Philipp Korell

Combinatorics of Valuations on Curve Singularities

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Betreuer

Prof. Dr. Mathias Schulze

Gutachter

Prof. Dr. Antonio Campillo (Universidad de Valladolid)

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Abstract

Using valuation theory we associate to a one-dimensional equidimensional semilocal Cohen–Macaulay ring R its semigroup of values, and to a fractional ideal of R we associate its value semigroup ideal. For a class of curve singularities (here called admissible rings) including algebroid curves the semigroups of values, respectively the value semigroup ideals, satisfy combinatorial properties defining good semigroups, respectively good semigroup ideals. Notably, the class of good semigroups strictly contains the class of value semigroups of admissible rings. On good semigroups we establish combinatorial versions of algebraic concepts on admissible rings which are compatible with their prototypes under taking values.

We give a definition for canonical semigroup ideals of good semigroups which characterizes canonical fractional ideals of an admissible ring in terms of their value semigroup ideals. Moreover, a canonical semigroup ideal induces a duality on the set of good semigroup ideals of a good semigroup. This duality is compatible with the Cohen–Macaulay duality on fractional ideals under taking values.

The properties of the semigroup of values of a quasihomogeneous curve singularity lead to a notion of quasihomogeneity on good semigroups which is compatible with its algebraic prototype. We give a combinatorial criterion which allows to construct from a quasihomogeneous semigroup S a quasihomogeneous curve singularity having S as semigroup of values.

Using the semigroup of values we compute endomorphism rings of maximal ideals of algebroid curves. This yields an explicit description of the intermediate steps in an algorithmic normalization of plane arrangements of smooth curves based on a criterion by Grauert and Remmert. Applying this result to hyperplane arrangements we determine the number of steps needed to compute the normalization of a the arrangement in terms of its Möbius function.

Contents

Overview

Chapter 2 In this chapter we introduce basic concepts for this thesis: fractional ideals, discrete valuation(ring)s, and fibre products.

Chapter 3 We use valuation theory on one-dimensional Cohen–Macaulay rings to associate to a class of so-called admissible rings including algebroid curves the semigroup of values. We prove the compatibility of the semigroup of values with localization as well as its invariance under completion.

Chapter 4 Based on the properties of the semigroup of values we introduce good semigroups as a combinatorial counterpart of admissible rings. We study the properties of good semigroups in particular in relation with the corresponding algebraic concepts.

Chapter 5 On good semigroups we establish a combinatorial counterpart of the Cohen–Macaulay duality on fractional ideals. We relate the dualities by taking values. In particular, we characterize canonical fractional ideals in terms of their value semigroup ideals.

Chapter 6 Extending a result by Kunz and Ruppert we want to describe quasihomogeneous curves in terms of their semigroups of values. An irreducible quasihomogeneous curve is determined by the semigroup ring of its semigroup of values. A quasihomogeneous curve with two branches can be reconstructed from its branches as a fibre product of their branches over their intersection. In general, however, this construction yields only an inclusion.

Chapter 7 Considering the properties of the semigroup of values of a quasihomogeneous curve derived in the previous section we establish a notion of quasihomogeneity on good semigroups which is compatible with its algebraic prototype. We introduce a closedness property on quasihomogeneous semigroups which characterizes those quasihomogeneous curves that can be reconstructed as a fibre product. Moreover, any good semigroup satisfying this property is the semigroup of values of a quasihomogeneous curve.

Chapter 8 Using the semigroup of values we compute explicitly the intermediate steps in a normalization algorithm based on a criterion by Grauert and Remmert for two kinds of arrangements: plane arrangements of smooth curves and hyperplane arrangements.

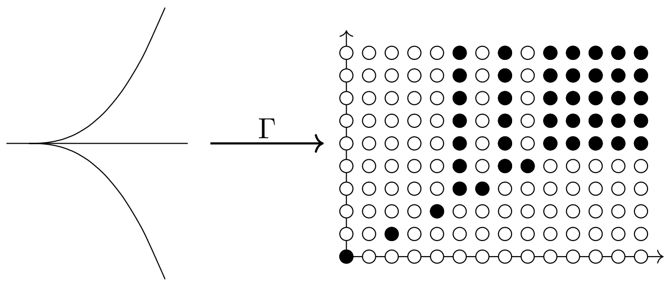
Notations

In this thesis, all rings under consideration will be commutative and unitary. We use the following notations.

$\ell_R(M)$	the length of a module M over a ring R
$\mathbf{e}_i$	the ith unit generator of a free module
$\operatorname{Spec}(R)$	the set of prime ideals of a ring R
$\operatorname{Min}(R)$	the set of minimal prime ideals of a ring R
$\operatorname{Max}(R)$	the set of maximal ideals of a ring R
$\widehat{R}$	the $\mathfrak{i}$-adic completion of a ring R at an ideal $\mathfrak{i}$ of R, where $\mathfrak{i}$ is the Jacobson radical of R if not specified otherwise
R^*	the set of units of a ring R
R^{reg}	the set of regular elements (non-zerodivisors) of a ring R
Q_R	the total ring of fractions of a ring R (see Section A.2)
$\overline{R}$	integral closure of R in Q_R (see Definition B.1)
$\mathfrak{I}^{\mathrm{reg}}$	$\mathfrak{I}^{\mathrm{reg}} = \mathfrak{I} \cap Q_R^{\mathrm{reg}}$ for an R-submodule $\mathfrak{I}$ of the total ring of fractions Q_R of a ring R
$\mathcal{R}_R$	the set of regular fractional ideals of a ring R (see Definition 2.5)
$\mathfrak{C}_{\mathfrak{I}}$	the conductor of a fractional ideal $\mathfrak{I}$ of a ring R (see Definition B.22)
$\mathcal{V}_R$	the set of valuation rings of Q_R containing R (see Definition D.1.(3))
$\mathfrak{m}_V$	the regular maximal ideal of a valuation ring V (see Remark D.5)
I_V	the infinite prime ideal of a valuation ring V (see Remark D.5)
$\mathfrak{q}_V$	see Proposition 3.13.(1)
μ_V	the valuation of a valuation ring (see Definition D.10)
V_ν	the ring of a valuation ν (see Definition D.23)
$\mathfrak{Q}^\alpha$, $\mathfrak{I}^\alpha$	see Definition 3.6
Γ_R	the semigroup of values of a one-dimensional equidimensional Cohen–Macaulay ring (see Definition 3.14)
$\Gamma_{\mathfrak{I}}$	the value semigroup ideal of a regular fractional ideal $\mathfrak{I} \in \mathcal{R}_R$ of a one-dimensional equidimensional Cohen–Macaulay ring (see Definition 3.14)
(E0), (E1), (E2)	see Definition 3.19
$\mathcal{G}_S$	the set of good semigroup ideals of a good semigroup S
M_S	the maximal ideal of a local good semigroup S
$E_J^{J'}$	see Definition 4.60
$\operatorname{Fib}(F)$	see Definition 2.29
$\operatorname{Fib}(\Gamma_R, \overline{w}, \zeta)$	see Theorem 6.2.(4) and Lemma 6.13
$\operatorname{Fib}(S, w, \zeta)$, $\operatorname{Fib}(S, w)$	see Definition 7.10

1. Introduction

The parametrization of a curve singularity allows for the definition of the semigroup of values associated to the singularity by taking the (multi)orders of regular elements.



$(x^5 - y^2)y = 0$ $\qquad$ $x \mapsto (t_1^2, t_2), \quad y \mapsto (t_1^5, 0)$

For a curve singularity C the normalization splits into a finite product of discrete valuation rings, i.e.

$$\widehat{\mathcal{O}_C} \hookrightarrow \overline{\widehat{\mathcal{O}_C}} \cong \prod_{i=1}^{s} \overline{\widehat{\mathcal{O}_C}/\mathfrak{p}_i} \cong \prod_{i=1}^{s} \mathbb{C}[[t_i]],$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the minimal prime ideals of $\widehat{\mathcal{O}_C}$. Then on the total ring of fractions $Q_{\widehat{\mathcal{O}_C}}$ of $\widehat{\mathcal{O}_C}$ we have the *order*

$$\mathrm{ord}_t \colon Q_{\widehat{\mathcal{O}_C}} \cong \prod_{i=1}^{s} \to (\mathbb{Z} \cup \{\infty\})^s,$$

and the multiplicative group $Q^{\mathrm{reg}}_{\widehat{\mathcal{O}_C}}$ of non-zerodivisors of $Q_{\widehat{\mathcal{O}_C}}$ maps onto the additive group $\mathbb{Z}^s$.

The semigroup of values of the curve is then a submonoid of $\mathbb{N}^s$. More generally, a semigroup of values as a submonoid of $\mathbb{N}^s$ for some s can be associated to a one-dimensional equidimensional semilocal Cohen–Macaulay ring R considering the (finitely many) discrete valuations of its total ring of fractions containing R.

In the last decades semigroups of values have been studied most intensively in the cases of irreducible or plane complex algebroid curves. For an irreducible plane curve singularity the semigroup of values is a numerical semigroup which is equivalent to other classical invariants like the characteristic exponents, the multiplicity sequence, or the resolution graph [1, 2]. Moreover, the semigroup of values can be interpreted as the set of intersection multiplicities of the curve singularity with all other plane curve singularities. Waldi showed that any plane algebroid curve is determined by its value semigroup up to equivalence in the sense of Zariski [3, 4]. More recently, the semigroup of values played a central role in the analytic classification of plane curve singularities with two branches by Hefez, Hernandes, and Hernandes [5].

Kunz characterized irreducible Gorenstein curve singularities by having a symmetric semigroup of values [6]. Later Delgado extended the notion of symmetry to non-numerical semigroups of values. This allowed for a generalization of Kunz' result to arbitrary curve singularities [7]. Using Delgado's symmetry condition D'Anna was able to characterize (suitably normalized) canonical ideals of a curve singularity by having a certain set of values [8].

The semigroup of values yields particularly strong constraints for quasihomogeneous curve singularities. Kunz and Ruppert showed that an irreducible quasihomogeneous complex curve singularity is determined completely by its semigroup of values. Moreover, they reconstructed a quasihomogeneous complex curve singularity with two branches from the semigroups of values of its branches and a certain coefficient map [9].

Semigroup of Values and Good Semigroups

The semigroup of values associated to a complex algebroid curve is a submonoid S of $\mathbb{N}^s$, where s is the number of branches of the curve. Delgado [7] described further combinatorial properties of the semigroup of values S of a complex algebroid curve:

(E0) There is an $\alpha \in S$ such that $\alpha + \mathbb{N}^s \subset S$.

(E1) For any $\alpha, \beta \in S$, also $\inf\{\alpha, \beta\} = (\min\{\alpha_1, \beta_1\}, \ldots, \min\{\alpha_s, \beta_s\}) \in S$.

(E2) If $\alpha, \beta \in S$ with $\alpha_i = \beta_i$ for some $i \in \{1, \ldots, s\}$, then there is a $\delta \in S$ with

$$\begin{aligned} &\delta_j > \alpha_i = \beta_i, \\ &\delta_j \geq \min\{\alpha_j, \beta_j\} \text{ for all } j = 1, \ldots, s, \\ &\delta_k = \min\{\alpha_k, \beta_k\} \text{ for every } k \in \{1, \ldots, s\} \text{ with } \alpha_k \neq \beta_k. \end{aligned}$$

Consider the curve singularity defined by $(x^5 - y^2)y = 0$. Embedding the ring

$$\widehat{\mathcal{O}_C} = \mathbb{C}[[x, y]] / \left\langle \left(x^5 - y^2\right) y \right\rangle$$

into its normalization

$$\overline{\widehat{\mathcal{O}_C}} \cong \mathbb{C}[[t_1]] \times \mathbb{C}[[t_2]]$$

it can be described by

$$\widehat{\mathcal{O}_C} \cong \mathbb{C}\left[\left[\left(t_1^2, t_2\right), \left(t_1^5, 0\right)\right]\right].$$

Then properties (E0), (E1), and (E2) can be understood in the following way:

(E0) Since the normalization $\overline{\widehat{\mathcal{O}_C}}$ is finite over $\widehat{\mathcal{O}_C}$, there is an $x \in \widehat{\mathcal{O}_C}$ such that $x\overline{\widehat{\mathcal{O}_C}} \subset \widehat{\mathcal{O}_C}$, for example

$$\left(t_1^9, t_2^5\right)(\mathbb{C}[[t_1]] \times \mathbb{C}[[t_2]]) \subset \mathbb{C}\left[\left[\left(t_1^2, t_2\right), \left(t_1^5, 0\right)\right]\right].$$

Taking orders this yields

$$(9,5)+\mathbb{N}^s \subset S.$$

(E1) Property (E1) is the result of generic linear combinations of power series, where generic means that in no component a term of least order is cancelled. For example, the power series $(t_1^5+t_1^{15}, t_2^7)$ and (t_1^8, t_2^4) correspond to the semigroup elements $(5,7)$ and $(8,4)$, and the sum

$$\left(t_1^5+t_1^{15}, t_2^7\right)+\left(t_1^8, t_2^4\right)=\left(t_1^5+t_1^8+t_1^{15}, t_2^4+t_2^7\right)$$

corresponds to the semigroup element

$$(5,4)=\inf\{(5,7),(8,4)\}.$$

(E2) Considering special linear combinations of power series which cause cancellations of terms of least order leads to an "inverse" of property (E1) which we denote by (E2). For example, taking now the power series $(t_1^5+t_1^{15}, t_2^7)$ and $(t_1^5+t_1^8+t_1^{15}, t_2^4+t_2^7)$ which have the values $(5,7)$ and $(5,4)$, the difference

$$\left(t_1^5+t_1^8+t_1^{15}, t_2^4+t_2^7\right)-\left(t_1^5+t_1^{15}, t_2^7\right)=\left(t_1^8, t_2^4\right)$$

has value

$$(8,4).$$

The first aim of Chapter 3 is to find general algebraic hypotheses leading to value semigroups and value semigroup ideals having these properties. We start with one-dimensional equidimensional semilocal Cohen–Macaulay rings. For such a ring R there are only finitely many valuations of the total ring of fractions containing R, and all of them are discrete. This allows for the definition of a semigroup of values. If $\widehat{R}$ is reduced, the normalization $\widehat{R}$ is finite, and hence R satisfies (E0). As illustrated above, for property (E1) we need "sufficiently large" residue fields. Finally, for the cancellation of terms of least order in (E2) we need the ring to be residually rational. This leads to the notion of *admissible rings*.

As an abstract version of value semigroups D'Anna introduced the class of good semigroups [8]. A *good semigroup* is a submonoid of $\mathbb{N}^s$ for some s satisfying (E0), (E1), and (E2). Then by definition the semigroup of values of an admissible ring is a good semigroup. However, Barucci, D'Anna, and Fröberg showed that these properties do not characterize semigroups of values; in fact, they gave an explicit example of a good semigroup which is not the semigroup of values of a ring [10]. Nevertheless, good semigroups can be regarded as combinatorial counterparts of admissible rings in many respects. It is a main motivation for this thesis to establish combinatorial versions of algebraic concepts on admissible rings which are compatible with their prototypes under taking values. In particular, we deal with localization, conductors, the length of a module, duality, and quasihomogeneity.

Ideals

A *fractional ideal* of an admissible ring R is an R-submodule $\mathfrak{I}$ of the total ring of fractions Q_R of R such that $x\mathfrak{I} \subset R$ for some non-zerodivisor $x \in R$. Analogously we define a *semigroup ideal* of a good semigroup $S \subset \mathbb{Z}^s$ to be a non-empty subset E of $\mathbb{Z}^s$ such that $E + S \subset E$ and $\alpha + E \subset R$ for some $\alpha \in S$. Moreover, we call E *good* if it satisfies (E1) and (E2). Then the *value semigroup ideal* of a fractional ideal of R (defined by taking the values of the elements of $\mathfrak{I}$ which are non-zerodivisors in Q_R) is a good semigroup ideal of Γ_R.

A drawback of this construction is that taking values does in general not relate compatibly the product and quotient of fractional ideals with their combinatorial counterparts, the sum and difference of good semigroup ideals. In fact, the set of good semigroup ideals of a good semigroup is in general not even closed under these operations. However, for example in the case of conductors or canonical ideals, taking the difference is a operation on the set of good semigroup ideals, and it is also compatible with the ideal quotient under taking values.

Dualities

A canonical module ω_R of a Cohen–Macaulay ring R induces a duality

$$M \mapsto \operatorname{Ext}_R^{\dim R - \dim M}(M, \omega_R).$$

If, for example, R is generically Gorenstein, the canonical module can be chosen to be a fractional ideal $\mathfrak{K}$, and on the fractional ideals of R the duality can be expressed in terms of the ideal quotient as

$$\mathfrak{I} \mapsto \mathfrak{K} : \mathfrak{I}.$$

This leads to the definition of a *canonical ideal* of a one-dimensional Cohen–Macaulay ring R as a fractional ideal of R satisfying $\mathfrak{I} = \mathfrak{K} : (\mathfrak{K} : \mathfrak{I})$ for all fractional ideals $\mathfrak{I}$ of R. Then a canonical ideal of R is a canonical module of R.

So a one-dimensional Cohen–Macaulay ring R is Gorenstein if it is a canonical ideal of itself. Kunz showed that an analytically irreducible and residually rational one-dimensional local ring R is Gorenstein if and only if its (numerical) semigroup of values Γ_R is *symmetric* [6]. Jäger used this symmetry condition to define a semigroup ideal K such that (suitably normalized) canonical ideals $\mathfrak{K}$ of R are characterized by having value semigroup ideal $\Gamma_{\mathfrak{K}} = K$ [11].

Waldi was the first to describe a symmetry property of the semigroup of values of a plane algebroid curve with two branches [3]. In analogy to Kunz' result, Delgado then characterized general Gorenstein algebroid curves in terms of a symmetry of their semigroups of values [12, 7]. Later Campillo, Delgado, and Kiyek extended Delgado's result to include analytically reduced and residually rational local rings with infinite residue field [13].

In the spirit of Jäger's approach, D'Anna turned Delgado's symmetry condition into an explicit formula for a *canonical semigroup ideal* K^0. He showed that any (suitably normalized) fractional ideal $\mathfrak{K}$ of an analytically reduced and residually rational one-dimensional local ring with infinite residue field is canonical if and only if $\Gamma_{\mathfrak{K}} = K^0$ [8]. More recently Pol computed explicitly the value semigroup ideal $\Gamma_{R:\mathfrak{I}}$ of the dual $R : \mathfrak{I}$ of any fractional ideal $\mathfrak{I}$ of a Gorenstein algebroid curve R as $\Gamma_{R:\mathfrak{I}} = \Gamma_R - \Gamma_{\mathfrak{I}}$ [14].

In Chapter 5 we unify and extend D'Anna's and Pol's results. In particular, we work in the more general class of admissible rings. First, however, we introduce a purely combinatorial version of duality.

Statement (See Theorem 5.14)**.** *Any good semigroup S admits a* canonical semigroup ideal*, that is a good semigroup ideal K of S inducing a duality $E \mapsto K - E$ on the good semigroup ideals of S. In particular, the set of good semigroup ideals is closed under taking duals, and*

$$K - (K - E) = E$$

for every good semigroup ideal E of S.

It turns out that our canonical semigroup ideals are exactly the translations of D'Anna's K^0. Moreover, using combinatorial properties we can relate the duality on fractional ideals to the duality on good semigroup ideals in the following way.

Statement (See Theorems 5.31 and 5.34)**.** *Let R be an analytically reduced one-dimensional equidimensional semilocal Cohen–Macaulay ring with sufficiently large residue fields and trivial residue field extensions. A fractional ideal $\mathfrak{K}$ of R is canonical if and only if its value semigroup ideal is a canonical semigroup ideal of the semigroup of values of R. Moreover, if $\mathfrak{K}$ is a canonical ideal of R, then there is a commutative diagram*

$$\begin{tikzcd}
\left\{\begin{matrix}\textit{regular fractional}\\ \textit{ideals of } R\end{matrix}\right\} \arrow[r, "\mathfrak{I}\mapsto\mathfrak{K}:\mathfrak{I}"] \arrow[d, "\mathfrak{I}\mapsto\Gamma_{\mathfrak{I}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \left\{\begin{matrix}\textit{regular fractional}\\ \textit{ideals of } R\end{matrix}\right\} \arrow[d, "\mathfrak{I}\mapsto\Gamma_{\mathfrak{I}}"] \\
\left\{\begin{matrix}\textit{good semigroup}\\ \textit{ideals of } \Gamma_R\end{matrix}\right\} \arrow[r, "E\mapsto\Gamma_{\mathfrak{K}}-E"'] & \left\{\begin{matrix}\textit{good semigroup}\\ \textit{ideals of } \Gamma_R\end{matrix}\right\},
\end{tikzcd}$$

where $\Gamma_{\mathfrak{I}}$ denotes the value semigroup ideal of a fractional ideal $\mathfrak{I}$.

Algorithmic Normalization

Endomorphism rings occur in the construction of blow ups [15] or non-commutative resolutions [16, 17]. A non-commutative crepant resolution of a curve can be computed [18] considering the intermediate steps of a normalization algorithm [19] which is based on a characterization of normality in terms of the endomorphism ring of a so-called *test ideal* [20]: a reduced Noetherian ring R is normal if and only if $R = \mathrm{End}_R(\mathfrak{i})$ for a test ideal $\mathfrak{i}$ of R. If R is a reduced one-dimensional Noetherian local ring, then the maximal ideal $\mathfrak{m}$ is the unique test ideal for R.

The above criterion by Grauert and Remmert can be turned into an algorithm for normalization computing successively endomorphism rings of test ideals. Following an idea by Böhm, Decker, and Schulze [21] we use the semigroup of values to determine the intermediate steps explicitly. In general, not much is known about the properties of sequences obtained by the Grauert–Remmert algorithm. As a step towards a more fundamental understanding, we prove in Chapter 5 the following result on Gorenstein algebroid curves.

Statement (See Theorems 5.42 and 5.56)**.** *Let R be a Gorenstein complex algebroid curve with maximal ideal $\mathfrak{m}$. Then $\operatorname{End}_R(\mathfrak{m})$ is Gorenstein if and only if R is of type $\mathrm{A_n}$ for some $n \in \mathbb{N}$ (see [22]).*

In Chapter 8 we apply the Grauert–Remmert algorithm to two kinds of arrangements. First we study plane arrangements of smooth curves.

Statement (See Theorems 8.1 and 8.2)**.** *Let C be a reduced plane curve. Suppose that C has only finitely many singular points, and assume that the analytic branches at the singular points of C are smooth and intersect transversally. Then the number of steps in the Grauert–Remmert algorithm which is needed to compute the normalization of C is determined by the maximal number of analytic branches intersecting in a singular point of C.*

Using Serre's criterion which allows for checking normality in codimension one, we apply this result to hyperplane arrangements. Geometrically, after localization in codimension one we look at "transversal slices" of the arrangement. This reduces the problem to plane line arrangements whose cardinalities are the numbers of hyperplanes intersecting the respective slices.

Statement (See Theorem 8.14)**.** *Let $(\mathcal{A}, V)$ be an arrangement of hyperplanes. Then the Grauert–Remmert algorithm computes the normalization of the arrangement after*

$$\max\{\mu_{\mathcal{A}}(V, X) \mid X \in L(\mathcal{A}) \text{ with } \operatorname{codim} X = 2\}$$

steps, where $L(\mathcal{A})$ is the set of intersections of hyperplanes of $\mathcal{A}$, and $\mu_{\mathcal{A}}$ is the Möbius function of the arrangement.

Quasihomogeneous Semigroups

Kunz and Ruppert gave a description of quasihomogeneous curve singularities with at most two branches in terms of the semigroups of values of their branches [9]. An irreducible quasihomogeneous curve singularity is determined completely by its semigroup of values, and a quasihomogeneous curve singularity with two branches can be reconstructed as a fibre product of its branches over their intersection from combinatorial and analytic data: the semigroups of values of the branches as well as certain value semigroup ideals and a coefficient map. We show that the combinatorial informations can be deduced from the semigroup of values of the curve singularity.

In Chapter 6 we give a generalization of this result to quasihomogeneous curve singularities with arbitrarily many branches. Here we use an extended notion of fibre products which is introduced in Chapter 2. This fibre product is determined by the semigroup of values of the curve singularity and a certain coefficient map. In general, however, the curve singularity is not completely described by the fibre product.

In Chapter 7 we transfer the concept of quasihomogeneity to good semigroups. First we define gradings on a good semigroup, then we consider properties of values of homogeneous elements of quasihomogeneous curve singularities. In fact, both approaches yield the same concept of quasihomogeneity on good semigroups, and this is compatible with the algebraic definition.

Statement (See Proposition 7.6)**.** *The semigroup of values of a quasihomogeneous curve singularity is quasihomogeneous.*

On quasihomogeneous semigroups we introduce a closedness property related to the weights of the grading. This allows to characterize those quasihomogeneous curve singularities which can be reconstructed as a fibre product.

Statement (See Theorem 7.23)**.** *A quasihomogeneous curve singularity is isomorphic to a fibre product if and only if its semigroup of values is closed.*

Moreover, this closedness allows to construct curve singularities from good semigroups.

Statement (See Theorem 7.24)**.** *A quasihomogeneous semigroup S is closed if and only if it is the semigroup of values of a quasihomogeneous curve singularity. If S is closed, then a quasihomogeneous curve singularity R with $\Gamma_R = S$ can be constructed as a fibre product solely from S.*

We show that a quasihomogeneous semigroup with two branches is always closed. This yields the result by Kunz and Ruppert. Finally, the results above imply that a closed quasihomogeneous semigroup can be reconstructed from information on its branches. In fact, we obtain a stronger statement.

Statement (See Theorem 7.27)**.** *Any quasihomogeneous semigroup S can be reconstructed from its branches and certain ideals of its branches (which are determined by S).*

2. Preliminaries

The purpose of this chapter is to provide the fundamental material for this thesis. In Section 2.1 we introduce the monoid of regular fractional ideals of a ring. This concept is important for the study of valuation rings (see Chapter D).

In Section 2.2 we deal with discrete valuation rings and discrete valuations. Later we use valuation theory on one-dimensional Cohen–Macaulay rings to relate algebra and combinatorics.

Finally, in Section 2.3 we introduce a generalization of the usual fibre product. This will be applied in the context of quasihomogeneous curves in Chapters 6 and 7.

2.1. Regular and Fractional Ideals

In this section we study the set of R-submodules of the total ring of fractions Q_R of a ring R. This set is a monoid with respect to the product, and it is closed under quotients (see Proposition 2.7). In particular, we are interested in fractional ideals of R, that is "ideals with a common denominator" (see Definition 2.5). The set of all regular fractional ideals is a submonoid of the monoid of regular R-submodules of Q_R, and it is also closed under quotients (see Proposition 2.7).

Definition 2.1. Let R be a ring, and let $\mathfrak{I}$ and $\mathfrak{J}$ be R-submodules of Q_R.

(1) The *product* of $\mathfrak{I}$ and $\mathfrak{J}$ is

$$\mathfrak{I}\mathfrak{J} = \left\{ \sum_{(x,y)\in\Lambda} xy \;\middle|\; \Lambda \subset \mathfrak{I} \times \mathfrak{J} \text{ finite} \right\}.$$

(2) The *quotient* of $\mathfrak{I}$ and $\mathfrak{J}$ in Q_R is

$$\mathfrak{I} :_{Q_R} \mathfrak{J} = \{x \in Q_R \mid x\mathfrak{J} \subset \mathfrak{I}\} \in \mathcal{R}_R.$$

We also write $\mathfrak{I} : \mathfrak{J}$ instead of $\mathfrak{I} :_{Q_R} \mathfrak{J}$.

Lemma 2.2. *Let R be a ring, and let $\mathfrak{I}$ and $\mathfrak{J}$ be R-submodules of Q_R. Then $\mathfrak{I}\mathfrak{J}$ and $\mathfrak{I} : \mathfrak{J}$ are R-submodule of Q_R.*

Proof. By definition we have $\mathfrak{I}\mathfrak{J} \subset Q_R$ and $\mathfrak{I} : \mathfrak{J} \subset Q_R$. Moreover, the set $\mathfrak{I}\mathfrak{J}$ is by definition an R-module. So let $x, y \in \mathfrak{I} : \mathfrak{J}$, and let $r, s \in R$. Then

$$(rx + sy)\mathfrak{I} = rx\mathfrak{I} + sy\mathfrak{I} = x\mathfrak{I} + y\mathfrak{I} \subset \mathfrak{J} + \mathfrak{J} = \mathfrak{J}$$

since $\mathfrak{I}$ and $\mathfrak{J}$ are R-modules. Thus, $\mathfrak{I} : \mathfrak{J}$ is an R-submodule of Q_R. □

Lemma 2.3. *Let R be a ring, let $x \in Q_R^{\mathrm{reg}}$, and let $\mathfrak{I}$, $\mathfrak{I}'$, $\mathfrak{J}$, $\mathfrak{J}'$, and $\mathfrak{H}$ be R-submodules of Q_R. Then*

(1) $(\mathfrak{I} : \mathfrak{J}) : \mathfrak{H} = \mathfrak{I} : (\mathfrak{J}\mathfrak{H}) = (\mathfrak{I} : \mathfrak{H}) : \mathfrak{J}$,

(2) $(x\mathfrak{I}) : \mathfrak{J} = x(\mathfrak{I} : \mathfrak{J}) = \mathfrak{I} : (x^{-1}\mathfrak{J})$,

(3) $\mathfrak{I} : \mathfrak{J}' \subset \mathfrak{I} : \mathfrak{J} \subset \mathfrak{I}' : \mathfrak{J}$ *if* $\mathfrak{I} \subset \mathfrak{I}'$ *and* $\mathfrak{J} \subset \mathfrak{J}'$, *and*

(4) $\mathfrak{I} : \mathfrak{J} = (\mathfrak{I} : A) : \mathfrak{J}$ *if A is a ring with $R \subset A \subset Q_R$ and $\mathfrak{J}$ is an A-module.*

Proof. (1) By the definition of the ideal quotient (see Definition 2.1) we have

$$
\begin{aligned}
(\mathfrak{I} : \mathfrak{J}) : \mathfrak{H} &= \{x \in Q_A \mid xy \in \mathfrak{I} : \mathfrak{J} \text{ for all } y \in \mathfrak{H}\} && (2.1)\\
&= \{x \in Q_A \mid xyz \subset \mathfrak{I} \text{ for all } y \in \mathfrak{H} \text{ and } z \in \mathfrak{J}\} && (2.2)\\
&= \{x \in Q_A \mid xz \subset \mathfrak{I} : \mathfrak{H} \text{ for all } z \in \mathfrak{J}\} && (2.3)\\
&= (\mathfrak{I} : \mathfrak{H}) : \mathfrak{J}. && (2.4)
\end{aligned}
$$

Let $x \in (\mathfrak{I} : \mathfrak{J}) : \mathfrak{H}$, and let $y, y' \in \mathfrak{H}$ and $z, z' \in \mathfrak{J}$. Then Equation (2.2) yields $xyz, xy'z' \in \mathfrak{I}$, and hence $x\,(yz + y'z') \in \mathfrak{I}$ since $\mathfrak{I} \in \mathcal{R}_A$. This implies $x\mathfrak{J}\mathfrak{H} \subset \mathfrak{I}$, and Equation (2.2) yields

$$
\begin{aligned}
(\mathfrak{I} : \mathfrak{J}) : \mathfrak{H} &= \{x \in Q_A \mid xyz \subset \mathfrak{I} \text{ for all } y \in \mathfrak{H} \text{ and } z \in \mathfrak{J}\}\\
&= \{x \in Q_A \mid x\mathfrak{J}\mathfrak{H} \subset \mathfrak{I}\}\\
&= \mathfrak{I} :\ (\mathfrak{J}\mathfrak{H})\,.
\end{aligned}
$$

(2) Since $Q_R^{\mathrm{reg}} = Q_R^*$, we have

$$(x\mathfrak{I}) : \mathfrak{J} = \{y \in Q_R \mid y\mathfrak{J} \subset x\mathfrak{I}\} = \Big\{y \in Q_R \;\Big|\; yx^{-1}\mathfrak{J} \subset \mathfrak{I}\Big\}.$$

(3) This follows immediately from Definition 2.1.

(4) Since A is an R-submodule of Q_R, and since $\mathfrak{J}$ is an A-module, (1) yields

$$\mathfrak{I} : \mathfrak{J} = \mathfrak{I} : (\mathfrak{J}A) = (\mathfrak{I} : A) : \mathfrak{J}. \qquad \square$$

Definition 2.4. Let R be a ring.

(1) An R-submodule $\mathfrak{I}$ of Q_R is called *regular* if $\mathfrak{I}^{\mathrm{reg}} = \mathfrak{I} \cap Q_R^{\mathrm{reg}} \neq \emptyset$, or, equivalently, $Q_R\mathfrak{I} = Q_R$.

(2) If every regular ideal $\mathfrak{i}$ of R is generated by $\mathfrak{i}^{\mathrm{reg}}$, then R is called a *Marot* ring.

Definition 2.5. Let R be a ring.

(1) A *fractional ideal* of R is an R-submodule $\mathfrak{I}$ of Q_A such that $x\mathfrak{I} \subset R$ for some $x \in R^{\mathrm{reg}}$.

(2) The set of regular fractional ideals of R is denoted by $\mathcal{R}_R$.

Remark 2.6. Let R be a ring.

(1) If R is Noetherian, then an R-submodule $\mathfrak{I}$ of Q_R is a fractional ideal of R if and only if it is finitely generated.

(2) If R is a Marot ring, then any regular fractional ideal $\mathfrak{I} \in \mathcal{R}_R$ is generated by $\mathfrak{I}^{\mathrm{reg}}$.

Proposition 2.7. *Let R be a ring.*

(1) The set of regular R-submodules of Q_R and the set $\mathcal{R}_R$ are a commutative monoids with respect to product of ideals (the neutral element is R).

(2) The set of regular R-submodules of Q_R and the set $\mathcal{R}_R$ are closed under ideal quotient, i.e. $(\mathfrak{I} : \mathfrak{J})^{\mathrm{reg}} \neq \emptyset$ for all regular R submodules $\mathfrak{I}$ and $\mathfrak{J}$ of Q_R, and with $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$ also $\mathfrak{I} : \mathfrak{J} \in \mathcal{R}_R$.

Proof. Let $\mathfrak{I}$ and $\mathfrak{J}$ be regular R-submodules of Q_R. Then $\mathfrak{I}^{\mathrm{reg}}, \mathfrak{J}^{\mathrm{reg}} \neq \emptyset$. Moreover, $\mathfrak{I}\mathfrak{J}$ and $\mathfrak{I} : \mathfrak{J}$ are R-submodules of Q_R by Lemma 2.2. If $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$, then there are $x, y \in R^{\mathrm{reg}}$ such that $x\mathfrak{I}, y\mathfrak{J} \subset R$.

(1) For any $x \in \mathfrak{I}^{\mathrm{reg}}$ and $y \in \mathfrak{J}^{\mathrm{reg}}$ we have $xy \in (\mathfrak{I}\mathfrak{J})^{\mathrm{reg}}$. Hence, $\mathfrak{I}\mathfrak{J}$ is a regular R-submodule of Q_R. Moreover, we obviously have $\mathfrak{H}R = \mathfrak{H}$ for any $\mathfrak{H} \in \mathcal{R}_R$ since $\mathfrak{H}$ is an R-module, and since $1 \in R$.

If $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$, then $yx\mathfrak{I}\mathfrak{J} \subset yR\mathfrak{J} \subset y\mathfrak{J} \subset R$, and hence $\mathfrak{I}\mathfrak{J} \in \mathcal{R}_R$.

(2) Obviously, $\mathfrak{I} : \mathfrak{J}$ is an R-submodule of Q_R. Let $a \in \mathfrak{I}^{\mathrm{reg}}$. Then $ay \in Q_R^{\mathrm{reg}}$ and $ay\mathfrak{J} \subset aR \subset \mathfrak{I}$. Thus, $\mathfrak{I} : \mathfrak{J}$ is regular.

Suppose now that $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$, and let $b \in \mathfrak{J}^{\mathrm{reg}}$. Then $byx \in R^{\mathrm{reg}}$ and $yxb(\mathfrak{I} : \mathfrak{J}) \subset yx\mathfrak{I} \subset yR \subset R$. Hence, $\mathfrak{I} : \mathfrak{J} \in \mathcal{R}_R$. □

Definition 2.8. Let R be a ring. An R-submodule $\mathfrak{I}$ of Q_R is called *invertible* if $\mathfrak{I}\mathfrak{J} = R$ for some R-submodule $\mathfrak{J}$ of Q_R which then is uniquely determined as $\mathfrak{J} = R : \mathfrak{I}$, see [23, Ch. II, Prop. 2.2.(1)]. For an invertible R-submodule $\mathfrak{I}$ of Q_R we write $\mathfrak{I}^{-1} = R : \mathfrak{I}$.

Remark 2.9. Let R be a ring.

(1) Let $x \in Q_R^{\mathrm{reg}}$ and $\mathfrak{I} \in \mathcal{R}_R$. Then $x\mathfrak{I} \in \mathcal{R}_R$.

(2) Every invertible R-submodule $\mathfrak{I}$ of Q_R is regular and finitely generated, see [23, Ch. II, Rem. 2.1.(3) and Prop. 2.2.(1),(2)]. In particular, if an R-submodule $\mathfrak{I}$ of Q_R is invertible, then $\mathfrak{I} \in \mathcal{R}_R$

(3) The set $\mathcal{R}_R^*$ of invertible (regular fractional) ideals of R is the largest submonoid of $\mathcal{R}_R$ which is also a group.

(4) If R is (quasi)semilocal, $\mathcal{R}_R^*$ consists of the regular principal fractional ideals of R, see [23, Ch. II, Prop. 2.2.(3)].

Lemma 2.10. *Let R be a (quasi)semilocal Marot ring, let $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$, and let $\mathfrak{H} \in \mathcal{R}_R^*$. Then $(\mathfrak{I}\mathfrak{J}) : \mathfrak{H} = \mathfrak{I} : (\mathfrak{J} : \mathfrak{H})$. In particular, $\mathfrak{I} : \mathfrak{H} = \mathfrak{I}\mathfrak{H}^{-1}$.*

Proof. Since $\mathfrak{H} \in \mathcal{R}_R^*$, there is by Remark 2.9.(4) an $x \in Q_R^{\mathrm{reg}}$ such that $\mathfrak{H} = xR$. Then Lemma 2.3.(2) yields

$$\begin{aligned}(\mathfrak{I}\mathfrak{J}) : \mathfrak{H} &= (\mathfrak{I}\mathfrak{J}) : (xR) \\ &= x^{-1}((\mathfrak{I}\mathfrak{J}) : R) \\ &= x^{-1}\mathfrak{I}\mathfrak{J} \\ &= x^{-1}\mathfrak{I}(\mathfrak{J} : R) \\ &= \mathfrak{I}(\mathfrak{J} : xR) \\ &= \mathfrak{I}(\mathfrak{J} : \mathfrak{H}).\end{aligned}$$

In particular, this implies

$$\mathfrak{I} : \mathfrak{H} = (\mathfrak{I}R) : \mathfrak{H} = \mathfrak{I}(R : \mathfrak{H}) = \mathfrak{I}\mathfrak{H}^{-1}.$$ □

Lemma 2.11. *Let R and A be rings such that $Q_R = Q_A$ and $A \in \mathcal{R}_R$. Then $\mathfrak{I} : A \in \mathcal{R}_R \cap \mathcal{R}_A$ for any $\mathfrak{I} \in \mathcal{R}_R$.*

Proof. Let $\mathfrak{I} \in \mathcal{R}_R$. Then $\mathfrak{I} : A \in \mathcal{R}_R$ by Proposition 2.7.(2). Therefore, $\mathfrak{I} : A$ is a regular A-submodule of $Q_R = Q_A$. Moreover, since $A \in \mathcal{R}_R$, we also have $A : R \in \mathcal{R}_A$ by Proposition 2.7.(2). Hence, there is an $x \in (A : R)^{\mathrm{reg}}$. As $\mathfrak{I} : A \in \mathcal{R}_R$, there is a $y \in R^{\mathrm{reg}}$ such that $y(\mathfrak{I} : A) \subset A$. This yields $xy \in A^{\mathrm{reg}}$ and

$$xy(\mathfrak{I} : A) \subset xR \subset A.$$

Thus, $\mathfrak{I} : A \in \mathcal{R}_A$. □

Lemma 2.12. *Let R and A be rings such that $R \subset A \subset Q_R$ and $A \in \mathcal{R}_R$. Then $\mathcal{R}_A \subset \mathcal{R}_R$.*

Proof. If $R \subset A \subset Q_R$, then $Q_R = Q_A$ by Lemma A.34. Let $\mathfrak{I} \in \mathcal{R}_A$. Then $\mathfrak{I}^{\mathrm{reg}} \neq \emptyset$, and $R\mathfrak{I} \subset A\mathfrak{I} \subset \mathfrak{I}$. Moreover, there is an $x \in A^{\mathrm{reg}}$ such that $x\mathfrak{I} \subset A$. Since $A \in \mathcal{R}_R$, there is a $y \in R^{\mathrm{reg}}$ such that $yA \subset R$. Then $xy \in R^{\mathrm{reg}}$, and

$$xy\mathfrak{I} \subset xA \subset R.$$

Thus, $\mathfrak{I} \in \mathcal{R}_R$. □

Lemma 2.13. *Let R be a ring, and let $\mathfrak{I}$ and $\mathfrak{J}$ be regular R-submodules of Q_R. Then there is a natural R-module isomorphism*

$$\begin{aligned}\phi_{\mathfrak{J}}^{\mathfrak{I}} \colon \operatorname{Hom}_R(\mathfrak{I}, \mathfrak{J}) &\to \mathfrak{J} : \mathfrak{I}, \\ \phi &\mapsto \frac{\phi(x)}{x},\end{aligned}$$

which is independent of the choice of a regular element $x \in \mathfrak{I}^{\mathrm{reg}}$. In particular, any $\phi \in \operatorname{Hom}_R(\mathfrak{I}, \mathfrak{J})$ is multiplication by an element of $\mathfrak{J} : \mathfrak{I}$, and it can be extended uniquely to an endomorphism of Q_R.

Proof. See [24, Lemma 2.1] and [19, Lemma 3.1]. □

Remark 2.14. Let R be a ring. With Lemma 2.13 we may define the *dual* of a regular fractional ideal $\mathfrak{I} \in \mathcal{R}_R$ as

$$\mathfrak{I}^\vee = \mathrm{Hom}_R(\mathfrak{I}, R) \cong R : \mathfrak{I}.$$

Note that if $\mathfrak{I} \in \mathcal{R}_R^*$, then $\mathfrak{I}^{-1} \cong \mathfrak{I}^\vee$.

Proposition 2.15. *Let R be a ring, and let $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$. Then*

$$\mathfrak{J} \subset \mathfrak{I} : (\mathfrak{I} : \mathfrak{J}).$$

Proof. Let $x \in \mathfrak{J}$. Then we have for all $y \in \mathfrak{I} : \mathfrak{J}$

$$xy \subset y\mathfrak{J} \subset \mathfrak{I}.$$

This implies $x \in \mathfrak{I} : (\mathfrak{I} : \mathfrak{J})$. □

Lemma 2.16 (See [25], Lemma 2.1.3)**.** *Let R and A be rings such that there is a flat ring homomorphism $\alpha \colon R \to A$. Then there is a ring homomorphism*

$$\begin{aligned} \phi \colon Q_R &\to Q_A, \\ \frac{x}{y} &\mapsto \frac{\alpha(x)}{\alpha(y)}. \end{aligned}$$

Moreover, the following hold:

(1) *Suppose α is injective. Then ϕ is injective, and*

$$\phi(\mathfrak{I})A = \mathfrak{I} \otimes_R A$$

for any R-submodule $\mathfrak{I}$ of Q_R.

(2) *For any fractional ideal $\mathfrak{I}$ of R we have*

$$\mathfrak{I} \otimes_R A = \phi(\mathfrak{I})A.$$

Moreover, if $\mathfrak{I} \in \mathcal{R}_R$, then $\mathfrak{I} \otimes_R A = \phi(\mathfrak{I})A \in \mathcal{R}_A$.

(3) *For any fractional ideals $\mathfrak{I}$ and $\mathfrak{J}$ of R we have*

$$\phi(\mathfrak{I} : \mathfrak{J})A = \phi(\mathfrak{I})A : \phi(\mathfrak{J})A.$$

(4) *If α is faithfully flat, then*

$$\phi(\mathfrak{I})A \cap Q_R = \mathfrak{I}$$

and

$$\phi(\mathfrak{I} \cap \mathfrak{J})A = \phi(\mathfrak{I})A \cap \phi(\mathfrak{J})A$$

for any R-submodules $\mathfrak{I}$ and $\mathfrak{J}$ of Q_R.

Proof. Since $\alpha\colon R \to A$ is flat, we have $\alpha(R^{\mathrm{reg}}) \subset A^{\mathrm{reg}}$ by Lemma A.7. Thus, Lemma A.29 yields a ring homomorphism

$$\begin{aligned} \phi\colon Q_R &\to Q_A, \\ \frac{x}{y} &\mapsto \frac{\alpha(x)}{\alpha(y)}. \end{aligned}$$

(1) The ring homomorphism ϕ is injective by Lemma 7.54.

For the bilinear map

$$\begin{aligned} \mathfrak{I} \times A &\to \phi(\mathfrak{I})A, \\ (x, y) &\mapsto \phi(x)y \end{aligned}$$

the universal property of the tensor product yields the R-module homomorphism

$$\begin{aligned} \beta\colon \mathfrak{I} \otimes_R A &\to \phi(\mathfrak{I})A, \\ x \otimes y &\mapsto \phi(x)y. \end{aligned}$$

Obviously, β is also an A-module homomorphism, and it is surjective.

Since $\mathfrak{I} \subset Q_R$, and since A is flat, we obtain $\mathfrak{I} \otimes_R A \subset Q_R \otimes_R A$. Moreover, setting $\mathfrak{I} = Q_R$ yields a surjective A-module homomorphism

$$\begin{aligned} \gamma\colon \mathfrak{I} \otimes_R A &\to \phi(\mathfrak{I})A, \\ \frac{x}{y} \otimes a &\mapsto \phi\left(\frac{x}{y}\right)a = \frac{\alpha(x)}{\alpha(y)}a. \end{aligned}$$

In particular, we obtain a commutative diagram

$$\begin{tikzcd} \mathfrak{I} \otimes_R A \arrow[r, two heads, "\beta"] \arrow[d, hook] & \phi(\mathfrak{I})A \arrow[d, hook] \\ Q_R \otimes_R A \arrow[r, two heads, "\gamma"] & \phi(Q_R)A. \end{tikzcd} \tag{2.5}$$

Since $Q_R = (R^{\mathrm{reg}})^{-1}R$, Theorem A.22 and Proposition A.38 yield an R-module isomorphism

$$\begin{aligned} \delta\colon Q_R &\to (\alpha(R^{\mathrm{reg}}))^{-1}A, \\ \frac{x}{y} \otimes a &\mapsto \frac{\alpha(x)a}{\alpha(y)}. \end{aligned}$$

Since $(\alpha(R^{\mathrm{reg}}))^{-1}A \subset Q_A$ by Lemma A.30 (recall that $\alpha(R^{\mathrm{reg}}) \subset A^{\mathrm{reg}}$ by Lemma A.7), we obtain with Diagram (2.5) a commutative diagram

$$\begin{tikzcd} \mathfrak{I} \otimes_R A \arrow[r, two heads, "\beta"] \arrow[d, hook] & \phi(\mathfrak{I})A \arrow[d, hook] \\ Q_R \otimes_R A \arrow[r, two heads, "\gamma"] \arrow[d, "\delta"', "\cong"] & \phi(Q_R)A \arrow[d, hook] \\ (\alpha(R^{\mathrm{reg}}))^{-1}A \arrow[r, hook] & Q_A. \end{tikzcd}$$

This implies that β is injective, and hence $\mathfrak{I} \otimes_R A = \phi(\mathfrak{I})A$.

(2) Let $\mathfrak{I}$ be a fractional ideal of R. Then for the bilinear map

$$\begin{aligned} \mathfrak{I} \times A &\to \phi(\mathfrak{I})A, \\ (x, y) &\mapsto \phi(x)y \end{aligned}$$

the universal property of the tensor product yields the surjective homomorphism

$$\begin{aligned} \epsilon\colon \mathfrak{I} \otimes_R A &\to \phi(\mathfrak{I})A, \\ x \otimes y &\mapsto \phi(x)y. \end{aligned}$$

Suppose that $\mathfrak{I} \subset R$. Since A is a flat R-module, we obtain a commutative diagram

$$\begin{tikzcd} \mathfrak{I} \otimes_R A \arrow[r, two heads, "\epsilon"] \arrow[d, hook] & \phi(\mathfrak{I})A \arrow[d, hook] \\ R \otimes_R A \arrow[r, "\cong"] & A. \end{tikzcd}$$

This implies that ϵ is also injective, and hence

$$\mathfrak{I} \otimes_R A = \phi(\mathfrak{I})A. \tag{2.6}$$

Let now $\mathfrak{I}$ be a general fractional ideal of R. Then there is an $x \in Q_R^{\mathrm{reg}}$ such that $x\mathfrak{I} \subset R$. Moreover, $\phi(x) \in Q_A^{\mathrm{reg}}$ since $Q_R^{\mathrm{reg}} = Q_R^*$ and $Q_A^{\mathrm{reg}} = Q_A^*$. Then Equation (2.6) yields (considering A-modules)

$$\begin{aligned} \phi(\mathfrak{I})A &= (\phi(x))^{-1}\phi(x)\phi(\mathfrak{I})A \\ &= (\phi(x))^{-1}\phi(x\mathfrak{I})A \\ &= (\phi(x))^{-1}(x\mathfrak{I} \otimes_R A) \\ &= (\phi(x))^{-1}(\mathfrak{I} \otimes_R \phi(x)A) \\ &= (\phi(x))^{-1}\phi(x)(\mathfrak{I} \otimes_R A) \\ &= \mathfrak{I} \otimes_R A. \end{aligned}$$

Finally, $\phi(\mathfrak{I})A$ is an A-submodule of Q_A, and $\phi(x)\phi(\mathfrak{I})A = \phi(x\mathfrak{I})A \subset \phi(R)A = A$. If $\mathfrak{I} \in \mathcal{R}_R$, there is a $y \in \mathfrak{I}^{\mathrm{reg}}$. Then $\phi(y) \in \phi(\mathfrak{I}) \cap Q_A^* = \phi(\mathfrak{I}) \cap Q_A^{\mathrm{reg}} = (\phi(\mathfrak{I}))^{\mathrm{reg}}$. Therefore, $\phi(\mathfrak{I})A \in \mathcal{R}_A$.

(3) For any fractional ideals $\mathfrak{I}$ and $\mathfrak{J}$ of R part (2) as well as Propositions A.40 and 2.7.(2) and Lemma 2.13 yield the following commutative diagram of isomorphisms

$$\begin{tikzcd} \operatorname{Hom}_R(\mathfrak{I}, \mathfrak{J}) \otimes_R A \arrow[r, "\cong"] \arrow[d, "\cong"'] & (\mathfrak{J} : \mathfrak{I}) \otimes_R A \arrow[r, "="] \arrow[d, dotted, "\cong"] & \phi(\mathfrak{J} : \mathfrak{I})A \arrow[d, dotted, "\cong"] \\ \operatorname{Hom}_A(\mathfrak{I} \otimes_R A, \mathfrak{J} \otimes_R A) \arrow[r, "\cong"] \arrow[d, "="'] & (\mathfrak{J} \otimes_R A) : (\mathfrak{I} \otimes_R A) \arrow[r, "="] & \phi(\mathfrak{J})A : \phi(\mathfrak{I})A \\ \operatorname{Hom}_A(\phi(\mathfrak{I})A, \phi(\mathfrak{J})). \arrow[urr, "\cong"'] & & \end{tikzcd}$$

(4) See [26, Chapitre I, § 3, no. 5, Proposition 10]. □

Lemma 2.17. *Let R be a Noetherian ring, and let $\mathfrak{I} \in \mathcal{R}_R$. Then $\mathfrak{I} : \mathfrak{I}$ is an integral extension of R. In particular,*

$$R \subset \mathfrak{I} : \mathfrak{I} \subset \overline{R}$$

Proof. Since $\mathfrak{I}$ is a fractional ideal of R, we have $\mathfrak{I}R \subset \mathfrak{I}$, and hence $R \subset \mathfrak{I} : \mathfrak{I}$. In particular, $1 \in \mathfrak{I} : \mathfrak{I}$. Let $x, y \in \mathfrak{I} : \mathfrak{I}$. Then

$$xy\mathfrak{I} \subset x\mathfrak{I} \subset \mathfrak{I}.$$

Since $\mathfrak{I} : \mathfrak{I}$ is an R-module by Lemma 2.2, this implies that it is a ring. Moreover, since R is Noetherian, $\mathfrak{I} : \mathfrak{I}$ is by Proposition 2.7.(2) and Remark 2.6.(1) finite over R. Thus, $\mathfrak{I} : \mathfrak{I}$ is by Theorem B.11 an integral extension of R. The particular claim follows with Proposition B.5. Also see [27, Lemma 3.6.1] and Lemma 2.13. □

2.2. Discrete Valuation(Ring)s

In order to relate algebra and combinatorics we apply valuation theory. A *valuation* of a ring A is a surjective map ν from A onto a totally ordered abelian monoid G^ν_∞ such that

$$\nu(xy) = \nu(x) + \nu(y), \tag{2.7}$$
$$\nu(x + y) \geq \min\{\nu(x), \nu(y)\} \tag{2.8}$$

for every $x, y \in A$, where G^ν is a totally ordered additive abelian group (the *value group* of ν) which we include into the totally ordered abelian monoid $G^\nu_\infty = G^\nu \cup \{\infty\}$ with $x + \infty = \infty$, $\infty + \infty = \infty$ and $\infty > x$ for all $x \in G^\nu$. To a valuation $\nu\colon A \to G^\nu_\infty$ we associate its *valuation ring*

$$V_\nu = \{x \in A \mid \nu(x) \geq 0\}.$$

For more on valuations see Section D.2.

Definition 2.18. Let A be a ring. A valuation ν of A is said to be a *discrete valuation* if there is an order preserving group isomorphism $\phi\colon G^\nu \to \mathbb{Z}$.

We may also start with rings of valuations. Let Q be a ring having a large Jacobson radical with $Q^{\mathrm{reg}} = Q^*$. A *valuation ring* of Q is a subring V of Q with $V \neq Q$ such that $Q \setminus V$ is multiplicatively closed. If V is a valuation ring of Q, then the group $\mathcal{R}^*_V$ is totally ordered by reverse inclusion. We include $\mathcal{R}^*_V$ into the totally ordered monoid $\mathcal{R}^*_{V,\infty} = \mathcal{R}^*_V \cup \{I_V\}$, where $I_V = V : Q$ is the *infinite prime ideal* of V. Then the *valuation* of V is the map

$$\begin{aligned} \mu_V \colon Q &\to \mathcal{R}^*_{V,\infty}, \\ x &\mapsto \mu_V(x) = \bigcap_{\substack{\mathfrak{I} \in \mathcal{R}^*_V \\ x \in \mathfrak{I}}} \mathfrak{I}. \end{aligned}$$

This map is surjective, and it satisfies

$$\mu_V(xy) = \mu_v(x)\mu_V(y),$$
$$\mu_V(x+y) \geq \min\{\mu_V(x), \mu_V(y)\}$$

for any $x, y \in Q$. Then

$$V = \{x \in Q \mid \mu_V(x) \geq V\},$$

and V has a unique regular maximal ideal

$$\mathfrak{m}_V = \{x \in Q \mid \mu_V(x) > V\}.$$

The infinite prime ideal of V is

$$I_V = \{x \in Q \mid \mu_V(x) = I_V\}.$$

For more on valuation rings see Section D.1. Note that by Corollary D.32 there is a bijection

$$V \mapsto \mu_V,$$
$$V_\nu \leftarrow\!\shortmid \nu$$

between the valuation rings and the valuations (up to equivalence, see Definition D.28) of Q.

Definition 2.19. Let Q be a ring having a large Jacobson radical with $Q^{\text{reg}} = Q^*$. A valuation ring V of Q with regular maximal ideal $\mathfrak{m}_V$ is called a *discrete valuation ring* of Q if $\mathfrak{m}_V \in \mathcal{R}_V^*$.

Remark 2.20. Let Q be a ring having a large Jacobson radical with $Q^{\text{reg}} = Q^*$. A valuation ring V of Q is by Remark D.4.(1) and (2) discrete if and only if its regular maximal ideal $\mathfrak{m}_V$ is finitely generated.

Proposition 2.21. *Let Q be a ring having a large Jacobson radical with $Q^{\text{reg}} = Q^*$, and let V be a discrete valuation ring of Q.*

(1) For the regular maximal ideal $\mathfrak{m}_V$ of V we have

$$\mathfrak{m}_V = \min\{\mathfrak{I} \in \mathcal{R}_V^* \mid V < \mathfrak{I}\} \in \mathcal{R}_V^*.$$

(2) There is an order preserving group isomorphism

$$\phi_V \colon \mathcal{R}_V^* \to \mathbb{Z},$$
$$\mathfrak{I} \mapsto \phi_V(\mathfrak{I}) = \max\left\{k \in \mathbb{Z} \,\middle|\, \mathfrak{m}_V^k \leq \mathfrak{I}\right\},$$
$$\mathfrak{m}_V^k \leftarrow\!\shortmid k.$$

Proof. (1) Let $\mathfrak{I} \in \mathcal{R}_V^*$ with $\mathfrak{I} > V$. Then $\mathfrak{I}^{\text{reg}} \subset \mathfrak{m}_V$ by Remark D.5, and hence $\mathfrak{I} \subset \mathfrak{m}_V$ since V is a Marot ring. The claim follows since $\mathfrak{m}_V \in \mathcal{R}_V^*$ by the definition of V.

(2) Let $\mathfrak{J} \in \mathcal{R}_V^*$. Since $\mathcal{R}_V$ is totally ordered by Remark D.4.(3), we have either $\mathfrak{J} \leq V$ or $\mathfrak{J} \geq V$. Suppose $\mathfrak{J} \geq V$, i.e. $\mathfrak{J} \subset V$. Since $\mathfrak{m}_V \in \mathcal{R}_V^*$, also $\mathfrak{m}_V^k \in \mathcal{R}_V^*$ for any $k \in \mathbb{N}$. Hence, for any $k \in \mathbb{N}$ Remark D.4.(3) yields either $\mathfrak{m}_V^k \leq \mathfrak{J}$ or $\mathfrak{m}_V^k \leq \mathfrak{J}$. Assume $\mathfrak{m}_V^k \leq \mathfrak{J}$ for all $k \in \mathbb{N}$. Then $\mathfrak{J} \subset \bigcap_{k=1}^{\infty} \mathfrak{m}_V^k \subset V \setminus V^{\mathrm{reg}}$ by Corollary A.5, contradicting $\mathfrak{J} \in \mathcal{R}^*$. We obtain $\bigcap_{k=1}^{\infty} \mathfrak{m}_V^k \subsetneq \mathfrak{J} \subset V$, and hence there is $\max\left\{k \in \mathbb{Z} \,\middle|\, \mathfrak{m}_V^k \leq \mathfrak{J}\right\}$.

Suppose now that $\mathfrak{J} \leq V$. Then $\mathfrak{J}^{-1} \geq V$. Arguing as above we find

$$\min\left\{l \in \mathbb{Z} \,\middle|\, \mathfrak{m}_V^l \geq \mathfrak{J}^{-1}\right\} = \max\left\{k \in \mathbb{Z} \,\middle|\, \mathfrak{m}_V^k \leq \mathfrak{J}\right\}.$$

These considerations show that for any $\mathfrak{J} \in \mathcal{R}_V^*$ there is a $k \in \mathbb{Z}$ such that $\mathfrak{m}_V^k \leq \mathfrak{J}$ and $\mathfrak{m}_V^{k+1} \not\leq \mathfrak{J}$. Then we have

$$V = \mathfrak{m}_V^k : \mathfrak{m}_V^k \leq \mathfrak{J} : \mathfrak{m}_V^k \tag{2.9}$$

by Lemma 2.3.(3).

Assume $\mathfrak{J} : \mathfrak{m}_V^k \geq \mathfrak{m}_V$. Since $\mathfrak{m}_V$ is invertible, this implies $\mathfrak{J} \geq \mathfrak{m}_V^{k+1}$ by Lemma 2.10 contradicting the assumption on k. Therefore, $\mathfrak{J} : \mathfrak{m}_V^k < \mathfrak{m}_V$ since $\mathfrak{J} : \mathfrak{m}_V^k \in \mathcal{R}_V^*$ and since $\mathcal{R}_V^*$ is totally ordered by Remark D.4.(3). Thus, (1) and Equation (2.9) yield $V = \mathfrak{J} : \mathfrak{m}_V^k$, i.e. $\mathfrak{J} = \mathfrak{m}_V^k$ by Lemma 2.10. □

Let V be a discrete valuation ring of Q. Embedding $\mathbb{Z}$ into the totally ordered monoid $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$ we may extend ϕ_V to an order preserving isomorphism of monoids

$$\phi_V \colon \mathcal{R}_{V,\infty}^* \to \mathbb{Z}_\infty$$

by setting $\phi_V(I_V) = \infty$. Then Proposition 2.21.(2) yields a commutative diagram

$$\begin{tikzcd} Q \arrow[d, two heads, "\mu_V"'] \arrow[dr, "\nu_V"] & \\ \mathcal{R}_{V,\infty}^* \arrow[r, "\cong", "\phi_V"'] & \mathbb{Z}_\infty, \end{tikzcd} \tag{2.10}$$

where ν_V is a discrete valuation of Q. In particular, μ_V and ν_V are equivalent, and hence $V = V_{\nu_V}$ is by Proposition D.29 the ring of a discrete valuation.

Proposition 2.22. *Let Q be a ring having a large Jacobson radical with $Q^{\mathrm{reg}} = Q^*$, and let V be a valuation ring of Q. Then the following are equivalent:*

(a) The ring V is a discrete valuation ring.

(b) V is the ring of a discrete valuation $\nu \colon Q \to \mathbb{Z}_\infty$.

(c) Every regular ideal of V is finitely generated.

(d) The regular maximal ideal $\mathfrak{m}_V$ is finitely generated, and $\mathfrak{m}_V$ is the only regular prime ideal of V.

Proof. See Propositions D.13.(1) and 2.21.(2) and [23, Chapter I, Proposition 2.15]. □

Proposition 2.23. *Let Q be a ring having a large Jacobson radical with $Q^{\mathrm{reg}} = Q^*$, and let V be a discrete valuation ring of Q.*

(1) *Any $\mathfrak{I} \in \mathcal{R}_V$ contains a regular element of minimal value, i.e. there is an $x \in \mathfrak{I}^{\mathrm{reg}}$ such that $\nu_V(x) \leq \nu_V(y)$ for all $y \in \mathfrak{I}$.*

(2) *Each $\mathfrak{I} \in \mathcal{R}_V$ is generated by any element $x \in Q^{\mathrm{reg}}$ with $\nu_V(x) = \min\{\nu_V(y) \mid y \in \mathfrak{I}\}$. In particular, $\mathfrak{m}_V$ is generated by any $t \in Q^{\mathrm{reg}}$ with $\nu_V(t) = 1$. Such a t is said to be a* uniformizing parameter *for V.*

(3) *Let $\mathfrak{I} \in \mathcal{R}_V$. Any finite generating set for $\mathfrak{I}$ contains an element $x \in Q$ with $\nu_V(x) = \min\{\nu_V(y) \mid y \in \mathfrak{I}\}$.*

(4) *If $\mathfrak{I} \in \mathcal{R}_V$, then*

$$\mathfrak{I} = \{x \in Q \mid \nu_V(x) \geq \min\{\nu_V(y) \mid y \in \mathfrak{I}\}\}.$$

(5) *Let $\mathfrak{I} \in \mathcal{R}_V^*$. Then*

$$\phi_V(\mathfrak{I}) = \min\{\nu_V(x) \mid x \in \mathfrak{I}\},$$

and for any $k \in \mathbb{Z}$ we have

$$\begin{aligned}\phi_V^{-1}(k) &= xV \textit{ for all } x \in Q^{\mathrm{reg}} \textit{ with } \nu_V(x) = k\\ &= \langle y \in Q^{\mathrm{reg}} \mid \nu(y) = k\rangle\\ &= \{y \in Q \mid \nu_V(y \geq k)\}\end{aligned}$$

(see Proposition 2.21.(2)).

Proof. (1) Since $\mathfrak{I} \in \mathcal{R}_V$, there is an $a \in V^{\mathrm{reg}}$ such that $a\mathfrak{I} \subset V$. This implies $\nu_V(ax) \geq 0$, and hence $\nu_V(x) \geq -\nu_V(a)$ for all $x \in \mathfrak{I}$. Thus, there is $y \in \mathfrak{I}$ such that $\nu(y) \leq \nu(x)$ for all $x \in \mathfrak{I}$.

Assume now that $y \in \mathfrak{I} \setminus \mathfrak{I}^{\mathrm{reg}}$. Since Q has a large Jacobson radical and $V \subset Q$, V is a Marot ring, see [23, Chapter I, Proposition 1.12]. Then $\mathfrak{I}$ is generated by $\mathfrak{I}^{\mathrm{reg}}$, and hence we find $x_1, \ldots, x_n \in \mathfrak{I}^{\mathrm{reg}}$ and $a_1, \ldots, a_n \in V$ such that $y = \sum_{i=1}^n a_i x_i$. By the definition of valuations we obtain

$$\nu_V(y) = \nu_V\left(\sum_{i=1}^n a_i x_i\right) \geq \min_{i=1,\ldots,s}\{\nu_V(a_i x_i)\}.$$

Thus, there is $x \in \mathfrak{I}^{\mathrm{reg}}$ and $a \in V$ such that

$$\nu_V(y) \geq \nu_V(ax) = \nu_V(a) + \nu_V(x).$$

Since $a \in V$ yields $\nu_V(a) \geq 0$, this implies $\nu_V(y) \geq \nu_V(x)$. Hence, $\nu_V(y) = \nu_V(x)$ as y is of minimal value in $\mathfrak{I}$.

