R_P = \frac{r}{t}$ *be its* P*-characteristic. Assume that the singular values of* $H_{R_P^*}$ *are* $1 > \mu_1 > \mu_2 > \ldots > \mu_k > 0$*, where* μ_i *is of multiplicity* n_i*,* $i = 1, \ldots, k$*,* $\sum_{j=1}^{k} n_j = n$*. Let* $\mathcal{B}$ *resp.* $\bar{\mathcal{B}}$ *denote the orthogonal bases of* X^t *resp.* X^d *as introduced in Theorem 5.2, and assume the normalization (80). Then the matrix representation of the shift realization of* g *with respect to the basis* $\bar{\mathcal{B}}$ *is in positive real balanced canonical form with positive real gramian* $\Sigma_P = diag(\mu_1 I_{n_1}, \ldots, \mu_k I_{n_k})$*. More specifically:*

$A = (A_{ij})$ *of the form* (81);

b *of the form* (82); $b_i = -\sqrt{2} s_i p_{i,n-1}^{(n_i)}$, $i = 1, \ldots, k$

c *of the form* (83; $c_i = \sqrt{2} p_{i,n-1}^{(n_i)}$, $i = 1, \ldots, k$

$d = 1$

and

$$a_{ji} = -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \frac{(1 + s_i\mu_i)(1 + s_j\mu_j)}{s_i s_j \mu_i + \mu_j} \tag{96}$$

for $i, j = 1, \ldots, k$.

Proof:
Denote by $(\bar{A}, \bar{B}, \bar{C}, 0)$ the bounded real balanced realization of $\mathcal{C}^{-1} g$ with respect to $\bar{\mathcal{B}}$ as obtained in Theorem 5.2. Now application of formula (92) yields the result; indeed

$$a_{ji} = -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \left\{ \frac{1 + s_i s_j \mu_i \mu_j}{s_i s_j \mu_i + \mu_j} \right\} - s_j p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} =$$

$$= -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \frac{1 + s_i s_j \mu_i \mu_j + s_i \mu_i + s_j \mu_j}{s_i s_j \mu_i + \mu_j} = -p_{i,n-1}^{(n_i)} p_{j,n-1}^{(n_j)} \frac{(1 + s_i\mu_i)(1 + s_j\mu_j)}{s_i s_j \mu_i + \mu_j}.$$

Comparison with Corollary 6.2 in [Ob 1991] shows that the system is in positive real balanced canonical form. ■

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