(2) By (1) there is an element $x \in \mathfrak{I}^{\mathrm{reg}}$ of minimal value in $\mathfrak{I}$. Let now $y \in \mathfrak{I}^{\mathrm{reg}}$. Then $\nu_V(y) \geq \nu_V(x)$, and hence by Lemma D.22.(2)

$$\nu_V\left(\frac{y}{x}\right) = \nu_V(y) + \nu\left(x^{-1}\right) = \nu(y) - \nu(x) \geq 0. \tag{2.11}$$

This implies $\frac{y}{x} \in V$, and therefore $y = x\frac{y}{x} \in xV$. Thus,

$$xV = \langle \mathfrak{I}^{\mathrm{reg}} \rangle = \mathfrak{I}$$

since V is a Marot ring (see above).

Let now $z \in Q^{\mathrm{reg}}$ such that $\nu_V(x) = \nu_V(z)$. Then we obtain as above with Lemma D.22.(2)

$$\nu_V\left(\frac{x}{z}\right) = \nu_V(x) + \nu_V\left(z^{-1}\right) = \nu_V(x) - \nu_V(z) = 0 = \nu_V\left(\frac{z}{x}\right),$$

and hence $\frac{x}{z}, \frac{z}{x} \in V$. This implies $zV = xV = \mathfrak{I}$.

(3) Let $\{x_1, \ldots, x_n\} \subset \mathfrak{I}$ be a generating set for $\mathfrak{I}$. Since $V = \{x \in Q \mid \nu_V(x) \geq V\}$, Equations (2.7) and (2.8) imply

$$\min\{\nu_V(x) \mid x \in \mathfrak{I}\} = \min\{\nu_V(x_1), \ldots, \nu_V(x_n)\}.$$

(4) By (2), $\mathfrak{I} = xV$ for any $x \in Q^{\mathrm{reg}}$ with $\nu_V(x) = \min\{\nu_V(y) \mid y \in \mathfrak{I}^{\mathrm{reg}}\}$. If $y \in Q$ with $\nu_V(y) \geq \nu_V(x)$, then Lemma D.22.(2) yield as in Equation (2.11)

$$\nu_V\left(\frac{y}{x}\right) = \nu_V(y) + \nu\left(x^{-1}\right) = \nu(y) - \nu(x) \geq 0,$$

and hence $\frac{y}{x} \in V$. This implies $y = x\frac{y}{x} \in xV = \mathfrak{I}$.

(5) By (2) $\mathfrak{I} = xV$ for any $x \in Q^{\mathrm{reg}}$ with $\nu_V(x) = \min\{\nu_V(y) \mid y \in \mathfrak{I}^{\mathrm{reg}}\}$. Then Remark D.14.(2) yields $\mu_V(x) = xV$, and we obtain by Proposition 2.21.(2)

$$\begin{aligned} \phi_V(\mathfrak{I}) &= \phi_V(xV) \\ &= \phi_V \circ \mu_V(x) \\ &= \nu_V(x) \\ &= \min\{\nu_V(y) \mid y \in \mathfrak{I}^{\mathrm{reg}}\}. \end{aligned}$$

Let now $k \in \mathbb{Z}$. By (2) the map

$$\begin{aligned} \psi \colon \mathbb{Z} &\to \mathcal{R}_V^* \\ k &\mapsto xV \text{ for some } x \in Q^{\mathrm{reg}} \text{ with } \nu_V(x) = k \end{aligned}$$

is well-defined, and by the considerations above we have $\psi = \phi^{-1}$. Then the equalities

$$\phi_V^{-1}(k) = \langle y \in Q^{\mathrm{reg}} \mid \nu_V(y) = k \rangle = \{y \in Q \mid \nu_V(y) \geq k\}$$

follow from (2) and (4). $\square$

In the remainder of this section we list some more properties of discrete valuation rings.

Proposition 2.24. *Let Q be a ring having a large Jacobson radical with $Q^{\mathrm{reg}} = Q^*$, and let V be a discrete valuation ring of Q.*

(1) Every regular fractional ideal of V is principal, i.e. $\mathcal{R}_V = \mathcal{R}_V^*$.

(2) Let now $t \in Q$ such that $\mathfrak{m}_V = tV$ (see Proposition 2.23.(2)). Then $t \in Q^{\mathrm{reg}}$.

(1) We have $Q = V\left[t^{-1}\right]$.

(2) Every element $x \in Q^{\mathrm{reg}}$ has a unique representation $x = at^k$, where $a \in V^*$ and $k \in \mathbb{Z}$.

(3) Any regular Q-submodule of V is of the form $t^k V$ for some $k \in \mathbb{Z}$.

(3) We have $I_V = \bigcap_{k\in\mathbb{N}} \mathfrak{m}_V^k$.

(4) There is no ring strictly between V and Q.

Proof. See Proposition 2.22 and [23, Chapter I, Proposition 2.15]. □

Corollary 2.25. *Let Q be a ring having a large Jacobson radical with $Q^{\mathrm{reg}} = Q^*$, let V be a discrete valuation ring of Q, and let $t \in Q$ such that $\mathfrak{m}_V = tV$ (see Proposition 2.23.(2)).*

(1) t is a uniformizing parameter for V.

(2) For any $x \in Q^{\mathrm{reg}}$ there is a unique $a \in V^*$ such that $x = at^{\nu_V(x)}$.

Proof. (1) By Proposition 2.24.(2) we have $t \in Q^{\mathrm{reg}}$. Then Diagram (2.10) and Proposition 2.21.(1) yield

$$\nu_V(t) = \phi_V \circ \mu_V(t) = \phi_V(tV) = \phi_V(\mathfrak{m}_V) = 1.$$

Hence, t is a uniformizing parameter for V.

(2) Let $x \in Q^{\mathrm{reg}}$. By Proposition 2.24.(2).(2) there is a unique $a \in V^*$ and a unique $k \in \mathbb{Z}$ such that $x = at^k$. Then

$$\nu_V(x) = \nu_V(a) + k\nu_V(t) = k$$

since $\nu_V(a) = 0$ by Proposition D.13.(2) and Corollary D.32, and since $\nu_V(t) = 1$ by (1). □

Theorem 2.26 (Approximation Theorem for Discrete Valuations)**.** *Let Q be a ring having a large Jacobson radical with $Q^{\mathrm{reg}} = Q^*$, and let $\mathcal{V}$ be a finite set of discrete valuation rings of Q. We set $R = \bigcap_{V\in\mathcal{V}} V$.*

(1) Every maximal ideal of R is regular, and there is a bijection

$$\begin{aligned} \operatorname{Max}\left(\overline{R}\right) &\to \mathcal{V}, \\ \mathfrak{m} &\mapsto \left((R \setminus \mathfrak{m})^{\mathrm{reg}}\right)^{-1} R, \\ \mathfrak{m}_{V_i} \cap R &\leftarrow\!\shortmid V_i \end{aligned}$$

such that $(\mathfrak{m}_V \cap R)V = \mathfrak{m}_V$ for every $V \in \mathcal{V}$.

(2) *For any* $(x_V)_{V\in\mathcal{V}} \in Q^{\mathcal{V}}$ *and any* $\alpha \in \mathbb{Z}^{\mathcal{V}}$ *there is an* $x \in Q$ *such that*

$$\nu_V(x - x_V) \geq \alpha_V$$

for every $V \in \mathcal{V}$.

(3) *For any* $\alpha \in \mathbb{Z}^{\mathcal{V}}$ *there is an* $x \in Q$ *such that*

$$\nu_V(x) = \alpha_V$$

for every $V \in \mathcal{V}$.

Proof. See [23, Chapter I, Theorem 2.20]. □

Corollary 2.27. *Let* Q *be a ring having a large Jacobson radical with* $Q^{\mathrm{reg}} = Q^*$, *let* $\mathcal{V}$ *be a finite set of discrete valuation rings of* Q, *and suppose that* $\{I_V \mid V \in \mathcal{V}\}$ *is the set of prime ideals of* Q.

(1) *Let* $x \in Q$. *Then* $x \in Q^{\mathrm{reg}}$ *if and only if* $\nu_V(x) < \infty$ *for every* $V \in \mathcal{V}$.

(2) *For any* $\alpha \in \mathbb{Z}^{\mathcal{V}}$ *there is an* $x \in Q^{\mathrm{reg}}$ *such that*

$$\nu_V(x) = \alpha_V$$

for every $V \in \mathcal{V}$.

(3) *Every regular ideal of the ring* $\bigcap_{V\in\mathcal{V}} V$ *is principal.*

Proof. See [23, Corollary 2.21]. □

2.3. Fibre Products

Let R be a reduced ring with two branches (see Definition A.69), say $\operatorname{Min}(R) = \{\mathfrak{p}, \mathfrak{q}\}$. Then R can be written as a fibre product

$$\begin{aligned} R &= R/\mathfrak{p} \times_{R/\mathfrak{p}+\mathfrak{q}} R/\mathfrak{q} \\ &= \{x \in R/\mathfrak{p} \times R/\mathfrak{q} \mid \pi_{\mathfrak{p}}(x) = \pi_{\mathfrak{q}}(x)\}, \end{aligned} \tag{2.12}$$

where $\pi_{\mathfrak{p}} \colon R \to R/\mathfrak{p}$ and $\pi_{\mathfrak{q}} \colon R \to R/\mathfrak{q}$ are the canonical surjections.

More generally, let $\mathcal{C}$ be a category, let $A, B, C \in \operatorname{Ob}\mathcal{C}$, and let $f \in \operatorname{Mor}_{\mathcal{C}}(A, C)$ and $g \in \operatorname{Mor}_{\mathcal{C}}(B, C)$. The fibre product of A and B over C is an object $A \times_C B \in \operatorname{Ob}\mathcal{C}$ with morphisms $f' \in \operatorname{Mor}_{\mathcal{C}}(A \times_C B, A)$ and $g' \in \operatorname{Mor}_{\mathcal{C}}(A \times_C B, B)$ such that the diagram

$$\begin{tikzcd} & A \times_C B \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

commutes, and it satisfies the following universal property: for any object $D \in \operatorname{Ob}\mathcal{C}$ with morphisms $f'' \in \operatorname{Mor}_{\mathcal{C}}(D, A)$ and $g'' \in \operatorname{Mor}_{\mathcal{C}}(D, B)$ such that the diagram

$$\begin{tikzcd} & D \arrow[dl, "f''"'] \arrow[dr, "g''"] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

commutes there is a unique morphism $h \in \operatorname{Mor}_{\mathcal{C}}(D, A \times_C B)$ such that the diagram

$$\begin{tikzcd} & C \arrow[d, dotted, "h"] \arrow[ddl, bend right, "f''"'] \arrow[ddr, bend left, "g''"] & \\ & A \times_C B \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd} \tag{2.13}$$

commutes.

This definition can easily be extended to more than two factors. However, to obtain a description as in Equation (2.12) for reduced rings with arbitrarily many branches we need more than one basis of the fibre product. In Definition 2.29 we introduce a more general notion of a fibre product as a limit of a certain functor. In fact, such a fibre product can equivalently be described by taking diagrams as (2.13) pairwise for all factors (see Lemma 2.31). Note, however, that in general the equality in Equation (2.12) will be merely an inclusion since we only consider pairwise relations of the branches of the ring R.

Definition 2.28. Let $D\colon \mathcal{I} \to \mathcal{C}$ be a diagram of type $\mathcal{I}$ for any category $\mathcal{C}$ and an index category $\mathcal{I}$. A *cone* to D is an object $C \in \mathcal{C}$ together with a family of morphisms $\phi_A \in \operatorname{Mor}_{\mathcal{C}}(C, D(A))$ indexed by $\operatorname{Ob}\mathcal{I}$ such that for any two objects $A, B \in \operatorname{Ob}\mathcal{I}$ and any morphism $f \in \operatorname{Mor}_{\mathcal{I}}(A, B)$ the diagram

$$\begin{tikzcd} & C \arrow[dl, "\phi_A"'] \arrow[dr, "\phi_B"] & \\ D(A) \arrow[rr, "D(f)"'] & & D(B) \end{tikzcd}$$

commutes.

A cone C to D is called *universal* if any cone to D factors through C. That is, a universal cone to D satisfies the following universal property: for any cone C' to D with morphisms $\phi'_A \in \operatorname{Mor}_{\mathcal{C}}(C', D(A))$ for $A \in \operatorname{Ob}\mathcal{I}$ there is a unique morphism $u \in \operatorname{Mor}_{\mathcal{C}}(C', C)$ such that the diagram

$$\begin{tikzcd}
& C' \arrow[d, dotted, "u"] \arrow[ddl, bend right, "\phi'_A"'] \arrow[ddr, bend left, "\phi'_B"] & \\
& C \arrow[dl, "\phi_A"'] \arrow[dr, "\phi_B"] & \\
D(A) \arrow[rr, "D(f)"'] & & D(B)
\end{tikzcd}$$

commutes for any two objects $A, B \in \operatorname{Ob} \mathcal{I}$ and any morphism $f \in \operatorname{Mor}_{\mathcal{I}}(A, B)$.

A universal cone to D is also called a *limit* of D.

Note. Being defined by a universal property, a limit (if it exists) is unique up to unique isomorphism.

Definition 2.29. Let $\mathcal{I}$ be a small category, let $\mathcal{C}$ be a category, and let $D\colon \mathcal{I} \to \mathcal{C}$ be a diagram of type $\mathcal{I}$. We define the category $\mathcal{J}$ by

$$\operatorname{Ob} \mathcal{J} = \operatorname{Ob} \mathcal{I} \times \operatorname{Ob} \mathcal{I}$$

and

$$\operatorname{Mor}_{\mathcal{J}}((A, B), (A', B')) = \begin{cases} \left\{\mathrm{id}_{(A,B)}\right\} & \text{if } (A, B) = (A', B'), \\ \{(A, B) \to (B, A)\} & \text{if } (A, B) = (B', A'), \\ \{(A, A) \to (A, B')\} & \text{if } A = B = A', \\ \emptyset & \text{else.} \end{cases}$$

Let $F\colon \mathcal{J} \to \mathcal{C}$ be a diagram of type $\mathcal{J}$ such that $F((A, A)) = D(A)$.

A *fibre product* in $\mathcal{C}$ over F is a limit of F, i.e. a fibre product is an object $C \in \mathcal{C}$ together with morphisms $\phi_{(A,B)} \in \operatorname{Mor}_{\mathcal{C}}(C, F(A, B))$ for all $(A, B) \in \operatorname{Ob} \mathcal{J}$ such that for any two objects $(A, B), (A', B') \in \operatorname{Ob} \mathcal{J}$ and any morphism $f \in \operatorname{Mor}_{\mathcal{J}}((A, B), (A', B'))$ the diagram

$$\begin{tikzcd}
& C \arrow[dl, "\phi_{(A,B)}"'] \arrow[dr, "\phi_{(A',B')}"] & \\
F((A, B)) \arrow[rr, "F(f)"'] & & F((A', B'))
\end{tikzcd} \tag{2.14}$$

commutes, and it satisfies the following universal property: if C' is a cone to F with morphisms $\phi_{(A,B)} \in \operatorname{Mor}_{\mathcal{C}}(C, F(A, B))$ for all $(A, B) \in \operatorname{Ob} \mathcal{J}$, then there is a unique morphism $u \in \operatorname{Mor}_{\mathcal{C}}(C', C)$ such that the diagram

$$\begin{tikzcd}
& C' \arrow[d, dotted, "u"] \arrow[ddl, bend right, "\phi'_{(A,B)}"'] \arrow[ddr, bend left, "\phi'_{(A',B')}"] & \\
& C \arrow[dl, "\phi_{(A,B)}"'] \arrow[dr, "\phi_{(A',B')}"] & \\
F((A, B)) \arrow[rr, "F(f)"'] & & F((A', B'))
\end{tikzcd} \tag{2.15}$$

commutes for any two objects $(A, B), (A', B') \in \operatorname{Ob} \mathcal{J}$ and any morphism

$$f \in \operatorname{Mor}_{\mathcal{J}}((A, B), (A', B')).$$

Since a fibre product over F is unique up to unique isomorphism, we denote it by $\operatorname{Fib}(F)$.

Remark 2.30. For any $(A, B) \in \operatorname{Ob} \mathcal{J}$ we have

$$((A, B) \to (B, A)) \circ ((B, A) \to (A, B)) \in \operatorname{Mor}_{\mathcal{J}}((B, A), (B, A)) = \left\{ \operatorname{id}_{(B,A)} \right\}$$

and

$$((B, A) \to (A, B)) \circ ((A, B) \to (B, A)) \in \operatorname{Mor}_{\mathcal{J}}((A, B), (A, B)) = \left\{ \operatorname{id}_{(A,B)} \right\}.$$

Therefore, $((B, A) \to (A, B))$, and hence also $F((B, A) \to (A, B))$ are isomorphisms for all $(A, B) \in \operatorname{Ob} \mathcal{J}$.

Lemma 2.31. *Let $\mathcal{C}$ be a category, and let $F \colon \mathcal{J} \to \mathcal{C}$ be a diagram of type $\mathcal{J}$ as in Definition 2.29. Let $C \in \operatorname{Ob} \mathcal{C}$ together with morphisms $\psi_A \in \operatorname{Mor}_{\mathcal{C}}(C, F((A, A)))$ for all $A \in \operatorname{Ob} \mathcal{I}$ such that for any two objects $A, B \in \operatorname{Ob} \mathcal{I}$ the diagram*

$$\begin{tikzcd}
 & C \arrow[dl, "\psi_A"'] \arrow[dr, "\psi_B"] & \\
F((A, A)) \arrow[d, "F((A,A)\to(A,B))"'] & & F((B, B)) \arrow[d, "F((B,B)\to(B,A))"] \\
F((A, B)) \arrow[rr, "F((A,B)\to(B,A))"'] & & F((B, A)).
\end{tikzcd} \tag{2.16}$$

commutes. Then C is a fibre product over F if and only if it satisfies the following universal property: if $C' \in \operatorname{Ob} \mathcal{C}$ satisfies Diagram (2.16) with morphisms $\psi'_A \in \operatorname{Mor}_{\mathcal{C}}(C', F((A, A)))$ for all $A \in \operatorname{Ob} \mathcal{I}$, then there is a unique morphism $u \in \operatorname{Mor}_{\mathcal{C}}(C', C)$ such that the diagram

$$\begin{tikzcd}
 & C' \arrow[d, dotted, "u"'] \arrow[ddl, bend right, "\psi'_A"'] \arrow[ddr, bend left, "\psi'_B"] & \\
 & C \arrow[dl, "\psi_A"'] \arrow[dr, "\psi_B"] & \\
F((A, A)) \arrow[d, "F((A,A)\to(A,B))"'] & & F((B, B)) \arrow[d, "F((B,B)\to(B,A))"] \\
F((A, B)) \arrow[rr, "F((A,B)\to(B,A))"'] & & F((B, A)).
\end{tikzcd} \tag{2.17}$$

commutes for any two objects $A, B \in \operatorname{Ob} \mathcal{I}$.

Proof. Let C be a fibre product over F. Then putting together three diagrams of type (2.14) we obtain a commutative diagram

$$\begin{tikzcd}
& C \arrow[dl, "\phi_{(A,A)}"'] \arrow[ddl, "\phi_{(A,B)}"] \arrow[dr, "\phi_{(B,B)}"] \arrow[ddr, "\phi_{(B,A)}"'] & \\
F((A,A)) \arrow[d, "F((A,A)\to(A,B))"'] & & F((B,B)) \arrow[d, "F((B,B)\to(B,A))"] \\
F((A,B)) \arrow[rr, "F((A,B)\to(B,A))"'] & & F((B,A))
\end{tikzcd}$$

for any two objects $A, B \in \operatorname{Ob}\mathcal{I}$. Thus, setting $\psi_A = \phi_{(A,A)}$, C satisfies Diagram (2.16). Now assume that C' satisfies Diagram (2.16), as well. Then for any two objects $A, B \in \operatorname{Ob}\mathcal{I}$ we have a commutative diagram

$$\begin{tikzcd}
& C' \arrow[ddl, bend right, "\psi'_A"'] \arrow[ddr, bend left, "\psi'_B"] & \\
& C \arrow[dl, "\phi_{(A,A)}"'] \arrow[ddl, "\phi_{(A,B)}"] \arrow[dr, "\phi_{(B,B)}"] \arrow[ddr, "\phi_{(B,A)}"'] & \\
F((A,A)) \arrow[d, "F((A,A)\to(A,B))"'] & & F((B,B)) \arrow[d, "F((B,B)\to(B,A))"] \\
F((A,B)) \arrow[rr, "F((A,B)\to(B,A))"'] & & F((B,A)).
\end{tikzcd}$$

Setting

$$\phi'_{(A,B)} = \begin{cases} \psi_A & if A = B, \\ F\left((A,A) \to (A,B)\right) \circ \psi'_{(A,A)} & \text{else} \end{cases}$$

for any $(A,B) \in \operatorname{Ob}\mathcal{J}$, we obtain a commutative diagram of type (2.15) for any two objects $(A,B), (A',B') \in \operatorname{Ob}\mathcal{J}$ and any morphism $f \in \operatorname{Mor}_{\mathcal{J}}\left((A,B), (A',B')\right)$. Hence, the universal property of the fibre product yields a unique morphism $u \in \operatorname{Mor}_{\mathcal{C}}(C', C)$ such that Diagram (2.17) commutes, i.e. C satisfies the universal property of the statement.

Let now $C \in \operatorname{Ob}\mathcal{C}$ satisfy the universal property of Diagram (2.17), and let $C' \in \operatorname{Ob}\mathcal{C}$ be a cone to F. Then we have a commutative diagram

$$\begin{tikzcd}
& & C' \arrow[dl, bend right, "\phi'_{(A,A)}"'] \arrow[dr, bend left, "\phi'_{(B,B)}"] \arrow[ddll, bend right, "\phi'_{(A,B)}"'] \arrow[ddrr, bend left, "\phi'_{(B,A)}"] & & \\
& F((A,A)) \arrow[dl, "F((A,A)\to(A,B))"] & C \arrow[l, "\phi_{(A,A)}"'] \arrow[r, "\phi_{(B,B)}"] & F((B,B)) \arrow[dr, "F((B,B)\to(B,A))"'] & \\
F((A,B)) \arrow[rrrr, "F((A,B)\to(B,A))"'] & & & & F((B,A)).
\end{tikzcd}$$

Setting $\psi'_A = \phi'_{(A,A)}$ for any $A \in \operatorname{Ob}\mathcal{I}$, we obtain a commutative diagram of type (2.17). Hence, the universal property of C yields a unique morphism $u \in \operatorname{Mor}_{\mathcal{C}}(C', C)$ such that Diagram (2.17) commutes. Moreover, setting

$$\phi_{(A,B)} = \begin{cases} \psi_A & \text{if } A = B, \\ F((A,A) \to (A,B)) \circ \psi_A & \text{else} \end{cases}$$

for all $(A,B) \in \operatorname{Ob}\mathcal{J}$, u is the unique morphism in $\operatorname{Mor}_{\mathcal{C}}(C', C)$ such that Diagram (2.15) commutes. Thus, C is a fibre product over F. $\square$

Theorem 2.32. *Let $\mathcal{C}$ be a category, and let $F\colon \mathcal{J} \to \mathcal{C}$ be a diagram of type $\mathcal{J}$ as in Definition 2.29.*

(1) Suppose that

$$\prod_{(A,B) \in \operatorname{Ob}\mathcal{J}} F((A,B)) \in \operatorname{Ob}\mathcal{C},$$

and let C be the subset of $\prod_{(A,B)\in\operatorname{Ob}\mathcal{J}} F((A,B))$ consisting of all elements $a \in \prod_{(A,B)\in\operatorname{Ob}\mathcal{J}} F((A,B))$ satisfying

$$F(f) \circ \operatorname{pr}_{(A,B)}(a) = \operatorname{pr}_{(A',B')}(a)$$

for any $(A,B), (A',B') \in \operatorname{Ob}\mathcal{J}$ and every morphism $f \in \operatorname{Mor}_{\mathcal{C}}(F(A,B), F(A',B'))$, where

$$\operatorname{pr}_{(A,B)}\colon \prod_{(A',B')\in\operatorname{Ob}\mathcal{J}} F((A',B')) \to F((A,B)),$$
$$(a_{A',B'})_{(A',B')\in\operatorname{Ob}\mathcal{J}} \mapsto a_{(A,B)}$$

is the projection for any $(A,B) \in \operatorname{Ob}\mathcal{J}$.

If $C \in \operatorname{Ob}\mathcal{C}$, then C together with the morphisms $\phi_{(A,B)} = \operatorname{pr}_{(A,B)}\big|_C$ for all $(A,B) \in \operatorname{Ob}\mathcal{J}$ is a fibre product over F.

(2) Suppose that

$$\prod_{A\in\operatorname{Ob}\mathcal{I}} F((A,A)) \in \operatorname{Ob}\mathcal{C},$$

and let D be the subset of the product $\prod_{A\in\operatorname{Ob}\mathcal{I}} F((A,A))$ consisting of all elements $a \in \prod_{A\in\operatorname{Ob}\mathcal{I}} F((A,A))$ satisfying

$$F((A,B) \to (B,A)) \circ F((A,A) \to (A,B)) \circ \operatorname{pr}_A(a) = F((B,B) \to (B,A)) \circ \operatorname{pr}_B(a)$$

for all $A, B \in \operatorname{Ob}\mathcal{I}$, where

$$\operatorname{pr}_A\colon \prod_{A'\in\operatorname{Ob}\mathcal{I}} F((A',A')) \to F((A,A)),$$
$$(a_{A'})_{(A')\in\operatorname{Ob}\mathcal{I}} \mapsto a_A$$

is the projection for any $A \in \operatorname{Ob}\mathcal{I}$.

If $D \in \operatorname{Ob}\mathcal{C}$, then D together with the morphisms $\psi_A = \operatorname{pr}_A|_D$ for all $A \in \operatorname{Ob}\mathcal{I}$ is a fibre product over F.

In particular, if $C, D \in \operatorname{Ob}\mathcal{C}$, then $C \cong D$.

Proof. (1) Assume that $C \in \operatorname{Ob}\mathcal{C}$. We have to show that C satisfies the universal property of Diagram (2.15). So let $C' \in \operatorname{Ob}\mathcal{C}$ together with morphisms $\phi'_{(A,B)} : C' \to F((A,B))$ for all $(A,B) \in \operatorname{Ob}\mathcal{J}$ such that the diagram

$$\begin{tikzcd} & C' \arrow[dl, "\phi'_{(A,B)}"'] \arrow[dr, "\phi'_{(A',B')}"] & \\ F((A,B)) \arrow[rr, "F(f)"'] & & F((A',B')) \end{tikzcd} \tag{2.18}$$

commutes for any two objects $(A,B),(A',B') \in \operatorname{Ob}\mathcal{J}$ and all morphisms $f \in \operatorname{Mor}_{\mathcal{C}}(F(A,B), F(A',B'))$.

Then the universal property of the product yields a unique morphism

$$u \colon C' \to \prod_{(A,B)\in\operatorname{Ob}\mathcal{J}} F((A,B))$$

such that the diagram

$$\begin{tikzcd} & C' \arrow[dl, "\phi'_{(A,B)}"'] \arrow[d, dotted, "u"] \arrow[dr, "\phi'_{(A',B')}"] & \\ F((A,B)) & \prod_{(A'',B'')\in\operatorname{Ob}\mathcal{J}} F((A'',B'')) \arrow[l, "\mathrm{pr}_{(A,B)}"] \arrow[r, "\mathrm{pr}_{(A',B')}"'] & F((A',B')) \end{tikzcd}$$

commutes for any two objects $(A,B),(A',B') \in \operatorname{Ob}\mathcal{J}$. Together with Diagram (2.18) we obtain

$$F(f) \circ \mathrm{pr}_{(A,B)} \circ u = F(f) \circ \phi'_{(A,B)} = \phi_{(A',B')} = \mathrm{pr}_{(A',B')} \circ u$$

for any two objects $(A,B),(A',B') \in \operatorname{Ob}\mathcal{J}$ and all morphisms

$$f \in \operatorname{Mor}_{\mathcal{C}}(F(A,B), F(A',B')).$$

By the definition of C this implies $u(C') \subset C$.

Thus, there is a unique morphism

$$u \in \operatorname{Mor}_{\mathcal{C}}(C', C)$$

such that the diagram

$$\begin{tikzcd} & C' \arrow[ddl, bend right, "\phi'_{(A,B)}"'] \arrow[d, dotted, "u"] \arrow[ddr, bend left, "\phi'_{(A',B')}"] & \\ & C \arrow[dl, "\phi_{(A,B)}"'] \arrow[dr, "\phi_{(A',B')}"] & \\ F((A,B)) \arrow[rr, "F(f)"'] & & F((A',B')) \end{tikzcd}$$

commutes for any two objects $(A,B),(A',B') \in \operatorname{Ob}\mathcal{J}$ and all morphisms $f \in \operatorname{Mor}_{\mathcal{C}}(F(A,B), F(A',B'))$. Therefore, C is a fibre product over F.

(2) Assume that $D \in \operatorname{Ob}\mathcal{C}$. Using Lemma 2.31, we have to show that D satisfies the universal property of Diagram (2.17). So let $D' \in \operatorname{Ob}\mathcal{C}$ together with morphisms $\psi'_A \colon D' \to F((A,A))$ for all $A \in \operatorname{Ob}\mathcal{I}$ such that the diagram

$$\begin{tikzcd} & D' \arrow[dl, "\psi'_A"'] \arrow[dr, "\psi'_B"] & \\ F((A,A)) \arrow[d, "F((A,A)\to(A,B))"'] & & F((B,B)) \arrow[d, "F((B,B)\to(B,A))"] \\ F((A,B)) \arrow[rr, "F((A,B)\to(B,A))"'] & & F((B,A)). \end{tikzcd} \tag{2.19}$$

commutes for for any two objects $A, b \in \operatorname{Ob}\mathcal{I}$.

Then the universal property of the product yields a unique morphism

$$v \colon D' \to \prod_{A \in \operatorname{Ob}\mathcal{I}} F((A,A))$$

such that the diagram

$$\begin{tikzcd} & D' \arrow[dl, "\psi'_A"'] \arrow[d, dotted, "v"] \arrow[dr, "\psi'_B"] & \\ F((A,A)) & \prod\limits_{A \in \operatorname{Ob}\mathcal{I}} F((A,A)) \arrow[l, "\mathrm{pr}_A"] \arrow[r, "\mathrm{pr}_B"'] & F((B,B)) \end{tikzcd}$$

commutes for any two objects $A, b \in \operatorname{Ob}\mathcal{I}$. Together with Diagram (2.19) we obtain

$$\begin{aligned} F((A,B) \to (B,A)) \circ F((A,A) \to (A,B)) \circ \mathrm{pr}_A \circ v \\ &= F((A,B) \to (B,A)) \circ F((A,A) \to (A,B)) \circ \psi'_A \\ &= F((B,B) \to (B,A)) \circ \psi'_B \\ &= F((B,B) \to (B,A)) \circ \mathrm{pr}_B \circ v \end{aligned}$$

for any two objects $A, B \in \operatorname{Ob}\mathcal{I}$. By the definition of D, this implies $v(D') \subset D$. Thus, there is a unique morphism

$$v \in \operatorname{Mor}_{\mathcal{C}}(D', D)$$

such that the diagram

$$\begin{tikzcd} & D' \arrow[ddl, bend right, "\psi'_A"'] \arrow[d, dotted, "u"] \arrow[ddr, bend left, "\psi'_B"] & \\ & D \arrow[dl, "\psi_A"'] \arrow[dr, "\psi_B"] & \\ F((A,A)) \arrow[d, "F((A,A)\to(A,B))"'] & & F((B,B)) \arrow[d, "F((B,B)\to(B,A))"] \\ F((A,B)) \arrow[rr, "F((A,B)\to(B,A))"'] & & F((B,A)). \end{tikzcd}$$

commutes for any two objects $A, B \in \operatorname{Ob}\mathcal{I}$. Therefore, D is by Lemma 2.31 a fibre product over F. $\square$

With (1) and (2) the particular claim follows from the universal property of the fibre product.

Theorem 2.33 (Mitchell's Embedding Theorem)**.** *Let $\mathcal{C}$ be a small abelian category. Then there exists a ring R and an exact fully faithful covariant functor $F\colon \mathcal{C} \to R\text{-Mod}$, where R-Mod is the category of left R-modules and R-homomorphisms.*

Proof. See [28, Theorem 1.12]. $\square$

Corollary 2.34. *Let $\mathcal{C}$ be a small abelian category, and let $F\colon \mathcal{J} \to \mathcal{C}$ be a diagram of type $\mathcal{J}$ as in Definition 2.29. Then the fibre product over F exists, and it is isomorphic to the subobject C of $\prod_{(A,B)\in\mathcal{J}} F((A,B))$ consisting of all elements $a \in \prod_{(A,B)\in\operatorname{Ob}\mathcal{J}} F((A,B))$ satisfying*

$$F(f) \circ \operatorname{pr}_{(A,B)}(a) = \operatorname{pr}_{(A',B')}(a) \tag{2.20}$$

for any $(A,B),(A',B') \in \operatorname{Ob}\mathcal{J}$ and every morphism $f \in \operatorname{Mor}_{\mathcal{C}}(F(A,B), F(A',B'))$, where

$$\begin{aligned} \operatorname{pr}_{(A,B)}\colon \prod_{(A',B')\in\operatorname{Ob}\mathcal{J}} & F((A',B')) \to F((A,B)), \\ & (a_{A',B'})_{(A',B')\in\operatorname{Ob}\mathcal{J}} \mapsto a_{(A,B)} \end{aligned}$$

is the projection for any $(A,B) \in \operatorname{Ob}\mathcal{J}$. Moreover, it is isomorphic to the subobject D of $\prod_{A\in\operatorname{Ob}\mathcal{I}} F((A,A))$ consisting of all elements $a \in \prod_{A\in\operatorname{Ob}\mathcal{I}} F((A,A))$ satisfying

$$F((A,B)\to(B,A))\circ F((A,A)\to(A,B))\circ \operatorname{pr}_A(a) = F((B,B)\to(B,A))\circ\operatorname{pr}_B(a) \tag{2.21}$$

for all $A, B \in \operatorname{Ob}\mathcal{I}$, where

$$\begin{aligned} \operatorname{pr}_A\colon \prod_{A'\in\operatorname{Ob}\mathcal{I}} & F((A',A')) \to F((A,A)), \\ & (a_{A'})_{(A')\in\operatorname{Ob}\mathcal{I}} \mapsto a_A \end{aligned}$$

is the projection for any $A \in \operatorname{Ob}\mathcal{I}$.

Proof. By Theorem 2.33 we only have to show the statement in the case that $\mathcal{C}$ is the category of left modules over a ring R. Since products exist in $\mathcal{C}$, we have to show that $C, D \in \operatorname{Ob}\mathcal{C}$. The statement follows then from Theorem 2.32.

As C, respectively D, is a subset of the R-module $\prod_{(A,B)\in\operatorname{Ob}\mathcal{J}} F((A,B))$, respectively $\prod_{A\in\operatorname{Ob}\mathcal{I}} F((A,A))$, we only have to show that C and D are closed under addition and multiplication with scalars. In fact, since the products are closed under these operations, we only have to show that they are compatible with Equations (2.20) and (2.21).

So let $r_1, r_2 \in R$, and let $c_1, c_2 \in C$. This implies

$$\begin{aligned} F(f) \circ \phi_{(A,B)}(c_1) &= \phi_{(A',B')}(c_1), \\ F(f) \circ \phi_{(A,B)}(c_2) &= \phi_{(A',B')}(c_2) \end{aligned}$$

for any two objects $(A,B),(A',B') \in \operatorname{Ob}\mathcal{J}$ and all morphisms

$$f \in \operatorname{Mor}_{\mathcal{C}}\left(F(A,B), F(A',B')\right).$$

Since $F(f)$, $\operatorname{pr}_{(A,B)}$ and $\operatorname{pr}_{(A',b')}$ are R-module homomorphisms, this yields

$$\begin{aligned}
F(f)\circ\phi_{(A,B)}(r_1c_1+r_2c_2) &= r_1F(f)\circ\operatorname{pr}_{(A,B)}(c_1)+r_2F(f)\circ\operatorname{pr}_{(A,B)}(c_2)\\
&= r_1\operatorname{pr}_{(A',B')}(c_1)+r_2\operatorname{pr}_{(A',B')}(c_2)\\
&= \operatorname{pr}_{(A',B')}(r_1c_1+r_2c_2)
\end{aligned}$$

for any two objects $(A,B),(A',B') \in \operatorname{Ob}\mathcal{J}$ and all morphisms

$$f \in \operatorname{Mor}_{\mathcal{C}}\left(F(A,B), F(A',B')\right).$$

Therefore, we have $r_1c_1+r_2c_2 \in C$, and hence $C \in \operatorname{Ob}\mathcal{C}$.

Let now $d_1, d_2 \in D$. Then

$$\begin{aligned}
F((A,B)\to(B,A))\circ F((A,A)\to(A,B))\circ\operatorname{pr}_A(d_1) &= F((B,B)\to(B,A))\circ\operatorname{pr}_B(d_1),\\
F((A,B)\to(B,A))\circ F((A,A)\to(A,B))\circ\operatorname{pr}_A(d_2) &= F((B,B)\to(B,A))\circ\operatorname{pr}_B(d_2)
\end{aligned}$$

for any two objects $A,B \in \operatorname{Ob}\mathcal{I}$. Since the maps $F((A,B)\to(B,A))$, $F((A,A)\to(A,B))$, $((B,B)\to(B,A))$, pr_A and pr_B are R-module homomorphisms, this yields

$$\begin{aligned}
&F((A,B)\to(B,A))\circ F((A,A)\to(A,B))\circ\operatorname{pr}_A(r_1d_1+r_2d_2)\\
&\qquad= r_1F((A,B)\to(B,A))\circ F((A,A)\to(A,B))\circ\operatorname{pr}_A(d_1)\\
&\qquad\qquad+ r_2F((A,B)\to(B,A))\circ F((A,A)\to(A,B))\circ\operatorname{pr}_A(d_2)\\
&\qquad= r_1F((B,B)\to(B,A))\circ\operatorname{pr}_B(d_1)\\
&\qquad\qquad+ r_2F((B,B)\to(B,A))\circ\operatorname{pr}_B(d_2)\\
&\qquad= F((B,B)\to(B,A))\circ\operatorname{pr}_B(r_1d_1+r_2d_2)
\end{aligned}$$

for any two objects $A,B \in \operatorname{Ob}\mathcal{I}$. Therefore, we have $r_1d_1+r_2d_2 \in D$, and hence $D \in \operatorname{Ob}\mathcal{C}$. $\square$

3. Valuations over One-dimensional Cohen–Macaulay Rings

In this Chapter we introduce the semigroup of values. This will lead to good semigroups as a combinatorial counterpart of curve singularities in Chapter 4. In Section 3.1 we start with the valuation theory on one-dimensional Cohen–Macaulay rings. This is based on a theorem which was proved by Matlis in the local case (see [29, Chapter VI]), and later generalized by Kiyek and Vicente (see [23, Chapter II, Theorem 2.11]): if R is a one-dimensional equidimensional semilocal Cohen–Macaulay ring, then the set $\mathcal{V}_R$ of valuation rings of Q_R which contain R is finite, and every $V \in \mathcal{V}_R$ is a discrete valuation ring (see Theorem 3.2). This allows us to introduce a discrete *multivaluation*

$$\nu \colon Q_R \to (\mathbb{Z} \cup \{\infty\})^{\mathcal{V}_R}.$$

In Section 3.1 we study the properties of the set $\mathcal{V}_R$ and the multivaluation ν, in particular its relations to the integral closure $\overline{R}$ of R in Q_R, and we introduce a filtration of Q_R, respectively of any fractional ideal of R, which is based on the valuation ν (see Definition 3.6). Moreover, we show that each valuation ring $V \in \mathcal{V}_R$ can be associated to a branch of R, i.e. a minimal prime ideal $\mathfrak{q}_V = I_V \cap R \in \operatorname{Min}(R)$. Then the corresponding valuation is constant along the other branches (see Proposition 3.13).

In Section 3.2 we associate to R its *semigroup of values* Γ_R as the subset of $\mathbb{N}^{\mathcal{V}_R}$ containing the values of all regular elements of R. Similarly, we can define value semigroup ideals for fractional ideals of R (see Definition 3.14). In fact, the value semigroup ideal of a fractional ideal is a semigroup ideal of Γ_R (see Proposition 3.22). As a first application we introduce a concept of locality on the semigroup of values of R which is equivalent to R being local (see Proposition 3.17). Particular algebraic hypotheses on R lead to properties of the semigroup of values and the value semigroup ideals of fractional ideals (see Proposition 3.22 and Corollary 3.30) which will characterize the class of good semigroups (see Chapter 4). We collect these hypotheses on R in the definition of *admissible rings* (see Definition 3.18). In Proposition 2.7 we saw that the set $\mathcal{R}_R$ is a monoid with respect to the product of ideals, and that it is closed under taking quotients. However, taking values is in general not compatible with these operations; for $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$ we may have strict inclusions

$$\Gamma_{\mathfrak{I}} + \Gamma_{\mathfrak{J}} \subsetneq \Gamma_{\mathfrak{I}\mathfrak{J}}$$

and

$$\Gamma_{\mathfrak{I}:\mathfrak{J}} \subsetneq \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}$$

(see Lemma 3.23 and Remark 3.24). In Chapters 4 and 5 we obtain equalities for two classes of ideals, that is conductors (see Proposition 4.57) and canonical ideals (see Theorem 5.34).

Moreover, in Section 3.2.1 we show that the value semigroup is compatible with localization, and in Section 3.2.2 we prove its invariance under completion.

An important example of admissible rings are algebroid curves (see Proposition 3.41). Algebroid curves occur as the completion of local rings of curve singularities. For an algebroid curve R there is a bijection between the set $\mathcal{V}_R$ of valuation rings of Q_R over R and the set $\operatorname{Min}(R)$ of minimal prime ideals of R. Using properties of discrete valuation rings we show that an algebroid curve admits a parametrization (see Theorem 3.44).

Section 3.4 is dedicated to integral extensions of admissible rings and algebroid curves. We show that an integral extension of an admissible ring in its total ring of fractions is an admissible ring, and that an integral extension of an algebroid curve over a field is an algebroid curve over the same field (see Theorem 3.45).

3.1. One-dimensional Cohen-Macaulay Rings

Remark 3.1. Let R be a one-dimensional semilocal Cohen–Macaulay ring. Then the total ring of fractions Q_R has a large Jacobson radical since $\dim Q_R = 0$ by Theorems A.72 and A.74.(1), and hence any prime ideal of Q_R is maximal, see Remark A.17.(1) and [30, Section 7, page 423].

Theorem 3.2. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring.*

(1) The set $\mathcal{V}_R$ of valuation rings of Q_R containing R is finite and non-empty, and each $V \in \mathcal{V}_R$ is a discrete valuation ring of Q_R.

(2) We have $\operatorname{Max}(Q_R) = \{I_V \mid V \in \mathcal{V}_R\}$.

(3) Let $\mathfrak{m} \in \operatorname{Max}(Q_R)$. *There is a bijection*

$$\begin{aligned}\{V \in \mathcal{V}_R \mid I_V = \mathfrak{m}\} &\to \mathcal{V}_{R/(\mathfrak{m}\cap R)}\\ V &\mapsto V/I_V,\end{aligned}$$

where $Q_{R/(\mathfrak{m}\cap R)} = Q_R/\mathfrak{m}$.

(4) The integral closure of R in Q_R is $\overline{R} = \bigcap_{V\in\mathcal{V}_R} V$.

(5) Any regular ideal of $\overline{R}$ is principal, and every regular prime ideal of $\overline{R}$ is maximal.

(6) There is a bijection

$$\begin{aligned}\operatorname{Max}\left(\overline{R}\right) &\to \mathcal{V}_R\\ \mathfrak{n} &\mapsto \left(\left(\overline{R}\setminus\mathfrak{n}\right)^{\mathrm{reg}}\right)^{-1}\overline{R}\\ \mathfrak{m}_V \cap \overline{R} &\leftarrow V.\end{aligned}$$

In particular, $\overline{R}/\left(\mathfrak{m}_V \cap \overline{R}\right) = V/\mathfrak{m}_V$ *for any* $V \in \mathcal{V}_R$.

Proof. See [23, Chapter II, Theorem 2.11]. □

Corollary 3.3. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and set $\nu = (\nu_V)_{V \in \mathcal{V}_R} \colon Q_R \to \mathbb{Z}^{\mathcal{V}_R}$. Then for any $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$ there is an $x \in Q_R^{\mathrm{reg}}$ such that $\nu(x) = \alpha$.*

Proof. This follows from Remark 3.1 Theorem 3.2.(1) and (2), and Corollary 2.27. □

Let R be a one-dimensional semilocal Cohen–Macaulay ring. Then Corollary 3.3 and Proposition 2.21.(2) (also cf. Diagram (2.10)) yield a commutative diagram

$$\begin{tikzcd} Q_R^{\mathrm{reg}} \arrow[d, two heads, "\mu"] \arrow[dr, two heads, "\nu"] & \\ \prod_{V \in \mathcal{V}_R} \mathcal{R}_{V,\infty}^* \arrow[r, "\cong", "\phi_R"'] & \mathbb{Z}_\infty^{\mathcal{V}_R}, \end{tikzcd} \tag{3.1}$$

where $\mu = (\mu_V)_{V \in \mathcal{V}_R}$, $\nu = (\nu_V)_{V \in \mathcal{V}_R}$ and $\phi = (\phi_V)_{V \in \mathcal{V}_R}$. Moreover, ϕ is compatible with the partial order on $\prod_{V \in \mathcal{V}_R} \mathcal{R}_V^*$ by reverse inclusion and the natural partial order on $\mathbb{Z}^{\mathcal{V}_R}$.

Lemma 3.4. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring. Then*

(1) $\overline{R} = \{x \in Q \mid \nu(x) \geq 0\}$,

(2) $Q^{\mathrm{reg}} = \left\{x \in Q \,\middle|\, \nu(x) \in \mathbb{Z}^{\mathcal{V}_R}\right\}$ *and* $\overline{R}^{\mathrm{reg}} = \left\{x \in Q \,\middle|\, \nu(x) \in \mathbb{N}^{\mathcal{V}_R}\right\}$,

(3) $\overline{R}^* = \{x \in Q \mid \nu(x) = 0\}$, *and*

(4) $R^* = \overline{R}^* \cap R = \{x \in R \mid \nu(x) = 0\}$.

Proof. (1) If $x \in \overline{R}$, then by Theorem 3.2.(4) $x \in V$, and hence $\nu_V \geq 0$ for all $V \in \mathcal{V}_R$.

Let now $x \in Q$ such that $\nu(x) \geq 0$, i.e. $\nu_V(x) \geq 0$ for all $V \in \mathcal{V}_R$. Then $x \in V$ for all $V \in \mathcal{V}_R$, and hence $x \in \bigcap_{V \in \mathcal{V}_R} V = \overline{R}$, see again Theorem 3.2.(4).

(2) If $x \in Q^{\mathrm{reg}}$, then $\nu(x) \in \mathbb{Z}^{\mathcal{V}_R}$ by definition. So let $x \in Q$ with $\nu(x) \in \mathbb{Z}^{\mathcal{V}_R}$, and assume $x \notin Q^{\mathrm{reg}}$. Then there is $\mathfrak{m} \in \operatorname{Max}(Q)$ such that $x \in \mathfrak{m}$. But then there is by Theorem 3.2.(2) a $V \in \mathcal{V}_R$ such that $\mathfrak{m} = I_V$, and this implies $\nu_V(x) = \infty$ by Proposition D.8.(2) and Diagram (3.1), contradicting our assumption.

Moreover, we have by (1)

$$\overline{R}^{\mathrm{reg}} = \overline{R} \cap Q^{\mathrm{reg}} = \left\{x \in Q \,\middle|\, \nu(x) \in \mathbb{N}^{\mathcal{V}_R}\right\}.$$

(3) Since $\overline{R} \subset Q_R$ and $V \subset Q_R$ for all $V \in \mathcal{V}_R$, Theorem 3.2.(4), Lemma A.11, and Proposition D.13.(2) yield

$$\begin{aligned}
\overline{R}^* &= \left(\bigcap_{V \in \mathcal{V}_R} V \right)^* \\
&= \bigcap_{V \in \mathcal{V}_R} V^* \\
&= \bigcap_{V \in \mathcal{V}_R} \{x \in Q^{\mathrm{reg}} \mid \nu_V(x) = 0\} \\
&= \{x \in Q^{\mathrm{reg}} \mid \nu(x) = \mathbf{0}\} \\
&= \{x \in Q \mid \nu(x) = \mathbf{0}\},
\end{aligned}$$

where the last equality follows from (2).

(4) This follows from (3) and Corollary B.4. □

We can also relate $\prod_{V \in \mathcal{V}_R} \mathcal{R}_V^*$ and Diagram (3.1) to $\mathcal{R}_{\overline{R}}$.

Proposition 3.5 (See [25], Section 3.1)**.** *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring. Then there is an order preserving group isomorphism*

$$\begin{aligned}
\psi\colon \mathcal{R}_{\overline{R}} &\to \prod_{V \in \mathcal{V}_R} \mathcal{R}_V^* \\
\mathfrak{I} &\mapsto (\mathfrak{I}V)_{V \in \mathcal{V}_R} \\
\bigcap_{V \in \mathcal{V}_R} \mathfrak{I}_V &\leftarrow\!\shortmid (\mathfrak{I}_V)_{V \in \mathcal{V}_R}
\end{aligned}$$

such that the diagram

$$\begin{tikzcd}
 & Q_R^{\mathrm{reg}} \arrow[dl, two heads] \arrow[d, two heads, "\mu"] \arrow[dr, two heads, "\nu"] & \\
\mathcal{R}_{\overline{R}} \arrow[r, "\cong", "\psi"'] & \prod_{V \in \mathcal{V}_R} \mathcal{R}_V^* \arrow[r, "\cong", "\phi"'] & \mathbb{Z}^{\mathcal{V}_R}
\end{tikzcd}$$

commutes, where $\mu = (\mu_V)_{V \in \mathcal{V}_R}$, $\nu = (\nu_V)_{V \in \mathcal{V}_R}$ and $\phi = (\phi_V)_{V \in \mathcal{V}_R}$.

Proof. By Theorem 3.2.(5) we have $\mathcal{R}_{\overline{R}} = \mathcal{R}^*_{\overline{R}}$, and for any $\mathfrak{I} \in \mathcal{R}_{\overline{R}}$ there is an $x \in Q^{\mathrm{reg}}$ such that $\mathfrak{I} = x\overline{R}$. Then Theorem 3.2.(4) yields

$$\bigcap_{V \in \mathcal{V}_R} \mathfrak{I}V = \bigcap_{V \in \mathcal{V}_R} xV = x \bigcap_{V \in \mathcal{V}_R} V = x\overline{R} = \mathfrak{I}.$$

Hence, ψ is injective, and considering Diagram (3.1) we obtain a commutative diagram

$$\begin{tikzcd}
 & Q_R^{\mathrm{reg}} \arrow[dl, two heads, "\alpha"'] \arrow[d, "\mu"] \arrow[dr, two heads, "\nu"] & \\
\mathcal{R}_{\overline{R}} \arrow[r, "\psi"'] & \prod_{V \in \mathcal{V}_R} \mathcal{R}_V^* \arrow[r, "\cong", "\phi"'] & \mathbb{Z}^{\mathcal{V}_R}.
\end{tikzcd}$$

The surjectivity of α follows from Theorem 3.2.(5), and the surjectivity of ν follows from Corollary 3.3. This implies the surjectivity of μ, and hence of ψ. Moreover, the isomorphisms ψ and ϕ preserve the partial orders on $\mathcal{R}_{\overline{R}}$ and $\prod_{V\in\mathcal{V}_R}\mathcal{R}_V^*$ by reverse inclusion and the natural partial order on $\mathbb{Z}^{\mathcal{V}_R}$. $\square$

Definition 3.6. Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring.

(1) We define a decreasing filtration $\mathfrak{Q}^\bullet$ on Q_R by setting

$$\mathfrak{Q}^\alpha = \{x \in Q_R \mid \nu(x) \geq \alpha\}$$

for any $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$.

(2) For any R-submodule $\mathfrak{I}$ of Q_R we define a decreasing filtration $\mathfrak{I}^\bullet$ on $\mathfrak{I}$ by setting

$$\mathfrak{I}^\alpha = \mathfrak{I} \cap \mathfrak{Q}^\alpha = \{x \in \mathfrak{I} \mid \nu(x) \geq \alpha\}$$

for any $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$.

Lemma 3.7. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring. For any $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$ there is an $x \in R^{\mathrm{reg}}$ with $\nu(x) \geq \alpha$.*

Proof. By Corollary 3.3 and Lemma 3.4.(2) there is a fraction $\frac{x}{y} \in Q_R^{\mathrm{reg}}$ with $\nu(x) - \nu(y) = \nu\left(\frac{x}{y}\right) = \alpha$, see Lemma D.22.(2). Since $\frac{x}{y} \in Q_R^{\mathrm{reg}}$, we have $x \in R^{\mathrm{reg}}$, and since $x, y \in R$, we have $\nu(x), \nu(y) \geq \mathbf{0}$. This implies $\nu(x) \geq \nu(x) - \nu(y) = \alpha$. $\square$

Proposition 3.8. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$. For any $\mathfrak{I} \in \mathcal{R}_R$ we have $\mathfrak{I}^\alpha \in \mathcal{R}_R$.*

Proof. Let $x \in \mathfrak{I}^\alpha$, and let $r \in R$. Then $\nu(rx) = \nu(r) + \nu(x) \geq \nu(x) \geq \alpha$. This implies $rx \in \mathfrak{I}^\alpha$ since $rx \in \mathfrak{I}$.

Let $y \in \mathfrak{I}^\alpha$. Then $\nu(x+y) \geq \inf\{\nu(x), \nu(y)\} \geq \alpha$. This implies $x + y \in \mathfrak{I}^\alpha$ since $x + y \in \mathfrak{I}$. Thus, $\mathfrak{I}^\alpha$ is an R-submodule of Q_R. Since $\mathfrak{I} \in \mathcal{R}_R$, there is an $r \in R^{\mathrm{reg}}$ such that $r\mathfrak{I}^\alpha \subset r\mathfrak{I} \subset R$. Thus, $\mathfrak{I}^\alpha$ is a fractional ideal of R.

Since $\mathfrak{I} \in \mathcal{R}_R$, there is an $x \in \mathfrak{I}^{\mathrm{reg}}$. Set $\beta = \alpha - \nu(x)$. Then Lemma 3.4.(2) yields $\beta \in \mathbb{Z}^{\mathcal{V}_R}$. By Lemma 3.7 there is an $r \in R^{\mathrm{reg}}$ with $\nu(x) \geq \beta$. Then $rx \in \mathfrak{I}^{\mathrm{reg}}$ with $\nu(rx) = \nu(r) + \nu(x) \geq \alpha - \nu(x) + \nu(x) = \alpha$. This implies $rx \in (\mathfrak{I}^\alpha)^{\mathrm{reg}}$. Therefore, $\mathfrak{I}^\alpha \in \mathcal{R}_R$. $\square$

Lemma 3.9 (See [25], Section 3.3)**.** *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring.*

(1) The isomorphism $\phi \circ \psi$ of Proposition 3.5 is given by

$$\begin{aligned} \phi \circ \psi \colon \mathcal{R}_{\overline{R}} &\to \mathbb{Z}^{\mathcal{V}_R} \\ \mathfrak{I} &\mapsto \left(\min\{\nu_V(x) \mid x \in \mathfrak{I}V\}\right)_{V\in\mathcal{V}_R} \\ \mathfrak{Q}^\alpha &\mapsfrom \alpha. \end{aligned}$$

(2) *Let* $\alpha, \beta \in \mathbb{Z}^{\mathcal{V}_R}$. *Then*

$$\begin{aligned} \mathfrak{Q}^\alpha \mathfrak{Q}^\beta &= \mathfrak{Q}^{\alpha+\beta}, \\ \mathfrak{Q}^\alpha : \mathfrak{Q}^\beta &= \mathfrak{Q}^{\alpha-\beta}, \\ (\mathfrak{Q}^\alpha)^{-1} &= \mathfrak{Q}^{-\alpha}. \end{aligned}$$

(3) *Let* $\mathfrak{I}$ *be an* R*-submodule of* Q_R. *For any* $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$ *we have*

$$\nu\left(\mathfrak{I}^\alpha\right) = \left\{\beta \in \nu\left(\mathfrak{I}\right) \mid \beta \geq \alpha\right\}.$$

Proof. (1) This follows from Propositions 2.21.(2), 2.23.(5), and 3.5.

(2) This follows immediately from (1).

(3) This follows immediately from Definition 3.6.(2). □

Proposition 3.10. *Let* R *be a one-dimensional equidimensional semilocal Cohen–Macaulay ring. Then*

$$\mathfrak{Q}^\alpha = \bigcap_{V \in \mathcal{V}_R} (\mathfrak{m}_V)^{\alpha_V}$$

for any $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$.

Proof. This follows from Propositions 3.5 and 2.21.(2) and Lemma 3.9.(1). □

Lemma 3.11. *Let* R *be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let* $\mathfrak{I} \in \mathcal{R}_{\overline{R}}$. *Then* $\mathfrak{I}$ *is generated by any* $x \in Q_R$ *having the multivalue*

$$\nu(x) = \left(\min\left\{\nu_V(x) \mid x \in \mathfrak{I}V\right\}\right)_{V \in \mathcal{V}_R}.$$

Moreover, any such x *is regular.*

Proof. Set $\alpha = \left(\min\left\{\nu_V(x) \mid x \in \mathfrak{I}V\right\}\right)_{V \in \mathcal{V}_R}$. Then

$$\mathfrak{I} = \mathfrak{Q}^\alpha$$

by Lemma 3.9.(1). Thus, there is an $x \in \mathfrak{I}$ with $\nu(x) = \alpha$. Since $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$, this implies $x \in \mathfrak{I}^{\mathrm{reg}}$ by Lemma 3.4.(2).

Let now $y \in \mathfrak{I} = \mathfrak{Q}^\alpha$. Then $\nu(y) \geq \nu(x) = \alpha$, and hence by Lemma D.22.(2)

$$\nu\left(\frac{y}{x}\right) = \nu(y) + \nu\left(x^{-1}\right) = \nu(y) - \nu(x) \geq 0.$$

This implies $\frac{y}{x} \in \overline{R}$ by Lemma 3.4.(1), and therefore $y \in x\overline{R}$. □

Lemma 3.12. *Let* R *be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let* $\mathfrak{I} \in \mathcal{R}_{\overline{R}}$. *For any* $x \in \mathfrak{I}$ *there is a* $y \in \mathfrak{I}^{\mathrm{reg}}$ *such that* $\nu_V(y) = \nu_V(x)$ *for all* $V \in \mathcal{V}_R$ *with* $x \notin I_V$.

Proof. Since $\mathfrak{C}_{\mathfrak{J}} \in \mathcal{R}_{\overline{R}}$ by Corollary C.16, there is by Lemma 3.9.(1) an $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$ such that $\mathfrak{C}_{\mathfrak{J}} = \mathfrak{Q}^{\alpha}$. Hence, by Lemma 3.4.(2) there is a $z \in \mathfrak{C}_{\mathfrak{J}}^{\mathrm{reg}}$ such that $\nu_V(z) > \nu_V(x)$ for all $V \in \mathcal{V}_R$ with $x \notin I_V$. Then Lemma D.22.(5) yields for any $V \in \mathcal{V}_R$

$$\nu_V(x+z) = \min\{\nu_V(x), \nu_V(z)\} = \begin{cases} \nu_V(x) & \text{if } x \notin I_V, \\ \nu_V(z) & \text{else.} \end{cases}$$

In particular, we have $\nu(x+z) \in \mathbb{Z}^{\mathcal{V}_R}$, and hence $y = x + z \in \mathfrak{I}^{\mathrm{reg}}$ by Lemma 3.4.(2). □

Proposition 3.13. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring.*

(1) For any $V \in \mathcal{V}_R$ we have

$$I_V = 0 \times \prod_{\mathfrak{p} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{q}_V\}} Q_{R/\mathfrak{p}},$$

where

$$\mathfrak{q}_V = I_V \cap R \in \mathrm{Min}\,(R).$$

(2) For any $\mathfrak{q} \in \mathrm{Min}\,(R)$ there is a bijection

$$\begin{aligned} \{V \in \mathcal{V}_R \mid \mathfrak{q}_V = \mathfrak{q}\} &\to \mathcal{V}_{R/\mathfrak{q}}, \\ V &\mapsto V/I_V, \\ \overline{V} \times \prod_{\mathfrak{p} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{q}\}} Q_{R/\mathfrak{p}} &\hookleftarrow \overline{V}. \end{aligned}$$

In particular, any valuation ring $V \in \mathcal{V}_R$ is of the form

$$V = V/I_V \times \prod_{\mathfrak{p} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{q}_V\}} Q_{R/\mathfrak{p}},$$

where $V/I_V \in \mathcal{V}_{R/\mathfrak{q}_V}$, and we have

$$\nu_V = \pi \circ \nu_{V/I_V},$$

where $\pi\colon Q_R \to Q_R/I_V$ is the canonical surjection.

(3) For any subset $J \subset \mathrm{Min}\,(R)$ there is a bijection

$$\begin{aligned} \{V \in \mathcal{V}_R \mid \mathfrak{q}_V \in J\} &\to \mathcal{V}_{R/\bigcap_{\mathfrak{p} \in J} \mathfrak{p}}, \\ V &\mapsto V/\bigcap_{\mathfrak{p} \in J} \mathfrak{p} Q_R, \\ V' \times \prod_{\mathfrak{q} \in \mathrm{Min}\,(R) \setminus J} Q_{R/\mathfrak{q}} &\hookleftarrow V'. \end{aligned}$$

Moreover,

$$\nu_V = \pi \circ \nu_{V/\bigcap_{\mathfrak{p} \in J} \mathfrak{p} Q_R}$$

for any $V \in \mathcal{V}_R$ with $I_V \in J$, where $\pi\colon Q_R \to Q_R/\bigcap_{\mathfrak{p} \in J} \mathfrak{p} Q_R$ is the canonical surjection.

Proof. (1) By Theorem 3.2.(2) we have $I_V \in \operatorname{Max}(Q_R)$, and hence the claim follows from Corollary A.75.

(2) By (1) and Theorem 3.2.(2) and (3) there is a bijection

$$\begin{aligned} \phi\colon \{V \in \mathcal{V}_R \mid \mathfrak{q}_V = \mathfrak{q}\} &\to \mathcal{V}_{R/\mathfrak{q}}, \\ V &\mapsto V/I_V. \end{aligned}$$

If $\mathfrak{q} \in \operatorname{Min}(R)$ and $\overline{V} \in \mathcal{V}_{R/\mathfrak{q}}$, then

$$V = \overline{V} \times \prod_{\mathfrak{p} \in \operatorname{Min}(R) \setminus \{\mathfrak{q}\}} Q_{R/\mathfrak{p}}$$

is by Theorems A.74.(2) and 3.2.(1) a discrete valuation ring of $Q_{R/\mathfrak{q}}$. Moreover, since by (1)

$$I_V = 0 \times \prod_{\mathfrak{p} \in \operatorname{Min}(R) \setminus \{\mathfrak{q}_v\}} Q_{R/\mathfrak{p}},$$

we have $\mathfrak{q}_V = \mathfrak{q}$ and $V/I_V = \overline{V}$. Hence, the map

$$\begin{aligned} \psi\colon \mathcal{V}_{R/\mathfrak{q}} &\to \{V \in \mathcal{V}_R \mid \mathfrak{q}_V = \mathfrak{q}\}, \\ \overline{V} &\mapsto \overline{V} \times \prod_{\mathfrak{p} \in \operatorname{Min}(R) \setminus \{\mathfrak{q}\}} Q_{R/\mathfrak{p}} \end{aligned}$$

is the inverse of ϕ. Also see [23, Chapter II, 2.12].

The remaining part of the statement follows from Proposition D.16.

(3) Let $V \in \mathcal{V}_R$ such that $I_V \cap R \in J$. Then by (2)

$$V = V/I_V \times \prod_{\mathfrak{p} \in \operatorname{Min}(R) \setminus \{\mathfrak{q}_V\}} Q_{R/\mathfrak{p}},$$

where $V/I_V \in \mathcal{V}_{R/\mathfrak{q}_V}$. Moreover, Corollary A.75 yields

$$\bigcap_{\mathfrak{p} \in J} \mathfrak{p} Q_R = \bigcap_{\mathfrak{p} \in J} \left(0 \times \prod_{\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}} Q_{R/\mathfrak{q}} \right) = \prod_{\mathfrak{p} \in J} 0 \times \prod_{\mathfrak{q} \in \operatorname{Min}(R) \setminus J} Q_{R/\mathfrak{q}}.$$

Since $\mathfrak{q}_V \in J$, this implies

$$V / \bigcap_{\mathfrak{p} \in J} \mathfrak{p} Q_R = V/I_V \times \prod_{\mathfrak{q} \in J \setminus \{\mathfrak{q}_V\}} Q_{R/\mathfrak{q}}. \tag{3.2}$$

Now note that the canonical surjection $\pi\colon R \to R' = R/\bigcap_{\mathfrak{p} \in J} \mathfrak{p}$ induces by Proposition A.10 an equality

$$\operatorname{Min}(R') = \left\{ \pi(\mathfrak{q}) = \mathfrak{q} + \bigcap_{\mathfrak{p} \in J} \mathfrak{p} / \bigcap_{\mathfrak{p} \in J} \mathfrak{p} \;\middle|\; \mathfrak{q} \in J \right\}, \tag{3.3}$$

and, moreover,

$$R/\mathfrak{q} = R'/\,\pi\,(\mathfrak{q}). \tag{3.4}$$

Then Equations (3.2), (3.3), and (3.4) yield

$$V/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}Q_R = V/I_V \times \prod_{\mathfrak{q}\in J\setminus\{\mathfrak{q}_V\}} Q_{R'/\,\pi\,(\mathfrak{q})} \tag{3.5}$$

$$= V/I_V \times \prod_{\mathfrak{q}\in \operatorname{Min}\,(R')\setminus\{\pi\,(\mathfrak{q}_V)\}} Q_{R'/\,\pi\,(\mathfrak{q})}. \tag{3.6}$$

Since $\mathfrak{q}_V \in J$, we obtain with Equation (3.4)

$$R'/\,\pi\,(\mathfrak{q}_V) = R/\mathfrak{q}_V \subset V/I_V \subset Q_{R/\mathfrak{q}_V} = Q_{R'/\,\pi\,(\mathfrak{q}_V)},$$

where $V/I_V \in \mathcal{V}_{R/\mathfrak{q}_V}$ by (1), and hence

$$V/I_V \in \mathcal{V}_{R'/\,\pi\,\mathfrak{q}_V}. \tag{3.7}$$

Thus, by (1) and Equations (3.6) and (3.7) there is a map

$$\begin{aligned}\phi\colon \{V \in \mathcal{V}_R \mid \mathfrak{q}_V \in J\} &\to \mathcal{V}_{R'},\\ V &\mapsto V/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}Q_R.\end{aligned}$$

Let now $V' \in \mathcal{V}_{R'}$. Then by (1) and Equations (3.3) and (3.4) we have

$$V' = V'/I_{V'} \times \prod_{\mathfrak{q}\in \operatorname{Min}\,(R')\setminus\{\mathfrak{q}_{V'}\}} Q_{R'/\mathfrak{q}} \tag{3.8}$$

$$= V'/I_{V'} \times \prod_{\mathfrak{q}\in J\setminus\{\pi^{-1}\,(\mathfrak{q}_{V'})\}} Q_{R'/\,\pi\,(\mathfrak{q})} \tag{3.9}$$

$$= V'/I_{V'} \times \prod_{\mathfrak{q}\in J\setminus\{\pi^{-1}\,(\mathfrak{q}_{V'})\}} Q_{R/\mathfrak{q}}. \tag{3.10}$$

By (1) we have $\mathfrak{q}_{V'} \in \operatorname{Min}\,(R')$, and hence $\pi^{-1}\,(\mathfrak{q}_{V'}) \in J$ by Equation (3.3). Since $V'/I_{V'} \in \mathcal{V}_{R'/(I_{V'}\cap R')}$ by (1), Equation (3.4) yields

$$R/\,\pi^{-1}\,(\mathfrak{q}_{V'}) = R'/\mathfrak{q}_{V'} \subset V'/I_{V'} \subset Q_{R'/\mathfrak{q}_{V'}} = Q_{R/\,\pi^{-1}\,(\mathfrak{q}_{V'})},$$

and hence

$$V'/I_{V'} \in \mathcal{V}_{R/\,\pi^{-1}\,(\mathfrak{q}_{V'})}. \tag{3.11}$$

Moreover, Equation (3.10) implies

$$\begin{aligned}V' \times \prod_{\mathfrak{q}'\in \operatorname{Min}\,(R)\setminus J} Q_{R/\mathfrak{q}'} &= V'/I_{V'} \times \prod_{\mathfrak{q}\in J\setminus\{\pi^{-1}\,(\mathfrak{q}_{V'})\}} Q_{R/\mathfrak{q}} \times \prod_{\mathfrak{q}'\in \operatorname{Min}\,(R)\setminus J} Q_{R/\mathfrak{q}'}\\ &= V'/I_{V'} \times \prod_{\mathfrak{q}\in \operatorname{Min}\,(R)\setminus\{\pi^{-1}\,(\mathfrak{q}_{V'})\}} Q_{R/\mathfrak{q}}.\end{aligned}$$

Thus, (1) and Equations (3.4) and (3.11) imply

$$V' \times \prod_{\mathfrak{q}' \in \operatorname{Min}(R) \setminus J} Q_{R/\mathfrak{q}'} \in \mathcal{V}_R$$

with

$$\mathfrak{q}_{V' \times \prod_{\mathfrak{q}' \in \operatorname{Min}(R) \setminus J} Q_{R/\mathfrak{q}'}} = \pi^{-1}(\mathfrak{q}_{V'}) \in J.$$

Hence, there is a map

$$\begin{aligned} \psi \colon \mathcal{V}_{R'} &\to \{V \in \mathcal{V}_R \mid \mathfrak{q}_V \in J\}, \\ V' &\mapsto V' \times \prod_{\mathfrak{q} \in \operatorname{Min}(R) \setminus J} Q_{R/\mathfrak{q}}. \end{aligned}$$

By construction, we obviously have $\phi \circ \psi = \mathrm{id}_{\mathcal{V}_{R'}}$ and $\psi \circ \phi = \mathrm{id}_{\{V \in \mathcal{V}_R \mid I_V \cap R \in J\}}$. Therefore, ϕ and ψ are bijective and mutually inverse maps.

With what we just showed, the remaining part of the statement follows from Proposition D.16. $\square$

3.2. Semigroup of Values

Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring. Theorem 3.2.(1) provides the basis for the definition of the semigroup of values of R. We consider the values in the finitely many discrete valuations of Q_R simultaneously. Similarly, we associate to a regular fractional ideal of R its value semigroup ideal. Studying the properties of these objects in relation to certain algebraic hypotheses (see Proposition 3.22 and Corollary 3.30) leads to the definition of admissible rings (see Definition 3.18) We decompose the semigroup of values and value semigroup ideals into local components (see Theorem 3.28), and we show their invariance under completion (see Theorem 3.34).

Definition 3.14. Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let $\mathcal{V}_R$ be the set of (discrete) valuation rings of Q_R over R (see Theorem 3.2.(1) and Definition D.1) with corresponding valuations

$$\nu_R = (\nu_V)_{V \in \mathcal{V}_R} \colon Q_R \to \mathbb{Z}_\infty^{\mathcal{V}_R}.$$

We will also write ν instead of ν_R.

(1) To a fractional ideal $\mathfrak{I} \in \mathcal{R}_R$ we associate its *value semigroup ideal*

$$\Gamma_{\mathfrak{I}} = \nu(\mathfrak{I}^{\mathrm{reg}}) \subset \mathbb{Z}^{\mathcal{V}_R}$$

(see Lemma 3.4.(2)).

(2) If $\mathfrak{I} = R$, then the monoid $\Gamma_R \subset \mathbb{N}^{\mathcal{V}_R}$ is called the *value semigroup* or *semigroup of values* of R.

(3) The value semigroup Γ_R is said to be *local* if $\mathbf{0}$ is the only element of Γ_R with a zero component in $\mathbb{Z}^{\mathcal{V}_R}$.

Remark 3.15 (See [25], Remark 3.1.10). Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let $\mathfrak{E}, \mathfrak{F} \in \mathcal{R}_R$. If $\mathfrak{E} \subset \mathfrak{F}$, then $\Gamma_{\mathfrak{E}} \subset \Gamma_{\mathfrak{F}}$.

Lemma 3.16. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, let $\mathfrak{I} \in \mathcal{R}_R$, and let $x \in \mathfrak{I}$. Then $x \in \mathfrak{I}^{\mathrm{reg}}$ if and only if $\nu(x) \in \Gamma_{\mathfrak{I}}$.*

Proof. If $x \in \mathfrak{I}^{\mathrm{reg}}$, then $\nu(x) \in \Gamma_{\mathfrak{I}}$ by Definition 3.14.

If $x \in \mathfrak{I} \setminus \mathfrak{I}^{\mathrm{reg}}$, then $\nu(x) \in (\mathbb{Z} \cup \{\infty\})^{\mathcal{V}_R} \setminus \mathbb{Z}^{\mathcal{V}_R}$ by Lemma 3.4.(2). Hence, $\nu(x) \in (\mathbb{Z} \cup \{\infty\})^{\mathcal{V}_R} \setminus \Gamma_R$. □

The following result was stated without prove in [7, (1.1.1)] and [10, Section 2].

Proposition 3.17 (See [25], Proposition 3.1.4)**.** *A one-dimensional equidimensional semilocal Cohen–Macaulay ring R is local if and only if its value semigroup Γ_R is local. If R is local, then the maximal ideal is*

$$\mathfrak{m}_R = \{x \in R \mid \nu(x) > \mathbf{0}\} = R^{\mathbf{1}}.$$

Proof. Suppose first that R is local with maximal ideal $\mathfrak{m}_R$. Then Theorem 3.2.(6) and Propositions B.3, B.15, and D.13.(3) imply

$$\mathfrak{m}_R \subset \bigcap_{\mathfrak{n} \in \operatorname{Max}\left(\overline{R}\right)} \mathfrak{n} = \bigcap_{V \in \mathcal{V}_R} \mathfrak{m}_V = \bigcap_{V \in \mathcal{V}_R} \{x \in Q_R \mid \nu_V(x) > 0\} = \{x \in \mathcal{V}_R \mid \nu(x) > \mathbf{0}\}.$$

The statement follows from Lemma 3.4.(4).

Suppose now that Γ_R is local. We want to show that

$$\mathfrak{m} = \{x \in R \mid \nu\left(x\right) > \mathbf{0}\}$$

is the unique maximal ideal of R.

We show that $\nu\left(x\right)$ has no zero component for any $x \in \mathfrak{m}$. Then

$$\mathfrak{m} = R^{\mathbf{1}},$$

and hence it is an ideal of R by Proposition 3.8.

So assume that there is $x \in \mathfrak{m}$ such that $\nu_{V_1}\left(x\right) = 0$ for some $V_1 \in \mathcal{V}_R$. Then $x \in R \setminus R^{\mathrm{reg}} \subset Q \setminus Q^{\mathrm{reg}}$ by the assumption on Γ_R and Lemma 3.16. Hence, by Theorem 3.2.(2) there is $V_2 \in \mathcal{V}_R$ such that $x \in I_{V_2}$, and Proposition D.8.(2) and Diagram (3.1) imply $V_1 \neq V_2$.

Since R is a one-dimensional Cohen–Macaulay ring, there is a $y \in R^{\mathrm{reg}} \setminus R^*$. Then $\nu(x) \in \Gamma_R$, and Lemma 3.4.(2) and (4) yield $\nu\left(x\right) > 0$ for every $V \in \mathcal{V}_R$. After replacing y by a suitable power, we may assume that $\nu_V\left(x\right) \neq \nu_V\left(y\right)$ for all $V \in \mathcal{V}_R$. Then

$$\nu\left(x + y\right) = \inf\left\{\nu\left(x\right), \nu\left(y\right)\right\} \in \mathbb{Z}^{\mathcal{V}_R}$$

by Lemma D.22.(5) and since $\nu(y) \in \mathbb{Z}^{\mathcal{V}_R}$. Thus, $x+y \in R^{\mathrm{reg}} = R \cap Q^{\mathrm{reg}}$ by Lemma 3.4.(2), and hence $\nu(x+y) \in \Gamma_R$.

Since $\nu_{V_1}(y) > 0$, we have

$$\nu_{V_1}(x+y) = \min\{\nu_{V_1}(x), \nu_{V_1}(y)\} = \nu_{V_1}(x) = 0.$$

By assumption on Γ_R, this implies $\nu(x+y) = \mathbf{0}$. Since $\nu_{V_2}(x) = \infty$, we obtain

$$0 = \nu_{V_2}(x+y) = \nu_{V_2}(y).$$

But this contradicts the choice of y.

Since $R^* = \overline{R}^* \cap R = \{x \in R \mid \nu(x) = \mathbf{0}\}$ by Lemma 3.4.(4), any proper ideal of R is contained in $\mathfrak{m}$. Therefore, $\mathfrak{m}$ is the unique maximal ideal of R. $\square$

Definition 3.18. Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring.

(1) We call R *analytically reduced* if $\widehat{R}$ is reduced or, equivalently, $\widehat{R_{\mathfrak{m}}}$ is reduced for all $\mathfrak{m} \in \operatorname{Max}(R)$ (see Lemma A.68).

(2) The ring R is called *residually rational* if $R/\mathfrak{m} = \overline{R}/\mathfrak{n}$ for any $\mathfrak{m} \in \operatorname{Max}(R)$ and $\mathfrak{n} \in \operatorname{Max}\left(\overline{R}\right)$ with $\mathfrak{n} \cap R = \mathfrak{m}$. Equivalently, $R/\mathfrak{m} = V/\mathfrak{m}_V$ for any $\mathfrak{m} \in \operatorname{Max}(R)$ and $V \in \mathcal{V}_R$ with $\mathfrak{m}_V \cap R = \mathfrak{m}$ (see Theorem 3.2.(6)).

(3) We say that R has *large residue fields* if $|R/\mathfrak{m}| \geq |\mathcal{V}_{R_{\mathfrak{m}}}|$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

(4) We call R *admissible* if it is analytically reduced and residually rational with large residue fields.

Definition 3.19. Let $\overline{S}$ be a partially ordered monoid, isomorphic to $\mathbb{N}^I$ with its natural partial order, where I is a finite set. We consider the following properties of a subset E of the *group of differences* $D_{\overline{S}} \cong \mathbb{Z}^I$ of $\overline{S}$ (see [7, Section 1] and [8, Section 2]).

(E0) There exists an $\alpha \in D_{\overline{S}}$ such that $\alpha + \overline{S} \subset E$.

(E1) If $\alpha, \beta \in E$, then $\inf\{\alpha, \beta\} = (\min\{\alpha_i, \beta_i\})_{i \in I} \in E$.

(E2) For any $\alpha, \beta \in E$ and $j \in I$ such that $\alpha_j = \beta_j$ there exists an $\epsilon \in E$ such that $\epsilon_j > \alpha_j = \beta_j$ and $\epsilon_i \geq \min\{\alpha_i, \beta_i\}$ for all $i \in I$, where equality is obtained whenever $\alpha_i \neq \beta_i$.

We call E *good* if it satisfies (E0), (E1), and (E2).

The *difference* of two subsets E and F of $D_{\overline{S}}$ is

$$E - F = \left\{\alpha \in D_{\overline{S}} \mid \alpha + F \subset E\right\}$$

Lemma 3.20 (See [25], Lemma 3.1.7)**.** *Any group isomorphism of Z^s preserving the partial order is defined by a permutation of the standard basis.*

Proof. Let φ be an automorphism of $\mathbb{Z}^s$ preserving the partial order. Then $(\varphi(\mathbf{e}_i))_{i\in\{1,\ldots,s\}}$ is a basis of $\mathbb{Z}^s$, and hence for $j \in \{1,\ldots,s\}$ there are $\lambda_i \in \mathbb{Z}$, $i = 1,\ldots,s$ such that $0 < \mathbf{e}_j = \sum_i \lambda_i \varphi(\mathbf{e}_i) = \varphi(\sum_{i=1}^s \lambda_i \mathbf{e}_i)$. Since φ preserves the order, this implies $\sum_{i=1}^s \lambda_i \mathbf{e}_i > 0$, and hence $\lambda_i \geq 0$ for all $i = 1,\ldots,s$. For the k-th component ($k \in \{1,\ldots,s\}$) we have

$$\sum_{i=1}^{s} \lambda_i (\varphi(\mathbf{e}_i))_k = \left(\sum_{i=1}^{s} \lambda_i \varphi(\mathbf{e}_i)\right)_k = (\mathbf{e}_j)_k = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{else.} \end{cases}$$

As φ is order preserving, we have $\phi(\mathbf{e}_i) > 0$ for every $i = 1,\ldots,s$. Therefore, $\mathbf{e}_j = \varphi(\mathbf{e}_i)$ for some $i \in \{1,\ldots,s\}$. $\square$

Lemma 3.21 (See [25], Lemma 3.1.8)**.** *Let R be a one-dimensional equidimensional analytically reduced semilocal Cohen–Macaulay ring, and let $\mathfrak{I} \in \mathcal{R}_R$. Then $\overline{R} \in \mathcal{R}_R$, and hence $\mathcal{R}_{\overline{R}} \subset \mathcal{R}_R$. In particular, $\mathfrak{C}_\mathfrak{I} \in \mathcal{R}_R \cap \mathcal{R}_{\overline{R}}$, and $\mathfrak{C}_\mathfrak{I} = x\overline{R}$ for some $x \in \mathfrak{C}_\mathfrak{I}^{\mathrm{reg}}$ with $\nu(x) + \mathbb{N}^{\mathcal{V}_R} \subset \Gamma_\mathfrak{I}$.*

Proof. Since R is analytically reduced, $\overline{R}$ is by Corollary C.15 a finite R-module. This implies $\overline{R} \in \mathcal{R}_R$ (see Remark 2.6.(1)), and hence $\mathfrak{C}_\mathfrak{I} = \mathfrak{I} : \overline{R} \in \mathcal{R}_R \cap \mathcal{R}_{\overline{R}}$ by Proposition 2.7.(2).

Moreover, $\mathfrak{C}_\mathfrak{I} \in \mathcal{R}_{\overline{R}}$ implies by Lemma 3.11 that there is an $x \in Q^{\mathrm{reg}}$ such that $\mathfrak{C}_\mathfrak{I} = x\overline{R}$. Since $1 \in \overline{R}$, this yields $x \in \mathfrak{C}_\mathfrak{I} \cap Q_R^{\mathrm{reg}} = \mathfrak{C}_\mathfrak{I}^{\mathrm{reg}}$. Finally, we obtain by Lemma 3.4.(2), Proposition 2.7.(1), and Remark 3.15

$$\nu(x)\mathbb{N}^{\mathcal{V}_R} = \Gamma_{x\overline{R}} \subset \Gamma_\mathfrak{I}$$

since ν is a group homomorphism, and since $x\overline{R} \subset \mathfrak{I}$. $\square$

If R is a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and if $\mathfrak{I} \in \mathcal{R}_R$, then the value semigroup ideal $\Gamma_\mathfrak{I}$ of $\mathfrak{I}$ is a semigroup ideal of Γ_R. Moreover, due to D'Anna (see [8]) certain algebraic hypotheses on R imply the properties (E0), (E1), and (E2) on $\Gamma_\mathfrak{I}$.

Proposition 3.22 (See [25], Proposition 3.1.9)**.** *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let $\mathfrak{I} \in \mathcal{R}_R$.*

(1) We have $\Gamma_\mathfrak{I} + \Gamma_R \subset \Gamma_\mathfrak{I}$.

(2) If R is analytically reduced, then $\Gamma_\mathfrak{I}$ satisfies (E0) with $I = \mathcal{V}_R$ and $\overline{S} = \Gamma_{\overline{R}} = \mathbb{N}^{\mathcal{V}_R}$.

(3) If R is local and analytically reduced with large residue field, then $\Gamma_\mathfrak{I}$ satisfies (E1).

(4) If R is local and residually rational, then $\Gamma_\mathfrak{I}$ satisfies (E2).

In particular, if R is local admissible, then $\Gamma_\mathfrak{I}$ satisfies (E0), (E1), and (E2).

Proof. (1) This follows from ν being a homomorphism of groups.

(2) By Lemma 3.21 there is an $x \in \mathfrak{C}_\mathfrak{I}^{\mathrm{reg}}$ such that

$$\nu(x) + \mathbb{N}^{\mathcal{V}_R} = \nu\left(x\overline{R}^{\mathrm{reg}}\right) = \nu((\mathfrak{C}_\mathfrak{I})^{\mathrm{reg}}) \subset \nu(\mathfrak{I}^{\mathrm{reg}}) = \Gamma_\mathfrak{I}$$

since ν is a group homomorphism and $\nu\left(\overline{R}^{\mathrm{reg}}\right) = \mathbb{N}^{\mathcal{V}_R}$ by Lemma 3.4.(2).

(3) See [25, Proposition 3.1.9.(c)].

(4) See [25, Proposition 3.1.9.(d)]. □

While taking the value semigroup preserves inclusions (see Remark 3.15), it is in general not compatible with the expected counterparts of products and quotients of ideals.

Lemma 3.23 (See [25], Lemma 5.3.1)**.** *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$.*

(1) If $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$, then

$$\Gamma_{\mathfrak{I}} + \Gamma_{\mathfrak{J}} \subset \Gamma_{\mathfrak{I}\mathfrak{J}}$$

and

$$\Gamma_{\mathfrak{I}:\mathfrak{J}} \subset \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}.$$

(2) If $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_{\overline{R}}$, then

$$\Gamma_{\mathfrak{I}\mathfrak{J}} = \Gamma_{\mathfrak{I}} + \Gamma_{\mathfrak{J}}$$

and

$$\Gamma_{\mathfrak{I}:\mathfrak{J}} = \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}.$$

Proof. (1) Let $\alpha \in \Gamma_{\mathfrak{I}} + \Gamma_{\mathfrak{J}}$. Then there is an $x \in \mathfrak{I}^{\mathrm{reg}}$ and a $y \in \mathfrak{J}^{\mathrm{reg}}$ such that $\nu(x) + \nu(y) = \alpha$. The claim follows since $xy \in (\mathfrak{I}\mathfrak{J})^{\mathrm{reg}}$, and since ν is a group homomorphism.

Let $\alpha \in \Gamma_{\mathfrak{I}:\mathfrak{J}}$. Then there is $x \in (\mathfrak{I} : \mathfrak{J})^{\mathrm{reg}}$ such that $\nu(x) = \alpha$. Since $x\mathfrak{J} \subset \mathfrak{I}$, this yields by Proposition D.11 and Diagram (2.10)

$$\alpha + \Gamma_{\mathfrak{J}} = \nu(x) + \nu(\mathfrak{J}^{\mathrm{reg}}) = \nu(x\mathfrak{J}^{\mathrm{reg}}) \subset \nu(\mathfrak{I}^{\mathrm{reg}}) = \Gamma_{\mathfrak{I}}.$$

Hence, $\alpha \in \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}$.

(2) This follows immediately from Definition 3.6.(1) and Lemma 3.9.(1) and (2). □

Remark 3.24 (See [25], Remark 3.1.10). Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring, and let $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$.

(1) The inclusion $\Gamma_{\mathfrak{I}} + \Gamma_{\mathfrak{J}} \subset \Gamma_{\mathfrak{I}\mathfrak{J}}$ (see Lemma 3.23.(1)) is in general not an equality, see Example 3.25 below.

(2) Similarly, the inclusion $\Gamma_{\mathfrak{I}:\mathfrak{J}} \subset \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}$ (see Lemma 3.23.(1)) is in general not an equality, see Example 3.26 below.

Example 3.25 (See [25], Example 4.1.3)**.** Consider the admissible ring (see Proposition 3.41)

$$R = \mathbb{C}\left[\left[\left(-t_1^4, t_2\right), \left(-t_1^3, 0\right), (0, t_2), \left(t_1^5, 0\right)\right]\right] \subset \mathbb{C}[[t_1]] \times \mathbb{C}[[t_2]] = \overline{R},$$

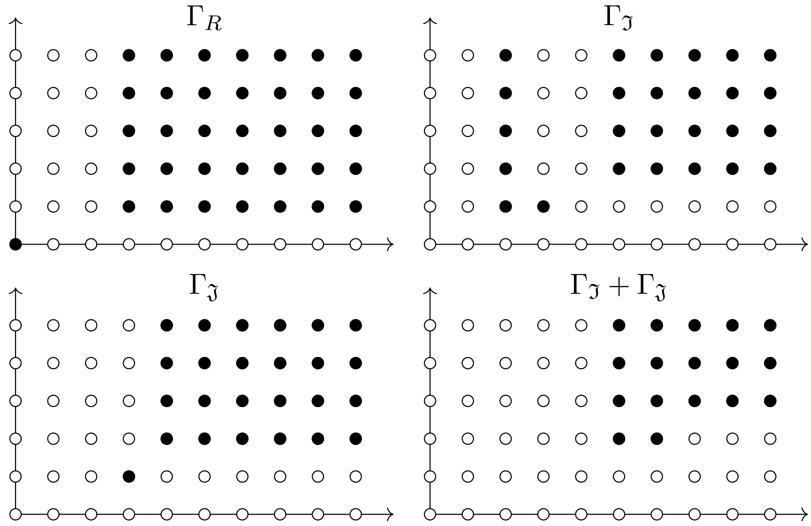


Figure 3.1.: The value semigroup (ideals) in Example 3.25.

and the R-submodules of Q_R

$$\mathfrak{I} = \left\langle \left(t_1^3, t_2\right), \left(t_1^2, 0\right) \right\rangle_R,$$
$$\mathfrak{J} = \left\langle \left(t_1^3, t_2\right), \left(t_1^4, 0\right), \left(t_1^5, 0\right) \right\rangle_R.$$

Then $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$ (see Remark 2.6.(1)). Moreover, Figure 3.1 shows that R is local (see Proposition 3.17), and that (E2) fails for $\Gamma_{\mathfrak{I}} + \Gamma_{\mathfrak{J}}$. Thus, $\Gamma_{\mathfrak{I}} + \Gamma_{\mathfrak{J}} \subsetneq \Gamma_{\mathfrak{I}\mathfrak{J}}$ by Proposition 3.22.

Example 3.26. Barucci, D'Anna and Fröberg showed in [10, Example 3.3] that for the local admissible ring (see Figure 3.2 and Propositions 3.17 and 3.41)

$$R = \mathbb{C}[[x_1, \ldots, x_{11}]]$$

with $x_1 = (t_1^7, t_2^6)$, $x_2 = (t_1^6, t_2^7)$, $x_3 = (t_1^9, t_2^{11})$, $x_4 = (t_1^{10}, t_2^{10})$, $x_5 = (t_1^{11}, t_2^9)$, $x_6 = (t_1^{11}, t_2^{10})$, $x_7 = (t_1^{12}, t_2^{12})$, $x_8 = (t_1^{13}, -t_2^{13})$, $x_9 = (t_1^{20}, t_2^{12})$, $x_{10} = (t_1^{16}, t_2^{20})$, $x_{11} = (t_1^{12}, t_2^{20})$ with maximal ideal $\mathfrak{m}_R$ property (E2) fails for the difference $\Gamma_{\mathfrak{m}_R} - \Gamma_R$, see Figure 3.2. Thus, $\Gamma_{\mathfrak{m}_R} - \Gamma_R \subsetneq \Gamma_{\mathfrak{m}_R : R}$ by Proposition 3.22.

3.2.1. Compatibility with Localization

We show the compatibility of the semigroup of values of a one-dimensional equidimensional reduced semilocal Cohen–Macaulay ring R with localization. Similarly, also the value semigroup ideals of fractional ideals of R decompose into local components (see Theorem 3.28). This enables us to extend the results of Proposition 3.22 to semilocal rings (see Corollary 3.30).

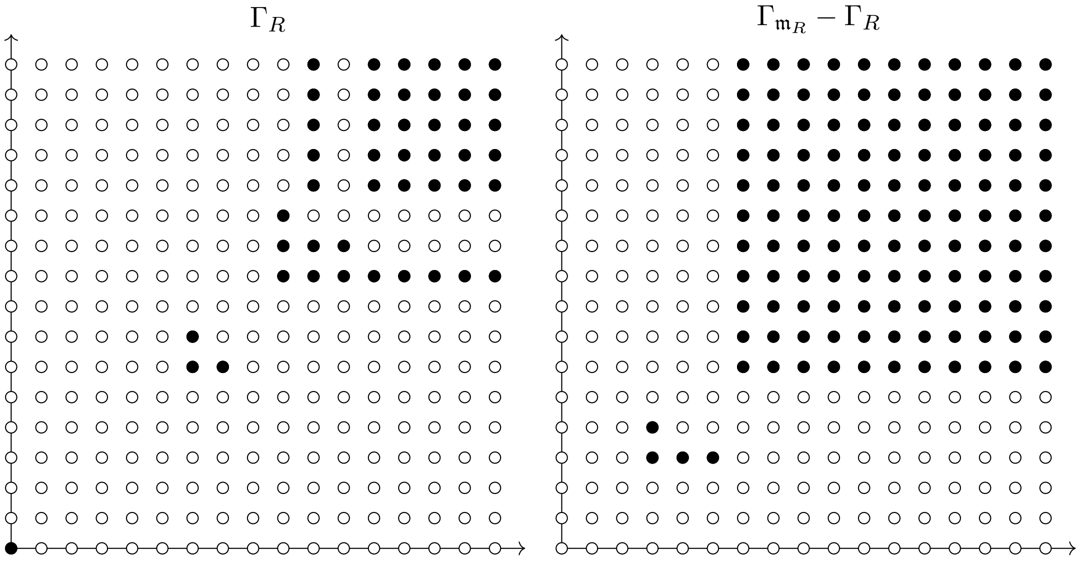


Figure 3.2.: The value semigroup (ideals) in Example 3.26, see [10, Example 3.3].

Proposition 3.27. *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring. Then $R_\mathfrak{m}$ is a one-dimensional reduced local Cohen–Macaulay ring for every $\mathfrak{m} \in \operatorname{Max}(R)$.*

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$. Then height $\mathfrak{m} = 1$ by Proposition B.27 since R is equidimensional. Hence, $\dim R_\mathfrak{m} = 1$ by Proposition A.20.(2). Moreover, $R_\mathfrak{m}$ is reduced by Lemma A.27 as R is reduced. Moreover, $R_\mathfrak{m}$ is Noetherian by Corollary A.21. Thus, $R_\mathfrak{m}$ is a one-dimensional reduced local Cohen–Macaulay by Proposition C.13. ☐

The first part of the following Theorem was stated by Barucci, D'Anna and Fröberg in [10, Section 1.1].

Theorem 3.28 (See [25], Theorem 3.2.2)**.** *Let R be a one-dimensional equidimensional reduced semilocal Cohen–Macaulay ring. Then there is a decomposition*

$$\Gamma_R = \prod_{\mathfrak{m} \in \operatorname{Max}(R)} \Gamma_{R_\mathfrak{m}}$$

of Γ_R into local value semigroups. Moreover, for any $\mathfrak{E} \in \mathcal{R}_R$ there is a decomposition

$$\Gamma_\mathfrak{J} = \prod_{\mathfrak{m} \in \operatorname{Max}(R)} \Gamma_{\mathfrak{J}_\mathfrak{m}}.$$

For the proof of Theorem 3.28 we need the following lemma.

Lemma 3.29 (See [25], Lemma 3.2.1)**.** *Let R be a one-dimensional equidimensional reduced semilocal Cohen–Macaulay ring. For any $\mathfrak{m} \in \operatorname{Max}(R)$ the localization map*

$\pi\colon Q_R \to (Q_R)_\mathfrak{m} = Q_{R_\mathfrak{m}}$ (see Proposition A.77 for the equality) induces a bijection

$$\begin{aligned}\rho_\mathfrak{m}\colon \{V \in \mathcal{V}_R \mid \mathfrak{m}_V \cap R = \mathfrak{m}\} &\to \mathcal{V}_{R_\mathfrak{m}},\\ V &\mapsto V_\mathfrak{m},\\ \pi^{-1}(W) &\leftarrow\!\shortmid W.\end{aligned}$$

In particular, $(\mathfrak{m}_V)_\mathfrak{m} = \mathfrak{m}_{\rho_\mathfrak{m}(V)}$ for every $V \in \mathcal{V}_R$.

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$, and let $V \in \mathcal{V}_R$ with $\mathfrak{m}_V \cap R = \mathfrak{m}$. Since localization is exact by Proposition A.24, we have

$$R_\mathfrak{m} \subset V_\mathfrak{m} \subset (Q_R)_\mathfrak{m} = Q_{R_\mathfrak{m}},$$

where the last equality follows from Proposition A.77. Since $R \setminus \mathfrak{m} \subset V \setminus \mathfrak{m}_V$ by assumption, and hence $\nu_V(x) = 0$ for all $x \in R \setminus \mathfrak{m}$ by Proposition D.13.(1) and (3), Lemma D.19 implies $V_\mathfrak{m} \in \mathcal{V}_{R_\mathfrak{m}}$ with $\pi^{-1}(V_\mathfrak{m}) = V$. Thus, $\rho_\mathfrak{m}$ is an injective map.

Let now $W \in \mathcal{V}_{R_\mathfrak{m}}$, and set $V = \pi^{-1}(W)$. Then $V_\mathfrak{m} = W \subsetneq Q_{R_\mathfrak{m}}$ by Lemma A.35.(2), and hence $R \subset V \subsetneq Q_R$.

Let $x, y \in Q_R \setminus V$, and suppose that $xy \in V$. Then $\pi(x), \pi(y) \in Q_{R_\mathfrak{m}} \setminus V_\mathfrak{m}$ yields $\pi(x)\pi(y) = \pi(xy) \in \pi(V) \subset W$ which is a contradiction to $Q_{R_\mathfrak{m}} \setminus W$ being multiplicatively closed as W is a valuation ring of $Q_{R_\mathfrak{m}}$. Thus, also $Q_R \setminus V$ is multiplicatively closed, and hence $V \in \mathcal{V}_R$.

Consider the commutative diagram of ring homomorphisms

$$\begin{tikzcd} R \arrow[r, "\pi|_R"] \arrow[d, hook] & R_\mathfrak{m} \arrow[d, hook] \\ V \arrow[r, "\pi"'] & W. \end{tikzcd}$$

Then $\pi^{-1}(\mathfrak{m}_W)$ is a prime ideal of V by Proposition A.20.(1), and Theorem 3.2.(6) and Propositions B.3 and B.15 yield

$$\begin{aligned}\pi^{-1}(\mathfrak{m}_W) \cap R &= \pi^{-1}(\mathfrak{m}_W) \cap \pi^{-1}(R_\mathfrak{m}) \cap R\\ &= \pi^{-1}(\mathfrak{m}_W \cap R_\mathfrak{m}) \cap R\\ &= (\pi|_R)^{-1}(\mathfrak{m}_W \cap R_\mathfrak{m})\\ &= (\pi|_R)^{-1}(\mathfrak{m}R_\mathfrak{m})\\ &= \mathfrak{m}.\end{aligned}$$

In particular, with $\mathfrak{m}$ also $\mathfrak{p}$ is regular, and hence $\mathfrak{p} = \mathfrak{m}_V$ by Theorem 3.2.(1) and Proposition 2.22.(d). □

Let R be a one-dimensional equidimensional reduced semilocal Cohen–Macaulay ring. By Theorem 3.2.(1) and Proposition 2.22.(d) the sets $\{V \in \mathcal{V}_R \mid \mathfrak{m}_V \cap R = \mathfrak{m}\}$, $\mathfrak{m} \in \operatorname{Max}(R)$, form a partition of $\mathcal{V}_R$. By Lemma 3.29 there is a bijection

$$\begin{aligned}\rho\colon \mathcal{V}_R &\to \coprod_{\mathfrak{m} \in \operatorname{Max}(R)} \mathcal{V}_{R_\mathfrak{m}},\\ V &\mapsto \rho_{\mathfrak{m}_V \cap R}(V) = V_{\mathfrak{m}_V \cap R}\end{aligned}$$

inducing an order preserving group homomorphism (see Lemma 2.16.(2) and Propositions A.38 and A.39)

$$\begin{aligned}\rho'\colon \prod_{V\in\mathcal{V}_R}\mathcal{R}_V^* &\to \prod_{\mathfrak{m}\in\operatorname{Max}(R)}\prod_{W\in\mathcal{V}_{R_\mathfrak{m}}}\mathcal{R}_W^*,\\ (\mathfrak{I}_V)_{V\in\mathcal{V}_R} &\mapsto \left(\left(\left(\mathfrak{I}_{\rho^{-1}(W)}\right)_{\rho^{-1}(W)\cap R}\right)_{W\in\mathcal{V}_{R_\mathfrak{m}}}\right)_{\mathfrak{m}\in\operatorname{Max}(R)}.\end{aligned}$$

Let $(\mathfrak{I}_V)_{V\in\mathcal{V}_R}\in\prod_{V\in\mathcal{V}_R}\mathcal{R}_V^*$. Then Proposition 2.21.(2) yields $\mathfrak{I}_V=\mathfrak{m}_V^{k_V}$ with $k_V=\max\left\{k\in\mathbb{Z}\mid \mathfrak{m}_V^k\le\mathfrak{I}\right\}$ for every $V\in\mathcal{V}_R$. So Lemma 3.29 implies

$$\begin{aligned}\rho'\left((\mathfrak{I}_V)_{V\in\mathcal{V}_R}\right)&=\rho'\left(\left(\mathfrak{m}_V^{k_V}\right)_{V\in\mathcal{V}_R}\right)\\ &=\left(\left(\left(\mathfrak{m}_{\rho^{-1}(W)}^{k_{\rho^{-1}(W)}}\right)_{\mathfrak{m}_{\rho^{-1}(W)}\cap R}\right)_{W\in\mathcal{V}_{R_\mathfrak{m}}}\right)_{\mathfrak{m}\in\operatorname{Max}(R)}\\ &=\left(\left(\left(\left(\mathfrak{m}_{\rho^{-1}(W)}\right)_{\mathfrak{m}_{\rho^{-1}(W)}\cap R}\right)^{k_{\rho^{-1}(W)}}\right)_{W\in\mathcal{V}_{R_\mathfrak{m}}}\right)_{\mathfrak{m}\in\operatorname{Max}(R)}\\ &=\left(\left(\mathfrak{m}_W^{k_{\rho^{-1}(W)}}\right)_{W\in\mathcal{V}_{R_\mathfrak{m}}}\right)_{\mathfrak{m}\in\operatorname{Max}(R)}.\end{aligned}$$

Since ρ also induces an isomorphism

$$\rho''\colon\mathbb{Z}^{\mathcal{V}_R}\to\prod_{\mathfrak{m}\in\operatorname{Max}(R)}\mathbb{Z}^{V_{R_\mathfrak{m}}},$$

we obtain with Diagram (3.1) a commutative diagram

$$\begin{tikzcd}
\prod\limits_{V\in\mathcal{V}_R}\mathcal{R}_V^* \arrow[r, "\phi_R", "\cong"'] \arrow[d, "\rho'"'] & \mathbb{Z}^{\mathcal{V}_R} \arrow[d, "\rho''", "\cong"'] \\
\prod\limits_{\mathfrak{m}\in\operatorname{Max}(R)}\prod\limits_{W\in\mathcal{V}_{R_\mathfrak{m}}}\mathcal{R}_W^* \arrow[r, "\cong", "\prod_{\mathfrak{m}\in\operatorname{Max}(R)}\phi_{R_\mathfrak{m}}"'] & \prod\limits_{\mathfrak{m}\in\operatorname{Max}(R)}\mathbb{Z}^{V_{R_\mathfrak{m}}},
\end{tikzcd}$$

and hence

$$\rho'=\left(\prod_{\mathfrak{m}\in\operatorname{Max}(R)}\phi_{R_\mathfrak{m}}\right)^{-1}\circ\rho''\circ\phi_R$$

is an isomorphism. With Proposition 3.5 it fits into a commutative diagram

$$\begin{tikzcd}
Q_R^{\mathrm{reg}} \arrow[rrr] \arrow[dr, two heads] \arrow[dddr, bend right, two heads, "\nu_R"'] & & & \prod\limits_{\mathfrak{m}\in\operatorname{Max}(R)} Q_{R_\mathfrak{m}}^{\mathrm{reg}} \arrow[dl, two heads] \arrow[dddl, bend left, two heads, "\prod_{\mathfrak{m}\in\operatorname{Max}(R)} \nu_{R_\mathfrak{m}}"] \\
& \mathcal{R}_{\overline{R}} \arrow[r, "\xi", "\cong"'] \arrow[d, "\psi_R"', "\cong"] & \prod\limits_{\mathfrak{m}\in\operatorname{Max}(R)} \mathcal{R}_{\overline{R_\mathfrak{m}}} \arrow[d, "\cong"', "\prod_{\mathfrak{m}\in\operatorname{Max}(R)} \psi_{R_\mathfrak{m}}"] & \\
& \prod\limits_{V\in\mathcal{V}_R} \mathcal{R}_V^* \arrow[r, "\rho'", "\cong"'] \arrow[d, "\phi_R"', "\cong"] & \prod\limits_{\mathfrak{m}\in\operatorname{Max}(R)} \prod\limits_{W\in\mathcal{V}_{R_\mathfrak{m}}} \mathcal{R}_W^* \arrow[d, "\prod_{\mathfrak{m}\in\operatorname{Max}(R)} \phi_{R_\mathfrak{m}}"', "\cong"] & \\
& \mathbb{Z}^{\mathcal{V}_R} \arrow[r, "\rho''", "\cong"'] & \prod\limits_{\mathfrak{m}\in\operatorname{Max}(R)} \mathbb{Z}^{\mathcal{V}_{R_\mathfrak{m}}}, &
\end{tikzcd}$$

where

$$\begin{aligned}
\xi\colon \mathcal{R}_{\overline{R}} &\to \prod_{\mathfrak{m}\in\operatorname{Max}(R)} \mathcal{R}_{\overline{R}_\mathfrak{m}} = \prod_{\mathfrak{m}\in\operatorname{Max}(R)} \mathcal{R}_{\overline{R_\mathfrak{m}}}, \\
\mathfrak{I} &\mapsto (\mathfrak{I}_\mathfrak{m})_{\mathfrak{m}\in\operatorname{Max}(R)},
\end{aligned}$$

see Lemma 2.16.(2), Proposition A.39, and Corollary B.8. This implies

$$\nu_R(x) = \left(\nu_{R_\mathfrak{m}}\left(\frac{x}{1}\right)\right)_{\mathfrak{m}\in\operatorname{Max}(R)} \tag{3.12}$$

for all $x \in Q_R^{\mathrm{reg}}$. To ease notation we identify $\mathbb{Z}^{\mathcal{V}_R}$ and $\prod\limits_{\mathfrak{m}\in\operatorname{Max}(R)} \mathbb{Z}^{\mathcal{V}_{R_\mathfrak{m}}}$ via ρ''.

Proof of Theorem 3.28. By Proposition A.20.(2) and Lemma A.27 $R_\mathfrak{m}$ is a one-dimensional reduced local Cohen–Macaulay ring, and hence $\Gamma_{R_\mathfrak{m}}$ is local by Proposition 3.17 for all $\mathfrak{m} \in \operatorname{Max}(R)$. To prove the Theorem we have to show the second decomposition of the statement.

So let $\mathfrak{E} \in \mathcal{R}_R$. Then for any $\mathfrak{m} \in \operatorname{Max}(R)$ Proposition A.39 and Lemma 2.16.(2) yield $\mathfrak{E}_\mathfrak{m} \in \mathcal{R}_{R_\mathfrak{m}}$, and by Equation (3.12) there is an inclusion

$$\Gamma_\mathfrak{E} \subset \prod_{\mathfrak{m}\in\operatorname{Max}(R)} \Gamma_{\mathfrak{E}_\mathfrak{m}}.$$

Let now

$$\alpha = (\alpha_\mathfrak{m})_{\mathfrak{m}\in\operatorname{Max}(R)} \in \prod_{\mathfrak{m}\in\operatorname{Max}(R)} \Gamma_{\mathfrak{E}_\mathfrak{m}}.$$

Then for any $\mathfrak{m} \in \mathrm{Max}\,(R)$ there is $\frac{x_\mathfrak{m}}{y_\mathfrak{m}} \in \mathfrak{E}_\mathfrak{m}$ such that

$$\nu_{R_\mathfrak{m}}\left(\frac{x_\mathfrak{m}}{y_\mathfrak{m}}\right) = \alpha_\mathfrak{m}.$$

Since $y_\mathfrak{m} \in R \setminus \mathfrak{m}$, and hence $\frac{y_\mathfrak{m}}{1} \in (R_\mathfrak{m})^*$ for every $\mathfrak{m} \in \mathrm{Max}\,(R)$, Lemma 3.4.(4) implies $\nu_{R_\mathfrak{m}}(y_\mathfrak{m}) = 0$. So after clearing denominators we may assume that $y_\mathfrak{m} = 1$ for all $\mathfrak{m} \in \mathrm{Max}\,(R)$.

The Chinese Remainder Theorem yields for any $\mathfrak{m} \in \mathrm{Max}\,(R)$ a

$$z_\mathfrak{m} \in \left(\bigcap_{\mathfrak{n} \in \mathrm{Max}\,(R) \setminus \{\mathfrak{m}\}} \mathfrak{n}\right) \setminus \mathfrak{m}.$$

Then Theorem 3.2.(6), Propositions B.3, B.15, and D.13.(1) and (3) imply for any $\mathfrak{m} \in \mathrm{Max}\,(R)$

$$\begin{aligned} &\nu_{R_\mathfrak{m}}\left(\frac{z_\mathfrak{m}}{1}\right) = 0, \\ &\nu_V\left(\frac{z_\mathfrak{m}}{1}\right) > 0 \text{ for all } V \in \mathcal{V}_{R_\mathfrak{n}} \text{ for all } V \in \mathcal{V}_{R_\mathfrak{n}} \text{ for every } \mathfrak{n} \in \mathrm{Max}\,(R) \setminus \{\mathfrak{m}\}. \end{aligned}$$

For every $\mathfrak{m} \in \mathrm{Max}\,(R)$ pick a

$$k_\mathfrak{m} > \max\left\{\nu_V\left(\frac{x_\mathfrak{n}}{1}\right) - \nu_V\left(\frac{x_\mathfrak{m}}{1}\right) \,\middle|\, V \in \mathcal{V}_{R_\mathfrak{n}},\ \mathfrak{n} \in \mathrm{Max}\,(R) \setminus \{\mathfrak{m}\}\right\}.$$

Then

$$z = \sum_{\mathfrak{m} \in \mathrm{Max}\,(R)} x_\mathfrak{m} z_\mathfrak{m}^{k_\mathfrak{m}} \in \mathfrak{E}$$

with

$$\nu_{R_\mathfrak{m}}\left(\frac{z}{1}\right) = \alpha_\mathfrak{m}$$

for all $\mathfrak{m} \in \mathrm{Max}\,(R)$. Thus,

$$\nu(z) = \alpha$$

by Equation (3.12), and the claim follows. □

With Theorem 3.28 we are able to generalize Proposition 3.22.(3) and (4) to the semilocal case.

Corollary 3.30 (See [25], Corollary 3.2.3)**.** *Let R be a one-dimensional equidimensional reduced semilocal Cohen–Macaulay ring with large residue fields, and let $\mathfrak{I} \in \mathcal{R}_R$.*

(1) If R is analytically reduced, then $\Gamma_\mathfrak{I}$ satisfies (E1).

(2) If R is residually rational, then $\Gamma_\mathfrak{I}$ satisfies (E2).

In particular, if R is admissible, then $\Gamma_\mathfrak{I}$ is good.

Proof. (1) This follows immediately from Theorem 3.28 and Proposition 3.22.(3).

(2) Let $\alpha, \beta \in \Gamma_{\mathfrak{J}}$ such that $\alpha_V = \beta_V$ for some $V \in V_R$, and set $\mathfrak{m} = \mathfrak{m}_V \cap R \in \operatorname{Max}(R)$ (see Theorem 3.2.(6) and Propositions B.3 and B.15). Then $\alpha_{\mathfrak{m}}, \beta_{\mathfrak{m}} \in \Gamma_{\mathfrak{J}_{\mathfrak{m}}}$ by Theorem 3.28, and Lemma 3.29 implies $(\alpha_{\mathfrak{m}})_{\rho_{\mathfrak{m}}(V)} = (\beta_{\mathfrak{m}})_{\rho_{\mathfrak{m}}(V)}$. Hence, Proposition 3.22.(4) yields an $\epsilon_{\mathfrak{m}} \in \Gamma_{\mathfrak{J}_{\mathfrak{m}}}$ such that

$$\begin{aligned}
(\epsilon_{\mathfrak{m}})_{\chi_{\mathfrak{m}}(V)} &> (\alpha_{\mathfrak{m}})_{\chi_{\mathfrak{m}}(V)} = (\beta_{\mathfrak{m}})_{\chi_{\mathfrak{m}}(V)}\,,\\
(\epsilon_{\mathfrak{m}})_{V'} &\geq \inf\left\{(\alpha_{\mathfrak{m}})_{V'}, (\beta_{\mathfrak{m}})_{V'}\right\} \text{ for all } V' \in \mathcal{V}_{R_{\mathfrak{m}}} \setminus \{\chi_{\mathfrak{m}}(V)\},\\
(\epsilon_{\mathfrak{m}})_{V''} &= \inf\left\{(\alpha_{\mathfrak{m}})_{V''}, (\beta_{\mathfrak{m}})_{V''}\right\} \text{ for all } V'' \in \mathcal{V}_{R_{\mathfrak{m}}} \setminus \{\chi_{\mathfrak{m}}(V)\}\\
&\qquad\qquad\qquad\qquad \text{with } (\alpha_{\mathfrak{m}})_{V''} \neq (\beta_{\mathfrak{m}})_{V''}\,.
\end{aligned}$$

Let now $\mathfrak{n} \in \operatorname{Max}(R) \setminus \{\mathfrak{m}\}$. Since R has large residue fields, there is by Proposition 3.22 an

$$\epsilon_{\mathfrak{n}} = \inf\{\alpha_{\mathfrak{n}}, \beta_{\mathfrak{n}}\} \in (\Gamma_{\mathfrak{J}})_{\mathfrak{n}}.$$

Hence, if we set $\epsilon = (\epsilon_{\mathfrak{m}})_{\mathfrak{m} \in \operatorname{Max}(R)}$, then $\epsilon \in \Gamma_{\mathfrak{J}}$ by Theorem 3.28, and

$$\begin{aligned}
\epsilon_V &> \alpha_V = \beta_V,\\
\epsilon_{V'} &\geq \inf\{\alpha_{V'}, \beta_{V'}\} \text{ for all } V' \in \mathcal{V}_R \setminus \{V\},\\
\epsilon_{V''} &= \inf\{\alpha_{V''}, \beta_{V''}\} \text{ for all } V'' \in \mathcal{V}_R \setminus \{V\} \text{ with } \alpha_{V''} \neq \beta_{V''}.
\end{aligned}$$

Thus, $\Gamma_{\mathfrak{J}}$ satisfies (E2).

The particular claim follows with Proposition 3.22. □

Remark 3.31. In the proof of Corollary 3.30.(2) we need to apply property (E1) in $(\Gamma_{\mathfrak{J}})_{\mathfrak{n}}$ only for those $\mathfrak{n} \in \operatorname{Max}(R)$ with $(\alpha_{\mathfrak{n}})_V \neq (\beta_{\mathfrak{n}})_V$ for all $V \in \mathcal{V}_{R_{\mathfrak{n}}}$. Otherwise property (E2) is sufficient to construct an $\epsilon \in \Gamma_{\mathfrak{J}}$ of the desired form.

The following corollary relates value semigroup ideals to jumps in the filtration induced by $\mathfrak{Q}^{\bullet}$, see Definition 3.6 and [13, Remark 4.3].

Corollary 3.32 (See [25], Lemma 3.3.4)**.** *Let R be a one-dimensional equidimensional analytically reduced semilocal Cohen–Macaulay ring with large residue fields, and let $\mathfrak{J}$ be an R-submodule of Q_R. For any $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$ we have $\alpha \in \Gamma_{\mathfrak{J}}$ if and only if $\mathfrak{J}^{\alpha}/\mathfrak{J}^{\alpha+\mathbf{e}_V} \neq 0$ for all $V \in \mathcal{V}_R$.*

Proof. We have $\mathfrak{J}^{\alpha}/\mathfrak{J}^{\alpha+\mathbf{e}_V} \neq 0$ for all $V \in \mathcal{V}_R$ if and only if for every $V \in \mathcal{V}_R$ there is an $x^{(V)} \in \mathfrak{J}$ with $\nu\left(x^{(V)}\right) \geq \alpha$ and $\nu_V\left(x^{(V)}\right) = \alpha_V$. In particular, we have $x^{(V)} \in \mathfrak{J}^{\alpha}$. Since R is a Marot ring by Corollary A.46, Theorem A.74.(1), and Remark A.17, and since therefore $\mathfrak{J}^{\alpha}$ is by Proposition 3.8 and Remark 2.6.(2) generated by $(\mathfrak{J}^{\alpha})^{\mathrm{reg}}$, we may assume that $x^{(V)} \in (\mathfrak{J}^{\alpha})^{\mathrm{reg}}$. Thus, there is for any $V \in \mathcal{V}_R$ an $x^{(V)} \in \mathfrak{J}$ with $\nu\left(x^{(V)}\right) \geq \alpha$ and $\nu_V\left(x^{(V)}\right) = \alpha_V$ if and only if for any $V \in \mathcal{V}_R$ there is a $\beta^{(V)} \in \Gamma_{\mathfrak{J}}$ with $\beta \geq \alpha$ and $\beta_V = \alpha_V$. Since $\Gamma_{\mathfrak{J}}$ satisfies (E1) by Corollary 3.30.(1), this is equivalent to $\alpha \in \Gamma_{\mathfrak{J}}$. □

Lemma 3.33. *A Noetherian semilocal ring R is admissible if and only if $R_\mathfrak{m}$ is admissible for every $\mathfrak{m} \in \operatorname{Max}(R)$.*

Proof. This follows from Propositions 3.27 and A.24, Corollary B.8, Lemma A.68, and Definition C.2. $\square$

3.2.2. Invariance under Completion

We show the invariance of the semigroup of values under completion. In the local case the following statement is due to D'Anna [8, Section 1].

Theorem 3.34 (See [25], Theorem 3.3.4)**.** *Let R be a one-dimensional equidimensional semilocal Cohen–Macaulay ring with large residue fields. If R is local or analytically reduced, then*

$$\Gamma_{\mathfrak{I}} = \Gamma_{\widehat{\mathfrak{I}}}$$

for any $\mathfrak{I} \in \mathcal{R}_R$.

For the proof of Theorem 3.34 we need the following Lemmas.

Lemma 3.35 (See [25], Lemma 3.3.1)**.** *With R also $\widehat{R}$ is a one-dimensional (semi)local Cohen–Macaulay ring.*

Proof. This follows from Theorem A.59.(2) and Corollaries A.64 and C.7. $\square$

Lemma 3.36 (See [25], Lemma 2.1.5)**.** *Let R be a one-dimensional local Cohen–Macaulay ring. Then $Q_R\widehat{R} = Q_{\widehat{R}}$, and there is an inclusion preserving group isomorphism*

$$\begin{aligned} \mathcal{R}_R &\to \mathcal{R}_{\widehat{R}}, \\ \mathfrak{I} &\mapsto \widehat{\mathfrak{I}}, \\ \mathfrak{J} \cap Q_R &\hookleftarrow \mathfrak{J}. \end{aligned}$$

Proof. By [23, Chapter II, (2.4)] we have

$$Q_{\widehat{R}} = \widehat{R}\left[\frac{1}{r}\right] = Q_R\widehat{R} \tag{3.13}$$

for any $r \in \mathfrak{m}^{\mathrm{reg}}$, where $\mathfrak{m}$ is the maximal ideal of R. Then Lemma 2.16.(2) and (4) and Theorems A.55 and A.60 yield an injective map

$$\begin{aligned} \mathcal{R}_R &\to \mathcal{R}_{\widehat{R}}, \\ \mathfrak{I} &\mapsto \mathfrak{I}\widehat{R} = \widehat{\mathfrak{I}} \end{aligned}$$

such that $\mathfrak{I} = \widehat{\mathfrak{I}} \cap Q_R$ for all $\mathfrak{I} \in \mathcal{R}_R$.

Let now $\mathfrak{J} \in \mathcal{R}_{\widehat{R}}$. Then there is an $x \in \widehat{R}^{\mathrm{reg}}$ such that $x\mathfrak{J} \subset \widehat{R}$, and by Equation (3.13) we may assume that $x \in R^{\mathrm{reg}}$. Now Theorem A.56 yields $x\mathfrak{J} \cap R \in \mathcal{R}_R$ with

$$(x\mathfrak{J} \cap R)\widehat{R} = x\mathfrak{J}.$$

Thus, we obtain $x^{-1}(x\mathfrak{J} \cap R) \in \mathcal{R}_R$ with

$$x^{-1}(x\mathfrak{J} \cap R)\widehat{R} = x^{-1}x\mathfrak{J} = \mathfrak{J}. \qquad \square$$

Theorem 3.37 (See [25], Theorem 3.3.2)**.** *Let R be a one-dimensional local Cohen–Macaulay ring. Then there is a bijection*

$$\begin{aligned} \sigma\colon \mathcal{V}_R &\to \mathcal{V}_{\widehat{R}} \\ V &\mapsto V\widehat{R} \\ W \cap Q_R &\leftarrow\!\shortmid W. \end{aligned}$$

In particular, $\mathfrak{m}_V\widehat{R} = \mathfrak{m}_{\sigma(V)}$ *for every* $V \in \mathcal{V}_R$*.*

Proof. See [23, Chapter II, Theorem 3.19.(2)] for the bijection $\sigma\colon \mathcal{V}_R \to \mathcal{V}_{\widehat{R}}$.

Let $\mathfrak{m}_R$ be the maximal ideal of R. By Theorem A.59.(3) we have $\widehat{R}/\mathfrak{m}_R\widehat{R} = R/\mathfrak{m}_R$, and hence

$$\widehat{R} = R + \mathfrak{m}_R\widehat{R}. \tag{3.14}$$

Let $V \in \mathcal{V}_R$. Since $\mathfrak{m}_R \subset \mathfrak{m}_V$ by Theorem 3.2.(6) and Proposition B.15, we have $\mathfrak{m}_R \subset \mathfrak{m}_V$. This implies $\mathfrak{m}_R V\widehat{R} \subset \mathfrak{m}_V V\widehat{R} = \mathfrak{m}_V\widehat{R}$. Therefore, we obtain with Equation (3.14)

$$V\widehat{R} = V\left(R + \mathfrak{m}_R\widehat{R}\right) = V + \mathfrak{m}_R V\widehat{R} = V + \mathfrak{m}_V\widehat{R},$$

where we use $\mathfrak{m}_V\widehat{R} \subset V\widehat{R}$ for the last equality. This yields

$$V\widehat{R}/\mathfrak{m}_V\widehat{R} = V/\left(\mathfrak{m}_V\widehat{R} \cap V\right).$$

Since $\mathfrak{m}_V\widehat{R} \cap V = \mathfrak{m}_V \cap Q_R \cap V = \mathfrak{m}_V \cap V = \mathfrak{m}_V$ by Lemma 2.16.(4) and Theorem A.60, this implies that

$$V\widehat{R}/\mathfrak{m}_V\widehat{R} = V/\mathfrak{m}_V$$

is a field. Therefore, $\mathfrak{m}_V\widehat{R}$ is a maximal ideal of $V\widehat{R}$.

Moreover, since

$$\emptyset \neq (\mathfrak{m}_R)^{\mathrm{reg}} = (\mathfrak{m}_V \cap R)^{\mathrm{reg}} \subset \left((\mathfrak{m}_V \cap R)\widehat{R}\right)^{\mathrm{reg}} = \left(\mathfrak{m}_V\widehat{R} \cap \widehat{R}\right)^{\mathrm{reg}}$$

by Lemmas 2.16.(4) and A.7, Theorems 3.2.(6) and A.60, and Proposition B.15, $\mathfrak{m}_V\widehat{R}$ is a regular maximal ideal of $V\widehat{R}$. Thus, $\mathfrak{m}_V\widehat{R} = \mathfrak{m}_{V\widehat{R}}$ by Remark D.5 since $V\widehat{R} \in \mathcal{V}_{\widehat{R}}$. □

Corollary 3.38 (See [25], Corollary 3.3.2)**.** *Let R be a one-dimensional local Cohen–Macaulay ring. Then $\overline{\widehat{R}} = \overline{R}\widehat{R}$. In particular, if $\overline{R}$ is finite over R, then $\overline{\widehat{R}} = \widehat{\overline{R}}$.*

Proof. Since $\widehat{R}$ is by Lemma 3.35 a one-dimensional local Cohen–Macaulay ring, Theorem 3.2.(4) yields with Lemma 2.16.(4) and Theorems 3.37 and A.60 (see [23, Chapter II, Theorem 3.19.(3)])

$$\overline{\widehat{R}} = \bigcap_{W \in \mathcal{V}_{\widehat{R}}} W = \bigcap_{V \in \mathcal{V}_R} V\widehat{R} = \left(\bigcap_{V \in \mathcal{V}_R} V\right)\widehat{R} = \overline{R}\widehat{R}.$$

Remark 2.6.(1), Lemma 2.16.(2) and Theorems A.52 and A.55 yield the particular claim. □

Let R be a one-dimensional local Cohen–Macaulay ring. The bijection $\sigma\colon \mathcal{V}_R \to \mathcal{V}_{\widehat{R}}$ of Theorem 3.37 induces an order preserving group isomorphism (see Lemma 2.16.(2) and Theorem A.54)

$$\begin{aligned}\sigma'\colon \prod_{V\in\mathcal{V}_R} \mathcal{R}_V^* &\to \prod_{W\in\mathcal{V}_{\widehat{R}}} \mathcal{R}_W^*,\\ (\mathfrak{I}_V)_{V\in\mathcal{V}_R} &\mapsto \left(\mathfrak{I}_{\sigma^{-1}(W)}\widehat{R}\right)_{W\in\mathcal{V}_{\widehat{R}}}.\end{aligned}$$

Let $(\mathfrak{I}_V)_{V\in\mathcal{V}_R} \in \prod_{V\in\mathcal{V}_R}\mathcal{R}_V^*$. Then Proposition 2.21.(2) yields $\mathfrak{I}_V = \mathfrak{m}_V^{k_V}$ with $k_V = \max\left\{k\in\mathbb{Z} \mid \mathfrak{m}_V^k \leq \mathfrak{I}\right\}$ for every $V\in\mathcal{V}_R$. So Theorem 3.37 implies

$$\begin{aligned}\sigma'\left((\mathfrak{I}_V)_{V\in\mathcal{V}_R}\right) &= \sigma'\left(\left(\mathfrak{m}_V^{k_V}\right)_{V\in\mathcal{V}_R}\right)\\ &= \left(\mathfrak{m}_V^{k_V}\widehat{R}\right)_{V\in\mathcal{V}_R}\\ &= \left(\mathfrak{m}_{\sigma^{-1}(W)}^{k_{\sigma^{-1}(W)}}\widehat{R}\right)_{W\in\mathcal{V}_{\widehat{R}}}\\ &= \left(\left(\mathfrak{m}_{\sigma^{-1}(W)}\widehat{R}\right)^{k_{\sigma^{-1}(W)}}\right)_{W\in\mathcal{V}_{\widehat{R}}}\\ &= \left(\mathfrak{m}_W^{k_{\sigma^{-1}(W)}}\right)_{W\in\mathcal{V}_{\widehat{R}}}.\end{aligned}$$

Since σ also induces an isomorphism

$$\sigma''\colon \mathbb{Z}^{\mathcal{V}_R} \to \mathbb{Z}^{\mathcal{V}_{\widehat{R}}},$$

we obtain with Diagram (3.1) a commutative diagram

$$\begin{array}{ccc}\prod_{V\in\mathcal{V}_R}\mathcal{R}_V^* & \xrightarrow[\cong]{\phi_R} & \mathbb{Z}^{\mathcal{V}_R}\\ {\scriptstyle\sigma'}\downarrow & & \cong\downarrow{\scriptstyle\sigma''}\\ \prod_{W\in\mathcal{V}_{\widehat{R}}}\mathcal{R}_W^* & \xrightarrow[\phi_{\widehat{R}}]{\cong} & \mathbb{Z}^{\mathcal{V}_{\widehat{R}}},\end{array}$$

and hence

$$\sigma' = \left(\phi_{\widehat{R}}\right)^{-1}\circ\sigma''\circ\phi_R$$

is an isomorphism. With Proposition 3.5 it fits into a commutative diagram

$$
\begin{tikzcd}
Q_R^{\mathrm{reg}} \arrow[rrr] \arrow[dr, two heads] \arrow[dddr, bend right, two heads, "\nu_R"'] & & & Q_{\widehat{R}}^{\mathrm{reg}} \arrow[dl, two heads] \arrow[dddl, bend left, two heads, "\prod_{\mathfrak{m}\in\operatorname{Max}(R)} \nu_{R_{\mathfrak{m}}}"] \\
& \mathcal{R}_{\overline{R}} \arrow[r, "\eta", "\cong"'] \arrow[d, "\psi_R"', "\cong"] & \mathcal{R}_{\overline{\widehat{R}}} \arrow[d, "\cong"', "\psi_{\widehat{R}}"] & \\
& \prod_{V\in\mathcal{V}_R} \mathcal{R}_V^* \arrow[r, "\sigma'", "\cong"'] \arrow[d, "\phi_R"', "\cong"] & \prod_{W\in\mathcal{V}_{\widehat{R}}} \mathcal{R}_W^* \arrow[d, "\cong"', "\phi_{\widehat{R}}"] & \\
& \mathbb{Z}^{\mathcal{V}_R} \arrow[r, "\sigma''", "\cong"'] & \mathbb{Z}^{\mathcal{V}_{\widehat{R}}}, &
\end{tikzcd}
\tag{3.15}
$$

where

$$
\begin{aligned}
\eta\colon \mathcal{R}_{\overline{R}} &\to \mathcal{R}_{\overline{R}\widehat{R}} = \mathcal{R}_{\overline{\widehat{R}}}, \\
\mathfrak{I} &\mapsto \mathfrak{I}\widehat{R},
\end{aligned}
$$

see Lemma 2.16.(2), Corollary 3.38, and Theorem A.54. This implies

$$\nu_R(x) = \nu_{\widehat{R}}(x)$$

for all $x \in Q_R^{\mathrm{reg}}$. To ease notation we identify $\mathbb{Z}^{\mathcal{V}_R}$ and $\mathbb{Z}^{\mathcal{V}_{\widehat{R}}}$ via σ''.

Proof of Theorem 3.34. Let $\mathfrak{I} \in \mathcal{R}_R$, and let $\mathfrak{m} \in \operatorname{Max}(R)$. Then Proposition A.38 yields

$$\widehat{\mathfrak{I}_{\mathfrak{m}}} = \mathfrak{I}\,\widehat{\otimes_R R_{\mathfrak{m}}}.$$

Since $\mathfrak{I} \otimes_R R_{\mathfrak{m}} \in \mathcal{R}_{R_{\mathfrak{m}}}$ by Lemma 2.16.(2) and Proposition A.39, Theorem A.55 implies

$$\widehat{\mathfrak{I}_{\mathfrak{m}}} = \mathfrak{I}\,\widehat{\otimes_R R_{\mathfrak{m}}} = \mathfrak{I} \otimes_R R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} \widehat{R_{\mathfrak{m}}} = \mathfrak{I} \otimes_R \widehat{R_{\mathfrak{m}}} = \mathfrak{I} \otimes_R \widehat{R}_{\widehat{\mathfrak{m}}},$$

where the last equality follows from Theorems A.55 and A.59.(2). So with Proposition A.38 and Theorem A.55 we obtain

$$\widehat{\mathfrak{I}_{\mathfrak{m}}} = \mathfrak{I} \otimes_R \widehat{R}_{\widehat{\mathfrak{m}}} = \mathfrak{I} \otimes_R \widehat{R} \otimes_{\widehat{R}} \widehat{R}_{\widehat{\mathfrak{m}}} = \widehat{\mathfrak{I}} \otimes_{\widehat{R}} \widehat{R}_{\widehat{\mathfrak{m}}} = \widehat{\mathfrak{I}}_{\widehat{\mathfrak{m}}}.$$

Therefore, using Theorem 3.28 we may assume that R is local.

So let R be a one-dimensional local Cohen–Macaulay ring, and let $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$. Then Lemmas 3.9.(1) and 3.35 and Diagram (3.15) yield

$$\begin{aligned}\mathfrak{Q}^\alpha \widehat{R} &= \eta(\mathfrak{Q}^\alpha) \\ &= \left(\phi_{\widehat{R}} \circ \psi_{\widehat{R}}\right)^{-1} \circ \sigma'' \circ (\phi_R \circ \psi_R)(\mathfrak{Q}^\alpha) \\ &= \left(\phi_{\widehat{R}} \circ \psi_{\widehat{R}}\right)^{-1} \circ \sigma''(\alpha) \\ &= Q^\alpha_{\widehat{R}}.\end{aligned}$$

So for any $\mathfrak{I} \in \mathcal{R}_R$ we obtain with Lemma 2.16.(2) and (4), Proposition 3.8 and Theorems A.55 and A.60

$$\widehat{\mathfrak{I}^\alpha} = \mathfrak{I}^\alpha \widehat{R} = (\mathfrak{I} \cap \mathfrak{Q}^\alpha)\widehat{R} = \mathfrak{I}\widehat{R} \cap \mathfrak{Q}^\alpha \widehat{R} = \widehat{\mathfrak{I}} \cap Q^\alpha_{\widehat{R}} = \widehat{\mathfrak{I}}^\alpha. \tag{3.16}$$

Now we have by Corollary 3.32 $\alpha \in \Gamma_{\mathfrak{I}}$ if and only if $\mathfrak{E}^\alpha / \mathfrak{E}^{\alpha + \mathbf{e}_V} \neq 0$ for all $V \in \mathcal{V}_R$. The claim follows since by Equation (3.16) and Theorem A.60 the latter condition commutes with completion. □

Remark 3.39. Let R be an analytically reduced one-dimensional local Cohen–Macaulay ring. Then $\widehat{R}$ is a reduced one-dimensional local Cohen–Macaulay ring by Lemma 3.35. Since $\overline{\widehat{R}} = \widehat{\overline{R}}$ by Corollary 3.38 and Theorem C.14, Corollary A.62 yields

$$\overline{\widehat{R}} = \widehat{\overline{R}} = \prod_{\mathfrak{n} \in \operatorname{Max}\left(\widehat{\overline{R}}\right)} \widehat{\overline{R}}_{\mathfrak{n}} = \prod_{\mathfrak{m} \in \operatorname{Max}\left(\overline{\widehat{R}}\right)} \overline{\widehat{R}}_{\mathfrak{m}}. \tag{3.17}$$

Since $\widehat{R}$ is equidimensional, also $\overline{\widehat{R}}$ is equidimensional by Proposition B.3 and Lemma 5.30. This implies $\operatorname{height} \mathfrak{m} = \dim \overline{\widehat{R}} = \dim \widehat{R} = 1$ by Proposition B.27 and Theorem B.14 for every $\mathfrak{m} \in \operatorname{Max}\left(\overline{\widehat{R}}\right)$. Thus, $\overline{\widehat{R}}_{\mathfrak{m}} = \overline{\overline{\widehat{R}}_{\mathfrak{m}}}$ is by Proposition B.5 and Corollary B.8 a one-dimensional integrally closed local ring. Moreover, $\overline{\widehat{R}}_{\mathfrak{m}}$ is reduced as a subring of the reduced ring $Q_{\widehat{R}}$ (see Lemma A.27). Thus, $\overline{\widehat{R}}_{\mathfrak{m}}$ is a one-dimensional integrally closed local Cohen–Macaulay ring, and hence a domain by [23, Chapter I, Proposition 3.29 and Chapter II, Proposition 2.5]. Therefore, there is by Equation (3.17) and Lemma A.6.(2) a bijection

$$\operatorname{Max}\left(\overline{\widehat{R}}\right) \to \operatorname{Min}\left(\overline{\widehat{R}}\right) \tag{3.18}$$

mapping $\mathfrak{m} \in \operatorname{Max}\left(\overline{\widehat{R}}\right)$ to the unique $\mathfrak{p} \in \operatorname{Min}\left(\overline{\widehat{R}}\right)$ contained in $\mathfrak{m}$.

Hence, Theorem 3.37, Theorem 3.2.(6), Equation (3.18), and Theorem A.72 yield a sequence of bijections

$$\mathcal{V}_R \to \mathcal{V}_{\widehat{R}} \to \operatorname{Max}\left(\overline{\widehat{R}}\right) \to \operatorname{Min}\left(\overline{\widehat{R}}\right) \to \operatorname{Min}\left(\widehat{R}\right) \tag{3.19}$$

mapping

$$V \mapsto \mathfrak{q}_{\widehat{V}}$$

(see Propositions 3.13 and D.13).

Suppose that $R = \widehat{R}$. Then

$$V/I_V = \overline{R/\mathfrak{q}_V}$$

by Equation (3.19), Theorem 3.2.(4), and Proposition 3.13.(2). Moreover, Corollary D.32 and Proposition D.16 yield

$$\nu_V = \nu_{\overline{R/\mathfrak{q}_V}} \circ \pi_V,$$

where $\pi_V : Q_R \to Q_{R/\mathfrak{q}_V} = Q_R/I_V$ is the canonical surjection (see Theorem 3.2.(2), Proposition 3.13.(1), and Theorem A.74.(1)). Therefore, the product

$$\left(\nu_{\overline{R/\mathfrak{q}_V}}\right)_{V\in\mathcal{V}_R} : Q_R = \prod_{V\in\mathcal{V}_R} Q_{R/\mathfrak{q}_V} \to \mathbb{Z}^{\mathcal{V}_R}$$

yields by Equation 3.19 and Theorem A.74.(2) the same notion of a semigroup of values as defined in Definition 3.14. This alternative approach is often used in the literature, see for example [31, 12, 7, 8].

3.3. Algebroid Curves

Definition 3.40. Let k be a field. An *algebroid curve* over k is a complete equidimensional reduced Noetherian semilocal k-algebra R of dimension one such that $|k| \geq |\mathrm{Min}\,(R)|$, and all residue fields of R are isomorphic to k (under the canonical surjections $R \to R/\mathfrak{m}$ for $\mathfrak{m} \in \mathrm{Max}\,(R)$).

Proposition 3.41. *An algebroid curve is an admissible ring.*

Proof. Let R be an algebroid curve. Then by definition and Proposition C.13 R is a one-dimensional equidimensional semilocal analytically reduced Cohen–Macaulay ring. Moreover, since there is a bijection between $\mathrm{Min}\,(R)$ and $\mathcal{V}_R$ (see Remark 3.39, Equation (3.19)), R also has large residue fields. Finally, R is residually rational by Lemma B.21 since its residue fields are isomorphic to k by assumption, and hence algebraically closed. □

Proposition 3.42. *Let k be a field, and let R be an algebroid curve over k. For any $\mathfrak{m} \in \mathrm{Max}\,(R)$ there is an $n_\mathfrak{m} \in \mathbb{N}$ and an ideal $\mathfrak{i}_\mathfrak{m}$ of $k\left[\left[x_1^{(\mathfrak{m})}, \ldots, x_{n_\mathfrak{m}}^{(\mathfrak{m})}\right]\right]$ such that*

$$R \cong \prod_{\mathfrak{m}\in\max(R)} k\left[\left[x_1^{(\mathfrak{m})}, \ldots, x_{n_\mathfrak{m}}^{(\mathfrak{m})}\right]\right]/\mathfrak{i}_\mathfrak{m}$$

Proof. Since R is a reduced complete semilocal ring, Theorem A.61 yields

$$R \cong \prod_{\mathfrak{m}\in\mathrm{Max}\,(R)} R_\mathfrak{m},$$

and $R_\mathfrak{m}$ is a reduced complete local ring for any $\mathfrak{m} \in \mathrm{Max}\,(R)$. For any maximal ideal $\mathfrak{m} \in \mathrm{Max}\,(R)$, let $\left(x_1^{(\mathfrak{m})}, \ldots, x_{n_\mathfrak{m}}^{(\mathfrak{m})}\right)$ be a family of generators of $\mathfrak{m}$. Since $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} = R/\mathfrak{m} = k$

and $k \subset R_{\mathfrak{m}}$ for any $\mathfrak{m} \in \operatorname{Max}(R)$, Theorem A.67 yields ideals $\mathfrak{i}_{\mathfrak{m}} \subset k\left[\left[x_1^{(\mathfrak{m})}, \ldots, x_{n_{\mathfrak{m}}}^{(\mathfrak{m})}\right]\right]$ such that

$$R_{\mathfrak{m}} \cong k\left[\left[x_1^{(\mathfrak{m})}, \ldots, x_{n_{\mathfrak{m}}}^{(\mathfrak{m})}\right]\right]/\mathfrak{i}_{\mathfrak{m}}$$

for any $\mathfrak{m} \in \operatorname{Max}(R)$. $\square$

Lemma 3.43. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, let V be a discrete valuation ring of Q, and assume that V is a complete domain.*

(1) The discrete valuation ring V is local with maximal ideal $\mathfrak{m}_V$.

(2) If $t \in Q$ is a uniformizing parameter of V, then there is an isomorphism

$$\begin{aligned} \phi\colon V &\to k[[T]], \\ t &\mapsto T, \end{aligned}$$

where $k = V/\mathfrak{m}_V$.

(3) The valuation of V is

$$\nu_V = \operatorname{ord}_T \circ \phi.$$

In particular, there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dd, "\cong"', "\phi"] & Q \arrow[dd, "\cong"'] \arrow[dr, two heads, "\nu"] & \\ & & \mathbb{Z}_\infty. \\ k[[T]] \arrow[r, hook] & k[[T]]\left[T^{-1}\right] \arrow[ur, two heads, "\operatorname{ord}_t"'] & \end{tikzcd}$$

Proof. (1) See Remark D.6.

(2) By Proposition 2.23.(2) the maximal ideal $\mathfrak{m}_V$ is generated by a uniformizing parameter t. Since V is complete by assumption and local by (1), Theorem A.67 yields

$$V \cong k[[T]]/\mathfrak{i}$$

for some ideal $\mathfrak{i} \in k[[T]]$, where $k = V/\mathfrak{m}_V$. As $\dim V = 1$ and V is a domain, we obtain $\mathfrak{i} = \langle 0 \rangle$, and hence

$$V \cong k[[T]].$$

(3) By Proposition 2.24.(1) we have

$$Q = V\left[T^{-1}\right] \cong k[[T]]\left[T^{-1}\right],$$

see (2). So let $f \in k[[T]]\left[T^{-1}\right]$. Then

$$f = T^{\operatorname{ord}_T f} \frac{f}{T^{\operatorname{ord}_T f}},$$

and $\frac{f}{T^{\mathrm{ord}_T f}} \in (k[[T]])^*$ since

$$\mathrm{ord}_T \frac{f}{T^{\mathrm{ord}_T f}} = 0.$$

Hence,

$$\nu_V = \mathrm{ord}_T \circ \phi$$

by Corollary 2.25.(2) since $\phi(t) = T$. $\square$

Theorem 3.44. *Let k be a field, and let R be an algebroid curve over k. The normalization $\overline{R}$ of R is a finite product of discrete valuation rings*

$$\overline{R} \cong \prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} \overline{R/\mathfrak{p}}.$$

For any $\mathfrak{p} \in \mathrm{Min}\,(R)$ there is an isomorphism

$$\begin{aligned} \phi_{\mathfrak{p}} \colon \overline{R/\mathfrak{p}} &\to k[[T_{\mathfrak{p}}]], \\ t_{\mathfrak{p}} &\mapsto T_{\mathfrak{p}}, \end{aligned}$$

where $t_{\mathfrak{p}}$ is a uniformizing parameter for $\overline{R/\mathfrak{p}}$. The valuation of $\overline{R/\mathfrak{p}}$ is

$$\nu_{\overline{R/\mathfrak{p}}} = \mathrm{ord}_{T_{\mathfrak{p}}} \circ \phi_{\mathfrak{p}}.$$

In particular, there is a commutative diagram

$$\begin{tikzcd}
\overline{R} \arrow[r, hook] \arrow[d, "\cong"', "(\phi_{\mathfrak{p}})_{\mathfrak{p}\in\mathrm{Min}\,(R)}"] & Q_R \arrow[dr, "\nu"] \arrow[d, "\cong"'] & \\
\prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} k[[T_{\mathfrak{p}}]] \arrow[r, hook] & \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} k[[T_{\mathfrak{p}}]]\left[T_{\mathfrak{p}}^{-1}\right] \arrow[r, "\left(\mathrm{ord}_{T_{\mathfrak{p}}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)}"'] & (\mathbb{Z}_\infty)^{\mathrm{Min}\,(R)}.
\end{tikzcd}$$

Proof. By Theorem B.42 we have

$$R = \prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} \overline{R/\mathfrak{p}}.$$

So let $\mathfrak{p} \in \mathrm{Min}\,(R)$. By Remark 3.39, $\overline{R/\mathfrak{p}}$ is a discrete valuation ring. Moreover, $\overline{R/\mathfrak{p}}$ is a domain by Corollary A.73. Since $R/\mathfrak{p}$ is a one-dimensional Cohen–Macaulay ring by Proposition C.13, $\overline{R/\mathfrak{p}}$ is complete by Theorems A.52, A.55, and C.14. Then Lemma 3.43 yields the statement since R is residually rational, and hence by Theorem B.42 and Lemma A.6.(2)

$$k = R/\mathfrak{m}_R = \overline{R}/\mathfrak{m}_{\overline{R}} = \left(\overline{R/\mathfrak{p}}\right)/\mathfrak{m}_{\overline{R/\mathfrak{p}}}$$

is a residue field for $\overline{R/\mathfrak{p}}$. $\square$

Let k be a field, and let R be an algebroid curve over k. By Proposition 3.42 we may assume that $R = k[[X_1, \ldots, X_n]]/\mathfrak{i}$ for some ideal $\mathfrak{i}$ of $k[[X_1, \ldots, X_n]]$. By Theorem 3.44 there is an isomorphism

$$\phi\colon \overline{R} \to \prod_{\mathfrak{p} \in \operatorname{Min}(R)} k[[t_\mathfrak{p}]],$$

where $k[[t_\mathfrak{p}]]$ is a discrete valuation ring with uniformizing parameter $t_\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Min}(R)$. This yields a *parametrization*

$$X_i \mapsto x_i(t) = \phi(X_i),$$

see [32, Chapter I, Section 3.1]. If $n = 2$, the map ϕ can be computed using using the Newton-Puiseux algorithm, see [32, Chapter I, Algorithm 3.6]. In the following, we may identify R with its image

$$\phi(R) \subset \prod_{\mathfrak{p} \in \operatorname{Min}(R)} k[[t_\mathfrak{p}]],$$

where

$$\phi(R) = k[[X_1(t), \ldots, X_n(t)]].$$

The total ring of fractions of R is by Theorem A.74.(2) and Proposition 2.24.(1)

$$Q_R = \prod_{\mathfrak{p} \in \operatorname{Min}(R)} k[[t_\mathfrak{p}]]\left[t_\mathfrak{p}^{-1}\right].$$

If $\mathfrak{I} \in \mathcal{R}_R$, then by Propositions 4.16.(2) and 4.56

$$\mathfrak{C}_\mathfrak{I} = t^{\gamma_{\Gamma_\mathfrak{I}}} \prod_{\mathfrak{p} \in \operatorname{Min}(R)} k[[t_\mathfrak{p}]],$$

where we use the multi-index notation, i.e. if $x \in \prod_{\mathfrak{p} \in \operatorname{Min}(R)} k[[t_\mathfrak{p}]]$ and $\alpha \in \mathbb{Z}^{\operatorname{Min}(R)}$, then

$$x^\alpha = \left(x_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p} \in \operatorname{Min}(R)}.$$

Also recall that by Corollary 2.25.(2) for any $x \in Q^{\mathrm{reg}}$ there is a unique element $a = (a_\mathfrak{p})_{\mathfrak{p} \in \operatorname{Min}(R)}$ such that

$$x = at^{\nu(x)}.$$

The multivaluation of R (see Section 3.1) is by Theorem 3.44

$$\nu = (\nu_\mathfrak{p})_{\mathfrak{p} \in \operatorname{Min}(R)} = (\operatorname{ord}_{t_\mathfrak{p}})_{\mathfrak{p} \in \operatorname{Min}(R)}.$$

So for any $\mathfrak{I} \in \mathcal{R}_R$, its value semigroup ideal is

$$\Gamma_\mathfrak{I} = \left\{ \left(\operatorname{ord}_{t_\mathfrak{p}} x_\mathfrak{p}(t_\mathfrak{p})\right)_{\mathfrak{p} \in \operatorname{Min}(R)} \,\middle|\, x = (x_\mathfrak{p})_{\mathfrak{p} \in \operatorname{Min}(R)} \in \mathfrak{I}^{\mathrm{reg}} \subset \prod_{\mathfrak{p} \in \operatorname{Min}(R)} k[[t_\mathfrak{p}]]\left[t_\mathfrak{p}^{-1}\right] \right\}$$
$$\subset \mathbb{Z}^{\operatorname{Min}(R)}.$$

3.4. Integral Extensions of Admissible Rings

Theorem 3.45. *Let R be a reduced ring, and let A be an integral extension of R in Q_R.*

(1) If R is admissible, then A is admissible. Moreover, $\mathcal{V}_R = \mathcal{V}_A$.

(2) If R is an algebroid curve over a field k, then A is an algebroid curve over k.

For the proof of Theorem 3.45 we need the following Lemmas.

Lemma 3.46. *Let R be an admissible ring, respectively an algebroid curve, and let A be an integral extension of R in Q_R. Then A is finite over R.*

Proof. Since A is an integral extension of R in Q_R, we have $A \subset \overline{R}$. Since R is by definition, respectively by Proposition 3.41, a one-dimensional analytically reduced semilocal Cohen–Macaulay ring, there is by Lemma 3.21 and Remark B.23.(1) an $x \in R^{\mathrm{reg}}$ such that

$$xA \subset x\overline{R} = \mathfrak{C}_R \subset R.$$

Since R is A Noetherian, and since $x \in R^{\mathrm{reg}}$, this implies that A is a finite R-module. □

Lemma 3.47. *Let R be an admissible ring, respectively an algebroid curve, and let A be an integral extension of R in Q_R. Then A is Noetherian.*

Proof. This follows from Lemma 3.46 and Theorem A.1. □

Lemma 3.48. *Let R be an admissible ring, and let A be an integral extension of R in Q_R. Then A is a Cohen–Macaulay ring.*

Proof. Since A is an integral extension of R, Theorem B.14 yields $\dim A = \dim R = 1$. Moreover, since R is reduced by definition, also Q_R is reduced by Lemma A.27. Therefore, A is reduced since $A \subset Q_R$. Since A is Noetherian by Lemma 3.47, it is a Cohen–Macaulay ring by Proposition C.13. □

Lemma 3.49. *Let R be an admissible ring, respectively an algebroid curve, and let A be an integral extension of R in Q_R. Then A is a semilocal ring.*

Proof. The inclusions $R \subset A \subset \overline{R}$ imply $Q_A = Q_R$ by Lemma A.34, and $\overline{A} = \overline{R}$ by Proposition B.5. By Theorem 3.2.(1) and (6), respectively by Theorem 3.2.(1) and (6) and Proposition 3.41, the set $\operatorname{Max}\left(\overline{R}\right) = \operatorname{Max}\left(\overline{A}\right)$ is finite. Since

$$\operatorname{Max}(A) = \left\{\mathfrak{m} \cap A \;\middle|\; \mathfrak{m} \in \operatorname{Max}\left(\overline{A}\right)\right\}$$

by Propositions B.3 and B.15 and Theorem B.12, A is semilocal. □

Proof of Theorem 3.45. (1) By Lemmas 3.48 and 3.49 and Theorem B.14 A is a one-dimensional semilocal Cohen–Macaulay ring. Since A is an integral extension of R in Q_R, we have inclusions $R \subset A \subset \overline{R}$. This implies $Q_A = Q_R$ by Lemma A.34, and

$\overline{A} = \overline{R}$ by Proposition B.5. Since $\overline{A} = \overline{R} \subset V$ for all $V \in \mathcal{V}_R$ and $\overline{R} = \overline{A} \subset W$ for all $W \in \mathcal{V}_A$ by Theorem 3.2.(4), and since $Q_R = Q_A$, we obtain $\mathcal{V}_R = \mathcal{V}_A$.

Since R is an analytically reduced one-dimensional semilocal Cohen–Macaulay ring, $\overline{R}$ is by Theorem C.14 a finitely generated R-module. Then the inclusions $R \subset A \subset \overline{R} = \overline{A}$ imply that $\overline{A}$ is a finite A-module. Thus, A is analytically reduced by Theorem C.14.

Let $\mathfrak{m} \in \operatorname{Max}(A)$, and let $\mathfrak{n} \in \operatorname{Max}\left(\overline{A}\right)$ with $\mathfrak{n} \cap A = \mathfrak{m}$. Then $\mathfrak{m} \cap R \in \operatorname{Max}(R)$ by Propositions B.3 and B.15, and Proposition B.6.(1) yields field extensions

$$R/(\mathfrak{m} \cap R) \subset A/\mathfrak{m} \subset \overline{A}/\mathfrak{n}.$$

Since $\overline{A} = \overline{R}$, and since R is residually rational, this implies

$$R/(\mathfrak{m} \cap R) = A/\mathfrak{m} = \overline{A}/\mathfrak{n}. \tag{3.20}$$

Hence, A is residually rational. Moreover, Equation (3.20) implies

$$|A/\mathfrak{m}| = |R/(\mathfrak{m} \cap R)| \geq |\mathcal{V}_R| = |\mathcal{V}_A|.$$

Therefore, A has large residue fields. Since A is equidimensional by Lemma B.31, it is admissible.

(2) Since A is an integral extension of R, Theorem B.14 yields $\dim A = \dim R = 1$. Moreover, since R is reduced by definition, also Q_R is reduced by Lemma A.27. Therefore, A is reduced since $A \subset Q_R$. Since $R \subset A$, and since R is a k-algebra, also A is a k-algebra. Moreover, A is Noetherian by Lemma 3.47.

Since R is by definition a complete semilocal ring, and since A is a finite R-module by Lemma 3.46, A is complete as R-module by Theorem A.55. As A is also semilocal by Lemma 3.49, the topology of A as R-module coincides by Theorem A.52 with the topology of A as a semilocal ring. Thus, A is a complete ring.

Let $\mathfrak{m} \in \operatorname{Max}(A)$. Then $\mathfrak{m} \cap R \in \operatorname{Max}(R)$ by Proposition B.15, and Lemma B.21 yields

$$A/\mathfrak{m} \cong R/(\mathfrak{m} \cap R) \cong k.$$

since R is an algebroid curve over k. Moreover, we have $|k| \geq |\operatorname{Min}(R)| = |\operatorname{Min}(A)|$ by Theorem A.72.

Thus, A is a complete reduced Noetherian k-algebra of dimension one such that $|k| \geq |\operatorname{Min}(A)|$, and all residue fields of A are isomorphic to k. Since A is equidimensional by Lemma B.31, it is an algebroid curve over k. □

4. Good Semigroups

Motivated by the properties of the semigroup of values of an admissible ring we introduce a combinatorial counterpart of curve singularities: *good semigroups*. Examples of good semigroups include the semigroups of values of admissible rings and numerical semigroups. In analogy to Definition 3.14 and Corollary 3.30 we define a good semigroup S as a submonoid of $\mathbb{N}^I$ (for a finite set I) satisfying properties (E0) (with $\overline{S} = \mathbb{N}^I$), (E1), and (E2) (see Definition 4.5). This Chapter is dedicated to the fundamental properties of good semigroups.

Barucci, D'Anna, and Fröberg showed that not any good semigroup is the semigroup of values of an admissible ring (see [10, Example 3.3] and Example 3.26). On good semigroups we want to introduce combinatorial counterparts of algebraic concepts on admissible rings. In Section 4.1 we define (good) semigroup ideals of good semigroups in analogy to fractional ideals of rings. Moreover, there is as in Theorem 3.28 a combinatorial version of localization for good semigroups and semigroup ideals (see Theorem 4.9) which is compatible under taking values (see Remark 4.10).

The "semigroup operation" corresponding to the quotient of fractional ideals is the difference of semigroup ideals (see Section 4.3). For a semigroup ideal E satisfying property (E1) of a good semigroup $S \in \mathbb{N}^I$ the difference $E - \mathbb{N}^I$ defines the *conductor ideal* of E (see Definition 4.26). We study properties of the conductor ideal in Section 4.4. In particular, if $\mathfrak{I}$ is a regular fractional ideal of an admissible ring R, then the value semigroup ideal of the conductor of $\mathfrak{I}$ is equal to the conductor of the value semigroup ideal $\Gamma_{\mathfrak{I}}$ of $\mathfrak{I}$ (see Proposition 4.56).

An important tool to relate good semigroups and good semigroup ideals to admissible rings and fractional ideals is the *distance* (see Definition 4.46). The properties of this function are examined in Section 4.5. Most importantly, it allows for computing the length of a quotient of two fractional ideals from their value semigroup ideals (see Proposition 4.51). In particular, we may check equality of fractional ideals using their value semigroup ideals (see Corollary 4.52).

Projections of the semigroup onto its components correspond to passing to the branches of an admissible ring (see Proposition 4.67). Moreover, from the semigroup of values of an admissible ring R we can directly deduce the value semigroup ideal of a minimal prime ideal $\mathfrak{p}$ of R on branches of R not corresponding to $\mathfrak{p}$ (see Proposition 4.69). For general good semigroups this construction allows for computing the conductor of the semigroup from information on its components (see Proposition 4.64). The results on branches of Section 4.6 will be important in the study of quasihomogeneous curves and semigroups in Chapters 6 and 7.

Important examples of good semigroups are *numerical semigroups*, i.e. submonoids of $\mathbb{N}$ with a finite complement (see Section 4.7). Among good semigroups the numerical semigroups have some particular properties. For example all semigroup ideals of numerical semigroup are good (see Remark 4.6.(2)), and a good semigroup is finitely generated if and only if it is a numerical semigroup (see [3, Bemerkung 1.2.14.(3)]). Given a numerical semigroup S and a semigroup ideal E of S we introduce a *quotient semigroup* S/E (see Definition 4.74). Then for any ring R the quotient of the semigroup ring $R[[S]]$ by the ideal $R[[E]]$ is given as the semigroup ring of the S/E modulo a certain relation (see Proposition 4.79). Finally, in Section 4.8 we study properties of semigroup rings over $\mathbb{C}$.

Before studying good semigroups we first discuss some general facts about monoids. Let S be a cancellative commutative monoid. Then S embeds into its (free abelian) group of differences D_S. If S is partially ordered, then D_S carries a natural induced partial order.

Let I be a finite set. On the group $\mathbb{Z}^I$ we consider the natural partial order given by $\alpha \leq \beta$ for $\alpha, \beta \in \mathbb{Z}^I$ if and only if $\alpha_i \leq \beta_i$ for all $i \in I$. We write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

Lemma 4.1. *A finite cancellative monoid S is a group.*

Proof. Let $0 \neq \alpha \in S$. Since S is finite, there are $m, n \in \mathbb{N}$ with $m < n$ such that

$$m\alpha = n\alpha.$$

As S is cancellative, this implies

$$0 = (n - m)\,\alpha.$$

Hence,

$$\alpha + (n - m - 1)\,\alpha = 0,$$

and therefore

$$-\alpha = (n - m - 1)\,\alpha. \qquad \square$$

Lemma 4.2. *Let S be a partially ordered monoid. If $\alpha \in S$ is a unit, then $\alpha > 0$ implies $-\alpha < 0$.*

Proof. Let $\alpha \in S^*$ such that $\alpha > 0$. Since S is a partially ordered monoid, we have

$$0 = \alpha - \alpha > 0 - \alpha = -\alpha. \qquad \square$$

Lemma 4.3. *Let S be a partially ordered group. If any $\alpha \in S$ is comparable to 0, then 0 is the only element of finite order in S.*

Proof. Assume there is $0 \neq \alpha \in S$ of finite order. Then there is $n \in \mathbb{N}$ with $n > 0$ such that $n\alpha = 0$. In particular, α is a unit. So by Lemma 4.2 we may assume without loss of generality that $\alpha > 0$. Since S is a partially ordered group, this yields the contradiction

$$0 = n\alpha = (n-1)\,\alpha + \alpha > (n-1)\,\alpha > (n-2)\,\alpha > \ldots > \alpha > 0. \qquad \square$$

Lemma 4.4. *Let S be a partially ordered cancellative commutative monoid, and suppose that D_S is generated by a finite set I such that there is an isomorphism*

$$\begin{aligned} \sigma\colon D_S &\to \mathbb{Z}^I \\ I \ni i &\mapsto \mathbf{e}_i. \end{aligned}$$

Assume that σ preserves the natural partial orders. Then I contains only positive elements. Moreover, if J is a finite set generating D_S such that there is an isomorphism

$$\begin{aligned}\tau\colon D_S &\to \mathbb{Z}^J\\ J \ni j &\mapsto \mathbf{e}_j\end{aligned}$$

preserving the natural partial orders, then $I = J$.

Proof. If σ preserves the natural partial orders, then $\mathbf{e}_i > 0$ implies $i = \sigma^{-1}(\mathbf{e}_i) > 0$ for all $i \in I$.

Let J be a finite set generating D_S such that the isomorphism $\tau\colon D_S \to \mathbb{Z}^J$ preserves the natural partial orders. Then there is a commutative diagram

$$\begin{tikzcd} & D_S \arrow[dl, "\sigma"', "\cong"] \arrow[dr, "\tau", "\cong"'] & \\ \mathbb{Z}^I \arrow[rr, "\cong", "\phi"'] & & \mathbb{Z}^J. \end{tikzcd} \tag{4.1}$$

Since ϕ is an isomorphism, we have $|I| = |J|$. Moreover, since σ and τ preserve the natural partial orders, also $\phi = \tau \circ \sigma^{-1}$ preserves the natural partial orders. Then there is by Lemma 3.20 a bijection $\overline{\phi}\colon I \to J$ such that

$$\phi(\mathbf{e}_i) = \mathbf{e}_{\overline{\phi}(i)}$$

for all $i \in I$. Therefore, the commutativity of Diagram (4.1) yields

$$i = \sigma^{-1}(\mathbf{e}_i) = (\tau)^{-1} \circ \phi(\mathbf{e}_i) = (\tau)^{-1}\left(\mathbf{e}_{\overline{\phi}(i)}\right) = \overline{\phi}(i)$$

for all $i \in I$. Thus, $I = J$. □

4.1. Good Semigroups and Their Ideals

Having Definition 3.14, Proposition 3.22, and Corollary 3.30 in mind we consider submonoids of $\mathbb{N}^I$ (for a finite set I) satisfying properties (E0), (E1), and (E2). These objects are called *good semigroups* by Barucci, D'Anna, and Fröberg [10]. We introduce local good semigroups (corresponding to Proposition 3.17), and we decompose good semigroups and their ideals into local components.

Definition 4.5. Let S be a partially ordered cancellative commutative monoid such that $\alpha \geq \mathbf{0}$ for all $\alpha \in S$. Assume that D_S is generated by a finite set I such that there is an isomorphism $D_S \cong \mathbb{Z}^I$ which preserves the natural partial orders. Note that I is then unique and contains only positive elements by Lemma 4.4. We set

$$\overline{S} := \{\alpha \in D_S \mid \alpha \geq 0\} \cong \mathbb{N}^I.$$

(1) We call S a *good semigroup* if it satisfies properties (E0), (E1), and (E2) (see Definition 3.19).

(2) A good semigroup S is said to be a *numerical semigroup* if $|I| = 1$.

(3) If $\mathbf{0}$ is the only element of S with a zero component in D_S, then S is called *local* (cf. Definition 3.14). The *maximal (semigroup) ideal* of a local good semigroup is

$$M_S = S \setminus \{\mathbf{0}\}.$$

(4) A *semigroup ideal* of a good semigroup S is a subset $\emptyset \neq E \subset D_S$ such that $E+S \subset E$ and $\alpha + E \in S$ for some $\alpha \in S$.

(5) A *good semigroup ideal* of a good semigroup S is a semigroup ideal E of S satisfying properties (E1) and (E2).

(6) For a good semigroup S we denote by $\mathcal{G}_S$ the set of all good semigroup ideals of S.

Remark 4.6 (See [25], Remark 4.1.2).

(1) If S is a good semigroup, any semigroup ideal E of S satisfies property (E0) since S does and $E + S \subset E$.

(2) Any numerical semigroup S is a local good semigroup. Moreover, $E \in \mathcal{G}_S$ for any semigroup ideal E of S.

(3) If S and S' are good semigroups with $S \subset S' \subset \overline{S}$, then $D_{S'} = D_S$, and hence $\overline{S'} = \overline{S}$. It follows that $\mathcal{G}_{S'} \subset \mathcal{G}_S$, and, in particular, $S' \in \mathcal{G}_S$.

(4) Let R be an admissible ring. Then by Definition 4.5 we have

$$\nu((Q_R)^{\mathrm{reg}}) = D_{\Gamma_R}.$$

(5) Let R be an admissible ring. Then by Lemma 3.21 there is an $\alpha \in \Gamma_R$ such that $\alpha + \mathbb{N}^{\mathcal{V}_R} \subset \Gamma_R$. It follows that $D_{\Gamma_R} = \mathbb{Z}^{\mathcal{V}_R}$. Moreover, Γ_R is a good semigroup with $\overline{\Gamma_R} = \Gamma_{\overline{R}} = \mathbb{N}^{\mathcal{V}}$ (see Lemma 3.4.(2)), and $\Gamma_{\mathfrak{J}} \in \mathcal{G}_{\Gamma_R}$ for every $\mathfrak{J} \in \mathcal{R}_R$ by Proposition 3.22.(1) and Corollary 3.30.

Lemma 4.7. *Let S be a local good semigroup. Then $M_S \in \mathcal{G}_S$.*

Proof. Since $M_S \subset S$, we have $S + M_S \subset S$. Moreover, $S \geq \mathbf{0}$ and $M_S > \mathbf{0}$ imply $M_S + S > \mathbf{0}$. Hence, M_S is a semigroup ideal of S.

Let $\alpha, \beta \in M_S \subset S$. Then $\inf\{\alpha, \beta\} \in S$ since S satisfies property (E1). Assume $\inf\{\alpha, \beta\} = \mathbf{0}$. Then there is an $i \in I$ such that without loss of generality $\alpha_i = 0$. Since S is local, this implies $\alpha = \mathbf{0}$, and hence $\alpha \notin M_S$. Therefore, M_S satisfies property (E2).

Assume there is $i \in I$ such that $\alpha_i = \beta_i$. Since S satisfies property (E2), there is an $\epsilon \in S$ such that

$$\begin{aligned} &\epsilon_i > \alpha_i = \beta_i, \\ &\epsilon_j \geq \inf\{\alpha_j, \beta_j\} \text{ for all } j \in I, \\ &\epsilon_k = \inf\{\alpha_k, \beta_k\} \text{ for all } k \in I \text{ with } \alpha_k \neq \beta_k. \end{aligned}$$

In particular, $\epsilon > \inf\{\alpha, \beta\} > \mathbf{0}$, where the second inequality follows since M_S satisfies property (E1). This implies $\epsilon \in M_S$, and hence M_S satisfies property (E2). $\square$

Remark 4.8. Let R be a local admissible ring with maximal ideal $\mathfrak{m}_R$. Then Γ_R is local, and

$$\Gamma_{\mathfrak{m}_R} = M_{\Gamma_R},$$

see Proposition 3.17 and Remark 4.6.(5).

Theorem 4.9 (See [25], Theorem 4.1.6). *Any good semigroup S decomposes uniquely and compatible with the partial orders as a finite direct product*

$$S = \prod_{m \in M} S_m$$

of good local semigroups S_m. Any semigroup ideal E of S satisfying (E1) decomposes as

$$E = \prod_{m \in M} E_m.$$

If $E \in \mathcal{G}_S$, then $E_m \in \mathcal{G}_{S_m}$ for all $m \in M$.

Proof. See [10, Theorem 2.5, Remark 2.6, and Proposition 2.12]. □

Remark 4.10 (See [25], Remark 4.1.7). The decompositions in Theorem 3.28 are special cases of those in Theorem 4.9 (see Corollary 3.30).

In the following, let $S \subset D_S$ be a good semigroup. By Definition 4.5 we may identify $D_S = \mathbb{Z}^I$ for some finite set I, and this identification is by Lemma 3.20 unique. So consider S now as a submonoid of $\mathbb{Z}^I$.

Notation. For any $J \subset I$ we write

$$\begin{aligned} \mathrm{pr}_J \colon \mathbb{Z}^I &\to \mathbb{Z}^J \cong \mathbb{Z}^{|J|} \\ \alpha = (\alpha_i)_{i \in I} &\mapsto \alpha_J = (\alpha_j)_{j \in J}. \end{aligned}$$

For a relative ideal E of S we denote

$$E_J = \mathrm{pr}_J(E).$$

If $J = \{j\}$ for some $j \in I$, we write $\mathrm{pr}_j = \mathrm{pr}_{\{j\}}$.

Lemma 4.11 (See [25], Remark 4.1.5). *Let M be a finite set, and let $(S_m)_{m \in M}$ be a family of good semigroups, and for any $m \in M$ let E_m be a semigroup ideal of S_m.*

(1) Then

$$S = \prod_{m \in M} S_m$$

is a good semigroup with

$$D_S = \prod_{m \in M} D_{S_m}$$

and

$$\overline{S} = \prod_{m \in M} \overline{S_m}.$$

(2) The set

$$E = \prod_{m \in M} E_m$$

is a semigroup ideal of S.

(3) If E_m satisfies property (E1) for every $m \in M$, then so does E.

(4) If $E_m \in \mathcal{G}_{S_m}$ for every $m \in M$, then $E \in \mathcal{G}_S$.

Proof. (1) For any $m \in M$ let $\alpha_m \colon S_m \to D_{S_m}$ be the canonical injection, and similarly let $\alpha colon S \to D_S$ be the canonical injection. Then the universal property of the group of differences D_S yields a unique group homomorphism

$$\beta \colon D_S \to \prod_{m \in M} D_{S_m}$$

such that the diagram

$$\begin{tikzcd}
S \arrow[d, "\alpha"'] \arrow[dr, "\prod_{m \in M} \alpha_m"] & \\
D_S \arrow[r, dotted, "\beta"'] & \prod\limits_{m \in M} D_{S_m}
\end{tikzcd}$$

commutes. So for any $n \in M$ there is a commutative diagram

$$\begin{tikzcd}
S_n \arrow[r, "\delta_n"] \arrow[ddr, "\alpha_n"'] & S \arrow[d, "\alpha"'] \arrow[dr, "\prod_{m \in M} \alpha_m"] & \\
& D_S \arrow[r, "\beta"] & \prod\limits_{m \in M} D_{S_m} \\
& D_{S_n}, &
\end{tikzcd}$$

where

$$\delta_n \colon S_n \to S = \prod_{m \in M} S_M$$

is the natural injection. Therefore, the universal property of the group of differences D_{S_n} yields a unique group homomorphism

$$\epsilon \colon D_{S_n} \to D_S$$

such that the diagram

$$\begin{tikzcd}
S_n \arrow[r, "\delta_n"] \arrow[ddr, "\alpha_n"'] & S \arrow[d, "\alpha"'] \arrow[dr, "\prod_{m \in M} \alpha_m"] & \\
& D_S \arrow[r, "\beta"] & \prod\limits_{m \in M} D_{S_m} \\
& D_{S_n} \arrow[u, dotted, "\epsilon"'] &
\end{tikzcd}$$

commutes. Since the projection

$$\zeta_n \colon \prod_{m\in M} D_{S_m} \to D_{S_n}$$

fits into the commutative diagram

$$\begin{tikzcd}
S_n \arrow[r, "\delta_n"] \arrow[ddr, "\alpha_n"'] & S \arrow[d, "\alpha"] \arrow[dr, "\prod_{m\in M}\alpha_m"] & \\
& D_S \arrow[r, "\beta"] & \prod\limits_{m\in M} D_{S_m} \arrow[dl, "\zeta_n"] \\
& D_{S_n} \arrow[u, "\epsilon"] &
\end{tikzcd}$$

for all $n \in M$, the universal property of the product $\prod_{m\in M} D_{S_m}$ yields a unique group homomorphism

$$\eta \colon \prod_{m\in M} D_{S_m} \to D_S$$

such that the diagram

$$\begin{tikzcd}
S_n \arrow[r, "\delta_n"] \arrow[ddr, "\alpha_n"'] & S \arrow[d, "\alpha"] \arrow[dr, "\prod_{m\in M}\alpha_m"] & \\
& D_S \arrow[r, shift left, "\beta"] & \prod\limits_{m\in M} D_{S_m} \arrow[dl, "\zeta_n"] \arrow[l, shift left, dotted, "\eta"] \\
& D_{S_n} \arrow[u, "\epsilon"] &
\end{tikzcd}$$

commutes. Therefore, $D_S = \prod_{m\in M} D_{S_m}$, and this is compatible with the partial order induced by that on S. Hence, it follows that $\overline{S} = \prod_{m\in M} \overline{S_m}$, and we have $\alpha \geq \mathbf{0}$ for all $\alpha \in S$.

Moreover, if I_m is for any $m \in M$ a finite set of generators of D_{S_m} such that $D_{S_m} \cong \mathbb{Z}^{I_m}$, then $I = \{\delta_m(i) \mid i \in I_m,\ m \in M\}$ is a finite set of generators of D_S such that $D_S \cong \mathbb{Z}^I$.

Since S_m satisfies (E0) for every $m \in M$, there is an $\alpha_m \in S_m$ such that $\alpha_m + \overline{S_m} \subset S_m$. Thus,

$$(\alpha_m)_{m\in M} + \overline{S} = (\alpha_m)_{m\in M} + \prod_{m\in M} \overline{S_m} = \prod_{m\in M} \left(\alpha + \overline{S_m}\right) \subset \prod_{m\in M} S_m = S,$$

and hence S satisfies (E0).

Let $\alpha, \beta \in S$. Then $\alpha_m, \beta_m \in S_m$ for all $m \in M$. Since S_m satisfies (E1) for any $m \in M$, we have $\inf(\alpha_m, \beta_m) \in S_m$ for every $m \in M$. This implies

$$\inf\{\alpha, \beta\} = (\inf\{\alpha_m, \beta_m\})_{m\in M} \in \prod_{m\in M} S_m = S,$$

and hence S satisfies (E1).

Suppose now there is an $m \in M$ and an $i \in I_m$ such that $\alpha_{\delta_n(i)} = \beta_{\delta_n(i)}$. Since S_m satisfies (E2), there is an $\epsilon_m \in S_m$ with

$$\begin{aligned}
&(\epsilon_m)_i > (\alpha_m)_i = (\beta_m)_i,\\
&(\epsilon_m)_j \geq \min\left\{(\alpha_m)_j, (\beta_m)_j\right\} \text{ for all } j \in I_m,\\
&(\epsilon_m)_k = \min\left\{(\alpha_m)_k, (\beta_m)_k\right\} \text{ for all } k \in I_m \text{ with } (\alpha_m)_k \neq (\beta_m)_k.
\end{aligned}$$

Setting $\epsilon_n = \inf \alpha_n, \beta_n \in S_m$ for every $n \in M$ and $\epsilon = (\epsilon_n)_{n \in M}$ we obtain $\epsilon \in S$ with

$$\begin{aligned}
\epsilon_{\delta_n(i)} &> \min \alpha_{\delta_n(i)} = \beta_{\delta_n(i)},\\
\epsilon_j &\geq \min\{\alpha_j, \beta_j\} \text{ for all } j \in I,\\
\epsilon_k &> \min\{\alpha_k, \beta_k\} \text{ for all } k \in I \text{ with } \alpha_k \neq \beta_k.
\end{aligned}$$

Thus, S is a good semigroup.

(2) Since E_m is a semigroup ideal of S_m for every $m \in M$, we have

$$\begin{aligned}
\left(\prod_{m \in M} E_m\right) + S &= \left(\prod_{m \in M} E_m\right) + \left(\prod_{m \in M} S_m\right)\\
&= \left(\prod_{m \in M} E_m + S_m\right) \subset \left(\prod_{m \in M} E_m\right)\\
&= E.
\end{aligned}$$

Moreover, for any $m \in M$ there is an $\alpha_m \in S_m$ such that $\alpha_m + E_m \subset S_m$, and hence

$$(\alpha_m)_{m \in M} + E = (\alpha_m)_{m \in M} + \prod_{m \in M} E_m = \prod_{m \in M} (\alpha_m + E_m) \subset \prod_{m \in M} S_m = S.$$

Thus, E is a semigroup ideal of S.

(3) Let $\alpha, \beta \in E$. Then $\alpha_m, \beta_m \in E_m$ for all $m \in M$. Since E_m satisfies (E1) for any $m \in M$, we have $\inf(\alpha_m, \beta_m) \in E_m$ for every $m \in M$. This implies

$$\inf\{\alpha, \beta\} = (\inf\{\alpha_m, \beta_m\})_{m \in M} \in \prod_{m \in M} E_m = S,$$

and hence E satisfies (E1).

(4) Suppose that $E_m \in \mathcal{G}_{S_m}$ for every $m \in M$. The E satisfies (E1) by (3). Let $\alpha, \beta \in E$, and assume that there is an $m \in M$ and an $i \in I_m$ such that $\alpha_{\delta_n(i)} = \beta_{\delta_n(i)}$. Since E_m satisfies (E2), there is an $\epsilon_m \in E_m$ with

$$\begin{aligned}
&(\epsilon_m)_i > (\alpha_m)_i = (\beta_m)_i,\\
&(\epsilon_m)_j \geq \min\left\{(\alpha_m)_j, (\beta_m)_j\right\} \text{ for all } j \in I_m,\\
&(\epsilon_m)_k = \min\left\{(\alpha_m)_k, (\beta_m)_k\right\} \text{ for all } k \in I_m \text{ with } (\alpha_m)_k \neq (\beta_m)_k.
\end{aligned}$$

Since E_n satisfies (E1), we have $\epsilon_n = \inf \alpha_n, \beta_n \in E_m$ for every $n \in M$. So setting $\epsilon = (\epsilon_n)_{n \in M}$ we obtain $\epsilon \in E$ with

$$\begin{aligned} \epsilon_{\delta_n(i)} &> \min \alpha_{\delta_n(i)} = \beta_{\delta_n(i)}, \\ \epsilon_j &\geq \min \{\alpha_j, \beta_j\} \text{ for all } j \in I, \\ \epsilon_k &> \min \{\alpha_k, \beta_k\} \text{ for all } k \in I \text{ with } \alpha_k \neq \beta_k. \end{aligned}$$

Hence, $E \in \mathcal{G}_S$. □

4.2. Minimal Elements

The group of differences D_S of a good semigroup is partially ordered. Hence, semigroup ideals of S are partially ordered. We show that any semigroup ideal E of S satisfying property (E1) has a unique minimal element, i.e. an element μ which is comparable to, and smaller than all other elements of E.

Lemma 4.12. *Let E be a semigroup ideal of S. If E satisfies property (E1), then there is a unique element $\mu \in E$ which is minimal with respect to the partial order on D_S, i.e. $\mu \leq \alpha$ for all $\alpha \in E$.*

Proof. By Definition 4.5.(4) there is an $\alpha \in D_S$ such that $\alpha + E \subset S$. Hence, the sets $E_i \subset \mathbb{Z}$ are bounded from below for all $i \in I$. This implies that there are

$$\beta_i^{(i)} = \min \{E_i\} \in \mathbb{Z}$$

for all $i \in I$. Thus, there are $\delta^{(i)} \in E$ with $\delta_i^{(i)} = \beta^{(i)}$ for all $i \in I$. Since E satisfies property (E1), this yields

$$\mu = \inf \left\{ \delta^{(i)} \;\middle|\; i \in I \right\} \in E,$$

and by the construction we have $\mu \leq \alpha$ for all $\alpha \in E$.

Now let $\mu' \in E$ such that $\mu' \leq \alpha$ for all $\alpha \in E$. Then $\mu \leq \mu'$ and $\mu' \leq \mu$ implies $\mu = \mu'$. □

Definition 4.13. Let E be a semigroup ideal of S satisfying property (E1). The *minimal element* of E is by Lemma 4.12 the unique element $\mu_E \in E$ satisfying $\mu_E \leq \alpha$ for all $\alpha \in E$.

Lemma 4.14. *Let E be a semigroup ideal of S satisfying (E1). Then $\mu_E = \mathbf{0}$ if and only if $S \subset E \subset \overline{S}$.*

Proof. Suppose that $\mu_E = \mathbf{0}$. Then $\alpha \geq \mu_E = \mathbf{0}$ for all $\alpha \in E$, and hence $E \subset \overline{S}$. Moreover, since E is a semigroup ideal and $\mu_E \in E$, we have

$$S = \mathbf{0} + S = \mu_E + S \subset E.$$

Conversely, if $S \subset E \subset \overline{S}$, then $\mathbf{0} = \mu_S \geq \mu_E \geq \mu_{\overline{S}} = \mathbf{0}$, and hence $\mu_E = \mathbf{0}$. □

Lemma 4.15. *Let R be an analytically reduced one-dimensional semilocal Cohen–Macaulay ring with large residue fields, and let $\mathfrak{I} \in \mathcal{R}_R$. Then $\mathfrak{I} \subset \mathfrak{Q}^{\mu_{\Gamma_\mathfrak{I}}}$.*

Proof. By Proposition 3.22.(1) and (2) and Corollary 3.30.(1), Γ_R is a good semigroup, and $\Gamma_{\mathfrak{I}}$ is a semigroup ideal of Γ_R satisfying property (E1). Hence, by Lemma 4.12 there exists a minimal element $\mu_{\Gamma_{\mathfrak{I}}}$ of $\Gamma_{\mathfrak{I}}$.

Let $x \in \mathfrak{I}^{\mathrm{reg}}$. Then $\nu(x) \in \Gamma_{\mathfrak{I}}$, and hence $\nu(x) \geq \mu_{\Gamma_{\mathfrak{I}}}$. This implies $\mathfrak{I}^{\mathrm{reg}} \subset \mathfrak{Q}^{\mu_{\Gamma_{\mathfrak{I}}}}$.

Now assume there is $y \in \mathfrak{I}$ such that $\nu(y) \not\geq \mu_{\Gamma_{\mathfrak{I}}}$. Then $y \in \mathfrak{I} \setminus \mathfrak{E}^{\mathrm{reg}}$, and there is $W \in \mathcal{V}_R$ such that $\nu_W(y) < (\mu_{\Gamma_{\mathfrak{I}}})_W$. By Remark 4.6.(1) we can choose an $\alpha \in \Gamma_{\mathfrak{I}}$ such that $\alpha_V \neq \nu_V(y)$ for all $V \in \mathcal{V}_R$. Then there is $z \in \mathfrak{I}^{\mathrm{reg}}$ such that $\nu(z) = \alpha$. Moreover, by Lemma D.22.(5) we have

$$\nu(y+z) = \inf\{\nu(y), \nu(z)\},$$

and hence $\nu_V(y+z) < \infty$ for all $V \in \mathcal{V}_R$. Lemma 3.4.(2) yields

$$y + z \in \mathfrak{I} \cap Q_R^{\mathrm{reg}} = \mathfrak{I}^{\mathrm{reg}},$$

and thus $\nu(y+z) \in \Gamma_{\mathfrak{I}}$. But since $\nu(z) \in \Gamma_{\mathfrak{I}}$, and hence $\nu(z) \geq \mu_{\Gamma_{\mathfrak{I}}}$, we have

$$\nu_W(y+z) = \nu_W(y) < (\mu_{\Gamma_{\mathfrak{I}}})_W,$$

contradicting the minimality of $\mu_{\Gamma_{\mathfrak{I}}}$ in $\Gamma_{\mathfrak{I}}$, see Lemma 4.12. □

Proposition 4.16. *Let R be a one-dimensional semilocal normal Cohen–Macaulay ring.*

(1) $\Gamma_R = \overline{\Gamma_R} = \mathbb{N}^{\mathcal{V}_R}$ *is a good semigroup.*

(2) $\Gamma_{\mathfrak{I}}$ *is a good semigroup ideal of* Γ_R *for any* $\mathfrak{I} \in \mathcal{R}_R$. *In particular, we have*

$$\mathfrak{I} = \mathfrak{Q}^{\mu_{\Gamma_{\mathfrak{I}}}}$$

and

$$\Gamma_{\mathfrak{I}} = \mu_{\Gamma_{\mathfrak{I}}} + \Gamma_R.$$

Proof. (1) Since $R = \overline{R}$, we have by Lemma 3.4.(2)

$$\Gamma_R = \nu(R^{\mathrm{reg}}) = \mathbb{N}^{\mathcal{V}_R}.$$

Hence, $\Gamma_R = \overline{\Gamma_R}$, and Γ_R satisfies properties (E0), (E1), and (E2).

(2) Let $\mathfrak{I} \in \mathcal{R}_R$. Since $R = \overline{R}$, there is by Lemma 3.9.(1) and $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$ such that $\mathfrak{I} = \mathfrak{Q}^{\alpha}$. This implies by Lemma 3.4.(2)

$$\Gamma_{\mathfrak{I}} = \nu(\mathfrak{I}^{\mathrm{reg}}) = \alpha + \mathbb{N}^{\mathcal{V}_R} \subset \mathbb{Z}^{\mathcal{V}_R} = D_{\Gamma_R}. \tag{4.2}$$

Then by Proposition 4.16

$$\Gamma_{\mathfrak{I}} + \Gamma_R = \alpha + \mathbb{N}^{\mathcal{V}_R} + \mathbb{N}^{\mathcal{V}_R} = \alpha + \mathbb{N}^{\mathcal{V}_R} = \Gamma_{\mathfrak{I}},$$

and

$$-\alpha + \Gamma_{\mathfrak{I}} = -\alpha + \alpha + \mathbb{N}^{\mathcal{V}_R} = \mathbb{N}^{\mathcal{V}_R} = \Gamma_R.$$

Thus, $\Gamma_{\mathfrak{I}}$ is a semigroup ideal of Γ_R. Moreover, $\Gamma_{\mathfrak{I}}$ obviously satisfies properties (E1) and (E2), and hence $\Gamma_{\mathfrak{I}} \in \mathcal{G}_{\Gamma_R}$. Therefore, $\Gamma_{\mathfrak{I}}$ has by Lemma 4.12 a unique minimal element $\mu_{\Gamma_{\mathfrak{I}}}$, and Equation (4.2) yields $\mu_{\Gamma_{\mathfrak{I}}} = \alpha$. □

4.3. Differences

The difference of semigroup ideals corresponds to the quotient of fractional ideals (see Definition 2.1.(2)).

Definition 4.17. Let E and F be semigroup ideals of S. We write

$$E - F = \{\alpha \in D_S \mid \alpha + F \subset E\}.$$

The set $\mathcal{R}_R$ of regular fractional ideals of a ring R is by Proposition 2.7.(2) closed under the quotient. Example 3.26 shows that for a good semigroup S the set $\mathcal{G}_S$ of good semigroup ideals (which correspond to fractional ideals by Corollary 3.30) is in general not closed under the difference. However, the property of being a semigroup ideal and property (E1) are always preserved under the difference.

Lemma 4.18 (See [25], Lemma 4.1.4)**.** *For any two semigroup ideals E and F of S also $E - F$ is a semigroup ideal of S. If E satisfies (E1), so does $E - F$, and $E - \overline{S} \in \mathcal{G}_S \cap \mathcal{G}_{\overline{S}}$.*

Proof. Since F is a semigroup ideal of S, we have

$$(E - F) + S + F = (E - F) + F \subset E,$$

and hence

$$(E - F) + S \subset E - F.$$

Since E is a semigroup ideal of S, there is $\alpha \in \mathbb{Z}^s$ such that $\alpha + E \subset \overline{S}$. Then we have for any $\beta \in F$,

$$\alpha + \beta + (E - F) \subset \alpha + E \subset \overline{S}.$$

Thus, $E - F$ is a semigroup ideal of S.

Assume now that E satisfies property (E1). Then for any $\alpha, \beta \in E - F$ and $\delta \in F$ we have

$$\inf\{\alpha, \beta\} + \delta = \inf\{\alpha + \delta, \beta + \delta\} \in E$$

since $\alpha + \delta, \beta + \delta \in E$. Hence, $\inf\{\alpha, \beta\} \in E - F$, and $E - F$ satisfies property (E1).

We have

$$\left(E - \overline{S}\right) + \overline{S} + \overline{S} = \left(E - \overline{S}\right) + \overline{S} \subset E,$$

and hence

$$\left(E - \overline{S}\right) + \overline{S} \subset E - \overline{S}.$$

Therefore, $E - \overline{S}$ is a semigroup ideal of $\overline{S}$.

As just shown $E - \overline{S}$ satisfies (E1), and hence

$$\inf\{\alpha, \beta\} + \overline{S} \subset E - \overline{S}$$

for any $\alpha, \beta \in E - \overline{S}$. Since $\overline{S} = \mathbb{N}^I$, it follows that $E - \overline{S}$ satisfies (E2). □

Lemma 4.19. *Let S be a good semigroup, and let $S = \prod_{m \in M} S_m$ be the decomposition of S into local good semigroups (see Theorem 4.9). Then for any two semigroup ideals E and F of S we have*

$$E - F = \prod_{m \in M} (E_m - F_m).$$

Proof. For any $\alpha \in D_S$ we have $\alpha \in E - F$ if and only if

$$\prod_{m \in M} (\alpha_m + F_m) = (\alpha_m)_{m \in M} + \prod_{m \in M} F_m = \alpha + F \subset E = \prod_{m \in M} E_m.$$

This is equivalent to $\alpha_m + F_m \subset E_m$ for all $m \in M$, and hence to $\alpha = (\alpha_m)_{m \in M} \in \prod_{m \in M} (E_m - F_m)$. □

Lemma 4.20. *Let E be a semigroup ideal with $E \subset S$. Then*

$$S \subset S - E.$$

Proof. If $E \subset S$, the claim follows from Definition 4.17 since by Definition 4.5.(4)

$$E + S \subset E \subset S$$

□

For good semigroups we have analogously to Lemma 2.3 the following.

Remark 4.21 (See [25], Remark 4.1.3). Let $\alpha \in D_S$.

(1) The map

$$\begin{aligned} \mathcal{G}_S &\to \mathcal{G}_S, \\ E &\mapsto \alpha + E \end{aligned}$$

is a bijection.

(2) For any two semigroup ideals E and F of S, we have

$$(\alpha + E) - F = \alpha + (E - F) = E - (-\alpha + F).$$

(3) Let E, E', F, and F' be semigroup ideals of S. If $E \subset E'$ and $F \subset F'$, then

$$E - F' \subset E - F \subset E' - F.$$

(4) For any $E \in \mathcal{G}_S$, we have $E - S = S$.

Lemma 4.22. *Let E, F and G be semigroup ideals of S. Then*

$$(E - F) - G = (E - G) - F = E - (F + G).$$

Proof. By Definition 4.17 we have

$$\begin{aligned} (E - F) - G &= \{\alpha \in D_S \mid \alpha + G \subset E - F\} \\ &= \{\alpha \in D_S \mid \alpha + F + G \subset E\} = E - (F + G) \\ &= \{\alpha \in D_S \mid \alpha + F \subset E - G\} = (E - G) - F. \end{aligned}$$

□

Remark 4.23. In general, $E - F$ does not satisfy property (E2) for $E, F \in \mathcal{G}_S$, see [10, Example 2.10] and Example 3.26.

Lemma 4.24. *Let E and F be semigroup ideals of S satisfying property (E1).*

(1) If $\mu_{E-F} = \mathbf{0}$, then $F \subset E$.

(2) If $E = F$, then $\mu_{E-F} = \mathbf{0}$.

(3) If $F \subset E$, then $\mu_{E-F} \leq \mathbf{0}$.

(4) If $E \subsetneq F$, then $\mu_{E-F} > \mathbf{0}$.

Proof. Since E and F satisfy property (E1), also $E - F$ satisfies property (E1) by Lemma 4.18. Hence, E, F, and $E - F$ have unique minimal elements, see Lemma 4.12.

(1) If $\mathbf{0} = \mu_{E-F} \in E - F$, then

$$F = \mu_{E-F} + F \subset E$$

by Definition 4.17.

(2) Let $E = F$. Then $\mathbf{0} + F = F = E$, and hence $\mathbf{0} \in E - F$. This implies $\mu_{E-F} \leq \mathbf{0}$. So assume $\mu_{E-F} < \mathbf{0}$. This yields

$$\mu_E = \mu_F > \mu_F + \mu_{E-F} \in E,$$

contradicting the minimality of μ_E in E.

(3) If $F \subset E$, then

$$\mathbf{0} + F = F \subset E,$$

and hence $\mathbf{0} \in E - F$ by Definition 4.17. This implies $\mu_{E-F} \leq \mathbf{0}$.

(4) Let $E \subsetneq F$, and assume $\mu_{E-F} \leq \mathbf{0}$. Then $\mu_{E-F} < \mathbf{0}$ as otherwise $F \subset E$ by (1). Since $\mu_{E-F} \in E - F$, we have $\mu_F + \mu_{E-F} \in E$ by Definition 4.17. This yields

$$\mu_E \geq \mu_F > \mu_F + \mu_{E-F} \geq \mu_E,$$

and we obtain a contradiction. Therefore, $\mu_{E-F} > \mathbf{0}$. $\square$

Proposition 4.25. *Let E be a semigroup ideal of S. Then $E - E$ is a partially ordered cancellative commutative monoid with $D_{E-E} = D_S$ and $S \subset E - E \subset \overline{E - E} = \overline{S}$. If E satisfies property (E1), so does $E - E$.*

Proof. Obviously, we have $\mathbf{0} \in E - E$. Moreover, $S \subset E - E$ since E is a semigroup ideal of S, and hence $E + S \subset E$.

Let $\alpha, \beta \in E - E$. Then

$$\alpha + E \subset E,$$
$$\beta + E \subset E.$$

Hence,

$$\alpha + \beta + E \subset \alpha + E \subset E.$$

This implies $\alpha+\beta \in E-E$. Thus, $E-E$ is a monoid. Since $E-E \subset D_S$ by Definition 4.17, it is partially ordered, cancellative and commutative. Moreover, as $E-E$ is a semigroup ideal of S, it satisfies property (E0), see Remark 4.6.(1). Hence, there is $\alpha \in E-E$ such that $\alpha + \overline{S} \subset E-E$, and therefore $D_{E-E} = D_S$.

Assume now that there is an $\alpha \in E-E$ with $\alpha \not\geq \mathbf{0}$, i.e. $\alpha_i < 0$ for some $i \in I$. Since $E-E$ is a semigroup ideal of S by Lemma 4.18, there is $\beta \in S$ such that $\beta + (E-E) \subset S$. In particular, this implies $\alpha + \beta \geq \mathbf{0}$. Let $n = \max\left(m \in \mathbb{N} \mid m\alpha_i + \beta \geq 0\right)$ (n exists since $\alpha_i < 0$). But then $(n+1)\alpha \in E-E$ since $E-E$ is a monoid, and $(\alpha+\beta)_i < 0$, contradicting $\beta + (E-E) \subset S$. Hence, $E-E \subset \overline{E-E} = \overline{S}$.

Finally, if E satisfies property (E1), then also $E-E$ satisfies property (E1) by Lemma 4.18. □

4.4. Conductor

An important case of the difference of semigroup ideals is the conductor. In analogy to Definition B.22 we define the following.

Definition 4.26. Let S be a good semigroup, and let E be a semigroup ideal of S satisfying property (E1). The *conductor ideal* of E is

$$C_E = E - \overline{S} = \left\{\alpha \in D_S \;\middle|\; \alpha + \overline{S} \subset E\right\},$$

and

$$\gamma_E = \mu_{C_E} = \inf\left\{\alpha \in D_S \;\middle|\; \alpha + \overline{S} \subset E\right\}$$

(see Lemma 4.18) is called the *conductor* of E. We abbreviate $\tau_E = \gamma_E - \mathbf{1}$.

Remark 4.27. Let S be a good semigroup, and let E be a semigroup ideal of S satisfying property (E1).

(1) Since $C_E \in \mathcal{G}_{\overline{S}}$ by Lemma 4.18, we have

$$C_E = \mu_{C_E} + \overline{S} = \gamma_E + \overline{S}.$$

(2) Since $\mathbf{0} \in \overline{S}$, we have

$$C_E \subset E.$$

Lemma 4.28. *Let S be a good semigroup, and let $S = \prod_{m\in M} S_m$ be the decomposition of S into local good semigroups (see Theorem 4.9). Then for any semigroup ideal $E \in \mathcal{G}_S$ we have*

$$C_E = \prod_{m\in M} C_{E_m},$$

and hence

$$\gamma_E = (\gamma_{E_m})_{m\in M}.$$

Proof. Since $E_m \in \mathcal{G}_{S_m}$ by Theorem 4.9, and since $\overline{S}_m = \overline{S_m}$ by Lemma 4.11.(1) for all $m \in M$, Lemma 4.19 yields

$$C_E = E - \overline{S} = \prod_{m\in M} \left(E_m - \overline{S}_m\right) = \prod_{m\in M} \left(E_m - \overline{S_m}\right) = \prod_{m\in M} C_{E_m}. \qquad \square$$

Lemma 4.29. *Let S be a good semigroup. Then for any $E \in \mathcal{G}_S$ we have*

$$\gamma_E - \mu_E \leq \gamma_S.$$

Proof. By Definition 4.5.(4) we have $E + S \subset E$. Since $C_S = \gamma_S + \overline{S} \subset S$ by Remark 4.27 and $\mu_E \in E$ by Definition 4.13, this yields

$$\mu_E + \gamma_S + \overline{S} = \mu_E + C_S \subset E + S \subset E.$$

Therefore, Definition 4.26 and Remark 4.27.(1) yield

$$\mu_E + \gamma_S \in E - \overline{S} = C_E = \gamma_E + \overline{S}.$$

Hence

$$\mu_E + \gamma_S \geq \gamma_E$$

since $\mu_{\overline{S}} = \mathbf{0}$. $\square$

Proposition 4.30. *Let S be a good semigroup, and let $E \in \mathcal{G}_S$ and $F \in \mathcal{G}_{\overline{S}}$. Then $F = C_F$, and*

$$E - F = C_{E-F}.$$

Proof. Since $F \in \mathcal{G}_{\overline{S}}$, we have

$$C_F = F - \overline{S} = F,$$

and Lemma 4.22 yields

$$E - F = E - \left(F - \overline{S}\right) = (E - F) - \overline{S} = C_{E-F}. \qquad \square$$

The following objects were introduced by Delgado [12, 7] for investigating the Gorenstein property on value semigroups.

Definition 4.31. Let S be a good semigroup, and let $\alpha \in D_S$.

(1) For $J \subset I$ we set

$$\Delta_J(\alpha) = \{\beta \in \mathbb{Z}^s \mid \alpha_j = \beta_j \text{ for all } j \in J \text{ and } \alpha_i < \beta_i \text{ for all } i \in I \setminus J\},$$

and we write

$$\Delta_j(\alpha) = \Delta_{\{j\}}(\alpha)$$

for any $j \in I$.

(2) Let $J \subset I$, and let E be a semigroup ideal of S. Then

$$\Delta^E_J(\alpha) = \Delta_J(\alpha) \cap E.$$

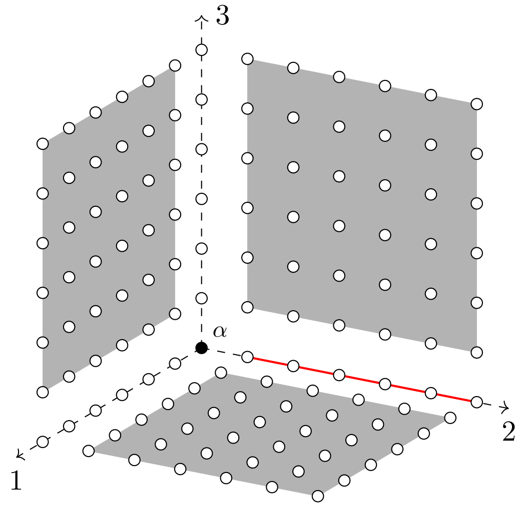


Figure 4.1.: The sets $\Delta_{1,3}(\alpha)$ (red) and $\Delta(\alpha) = \bigcup_{i\in\{1,2,3\}} \Delta_i(\alpha)$ (grey).

(3) We write

$$\Delta(\alpha) = \bigcup_{i\in I} \Delta_i(\alpha).$$

(4) If E is a semigroup ideal of S, then

$$\Delta^E(\alpha) = \Delta(\alpha) \cap E.$$

See Figure 4.1.

Lemma 4.32 (See [25], Lemma 4.1.9)**.** *Let S be a good semigroup, let $E \in \mathcal{G}_S$, and assume that there is an $\alpha \in E$ and $J \subset I$ such that $\alpha_j \geq (\gamma_E)_j$ for all $j \in J$. Then for any $j \in J$ we have $\alpha + \mathbf{e}_j \in E$.*

Proof. Let $j \in J$, and choose $\beta \in D_S$ with

$$\begin{aligned}
&\beta_j = \alpha_j,\\
&\beta_i > \alpha_i \text{ for all } i \in J,\\
&\beta_k > \max\{(\gamma_E)_k, \alpha_k\} \text{ for all } k \in I \setminus J.
\end{aligned}$$

Then $\beta \geq \gamma_E$, and hence $\beta \in E$. Applying property (E2) to α and β we obtain a $\delta \in E$ with

$$\begin{aligned}
&\delta_j > \alpha_j = \beta_j,\\
&\delta_i = \min\{\alpha_i, \beta_i\} = \alpha_i \text{ for all } i \in J \setminus \{j\}.
\end{aligned}$$

Now let $\epsilon \in D_S$ with

$$\begin{aligned} &\epsilon_j = \alpha_j + 1, \\ &\epsilon_i > \alpha_i \text{ for all } i \in J, \\ &\epsilon_k > \max\left\{(\gamma_E)_k, \alpha_k\right\} \text{ for all } k \in I \setminus J. \end{aligned}$$

Then $\epsilon \geq \gamma_E$, and hence $\epsilon \in E$. Applying property (E1) to δ and ϵ yields

$$\alpha + \mathbf{e}_j = \inf\{\delta, \epsilon\} \in E. \qquad \square$$

Lemma 4.33 (See [25], Lemma 4.1.9)**.** *Let S be a good semigroup, let $E \in \mathcal{G}_S$, and assume that there is $\alpha \in E$ and $J \subset I$ such that $\alpha_J \geq (\gamma_E)_J$. If $\delta \in D_S$ with $\delta_J \geq (\gamma_E)_J$ and $\delta_{I\setminus J} = \alpha_{I\setminus J}$, i.e.*

$$\begin{aligned} &\delta_j \geq (\gamma_E)_j \ \textit{for all } j \in J, \\ &\delta_k = \alpha_k \ \textit{for all } k \in I \setminus J, \end{aligned}$$

then $\delta \in E$.

Proof. Repeatedly applying Lemma 4.32 we obtain $(n_j)_{j\in J} \in \mathbb{N}^J$ such that

$$\delta \leq \alpha + \sum_{j\in J} n_j \mathbf{e}_j \in E.$$

Hence, we may assume that $\alpha \geq \delta$.

Pick $\epsilon \in D_S$ with

$$\begin{aligned} &\epsilon_j = \delta_j \text{ for all } j \in J, \\ &\epsilon_k > \max\left\{(\gamma_E)_k, \delta_k\right\} \text{ for all } k \in I \setminus J. \end{aligned}$$

In particular, $\epsilon \geq \gamma_E$, and hence $\epsilon \in E$. Thus, $\delta = \min\{\epsilon, \alpha\} \in E$ since E satisfies (E1). $\square$

Lemma 4.34 (See [25], Lemma 4.1.10)**.** *Let S be a good semigroup, and let $E \in \mathcal{G}_S$. Then*

$$\Delta^E(\tau_E) = \emptyset.$$

Proof. Assume that $\Delta^E(\tau_E) \neq \emptyset$. Then there is $i \in I$ with a $\beta \in \Delta_i^E(\tau_E)$, i.e.

$$\begin{aligned} &\beta_i = (\gamma_E)_i - 1, \\ &\beta_j \geq (\gamma_E)_j \text{ for all } j \in I \setminus \{i\}. \end{aligned}$$

Thus, Lemma 4.33 implies $\beta + \overline{S} \subset E$, and hence $\gamma_E > \beta \in C_E$ contradicting the minimality of γ_E in C_E. $\square$

We show the the analogues to Propositions 2.15, B.24 and B.25 for good semigroup ideals.

Lemma 4.35 (See [25], Lemma 4.1.11)**.** *Let S be a good semigroup, and let E and F be semigroup ideals of S satisfying property (E1). Then*

$$\gamma_{E-F} = \gamma_E - \mu_F.$$

Proof. Note that γ_{E-F} is defined since $E - F$ satisfies property (E1) by Lemma 4.18.
Since $F - \mu_F \subset \overline{S}$ and $\gamma_E + \overline{S} \subset E$, we have by definition

$$\gamma_E - \mu_F + \overline{S} + F \subset \gamma_E + \overline{S} \subset E.$$

This implies $\gamma_E - \mu_F + \overline{S} \subset E - F$, and hence $\gamma_E - \mu_F \geq \gamma_{E-F}$.
Conversely,

$$\gamma_{E-F} + \mu_F + \overline{S} = \gamma_{E-F} + \mu_F - \mu_F + F + \overline{S} = \gamma_{E-F} + F + \overline{S} \subset E$$

implies $\gamma_{E-F} + \mu_F \geq \gamma_E$. □

Corollary 4.36. *Let S be a good semigroup, and let F be a semigroup ideal of S satisfying property (E1), and let $E \in \mathcal{G}_S$. Then*

$$\Delta^{E-F}\left(\tau_{E-F}\right) = \emptyset.$$

Proof. Note that τ_{E-F} is defined since $E - F$ satisfies property (E1) by Lemma 4.18.
Now assume $\Delta^{E-F}\left(\tau_{E-F}\right) \neq \emptyset$, and let $\beta \in \Delta^{E-F}\left(\tau_{E-F}\right)$. Then Lemma 4.35 yields

$$\begin{aligned}\beta + \mu_F &\in \Delta\left(\tau_{E-F}\right) + \mu_F \\ &= \Delta\left(\tau_{E-F} + \mu_F\right) \\ &= \Delta\left(\tau_E\right).\end{aligned}$$

Moreover, $\beta \in E - F$ implies $\beta + \mu_F \in E$, and hence $\beta \in \Delta^E\left(\tau_E\right)$ contradicting $\Delta^E\left(\tau_E\right) = \emptyset$ by Lemma 4.34. □

Lemma 4.37. *Let S be a good semigroup, and let E and F be semigroup ideals of S satisfying property (E1). Then*

$$\mu_{E-F} \geq \gamma_E - \gamma_F.$$

Proof. Note that τ_{E-F} is defined since $E - F$ satisfies property (E1) by Lemma 4.18.
By definition we have $\gamma_F + \overline{S} \subset F$, and therefore

$$\mu_{E-F} + \gamma_F + \overline{S} \subset \mu_{E-F} + F \subset E.$$

This implies

$$\mu_{E-F} + \gamma_F \geq \gamma_E.$$ □

Proposition 4.38. *Let S is a local good semigroup. Then*

$$S - M_S = M_S - M_S.$$

Proof. Lemmas 4.7 and 4.37 yield

$$\mu_{S-M_S} \geq \gamma_S - \gamma_{M_S} = \gamma_S - \gamma_S = \mathbf{0}$$

(see Definition 4.5.(3)). This implies

$$(S - M_S) + M_S \subset \{\alpha \in S \mid \alpha \geq \mu_{M_S}\} = M_S,$$

and hence

$$S - M_S \subset M_S - M_S.$$

Since also

$$M_S - M_S \subset S - M_S$$

by Remark 4.21.(3), this yields

$$S - M_S = M_S - M_S. \qquad \square$$

Lemma 4.39. *Let S be a good semigroup.*

(1) We have

$$S - C_S = \overline{S}.$$

(2) If E is a semigroup ideal of S with $C_S \subset E \subset S$, then

$$S \subset S - E \subset \overline{S}.$$

Proof. (1) Since $C_S = S - \overline{S} \in \mathcal{G}_S$ by Lemma 4.18, Proposition 4.30 yields $S - C_S = C_{S-C_S}$. As $S - C_S$ satisfies property (E1) by Lemma 4.18, we obtain by Lemma 4.35

$$\mu_{S-C_S} = \mu_{C_{S-C_S}} = \gamma_{C_{S-C_S}} = \gamma_{S-C_S} = \gamma_S - \mu_{C_S} = \gamma_s - \gamma_S = \mathbf{0}.$$

Then Remark 4.27.(1) and Proposition 4.30 yield

$$S - C_S = C_{S-C_S} = \mu_{C_{S-C_S}} + \overline{S} = \mu_{S-C_S} + \overline{S} = \overline{S}.$$

(2) By (1) and Remarks 4.21.(3), (4) and 4.27.(2) we have

$$S = S - S \subset S - E \subset S - C_S = \overline{S}. \qquad \square$$

Lemma 4.40. *Let S be a good semigroup, and let E and F be semigroup ideals of S. Then*

(1) $F \subset E - (E - F)$.

(2) If E and F satisfy property (E1), $E \subsetneq F$, and $\gamma_E = \gamma_F$, then

$$F \subsetneq E - (E - F).$$

Proof. (1) Let $\alpha \in F$, and let $\beta \in E - F = \{\delta \in D_S \mid \delta + F \subset E\}$. This implies $\alpha + \beta \in E$, and hence

$$\alpha \in \{\delta \in D_S \mid \delta + (E - F) \subset E\} = E - (E - F).$$

(2) Since E and F satisfy property (E1), also $E-F$ and $E-(E-F)$ satisfies property (E1) by Lemma 4.18. Hence, E, F, $E-F$, and $E-(E-F)$ have unique minimal elements and conductors, see Lemma 4.12.

If $F \subsetneq E$, then $\mu_{E-F} > \mathbf{0}$ by Lemma 4.24.(4). Hence, Lemma 4.35 yields

$$\gamma_{E-(E-F)} = \gamma_E - \mu_{E-F} < \gamma_E = \gamma_F.$$

Then the claim follows since $F \subset E - (E-F)$ by (2). □

4.5. Distance and Length

The combinatorial counterpart of the relative length of two fractional ideal is the *distance* between two good semigroup ideals (see Definition 4.46). It serves as a main tool to relate algebra and combinatorics (see Proposition 4.51).

First we introduce the notion of chains in partially ordered sets.

Definition 4.41. Let E be a partially ordered set.

(1) A *chain* in E is a finite subset $C \subset E$ which is totally ordered with respect to the order induced by the partial order on D_S. The *length* of a chain E is $|E|-1$.

(2) Let $\alpha, \beta \in E$ with $\alpha \leq \beta$. A *chain* in E between α and β is a chain C in E with $\min C = \alpha$ and $\max C = \beta$.

(3) A chain C in E is called *saturated* if for any chain C' in E with $C \subset C'$, $\min C = \min C'$, and $\max C = \max C'$ we have $C = C'$.

(4) Two elements $\alpha, \beta \in E$ with $\alpha < \beta$ are called *consecutive in* E if there is no $\delta \in E$ with $\alpha < \delta < \beta$.

Remark 4.42. Let E be a partially ordered set. A chain C in E is saturated if and only if for any $\alpha \in C \setminus \{\max C\}$ there is a $\beta \in C$ such that α and β are consecutive in E.

Definition 4.43. Let S be a good semigroup, and let $E \subset D_S$. Additionally to the properties in Definition 3.19 we consider the following property.

(E4) For any fixed $\alpha, \beta \in E$ every two saturated chains in E between α and β have the same length.

Definition 4.44. Let S be a good semigroup, let $E \subset D_S$, let $\alpha, \beta \in E$ with $\alpha \leq \beta$, and suppose that E satisfies property (E4). The *distance* $d_E(\alpha, \beta)$ of α and β in E is the length of any saturated chain between α and β in E.

Proposition 4.45. *Let S be a good semigroup. Then any $E \in \mathcal{G}_S$ satisfies property (E4).*

Proof. See [8, Proposition 2.3]. □

Definition 4.46. Let S be a good semigroup, and let E and F be be two semigroup ideals of S satisfying property (E4) with $E \subset F$. Then we call

$$d\left(F \setminus E\right) = d_F\left(\mu_F, \gamma_E\right) - d_E\left(\mu_E, \gamma_E\right)$$

the *distance* between E and F.

Remark 4.47. Let S be a good semigroup. By Definitions 4.44 and 4.46 the distance has the following properties.

(1) Let $E \subset D_S$ satisfy property (E4). Then for any $\alpha, \beta \in E$ with $\alpha < \beta$ we have

$$d(\alpha, \beta) \in \mathbb{N}.$$

(2) Let $E \subset F$ be semigroup ideals of S satisfying property (E4). Then

$$d(F \setminus E) \in \mathbb{N}.$$

Remark 4.48 (See [25], Remark 4.2.3). Let S be a good semigroup, and let E and F be semigroup ideals of S satisfying properties (E1) and (E4) with $E \subset F$.

(1) d_E is additive with respect to composition of chains. That is, for any $\alpha, \beta, \gamma \in E$ with $\alpha \leq \beta \leq \delta$ we have

$$d\left(\alpha, \delta\right) = d\left(\alpha, \beta\right) + d\left(\beta, \delta\right).$$

(2) $d_E(\alpha, \beta) \leq d_F(\alpha, \beta)$ for all $\alpha, \beta \in E$.

(3) $d(E \setminus F) = d(\alpha + F \setminus \alpha + E)$ for all $\alpha \in D_S$.

(4) With the notation of Theorem 4.9 we have

$$d\left(F \setminus E\right) = \sum_{m \in M} d\left(F_m \setminus E_m\right),$$

see [10, Proposition 2.12].

(5) If $\epsilon \geq \gamma_E$, then (1) implies

$$\begin{aligned} d(F \setminus E) &= d_F(\mu_F, \gamma_E) - d_E(\mu_E, \gamma_E) \\ &= d_F(\mu_F, \gamma_E) + d_F(\gamma_E, \epsilon) - d_E(\mu_E, \gamma_E) - d_E(\gamma_E, \epsilon) \\ &= d_F(\mu_F, \epsilon) - d_E(\mu_E, \epsilon) \end{aligned}$$

since $d_F(\gamma_E, \epsilon) = d_E(\gamma_E, \epsilon)$.

Lemma 4.49. *Let $E \subset F \subset G$ be semigroup ideals of S satisfying property (E4). Then*

$$d\left(G \setminus E\right) = d\left(G \setminus F\right) + d\left(F \setminus E\right).$$

Proof. See [8, Proposition 2.7]. □

The following result was first stated in [8, Proposition 2.8].

Proposition 4.50. *Let $E, F \in \mathcal{G}_S$ with $E \subset F$. Then $E = F$ if and only if $d(F \setminus E) = 0$.*

Proof. See [25, Proposition 4.2.6]. □

The following result relates the length of quotients of fractional ideals with the distance of their value semigroups.

Proposition 4.51. *Let R be an admissible ring. If $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$ with $\mathfrak{I} \subset \mathfrak{J}$, then*

$$\ell_R(\mathfrak{J}/\mathfrak{I}) = d(\Gamma_{\mathfrak{J}} \setminus \Gamma_{\mathfrak{I}}).$$

Proof. See [25, Proposition 4.2.7]. □

Corollary 4.52 (See [25], Corollary 4.2.8)**.** *Let R be an admissible ring, and let $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$ with $\mathfrak{I} \subset \mathfrak{J}$. Then $\mathfrak{I} = \mathfrak{J}$ if and only if $\Gamma_{\mathfrak{I}} = \Gamma_{\mathfrak{J}}$.*

Proof. By Remark 4.6.(5) Γ_R is a good semigroup, and $\Gamma_{\mathfrak{I}}, \Gamma_{\mathfrak{J}} \in \mathcal{G}_{\Gamma_R}$. Hence, Proposition 4.51 yields $\mathfrak{I} = \mathfrak{J}$ if and only if $0 = \ell_R(\mathfrak{J}/\mathfrak{I}) = d(\Gamma_{\mathfrak{J}} \setminus \Gamma_{\mathfrak{I}})$, and by Proposition 4.50 this is equivalent to $\Gamma_{\mathfrak{I}} = \Gamma_{\mathfrak{J}}$. Also see [8, Proposition 2.5]. □

Lemma 4.53. *Let R be an admissible ring, and let $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$. If there is an $\mathfrak{H} \in \mathcal{R}_R$ such that*

$$\mathfrak{H}\mathfrak{J} \subset \mathfrak{I}$$

and

$$\Gamma_{\mathfrak{H}} = \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}},$$

then

$$\mathfrak{H} = \mathfrak{I} : \mathfrak{J}.$$

Proof. If $\mathfrak{H}\mathfrak{J} \subset \mathfrak{I}$, then $\mathfrak{H} \subset \mathfrak{I} : \mathfrak{J}$. This implies $\Gamma_{\mathfrak{H}} \subset \Gamma_{\mathfrak{I}:\mathfrak{J}}$. Moreover, we have $\Gamma_{\mathfrak{I}:\mathfrak{J}} \subset \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}} = \Gamma_{\mathfrak{H}}$ by Lemma 3.23.(1) and the assumption. Thus, $\Gamma_{\mathfrak{H}} = \Gamma_{\mathfrak{I}:\mathfrak{J}}$, and Corollary 4.52 yields $\mathfrak{H} = \mathfrak{I} : \mathfrak{J}$. □

Lemma 4.54. *Let R and R' be a admissible rings such that $\mathfrak{C}_{R'} \subset R \subset R' \subset Q_R$.*

(1) $R' \in \mathcal{R}_R$.

(2) If $\Gamma_R = \Gamma'_R$, then $R = R'$.

Proof. (1) Let $x \in \mathfrak{C}_{R'}^{\mathrm{reg}} \subset R^{\mathrm{reg}}$. Then $xR' \subset \mathfrak{C}_{R'} \subset R$. Since $\emptyset \neq R^{\mathrm{reg}} \subset (R')^{\mathrm{reg}}$, this yields $R' \in \mathcal{R}_R$.

(2) Since $R' \in \mathcal{R}_R$ by (1), and since $R \in \mathcal{R}_R$, Corollary 4.52 yields $R = R'$. □

Lemma 4.55. *Let R be an admissible ring, and let $\mathfrak{I} \in \mathcal{R}_R$. Then:*

(1) $\mathfrak{C}_{\mathfrak{I}} \subset \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}$.

(2) $\mathfrak{C}_{\mathfrak{I}} = \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}$ *if and only if* $\Gamma_{\mathfrak{C}_{\mathfrak{I}}} = C_{\Gamma_{\mathfrak{I}}}$.

Proof. (1) Since $\mathfrak{C}_{\mathfrak{I}} \in \mathcal{R}_{\overline{R}}$, there is by Lemma 3.9.(1) an $\alpha \in \mathbb{Z}^{\mathcal{V}_R}$ such that $\mathfrak{C}_{\mathfrak{I}} = \mathfrak{Q}^{\alpha}$. Then Lemmas 3.4.(2), 3.9.(3), and 3.23.(1) yield

$$\alpha + \mathbb{N}^{\mathcal{V}_R} = \Gamma_{\mathfrak{C}_{\mathfrak{I}}} = \Gamma_{\mathfrak{I}:\overline{R}} \subset \Gamma_{\mathfrak{I}} - \Gamma_{\overline{R}} = \Gamma_{\mathfrak{I}} - \mathbb{N}^{\mathcal{V}_R} = C_{\Gamma_{\mathfrak{I}}} = \gamma_{\Gamma_{\mathfrak{I}}} + \mathbb{N}^{\mathcal{V}_R}.$$

Hence, $\alpha \geq \gamma_{\Gamma_{\mathfrak{I}}}$, and $\mathfrak{C}_{\mathfrak{I}} = \mathfrak{Q}^{\alpha} \subset \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}$.

(2) Assume that $\mathfrak{C}_{\mathfrak{I}} = \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}$. Then Lemma 3.9.(3) yields

$$\Gamma_{\mathfrak{C}_{\mathfrak{I}}} = \Gamma_{\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}} = \gamma_{\Gamma_{\mathfrak{I}}} + \mathbb{N}^{\mathcal{V}_R} = C_{\Gamma_{\mathfrak{I}}}.$$

Now suppose that $\Gamma_{\mathfrak{C}_{\mathfrak{I}}} = C_{\Gamma_{\mathfrak{I}}}$. In particular, this implies

$$\mu_{\Gamma_{\mathfrak{C}_{\mathfrak{I}}}} = \mu_{C_{\Gamma_{\mathfrak{I}}}} = \gamma_{\Gamma_{\mathfrak{I}}}.$$

Since $\mathfrak{C}_{\mathfrak{I}} \in \mathcal{R}_{\overline{R}}$, Lemma 3.9.(1) yields then

$$\mathfrak{C}_{\mathfrak{I}} = \mathfrak{Q}^{\mu_{\Gamma_{\mathfrak{C}_{\mathfrak{I}}}}} = \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}. \qquad \square$$

Proposition 4.56. *Let R be an admissible ring, and let $\mathfrak{I} \in \mathcal{R}_R$. Then*

$$\mathfrak{C}_{\mathfrak{I}} = \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}},$$

and hence

$$\Gamma_{\mathfrak{C}_{\mathfrak{I}}} = C_{\Gamma_{\mathfrak{I}}}$$

(see Lemma 4.55.(2)).

Proof. By Lemma 4.55.(1) we have

$$\mathfrak{C}_{\mathfrak{I}} \subset \{x \in Q_R \mid \nu(x) \geq \gamma_{\Gamma_{\mathfrak{I}}}\} = \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}.$$

Moreover, Lemma 3.9.(3) yields

$$C_{\Gamma_{\mathfrak{I}}} = \gamma_{\Gamma_{\mathfrak{I}}} + \mathbb{N}^{\mathcal{V}_R} = \Gamma_{\mathfrak{I}^{\gamma_{\Gamma_{\mathfrak{I}}}}} = \Gamma_{\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}}.$$

Since $\mathfrak{I}^{\gamma_{\Gamma_{\mathfrak{I}}}} \subset \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}$, we obtain by Corollary 4.52

$$\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}} = \mathfrak{I}^{\gamma_{\Gamma_{\mathfrak{I}}}} \subset \mathfrak{I}.$$

As $\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}} \in \mathcal{R}_{\overline{R}}$, this implies

$$\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}} = \mathfrak{C}_{\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}}} \subset \mathfrak{C}_{\mathfrak{I}},$$

and hence

$$\mathfrak{C}_{\mathfrak{I}} \subset \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}} \subset \mathfrak{C}_{\mathfrak{I}}. \qquad \square$$

By Proposition 4.56 taking value semigroup ideals commutes with conductors in the sense that for an admissible ring R there is a commutative diagram

$$\begin{tikzcd}
\mathcal{R}_R \arrow[r, "\mathfrak{I}\mapsto\mathfrak{C}_{\mathfrak{I}}"] \arrow[d, "\mathfrak{I}\mapsto\Gamma_{\mathfrak{I}}"'] & \mathcal{R}_{\overline{R}} \arrow[d, "\mathfrak{C}\mapsto\Gamma_{\mathfrak{C}}"] \\
\mathcal{G}_{\Gamma_R} \arrow[r, "E\mapsto C_E"'] & \mathcal{G}_{\Gamma_{\overline{R}}}.
\end{tikzcd} \tag{4.3}$$

We can generalize Proposition 4.56 as follows.

Proposition 4.57. *Let R be an admissible ring, and let $\mathfrak{I} \in \mathcal{R}_R$ and $\mathfrak{J} \in \mathcal{R}_{\overline{R}}$. Then $\mathfrak{J} = \mathfrak{C}_{\mathfrak{J}}$, $\Gamma_{\mathfrak{J}} = C_{\Gamma_{\mathfrak{J}}}$, and*

$$\Gamma_{\mathfrak{I}:\mathfrak{J}} = \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}.$$

Proof. Since $\mathfrak{J} \in \mathcal{R}_{\overline{R}}$, we have

$$\mathfrak{C}_{\mathfrak{J}} = \mathfrak{J} : \overline{R} = \mathfrak{J}, \tag{4.4}$$

and Lemma 2.3.(1) yields

$$\mathfrak{I} : \mathfrak{J} = \mathfrak{I} : \left(\mathfrak{J} : \overline{R}\right) = (\mathfrak{I} : \mathfrak{J}) : \overline{R} = \mathfrak{C}_{\mathfrak{I}:\mathfrak{J}}. \tag{4.5}$$

Then we have by Equation (4.5), Proposition B.24, Lemma 3.23.(2) and Equation (4.4), Proposition 4.56, Definition 4.26, Lemma 4.22, and Proposition 4.30

$$\begin{aligned}\Gamma_{\mathfrak{I}:\mathfrak{J}} = \Gamma_{\mathfrak{C}_{\mathfrak{I}:\mathfrak{J}}} = \Gamma_{\mathfrak{C}_{\mathfrak{I}}:\mathfrak{J}} = \Gamma_{\mathfrak{C}_{\mathfrak{I}}} - \Gamma_{\mathfrak{J}} = C_{\Gamma_{\mathfrak{I}}} - \Gamma_{\mathfrak{J}} &= \left(\Gamma_{\mathfrak{I}} - \overline{\Gamma_R}\right) - \Gamma_{\mathfrak{J}} \\ &= (\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}) - \overline{\Gamma_R} = C_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}} = \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}. \qquad \square\end{aligned}$$

Lemma 4.58. *Let R be an admissible ring, and let $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$. Then*

$$\gamma_{\Gamma_{\mathfrak{I}:\mathfrak{J}}} = \gamma_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}} = \gamma_{\Gamma_{\mathfrak{I}}} - \mu_{\Gamma_{\mathfrak{J}}}.$$

Proof. By Lemma 3.23.(1) we have $\Gamma_{\mathfrak{I}:\mathfrak{J}} \subset \Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}$. Thus,

$$C_{\Gamma_{\mathfrak{I}:\mathfrak{J}}} = \Gamma_{\mathfrak{I}:\mathfrak{J}} - \overline{\Gamma_R} \subset (\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}) - \overline{\Gamma_R} = C_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}}$$

by Remark 4.21.(3), and hence

$$\gamma_{\Gamma_{\mathfrak{I}:\mathfrak{J}}} \geq \gamma_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}}.$$

Moreover, Remark 4.6.(5), Lemma 4.35, and Proposition 4.56 imply

$$\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}}}\mathfrak{J} \subset \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}}}\mathfrak{Q}^{\mu_{\Gamma_{\mathfrak{J}}}} = \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}} + \mu_{\Gamma_{\mathfrak{J}}}} = \mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}}}} = \mathfrak{C}_{\mathfrak{I}}.$$

Thus, we have $\mathfrak{Q}^{\gamma_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}}} \subset \mathfrak{C}_{\mathfrak{I}} : \mathfrak{J} = \mathfrak{C}_{\mathfrak{I}:\mathfrak{J}}$ by Proposition B.24. Hence, Proposition 4.56 yields

$$\gamma_{\Gamma_{\mathfrak{I}} - \Gamma_{\mathfrak{J}}} \geq \gamma_{\Gamma_{\mathfrak{I}:\mathfrak{J}}}. \qquad \square$$

4.6. Branches

The following result was proved in [10, Proposition 2.2] for good semigroups. Here we generalize it to semigroup ideals of a good semigroup S.

Proposition 4.59. *Let S be a good semigroup, let $J \subset I$, and let E be a semigroup ideal of S.*

(1) The projection of S onto J,

$$S_J = \mathrm{pr}_J(S) = \left\{\alpha_J \in \mathbb{Z}^J \,\middle|\, \text{there is } \beta \in S \text{ such that } \beta_j = \alpha_j \text{ for all } j \in J\right\},$$

is a good semigroup in $\mathbb{Z}^J$.

(2) The projection of E onto J,

$$E_J = \mathrm{pr}_J(E) = \left\{\alpha_J \in \mathbb{Z}^J \,\middle|\, \text{there is } \beta \in E \text{ such that } \beta_j = \alpha_j \text{ for all } j \in J\right\},$$

is a semigroup ideal of $S_J = \mathrm{pr}_J(S)$.

(3) If E satisfies property (E1) in $\mathbb{Z}^I$, then E_J satisfies property (E1) in $\mathbb{Z}^J$.

(4) If E satisfies property (E2) in $\mathbb{Z}^I$, then E_J satisfies property (E2) in $\mathbb{Z}^J$.

Proof. (1) See [10, Proposition 2.2].

(2) By (1) S_J is a good semigroup in $\mathbb{Z}^J$. Since E is a semigroup ideal of S, we have $E + S \subset E$, and hence

$$\mathrm{pr}_J(E) + \mathrm{pr}_J(S) = \mathrm{pr}_J(E + S) \subset \mathrm{pr}_J(E).$$

Moreover, there is an $\alpha \in \mathbb{Z}^s$ such that $\alpha + E \subset S$. This implies

$$\mathrm{pr}_J(\alpha) + \mathrm{pr}_J(E) = \mathrm{pr}_J(\alpha + E) \subset \mathrm{pr}_J(S).$$

Thus, E_J is a semigroup ideal of S_J.

(3) Let $\alpha, \beta \in E_J$. Then there are $\alpha', \beta' \in E$ such that $\mathrm{pr}_J(\alpha') = \alpha$ and $\mathrm{pr}_J(\beta') = \beta$. Since E satisfies property (E1), we also have $\inf\{\alpha', \beta'\} \in E$, and hence

$$\inf\{\alpha, \beta\} = \inf\{\mathrm{pr}_J(\alpha'), \mathrm{pr}_J(\beta')\} = \mathrm{pr}_J(\inf\{\alpha', \beta'\}) \in E.$$

(4) Let $\alpha, \beta \in E_J$ with $\alpha_j = \beta_j$ for some $j \in J$. Then there are $\alpha', \beta' \in E$ such that $\mathrm{pr}_J(\alpha') = \alpha$, $\mathrm{pr}_J(\beta') = \beta$ and $\alpha'_j = \beta'_j$. Since E satisfies property (E2), there is an $\epsilon \in E$ such that

$$\begin{aligned}
&\epsilon_j > \alpha'_j = \beta'_j \\
&\epsilon_i \geq \min\{\alpha'_i, \beta'_i\} \text{ for all } i \in I \\
&\epsilon_k = \min\{\alpha'_i, \beta'_i\} \text{ for all } k \in I \text{ with } \alpha'_k \neq \beta'_k.
\end{aligned}$$

This implies $\epsilon_J \in E_J$ with

$$\begin{aligned}(\epsilon_J)_j &> \alpha'_j = \beta'_j\\ (\epsilon_J)_i &\geq \min\{\alpha'_i, \beta'_i\} \text{ for all } i \in I\\ (\epsilon_J)_k &= \min\{\alpha'_i, \beta'_i\} \text{ for all } k \in I \text{ with } \alpha'_k \neq \beta'_k.\end{aligned}$$

Hence, E_J satisfies property (E2). □

Definition 4.60. Let S be a good semigroup, let E be a semigroup ideal of S satisfying property (E1). For $J, J' \subset I$ we define

$$E_J^{J'} = \{\beta \in D_{S_J} \mid \text{there is an } \alpha \in E \text{ such that } \alpha_J = \beta \text{ and } \alpha_{J'} \geq (\gamma_E)_{J'}\} \subset E_J.$$

If $J = \{i\}$ for some $i \in I$, we write $E_i^{J'} = E_{\{i\}}^{J'}$, if $J' = \{j\}$ for some $j \in I$, we write $E_J^j = E_J^{\{j\}}$, and we write $E_i^j = E_{\{i\}}^{\{j\}}$ for any $i, j \in I$ with $i \neq j$.

Remark 4.61. Let S be a good semigroup, let $J, J' \subset I$, let $E \in \mathcal{G}_S$, and let $\alpha \in E_J$. Then we have by Lemma 4.33 $\alpha \in E_J^{J'}$ if and only if $\beta \in E$ for any $\beta \in D_S$ with

$$\begin{aligned}\beta_J &= \alpha_J,\\ \beta_{J'} &\geq (\gamma_E)_{J'}.\end{aligned}$$

Lemma 4.62. *Let S be a good semigroup, let $J, J' \subset I$, and let E be a semigroup ideal of S satisfying property (E1). Then $E_J^{J'}$ is a semigroup ideal of S_J satisfying property (E1). Moreover, if $E \in \mathcal{G}_S$, then $E_J^{J'} \in \mathcal{G}_{S_J}$.*

Proof. Let $\alpha \in E_J^{J'}$, and let $\beta \in S_J$. Then there is a $\delta \in E$ such that

$$\delta_J = \alpha, \tag{4.6}$$
$$\delta_{J'} \geq (\gamma_E)_{J'}, \tag{4.7}$$

and there is an $\epsilon \in S$ such that $\epsilon_J = \beta$. Since $S \subset \overline{S}$, we have $\epsilon \geq \mathbf{0}$. This implies with Equations (4.6) and (4.7)

$$(\delta + \epsilon)_{J'} = \delta_{J'} + \epsilon_{J'} \geq \delta_{J'} \geq (\gamma_E)_{J'},$$

and hence

$$\alpha + \beta = \delta_J + \epsilon_J = (\delta + \epsilon)_J \in E_J^{J'}.$$

Therefore, $E_J^{J'} + S_J \subset E_J^{J'}$. Since $E_J^{J'} \subset (D_S)_J = D_{S_J}$ by Definition 4.60, $E_J^{J'}$ is a semigroup ideal of S_J.

Let now $\alpha, \beta \in E_J^{J'}$. Then there are $\delta, \epsilon \in E$ such that $\delta_J = \alpha$, $\epsilon_J = \beta$ and $\delta_{J'}, \epsilon_{J'} \geq (\gamma_E)_{J'}$. This implies

$$\inf\{\delta_{J'}, \epsilon_{J'}\} = (\inf\{\delta, \epsilon\})_{J'} \geq (\gamma_E)_{J'},$$

and hence

$$\inf\{\alpha, \beta\} = \inf\{\delta_J, \epsilon_J\} = (\inf\{\delta, \epsilon\})_J \in E_J^{J'}.$$

Thus, $E_J^{J'}$ satisfies property (E1).

Finally, let $E \in \mathcal{G}_S$, and assume there is an $i \in J$ such that $\alpha_i = \beta_i$. Then $\delta_i = \epsilon_i$, and since E satisfies property (E2), there is a $\zeta \in E$ such that

$$\begin{aligned}
&\zeta_i > \delta_i = \epsilon_i,\\
&\zeta_j \geq \min\{\delta_j, \epsilon_j\} \text{ for all } j \in I,\\
&\zeta_k = \min\{\delta_k, \epsilon_k\} \text{ for all } k \in I \text{ with } \delta_k \neq \epsilon_k.
\end{aligned}$$

This implies $\zeta_{J'} \geq (\gamma_E)_{J'}$, and hence $\zeta_J \in E_J^{J'}$ with

$$\begin{aligned}
(\zeta_J)_i &= \zeta_i > \delta_i = \epsilon_i = \beta_i = \alpha_i,\\
(\zeta_J)_j &= \zeta_j \geq \min\{\delta_j, \epsilon_j\} = \min\{\alpha_j, \beta_j\} \text{ for all } j \in J,\\
(\zeta_J)_k &= \zeta_k = \min\{\delta_k, \epsilon_k\} = \min\{\alpha_k, \beta_k\} \text{ for all } k \in J \text{ with } \delta_k \neq \epsilon_k.
\end{aligned}$$

Therefore, $E_J^{J'}$ satisfies property (E2), and hence $E_J^{J'} \in \mathcal{G}_{S_J}$. $\square$

Lemma 4.63. *Let S be a good semigroup. Then for any $i \in I$ we have $(C_S)_i \subset S_i^J$ for every $J \subset I$.*

Proof. Let $\alpha \in (C_S)_i$, i.e. there is $\beta \in C_S$ with $\beta_i = \alpha$. In particular, we have $\alpha_J \geq (\gamma_S)_J$ for every $J \subset I$. $\square$

Proposition 4.64. *Let S be a good semigroup. Then for any $E \in \mathcal{G}_S$ we have*

$$\gamma_E = \left(\gamma_{E_i^{I\setminus\{i\}}}\right)_{i\in I}.$$

Proof. Set

$$\gamma = \left(\gamma_{E_i^{I\setminus\{i\}}}\right)_{i\in I}.$$

Since $\gamma_E + \overline{S} \subset E$ by Definition 4.26, we have for any $i \in I$

$$(\gamma_E)_i \geq \gamma_{E_i^{I\setminus\{i\}}}.$$

Thus, $\gamma_E \geq \gamma$.

Let now $\alpha \in \overline{S}$, in particular $\alpha \geq \mathbf{0}$. Then we have for any $i \in I$

$$\gamma_i + \alpha_i \geq \gamma_{E_i^{I\setminus\{i\}}},$$

and hence

$$\gamma_i + \alpha_i \in E_i^{I\setminus\{i\}}.$$

Therefore, for any $i \in I$ there is a $\beta^{(i)} \in E$ such that

$$\begin{aligned}
\beta_i^{(i)} &= \gamma_i + \alpha_i,\\
\beta_{I\setminus\{i\}}^{(i)} &\geq (\gamma_E)_{I\setminus\{i\}}.
\end{aligned}$$

By Lemma 4.33 we may assume that $\beta^{(i)}_{I\setminus\{i\}} \geq \gamma_{I\setminus\{i\}} + \alpha_{I\setminus\{i\}}$ for any $i \in I$. Since E satisfies property (E1), this implies

$$\gamma + \alpha = \inf_{i\in I} \beta^{(i)} \in E,$$

and hence $\gamma + \overline{S} \subset E$. Thus, $\gamma \geq \gamma_E$. □

Proposition 4.65. *Let S be a good semigroup. Then S is local if and only if $0 \in S_i \setminus S_i^j$ for every $i, j \in I$ with $i \neq j$.*

Proof. Suppose there are $i, j \in I$ with $i \neq j$ such that $0 \in S_i^j$. Then there is an $\alpha \in S$ with $\alpha_i = 0$ and $\alpha_j \geq (\gamma_S)_j$ (see Definition 4.60). Using Lemma 4.33 we may assume that $\alpha_j > 0$. This implies that S is not local, see Definition 4.5.(3).

Now suppose that S is not local. Then there is an $\alpha \in S$ with $\alpha_i = 0$ and $\alpha_j > 0$ for some $i, j \in I$. Moreover, we can find an $n \in \mathbb{N}$ such that $n\alpha_j \geq (\gamma_S)_j$. Since S is a semigroup, and hence $n\alpha \in S$ with $(n\alpha)_i = n\alpha_i = 0$ and $(n\alpha)_j = n\alpha_j \geq (\gamma_S)_j$, this yields $0 \in S_i^j$. □

Lemma 4.66. *Let S be a good semigroup, let $E \in \mathcal{G}_S$, and let $J \subset I$. Then $\delta \in E$ for all $\delta \in D_S$ with*

$$\delta_J \in E_J^{I\setminus J},$$
$$\delta_{I\setminus J} \geq (\gamma_E)_{I\setminus J}.$$

Proof. Let $\alpha \in E_J^{I\setminus J}$. Then there is $\beta \in E$ with

$$\beta_J = \alpha,$$
$$\beta_{I\setminus J} \geq (\gamma_E)_{I\setminus J}.$$

Thus, Lemma 4.33 yields the statement. □

Proposition 4.67. *Let R be an admissible ring, let $J \subset \operatorname{Min}(R)$, and let*

$$J' = \{V \in \mathcal{V}_R \mid \mathfrak{q}_V \in J\}.$$

We denote by

$$\pi\colon Q_R \to Q_R / \bigcap_{\mathfrak{p}\in J} \mathfrak{p} Q_R = Q_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}$$

the canonical surjection (see Theorem A.74.(2) for the equality), and

$$\mathrm{pr}'\colon (\mathbb{Z} \cup \{\infty\})^{\mathcal{V}_R} \to (\mathbb{Z} \cup \{\infty\})^{\mathcal{V}_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}}$$

is the composition of the isomorphism $(\mathbb{Z} \cup \{\infty\})^{\mathcal{V}_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}} \cong (\mathbb{Z} \cup \{\infty\})^{J'}$ induced by the bijection $\mathcal{V}_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}} \to J'$ (see Proposition 3.13.(3)) and the projection $(\mathbb{Z} \cup \{\infty\})^{\mathcal{V}_R} \to (\mathbb{Z} \cup \{\infty\})^{J'}$.

(1) There is a commutative diagram

$$\begin{array}{ccc} Q_R & \xrightarrow{\quad\nu\quad} & (\mathbb{Z}\cup\{\infty\})^{\mathcal{V}_R} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}'} \\ Q_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}} & \xrightarrow[\overline{\nu}]{} & (\mathbb{Z}\cup\{\infty\})^{\mathcal{V}_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}}, \end{array}$$

where $\overline{\nu}$ is the multivaluation of $Q_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}$.

(2) For any $\mathfrak{I}\in\mathcal{R}_R$ we have

$$\Gamma_{\pi(\mathfrak{I})} = (\Gamma_{\mathfrak{I}})_{J'}$$

(see Lemma 4.68.(2)).

For the proof of Proposition 4.67 we need the following Lemma.

Lemma 4.68. *With the assumptions as in Proposition 4.67 we have the following:*

(1) If $x\in(Q_R)^{\mathrm{reg}}$, then $\pi(x)\in\left(Q_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}\right)^{\mathrm{reg}}$.

(2) Let $\mathfrak{I}\in\mathcal{R}_R$. Then $\pi(\mathfrak{I})\in\mathcal{R}_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}$.

Proof. 1. Assume $\pi(x)$ is not regular. Then there is a $y\in R$ such that $\pi(y)\neq 0$ and

$$\pi(xy) = \pi(x)\,\pi(y) = 0.$$

This implies

$$xy\in\bigcap_{\mathfrak{p}\in J}\mathfrak{p}Q_R.$$

Thus, either $x\in\mathfrak{p}Q_R$ or $y\in\mathfrak{p}Q_R$ for any $\mathfrak{p}\in J$. Since $\mathfrak{p}Q_R\in\operatorname{Max}(Q_R)$ for all $\mathfrak{p}\in J$ by Theorem A.74.(1), and since $x\in(Q_R)^{\mathrm{reg}}$, this yields $y\in\mathfrak{p}Q_R$ for all $\mathfrak{p}\in J$. But then $\pi(y)=0$, contradicting the choice of y.

2. Obviously, $\pi(\mathfrak{I})$ is an $R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}$-submodule of $Q_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}$, and it is regular by (1). Since $\mathfrak{I}$ is a fractional ideal of R, there is $x\in R^{\mathrm{reg}}$ such that $x\mathfrak{I}\subset R$. This implies

$$\pi(x)\,\pi(\mathfrak{I}) = \pi(x\mathfrak{I})\subset\pi(R) = R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p},$$

and we have $\pi(x)\in\left(R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}\right)^{\mathrm{reg}}$ by (1). Thus, $\pi(\mathfrak{I})\in\mathcal{R}_{R/\bigcap_{\mathfrak{p}\in J}\mathfrak{p}}$. ☐

Proof of Proposition 4.67. (1) This follows from Proposition 3.13.(3).

(2) Let $\alpha \in (\Gamma_{\mathfrak{I}})_{J'}$. Then there is an $x \in \mathfrak{I}$ such that (1) yields

$$\overline{\nu}(\pi(x)) = (\nu(x))_{J'} = \alpha.$$

Hence, $\pi(x) \in (\pi(\mathfrak{I}))^{\mathrm{reg}}$ by Lemma 3.4.(2) and Proposition 3.13.(3). This implies $\alpha \in \Gamma_{\pi(\mathfrak{I})}$.

Let now $\alpha \in \Gamma_{\pi(\mathfrak{I})}$. Then there is an $x \in (\pi(\mathfrak{I}))^{\mathrm{reg}}$ with $\overline{\nu}(x) = \alpha$. This implies that there is a $y \in \mathfrak{I}$ with $\pi(y) = x$, and

$$(\nu(y))_{J'} = \overline{\nu}(\pi(y)) = \overline{\nu}(x) = \alpha$$

by (1). Let now $z \in (\mathfrak{C}_{\mathfrak{I}})^{\mathrm{reg}}$ with

$$\nu_V(z) \neq \nu_V(y) \text{ for all } V \in \mathcal{V}_R,$$
$$\nu_W(z) > \nu_W(y) \text{ for all } W \in J'.$$

Then

$$\nu_V(y+z) = \min\{\nu_V(y), \nu_V(z)\} < \infty$$

for all $V \in \mathcal{V}_R$ (see Remark D.14.(1) and Lemma 3.4.(2)), and Lemma 3.4.(2) yields $y + z \in \mathfrak{I}^{\mathrm{reg}}$. Therefore,

$$\nu(y+z) \in \Gamma_{\mathfrak{I}}$$

with

$$(\nu(x+y))_{J'} = (\nu(y))_{J'} = \alpha.$$

Hence, $\alpha \in (\Gamma_{\mathfrak{I}})_{J'}$. ☐

Proposition 4.69. *Let R be an admissible ring, let $\mathfrak{p} \in \operatorname{Min}(R)$, let $I \subset \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$, and let*

$$J = \{V \in \mathcal{V}_R \mid \mathfrak{q}_V = \mathfrak{p}\},$$
$$J' = \{V \in \mathcal{V}_R \mid \mathfrak{q}_V \in I\}.$$

Then

$$(\Gamma_R)_J^{J'} = \Gamma_{\bigcap_{\mathfrak{q}\in I} \mathfrak{q}+\mathfrak{p}/\mathfrak{p}}$$

(note that $\bigcap_{\mathfrak{q}\in I} \mathfrak{q} + \mathfrak{p}/\mathfrak{p} \in \mathcal{R}_{R/\mathfrak{p}}$ since $\bigcap_{\mathfrak{q}\in I} \mathfrak{q}$ is an ideal of R not contained in $\mathfrak{p}$ and $R/\mathfrak{p}$ is a domain).

Proof. Let $\alpha \in (\Gamma_R)_J^{J'}$. Then there is an $x \in R^{\mathrm{reg}}$ with

$$(\nu(x))_J = \alpha,$$
$$(\nu(x))_{J'} \geq (\gamma_{\Gamma_R})_{J'}.$$

Since $Q_R = \prod_{\mathfrak{p}' \in \operatorname{Min}(R)} Q_R/\mathfrak{p}'Q_R$ by Theorem A.74.(2), and since $I \subset \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$, there is by Proposition 3.13.(3) a $y \in Q_R$ with

$$x - y \in \mathfrak{q}_V \text{ for all } V \in J', \tag{4.8}$$
$$\nu_W(y) > \max\{\nu_W(x), (\gamma_{\Gamma_R})_W\} \text{ for all } W \in J, \tag{4.9}$$
$$\nu_{W'}(y) \geq (\gamma_{\Gamma_R})_{W'} \text{ for all } W' \in \mathcal{V}_R \setminus (J \cup J'). \tag{4.10}$$

Then Proposition D.15 implies

$$\nu_V(x) = \nu_V(y) \geq (\gamma_{\Gamma_R})_V$$

for all $V \in J'$. Thus,

$$\nu(y) \geq \gamma_{\Gamma_R},$$

and hence

$$y \in Q^{\gamma_{\Gamma_R}} = \mathfrak{C}_R \subset R$$

by Proposition 4.56. This yields $x - y \in R$ with

$$\nu_V(x-y) = \min\{\nu_V(x), \nu_V(y)\}$$

for all $V \in J$, see Remark D.14.(1). Therefore,

$$\overline{\nu}(x - y + \mathfrak{p}) = (\nu(x-y))_J = \alpha$$

by Proposition 4.67.(1), where $\overline{\nu}$ is the multivaluation of $Q_{R/\mathfrak{p}}$. Hence, Equation (4.8) yields

$$x - y + \mathfrak{p} \in \left(\bigcap_{\mathfrak{q} \in I} \mathfrak{q} + \mathfrak{p}/\mathfrak{p}\right)^{\mathrm{reg}}$$

by Lemma 3.4.(2) and Proposition 3.13.(3). Thus,

$$\alpha \in \Gamma_{\bigcap_{\mathfrak{q}\in I} \mathfrak{q}+\mathfrak{p}/\mathfrak{p}}.$$

Let now $\alpha \in \Gamma_{\bigcap_{\mathfrak{q}\in I} \mathfrak{q}+\mathfrak{p}/\mathfrak{p}}$. Then there is an $x \in \left(\bigcap_{\mathfrak{q}\in I} \mathfrak{q} + \mathfrak{p}/\mathfrak{p}\right)^{\mathrm{reg}}$ with $\overline{\nu}(x) = \alpha$. Thus, there is a $y \in \bigcap_{\mathfrak{q}\in I} \mathfrak{q}$ such that $y + \mathfrak{p} = x$, and Proposition 4.67.(1) yields

$$(\nu(y))_J = \overline{\nu}(x) = \alpha.$$

Since $y \in \bigcap_{\mathfrak{q}\in I} \mathfrak{q}$, we have

$$\nu_V(y) = \infty$$

for all $V \in J'$. So let $z \in (\mathfrak{C}_R)^{\mathrm{reg}}$ with

$$\nu_V(z) \neq \nu_V(y) \text{ for all } V \in \mathcal{V}_R,$$
$$\nu_W(z) > \nu_W(y) \text{ for all } W \in J.$$

Then

$$\nu_V(y+z) = \min\{\nu_V(y), \nu_V(z)\} < \infty$$

for all $V \in \mathcal{V}_R$ (see Remark D.14.(1) and Lemma 3.4.(2)), and Lemma 3.4.(2) yields $y + z \in R^{\mathrm{reg}}$. Therefore,

$$\nu(y+z) \in \Gamma_R$$

with

$$(\nu(y+z))_J = (\nu(y))_J = \alpha,$$
$$(\nu(x+y))_{J'} \geq (\gamma_{\Gamma_R})_{J'}$$

by Proposition 4.56. Hence, $\alpha \in (\Gamma_R)_J^{J'}$. □

Example 4.70. Let

$$\begin{aligned} R &= \mathbb{C}[[X,Y]]/\Big\langle \Big(X^5 - Y^2\Big)Y \Big\rangle = \mathbb{C}[[X,Y]]/\Big(\Big\langle X^5 - Y^2 \Big\rangle \cap \langle Y \rangle\Big) \\ &= \mathbb{C}\Big[\Big[\Big(-t_1^2, t_2\Big), \Big(t_1^5, 0\Big)\Big]\Big]. \end{aligned}$$

The semigroup of values Γ_R of R is depicted in Figure 4.2. Then

$$R/\Big\langle X^5 - Y^2 \Big\rangle = \mathbb{C}[[X,Y]]/\Big\langle X^5 - Y^2 \Big\rangle = \mathbb{C}\Big[\Big[t_1^2, t_1^5\Big]\Big]$$

with semigroup of values

$$\Gamma_{R/\langle X^5 - Y^2 \rangle} = (\Gamma_R)_1 = \langle 2, 5 \rangle,$$

and

$$R/\langle Y \rangle = \mathbb{C}[[X,Y]]/\langle Y \rangle = \mathbb{C}[[t_2]]$$

with semigroup of values

$$\Gamma_{R/\langle Y \rangle} = (\Gamma_R)_2 = \mathbb{N}$$

(see Proposition 4.67).

The value semigroup ideals of the ideals

$$\langle Y \rangle + \Big\langle X^5 - Y^2 \Big\rangle/\Big\langle X^5 - Y^2 \Big\rangle = t_1^5\mathbb{C}\Big[\Big[t_1^2, t_1^5\Big]\Big],$$

respectively

$$\Big\langle X^5 - Y^2 \Big\rangle + \langle Y \rangle/\langle Y \rangle = t_2^5\mathbb{C}[[t_2]],$$

are

$$\Gamma_{\langle Y \rangle + \langle X^5 - Y^2 \rangle/\langle X^5 - Y^2 \rangle} = (\Gamma_R)_1^2 = 5 + \langle 2, 5 \rangle,$$

respectively

$$\Gamma_{\langle X^5 - Y^2 \rangle + \langle Y \rangle/\langle Y \rangle} = (\Gamma_R)_2^1 = 5 + \mathbb{N},$$

see Proposition 4.69.

4.7. Numerical Semigroups

Numerical semigroups (see Definition 4.5.(2)) are particularly important examples of good semigroups. Here we study some of their special properties. In particular, we consider semigroup rings (see Definition 4.73), and we introduce quotients on numerical semigroups (see Definition 4.74). We show that taking quotients "commutes" with the construction of semigroup rings (see Proposition 4.79).

Proposition 4.71. *A submonoid S of $\mathbb{N}$ is a numerical semigroup if and only if $\mathbb{N} \setminus S$ is finite.*

Proof. If S is a numerical semigroup, then $\mathbb{N} \setminus S$ is finite since S satisfies property (E0).

Conversely, if $\mathbb{N} \setminus S$ is finite, then there is an $\alpha \in S$ such that $\alpha + \mathbb{N} \subset S$. Hence, $D_S = \mathbb{Z}$, and S satisfies (E0). Since $|I| = 1$, S also satisfies properties (E1) and (E2). Thus, S is a numerical semigroup. $\square$

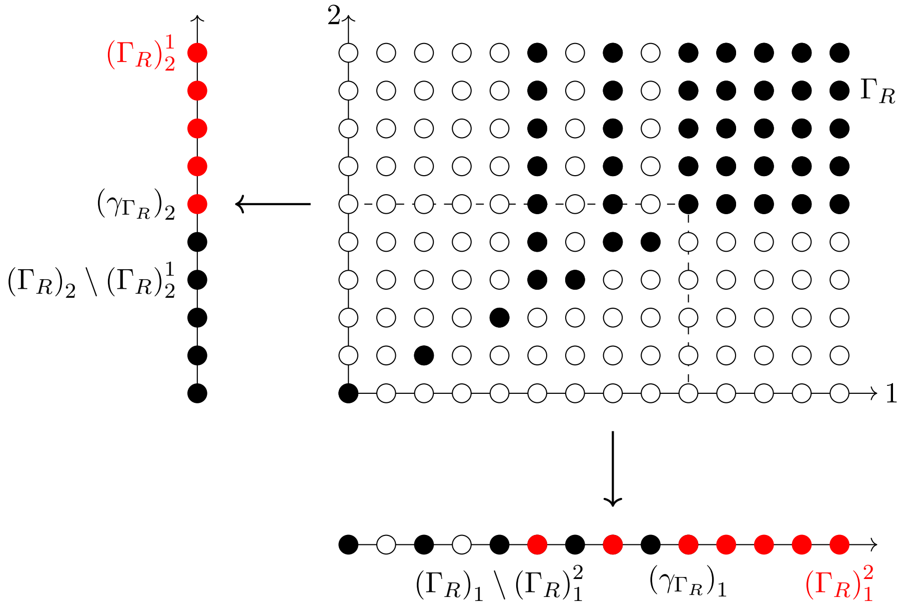


Figure 4.2.: The semigroup of values of the admissible ring $R = \mathbb{C}[[X,Y]]/\langle (X^5 - Y^2)Y \rangle$ of Example 4.70.

Proposition 4.72. *A numerical semigroup is finitely generated.*

Proof. Let $G = \{\alpha \in S \mid 0 < \alpha < 2\gamma_S\}$. Then G is finite, and S is generated by G as a monoid. □

Definition 4.73. Let S be a numerical semigroup, and let R be a ring. We denote by $R\left[t^S\right]$ the subset of $R[t]$ consisting of all polynomials $\sum_{\alpha\in S} r_\alpha t^\alpha$, where only finitely many coefficients r_α are different from 0. With the usual addition

$$\sum_{\alpha\in S} r_\alpha t^\alpha + \sum_{\beta\in S} s_\beta t^\beta = \sum_{\alpha\in S} (r_\alpha + s_\alpha) t^\alpha$$

and the multiplication

$$\left(\sum_{\alpha\in S} r_\alpha t^\alpha\right)\left(\sum_{\beta\in S} s_\beta t^\beta\right) = \sum_{\delta\in S} \sum_{\substack{\alpha,\beta\in S,\\ \alpha+\beta=\delta}} r_\alpha s_\beta t^\delta$$

for $\sum_{\alpha\in S} r_\alpha t^\alpha \in R\left[\left[t^S\right]\right]$ and $\sum_{\beta\in S} s_\beta t^\beta \in R\left[\left[t^S\right]\right]$, the set $R\left[\left[t^S\right]\right]$ is a ring, the *semigroup ring of S over R*.

Definition 4.74. Let S be a numerical semigroup, and let $E \in \mathcal{G}_S$ with $E \subset S$. The *quotient semigroup* of S by E, denoted by S/E, is the set $(S \setminus E) \cup \{\infty\}$ together with the

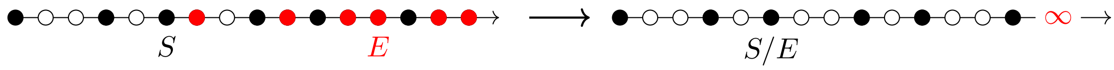


Figure 4.3.: The semigroup $S = \langle 3, 5 \rangle$, the semigroup ideal $E = 6 + S \in \mathcal{G}_S$ (red), and the quotient semigroup S/E of Example 4.76.

operation defined by

$$\alpha * \beta = \begin{cases} \alpha + \beta & \text{if } \alpha, \beta \in S \setminus E \text{ and } \alpha + \beta \in E, \\ \infty & \text{else} \end{cases}$$

for any $\alpha, \beta \in S/E$. We will also write $+$ for the "addition" in S/E.

Remark 4.75. Let S be a numerical semigroup, and let $E \in \mathcal{G}_S$ with $E \subset S$. Then $S \setminus E$ and hence also S/E are finite since E satisfies property (E0). Moreover, S/E is indeed a commutative monoid.

Example 4.76. Consider the numerical semigroup $S = \langle 3, 5 \rangle$ and the semigroup ideal $E = 6 + S \in \mathcal{G}_S$. Then the quotient semigroup S/E is given by the set $\{0, 3, 5, 8, 10, 13, \infty\}$, see Figure 4.3. In S/E we have for example $5 + 8 = 13$ and $3 + 13 = \infty$.

Definition 4.77. Let S be a numerical semigroup, let $E \in \mathcal{G}_S$ with $E \subset S$, and let R be a ring.

(1) We denote by $R\Big[\Big[t^E\Big]\Big]$ the set of all formal sums $\sum_{\alpha \in E} r_\alpha t^\alpha$ with $(r_\alpha)_{\alpha \in E} \in R^E$.

(2) If $\sum_{\alpha \in E} r_\alpha t^\alpha \in R\Big[\Big[t^E\Big]\Big]$ with $r_\alpha = 0$ for all $\alpha \in F$ for some subset F of E, we write $\sum_{\alpha \in E \setminus F} r_\alpha t^\alpha = \sum_{\alpha \in E} r_\alpha t^\alpha$.

(3) We write $R\Big[t^{S/E}\Big]$ for the set of formal sums $\sum_{\alpha \in S/E} r_\alpha t^\alpha$ with $(r_\alpha)_{\alpha \in S/E} \in R^{S/E}$ modulo the relation $t^\infty = 0$. In particular, for any element of $R\Big[t^{S/E}\Big]$ we find a representative of the form $\sum_{\alpha \in S \setminus E} r_\alpha t^\alpha$ with $(r_\alpha)_{\alpha \in S \setminus E} \in R^{S \setminus E}$.

Remark 4.78. Let S be a numerical semigroup, let $E, F \in \mathcal{G}_S$ with $E, F \subset S$, and let R be a ring.

(1) With the usual addition

$$\sum_{\alpha \in S} r_\alpha t^\alpha + \sum_{\beta \in S} s_\beta t^\beta = \sum_{\alpha \in E \cup F} (r_\alpha + s_\alpha) t^\alpha$$

and the multiplication

$$\left(\sum_{\alpha \in S} r_\alpha t^\alpha \right) \left(\sum_{\beta \in S} s_\beta t^\beta \right) = \sum_{\delta \in S} \sum_{\substack{\alpha, \beta \in S, \\ \alpha + \beta = \delta}} r_\alpha s_\beta t^\delta$$

for $\sum_{\alpha\in S} r_\alpha t^\alpha \in R\Big[\Big[t^S\Big]\Big]$ and $\sum_{\beta\in S} s_\beta t^\beta \in R\Big[\Big[t^S\Big]\Big]$, the set $R\Big[\Big[t^S\Big]\Big]$ is an R-algebra. Moreover, $E \in \mathcal{G}_S$ with $E \subset S$ implies that $R\Big[\Big[t^E\Big]\Big]$ is an ideal of $R\Big[\Big[t^S\Big]\Big]$.

(2) Similarly, also the set $R\Big[t^{S/E}\Big]$ is an R-algebra.

Proposition 4.79. *Let S be a numerical semigroup, let $E \in \mathcal{G}_S$ with $E \subset S$, and let R be a ring. There is a surjective R-algebra homomorphism*

$$\begin{aligned}\Psi\colon R\Big[\Big[t^S\Big]\Big] &\to R\Big[t^{S/E}\Big],\\ \sum_{\alpha\in S} r_\alpha t^\alpha &\mapsto \overline{\sum_{\alpha\in S\setminus E} r_\alpha t^\alpha}\end{aligned}$$

inducing an R-algebra isomorphism

$$\begin{aligned}\psi\colon R\Big[\Big[t^S\Big]\Big]/R\Big[\Big[t^E\Big]\Big] &\to R\Big[t^{S/E}\Big],\\ \sum_{\alpha\in S} r_\alpha t^\alpha + R\Big[\Big[t^E\Big]\Big] &\mapsto \overline{\sum_{\alpha\in S\setminus E} r_\alpha t^\alpha}.\end{aligned}$$

In particular, there is a commutative diagram

$$\begin{tikzcd} R\Big[\Big[t^S\Big]\Big] \arrow[d, two heads, "\pi"'] \arrow[dr, two heads, "\Psi"] & \\ R\Big[\Big[t^S\Big]\Big]/R\Big[\Big[t^E\Big]\Big] \arrow[r, "\psi", "\cong"'] & R\Big[t^{S/E}\Big], \end{tikzcd}$$

where $\pi\colon R\Big[\Big[t^S\Big]\Big] \to R\Big[\Big[t^S\Big]\Big]/R\Big[\Big[t^E\Big]\Big]$ is the canonical surjection.

Proof. Let $\sum_{\alpha\in S} r_\alpha t^\alpha, \sum_{\beta\in S} s_\beta t^\beta \in R\Big[\Big[t^S\Big]\Big]$. Then

$$\begin{aligned}\psi\left(\sum_{\alpha\in S} r_\alpha t^\alpha + \sum_{\beta\in S} s_\beta t^\beta\right) &= \psi\left(\sum_{\alpha\in S} (r_\alpha + s_\alpha) t^\alpha\right)\\ &= \overline{\sum_{\alpha\in S\setminus E} (r_\alpha + s_\alpha) t^\alpha}\\ &= \overline{\sum_{\alpha\in S\setminus E} r_\alpha t^\alpha} + \overline{\sum_{\beta\in S\setminus E} s_\beta t^\beta}\\ &= \psi\left(\sum_{\alpha\in S} r_\alpha t^\alpha\right) + \psi\left(\sum_{\beta\in S} s_\beta t^\beta\right)\end{aligned}$$

and

$$
\begin{aligned}
\psi\left(\left(\sum_{\alpha\in S} r_\alpha t^\alpha\right)\left(\sum_{\beta\in S} s_\beta t^\beta\right)\right) &= \psi\left(\sum_{\delta\in S}\sum_{\substack{\alpha,\beta\in S\\ \alpha+\beta=\delta}} r_\alpha s_\beta t^\delta\right)\\
&= \overline{\sum_{\delta\in S\setminus E}\sum_{\substack{\alpha,\beta\in S\\ \alpha+\beta=\delta}} r_\alpha s_\beta t^\delta}\\
&= \overline{\sum_{\delta\in S\setminus E}\sum_{\substack{\alpha,\beta\in S\setminus E\\ \alpha+\beta=\delta}} r_\alpha s_\beta t^\delta} \qquad (4.11)\\
&= \left(\overline{\sum_{\alpha\in S\setminus E} r_\alpha t^\alpha}\right)\left(\overline{\sum_{\beta\in S\setminus E} s_\beta t^\beta}\right)\\
&= \left(\psi\left(\sum_{\alpha\in S} r_\alpha t^\alpha\right)\right)\left(\psi\left(\sum_{\beta\in S} s_\beta t^\beta\right)\right),
\end{aligned}
$$

where the equality in Equation (4.11) follows since E satisfies property (E0), i.e. $E+S\subset E$. Moreover, since ψ is obviously R-linear, it is an R-algebra homomorphism.

Let now $\sum_{\alpha\in S} r_\alpha t^\alpha \in R\left[\left[t^S\right]\right]$ with

$$\overline{\sum_{\alpha\in S\setminus E} r_\alpha t^\alpha} = \psi\left(\sum_{\alpha\in S} r_\alpha t^\alpha\right) = \overline{0}.$$

Then $r_\alpha = 0$ for all $\alpha \in S\setminus E$ (see Definitions 4.74 and 4.77.(3)), and hence $\sum_{\alpha\in S} r_\alpha t^\alpha \in R\left[\left[t^E\right]\right]$. Therefore, $\ker\Psi \subset R\left[\left[t^E\right]\right]$.

Moreover, if $\sum_{\alpha\in E} r_\alpha t^\alpha \in R\left[\left[t^E\right]\right]$, then

$$\Psi\left(\sum_{\alpha\in E} r_\alpha t^\alpha\right) = \overline{0}.$$

This yields $\ker\Psi = R\left[\left[t^E\right]\right]$.

Finally, let $\overline{\sum_{\alpha\in S\setminus E} r_\alpha t^\alpha} \in R\left[t^{S/E}\right]$ (note that we can write any element of $R\left[t^{S/E}\right]$ in this form, see Definition 4.77.(3)). Then by setting $r_\alpha = 0$ for all $\alpha\in E$ we obtain $\sum_{\alpha\in S} r_\alpha t^\alpha \in R\left[\left[t^S\right]\right]$ and

$$\psi\left(\sum_{\alpha\in S} r_\alpha t^\alpha\right) = \overline{\sum_{\alpha\in S\setminus E} r_\alpha t^\alpha}.$$

Hence, Ψ is also surjective, and the homomorphism theorem yields the statement. $\square$

4.8. Semigroup Rings over $\mathbb{C}$

Considering quasihomogeneous curves in Chapters 6 and 7 we will deal with semigroup rings over $\mathbb{C}$. Here we study some basic properties.

Proposition 4.80. *Let S be a numerical semigroup. Then $\mathbb{C}\left[\left[t^S\right]\right]$ is a local admissible ring with maximal ideal*

$$\mathfrak{m}_{\mathbb{C}[[t^S]]} = \left\{x \in \mathbb{C}\left[\left[t^S\right]\right] \,\middle|\, \mathrm{ord}_t\left(x\right) \geq 0\right\}.$$

Moreover, we have $\mathcal{V}_{\mathbb{C}[[t^S]]} = \{\mathbb{C}[[t]]\}$, the corresponding valuation is ord_t, and

$$\Gamma_{\mathbb{C}[[t^S]]} = S.$$

Proposition 4.81. *Let S be a numerical semigroup. The set*

$$\mathfrak{m}_{\mathbb{C}[t^S]} = \left\{x \in \mathbb{C}\left[t^S\right] \,\middle|\, \mathrm{ord}_t\left(x\right) > 0\right\}$$

is a maximal ideal of $\mathbb{C}\left[t^S\right]$, and $\mathbb{C}\left[\left[t^S\right]\right]$ is the $\mathfrak{m}_{\mathbb{C}[t^S]}$-adic completion of $\mathbb{C}\left[t^S\right]$.

Proof. Obviously, $\mathfrak{m}_{\mathbb{C}[t^S]}$ is an ideal of $\mathbb{C}\left[t^S\right]$, and $\mathbb{C}\left[t^S\right]/\mathfrak{m}_{\mathbb{C}[t^S]} \cong \mathbb{C}$. Thus, $\mathfrak{m}_{\mathbb{C}[t^S]}$ is a maximal ideal of $\mathbb{C}\left[t^S\right]$. Then it is also easy to see that $\mathbb{C}\left[\left[t^S\right]\right]$ is the $\mathfrak{m}_{\mathbb{C}[t^S]}$-adic completion of $\mathbb{C}\left[t^S\right]$. ☐

Lemma 4.82. *Let S be a numerical semigroup. Then $\mathbb{C}\left[t^S\right]$ is Noetherian.*

Proof. By Proposition 4.72 S admits a finite set G of generators. Then $\mathbb{C}\left[t^S\right]$ is generated as a $\mathbb{C}$-algebra by $\{t^\alpha \mid \alpha \in G\}$. Thus, $\mathbb{C}\left[t^S\right]$ is Noetherian by Theorem A.1. ☐

Corollary 4.83. *Let S be a numerical semigroup. Then $\mathbb{C}\left[\left[t^S\right]\right]$ is Noetherian.*

Proof. This follows from Proposition 4.80, Lemma 4.82 and Theorem A.53. ☐

Lemma 4.84. *Let S be a numerical semigroup. Then $\overline{\mathbb{C}[[t^S]]} = \mathbb{C}[[t]]$. In particular, $\dim \mathbb{C}\left[\left[t^S\right]\right] = 1$.*

Proof. Since S is a good semigroup (see Remark 4.6.(2)), we have

$$t^{\gamma_S}\mathbb{C}[[t]] \subset \mathbb{C}\left[\left[t^S\right]\right]. \tag{4.12}$$

This implies $Q_{\mathbb{C}[[t^S]]} = \mathbb{C}[[t]]\left[t^{-1}\right]$. Moreover, Equation (4.12) and Lemma 4.82 imply that $\mathbb{C}[[t]]$ is finite over $\mathbb{C}\left[\left[t^S\right]\right]$. Thus $\mathbb{C}[[t]]$ is generated by integral elements over $\mathbb{C}\left[\left[t^S\right]\right]$ by Theorem B.11, and hence $\mathbb{C}[[t]]$ is integral over $\mathbb{C}\left[\left[t^S\right]\right]$ by Theorem B.10. Since $\mathbb{C}[[t]]$ is integrally closed in $\mathbb{C}[[t]]\left[t^{-1}\right]$, this implies $\overline{\mathbb{C}[[t^S]]} = \mathbb{C}[[t]]$. Moreover, Theorem B.14 yields $\dim \mathbb{C}\left[\left[t^S\right]\right] = \dim \mathbb{C}[t] = 1$. ☐

Lemma 4.85. *Let S be a numerical semigroup. Then $\mathbb{C}\left[\left[t^S\right]\right]$ is local with maximal ideal $\mathfrak{m} = \left\{x \in \mathbb{C}\left[\left[t^S\right]\right] \mid \operatorname{ord}_t(x) > 0\right\}$. Moreover, if M is a finite set of generators of S (see Proposition 4.72) not containing 0, then*

$$\mathfrak{m} = \langle t^\alpha \mid \alpha \in M \rangle.$$

Proof. By Lemma 4.84 we have $\overline{\mathbb{C}[[t^S]]} = \mathbb{C}[[t]]$, and $\mathbb{C}[[t]]$ is local with maximal ideal $t\mathbb{C}[[t]]$. Thus, the statement follows from Propositions B.3 and B.15.

By Remark 4.6.(2) S is local, and hence $\mu_{M_S} = \min\{\alpha \in S \mid \alpha > 0\}$. Since M is a set of generators of S, this implies $\mu_{M_S} \in M$. Moreover, $\langle t^\alpha \mid \alpha \in M \rangle \subset \mathfrak{m}$, and for any $\alpha \in M_S = S \setminus \{0\}$ there are $\beta_1^{(\alpha)}, \ldots, \beta_{n_\alpha}^{(\alpha)} \in M$ with $n_\alpha \geq 1$ such that $\alpha = \sum_{i=1}^{n_\alpha} \beta_i^{(\alpha)}$.

Let now $\sum_{\alpha \in S} a_\alpha t^\alpha \in \mathfrak{m}$. Then $a_0 = 0$, and we can write

$$\begin{aligned}
\sum_{\alpha \in M_S} a_\alpha t^\alpha &= \sum_{\substack{\alpha \in S \\ \alpha < \mu_{M_S} + \gamma_S}} a_\alpha t^\alpha + \sum_{\substack{\alpha \in S \\ \alpha \geq \mu_{M_S} + \gamma_S}} a_\alpha t^\alpha \\
&= \sum_{\substack{\alpha \in S \\ \alpha < \mu_{M_S} + \gamma_S}} a_\alpha t^{\sum_{i=1}^{n_\alpha} \beta_i^{(\alpha)}} + t^{\mu_{M_S}} \sum_{\substack{\alpha \in S \\ \alpha \geq \mu_{M_S} + \gamma_S}} a_\alpha t^{\alpha - \mu_{M_S}} \\
&= \sum_{\substack{\alpha \in S \\ \alpha < \mu_{M_S} + \gamma_S}} a_\alpha \prod_{i=1}^{n_\alpha} t^{\beta_i^{(\alpha)}} + t^{\mu_{M_S}} \sum_{\substack{\alpha \in S \\ \alpha \geq \mu_{M_S} + \gamma_S}} a_\alpha t^{\alpha - \mu_{M_S}}.
\end{aligned}$$

Since $a_\alpha \in \mathbb{C}$ for all $\alpha \in S$, since $t^{\mu_{M_S}}, t^{\beta_i^{(\alpha)}} \in \mathfrak{m}$ for every $\alpha \in M$ and for all $i = 1, \ldots, n_\alpha$, and since $\sum_{\substack{\alpha \in S \\ \alpha \geq \mu_{M_S} + \gamma_S}} a_\alpha t^{\alpha - \mu_{M_S}} \subset t^\gamma \mathbb{C}[[t]] \subset \mathbb{C}\left[\left[t^S\right]\right]$, this yields the claim. ☐

Lemma 4.86. *Let S be a numerical semigroup. Then $\mathbb{C}\left[\left[t^S\right]\right]$ is Cohen–Macaulay.*

Proof. Since $\dim \mathbb{C}\left[\left[t^S\right]\right] = 1$ by Lemma 4.84, since $\mathbb{C}\left[\left[t^S\right]\right]$ is Noetherian by Corollary 4.83, and since $\mathbb{C}\left[\left[t^S\right]\right]$ is reduced by definition, the statement follows from Proposition C.13. ☐

Proof of Proposition 4.80. By Lemmas 4.84 and 4.86 $\mathbb{C}\left[\left[t^S\right]\right]$ is a one-dimensional ring. Moreover, $\mathbb{C}\left[\left[t^S\right]\right]$ is by construction reduced and complete, hence it is analytically reduced. By Lemma 4.85 $\mathbb{C}\left[\left[t^S\right]\right]$ is local with maximal ideal

$$\mathfrak{m} = \left\{x \in \mathbb{C}\left[\left[t^S\right]\right] \,\middle|\, \operatorname{ord}_t(x) > 0\right\}.$$

This implies

$$\mathbb{C}\left[\left[t^S\right]\right]/\mathfrak{m} \cong \mathbb{C} \cong \mathbb{C}[[t]]/t\mathbb{C}[[t]].$$

Therefore, $\mathbb{C}\left[\left[t^S\right]\right]$ is residually rational as $\overline{\mathbb{C}[[t^S]]} = \mathbb{C}[[t]]$ by Lemma 4.84, and $\mathbb{C}[[t]]$ is local with maximal ideal $t\mathbb{C}[[t]]$. Since $\operatorname{char}\left(\mathbb{C}\left[\left[t^S\right]\right]/\mathfrak{m}\right) = \operatorname{char}(\mathbb{C}) = 0$, $\mathbb{C}\left[\left[t^S\right]\right]$ has a large residue field. Thus, $\mathbb{C}\left[\left[t^S\right]\right]$ is admissible.

Obviously, ord_t is a valuation of $Q_{\mathbb{C}[[t^S]]} = \mathbb{C}[[t]][t^{-1}]$ with $\operatorname{ord}_t(x) \geq 0$ for all $x \in \mathbb{C}\left[\left[t^S\right]\right]$. Since $\mathbb{C}\left[\left[t^S\right]\right]$ is analytically irreducible, we have $\left|\mathcal{V}_{\mathbb{C}[[t^S]]}\right| = 1$, see Remark 3.39 (Equation (3.19)). Thus, $\mathcal{V}_{\mathbb{C}[[t^S]]} = \{\operatorname{ord}_t\}$, and the valuation ring of the valuation ord_t is

$$\left\{x \in Q_{\mathbb{C}[[t^S]]} \,\middle|\, \operatorname{ord}_t(x) \geq 0\right\} = \mathbb{C}[[t]].$$

This implies

$$\Gamma_{\mathbb{C}[[t^S]]} = \operatorname{ord}_t\left(\mathbb{C}\left[\left[t^S\right]\right] \setminus \{0\}\right) = S. \qquad \square$$

5. Duality and Gorenstein Property

The canonical module ω_R of a Cohen–Macaulay ring R is characterized by the duality

$$M \mapsto \operatorname{Ext}_R^{\dim R - \dim M}(M, \omega_R)$$

on the Cohen–Macaulay modules of R (see Theorem C.22). Equivalently, there is a duality on the maximal Cohen–Macaulay modules of R given by

$$M \mapsto \operatorname{Hom}_R(M, \omega_R). \tag{5.1}$$

If R is generically Gorenstein, then ω_R can by Proposition C.23 be identified with a (regular) fractional ideal $\mathfrak{K}$ of R. If R is one-dimensional, then all regular fractional ideals of R are maximal Cohen–Macaulay modules. Therefore, Equation (5.1) induces with Lemma 2.13 a duality

$$\mathfrak{I} \mapsto \mathfrak{K} : \mathfrak{I}$$

on $\mathcal{R}_R$. This leads to the definition of a *canonical ideal* of a one-dimensional Cohen–Macaulay ring as a dualizing object on the fractional ideals, i.e. a regular fractional ideal $\mathfrak{K}$ of R such that

$$\mathfrak{I} = \mathfrak{K} : (\mathfrak{K} : \mathfrak{I})$$

for all $\mathfrak{I} \in \mathcal{R}_R$. In fact, a canonical ideal of a one-dimensional Cohen–Macaulay ring R is a canonical module of R (see Section 5.1).

This Chapter is dedicated to a combinatorial version of this duality on the good semigroup ideals of a good semigroup and its relation to the duality on fractional ideals. In Section 5.2 we define a *canonical semigroup ideal* K of a good semigroup S as a dualizing object on the good semigroup ideals of S, i.e. $K - E$ is a good semigroup ideal, and

$$E = K - (K - E) \tag{5.2}$$

for every good semigroup ideal E of S. Moreover, if R is an admissible ring, then canonical ideals of its semigroup of values Γ_R characterize the canonical (fractional) ideals of R in terms of their value semigroup ideals (see Section 5.3). This unifies and extends results by D'Anna [8] and Pol [14].

A Cohen–Macaulay ring R is by Theorem C.26 a Gorenstein ring if and only if R is a canonical module of R. Historically, the first step in describing the value semigroup ideals of canonical ideals was a characterization of the semigroups of values of Gorenstein rings.

Kunz showed that an analytically irreducible and residually rational one-dimensional local ring R is Gorenstein if and only if its (numerical) semigroup of values is symmetric [6], i.e. if and only if

$$\Gamma_R = \{\alpha \in D_{\Gamma_R} \mid \tau_{\Gamma_R} - \alpha \notin \Gamma_R\}. \tag{5.3}$$

Jäger used this symmetry condition to define a canonical semigroup ideal

$$K = \{\alpha \in D_{\Gamma_R} \mid \tau_{\Gamma_R} - \alpha \notin \Gamma_R\}$$

such that a fractional ideal $\mathfrak{K}$ of R with $R \subset \mathfrak{K} \subset \overline{R}$ is a canonical ideal of R if and only if $\Gamma_{\mathfrak{K}} = K$ [11].

Waldi was the first to describe a symmetry property of the semigroup of values of a plane algebroid curve with two branches [3]. Note that plane algebroid curves are always Gorenstein (see [1, Corollary 5.2.9]). Delgado extended this symmetry to plane algebroid curves with arbitrarily many branches [12]. Later he generalized the symmetry of numerical semigroups to good semigroups (see Definition 5.36), and in analogy to Kunz' result he characterized Gorenstein algebroid curves by the symmetry of their semigroups of values [7]. In his setup the symmetry of the semigroup of values of an algebroid curve can be written as

$$\Gamma_R = \left\{\alpha \in D_{\Gamma_R} \;\middle|\; \Delta^{\Gamma_R}(\tau_{\Gamma_R} - \alpha) = \emptyset\right\} \tag{5.4}$$

(see Definition 4.31). Note that in the irreducible case Equation (5.4) reduces to Equation (5.3). Later Campillo, Delgado, and Kiyek extended Delgado's result to include analytically reduced and residually rational local rings with infinite residue field [13].

Starting from this result D'Anna followed Jäger's approach by turning Delgado's symmetry condition into an explicit formula for a canonical semigroup ideal

$$K_S^0 = \left\{\alpha \in D_S \;\middle|\; \Delta^S(\tau_S - \alpha) = \emptyset\right\}$$

of a good semigroup S (see Definition 5.8). In analogy to Jäger's result he showed that a fractional ideal $\mathfrak{K}$ of an analytically reduced and residually rational one-dimensional local ring R (having arbitrarily many branches) with $R \subset \mathfrak{K} \subset \overline{R}$ is a canonical ideal of R if and only if $\Gamma_{\mathfrak{K}} = K^0_{\Gamma_R}$ [8]. In Section 5.2 we give an intrinsic definition of a canonical ideal of a good semigroup (see Definition 5.10). For a good semigroup ideal of a good semigroup this definition is equivalent to satisfying the duality of Equation (5.2) and to being a shift of D'Anna's K^0 (see Theorem 5.14).

In Section 5.3 we relate the duality on good semigroups to the duality on fractional ideals. We show that D'Anna's characterization of canonical ideals by their value semigroup ideal applies also for admissible rings. Moreover, with our definition of a canonical semigroup ideal allowing for shifts we can prove that any fractional ideal $\mathfrak{K}$ of an admissible ring R is a canonical ideal of R if and only if its value semigroup ideal $\Gamma_{\mathfrak{K}}$ is a canonical ideal of the semigroup of values Γ_R of R (see Theorem 5.31).

While giving a further characterization of local Gorenstein algebroid curves, Pol computed explicitly the value semigroup ideal of the dual $R : \mathfrak{J}$ of a fractional ideal $\mathfrak{J}$ of a Gorenstein algebroid curve R [33, 14]. Using Delgado's characterization of Gorenstein algebroid curves in terms of their semigroups of values Pol's formula can be written as

$$\Gamma_{R:\mathfrak{J}} = \Gamma_R - \Gamma_{\mathfrak{J}}. \tag{5.5}$$

Since R is Gorenstein, it is a canonical ideal of itself. Therefore, Γ_R is a canonical semigroup ideal of itself. Using properties of canonical semigroup ideals one can prove that

Equation (5.5) is valid in any admissible ring if R is replaced by a canonical ideal $\mathfrak{K}$ of R (see Theorem 5.34). This shows that the duality on good semigroup ideals is compatible with the duality on fractional ideals under taking values in the following sense: a regular fractional ideal $\mathfrak{K}$ of an admissible ring is a canonical ideal of R if and only if its value semigroup ideal $\Gamma_{\mathfrak{K}}$ is a canonical ideal of Γ_R, and we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{R}_R \arrow[r, "\mathfrak{I}\mapsto\mathfrak{K}:\mathfrak{I}"] \arrow[d, "\mathfrak{I}\mapsto\Gamma_{\mathfrak{I}}"'] & \mathcal{R}_R \arrow[d, "\mathfrak{I}\mapsto\Gamma_{\mathfrak{I}}"] \\ \mathcal{G}_{\Gamma_R} \arrow[r, "E\mapsto\Gamma_{\mathfrak{K}}-E"'] & \mathcal{G}_{\Gamma_R}. \end{tikzcd}$$

As a consequence of this result we extend in Section 5.4 Delgado's and Pol's characterizations of Gorensteinness from local algebroid curves to admissible rings (see Corollaries 5.37 and 5.41).

With a view towards the Grauert–Remmert algorithm for normalization presented in Section B.5.2 (also see Chapter 8) we study in Section 5.6 the endomorphism ring $\mathfrak{m}_R : \mathfrak{m}_R$ for a local Gorenstein algebroid curve R with maximal ideal $\mathfrak{m}_R$. We show that $\mathfrak{m}_R : \mathfrak{m}_R$ is Gorenstein if and only if R is of type $\mathrm{A_n}$ (see [22]) for some $n \in \mathbb{N}$ (see Theorem 5.56). In the proof we use the corresponding statement for good semigroups: a good local semigroup S and $M_S - M_S$ are symmetric if and only if S is the semigroup of values of an algebroid curve of type $\mathrm{A_n}$ (see Theorem 5.42).

5.1. Cohen–Macaulay Duality on One-dimensional Rings

Let R be a one-dimensional equidimensional Cohen–Macaulay ring. Then Equation (5.1) and Lemma 2.13 lead to the following definition of a dualizing object on $\mathcal{R}_R$.

Definition 5.1. Let R be a one-dimensional equidimensional Cohen–Macaulay ring. A regular fractional ideal $\mathfrak{K} \in \mathcal{R}_R$ is called a *canonical (fractional) ideal* of R if

$$\mathfrak{I} = \mathfrak{K} : (\mathfrak{K} : \mathfrak{I})$$

for all $\mathfrak{I} \in \mathcal{R}_R$.

Being a canonical ideal is a local property in the following sense.

Lemma 5.2 (See [25], Lemma 5.1.3)**.** *Let R be a one-dimensional equidimensional Cohen–Macaulay ring, and let $\mathfrak{K} \in \mathcal{R}_R$. Then R is a canonical ideal of R if and only if $\mathfrak{K}_{\mathfrak{m}} = \mathfrak{K}R_{\mathfrak{m}} \in \mathcal{R}_{R_{\mathfrak{m}}}$ is a canonical ideal of $R_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Max}\,(R)$.*

Proof. This follows from Lemma 2.16.(2) and (3) and Proposition A.39 since equality is a local property (see [24, Lemma 2.6]). □

In fact, if a one-dimensional Cohen–Macaulay ring R has a canonical ideal $\mathfrak{K}$, then $\mathfrak{K}$ is a canonical module of R.

Remark 5.3. Let R be a one-dimensional equidimensional Cohen–Macaulay ring. Then a canonical ideal of R is a canonical module of R, see [25, Remark 5.1.4].

Canonical ideals are unique "up to multiplication by units".

Proposition 5.4. *Let R be a one-dimensional equidimensional Cohen–Macaulay ring, and let $\mathfrak{K}$ be a canonical ideal of R. Then $\mathfrak{K}' \in \mathcal{R}_R$ is a canonical ideal of R if and only if $\mathfrak{K}' = \mathfrak{I}\mathfrak{K}$ for some invertible fractional ideal $\mathfrak{I}$ of R. If R is semilocal, then $\mathfrak{K}'$ is a canonical ideal of R if and only if $\mathfrak{K}' = x\mathfrak{K}$ for some $x \in Q_R^{\mathrm{reg}}$.*

Proof. See [25, Proposition 5.1.5]. □

The existence of canonical ideals for one-dimensional Cohen–Macaulay rings can be characterized as follows.

Theorem 5.5. *A one-dimensional local Cohen–Macaulay ring R has a canonical ideal if and only if $\widehat{R}$ is generically Gorenstein. In particular, any one-dimensional analytically reduced local ring has a canonical ideal.*

Proof. See [24, Korollar 2.12 and Satz 6.21]. □

Note that the particular claim of Theorem 5.5 includes local admissible rings (see Definition 3.18). Moreover, for a local admissible ring we can choose a "normalized" canonical ideal.

Corollary 5.6. *Any one-dimensional analytically reduced local Cohen–Macaulay ring R with large residue field has a canonical ideal $\mathfrak{K}$ such that $R \subset \mathfrak{K} \subset \overline{R}$. It is unique up to multiplication by $\overline{R}^*$ with unique value semigroup ideal.*

Proof. See [25, Corollary 5.1.7]. □

Finally, as in Theorem C.21 canonical ideals propagate along finite ring extensions.

Lemma 5.7. *Let R and R' be one-dimensional local Cohen–Macaulay rings, and let $\phi\colon R \to R'$ be a local homomorphism such that R' is a finite R-module and $Q_R = Q_{R'}$. If $\mathfrak{K}$ is a canonical ideal of R, then $\mathfrak{K} : R'$ is a canonical ideal of R'.*

Proof. See [25, Lemma 5.1.8]. □

5.2. Duality on Good Semigroups

Motivated by a result by Jäger in the irreducible case (see [11, Hilfssatz 5]) D'Anna introduced the following semigroup ideal (see [8, Section 3]) based on a symmetry condition on the semigroup of values of Gorenstein algebroid curves by Delgado (see [7, Theorem 2.8]) to characterize canonical ideals in terms of their value semigroup ideals (see [8, Theorem 4.1] and Theorem 5.30).

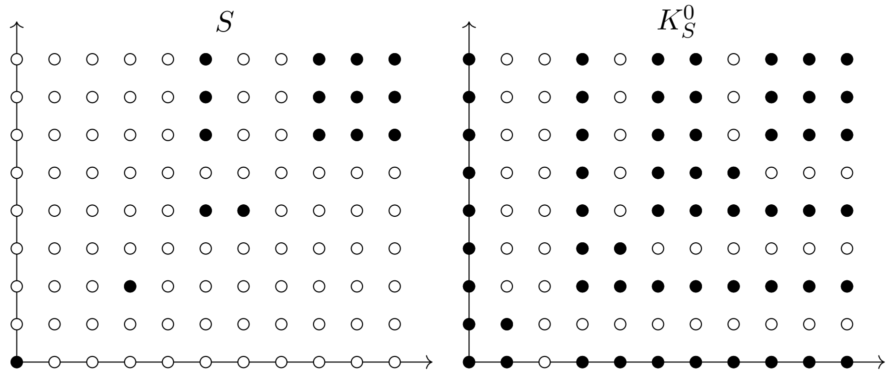


Figure 5.1.: A good semigroup S with canonical ideal K_S^0.

Definition 5.8. Let S be a good semigroup. The set

$$K_S^0 = \left\{\alpha \in D_S \;\middle|\; \Delta^S\left(\tau_S - \alpha\right) = \emptyset\right\}$$

is called the *(normalized) canonical (semigroup) ideal* of S (see Figure 5.1).

Lemma 5.9 (see [25], Lemma 5.2.2)**.** *Let S be a good semigroup.*

(1) The set K_S^0 is a semigroup ideal of S satisfying property (E1)

(2) The minimal element of K_S^0 is $\mu_{K_S^0} = \mu_S = \mathbf{0}$. In particular, $S \subset K_S^0 \subset \overline{S}$.

(3) The conductor of K_S^0 is $\gamma_{K_S^0} = \gamma_S$.

Proof. (1) See [8, Proposition 3.2].

(2) Since K_S^0 is a semigroup ideal of S satisfying property (E1) by (1), it has by Lemma 4.12 a minimal element. By Lemma 4.34 we have

$$\Delta^S\left(\tau_S - \mathbf{0}\right) = \Delta^S\left(\tau_S\right) = \emptyset,$$

and hence $\mathbf{0} \in K_S^0$ by Definition 5.8.

Now assume there is $\alpha \in K_S^0$ with $\alpha \not\geq \mathbf{0}$. Then there is $i \in I$ such that $\alpha_i < 0$. Using (1) to apply property (E1) in K_S^0 to α and $\mathbf{0}$ yields a $\beta \in K_S^0$ with $\beta < \mathbf{0}$ and $\beta_i < 0$. Therefore, $\left(\tau_S - \beta\right)_i \geq \gamma_S$, and hence $\Delta_i\left(\tau_S - \beta\right) \neq \emptyset$. This implies $\beta \neq K_S^0$, contradicting the assumption $\alpha \in K_S^0$. The particular claim follows from (1) and Lemma 4.14.

(3) By (1) and Lemmas 4.12 and 4.18, K_S^0 also has a conductor. Since $K_S^0 + S \subset K_S^0$ by (1) and $0 \in K_S^0$ by (2), we have $\gamma_{K_S^0} \leq \gamma_S$.

Now let $\alpha \in D_S$ with $\alpha \geq \gamma_S$. Then $\tau_S - \alpha < \mathbf{0}$, and hence $\Delta^S\left(\tau_S - \alpha\right) = \emptyset$ since $\mu_S = \mathbf{0}$. This implies $\gamma_{K_S^0} \geq \gamma_S$. $\square$

The following definition of a canonical semigroup ideal relies on the inclusion relations of good semigroup ideals and avoids a fixed conductor.

Definition 5.10. Let S be a good semigroup. A good semigroup ideal $K \in \mathcal{G}_S$ is called a *canonical (semigroup) ideal* of S if $K \subset E$ implies $K = E$ for all $E \in \mathcal{G}$ with $\gamma_K = \gamma_E$.

Remark 5.11 (See [25], Remark 5.2.6). If K is a canonical ideal of S, then also $\alpha + K$ is a canonical ideal of S for any $\alpha \in D_S$. This follows immediately from Definition 5.10 and Remark 4.21.(1).

Proposition 5.12. *Let S be a good semigroup. Then for any $\alpha \in D_S$ there is a canonical ideal K of S having conductor $\gamma_K = \alpha$.*

Proof. First we show that there is a canonical ideal K of S with conductor $\gamma_K = \gamma_S$. By Remark 4.6.(3) we have $S \in \mathcal{G}_S$. So there is a good semigroup ideal of S with conductor γ_S. Now assume that S does not have a canonical ideal with conductor γ_S. Then for any $E \in \mathcal{G}_S$ with $\gamma_E = \gamma_S$ there is an $E' \in \mathcal{G}_S$ with $\gamma_{E'} = \gamma_S$ and $E \subsetneq E'$. Then starting with some $E^{(0)} \in \mathcal{G}_S$ with $\gamma_{E^{(0)}} = \gamma_S$ we find a sequence

$$\left(E^{(i)}\right)_{i\in\mathbb{N}} \in (\mathcal{G}_S)^{\mathbb{N}},$$

where $\gamma_{E^{(i)}} = \gamma_S$ and $E^{(i)} \subset E^{(i+1)}$ for all $i \in \mathbb{N}$. This yields

$$d\left(E^{(i+1)} \setminus E^{(i)}\right) > 0 \tag{5.6}$$

for all $i \in \mathbb{N}$ by Remark 4.47.(2) and Proposition 4.50. Moreover, Lemma 4.29 implies

$$\mu_{E^{(i)}} \geq \gamma_{E^{(i)}} - \gamma_S = 0$$

for all $i \in \mathbb{N}$, and hence $E^{(i)} \subset \overline{S}$.

Therefore, we obtain with Equation (5.6), Lemma 4.49 and Remark 4.47.(2) for any $i \in \mathbb{N}$

$$d\left(\overline{S} \setminus E^{(i)}\right) < d\left(\overline{S} \setminus E^{(i)}\right) - d\left(E^{(i+1)} \setminus E^{(i)}\right) = d\left(\overline{S} \setminus E^{(i+1)}\right) \in \mathbb{N}. \tag{5.7}$$

Then by induction there is a $j \in \mathbb{N}$ such that

$$d\left(\overline{S} \setminus E^{(j)}\right) < 0,$$

contradicting Equation (5.7). Thus, there is a canonical ideal K of S with conductor $\gamma_K = \gamma_S$.

Set $\beta = \alpha - \gamma_S$. Then $\beta + K$ is by Remark 5.11 a canonical ideal of S with conductor $\gamma_{\beta+K} = \beta + \gamma_K = \alpha + \gamma_S - \gamma_S = \alpha$. ☐

The following result was stated by Barucci, D'Anna, and Fröberg in for the normalized canonical semigroup ideal (see [10, Proposition 2.15]).

Proposition 5.13 (See [25], Proposition 5.2.5)**.** *Let S be a good semigroup, and let $S = \prod_{m\in M} S_m$ be the decomposition of S into local good semigroups (see Theorem 4.9). Then a good semigroup ideal $K \in \mathcal{G}_S$ is a canonical ideal of S if and only if K_m is a canonical ideal of S_m for every $m \in M$.*

Proof. First note that $K_m \in \mathcal{G}_{S_m}$ for any $m \in M$ by Theorem 4.9. Suppose that K is a canonical ideal of S. Let $m \in M$, and assume that K_m is not a canonical ideal of S_m. Then there is an $E_m \in \mathcal{G}_{S_m}$ with $\gamma_{E_m} = \gamma_{K_m}$ and $K_m \subsetneq E_m$. Now Lemmas 4.11.(4) and 4.28 yield

$$E = E_m \times \prod_{n \in M \setminus \{m\}} K_n \in \mathcal{G}_S$$

with $\gamma_E = \gamma_K$ and $K \subsetneq E$, contradicting E being a canonical ideal.

Suppose now that K_n is a canonical ideal of S_m for all $m \in M$. Let $E \in \mathcal{G}_S$ with $\gamma_E = \gamma_K$ and $E \subset K$. Then for every $m \in M$ Theorem 4.9 and Lemma 4.28 yield $E_m \in \mathcal{G}_{S_m}$ with $\gamma_{E_m} = \gamma_{K_m}$ and $K_m \subset E_m$. Since K_m is a canonical ideal of S_m, this implies $K_m = E_m$ for every $m \in M$, and hence $E = K$. Thus, K is a canonical ideal of S. □

Our aim in this section is to establish the following result on canonical semigroup ideals in analogy with the properties of canonical ideals of admissible rings (see Theorems C.20, C.21, and C.22).

Theorem 5.14 (See [25], Theorem 5.2.6)**.** *Any good semigroup S has a canonical ideal. Moreover, for any $K \in \mathcal{G}_S$ the following are equivalent:*

(a) *K is a canonical ideal of S.*

(b) *There is an $\alpha \in D_S$ such that $\alpha + K = K_S^0$.*

(c) *For all $E \in \mathcal{G}_S$ we have $K - (K - E) = E$.*

If K is a canonical ideal of S, then the following hold:

(1) *$S \subset K \subset \overline{S}$ if and only if $K = K_S^0$.*

(2) *If $E \in \mathcal{G}_S$, then $K - E \in \mathcal{G}_S$.*

(3) *$K - K = S$.*

(4) *If S' is a good semigroup with $S \subset S' \subset \overline{S}$, then $K' = K - S'$ is a canonical ideal of S'.*

Proof. For the existence of a canonical ideal see Proposition 5.19.

(a) $\Longrightarrow$ (b) See Proposition 5.19.

(b) $\Longrightarrow$ (c) See Corollary 5.28.

(c) $\Longrightarrow$ (a) See Proposition 5.24.

(1) See Corollary 5.20.

(2) See Corollary 5.21.

(3) See Corollary 5.29.

(4) See Corollary 5.23. ☐

In particular, the equivalent statements of Theorem 5.14 show that for a good semigroup S the normalized canonical ideal K_S^0 is a canonical ideal of S in the sense of Definition 5.10, and hence a good semigroup ideal of S. This was stated in [10, Proposition 2.14].

Remark 5.15 (See [25], Remark 5.2.7). As the example given in Figure 5.2 shows, the assumption $E \in \mathcal{G}_S$ in Theorem 5.14.(c) and (2) is necessary.

The rest of this section is concerned with the proof of Theorem 5.14. To keep the notations shorter, we will write τ for τ_S and γ for γ_S if S is a good semigroup.

First we deal with the statement of Theorem 5.14.(2) in the case $K = K_S^0$. For this we collect some properties of K_S^0.

Lemma 5.16 (See [25], Lemma 5.2.8)**.** *Let S be a good semigroup. Then the semigroup ideal K_S^0 of S (see Lemma 5.9.(1)) has the following properties:*

(1) If E is a semigroup ideal of S, then

$$K_S^0 - E = \left\{ \alpha \in D_S \;\middle|\; \Delta^E(\tau - \alpha) = \emptyset \right\}.$$

(2) $\Delta^{K_S^0}(\tau) = \emptyset$.

(3) If S is local (see Definition 4.5.(3)) and $|I| \geq 2$, then

$$\tau + \bigcup_{\substack{J \subset I \\ |J| \leq |I|-2}} \sum_{j \in J} \mathbb{N}\mathbf{e}_j \subset K_S^0.$$

(4) If S is local and $|I| \geq 3$, then

$$\tau + \bigcup_{i \in I} \mathbb{N}\mathbf{e}_i \subset K_S^0.$$

Proof. (1) See [8, Computation 3.3].

(2) Let $\alpha \in \Delta^{K_S^0}(\tau)$. Then there is $i \in I$ such that

$$\begin{aligned} &\alpha_i = \tau_i, \\ &\alpha_j > \tau_j \text{ for all } j \in I \setminus \{j\}. \end{aligned}$$

This implies

$$\begin{aligned} &\tau_i - \alpha_i = 0, \\ &\tau_j - \alpha_j < 0 \text{ for all } j \in I \setminus \{j\}, \end{aligned}$$

and hence

$$0 \in \Delta_i(\tau - \alpha).$$

Since also $0 \in S$, this yields $\Delta^S(\tau - \alpha) \neq \emptyset$. Therefore, $\alpha \notin K_S^0$ by Definition 5.8.

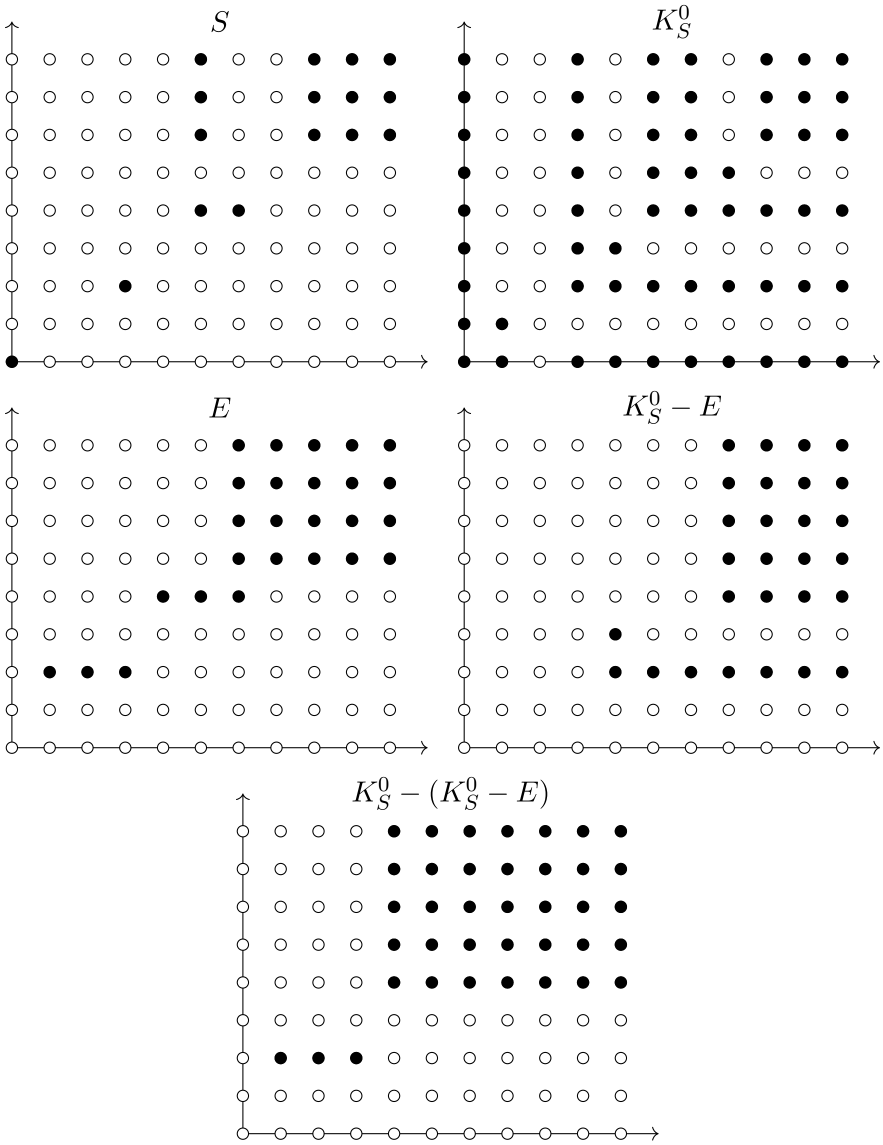


Figure 5.2.: A good semigroup S and semigroup ideal E of S satisfying property (E1) but not (E2), where $K_S^0 - E \notin \mathcal{G}_S$ and $E \subsetneq K_S^0 - \left(K_S^0 - E\right)$.

(3) Let S be a local good semigroup, and let

$$\alpha \in \bigcup_{\substack{J \subset I \\ |J| \leq |I|-2}} \tau + \sum_{j \in J} \mathbb{N}\mathbf{e}_j.$$

This means there is $J \subset I$ with $|J| \leq |I| - 2$, and for all $j \in J$ there are $n_j \in \mathbb{N}$ such that

$$\alpha = \tau + \sum_{j \in J} n_j \mathbf{e}_j.$$

This implies

$$\tau - \alpha = -\sum_{j \in J} n_j \mathbf{e}_j \leq 0.$$

Now assume there is

$$\beta \in \Delta^S (\tau - \alpha) \neq \emptyset,$$

i.e. there is $i \in I$ such that

$$\begin{aligned} \beta_i &= \tau_i - \alpha_i, \\ \beta_j &> \tau_j - \alpha_j \text{ for all } j \in I \setminus \{i\}. \end{aligned}$$

In particular, $\beta_i \leq 0$. Thus, $\beta \in S$ implies $\beta_i = 0$ as $\mu_s = \mathbf{0}$. Then $\beta = 0$ since S is local, and hence $\alpha_i = \tau_i$.

However, since $|I| \geq 2$ and $|J| \leq |I| - 2$, there is $k \in I \setminus \{i\}$ such that

$$\beta_k > \tau_k - \alpha_k = 0,$$

and hence

$$0 \notin \Delta^S (\tau - \alpha).$$

This yields

$$\Delta (\tau - \alpha) = \emptyset,$$

and thus $\alpha \in K_S^0$ by Definition 5.8.

(4) Let S be local, and let

$$\alpha \in \tau + \bigcup_{i \in I} \mathbb{N}\mathbf{e}_i.$$

Then there is $i \in I$ and $n \in \mathbb{N}$ such that $\alpha = \tau + n\mathbf{e}_i$. This implies

$$\tau - \alpha = -n\mathbf{e}_i.$$

Therefore,

$$\Delta_i^S (\tau - \alpha) = \Delta_i^S (-n\mathbf{e}_i) = \{\beta \in D_S \mid \beta_i = -n \text{ and } \beta_j > 0 \text{ for all } j \in I \setminus \{i\}\} = \emptyset$$

since $\mu_S = \mathbf{0}$, and

$$\Delta_j^S (\tau - \alpha) = \Delta_j^S (-n\mathbf{e}_i) = \emptyset$$

for all $j \in I \setminus \{i\}$ since S is local, and hence $\mathbf{0}$ is the only element of S with a zero component but

$$\mathbf{0} \notin \Delta_j^S(-n\mathbf{e}_i) = \{\beta \in D_S \mid \beta_j = 0, \beta_i > -n \text{ and } \beta_k > 0 \text{ for all } k \in I \setminus \{i,j\}\}.$$

Hence, $\alpha \in K_S^0$ by Definition 5.8. □

The proof of Theorem 5.14.(2) is obtained by the following Proposition. In particular, it shows that K_S^0 is a good semigroup ideal of S. D'Anna established a weaker statement, where (E2) is replaced by a certain property (E3) (see [8, Theorem 3.6]). This property (E3) follows from (E2) (see [8, Proposition 2.3]).

Proposition 5.17 (See [25], Proposition 5.2.9)**.** *Let S be a good semigroup. Then $K_S^0 - E \in \mathcal{G}_S$ for any $E \in \mathcal{G}_S$. In particular, $K_S^0 \in \mathcal{G}_S$.*

Proof. The idea of the following proof is illustrated in Figure 5.3.

Let $E \in \mathcal{G}_S$, and suppose that $K_S^0 - E \notin \mathcal{G}_S$. Since $K_S^0 - E$ is a semigroup ideal of S satisfying property (E1) by Lemmas 4.18 and 5.9.(1), it then has to violate property (E2). This means that there are $\alpha, \beta \in K_S^0 - E$ with

$$\emptyset \neq J = \{j \in I \mid \alpha_j \neq \beta_j\} \subset I$$

such that

$$\zeta^{(0)} = \inf\{\alpha, \beta\} \in K_S^0 - E,$$

and there is an $l_0 \in I \setminus J$ such that

$$\zeta \notin K_S^0 - E$$

for any $\zeta \in D_S$ with

$$\begin{aligned} \zeta_{l_0} &> \zeta_{l_0}^{(0)}, \\ \zeta_i &\geq \zeta_i^{(0)} \text{ for all } i \in I, \\ \zeta_j &\geq \zeta_j^{(0)} \text{ for all } j \in J. \end{aligned}$$

In particular, any choice of a sequence $(l_r)_{r \in \mathbb{N}}$ in $I \setminus J$ yields a sequence $\left(\zeta^{(r)}\right)_{r \in \mathbb{N}}$ in D_S with

$$\zeta^{(0)} \in K_S^0 - E, \tag{5.8}$$

$$\zeta^{(r)} = \zeta^{(r-1)} + \mathbf{e}_{l_{r-1}} \notin K_S^0 - E. \tag{5.9}$$

By Lemma 5.16.(1) this means that

$$\Delta^E\left(\tau - \zeta^{(0)}\right) = \emptyset, \tag{5.10}$$

and that for any $r \geq 1$ there is an $i \in I$ such that

$$\Delta_i^E\left(\tau - \zeta^{(r)}\right) \neq \emptyset. \tag{5.11}$$

We construct a sequence as above by induction on r with the additional property that for $r \geq 1$ we have

$$\Delta_j^E\left(\tau - \zeta^{(r)}\right) = \emptyset \tag{5.12}$$

for all $j \in J$, where in each step we pick an $l_r \in I \setminus J$ and a

$$\delta^{(r)} \in \Delta_{l_r}^E\left(\tau - \zeta^{(r)}\right). \tag{5.13}$$

Assume we have a sequence $(l_t)_{t=1,\ldots,r-1}$ in $I \setminus J$ satisfying Equations (5.8), (5.9), (5.10), (5.11), (5.12), and (5.13), and suppose there is a $j \in J$ such that

$$\Delta_j^E\left(\tau - \zeta^{(r)}\right) \neq \emptyset.$$

In particular, we then have $j \neq l_{r-1}$. By Equation (5.11), and since $\zeta^{(r)} = \zeta^{(r-1)} + \mathbf{e}_{l_{r-1}}$ (see Equation (5.9)), this implies that there is a

$$\begin{aligned}
\delta &\in \Delta_j^E\left(\tau - \zeta^{(r)}\right) &(5.14)\\
&= \Delta_j^E\left(\tau - \zeta^{(r-1)}\right) \cup \Delta_{\{j,l_{r-1}\}}^E\left(\tau - \zeta^{(r-1)}\right) &(5.15)\\
&= \Delta_{\{j,l_{r-1}\}}^E\left(\tau - \zeta^{(r-1)}\right), &(5.16)
\end{aligned}$$

where the union in Equation (5.15) is disjoint, and the equality in Equation (5.16) follows from the induction hypothesis $\Delta_j^E\left(\tau - \zeta^{(r-1)}\right) = \emptyset$ for all $j \in J$ (see Equation (5.12)). We deduce contradictions with different arguments for $r = 1$ and $r \geq 2$, respectively.

First consider the case $r = 1$. As $j \in J$, we may assume that

$$\beta_j > \alpha_j = \zeta_j^{(0)}, \tag{5.17}$$

and we have

$$\beta_{l_0} = \zeta_{l_0}^{(0)} \tag{5.18}$$

by the choice of $l_0 \in I \setminus J$. Since $\beta \in K_S^0 - E$ and $\delta \in E$, we get

$$\delta + \beta \in K_S^0. \tag{5.19}$$

Then Equations (5.14), (5.15), and (5.16) yield

$$\delta + \zeta^{(0)} \in \Delta_{\{j,l_0\}}(\tau),$$

and this implies with Equations (5.17), (5.18), and (5.19)

$$\delta + \beta \in \Delta_{l_0}^{K_S^0}(\tau),$$

contradicting Lemma 5.16.(2).

Assume now that $r \geq 2$. By Equations (5.13) and (5.16), and since $j \neq l_{r-1}$, we obtain

$$\delta_{l_{r-1}} = \tau_{l_{r-1}} - \zeta_{l_{r-1}}^{(r-1)} = \delta_{l_{r-1}}^{(r-1)}$$

and
$$\delta_j = \tau_j - \zeta_j^{(r-1)} < \delta_j(r-1).$$
Since $E \in \mathcal{G}_S$, property (E2) applied to $\delta^{(r-1)}, \delta^{(r)} \in E$ yields an $\epsilon \in E$ with
$$\begin{aligned}\epsilon_{l_{r-1}} &> \delta^{(r-1)}_{l_{r-1}} = \delta_{l_{r-1}},\\ \epsilon_i &\geq \min\left\{\delta_i^{(r-1)}, \delta_i\right\} \text{ for all } i \in I,\\ \epsilon_k &= \min\left\{\delta_k^{(r-1)}, \delta_k\right\} \text{ for all } k \in I \text{ with } \delta_k^{(r-1)} \neq \delta_k.\end{aligned}$$
In particular, we have
$$\epsilon_j = \delta_j = \tau_j - \zeta_j^{(r-1)}.$$
Then Equation (5.16) yields
$$\epsilon \in \Delta_j\left(\tau - \zeta^{(r-1)}\right),$$
contradicting the induction hypothesis (see Equation (5.12)).

Now pick
$$r > \sum_{k \in I \setminus J} \left|\tau_k - \zeta_k^{(1)} - (\mu_E)_k\right|.$$
Then Equation (5.13) yields
$$\delta^{(r)}_{l_r} = \tau_{l_r} - \zeta^{(r)}_{l_r} < (\mu_E)_{l_r}.$$
Since $\delta^{(r)} \in E$ by Equation (5.11), this contradicts the minimality of μ_E in E. Thus, it follows that $K_S^0 - E \in \mathcal{G}_S$.

In particular, since K_S^0 is a semigroup ideal of S by Lemma 5.9.(1), $\mathbf{0} \in S$, and $S \in \mathcal{G}_S$ by Remark 4.6.(3), this yields
$$K_S^0 = K_S^0 - S \in \mathcal{G}_S. \qquad \square$$

We can now relate our definition of canonical semigroup ideals (see Definition 5.10) to D'Anna's definition of normalized canonical semigroup ideals (see Definition 5.8).

Lemma 5.18 (See [25], Proposition 5.2.10)**.** *Let S be a good semigroup. Then $E \subset K_S^0$ for any $E \in \mathcal{G}_S$ with $\gamma_E = \gamma$.*

Proof. Let $E \in \mathcal{G}_S$ with conductor $\gamma_E = \gamma$, and assume there is a $\beta \in E \setminus K_S^0$. Then Definition 5.8 implies that there is a $\delta \in \Delta^S(\tau - \beta)$. Hence, $\beta + \delta \in \Delta^E(E)$. However, this contradicts Lemma 4.34, and therefore $E \subset K_S^0$. $\square$

Proposition 5.19 (See Theorem 5.14.(a) $\Longrightarrow$ (b) and [25], Proposition 5.2.10)**.** *Let S be a good semigroup, and let $K \in \mathcal{G}_S$. Then K is a canonical ideal of S if and only if there is an $\alpha \in D_S$ such that $K = \alpha + K_S^0$. In particular, for any $\delta \in D_S$, there is a unique canonical ideal K of S with $\gamma_K = \delta$.*

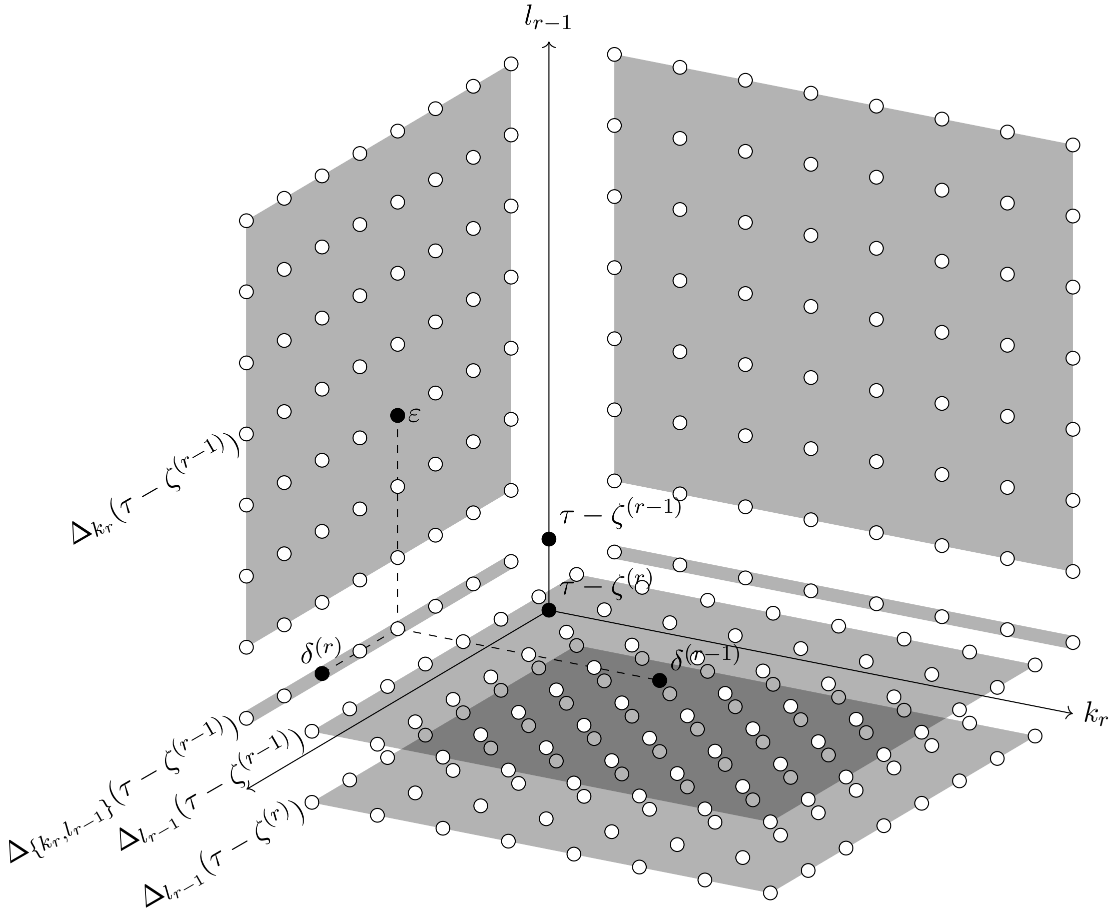


Figure 5.3.: Induction step in the proof of Proposition 5.17 in case $I \setminus J = \{l_{r-1}\}$.

Proof. Using Remark 5.11, it suffices to show that K_S^0 is the unique canonical ideal of S with conductor $\gamma_{K_S^0} = \gamma$ (see Lemma 5.9.(3)). For any $E \in \mathcal{G}_S$ with $\gamma_E = \gamma$, Lemma 5.18 yields $E \subset K_S^0$. Since $K_S^0 \in \mathcal{G}_S$ by Proposition 5.17, this implies that K_S^0 is a canonical ideal of S.

If $K \in \mathcal{G}_S$ is a canonical ideal of S with $\gamma_K = \gamma$, then Lemma 5.18 yields $K \subset K_S^0$, and hence $K = K_S^0$ by Definition 5.10. Thus, K_S^0 is the unique canonical ideal of S with conductor $\gamma_{K_S^0} = \gamma$. □

As a consequence we deduce the combinatorial counterpart of Lemma 5.7 on good semigroups.

Corollary 5.20 (See Theorem 5.14.(1))**.** *Let S be a good semigroup. If K is a canonical ideal of S with $S \subset K \subset \overline{S}$, then $K = K_S^0$.*

Proof. By Proposition 5.19 there is an $\alpha \in D_S$ such that $K = \alpha + K_S^0$. Then Lemma 5.9.(2) yields $\mu_K = \alpha + \mu_{K_S^0} = \alpha$. Since $S \subset K \subset \overline{S}$, we have

$$\mathbf{0} = \mu_S \geq \mu_K + \alpha \geq \mu_{\overline{S}} = \mathbf{0}.$$

Thus, $K = K_S^0$. □

The relation of Proposition 5.19 between general canonical ideals of S and the normalized canonical ideal of S allows for deducing the statements of Theorem 5.14 from results on K_S^0.

Corollary 5.21 (See Theorem 5.14.(2))**.** *Let S be a good semigroup, and let K be a canonical ideal of S. Then $K - E \in \mathcal{G}_S$ for all $E \in \mathcal{G}_S$.*

Proof. By Proposition 5.19 there is an $\alpha \in D_S$ such that $K = \alpha + K_S^0$. Then

$$K - E = \left(\alpha + K_S^0\right) - E = \alpha + \left(K_S^0 - E\right)$$

by Remark 4.21.(2). Since $K_S^0 - E \in \mathcal{G}_S$ by Proposition 5.17, Remark 4.21.(1) yields $K - E \in \mathcal{G}_S$. □

Corollary 5.22. *Let S be a good semigroup, and let K be a canonical ideal of S. Then $E \subset K$ for all $E \in \mathcal{G}_S$ with $\gamma_E = \gamma_K$.*

Proof. This follows from Remark 4.21.(1), Lemma 5.18 and Proposition 5.19. □

Corollary 5.23 (See Theorem 5.14.(4) and [25], Corollary 5.2.11)**.** *Let S and S' be good semigroups such that $S \subset S' \subset \overline{S}$. If K is a canonical ideal of S, then $K' = K - S'$ is a canonical ideal of S'.*

Proof. By Remark 4.6.(3) we have $S' \in \mathcal{G}_S$, and Proposition 5.19 implies $K = \alpha + K_S^0$ for some $\alpha \in D_S$. Then Lemma 5.16.(1) and Remark 4.21.(2) yield

$$\begin{aligned} K' &= \left(\alpha + K_S^0\right) - S' \\ &= \alpha + \left(K_S^0 - S'\right) \\ &= \alpha + \left\{\beta \in D_S \;\middle|\; \Delta^{S'}(\tau - \beta) = \emptyset\right\} \\ &= \alpha + \tau - \tau_{S'} + \left\{\delta \in D_S \;\middle|\; \Delta^{S'}(\tau_{S'} - \delta) = \emptyset\right\} \\ &= \alpha + \tau - \tau_{S'} + K_{S'}^0. \end{aligned}$$

Thus, K' is a canonical ideal of S' by Proposition 5.19. □

In the following two Propositions 5.24 and 5.26 we establish an equivalent definition of canonical semigroup ideals (see Theorem 5.14.(c)) which corresponds to the definition of canonical fractional ideals (see Definition 5.1).

Proposition 5.24 (See Theorem 5.14.(c) $\Longrightarrow$ (a) and [25], Proposition 5.2.13)**.** *Let S be a good semigroup. If $K \in \mathcal{G}_S$ with $K - (K - E) = E$ for all $E \in \mathcal{G}_S$, then K is a canonical ideal of S.*

Proof. Assume that K is not a canonical ideal of S. Then there is an $E \in \mathcal{G}_S$ with $\gamma_E = \gamma_K$ and $K \subsetneq E$ (see Definition 5.10). Then Lemma 4.40.(2) yields the contradiction

$$E \subsetneq K - (K - E) = E.$$

Hence, K is a canonical ideal of S. □

Lemma 5.25 (See [25], Lemma 5.2.14)**.** *Let S be a good semigroup, let E be a semigroup ideal of S, and let $\alpha \in K_S^0 - \left(K_S^0 - E\right)$. If $\zeta \in D_S$ satisfies $\Delta^E(\tau - \zeta) = \emptyset$, then*

$$\Delta^S(\tau - \zeta - \alpha) = \emptyset.$$

Equivalently, if $\beta \in D_S$ satisfies $\Delta^S(\tau - \beta) \neq \emptyset$, then

$$\Delta^E(\tau - \beta + \alpha) \neq \emptyset.$$

Proof. Using Lemma 5.16.(1) we have

$$\begin{aligned} K_S^0 - \left(K_S^0 - E\right) &= \left\{\alpha \in D_S \;\middle|\; \alpha + \left(K_S^0 - E\right) \subset K_S^0\right\} \\ &= \left\{\alpha \in D_S \;\middle|\; \alpha + \left\{\zeta \in D_S \;\middle|\; \Delta^E(\tau - \zeta) = \emptyset\right\} \subset K_S^0\right\} \\ &= \left\{\alpha \in D_S \;\middle|\; \Delta^S(\tau - \zeta - \alpha) = \emptyset \text{ for all } \zeta \in D_S \text{ with } \Delta^E(\tau - \zeta) = \emptyset\right\}. \end{aligned}$$

Thus, if $\zeta \in D_S$ satisfies $\Delta^E(\tau - \zeta) = \emptyset$, then

$$\Delta^S(\tau - \zeta - \alpha) = \emptyset$$

for all $\alpha \in K_S^0 - (K_S^0 - E)$. By setting $\zeta = \beta - \alpha$ for $\alpha \in K_S^0 - (K_S^0 - E)$, we obtain equivalently

$$\Delta^E (\tau - \beta + \alpha) \neq \emptyset$$

if $\beta \in D_S$ satisfies $\Delta^S (\tau - \beta) \neq \emptyset$. □

Proposition 5.26 (See [25], Proposition 5.2.15)**.** *Let S be a good semigroup. Then*

$$K_S^0 - \left(K_S^0 - E\right) = E$$

for any $E \in \mathcal{G}_S$. In particular, $K_S^0 - K_S^0 = S$.

Proof. Note that the inclusion $E \subset K_S^0 - (K_S^0 - E)$ holds trivially by Lemma 4.40.(1). So assume that

$$E \subsetneq K_S^0 - \left(K_S^0 - E\right).$$

By Lemmas 4.18 and 5.9.(1), $K_S^0 - (K_S^0 - E)$ is a semigroup ideal of S satisfying property (E1), and hence it also satisfies property (E0) by Remark 4.6.(1). Thus, there is an

$$\alpha \in \left(K_S^0 - \left(K_S^0 - E\right)\right) \setminus E$$

which is minimal with respect to the partial order on D_S.

Since E satisfies property (E1), and since $\alpha \notin E$, there is a $k \in I$ such that no $\epsilon \in E$ satisfies

$$\epsilon_k = \alpha_k, \tag{5.20}$$
$$\epsilon_i \geq \alpha_i \text{ for all } i \in I \setminus \{k\}. \tag{5.21}$$

We set $\beta = \gamma - \mathbf{e}_k$, i.e.

$$\beta_k = \tau_k, \tag{5.22}$$
$$\beta_i = \gamma_i \text{ for all } i \in I \setminus \{k\}. \tag{5.23}$$

Then $\mathbf{0} \in \Delta_k^S (\tau - \beta) \neq \emptyset$, and Lemma 5.25 yields a

$$\zeta \in \Delta^E (\tau - \beta + \alpha) \neq \emptyset.$$

This means there is a $j \in I$ such that $\zeta \in E$ with

$$\zeta_j = \tau_j - \beta_j + \alpha_j,$$
$$\zeta_i > \tau_i - \beta_i + \alpha_i \text{ for all } i \in I \setminus \{j\}.$$

Now $j = k$ contradicts the choice of k as then, using Equations (5.22) and (5.23),

$$\zeta_k = \alpha_k,$$
$$\zeta_i \geq \alpha_i \text{ for all } i \in I \setminus \{k\},$$

see Equations (5.20) and (5.21) with $\epsilon = \zeta$. Thus, we have $j \neq k$ and

$$\zeta_j = \alpha_j - 1, \tag{5.24}$$
$$\zeta_k > \alpha_k, \tag{5.25}$$
$$\zeta_i \geq \alpha_i \text{ for all } i \in I \setminus \{j, k\}. \tag{5.26}$$

Since $\zeta \in E \subset K_S^0 - (K_S^0 - E)$ by Lemma 4.40.(1), $\alpha \in K_S^0 - (K_S^0 - E)$ by assumption, and since $K_S^0 - (K_S^0 - E)$ satisfies property (E1) by Lemmas 4.18 and 5.9.(1), we obtain, using Equations (5.24), (5.25), and (5.26),

$$\alpha > \alpha - \mathbf{e}_j = \inf\{\alpha, \zeta\} \in K_S^0 - \left(K_S^0 - E\right).$$

Now set $\alpha' = \inf\{\alpha, \zeta\}$, and assume that $\alpha' \in E$. Then applying property (E2) to α' and ζ in E yields an $\epsilon \in E$ with

$$\begin{aligned}
&\epsilon_j > \alpha' = \zeta = \alpha - 1, \\
&\epsilon_k = \min\{\alpha'_k, \zeta_k\} = \alpha_k, \\
&\epsilon_i \geq \min\{\alpha'_i, \zeta_i\} = \alpha_i \text{ for all } i \in I \setminus \{j, k\}.
\end{aligned}$$

However, this is a contradiction to the choice of k, see Equations (5.20) and (5.21). Thus,

$$\alpha > \alpha' \in \left(K_S^0 - \left(K_S^0 - E\right)\right) \setminus E,$$

contradicting the minimality of α. Therefore, we obtain

$$K_S^0 - \left(K_S^0 - E\right) = E.$$

Then setting $E = S$ the particular claim follows from Remark 4.21.(3) and (4) and Lemma 5.9.(1). □

Remark 5.27. In case $|I| = 1$, there is an easier proof of Proposition 5.26: Let $\alpha \in (K_S^0 - (K_S^0 - E)) \setminus E$, and set $\beta = \tau$. Since

$$\Delta^S(\tau - \beta) = \Delta(0) \cap S = \{0\} \cap S = \{0\} \neq \emptyset,$$

Lemma 5.25 yields

$$\emptyset \neq \Delta^E(\tau - \beta + \alpha) = \Delta(\alpha) \cap E = \{\alpha\} \cap E,$$

and hence $\alpha \in E$. Thus, $E = K_S^0 - (K_S^0 - E)$.

Corollary 5.28 (See Theorem 5.14.(b) $\Longrightarrow$ (c))**.** *Let $\alpha \in D_S$, and let $K = \alpha + K_S^0 \in \mathcal{G}_S$ (see Remark 4.21.(1) and Proposition 5.17). Then*

$$K - (K - E) = E$$

for any $E \in \mathcal{G}_S$.

Proof. By Remark 4.21.(2) and Proposition 5.26 we have

$$\begin{aligned} K-(K-E) &= \left(\alpha+K_S^0\right)-\left(\left(\alpha+K_S^0\right)-E\right) \\ &= \alpha-\alpha+\left(K_S^0-\left(K_S^0-E\right)\right) \\ &= E. \end{aligned}$$ □

Corollary 5.29 (See Theorem 5.14.(3))**.** *Let S be a good semigroup, and let K be a canonical ideal of S. Then $K-K=S$.*

Proof. By Proposition 5.19 there is $\alpha \in D_S$ such that $K=\alpha+K_S^0$. Since $S \in \mathcal{G}_S$ by Definition 4.5, Remark 4.21.(4) and Corollary 5.28 yield

$$K-K=K-(K-S)=S.$$ □

5.3. Relation of Dualities

In this section we relate the duality on good semigroup ideals (see Section 5.2) to the Cohen–Macaulay duality on fractional ideals (see Section 5.1). D'Anna characterized normalized canonical ideals of a local admissible ring in the following way.

Theorem 5.30. *Let R be a local admissible ring. Then a regular fractional ideal $\mathfrak{K}$ of R is canonical if and only if $\Gamma_{\mathfrak{K}}=K^0_{\Gamma_R}$ (see Definition 5.8).*

Proof. See [8, Theorem 4.1]. □

Note that $K^0_{\Gamma_R}$ is a canonical semigroup ideal of Γ_R by Theorem 5.14. We extend Theorem 5.30 to admissible rings dropping the normalization of canonical ideals.

Theorem 5.31 (See [25], Theorem 5.3.2)**.** *Let R be an admissible ring. Then $\mathfrak{K} \in \mathcal{R}_R$ is a canonical ideal of R if and only if $\Gamma_{\mathfrak{K}}$ is a canonical ideal of Γ_R (see Definition 5.10).*

Proof. First suppose that R is local. By Proposition 5.4 and Corollary 5.6 $\mathfrak{K}$ is a canonical ideal of R if and only if there is an $x \in Q_R^{\mathrm{reg}}$ such that $x\mathfrak{K}$ is a canonical ideal of R with $R \subset x\mathfrak{K} \subset \overline{R}$. By Theorem 5.30 this is equivalent to

$$K^0_{\Gamma_R}=\Gamma_{x\mathfrak{K}}=\nu(x)+\Gamma_{\mathfrak{K}}.$$

By Theorem 5.14.(a) $\Longleftrightarrow$ (b) this is the case if and only if $\Gamma_{\mathfrak{K}}$ is a canonical ideal of Γ_R.

Let now R be semilocal. By Lemma 5.2 $\mathfrak{K}$ is a canonical ideal of R if and only if $\mathfrak{K}_{\mathfrak{m}}$ is a canonical ideal of $R_{\mathfrak{m}}$ for every $\mathfrak{m} \in \operatorname{Max}(R)$. By Lemma 3.33 and the local case this is equivalent to $(\Gamma_{\mathfrak{K}})_{\mathfrak{m}}=\Gamma_{\mathfrak{K}_{\mathfrak{m}}}$ being a canonical ideal of $(\Gamma_R)_{\mathfrak{m}}=\Gamma_{R_{\mathfrak{m}}}$ (see Theorem 4.9 and Remark 4.10). By Remark 4.10 and Proposition 5.13 this is the case if and only if $\Gamma_{\mathfrak{K}}$ is a canonical ideal of Γ_R. □

Example 5.32. Barucci, D'Anna, and Fröberg gave in [10, Example 2.16] the following example of a good semigroup which is not the value semigroup of an admissible ring: Consider the good semigroup S depicted in Figure 5.4, and suppose that there is an admissible ring R with $\Gamma_R = S$. Then R is local by Proposition 3.17. Thus, there is by Corollary 5.6 a canonical ideal $\mathfrak{K}$ of R with $R \subset \mathfrak{K} \subset \overline{R}$. Theorem 5.30 yields $\Gamma_{\mathfrak{K}} = K_S^0$.

Consider the maximal chains

$$\begin{aligned}\{\mu_S = (0,0), (6,7), (9,7), (12,7),& \\ &(12,14), (15,14), (16,14), (18,14), (19,14), (20,14),\\ &(21,14), (22,14), (23,14), (24,14), (25,14), (26,14), (27,14),\\ &(27,15), (27,18), (27,19), (27,21), (27,22), (27,23), (27,25) = \gamma_S\}\end{aligned}$$

in S and

$$\begin{aligned}\Big\{\mu_{K_S^0} = (0,0), (6,7), (9,7), (12,7), (12,11),& \\ &(12,14), (13,14), (15,14), (16,14), (18,14), (19,14), (20,14),\\ &(21,14), (22,14), (23,14), (24,14), (25,14), (26,14), (27,14),\\ &(27,15), (27,18), (27,19), (27,21), (27,22), (27,23), (27,25) = \gamma_{K_S^0}\Big\}\end{aligned}$$

in K_S^0. Then Proposition 4.51 yields

$$\ell_R(\mathfrak{K}/R) = d\Big(K_S^0 \setminus S\Big) = 2.$$

Thus, there is an $\mathfrak{J} \in \mathcal{R}_R$ with $R \subsetneq \mathfrak{J} \subsetneq \mathfrak{K}$. Remark 3.15 and Corollary 4.52 imply

$$S = \Gamma_R \subsetneq \Gamma_{\mathfrak{J}} \subsetneq \Gamma_{\mathfrak{K}} = K_S^0. \tag{5.27}$$

However, it is easy to see that for any $E \in \mathcal{G}_S$ with $S \subset E$ which contains a point of $K_S^0 \setminus S$ we have $K_S^0 \subset E$. This is a contradiction to Equation (5.27) since $\Gamma_{\mathfrak{J}} \in \mathcal{G}_S$ by Proposition 3.22.(4).

Let R be a local plane algebroid curve, and let $\mathfrak{J} \in \mathcal{R}_R$. Pol gave an explicit formula for the value semigroup ideal $\Gamma_{R:\mathfrak{J}}$ of the dual $R : \mathfrak{J}$ of $\mathfrak{J}$.

Theorem 5.33. *Let R be a local plane algebroid curve. Then*

$$\Gamma_{R:\mathfrak{J}} = \Big\{\alpha \in D_{\Gamma_R} \;\Big|\; \Delta^{\Gamma_{\mathfrak{J}}}(\tau_{\Gamma_R} - \alpha) = \emptyset\Big\}. \tag{5.28}$$

Proof. See [33, Theorem 2.4]. □

Replacing $\mathfrak{J}$ by R in Equation (5.28), Theorem 5.33 implies that for a local plane algebroid curve R we have

$$\Gamma_R = \Big\{\alpha \in D_{\Gamma_R} \;\Big|\; \Delta^{\Gamma_R}(\tau_{\Gamma_R} - \alpha) = \emptyset\Big\} = K_{\Gamma_R}^0 \tag{5.29}$$

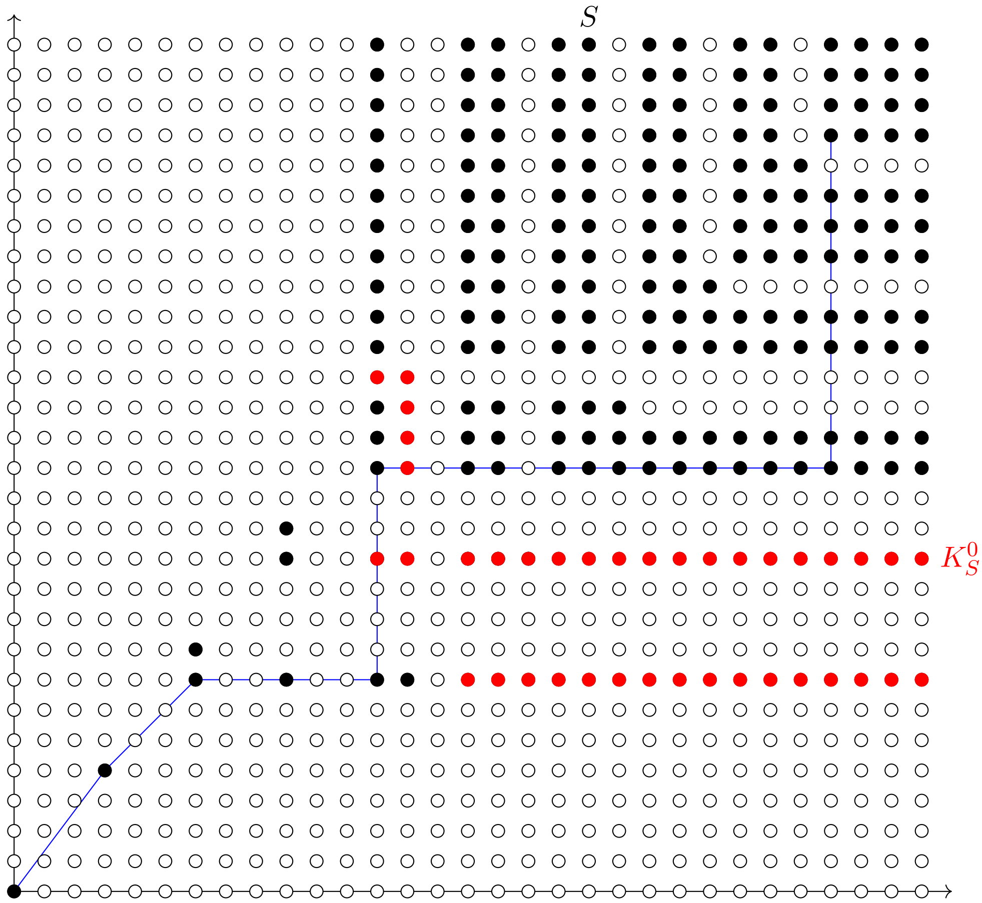


Figure 5.4.: A good semigroup S which is not the semigroup of values of an admissible ring (see Example 5.32). The canonical ideal K_S^0 of S consists of S together with the red points. The distance $d\big(K_S^0 \setminus S\big) = 2$ can be computed along the blue path. Moreover, using properties (E1) and (E2) we see that any good semigroup ideal E of S which contains a point of $K_S^0 \setminus S$ has to contain K_S^0. See [10, Example 2.16].

(see Definition 5.8) for the second equality). In fact, due to Delgado Equation (5.29) characterizes Gorensteinness of local algebroid curves (see [7, Theorem 2.8]). Then Lemma 5.16.(1), Theorem 5.33, and Equation (5.29) imply

$$\Gamma_{R:\mathfrak{I}} = K^0_{\Gamma_R} - \Gamma_{\mathfrak{I}} = \Gamma_R - \Gamma_{\mathfrak{I}}. \tag{5.30}$$

Note that R is a canonical ideal by Theorem 5.31. We extend Equation (5.30) to admissible rings replacing R by a canonical ideal $\mathfrak{K}$ of R.

Theorem 5.34. *Let R be an admissible ring with canonical ideal $\mathfrak{K}$. Then*

$$\Gamma_{\mathfrak{K}:\mathfrak{I}} = \Gamma_{\mathfrak{K}} - \Gamma_{\mathfrak{I}}$$

for any $\mathfrak{I} \in \mathcal{R}_R$, and

$$d(\Gamma_{\mathfrak{K}} - \Gamma_{\mathfrak{I}} \setminus \Gamma_{\mathfrak{K}} - \Gamma_{\mathfrak{J}}) = d(\Gamma_{\mathfrak{J}} \setminus \Gamma_{\mathfrak{I}})$$

for any $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_R$ with $\mathfrak{I} \subset \mathfrak{J}$.

Proof. See [25, Theorem 5.3.5]. □

5.4. Gorenstein Property and Symmetry of Good Semigroups

In this Section we give a characterization of Gorenstein (see Definition C.24) admissible rings in terms of their semigroup of values. A Cohen–Macaulay is by Theorem C.26 Gorenstein if and only if it is a canonical module of itself.

Let R be a one-dimensional Cohen–Macaulay ring. If R is a canonical module of R, then it is a canonical ideal of R. Since, moreover, any canonical ideal of R is a canonical module of R by Remark 5.3, this yields the following characterization of one-dimensional Gorenstein rings in terms of canonical ideals.

Theorem 5.35. *A one-dimensional Cohen–Macaulay ring R is Gorenstein if and only if R is a canonical ideal.*

Proof. See [24, Korollar 3.4]. □

Theorem 5.31 leads to the following definition for good semigroups.

Definition 5.36. A good semigroup S is called *symmetric* if S is a canonical ideal of itself, i.e. if

$$S = K^0_S = \left\{\alpha \in D_S \mid \Delta^S(\tau_S - \alpha) = \emptyset\right\}$$

(see Theorem 5.14.(1)).

This symmetry condition was introduced by Kunz in the irreducible case (see [6]), and by Delgado for algebroid curves with arbitrarily many branches (see [7, Theorem 2.8]), to characterize Gorenstein curves. Here we extend this result to admissible rings.

Corollary 5.37 (See [25], Proposition 5.3.6)**.** *Let R be an admissible ring. Then R is Gorenstein if and only if Γ_R is symmetric.*

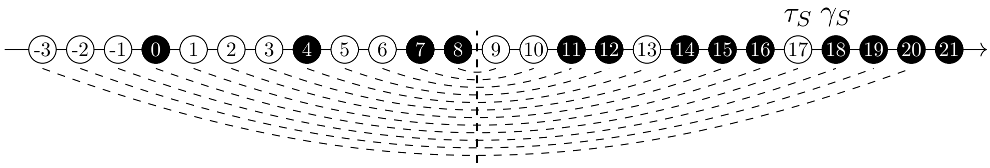


Figure 5.5.: The symmetric semigroup $S = \langle 4, 7\rangle$.

Proof. Gorensteinness of R is by Theorem 5.35 equivalent to R being a canonical ideal of R, and hence to Γ_R being a canonical semigroup ideal of Γ_R by Theorem 5.31. □

Remark 5.38. Let S be a good semigroup. If $|I| = 1$, then

$$S = \left\{\alpha \in D_S \;\middle|\; \tau_S - \alpha \notin S\right\}.$$

So the symmetry condition above indeed means a symmetry of gaps and non-gaps in the semigroup, see Example 5.39 below.

Example 5.39. Let $S = \langle 4, 7\rangle$. Then for any $\alpha \in D_S$ we have $\alpha \in S$ if and only if $\tau_S - \alpha \notin S$. So S is symmetric according to Definition 5.36, and there is a symmetry of gaps and non-gaps of S, see Figure 5.5.

Example 5.40. Consider the admissible ring $R = \mathbb{C}[[x, y]]/\langle x^5 y - y^3\rangle \cong \mathbb{C}[[(t_1^2, t_2), (t_1^5, 0)]]$. Then Γ_R is symmetric (see Figure 5.6), and hence R is Gorenstein by Corollary 5.37. This follows also from R being a plane algebroid curve (see [1, Corollary 5.2.9]).

Pol generalized Theorem 5.33 showing that local Gorenstein algebroid curves are characterized by satisfying Equation (5.28) for every regular fractional ideal (see [14, Théorème 5.2.1]). We extend Pol's result to admissible rings.

Corollary 5.41 (See [25], Proposition 5.3.7)**.** *Let R be an admissible ring. Then R is Gorenstein if and only if*

$$\Gamma_{R:\mathfrak{I}} = \Gamma_R - \Gamma_{\mathfrak{I}} = \left\{\alpha \in D_{\Gamma_R} \;\middle|\; \Delta^{\Gamma_R}(\tau_{\Gamma_R} - \alpha) = \emptyset\right\} \tag{5.31}$$

for any $\mathfrak{I} \in \mathcal{R}_R$.

Proof. Suppose that R is Gorenstein. Then Γ_R is a canonical ideal by Corollary 5.37. Thus, Lemma 5.16.(1) and Theorems 5.14.(1) and 5.34 yield

$$\Gamma_{R:\mathfrak{I}} = \Gamma_R - \Gamma_{\mathfrak{I}} = K^0_{\Gamma_R} - \Gamma_{\mathfrak{I}} = \left\{\alpha \in D_{\Gamma_R} \;\middle|\; \Delta^{\Gamma_R}(\tau_{\Gamma_R} - \alpha) = \emptyset\right\}$$

for every $\mathfrak{I} \in \mathcal{R}_{\mathfrak{I}}$.

Conversely, suppose that Equation (5.31) is satisfied for every $\mathfrak{I} \in \mathcal{R}_R$. Since $R \in \mathcal{R}_R$ with $R : R = R$ (see Section 2.1), this implies

$$\Gamma_R = \left\{\alpha \in D_{\Gamma_R} \;\middle|\; \Delta^{\Gamma_R}(\tau_{\Gamma_R} - \alpha) = \emptyset\right\} = K^0_{\Gamma_R}.$$

Thus, R is Gorenstein by Corollary 5.37. □

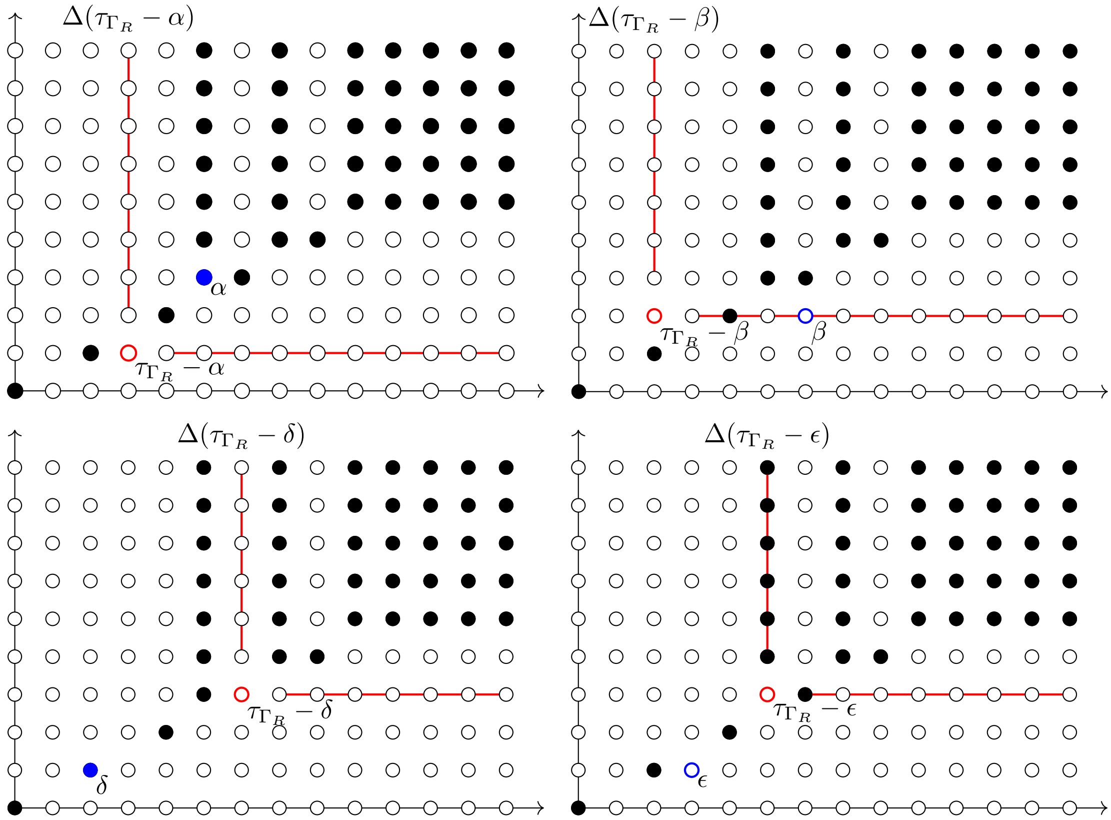


Figure 5.6.: The admissible ring $\mathbb{C}[[x,y]]/\langle x^5y - y^3\rangle \cong \mathbb{C}[[(t_1^2, t_2), (t_1^5, 0)]]$ of Example 5.40 is Gorenstein with symmetric semigroup of values Γ_R (see Corollary 5.37). For instance, we have $\alpha \in \Gamma_R$ and $\Delta^{\Gamma_R}(\tau_{\Gamma_R} - \alpha) = \emptyset$, $\beta \notin \Gamma_R$ and $\Delta^{\Gamma_R}(\tau_{\Gamma_R} - \beta) \neq \emptyset$, $\delta \in \Gamma_R$ and $\Delta^{\Gamma_R}(\tau_{\Gamma_R} - \delta) = \emptyset$, and $\epsilon \notin \Gamma_R$ and $\Delta^{\Gamma_R}(\tau_{\Gamma_R} - \epsilon) \neq \emptyset$.

5.5. Symmetric Semigroups

In this section we study local symmetric semigroups. A local semigroup S has by Lemma 4.7 a maximal ideal $M_S \in \mathcal{G}_S$ (see Definition 4.5.(3)). We show that for a local symmetric semigroup S the semigroup ideal $M_S - M_S$ is a good semigroup with $S \subset M_S - M_S \subset \overline{M_S - M_S} = \overline{S}$ (see Proposition 5.43).

As the main result of this section we give a characterization of the case when $M_S - M_S$ is also a symmetric semigroup.

Theorem 5.42. *Let S be a local good semigroup. Then the following are equivalent:*

(a) S and $M_S - M_S$ are symmetric semigroups.

(b) We have $|I| \leq 2$. If $|I| = 1$, then there is an $n \in 2\mathbb{N}$ such that

$$S = \langle 2, n+1 \rangle,$$

and if $|I| = 2$, then there is an $n \in 1 + 2\mathbb{N}$ such that

$$\begin{aligned} S &= \langle (1)_{i \in I} \rangle \cup \left(\left(\frac{n+1}{2} \right)_{i \in I} + \mathbb{N}^I \right) \\ &\cong \langle (1,1) \rangle \cup \left(\left(\frac{n+1}{2}, \frac{n+1}{2} \right) + \mathbb{N}^2 \right). \end{aligned}$$

Proposition 5.43. *Let S be a local symmetric semigroup. Then $M_S - M_S \in \mathcal{G}_S$, and $M_S - M_S$ is a good semigroup with $D_{M_S - M_S} = D_S$ and $S \subset M_S - M_S \subset \overline{M_S - M_S} = \overline{S}$.*

Proof. Since S is a local symmetric semigroup, Proposition 4.38 and Theorem 5.14.(2) yield

$$M_S - M_S = S - M_S \in \mathcal{G}_S.$$

Then $M_S - M_S$ satisfies property (E0) by Remark 4.6.(1) and properties (E1) and (E2) by definition. Moreover, $M_S - M_S$ is by Lemma 4.7 and Proposition 4.25 a partially ordered cancellative commutative monoid with $D_{M_S - M_S} = D_S$ and $S \subset M_S - M_S \subset \overline{M_S - M_S} = \overline{S}$. $\square$

In the remainder of the section we prove Theorem 5.42. First we show the implication (b) $\Longrightarrow$ (a) in Proposition 5.45, then we show the implication (a) $\Longrightarrow$ (b) for the case $|I| = 1$ in Proposition 5.47 and for the case $|I| = 2$ in Proposition 5.52.

Lemma 5.44. *Let S be a local symmetric semigroup. Then*

$$M_S - M_S = S \cup \Delta(\tau_S).$$

Proof. By Proposition 4.25 we have $S \subset M_S - M_S$. Since S is local, we have $\mu_{M_S} \geq \mathbf{1}$ (see Lemma 4.7 and Definition 4.13). Then Lemma 4.35 yields

$$\gamma_{M_S - M_S} = \gamma_{M_S} - \mu_{M_S} \leq \gamma_S - \mathbf{1} = \tau_S,$$

and hence $\Delta(\tau_S) \subset M_S - M_S$.

Assume now that there is an

$$\alpha \in (M_S - M_S) \setminus (S \cup \Delta(\tau_S)).$$

Since S is symmetric, this implies that there is a

$$\beta \in \Delta^S(\tau_S - \alpha).$$

Therefore, we have $\alpha + \beta \in \Delta(\tau_S)$.

As $S \cup \Delta(\tau_S) = S \cup \left(\tau_S + \overline{S}\right)$ by Lemma 5.16.(3) and (4), we have $\alpha \leq \tau_S - \mathbf{1}$. Therefore, $\beta \in M_S$. This yields the contradiction

$$\alpha + \beta \in \Delta(\tau_S) \cap M_S \subset \Delta^S(\tau_S) = \emptyset,$$

where the last equality follows from Lemma 5.16.(2) since S is symmetric. Also see [8, Lemma 3.5]. $\square$

Next we show the implication (b) $\Longrightarrow$ (a) of Theorem 5.42.

Proposition 5.45. *Let S be a good semigroup.*

(1) If

$$S = \langle 2, n+1 \rangle$$

for some $n \in 2\mathbb{N}$, then

$$M_S - M_S = \langle 2, n-1 \rangle.$$

In particular, S and $M_S - M_S$ are symmetric semigroups.

(2) If

$$S = \langle (1,1) \rangle \cup \left(\left(\frac{n+1}{2}, \frac{n+1}{2} \right) + \mathbb{N}^2 \right)$$

for some $n \in 1 + 2\mathbb{N}$, then

$$M_S - M_S = \langle (1,1) \rangle \cup \left(\left(\frac{n-1}{2}, \frac{n-1}{2} \right) + \mathbb{N}^2 \right).$$

In particular, S and $M_S - M_S$ are symmetric semigroups.

Proof. (1) Obviously, $\gamma_S = n$, and hence $\tau_S = n - 1$. Then Lemma 5.44 yields

$$M_S - M_S = S \cup \{\tau_S\} = \begin{cases} \langle 2, n-1 \rangle & \text{if } n > 0 \\ \overline{S} = \mathbb{N} & \text{if } n = 0. \end{cases}$$

Clearly, S and $M_S - M_S$ are symmetric.

(2) Obviously,
$$\gamma_S = \left(\frac{n+1}{2}, \frac{n+1}{2}\right),$$
and hence
$$\tau_S = \gamma_S - \mathbf{1} = \left(\frac{n+1}{2}, \frac{n+1}{2}\right) - (1,1) = \left(\frac{n-1}{2}, \frac{n-1}{2}\right).$$
Then Lemma 5.44 yields
$$M_S - M_S = S \cup \Delta\left(\tau_S\right) = \langle(1,1)\rangle \cup \left(\left(\frac{n-1}{2}, \frac{n-1}{2}\right) + \mathbb{N}^2\right).$$
Clearly, S and $M_S - M_S$ are symmetric semigroups. $\square$

The following statement is well-known, see for example [6] or [1, Theorem 5.2.4].

Lemma 5.46. *Let S be a local symmetric semigroup with $|I| = 1$. Then*
$$\gamma_S = 2d\left(\overline{S} \setminus S\right).$$

Proof. First note that $|I| = 1$ implies
$$d\left(\overline{S} \setminus S\right) = \left|\overline{S} \setminus S\right|.$$
Now for any $\alpha \in D_S$ we have by Remark 5.38 and Definitions 4.31 and 5.8 $\alpha \in S$ if and only if $\tau_S - \alpha \notin S$ since S is symmetric. As $C_S = \gamma_S + \overline{S} \subset S$ and $\gamma = \tau + \mathbf{1}$, this yields
$$\left|\overline{S} \setminus S\right| = \left|S \setminus C_S\right|.$$
Therefore, $\gamma_S = 2d\left(\overline{S} \setminus S\right)$. $\square$

We can already prove the converse of Proposition 5.45.(1).

Proposition 5.47. *Let S be a local symmetric semigroup with $|I| = 1$. If $M_S - M_S$ is a symmetric semigroup, then there is an $n \in 2\mathbb{N}$ such that*
$$S = \langle 2, n+1\rangle.$$

Proof. By Lemma 5.44 we have $M_S - M_S = S \cup \{\tau_S\}$. This implies
$$d\left(\overline{S} \setminus (M_S - M_S)\right) = d\left(\overline{S} \setminus S\right) - 1.$$
So if $M_S - M_S$ is a symmetric semigroup, Lemma 5.46 yields
$$\gamma_{M_S - M_S} = 2d\left(\overline{S} \setminus (M_S - M_S)\right) = 2\left(d\left(\overline{S} \setminus S\right) - 1\right) = \gamma_S - 2.$$
Thus, we obtain by Lemmas 4.7 and 4.35
$$2 = \gamma_S - \gamma_{M_S - M_S} = \mu_{M_S} \in M_S \subset S.$$
This implies
$$S = \langle 2, \gamma_S + 1\rangle,$$
and $\gamma_S = 2d\left(\overline{S} \setminus S\right)$ is even by Lemma 5.46. $\square$

Lemma 5.48. *Let S be a local symmetric semigroup. If $M_S - M_S$ is a symmetric semigroup, then*
$$\mu_{M_S} + (M_S - M_S) = M_S.$$

Proof. Note that M_S satisfies property (E1) by Lemma 4.7 since S is local. Hence, μ_{M_S} is defined by Lemma 4.12.

Let $\beta \in D_S$. First assume that $\beta \in \mu_{M_S} + (M_S - M_S)$, i.e. $\beta - \mu_{M_S} \in M_S - M_S$. Since $M_S - M_S$ is symmetric, Remark 5.38, Definitions 4.31 and 5.8, and Lemma 4.7 yield

$$\Delta^{M_S - M_S}\left(\tau_{M_S - M_S} + \mu_{M_S} - \beta\right) = \emptyset.$$

By Lemma 4.35 we have

$$\tau_{M_S - M_S} + \mu_{M_S} = \gamma_{M_S - M_S} + \mu_{M_S} - \mathbf{1} = \gamma_S - \mathbf{1} = \tau_S,$$

and hence we obtain
$$\Delta^{M_S - M_S}\left(\tau_S - \beta\right) = \emptyset.$$

Since $S \subset M_S - M_S$ by Lemmas 4.7 and 4.20, this yields

$$\Delta^S\left(\tau_S - \beta\right) = \emptyset.$$

As S is symmetric, we obtain $\beta \in S$ by Remark 5.38 and Definitions 4.31 and 5.8. Moreover, $M_S - M_S \subset \overline{S}$ by Lemma 4.39.(2) since S is local, and hence $C_S \subset M_S \subset S$. This yields $\beta \geq \mu_{M_S}$, i.e. $\beta \in M_S$ by Definition 4.5.(3).

Suppose now that $\beta \notin \mu_{M_S} + M_S - M_S$, i.e. $\beta - \mu_{M_S} \notin M_S - M_S$. Since $M_S - M_S$ is symmetric, this implies

$$\Delta^{M_S - M_S}(\tau_{M_S - M_S} - (\beta - \mu_{M_S - M_S})) \neq \emptyset. \tag{5.32}$$

Since $S \subset M_S - M_S$ by Lemmas 4.7 and 4.20, and since

$$\mu_{M_S} = \gamma_S - (\gamma_S - \mu_{M_S}) = \gamma_S - (\gamma_{M_S} - \mu_{M_S}) = \gamma_S - \gamma_{M_S - M_S} = \tau_S - \tau_{M_S - M_S}$$

by Lemma 4.35, Equation (5.32) yields

$$\begin{aligned}\emptyset &\neq \Delta^{M_S - M_S}(\tau_{M_S - M_S} - (\beta - \mu_{M_S - M_S}))\\ &= \Delta^S(\tau_{M_S - M_S} - (\beta - (\tau_S - \tau_{M_S - M_S})))\\ &= \Delta^S(\tau_S - \beta).\end{aligned}$$

Since S is symmetric, this implies $\beta \notin S$, and hence $\beta \notin M_S$. Thus,

$$\mu_{M_S} + (M_S - M_S) = M_S. \qquad \square$$

Lemma 5.49. *Let S be a local symmetric semigroup, and let $\alpha \in M_S \setminus C_S$. If $M_S - M_S$ is a symmetric semigroup, then*
$$\alpha - \mu_{M_S} \in S.$$

Proof. By Lemmas 5.44 and 5.48 we have

$$\begin{aligned}\alpha \in M_S &= \mu_{M_S} + (M_S - M_S)\\ &= \mu_{M_S} + (S \cup \Delta(\tau_S))\\ &= (\mu_{M_S} + S) \cup \Delta(\tau_S + \mu_{M_S}).\end{aligned}$$

Assume $\alpha \in \Delta(\tau_S + \mu_{M_S})$. Then $\alpha \geq \gamma_S$ since $\mu_{M_S} \geq \mathbf{1}$ as S is local. But this is a contradiction to the choice of $\alpha \notin C_S$. Hence, $\alpha - \mu_{M_S} \in S$. $\square$

Lemma 5.50. *Let S be a local symmetric semigroup, and let $\alpha \in M_S \setminus C_S$. If $M_S - M_S$ is a symmetric semigroup, then there is an $n \in \mathbb{N}$ such that*

$$\alpha = n\mu_{M_S}.$$

In particular,

$$S = \langle \mu_{M_S} \rangle \cup C_S.$$

Proof. Since S is local, and since $C_S = \gamma_S + \overline{S}$ by Remark 4.27.(1), repeatedly applying Lemma 5.49 yields

$$\alpha - m\mu_{M_S} \in S \tag{5.33}$$

for all $m \in \mathbb{N}$ satisfying

$$(m-1)\,\mu_{M_S} < \alpha. \tag{5.34}$$

Since α is finite, there is

$$n = \max\left\{m \in \mathbb{N} \mid (m-1)\,\mu_{M_S} < \alpha\right\}.$$

Then

$$n\mu_{M_S} \not< \alpha \tag{5.35}$$

by definition, and

$$\alpha - n\mu_{M_S} \in S \tag{5.36}$$

by Equations (5.33) and (5.34). Since $\mu_S = \mathbf{0}$, Equation (5.36) implies $\alpha - n\mu_{M_S} \geq \mathbf{0}$. However, $\alpha - n\mu_{M_S} > \mathbf{0}$ contradicts the choice of n (see Equation (5.35)), and hence we obtain $\alpha - n\mu_{M_S} = \mathbf{0}$. $\square$

Lemma 5.51. *Let S be a local symmetric semigroup. If $M_S - M_S$ is a symmetric semigroup, then $|I| \leq 2$.*

Proof. Assume that $|I| \geq 3$. Then

$$\tau_S + \bigcup_{i \in I} \mathbb{N}\mathbf{e}_i \subset S$$

by Lemma 5.16.(4). Since

$$\left(\tau_S + \bigcup_{i \in I} \mathbb{N}\mathbf{e}_i\right) \cap C_S = \emptyset,$$

this is a contradiction to Lemma 5.50. $\square$

To complete the proof of Theorem 5.42 we show the converse of Proposition 5.45.(2).

Proposition 5.52. *Let S be a local symmetric semigroup with $|I| = 2$. If $M_S - M_S$ is a symmetric semigroup, then*

$$\mu_{M_S} = (1,1)\,.$$

Moreover, there is an $n \in 2\mathbb{N}+1$ such that

$$S = \langle (1,1) \rangle \cup \left(\left(\frac{n+1}{2}, \frac{n+1}{2} \right) + \mathbb{N}^2 \right).$$

Proof. Since $|I| = 2$, we may assume $I = 1, 2$. Note that M_S satisfies property (E1) by Lemma 4.7 since S is local. Hence, μ_{M_S} is defined by Lemma 4.12. Moreover, we have $\mu_{M_S} \geq \mathbf{1}$ by Definition 4.5.(3). Hence, Lemma 4.35 yields

$$\gamma_{M_S - M_S} = \gamma_{M_S} - \mu_{M_S} = \gamma_S - \mu_{M_S} \leq \gamma_S - \mathbf{1} = \tau_S. \tag{5.37}$$

Suppose that $\gamma_{M_S - M_S} < \tau_S$. Then

$$\Delta(\tau_S - \mathbf{1}) = (\tau_S - \mathbf{1}) + \bigcup_{i \in I} \mathbb{N}\mathbf{e}_i \subset M_S - M_S$$

since $|I| = 2$. However,

$$\Delta(\tau_S - \mathbf{1}) \not\subset S$$

by Lemma 5.50, and

$$\Delta(\tau_S - \mathbf{1}) \cap \Delta(\tau_S) = \emptyset$$

by Definition 4.31. Using Lemma 5.44 this yields the contradiction

$$\Delta(\tau_S - \mathbf{1}) \not\subset S \cup \Delta(\tau_S) = M_S - M_S.$$

Hence, we obtain with Equation (5.37)

$$\tau_S = \gamma_{M_S - M_S}.$$

So Lemma 4.35 yields

$$\mu_{M_S} = \gamma_S - \gamma_{M_S - M_S} = \gamma_S - \tau_S = \mathbf{1}. \tag{5.38}$$

Therefore,

$$S = \langle (1,1) \rangle \cup C_S \tag{5.39}$$

by Lemma 5.50.

Assume $(\gamma_S)_1 \neq (\gamma_S)_2$. Then without loss of generality

$$(\gamma_S)_1 < (\gamma_S)_2\,. \tag{5.40}$$

By Equation (5.39) we have

$$(\alpha, \alpha) \in S \tag{5.41}$$

for any

$$(\gamma_S)_1 \leq \alpha \in \mathbb{N}. \tag{5.42}$$

This implies by Lemma 4.33

$$(\alpha, \alpha) + \mathbb{N}\mathbf{e}_1 \subset S. \tag{5.43}$$

Let now $n \in \mathbb{N}$ and

$$\beta = (\alpha, \alpha + n) \in (\alpha, \alpha) + \mathbb{N}\mathbf{e}_2.$$

Then we have

$$(\alpha + n, \alpha + n) \in S$$

by Equations (5.41) and (5.42). Now $\alpha \geq (\gamma_S)_1$ and $\max\{\alpha + n, (\gamma_S)_2\} \geq (\gamma_S)_2$ imply

$$(\alpha, \max\{\alpha + n, (\gamma_S)_2\}) \in S.$$

Since S satisfies property (E1), this yields

$$\beta = \min\{(\alpha + n, \alpha + n), (\alpha, \max\{\alpha + n, (\gamma_S)_2\})\} \in S,$$

and hence

$$(\alpha, \alpha) + \mathbb{N}\mathbf{e}_2 \subset S. \tag{5.44}$$

Thus, Equations (5.43) and (5.44) imply

$$(\alpha, \alpha) + \bigcup_{i \in I} \mathbb{N}\mathbf{e}_i \subset S,$$

for any $(\gamma_S)_1 \leq \alpha \in \mathbb{N}$, and hence

$$((\gamma_S)_1, (\gamma_S)_1) + \mathbb{N}^I \subset S,$$

contradicting the assumption $(\gamma_S)_1 < (\gamma_S)_2$ (see Equation (5.40)).

Therefore, setting

$$n = (2\gamma_S)_1 - 1 \in 2\mathbb{N} + 1$$

(note that $\gamma_S \geq \mathbf{1}$ since S is local) we obtain

$$C_S = \left(\frac{n+1}{2}, \frac{n+1}{2}\right) + \mathbb{N}^2,$$

and hence

$$S = \langle(1,1)\rangle \cup \left(\left(\frac{n+1}{2}, \frac{n+1}{2}\right) + \mathbb{N}^2\right)$$

by Equation (5.39). □

Combining Propositions 5.45, 5.47, and 5.52 yields the proof of Theorem 5.42.

Proof of Theorem 5.42. (a) $\Longrightarrow$ (b) Let S be a local symmetric semigroup. If $M_S - M_S$ is a symmetric semigroup, then $|I| \leq 2$ by Lemma 5.51.

If $|I| = 1$, then by Proposition 5.47 there is an $n \in 2\mathbb{N}$ such that

$$S = \langle 2, n+1 \rangle.$$

If $|I| = 2$, then by Proposition 5.52 there is an $n \in 2\mathbb{N}+1$ such that

$$S = \langle (1,1) \rangle \cup \left(\left(\frac{n+1}{2}, \frac{n+1}{2} \right) + \mathbb{N}^2 \right).$$

(b) $\Longrightarrow$ (a) See Proposition 5.45. $\square$

5.6. Gorenstein Algebroid Curves

In Section 5.5 we characterized the class of good semigroups S satisfying the property that S and $M_S - M_S$ are symmetric (see Theorem 5.42). This class equals the class of semigroups of values of curve singularities of type $\mathrm{A_n}$ (see [34] and Proposition 5.54). Conversely, we show that having semigroup of values $\mathrm{A_n}$ determines an algebroid curve to be of type $\mathrm{A_n}$ (see Proposition 5.57).

In analogy to Theorem 5.42 we characterize the class of local algebroid curves R (with maximal ideal $\mathfrak{m}_R$) satisfying the property that R and $\mathfrak{m}_R : \mathfrak{m}_R$ are Gorenstein as the class of curve singularities of type $\mathrm{A_n}$ (see Theorem 5.56).

In dependence on the classification of singularities by Arnold (see [34, 22]) we use the following notation.

Definition 5.53. Let $n \in \mathbb{N}$.

(1) Let k be a field, and let R be an algebroid curve over k. Then R is said to be *of type* $\mathrm{A_n}$ if there is a surjective k-algebra homomorphism

$$\phi\colon k[[x,y]] \to R$$

with

$$\ker\phi = \left\langle x^2 - y^{n+1} \right\rangle.$$

(2) A good semigroup S is said to be *of type* $\mathrm{A_n}$ if

$$S = \begin{cases} \langle 2, n+1 \rangle & \text{if } n \in 2\mathbb{N} \text{ and } |I| = 1, \\ \langle (1)_{i\in I} \rangle \cup \left(\left(\frac{n+1}{2} \right)_{i \in I} + \mathbb{N}^I \right) & \text{if } n \in 2\mathbb{N}+1 \text{ and } |I| = 2. \end{cases}$$

We relate algebroid curves of type $\mathrm{A_n}$ to good semigroups of type $\mathrm{A_n}$.

Proposition 5.54. *Let k be a field, let R be an algebroid curve over k, and suppose that R is of type* $\mathrm{A_n}$ *for some $n \in \mathbb{N}$.*

(1) If n is even, then

$$R = k\left[\left[t^2, t^{n+1}\right]\right] \subset k[[t]] = \overline{R}.$$

(2) If n is odd, then

$$R = k\left[\left[(t_1, t_2), \left(-t_1^{\frac{n+1}{2}}, t_2^{\frac{n+1}{2}}\right)\right]\right] \subset k[[t_1]] \times k[[t_2]] = \overline{R}.$$

In particular, a good semigroup S is of type $\mathrm{A_n}$ if and only if there is an algebroid curve A of type $\mathrm{A_n}$ with $\Gamma_A = S$.

Proof. Let k be a field, and let R be an algebroid curve over k of type $\mathrm{A_n}$ for some $n \in \mathbb{N}$. Then we may assume that

$$R = k[[X, Y]]/\left\langle X^2 - Y^{n+1}\right\rangle$$

(see Definition 5.53.(1)). We prove the claim by constructing the normalization of R.

(1) Suppose that $n \in 2\mathbb{N}$. We write $\pi \colon k[[X, Y]] \to R$ for the canonical surjection, and we set $x = \pi(X)$ and $y = \pi(Y)$. Since $y \in R^{\mathrm{reg}}$, we have

$$t = \frac{x}{y^{n/2}} \in Q_R$$

with

$$t^2 = \left(\frac{x}{y^{n/2}}\right)^2 = \frac{x^2}{y^n} = \frac{y^{n+1}}{y^n} = y. \tag{5.45}$$

This implies

$$t^{2(n+1)} - xy^{n/2}t = \left(t^2\right)^{n+1} - xy^{n/2}\frac{x}{y^{n/2}} = y^{n+1} - x^2 = 0,$$

and hence

$$t \in \overline{R}. \tag{5.46}$$

Moreover, we have

$$t^{n+1} = \left(\frac{x}{y^{n/2}}\right)^{n+1} = \frac{x^{n+1}}{\left(y^{n+1}\right)^{n/2}} = \frac{x^{n+1}}{\left(x^2\right)^{n/2}} = x. \tag{5.47}$$

Therefore, Equations (5.45), (5.46), and (5.47) yield

$$\overline{R} \supset R[t] = k\left[\left[t^{n+1}, t^2\right]\right][t] = k[[t]],$$

and hence $R = k[[t]]$ since $\mathbb{C}[[t]]$ is integrally closed in

$$Q_R = Q_{k[[t]]} = k[[t]]\left[t^{-1}\right]$$

(see Lemma A.34 and Proposition B.5). Then Proposition B.3 implies

$$\begin{aligned}R &= k[[X,Y]]/\left\langle X^2 - Y^{n+1}\right\rangle \\ &= k[[x,y]] \\ &= k\left[\left[t^{n+1}, t^2\right]\right] \\ &\subset k[[t]] \\ &= \overline{R}.\end{aligned}$$

(2) Suppose that $n \in 2\mathbb{N}+1$. Then

$$\operatorname{Min}(R) = \left\{\left\langle X + Y^{\frac{n+1}{2}}\right\rangle_R, \left\langle X - Y^{\frac{n+1}{2}}\right\rangle_R\right\}.$$

Set

$$R_1 = R/\left\langle X + Y^{\frac{n+1}{2}}\right\rangle_R = k[[X,Y]]/\left\langle X + Y^{\frac{n+1}{2}}\right\rangle$$

and

$$R_2 = R/\left\langle X + Y^{\frac{n+1}{2}}\right\rangle_R = k[[X,Y]]/\left\langle X - Y^{\frac{n+1}{2}}\right\rangle.$$

Then Theorem B.42 implies

$$\overline{R} = \overline{R_1} \times \overline{R_2}. \tag{5.48}$$

We write

$$\begin{aligned}t_1 &= Y + \left\langle X + Y^{\frac{n+1}{2}}\right\rangle \in R_1, \\ t_2 &= Y + \left\langle X - Y^{\frac{n+1}{2}}\right\rangle \in R_2.\end{aligned}$$

Then

$$\begin{aligned}X + \left\langle X + Y^{\frac{n+1}{2}}\right\rangle &= -t_1^{\frac{n+1}{2}}, \\ X + \left\langle X - Y^{\frac{n+1}{2}}\right\rangle &= t_1^{\frac{n+1}{2}}.\end{aligned}$$

This implies

$$R_1 = k\left[\left[-t_1^{\frac{n+1}{2}}, t_1\right]\right] = k[[t_1]] = \overline{R_1}$$

and

$$R_2 = k\left[\left[t_2^{\frac{n+1}{2}}, t_2\right]\right] = k[[t_2]] = \overline{R_2}.$$

Thus, Equation (5.48) and Proposition B.3 imply

$$\begin{aligned}R &= k[[X,Y]]/\left\langle X^2 - Y^{n+1}\right\rangle \\ &= k[[X,Y]]/\left(\left\langle X - Y^{\frac{n+1}{2}}\right\rangle \cap \left\langle X + Y^{\frac{n+1}{2}}\right\rangle\right) \\ &= k\left[\left[\left(-t_1^{\frac{n+1}{2}}, t_2^{\frac{n+1}{2}}\right), (t_1, t_2)\right]\right] \\ &\subset k[[t_1]] \times k[[t_2]] \\ &= \overline{R_1} \times \overline{R_2} \\ &= \overline{R}.\end{aligned}$$

The particular claim follows since by Theorem 3.44 the valuation of Q_R containing R is ord_t. □

Corollary 5.55. *Let k be a field, and let R be an algebroid curve over k. If R is of type $\mathrm{A_n}$ for some $n \in \mathbb{N}$, then R is Gorenstein.*

Proof. This follows from Corollary 5.37, Theorem 5.42, and Proposition 5.54 (also see Definition 5.53.(2)). □

Theorem 5.56. *Let k be an algebraically closed field, and let R be a local algebroid curve over k with maximal ideal $\mathfrak{m}_R$. Then R and $\mathfrak{m}_R : \mathfrak{m}_R$ (see Remark B.49 and Proposition B.57) are Gorenstein if and only if R is of type $\mathrm{A_n}$ for some $n \in \mathbb{N}$.*

To prove Theorem 5.56 we start with showing that over an algebraically closed ground field also the converse of Proposition 5.54 is valid.

Proposition 5.57. *Let k be an algebraically closed field, and let R be an algebroid curve over k. Then R is of type $\mathrm{A_n}$ for some n if and only if Γ_R is of type $\mathrm{A_n}$, i.e.*

$$\Gamma_R = \langle 2, n+1 \rangle$$

with $n \in 2\mathbb{N}$, respectively

$$\Gamma_R = \langle (1,1) \rangle \cup \left(\left(\frac{n+1}{2}, \frac{n+1}{2} \right) + \mathbb{N}^2 \right)$$

with $n \in 2\mathbb{N} + 1$.

For the proof of Proposition 5.57 we need the following Lemmas.

Lemma 5.58 (See [35], Lemma 4.25)**.** *Let k be an algebraically closed field, and let R be an irreducible algebroid curve over k. If Γ_R is of type $\mathrm{A_n}$ with $n \in 2\mathbb{N}$, i.e.*

$$\Gamma = \langle 2, n+1 \rangle,$$

then R is of type $\mathrm{A_n}$.

Proof. First note that the conductor of Γ_R is $\gamma_{\Gamma_R} = n$.

Since $2 \in \Gamma_R$, there is an $x \in R$ with $\nu(x) = 2$. Hence, identifying $\overline{R} \cong k[[t]]$ (see Theorem 3.44), there are $a \in k \setminus \{0\}$ and $b_i \in k$ for $i \in \mathbb{N}_{>0}$ such that

$$x = at^2 + \sum_{i=1}^{\infty} b_i t^{2+i} = ut^2,$$

where

$$u = a + \sum_{i=1}^{\infty} b_i t^i \in (k[[t]])^*.$$

Since k is algebraically closed, there is a $w \in (k[[t]])^*$ such that $w^2 = u$ and $x = (wt)^2$. This yields a k-automorphism

$$\begin{aligned} k[[t]] &\to k[[t]], \\ t &\mapsto w^{-1}t, \end{aligned}$$

sending x to t^2. So we may assume that $x = t^2$. Note that this assumption corresponds to a suitable choice of a uniformizing parameter of $\overline{R}$ in the construction of the isomorphism $\overline{R} \cong k[[t]]$, see Proposition 2.23.(2), Lemma 3.43 and Theorem 3.44.

Since

$$\mathfrak{C}_R = t^{\gamma_{\Gamma_R}} k[[t]] \subset R$$

by Propositions 4.16.(2) and 4.56, we have

$$y = t^{\gamma_{\Gamma_R}+1} \in R$$

and

$$\nu((k[[x,y]])^{\mathrm{reg}}) = \Gamma_R. \tag{5.49}$$

Moreover, $R' = k[[x,y]]$ is an algebroid curve over k with $Q_R = Q_{R'}$ and $\mathcal{V}_R = \mathcal{V}_{R'}$. Thus, Proposition 4.56 and Equation (5.49) yield

$$\mathfrak{C}_R \subset k[[x,y]] \subset R.$$

Then we obtain by Lemma 4.54.(2) $R = k[[x,y]]$. Hence, R is of type A_γ. $\square$

Lemma 5.59. *Let k be an algebraically closed field, and let R be an algebroid curve over k. If Γ_R is of type $\mathrm{A_n}$ with $n \in 2\mathbb{N}+1$, i.e.*

$$\Gamma_R = \langle(1,1)\rangle \cup \left(\left(\frac{n+1}{2}, \frac{n+1}{2}\right) + \mathbb{N}^2\right),$$

then R is of type $\mathrm{A_n}$.

Proof. We set $\delta = \frac{n+1}{2} \in \mathbb{N}$. Then the conductor of Γ_R is $\gamma_{\Gamma_R} = (\delta, \delta)$.

Since $(1,1) \in \Gamma_R$, there is an $x \in A$ with $\nu(x) = (1,1)$. Hence, identifying $\overline{R} \cong k[[t_1]] \times k[[t_2]]$, there are $a_1, a_2 \in k \setminus \{0\}$ and $b_{1,i}, b_{2,i} \in k$ for $i \in \mathbb{N}_{>0}$ such that

$$x = \left(a_1t_1 + \sum_{i=1}^{\infty} b_{1,i}t_1^{1+i},\ a_2t_2 + \sum_{i=1}^{\infty} b_{2,i}t_2^{1+i}\right) = ut,$$

where

$$u = \left(a_1 + \sum_{i=1}^{\infty} b_{1,i}t_1^{i},\ a_2 + \sum_{i=1}^{\infty} b_{2,i}t_2^{i}\right) \in (k[[t_1]] \times k[[t_2]])^*.$$

Thus, there is a k-automorphism

$$\begin{aligned} k[[t_1]] \times k[[t_2]] &\to k[[t_1]] \times k[[t_2]], \\ t &\mapsto u^{-1}t, \end{aligned}$$

sending x to $t = (t_1, t_2)$. So we may assume $x = t$.

Since

$$\mathfrak{C}_R = t^{\gamma_{\Gamma_R}}(k[[t_1]] \times k[[t_2]]) \subset R$$

by Propositions 4.16.(2) and 4.56, we have

$$y = \left(t_1^\delta, -t_2^\delta\right) \in R$$

and

$$\nu((k[[x,y]])^{\mathrm{reg}}) = \Gamma_R. \tag{5.50}$$

Moreover, $R' = k[[x,y]]$ is an algebroid curve over k with $Q_R = Q_{R'}$ and $\mathcal{V}_R = \mathcal{V}_{R'}$. Thus, Proposition 4.56 and Equation (5.50) yield

$$\mathfrak{C}_R \subset k[[x,y]] \subset R.$$

Then we obtain by Lemma 4.54.(2) $R = k[[x,y]]$. Hence, R is of type $\mathrm{A_n}$ with $n = 2\delta - 1$. $\square$

Proof of Proposition 5.57. This follows from Proposition 5.54 and Lemmas 5.58 and 5.59. $\square$

Remark 5.60. Let R be Gorenstein. Then Γ_R is a symmetric semigroup by Corollary 5.37, and hence Proposition B.60, Theorem 5.34, Remark 4.8, and Proposition 4.38 yield

$$\Gamma_{\mathfrak{m}_R:\mathfrak{m}_R} = \Gamma_{R:\mathfrak{m}_R} = \Gamma_R - \Gamma_{\mathfrak{m}_R} = \Gamma_R - M_{\Gamma_R} = M_{\Gamma_R} - M_{\Gamma_R}.$$

Proof of Theorem 5.56. First note that $\mathfrak{m}_R : \mathfrak{m}_R$ is by Lemma 2.17 an integral extension of R, and hence an admissible ring with $\mathcal{V}_R = \mathcal{V}_{\mathfrak{m}_R:\mathfrak{m}_R}$ by Theorem 3.45.(1). Let R and $\mathfrak{m}_R : \mathfrak{m}_R$ be Gorenstein. Then Γ_R and $\Gamma_{\mathfrak{m}_R:\mathfrak{m}_R} = M_{\Gamma_R} - M_{\Gamma_R}$ (see Remark 5.60) are symmetric semigroups by Corollary 5.37. Thus, Theorem 5.42 implies that Γ_R is of type $\mathrm{A_n}$ for some $n \in \mathbb{N}$ (see Definition 5.53.(2)), and therefore R is of type $\mathrm{A_n}$ by Proposition 5.57.

Let now R be of type $\mathrm{A_n}$ for some $n \in \mathbb{N}$. Then Γ_R is of type $\mathrm{A_n}$ by Proposition 5.57. Hence, Γ_R and $M_{\Gamma_R} - M_{\Gamma_R} = \Gamma_{\mathfrak{m}_R:\mathfrak{m}_R}$ (see Remark 5.60) are symmetric semigroups by Theorem 5.42. Thus Corollary 5.37 implies that R and $\mathfrak{m}_R : \mathfrak{m}_R$ are Gorenstein. $\square$

6. Quasihomogeneous Curves

In this Chapter we describe quasihomogeneous curves in terms of their semigroups of values and a coefficient map.

Definition 6.1. Let R be a local complex algebroid curve, and let $w \in \mathbb{N}^n$ for some $n \in \mathbb{N}$ with $w_i > 0$ for all $i = 1, \dots, n$. Then R is called *quasihomogeneous* (of type w) if there is a $\mathbb{C}$-derivation $\mathfrak{d}$ of R and a generating system $(x_i)_{i=1}^n$ for the maximal ideal $\mathfrak{m}_R$ of R with $\mathfrak{d}(x_i) = w_i x_i$ for every $i = 1, \dots, n$. Equivalently, there is a surjective homomorphism $\phi\colon \mathbb{C}[[X_1, \dots, X_n]] \to R$ such that $\ker\phi$ is homogeneous with respect to weighted polynomial degree with weight w (see Theorems A.67 and E.13).

Since a quasihomogeneous curves is an algebroid curve by definition, it is an admissible ring by Proposition 3.41. Kunz and Ruppert proved that an irreducible quasihomogeneous curve R is isomorphic to the semigroup ring of its semigroup of values, i.e.

$$R \cong \mathbb{C}\left[\left[t^{\Gamma_R}\right]\right], \tag{6.1}$$

see [9, Satz 3.1]. In Section 6.1 we re-prove this statement (see Theorem 6.9).

Let R be a quasihomogeneous curve with two branches. We write $\operatorname{Min}(R) = \{\mathfrak{p}, \mathfrak{q}\}$. Then R can be written as the fibre product of its branches over their intersection. The branches are irreducible quasihomogeneous curve, and hence they can be expressed in terms of their semigroup of values. Moreover, Kunz and Ruppert show that the intersection of the branches can be described by the value semigroup ideal of a minimal prime ideal of R in the branch corresponding to the other minimal prime ideal, i.e.

$$R/\mathfrak{p}+\mathfrak{q} \cong \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right]/\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right] \cong \mathbb{C}\left[\left[t_{\mathfrak{q}}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{p}+\mathfrak{q}/\mathfrak{q}}}\right]\right], \tag{6.2}$$

see [9, Satz 4.2]. The quotient semigroup is defined in Definition 4.74, and its semigroup ring is defined in Definition 4.77.(3).

So Equations (6.1) and (6.2) yield a commutative diagram

$$
\begin{tikzcd}
 & & R \arrow[dl, two heads] \arrow[dr, two heads] & & \\
\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right] \arrow[dddd, two heads] & R/\mathfrak{p} \arrow[l, "\cong"'] \arrow[dr, two heads] & & R/\mathfrak{q} \arrow[dl, two heads] \arrow[r, "\cong"] & \mathbb{C}\left[\left[t_{\mathfrak{q}}^{\Gamma_{R/\mathfrak{q}}}\right]\right] \arrow[dddd, two heads] \\
 & & R/\mathfrak{p}+\mathfrak{q} \arrow[ddll, "\cong"'] \arrow[ddrr, "\cong"] & & \\
 & & & & \\
\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right] \arrow[rrrr, "\alpha", "\cong"'] & & & & \mathbb{C}\left[\left[t_{\mathfrak{q}}^{\Gamma_{R/\mathfrak{q}}/\Gamma_{\mathfrak{p}+\mathfrak{q}/\mathfrak{q}}}\right]\right].
\end{tikzcd}
\tag{6.3}
$$

Hence,

$$
R \cong \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right] \times_{\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right]} \mathbb{C}\left[\left[t_{\mathfrak{q}}^{\Gamma_{R/\mathfrak{q}}}\right]\right], \tag{6.4}
$$

where we take the fibre product with respect to the composition of the canonical surjection $\mathbb{C}\left[\left[t_{\mathfrak{q}}^{\Gamma_{R/\mathfrak{q}}}\right]\right] \to \mathbb{C}\left[\left[t_{\mathfrak{q}}^{\Gamma_{R/\mathfrak{q}}/\Gamma_{\mathfrak{p}+\mathfrak{q}/\mathfrak{q}}}\right]\right]$ and the isomorphism α^{-1}, see [9, Satz 4.2]).

The isomorphism α can be described more explicitly. First Kunz and Ruppert noted that for a quasihomogeneous curve R' there is a $w \in \mathbb{N}^{\mathrm{Min}\,(R)}$ with $w_{\mathfrak{p}} > 0$ for all $\mathfrak{p}' \in \mathrm{Min}\,(R)$ such that for a homogeneous element x of a R' we have

$$
\deg(x) = w_{\mathfrak{p}} \nu_{\mathfrak{p}}(x) \tag{6.5}
$$

for all $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $x \notin \mathfrak{p}$, see [9, Section 3]. So considering the values of homogeneous elements of R which are neither contained in $\mathfrak{p}$ nor in $\mathfrak{q}$ Kunz and Ruppert obtained a bijection

$$
\begin{aligned}
\tau_{\mathfrak{p}\mathfrak{q}} \colon \Gamma_{R/\mathfrak{p}} \setminus \Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}} &\to \Gamma_{R/\mathfrak{q}} \setminus \Gamma_{\mathfrak{p}+\mathfrak{q}/\mathfrak{q}}, \\
\alpha &\mapsto \frac{w_{\mathfrak{p}}\alpha}{w_{\mathfrak{q}}}.
\end{aligned}
$$

Then α is induced by this bijection $\tau_{\mathfrak{p}\mathfrak{q}}$, see [9, Satz 4.1].

In the following we want to extend these results in two ways: we will drop the restriction on the number of branches, and we will deduce the combinatorial data determining a quasihomogeneous curve only from its semigroup of values. Passing to an arbitrary number of branches we use the generalized notion of a fibre product introduced in Section 2.3. Then considering the branches pairwise we obtain again diagrams as (6.3). However, in general R is only contained in the fibre product of its branches but not isomorphic to it anymore. In Chapter 7 we will give a criterion on the value semigroup of values which determines this inclusion to be an isomorphism (see Theorem 7.23).

In order to deduce the combinatorial informations from the semigroup of values Γ_R of a quasihomogeneous curve R, we first note that for any minimal prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ with $\mathfrak{p} \neq \mathfrak{q}$ we have

$$
\Gamma_{R/\mathfrak{p}} = (\Gamma_R)_{\mathfrak{p}}
$$

by Proposition 4.67 and

$$\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}} = (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$$

by Proposition 4.69.

Theorem 6.2. *Let R be a quasihomogeneous curve.*

(1) There is a $(\overline{w}_{\mathfrak{p}})_{\mathfrak{p}\in\mathrm{Min}\,(R)} \in \mathbb{N}^{\mathrm{Min}\,(R)}$ with $\overline{w}_{\mathfrak{p}} > 0$ for every $\mathfrak{p} \in \mathrm{Min}\,(R)$ such that for any homogeneous element $x \in R$ we have

$$\deg\,(x) = \overline{w}_{\mathfrak{p}} \nu_{\mathfrak{p}}(x)$$

for all $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $x \notin \mathfrak{p}$.

(2) For any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}\,(R)$ with $\mathfrak{p} \neq \mathfrak{q}$ there is a bijection

$$\begin{aligned} \tau_{\mathfrak{pq}} \colon (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} &\to (\Gamma_R)_{\mathfrak{q}} \setminus (\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}, \\ \alpha &\mapsto \frac{\overline{w}_{\mathfrak{p}}\alpha}{\overline{w}_{\mathfrak{q}}} \end{aligned}$$

(see Definition 4.60).

(3) For any $\mathfrak{p} \in \mathrm{Min}\,(R)$ the isomorphism $\overline{R} \to \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}[[t_{\mathfrak{p}}]]$ of Theorem 3.44 induces a homogeneous surjective homomorphism (see Definition E.8)

$$\psi_{\mathfrak{p}} \colon R \to \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

with

$$\nu_{\mathfrak{p}}|_R = \mathrm{ord}_t \circ \psi_{\mathfrak{p}}, \tag{6.6}$$

and for any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}\,(R)$ with $\mathfrak{p} \neq \mathfrak{q}$ there is a map

$$\zeta_{\mathfrak{pq}} \colon (\Gamma_R)_{\mathfrak{p}} / (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \to \mathbb{C}$$

(see Definition 4.74) with

$$\zeta_{\mathfrak{pq}}(\alpha + \beta) = \zeta_{\mathfrak{pq}}(\alpha)\, \zeta_{\mathfrak{pq}}(\beta) \tag{6.7}$$

for all $\alpha, \beta \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ with $\alpha+\beta \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$, and a homogeneous isomorphism

$$\begin{aligned} \sigma_{\mathfrak{pq}} \colon \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right] &\to \mathbb{C}\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}}\right], \\ \overline{t_{\mathfrak{p}}^{\alpha}} &\mapsto \zeta_{\mathfrak{pq}}(\alpha)\, \overline{t_{\mathfrak{q}}^{\tau_{\mathfrak{pq}}(\alpha)}} \end{aligned} \tag{6.8}$$

induced by $\tau_{\mathfrak{p}\mathfrak{q}}$ and $\zeta_{\mathfrak{p}\mathfrak{q}}$ such that there is a commutative diagram

$$\begin{tikzcd} & R \arrow[dl, two heads, "\psi_{\mathfrak{p}}"'] \arrow[dr, two heads, "\psi_{\mathfrak{q}}"] & \\ \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right] \arrow[d, two heads, "\chi_{\mathfrak{p}\mathfrak{q}}"'] & & \mathbb{C}\left[\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}}\right]\right] \arrow[d, two heads, "\chi_{\mathfrak{q}\mathfrak{p}}"] \\ \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right] \arrow[rr, "\cong", "\sigma_{\mathfrak{p}\mathfrak{q}}"'] & & \mathbb{C}\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}}\right], \end{tikzcd}$$

where $\chi_{\mathfrak{p}\mathfrak{q}}$ and $\chi_{\mathfrak{q}\mathfrak{p}}$ denote the homogeneous surjective homomorphisms of Proposition 4.79.

(4) *With $\zeta = \left((\zeta_{\mathfrak{p},\mathfrak{q}})_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)}$ we denote by $\mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$ the $\mathbb{C}$-subalgebra of $\prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$ consisting of the elements*

$$\left(\sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}} a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)} \in \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

with

$$a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} = \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_{\mathfrak{p}})\, a_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_{\mathfrak{p}})}^{(\mathfrak{q})} \tag{6.9}$$

for any $\mathfrak{p} \in \mathrm{Min}\,(R)$, for every $\mathfrak{q} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{p}\}$, and for all $\alpha_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$. Then the $\mathbb{C}$-algebra isomorphism

$$Q_R \to \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}[[t_{\mathfrak{p}}]]\left[t_{\mathfrak{p}}^{-1}\right]$$

of Theorem 3.44 restricts to an injective homogeneous $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \Psi\colon R &\to \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta), \\ x &\mapsto (\psi_{\mathfrak{p}}(x))_{\mathfrak{p}\in\mathrm{Min}\,(R)} \end{aligned}$$

with $\psi_{\mathfrak{p}}$ as in (3) for all $\mathfrak{p} \in \mathrm{Min}\,(R)$. Moreover, for any $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Min}\,(R)$ there is a commutative diagram

$$\begin{tikzcd} & R \arrow[dl, two heads, "\psi_{\mathfrak{q}}"'] \arrow[d, hook, "\Psi"'] \arrow[dr, two heads, "\psi_{\mathfrak{q}'}"] & \\ \mathbb{C}\left[\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}}\right]\right] & \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta) \arrow[l, two heads, "\mathrm{pr}_{\mathfrak{q}}"] \arrow[r, two heads, "\mathrm{pr}_{\mathfrak{q}'}"'] & \mathbb{C}\left[\left[t_{\mathfrak{q}'}^{(\Gamma_R)_{\mathfrak{q}'}}\right]\right]. \end{tikzcd}$$

(5) *Let $x \in R$, and write $\Psi(x) = \left(\sum_{\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p}} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p} \in \mathrm{Min}\,(R)}$. Then for any $d \in \mathbb{Z}$ and for every $\mathfrak{p} \in \mathrm{Min}\,(R)$ we have*

$$((\Psi(x))_d)_\mathfrak{p} = (\psi_\mathfrak{p}(x))_d = \begin{cases} a_{d/\overline{w}_\mathfrak{p}}^{(\mathfrak{p})} t_\mathfrak{p}^{\frac{d}{\overline{w}_\mathfrak{p}}} & \text{if there is an } \alpha \in (\Gamma_R)_\mathfrak{p} \text{ with } \overline{w}_\mathfrak{p}\alpha = d, \\ 0 & \text{else.} \end{cases}$$

Moreover,

$$\mathrm{ord}_t\left((\Psi(x))_d\right) \geq \nu(x)$$

for all $d \in \mathbb{Z}$.

Proof. See Section 6.2. □

To ease notation in future constructions of fibre products we introduce the following.

Definition 6.3. Let R be a quasihomogeneous curve. Using the notation of Theorem 6.2, we call $\overline{w}$ *normal weights* and $\zeta = \left((\zeta_{\mathfrak{p}\mathfrak{q}})_{\mathfrak{q} \in \mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p} \in \mathrm{Min}\,(R)}$ *connecting maps* for R.

Unlike in the case $|\mathrm{Min}\,(R)| \leq 2$ which was treated by Kunz and Ruppert, in general the homomorphism $\Psi\colon R \to \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$ of Theorem 6.2.(4) is only an inclusion. We give a name to the special case when Ψ is an isomorphism.

Definition 6.4. Let R be a quasihomogeneous curve. We say that R is a *fibre product* if the homomorphism $\Psi\colon R \to \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$ of Theorem 6.2.(4) is an isomorphism.

Remark 6.5. Let R be a quasihomogeneous curve with normal weights $\overline{w} \in \mathbb{N}^{\mathrm{Min}\,(R)}$ and connecting maps ζ (see Definition 6.3). Theorem 6.2.(4) and the following Proposition 6.6 show that the fibre product $\mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$ is a "closure" of R in the following sense: It is the "largest" quasihomogeneous curve contained in $\prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_\mathfrak{p}^{(\Gamma_R)_\mathfrak{p}}\right]\right]$ with normal weights $\overline{w}$ and connecting maps ζ.

Proposition 6.6. *Let R be a quasihomogeneous curve with normal weights $\overline{w} \in \mathbb{N}^{\mathrm{Min}\,(R)}$ and connecting maps ζ (see Definition 6.3), and set*

$$A = \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta).$$

(1) *A is a quasihomogeneous curve.*

(2) *There is a bijection*

$$\eta\colon\ \mathrm{Min}\,(R) \to \mathrm{Min}\,(A),$$

and A has normal weights

$$\overline{w}^{(A)} = \left(\overline{w}_{\eta^{-1}(\mathfrak{p})}\right)_{\mathfrak{p} \in \mathrm{Min}\,(A)} \in \mathbb{N}^{\mathrm{Min}\,(A)}$$

and connecting maps

$$\zeta^{(A)} = \left(\left(\zeta_{\eta^{-1}(\mathfrak{p}),\eta^{-1}(\mathfrak{q})}\right)_{\mathfrak{q} \in \mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p} \in \mathrm{Min}\,(R)}.$$

(3) *For any* $\mathfrak{p} \in \mathrm{Min}\,(A)$ *we have (considering* $(\Gamma_A)_{\mathfrak{p}}$, $(\Gamma_R)_{\eta^{-1}(\mathfrak{p})}$, $(\Gamma_A)_{\mathfrak{p}}^{\mathfrak{q}}$ *and* $(\Gamma_R)_{\eta^{-1}(\mathfrak{p})}^{\eta^{-1}(\mathfrak{q})}$ *as subsets of* $\mathbb{N}$*)*

$$(\Gamma_A)_{\mathfrak{p}} = (\Gamma_R)_{\eta^{-1}(\mathfrak{p})},$$

and

$$(\Gamma_A)_{\mathfrak{p}}^{\mathfrak{q}} = (\Gamma_R)_{\eta^{-1}(\mathfrak{p})}^{\eta^{-1}(\mathfrak{q})}$$

for every $\mathfrak{q} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{p}\}$*. Moreover,* A *is a fibre product (see Definition 6.4), i.e.*

$$A = \mathrm{Fib}\left(\Gamma_A, \overline{w}^{(A)}, \zeta^{(A)}\right).$$

(4) *Let* $\overline{\Psi}\colon \overline{R} \to \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}[[t_{\mathfrak{p}}]]$ *be the isomorphism of Theorem 3.44. Then*

$$\overline{\Psi}^{-1}(A) \in \mathcal{R}_R,$$

and η *induces a bijection*

$$\begin{aligned} \Gamma_{\overline{\Psi}^{-1}(A)} &\to \Gamma_A, \\ (\alpha_{\mathfrak{p}})_{\mathfrak{p}\in\mathrm{Min}\,(R)} &\mapsto \left(\alpha_{\eta^{-1}(\mathfrak{q})}\right)_{\mathfrak{q}\in\mathrm{Min}\,(A)}. \end{aligned}$$

Proof. See Section 6.3. □

Finally, we show some important properties of the connecting maps of a quasihomogeneous curve.

Lemma 6.7. *Let* R *be a quasihomogeneous curve with connecting maps*

$$\left((\zeta_{\mathfrak{p}\mathfrak{q}})_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)},$$

and let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}\,(R)$ *with* $\mathfrak{p} \neq \mathfrak{q}$.

(1) *For any* $\alpha \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ *we have* $\zeta_{\mathfrak{p}\mathfrak{q}}(\alpha) \neq 0$.

(2) *We have* $\zeta_{\mathfrak{p}\mathfrak{q}}(0) = 1$.

Proof. (1) The rings $\mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right]$ and $\mathbb{C}\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}}\right]$ are $\mathbb{C}$-vector spaces with bases $\left(t_{\mathfrak{p}}^{\alpha}\right)_{\alpha\in(\Gamma_R)_{\mathfrak{p}}\setminus(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}$ and $\left(t_{\mathfrak{q}}^{\beta}\right)_{\beta\in(\Gamma_R)_{\mathfrak{q}}\setminus(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}}$, respectively. Since

$$\sigma_{\mathfrak{p}\mathfrak{q}}\colon \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right] \to \mathbb{C}\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}}\right]$$

is by Theorem 6.2.(3) a $\mathbb{C}$-vector space isomorphism, this implies $\zeta_{\mathfrak{p}\mathfrak{q}}(\alpha) \neq 0$ for all $\alpha \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ (see Equation (6.8)).

(2) Since R is local, we have $0 \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$ by Theorem 4.9 and Proposition 4.65. Therefore, Theorem 6.2.(3) yields

$$\zeta_{\mathfrak{pq}}(0) = \zeta_{\mathfrak{pq}}(0+0) = \zeta_{\mathfrak{pq}}(0)\,\zeta_{\mathfrak{pq}}(0)$$

(see Equation (6.7)). Thus, we have either $\zeta_{\mathfrak{pq}}(0) = 0$ or $\zeta_{\mathfrak{pq}}(0) = 1$, and (1) yields the claim. $\square$

Remark 6.8. Let R be a quasihomogeneous curve with normal weights $\overline{w} \in \mathbb{N}^{\mathrm{Min}\,(R)}$ and connecting maps $\zeta = \left((\zeta_{\mathfrak{pq}})_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)}$ (see Definition 6.3). Then Lemma 6.7.(2) implies that if $x \in \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$, then all components of x have the same constant term. Thus, with Theorem 6.2.(4) we have inclusions

$$\Psi(R) \subset \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta) \subset \left(\prod_{\mathfrak{p}\in\mathrm{Min}\,(R)}\right)_{\mathbb{C}} \mathbb{C}\left[\left[t_\mathfrak{p}^{(\Gamma_R)_\mathfrak{p}}\right]\right].$$

This was also shown by Kunz and Ruppert in [9, Satz 3.4].

6.1. Irreducible Curves

Before we treat the case of general quasihomogeneous curves, we investigate irreducible curves. More precisely, we prove the following.

Theorem 6.9. *Let R be an irreducible quasihomogeneous curve of type $w \in \mathbb{N}^n$ with respect to $\mathfrak{d}_R \in \mathrm{Der}_\mathbb{C}\,(R)$. This means that R is $\mathbb{Z}$-graded.*

(1) The normalization $\overline{R}$ is quasihomogeneous with respect to a derivation $\mathfrak{d}_{\overline{R}} \in \mathrm{Der}_\mathbb{C}\left(\overline{R}\right)$ with $\mathfrak{d}_{\overline{R}}|_R = \mathfrak{d}_R$. In particular, there is a uniformizing parameter $t \in \overline{R}$ and a $\overline{w} \in \mathbb{N}$ such that $\mathfrak{d}_{\overline{R}}(t) = \overline{w}t$.

(2) For any homogeneous element $x \in R^{\mathrm{reg}}$ we have

$$\deg\,(x) = \overline{w}\nu(x).$$

(3) The isomorphism

$$\phi\colon \overline{R} \to \mathbb{C}[[T]],$$
$$t \mapsto T$$

of Theorem 3.44 is homogeneous, and it restricts to a homogeneous isomorphism

$$\phi'\colon R \to \mathbb{C}\left[\left[T^{\Gamma_R}\right]\right],$$

and

$$\nu = \mathrm{ord}_T \circ \phi$$

if we extend ϕ to an isomorphism $Q_R \to \mathbb{C}[[T]]\,[T^{-1}]$.

(4) *Let R be also quasihomogeneous of type $w' \in \mathbb{N}^n$ with respect to $\mathfrak{d}'_R \in \operatorname{Der}_{\mathbb{C}}(R)$. Let $\mathfrak{d}'_{\overline{R}} \in \operatorname{Der}_{\mathbb{C}}\left(\overline{R}\right)$ with $\mathfrak{d}'_{\overline{R}}\Big|_R = \mathfrak{d}'$, $s \in \overline{R}$ a uniformizing parameter and $\overline{w}' \in \mathbb{N}$ such that $\mathfrak{d}'_{\overline{R}}(s) = \overline{w}'s$ as in (1). Then with the isomorphism*

$$\begin{aligned} \phi\colon \overline{R} &\to \mathbb{C}[[S]], \\ s &\mapsto S \end{aligned}$$

of Theorem 3.44 we obtain commutative diagrams

$$\begin{tikzcd}
 & \overline{R} \arrow[dl, bend right, "\phi"', "\cong"] \arrow[d, two heads, "\nu|_{\overline{R}}"] \arrow[dr, bend left, "\psi", "\cong"'] & \\
 & \overline{\Gamma_R} \cup \{\infty\} & \\
\mathbb{C}[[T]] \arrow[ur, "\mathrm{ord}_T"'] \arrow[rr, "\cong"'] & & \mathbb{C}[[S]], \arrow[ul, "\mathrm{ord}_S"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & R \arrow[dl, bend right, "\phi'"', "\cong"] \arrow[d, two heads, "\nu|_{R}"] \arrow[dr, bend left, "\psi'", "\cong"'] & \\
 & \Gamma_R \cup \{\infty\} & \\
\mathbb{C}\left[\left[T^{\Gamma_R}\right]\right] \arrow[ur, "\mathrm{ord}_T"'] \arrow[rr, "\cong"'] & & \mathbb{C}\left[\left[S^{\Gamma_R}\right]\right]. \arrow[ul, "\mathrm{ord}_S"]
\end{tikzcd}$$

Moreover, there is a unit $u \in (\mathbb{C}[[S]])^*$ such that

$$\begin{aligned} \psi \circ \phi^{-1}\colon \mathbb{C}\left[\left[T^{\Gamma_R}\right]\right] &\to \mathbb{C}\left[\left[S^{\Gamma_R}\right]\right], \\ T &\mapsto uS. \end{aligned}$$

Note that the isomorphisms ϕ, ϕ', ψ, and ψ' are homogeneous but the isomorphisms $\mathbb{C}[[T]] \to \mathbb{C}[[S]]$ and $\mathbb{C}\left[\left[T^{\Gamma_R}\right]\right] \to \mathbb{C}\left[\left[S^{\Gamma_R}\right]\right]$ are in general not homogeneous.

(5) *Let $\mathfrak{i}$ be a homogeneous non-zero ideal of R. Then*

$$\phi(\mathfrak{i}) = \mathbb{C}\left[\left[T^{\Gamma_{\mathfrak{i}}}\right]\right].$$

To prove Theorem 6.9 we need the following Lemmas.

Lemma 6.10. *Let $\mathfrak{i} \subset \mathbb{C}[[X_1, \dots, X_n]]$ be an ideal, and let $\mathbb{C}[[X_1, \dots, X_n]]/\mathfrak{i}$ be quasihomogeneous of type $w \in \mathbb{N}^n$ with respect to $\mathfrak{d} \in \operatorname{Der}_{\mathbb{C}}(\mathbb{C}[[x_1, \dots, x_n]]/\mathfrak{i})$ such that $\mathfrak{d}(x_i) = w_i x_i$, where $x_i = X_i + \mathfrak{i}$, for all $i = 1, \dots, n$.*

(1) There is a commutative diagram

$$\begin{array}{ccc}
\mathbb{C}[[X_1,\dots,X_n]] & \xrightarrow{\sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}} & \mathbb{C}[[X_1,\dots,X_n]] \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{pr}} \\
\mathbb{C}[[X_1,\dots,X_n]]/\mathfrak{i} & \xrightarrow[\mathfrak{d}]{} & \mathbb{C}[[X_1,\dots,X_n]]/\mathfrak{i},
\end{array}$$

where $\pi\colon \mathbb{C}[[X_1,\dots,X_n]] \to \mathbb{C}[[X_1,\dots,X_n]]/\mathfrak{i}$ *is the canonical surjection. Moreover,*

$$\sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}(\mathfrak{i}) \subset \mathfrak{i}.$$

(2) An element $y \in \mathbb{C}[[X_1,\dots,X_n]]/\mathfrak{i}$ *is homogeneous with respect to* $\mathfrak{d}$ *if and only if for any* $\alpha \in \mathbb{N}^n$ *with* $|\alpha|_w = \deg(y)$ *there is an* $a_\alpha \in \mathbb{C}$ *such that*

$$y = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|_w = \deg(y)}} a_\alpha \overline{X}^\alpha,$$

where we write $|\alpha|_w = \sum_{i=1}^n w_i\alpha_i$.

Proof. (1) By Theorem E.13 there is a $\mathbb{C}$-derivation $\mathfrak{d}'$ of $\mathbb{C}[[X_1,\dots,X_n]]$ such that $\mathfrak{d}\circ\pi = \pi\circ\mathfrak{d}'$ and $\mathfrak{d}(\mathfrak{i}) \subset \mathfrak{i}$. Moreover, Theorem E.13 yields $\mathfrak{d}'(X_i) = w_i X_i$ for all $i = 1,\dots,n$. This implies

$$\mathfrak{d}' = \sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}.$$

(2) By Theorem E.13 an element $y \in \mathbb{C}[[X_1,\dots,X_n]]/\mathfrak{i}$ is homogeneous if and only if there is a homogeneous element $Y \in (\mathbb{C}[[X_1,\dots,X_n]])_{\deg(y)}$ with $\mathrm{pr}(Y) = y$, where on $\mathbb{C}[[X_1,\dots,X_n]]$ we consider the grading corresponding to the $\mathbb{C}$-derivation $\sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}$ (see Theorem E.11.(1)). Write $Y = \sum_{\alpha\in\mathbb{N}^s} a_\alpha X^\alpha$. Then Y is by Theorem E.11.(1) homogeneous of degree $\deg(y)$ if and only if

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}^s} \deg(y) a_\alpha X^\alpha = \deg(y) Y
&= \sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}(Y) \\
&= \sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}\left(\sum_{\alpha\in\mathbb{N}^n} y_\alpha X^\alpha\right) \\
&= \sum_{\alpha\in\mathbb{N}^n} y_\alpha \sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}(X^\alpha) \\
&= \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\neq 0}} y_\alpha \sum_{i=1}^n w_i \alpha_i X^\alpha \\
&= \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\neq 0}} y_\alpha |\alpha|_w X^\alpha.
\end{aligned}$$

Comparing coefficients we see that this is the case if and only if

$$Y = \sum_{\substack{\alpha \in \mathbb{N}^s \\ |\alpha|_w = \deg(y)}} a_\alpha X^\alpha.$$

Therefore, y is homogeneous if and only if for any $\alpha \in \mathbb{N}^s$ with $|\alpha|_w = \deg(y)$ there is an $a_\alpha \in \mathbb{C}$ such that

$$y = \pi\left(\sum_{\substack{\alpha \in \mathbb{N}^s \\ |\alpha|_w = \deg(y)}} a_\alpha X^\alpha\right) = \sum_{\substack{\alpha \in \mathbb{N}^s \\ |\alpha|_w = \deg(y)}} a_\alpha x^\alpha. \qquad \square$$

Lemma 6.11. *Let R be an irreducible quasihomogeneous curve of type $w \in \mathbb{N}^n$.*

(1) The normalization $\overline{R}$ is quasihomogeneous with respect to a derivation $\mathfrak{d}_{\overline{R}} \in \operatorname{Der}_{\mathbb{C}}\left(\overline{R}\right)$ with $\mathfrak{d}_{\overline{R}}\big|_R = \mathfrak{d}_R$. In particular, there is a uniformizing parameter $t \in \overline{R}$ and a $\overline{w} \in \mathbb{N}$ such that $\mathfrak{d}_{\overline{R}}(t) = \overline{w}t$. Moreover, the isomorphism

$$\begin{aligned} \phi\colon \overline{R} &\to \mathbb{C}[[T]], \\ t &\mapsto T, \end{aligned}$$

of Theorem 6.9.(3) is homogeneous if we consider on $\mathbb{C}[[T]]$ the grading corresponding to the $\mathbb{C}$-derivation $\overline{w}t\frac{\partial}{\partial t}$ (see Theorem E.11.(1)).

(2) Let $(y_d)_{d \in \overline{w}\mathbb{Z}} \in \prod_{d \in \overline{w}\mathbb{Z}} R_d$. Then $\sum_{d \in \overline{w}\mathbb{Z}} y_d \in R$. Moreover, $R \cong \prod_{d \in \overline{w}\mathbb{Z}} R_d$.

(3) For any $d \in \mathbb{Z}$ we have

$$\overline{R}_d = \begin{cases} \phi^{-1}\left(\mathbb{C} \cdot T^{\frac{d}{\overline{w}}}\right) & \text{if } d \in \overline{w}\mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

(4) Let $\alpha \in \mathbb{N}$, and let $x \in R_{\overline{w}\alpha}$. Then $\nu(x) = \alpha$.

(5) For any $\alpha \in \Gamma_R$ there is an $x \in R_{\overline{w}\alpha}$ with $\nu(x) = \alpha$. In particular, $x \neq 0$.

(6) For any $\alpha \in \mathbb{N}$ we have

$$R_{\overline{w}\alpha} = \begin{cases} \overline{R}_{\overline{w}\alpha} = \phi^{-1}(\mathbb{C} \cdot T^\alpha), & \text{if } \alpha \in \Gamma_R, \\ 0, & \text{else.} \end{cases}$$

Proof. (1) By Theorem E.11.(2) the grading of R corresponds to a derivation $\mathfrak{d}_R \in \operatorname{Der}_{\mathbb{C}}(R)$. Then by [36, Satz 2.12] $\overline{R}$ is quasihomogeneous with respect to a $\mathbb{C}$-derivation $\mathfrak{d}_{\overline{R}} \in \operatorname{Der}_{\mathbb{C}}\left(\overline{R}\right)$ with $\mathfrak{d}_{\overline{R}}\big|_R = \mathfrak{d}_R$. Therefore, there are generators $x_1, \ldots, x_n$ of the maximal ideal $\mathfrak{m}_{\overline{R}}$ of $\overline{R}$ and weights $w_1, \ldots, w_n \in \mathbb{N}_{>0}$ such that $\mathfrak{d}_{\overline{R}}(x_i) = w_i x_i$ for every $i = 1, \ldots, n$. Since $\overline{R}$ is by Remark 3.39 a discrete valuation ring and a domain by Corollary A.73, there is by Proposition 2.23.(2) and (3) a uniformizing parameter $t \in \overline{R}$ such that $\mathfrak{d}_{\overline{R}}(t) = \overline{w}t$ for some $\overline{w} \in \mathbb{N}$.

(2) Since R is quasihomogeneous, there are generators $x_1, \dots, x_n$ of the maximal ideal $\mathfrak{m}_{\overline{R}}$ of $\overline{R}$ and weights $w_1, \dots, w_n \in \mathbb{N}_{>0}$ such that x_i is homogeneous with $\deg(x_i) = w_i$ for any $i = 1, \dots, n$. By Theorem A.67 there is a surjective $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \phi\colon \mathbb{C}[[X_1, \dots, X_s]] &\to R, \\ X_i &\mapsto x_i \text{ for all } i = 1, \dots, n. \end{aligned}$$

By Theorem E.13 $\ker\phi$ is homogeneous with respect to the grading on $\mathbb{C}[[X_1, \dots, X_n]]$ corresponding to the $\mathbb{C}$-derivation $\sum_{i=1}^n w_i X_i \frac{\partial}{\partial X_i}$ (see Theorem E.11.(1)). Moreover, the grading on $A = \mathbb{C}[[X_1, \dots, X_s]]/\ker\phi$ induced via ψ^{-1} agrees with the grading induced by that on $\mathbb{C}[[X_1, \dots, X_n]]$, where $\psi\colon A \to R$ is the isomorphism induced by ϕ.

Let $(y_d)_{d\in G} \in \prod_{d\in G} R_d$. Then $\phi(y_d)$ is homogeneous in A for any $d \in G$, and hence by Lemma 6.10.(2) there is

$$(z_\alpha)_{\alpha\in\left\{\beta\in\mathbb{N}^n \mid |\beta|_w = d\right\}} \in \mathbb{C}^{\left\{\beta\in\mathbb{N}^n \mid |\beta|_w = d\right\}}$$

such that

$$\psi^{-1}(y_d) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|_w = d}} z_\alpha x^\alpha.$$

Let now $m \in \mathbb{N}$ and $v = \max\{w_i \mid i = 1, \dots, n\}$. Then for any $\alpha \in \mathbb{N}^n$ with $|\alpha|_w = m$ we have

$$m = |\alpha|_w = \sum_{i=1}^n w_i\alpha_i \le \sum_{i=1}^n v\alpha_i = v\sum_{i=1}^n \alpha_i,$$

and hence

$$\sum_{i=1}^n \alpha_i \ge \frac{m}{v}.$$

This implies that for any $d \in G$ we have

$$\psi^{-1}(y_d) \in \left(\langle x_1, \dots, x_n\rangle_{\mathbb{C}[[x_1,\dots,x_n]]/\mathrm{i}}\right)^r,$$

and hence

$$y_d \in (\mathfrak{m}_R)^r \tag{6.10}$$

for all $r \in \mathbb{N}$ with $vr \le d$.

For $g \in \mathbb{N}$ we write

$$y^{(g)} = \sum_{\substack{d\in G \\ d\le g}} y_d,$$

and we consider the sequence

$$\left(y^{(g)}\right)_{g\in\mathbb{N}} \in R^{\mathbb{N}}.$$

Let now $e \in \mathbb{N}$. Then for any $g, g' \in \mathbb{N}$ with $g, g' \geq ve$ we have with Equation (6.10)

$$y^{(g)} - y^{(g')} = \sum_{\substack{d \in G \\ d \leq g}} y_d - \sum_{\substack{d \in G \\ d \leq g'}} y_d = \sum_{\substack{d \in G \\ \min(g,g') < d \leq \max(g,g')}} y_d \in (\mathfrak{m}_R)^e.$$

Hence, $\left(y^{(g)}\right)_{g \in \mathbb{N}}$ is a Cauchy sequence in R, and since R is complete, this implies

$$\sum_{d \in G} y_d = \lim_{g \in \mathbb{N}} y^{(g)} \in R.$$

Moreover, since by Proposition E.4 for any $x \in R$ there is $(z_d)_{d \in G} \in \prod_{d \in G} R_d$ such that $z = \sum_{d \in G} z_d$, we obtain $R \cong \prod_{d \in G} R_d$.

(3) Since ϕ is a homogeneous isomorphism by (1), we have

$$\overline{R}_d = \phi^{-1}((\mathbb{C}[[T]])_d)$$

for any $d \in \mathbb{Z}$, where the grading on $\mathbb{C}[[T]]$ corresponds to the $\mathbb{C}$-derivation $\overline{w}t\frac{\partial}{\partial_t}$ (see Theorem E.11.(1)). The statement follows from Lemma 6.10.(2).

(4) Since ϕ is homogeneous by (1), there is an $a \in \mathbb{C}$ such that $\phi(x) = aT^{\alpha}$. Since $\nu = \operatorname{ord}_T \circ \phi$ by Theorem 3.44, this implies $\nu(x) = \alpha$.

(5) Let $\alpha \in \Gamma_R$. Then there is an $x \in R$ with $\nu(x) = \alpha$. Now let

$$(x_d)_{d \in \overline{w}\mathbb{Z}} \in \prod_{d \in \overline{w}\mathbb{Z}} R_d$$

such that $x = \sum_{d \in \overline{w}\mathbb{Z}} x_d$. Then by Lemma 6.10.(2) and Proposition E.9 there is $(a_d)_{d \in \overline{w}\mathbb{Z}} \in \mathbb{C}^{\overline{w}\mathbb{Z}}$ such that

$$\phi(x) = \phi\left(\sum_{d \in \overline{w}\mathbb{Z}} x_d\right) = \sum_{d \in \overline{w}\mathbb{Z}} \phi(x_d) = \sum_{d \in \overline{w}\mathbb{Z}} a_d T^{\frac{d}{\overline{w}}}.$$

Since $\nu = \operatorname{ord}_T \circ \phi$ by Theorem 3.44, this implies $a_{\overline{w}\alpha} \neq 0$. Hence, $x_{\overline{w}\alpha} \neq 0$, and

$$\nu(x_{\overline{w}\alpha}) = \operatorname{ord}_T \circ \phi(x_{\overline{w}\alpha}) = \operatorname{ord}_T\left(a_{\overline{w}\alpha} T^{\frac{\overline{w}\alpha}{\overline{w}}}\right) = \alpha.$$

(6) By (1) and Lemma 6.10.(2) we have for any $\alpha \in \mathbb{N}$

$$R_{\overline{w}\alpha} \subset \overline{R}_{\overline{w}\alpha} = \phi^{-1}(\mathbb{C} \cdot T^{\alpha}).$$

Moreover, if $\alpha \in \mathbb{N} \setminus \Gamma_R$, then $R_{\overline{w}\alpha} = 0$ by (4). It remains to show that $\phi^{-1}(aT^{\alpha}) \in R_{\overline{w}\alpha}$ for any $\alpha \in \Gamma_R$ and for any $a \in \mathbb{C}$.

So let $\alpha \in \Gamma_R$. Then by (5) there is $x \in R_{\overline{w}\alpha} \setminus \{0\}$, and Lemma 6.10.(2) yields a $b \in \mathbb{C}$ such that $\phi(x) = bT^{\alpha}$. So for any $a \in \mathbb{C}$ we have $\frac{a}{b}x \in R_{\overline{w}\alpha}$ and

$$\phi\left(\frac{a}{b}x\right) = \frac{a}{b}\phi(x) = \frac{a}{b}bT^{\alpha} = aT^{\alpha}. \qquad \square$$

Proof of Theorem 6.9. (1) See Lemma 6.11.(1).

(2) Let $x \in R^{\mathrm{reg}}$ be homogeneous of degree $\deg(x)$, and let

$$\phi\colon \overline{R} \to \mathbb{C}[[T]],\\ t \mapsto T$$

be the isomorphism of Theorem 3.44. Then ϕ is homogeneous by Lemma 6.11.(1) if we consider on $\mathbb{C}[[T]]$ the grading corresponding to the $\mathbb{C}$-derivation $\overline{w}t\frac{\partial}{\partial_t}$ (see Theorem E.11.(1)), and hence $\phi(x) \in (\mathbb{C}[[T]])_{\deg(x)}$. So by Lemma 6.10.(2) there is an $a \in \mathbb{C}$ such that $\phi(x) = aT^{\frac{\deg(x)}{\overline{w}}}$. Thus, Theorem 3.44 yields

$$\nu(x) = \mathrm{ord}_T \circ \phi(x) = \frac{\deg(x)}{\overline{w}}.$$

(3) The isomorphism ϕ is homogeneous by Lemma 6.11.(1), and Lemma 6.11.(2) and (6) yield the homogeneous restriction ϕ'.

(4) The commutative diagrams follow immediately from (3). Now the isomorphism $\psi \circ \phi^{-1}$ is determined by

$$\psi \circ \phi^{-1}(S) = f$$

for some power series $f \in \mathbb{C}[[T]]$. Since

$$1 = \mathrm{ord}_S(S) = \mathrm{ord}_T\left(\psi \circ \phi^{-1}(S)\right) = \mathrm{ord}_T(f),$$

we obtain

$$f = Tg,$$

where $g \in \mathbb{C}[[T]]$ with $\mathrm{ord}_T(g) = 0$. As $\mathrm{ord}_T = \nu \circ \phi^{-1}$, this implies $g \in (\mathbb{C}[[T]])^*$ by Lemma 3.4.(3).

(5) Let $\mathfrak{i}$ be a homogeneous non-zero ideal of R. Since R is Noetherian, $\mathfrak{i}$ is by Proposition E.6.(1) generated by finitely many homogeneous elements $y_1, \ldots, y_m$. For any $i = 1, \ldots, m$ there is by Lemma 6.10.(2) an $b_i \in \mathbb{C}$ such that

$$\phi(y_i) = b_i T^{\frac{\deg(y_i)}{\overline{w}}}.$$

Now let $x \in R$. Then there is $(x_d)_{d\in G} \in \prod_{d\in G} R_d$ such that $x = \sum_{d\in G} x_d$. Moreover, by Lemma 6.10.(2) there is $(a_d)_{d\in G} \in \mathbb{C}^G$ such that

$$\phi(x_d) = a_d T^{\frac{d}{\overline{w}}}$$

for all $d \in G$. So for any $i = 1, \ldots, m$ and any $d \in G$ we have

$$\phi(y_i x_d) = \phi(y_i)\,\phi(x_d) = b_i a_d T^{\frac{\deg(y_i)+d}{\overline{w}}},$$

and Remark 4.6.(5) and Theorem 3.44 yield

$$\frac{\deg(y_i)+d}{\overline{w}} = \operatorname{ord}_T \circ \phi(y_i x_d) = \nu(y_i x_d) = \nu(y_i) + \nu(x_d) \in \Gamma_{\mathfrak{i}}.$$

Therefore, we obtain with Proposition E.9

$$\phi(y_i x) = \phi\left(y_i \sum_{d\in G} x_d\right) = \phi\left(\sum_{d\in G} y_i x_d\right) = \sum_{d\in G} \phi(y_i x_d) \in \mathbb{C}\left[\left[T^{\Gamma_{\mathfrak{i}}}\right]\right].$$

This implies

$$\phi(\mathfrak{i}) \subset \mathbb{C}\left[\left[T^{\Gamma_{\mathfrak{i}}}\right]\right] \tag{6.11}$$

Since $\phi(R) = \mathbb{C}\left[\left[T^{\Gamma_R}\right]\right]$ by (3), since $\mathbb{C}\left[\left[T^{\Gamma_{\mathfrak{i}}}\right]\right] \in \mathcal{R}_{\mathbb{C}[[T^{\Gamma_R}]]}$ (see Remark 4.78.(1)), and since $\Gamma_{\mathfrak{i}} = \Gamma_{\phi(\mathfrak{i})} = \Gamma_{\mathbb{C}[[T^{\Gamma_{\mathfrak{i}}}]]}$ by Theorem 3.44, Equation (6.11) yields with Corollary 4.52

$$\phi(\mathfrak{i}) = \mathbb{C}\left[\left[T^{\Gamma_{\mathfrak{i}}}\right]\right].$$ □

We conclude this section with a lemma we will use later on.

Lemma 6.12. *Let S be a numerical semigroup, and let $E \in \mathcal{G}_S$ with $E \subset S$.*

(1) Let M be a finite set of generators of S not containing 0 (see Proposition 4.72 and Lemma 4.85). Then $\mathbb{C}\left[\left[T^S\right]\right]$ is quasihomogeneous of type $(\alpha)_{\alpha\in M} \in \mathbb{N}^M$.

(2) The ideal $\mathbb{C}\left[\left[t^E\right]\right]$ of $\mathbb{C}\left[\left[t^S\right]\right]$ (see Remark 4.78.(1)) is homogeneous.

(3) An element

$$\overline{\sum_{\alpha\in S} a_\alpha t^\alpha} \in \mathbb{C}\left[\left[t^S\right]\right]/\mathbb{C}\left[\left[t^E\right]\right]$$

is homogeneous in the induced grading on $\mathbb{C}\left[\left[t^S\right]\right]/\mathbb{C}\left[\left[t^E\right]\right]$ (see (1) and (2), Remark 4.78.(1) and Proposition E.6.(3)) if and only if there is a $\beta \in S \setminus E$ such that $a_\alpha = 0$ for all $\alpha \in (S\setminus E)\setminus\{\beta\}$.

Proof. (1) First note that $\mathbb{C}\left[\left[T^S\right]\right]$ is a local admissible ring by Proposition 4.80, and it is complete by Proposition 4.81. Hence, it is a complex local algebroid curve.

Consider the $\mathbb{C}$-derivation

$$\mathfrak{d} = t\partial_t \colon \mathbb{C}\left[\left[t^S\right]\right] \to \mathbb{C}\left[\left[t^S\right]\right].$$

Indeed we have for any $\sum_{\alpha\in S} a_\alpha t^\alpha$

$$\mathfrak{d}\left(\sum_{\alpha\in S} a_\alpha t^\alpha\right) = \sum_{\alpha\in S} a_\alpha t\partial_t t^\alpha = \sum_{\alpha\in S} \alpha a_\alpha t^\alpha \in \mathbb{C}\left[\left[t^S\right]\right].$$

In particular, for any $\alpha \in M$ we obtain

$$\mathfrak{d}(t^\alpha) = t\partial_t t^\alpha = \alpha t^\alpha.$$

Since

$$\langle t^\alpha \mid \alpha \in M \rangle = \mathfrak{m}_{\mathbb{C}[[t^S]]}$$

by Lemma 4.85, the ring $\mathbb{C}\left[\left[t^S\right]\right]$ is quasihomogeneous (see Definition 6.1).

(2) Let $\sum_{\alpha\in E} a_\alpha t^\alpha \in \mathbb{C}\left[\left[t^E\right]\right]$. Then

$$\mathfrak{d}\left(\sum_{\alpha\in E} a_\alpha t^\alpha\right) = \sum_{\alpha\in E} \alpha a_\alpha t^\alpha.$$

Hence, the ideal $\mathbb{C}\left[\left[t^E\right]\right]$ is homogeneous by Lemma E.15.

(3) An element $\overline{\sum_{\alpha\in S} a_\alpha t^\alpha} \in \mathbb{C}\left[\left[t^S\right]\right]/\mathbb{C}\left[\left[t^E\right]\right]$ is homogeneous with respect to the induced grading if and only if

$$\sum_{\alpha\in S} \alpha a_\alpha t^\alpha = \mathfrak{d}\left(\sum_{\alpha\in S} a_\alpha t^\alpha\right) \in \sum_{\alpha\in S} a_\alpha t^\alpha + \mathbb{C}\left[\left[t^E\right]\right].$$

The statement follows. □

6.2. Proof of Theorem 6.2

Let R be a complex algebroid curve which is quasihomogeneous of type $w \in \mathbb{N}^n$. Then every $\mathfrak{p} \in \operatorname{Min}(R)$ is homogeneous by Proposition E.17, and hence also $\mathfrak{p} + \mathfrak{q}$ is homogeneous for any $\mathfrak{p}, \mathfrak{q} \in \operatorname{Min}(R)$. This implies that $R/\mathfrak{p}$ and $R/\mathfrak{p} + \mathfrak{q}$ are quasihomogeneous of type w with the induced grading by Proposition E.6.(3).

Moreover, also $\mathfrak{q} + \mathfrak{p}/\mathfrak{p}$ is a homogeneous ideal of $R/\mathfrak{p}$. Hence, $(R/\mathfrak{p})/(\mathfrak{q} + \mathfrak{p}/\mathfrak{p})$ is quasihomogeneous with respect to w with the induced grading by Proposition E.6.(3), and this grading corresponds to that on $R/\mathfrak{p} + \mathfrak{q}$.

Since by Theorem E.11 any grading corresponds to a derivation, we obtain for any $\mathfrak{p}, \mathfrak{q} \in \operatorname{Min}(R)$ a commutative diagram

$$\begin{tikzcd}
 & R \arrow[r, "\mathfrak{d}"] \arrow[d, two heads, "\pi_{\mathfrak{p}}"] \arrow[dddl, two heads, "\pi_{\mathfrak{p}+\mathfrak{q}}"'] & R \arrow[d, two heads, "\pi_{\mathfrak{p}}"'] \arrow[dddr, two heads, "\pi_{\mathfrak{p}+\mathfrak{q}}"] & \\
 & R/\mathfrak{p} \arrow[r, "\mathfrak{d}_{\mathfrak{p}}"] \arrow[d, two heads, "\pi_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}"] & R/\mathfrak{p} \arrow[d, two heads, "\pi_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}"'] & \\
 & (R/\mathfrak{p})/(\mathfrak{q}+\mathfrak{p}/\mathfrak{p}) \arrow[r, "\mathfrak{d}_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}"'] & (R/\mathfrak{p})/(\mathfrak{q}+\mathfrak{p}/\mathfrak{p}) & \\
R/\mathfrak{p}+\mathfrak{q} \arrow[rrr, "\mathfrak{d}_{\mathfrak{p}+\mathfrak{q}}"'] \arrow[ur, "\cong"'] & & & R/\mathfrak{p}+\mathfrak{q}, \arrow[ul, "\cong"]
\end{tikzcd} \tag{6.12}$$

where $\pi_\mathfrak{p}$, $\pi_{\mathfrak{p}+\mathfrak{q}}$ and $\pi_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}$ are the canonical surjections, and $\mathfrak{d}$, $\mathfrak{d}_\mathfrak{p}$, $\mathfrak{d}_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}$ and $\mathfrak{d}_{\mathfrak{p}+\mathfrak{q}}$ are the $\mathbb{C}$-derivations of R, $R/\mathfrak{p}$, $(R/\mathfrak{p})/(\mathfrak{q}+\mathfrak{p}/\mathfrak{p})$ and $R/\mathfrak{p}+\mathfrak{q}$ corresponding to the respective gradings.

So by Theorem 6.9.(4) and (5), Remark 3.39 (Equation (3.19)) and Propositions 4.67.(2), 4.69, and 4.79 there is a commutative diagram of homogeneous homomorphisms

$$\begin{tikzcd}
 & \mathbb{C}\left[\left[t_\mathfrak{p}^{(\Gamma_R)_\mathfrak{p}}\right]\right] \arrow[rr, two heads] & & \mathbb{C}\left[t_\mathfrak{p}^{(\Gamma_R)_\mathfrak{p}/(\Gamma_R)_\mathfrak{p}^\mathfrak{q}}\right] \arrow[dddddd, "\cong"] \\
 & \mathbb{C}\left[\left[t_\mathfrak{p}^{\Gamma_{R/\mathfrak{p}}}\right]\right] \arrow[u, "\cong"'] \arrow[r, two heads] \arrow[dr, two heads] & \mathbb{C}\left[t_\mathfrak{p}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right] \arrow[ur, "\cong"'] & \\
 & R/\mathfrak{p} \arrow[u, "\cong"'] \arrow[dr, two heads, "\pi_{\mathfrak{q}+\mathfrak{p}}"'] & \mathbb{C}\left[\left[t_\mathfrak{p}^{\Gamma_{R/\mathfrak{p}}}\right]\right]/\mathbb{C}\left[\left[t_\mathfrak{p}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right] \arrow[u, "\cong"'] & \\
R \arrow[ur, two heads, "\pi_\mathfrak{p}"] \arrow[dr, two heads, "\pi_\mathfrak{q}"'] & & R/\mathfrak{p}+\mathfrak{q} \arrow[u, "\cong"'] \arrow[d, "\cong"'] & \\
 & R/\mathfrak{q} \arrow[ur, two heads, "\pi_{\mathfrak{p}+\mathfrak{q}}"] \arrow[d, "\cong"'] & \mathbb{C}\left[\left[t_\mathfrak{q}^{\Gamma_{R/\mathfrak{q}}}\right]\right]/\mathbb{C}\left[\left[t_\mathfrak{q}^{\Gamma_{\mathfrak{p}+\mathfrak{q}/\mathfrak{q}}}\right]\right] \arrow[d, "\cong"'] & \\
 & \mathbb{C}\left[\left[t_\mathfrak{q}^{\Gamma_{R/\mathfrak{q}}}\right]\right] \arrow[d, "\cong"'] \arrow[r, two heads] \arrow[ur, two heads] & \mathbb{C}\left[t_\mathfrak{q}^{\Gamma_{R/\mathfrak{q}}/\Gamma_{\mathfrak{p}+\mathfrak{q}/\mathfrak{q}}}\right] \arrow[dr, "\cong"] & \\
 & \mathbb{C}\left[\left[t_\mathfrak{q}^{(\Gamma_R)_\mathfrak{q}}\right]\right] \arrow[rr, two heads] & & \mathbb{C}\left[t_\mathfrak{q}^{(\Gamma_R)_\mathfrak{q}/(\Gamma_R)_\mathfrak{q}^\mathfrak{p}}\right].
\end{tikzcd} \tag{6.13}$$

This leads to the proof of Theorem 6.2.

Proof of Theorem 6.2. Let R be quasihomogeneous of type $w \in \mathbb{N}^n$. Then every $\mathfrak{p} \in \operatorname{Min}(R)$ is homogeneous by Proposition E.17, and hence $R/\mathfrak{p}$ is quasihomogeneous of type w with induced grading by Proposition E.6.(3). Hence, Theorem 6.9.(1) yields a $\overline{w} = (\overline{w}_\mathfrak{p})_{\mathfrak{p}\in\operatorname{Min}(R)} \in \mathbb{N}^{\operatorname{Min}(R)}$ such that $\overline{R/\mathfrak{p}}$ is quasihomogeneous of type $\overline{w}_\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Min}(R)$, i.e. $R/\mathfrak{p}$ is $G_\mathfrak{p}$-graded with $G_\mathfrak{p} = \overline{w}_\mathfrak{p}\mathbb{Z}$.

(1) Let $x \in R$ homogeneous. Then $\pi_\mathfrak{p}(x)$ is homogeneous in $R/\mathfrak{p}$, and if $\pi(x) \neq 0$, Remark 3.39, Proposition E.6.(3) and Theorem 6.9.(2) yield

$$\deg(x) = \deg(\pi(x)) = \overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x).$$

(2) Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Min}(R)$ with $\mathfrak{p} \neq \mathfrak{q}$, and let $\alpha \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$. Since $(\Gamma_R)_\mathfrak{p} = \Gamma_{R/\mathfrak{p}}$ by Remark 3.39 (Equation (3.19)) and Proposition 4.67.(2) and $(\Gamma_R)_\mathfrak{p}^\mathfrak{q} = \Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}$ by

Proposition 4.69, there is by Lemma 6.11.(5) an

$$x \in (R/\mathfrak{p})_{\overline{w}_{\mathfrak{p}}\alpha} \setminus \Big((\mathfrak{q} + \mathfrak{p}/\mathfrak{p}) \cap (R/\mathfrak{p})_{\overline{w}_{\mathfrak{p}}\alpha}\Big)$$

with $\nu_{\overline{R/\mathfrak{p}}}(x) = \alpha$ (see Remark 3.39). In particular, $x \neq 0$. Then by Lemma E.7 there is an

$$X \in R_{\overline{w}_{\mathfrak{p}}\alpha} \setminus ((\mathfrak{p} + \mathfrak{q}) \cap R_{\overline{w}_{\mathfrak{p}}\alpha})$$

with $\pi_{\mathfrak{p}}(X) = x$ and $\nu_{\mathfrak{p}}(X) = \nu_{\overline{R/\mathfrak{p}}} \circ \pi_{\mathfrak{p}}(X) = \alpha$ (see Remark 3.39).

Now Lemma E.7 yields

$$\pi_{\mathfrak{q}}(X) \in (R/\mathfrak{q})_{\overline{w}_{\mathfrak{p}}\alpha} \setminus \Big((\mathfrak{p} + \mathfrak{q}/\mathfrak{q}) \cap (R/\mathfrak{q})_{\overline{w}_{\mathfrak{p}}\alpha}\Big).$$

In particular, $\pi_{\mathfrak{q}}(X) \neq 0$. So $\nu_{\mathfrak{q}}(X) \in (\Gamma_R) \setminus (\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}$ by Remark 3.39 (Equation (3.19)) and Propositions 4.67.(2) and 4.69. Moreover, by (1) we obtain

$$\overline{w}_{\mathfrak{p}}\alpha = \overline{w}_{\mathfrak{p}}\nu_{\mathfrak{p}}(X) = \deg(X) = \overline{w}_{\mathfrak{q}}\nu_{\mathfrak{q}}(X).$$

This yields a map

$$\begin{aligned} \tau_{\mathfrak{p}\mathfrak{q}} \colon (\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} &\to (\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}, \\ \alpha &\mapsto \frac{\overline{w}_{\mathfrak{p}}\alpha}{\overline{w}_{\mathfrak{q}}}. \end{aligned}$$

By symmetry, there is also a map $\tau_{\mathfrak{q}\mathfrak{p}} \colon (\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \to (\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}$, and for any $\alpha \in (\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ we have

$$\tau_{\mathfrak{q}\mathfrak{p}} \circ \tau_{\mathfrak{p}\mathfrak{q}}(\alpha) = \tau_{\mathfrak{q}\mathfrak{p}}\left(\frac{\overline{w}_{\mathfrak{p}}\alpha}{\overline{w}_{\mathfrak{q}}}\right) = \frac{\overline{w}_{\mathfrak{q}}\frac{\overline{w}_{\mathfrak{p}}\alpha}{\overline{w}_{\mathfrak{q}}}}{\overline{w}_{\mathfrak{p}}} = \alpha.$$

Hence, $\tau_{\mathfrak{p}\mathfrak{q}}$ is surjective, and therefore bijective as $(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ and $(\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}$ are finite (see Remark 4.75).

(3) Let $\mathfrak{p} \in \operatorname{Min}(R)$. By Theorem 6.9.(3) there is an isomorphism

$$\phi_{\mathfrak{p}} \colon R/\mathfrak{p} \to \mathbb{C}\Big[\Big[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\Big]\Big]$$

such that $\nu_{\mathfrak{p}} = \operatorname{ord}_t \circ \phi_{\mathfrak{p}}$ by Remark 3.39 (Equation (3.19)). Since $\Gamma_{R/\mathfrak{p}} = (\Gamma_R)_{\mathfrak{p}}$ by Remark 3.39 (Equation (3.19)) and Proposition 4.67.(2), we obtain a natural isomorphism

$$\eta_{\mathfrak{p}} \colon \mathbb{C}\Big[\Big[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\Big]\Big] \to \mathbb{C}\Big[\Big[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\Big]\Big].$$

This yields a surjective homomorphism

$$\psi_{\mathfrak{p}} = \eta_{\mathfrak{p}} \circ \phi_{\mathfrak{p}} \circ \pi_{\mathfrak{p}} \colon R \to \mathbb{C}\Big[\Big[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\Big]\Big]$$

satisfying

$$\nu_{\mathfrak{p}} = \operatorname{ord}_{t_{\mathfrak{p}}} \circ \psi_{\mathfrak{p}}. \tag{6.14}$$

Let now $\mathfrak{q} \in \operatorname{Min}(R)$ with $\mathfrak{p} \neq \mathfrak{q}$. Then $\mathfrak{q} + \mathfrak{p}/\mathfrak{p} \in \mathcal{R}_{R/\mathfrak{p}}$ since $\mathfrak{q}$ is an ideal of R not contained in $\mathfrak{p}$ (as $\mathfrak{p}, \mathfrak{q} \in \operatorname{Min}(R)$ with $\mathfrak{p} \neq \mathfrak{q}$), and since $R/\mathfrak{p}$ is a domain. Moreover, $\mathfrak{q} + \mathfrak{p}/\mathfrak{p}$ is homogeneous since $\mathfrak{q} \in \operatorname{Min}(R)$ is homogeneous by Proposition E.17, and since the grading on $R/\mathfrak{p}$ is induced by that on R. Therefore, Theorem 6.9.(5) yields

$$\phi_{\mathfrak{p}}(\mathfrak{q} + \mathfrak{p}/\mathfrak{p}) = \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right].$$

Thus, we obtain an isomorphism

$$\phi_{\mathfrak{p}\mathfrak{q}} \colon (R/\mathfrak{p})/(\mathfrak{q} + \mathfrak{p}/\mathfrak{p}) \to \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right]/\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right]$$

such that $\phi_{\mathfrak{p}\mathfrak{q}} \circ \pi_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}} = \theta_{\mathfrak{p}\mathfrak{q}} \circ \phi_{\mathfrak{p}}$, where

$$\pi_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}} \colon R/\mathfrak{p} \to (R/\mathfrak{p})/(\mathfrak{q} + \mathfrak{p}/\mathfrak{p})$$

and

$$\theta_{\mathfrak{p}\mathfrak{q}} \colon \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right] \to \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right]/\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right]$$

are the canonical surjections. Now Proposition 4.79 yields a isomorphism

$$\mu_{\mathfrak{p}\mathfrak{q}} \colon \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right]/\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right] \to \mathbb{C}\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]$$

and a surjective homomorphism

$$\vartheta_{\mathfrak{p}\mathfrak{q}} = \mu_{\mathfrak{p}\mathfrak{q}} \circ \theta_{\mathfrak{p}\mathfrak{q}} \colon \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right] \to \mathbb{C}\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right].$$

Since $\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}} = (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ by Proposition 4.69, there is a natural isomorphism

$$\lambda_{\mathfrak{p}\mathfrak{q}} \colon \mathbb{C}\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right] \to \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right].$$

So setting

$$\chi_{\mathfrak{p}\mathfrak{q}} = \lambda_{\mathfrak{p}\mathfrak{q}} \circ \vartheta_{\mathfrak{p}\mathfrak{q}} \circ (\eta_{\mathfrak{p}})^{-1} \colon \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right] \to \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right]$$

we obtain a commutative diagram

$$\begin{tikzcd}
 & & R/\mathfrak{p}+\mathfrak{q} \arrow[d, "\kappa_{\mathfrak{pq}}", "\cong"'] \\
R \arrow[urr, two heads, "\pi_{\mathfrak{p}+\mathfrak{q}}"] \arrow[r, two heads, "\pi_{\mathfrak{p}}"'] \arrow[dddr, two heads, "\psi_{\mathfrak{p}}"'] & R/\mathfrak{p} \arrow[r, two heads, "\pi_{\mathfrak{q}+\mathfrak{p}/\mathfrak{q}}"'] \arrow[d, "\phi_{\mathfrak{p}}", "\cong"'] & (R/\mathfrak{p})/(\mathfrak{q}+\mathfrak{p}/\mathfrak{p}) \arrow[d, "\phi_{\mathfrak{pq}}", "\cong"'] \\
 & \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right] \arrow[r, two heads, "\theta_{\mathfrak{pq}}"] \arrow[dr, two heads, "\vartheta_{\mathfrak{pq}}"'] \arrow[dd, "\eta_{\mathfrak{p}}", "\cong"'] & \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right]/\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right] \arrow[d, "\mu_{\mathfrak{pq}}", "\cong"'] \\
 & & \mathbb{C}\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}/\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right] \arrow[d, "\lambda_{\mathfrak{pq}}", "\cong"'] \\
 & \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right] \arrow[r, two heads, "\chi_{\mathfrak{pq}}"] & \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right],
\end{tikzcd} \tag{6.15}$$

where $\pi_{\mathfrak{p}+\mathfrak{q}}\colon R \to R/\mathfrak{p}+\mathfrak{q}$ denotes the canonical surjection, and $\kappa_{\mathfrak{pq}}\colon R/\mathfrak{p}+\mathfrak{q} \to (R/\mathfrak{p})/(\mathfrak{q}+\mathfrak{p}/\mathfrak{p})$ denotes the natural isomorphism, see Diagram (6.12). Moreover, since all gradings are induced from R, all maps in Diagram (6.15) are homogeneous. So interchanging $\mathfrak{p}$ and $\mathfrak{q}$ we obtain a homogeneous isomorphism

$$\begin{aligned}\sigma_{\mathfrak{pq}} = \kappa_{\mathfrak{qp}} \circ \phi_{\mathfrak{qp}} \circ \mu_{\mathfrak{qp}} \circ \lambda_{\mathfrak{qp}} \circ (\kappa_{\mathfrak{pq}} \circ \phi_{\mathfrak{pq}} \circ \mu_{\mathfrak{pq}} \circ \lambda_{\mathfrak{pq}})^{-1}\colon& \\ \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right] \to \mathbb{C}\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}}\right]&.\end{aligned}$$

such that the diagram

$$\begin{tikzcd}
 & R \arrow[dl, two heads, "\psi_{\mathfrak{p}}"'] \arrow[dr, two heads, "\psi_{\mathfrak{q}}"] & \\
\mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right] \arrow[d, two heads, "\chi_{\mathfrak{pq}}"'] & & \mathbb{C}\left[\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}}\right]\right] \arrow[d, two heads, "\chi_{\mathfrak{qp}}"] \\
\mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right] \arrow[rr, "\cong", "\sigma_{\mathfrak{pq}}"'] & & \mathbb{C}\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}/(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}}\right],
\end{tikzcd} \tag{6.16}$$

commutes. It remains to prove that this isomorphism is induced by $\tau_{\mathfrak{pq}}$, i.e. that $\sigma_{\mathfrak{pq}}\left(\overline{t_{\mathfrak{p}}^{\alpha}}\right) = c_{\mathfrak{pq}}\overline{t_{\mathfrak{p}}^{\tau_{\mathfrak{pq}}(\alpha)}}$ for some $c_{\mathfrak{pq}} \in \mathbb{C}$ and for every $\alpha \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$.

So let $\alpha \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$, and let $\overline{t_{\mathfrak{p}}^{\alpha}} \in \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right]$. Then

$$\overline{t_{\mathfrak{p}}^{\alpha}} \in \left(\mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right)_{\overline{w}_{\mathfrak{p}}\alpha}$$

by Lemma E.7 since $t_{\mathfrak{p}}^{\alpha} \in \left(\mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_R}\right]\right]\right)_{\overline{w}_{\mathfrak{p}}\alpha}$ by Lemma 6.11.(6) with $\chi_{\mathfrak{p}\mathfrak{q}}\left(t_{\mathfrak{p}}^{\alpha}\right) = \overline{t_{\mathfrak{p}}^{\alpha}}$. Moreover, we have

$$t_{\mathfrak{p}}^{\alpha} \in \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{R/\mathfrak{p}}}\right]\right] \setminus \mathbb{C}\left[\left[t_{\mathfrak{p}}^{\Gamma_{\mathfrak{q}+\mathfrak{p}/\mathfrak{p}}}\right]\right]$$

by Proposition 4.79. Hence,

$$(\eta_{\mathfrak{p}} \circ \phi_{\mathfrak{p}})^{-1}\left(t_{\mathfrak{p}}^{\alpha}\right) \in (R \setminus \mathfrak{p})_{\overline{w}_{\mathfrak{p}}\alpha} \setminus \left(\mathfrak{q} + \mathfrak{p}/\mathfrak{p} \cap (R \setminus \mathfrak{p})_{\overline{w}_{\mathfrak{p}}\alpha}\right)$$

by Proposition 4.69 and Theorem 6.9.(5). Then there is by Lemma E.7 an $x \in R_{\overline{w}_{\mathfrak{p}}\alpha}$ with $\psi_{\mathfrak{p}}(x) = t_{\mathfrak{p}}^{\alpha}$ and $\pi_{\mathfrak{q}}(x) \neq 0$. Since $\nu_{\mathfrak{p}}(x) = \alpha$ by Equation (6.14), (1) yields

$$\nu_{\mathfrak{q}}(x) = \frac{\overline{w}_{\mathfrak{p}}\alpha}{\overline{w}_{\mathfrak{q}}} = \tau_{\mathfrak{p}\mathfrak{q}}(\alpha).$$

So Equation (6.14) yields with Lemma 6.10.(2) $\psi_{\mathfrak{q}}(x) = a_{\alpha} t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha)}$ for some $a_{\alpha} \in \mathbb{C}$. Then Diagram (6.16) implies

$$\sigma_{\mathfrak{p}\mathfrak{q}}\left(\overline{t_{\mathfrak{p}}^{\alpha}}\right) = \sigma_{\mathfrak{p}\mathfrak{q}} \circ \chi_{\mathfrak{p}\mathfrak{q}} \circ \psi_{\mathfrak{p}}(x) = \chi_{\mathfrak{q}\mathfrak{p}} \circ \psi_{\mathfrak{q}}(x) = \chi_{\mathfrak{q}\mathfrak{p}}\left(a_{\alpha} t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha)}\right) = \overline{a_{\alpha} t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha)}}.$$

Moreover, since $\sigma_{\mathfrak{p}\mathfrak{q}}$ is a $\mathbb{C}$-algebra isomorphism, we have for any $\beta \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$

$$\begin{aligned}
\overline{a_{\alpha+\beta} t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha+\beta)}} &= \sigma_{\mathfrak{p}\mathfrak{q}}\left(\overline{t_{\mathfrak{p}}^{\alpha+\beta}}\right) \\
&= \sigma_{\mathfrak{p}\mathfrak{q}}\left(\overline{t_{\mathfrak{p}}^{\alpha} t_{\mathfrak{p}}^{\beta}}\right) \\
&= \sigma_{\mathfrak{p}\mathfrak{q}}\left(\overline{t_{\mathfrak{p}}^{\alpha}}\right) \sigma_{\mathfrak{p}\mathfrak{q}}\left(\overline{t_{\mathfrak{p}}^{\beta}}\right) \\
&= \overline{a_{\alpha} a_{\beta} t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha)} t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha)}} \\
&= \overline{a_{\alpha} a_{\beta} t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha+\alpha)}}.
\end{aligned}$$

Hence, we may define a map

$$\begin{aligned}
\zeta_{\mathfrak{p}\mathfrak{q}} \colon (\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} &\to \mathbb{C}, \\
\alpha &\mapsto a_{\alpha}
\end{aligned}$$

with

$$\zeta_{\mathfrak{p}\mathfrak{q}}(\alpha + \beta) = \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha)\, \zeta_{\mathfrak{p}\mathfrak{q}}(\beta)$$

for all $\alpha, \beta \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ with $\alpha + \beta \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ such that $\sigma_{\mathfrak{p}\mathfrak{q}}$ is induced by $\tau_{\mathfrak{p}\mathfrak{q}}$ and $\zeta_{\mathfrak{p}\mathfrak{q}}$.

(4) Let $\mathcal{C}$ be the category of $\mathbb{C}$-algebras, let $\mathcal{I}$ be a category with $\operatorname{Ob}\mathcal{I} = \operatorname{Min}(R)$, and let $D \colon \mathcal{I} \to \mathcal{C}$ be a diagram of type $\mathcal{I}$ with $D(\mathfrak{p}) = \mathbb{C}\left[\left[t^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$ for any $\mathfrak{p} \in \operatorname{Min}(R)$.

Let $\mathcal{J}$ and $F\colon \mathcal{J} \to \mathcal{C}$ be as in Definition 2.29, where for any $(\mathfrak{p},\mathfrak{q}) \in \operatorname{Ob}\mathcal{J}$ with $\mathfrak{p} \neq \mathfrak{q}$ we have

$$F((\mathfrak{p},\mathfrak{q})) = \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right],$$

$$F((\mathfrak{p},\mathfrak{p}) \to (\mathfrak{q},\mathfrak{p})) = \chi_{\mathfrak{p}\mathfrak{q}}\colon \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right] \to \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right],$$

and

$$F((\mathfrak{p},\mathfrak{q}) \to (\mathfrak{q},\mathfrak{p})) = \sigma_{\mathfrak{p}\mathfrak{q}}.$$

Then by Corollary 2.34 there is a $\mathbb{C}$-algebra isomorphism

$$\Phi\colon \operatorname{Fib}(F) \to A,$$

where A is the $\mathbb{C}$-subalgebra of $\prod_{\mathfrak{p}\in\operatorname{Min}(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$ consisting of all elements

$$\left(\sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}} a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}\right)_{\mathfrak{p}\in\operatorname{Min}(R)} \in \prod_{\mathfrak{p}\in\operatorname{Min}(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

with

$$\begin{aligned}
\overline{\sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}\setminus(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}} a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_{\mathfrak{p}})\, t_{\mathfrak{q}}^{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_{\mathfrak{p}})}} &= \sigma_{\mathfrak{p}\mathfrak{q}}\left(\overline{\sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}\setminus(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}} a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}}\right) \\
&= \sigma_{\mathfrak{p}\mathfrak{q}} \circ \chi_{\mathfrak{p}\mathfrak{q}}\left(\sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}} a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}\right) \\
&= \chi_{\mathfrak{q}\mathfrak{p}}\left(\sum_{\alpha_{\mathfrak{q}}\in(\Gamma_R)_{\mathfrak{q}}} a_{\alpha_{\mathfrak{q}}}^{(\mathfrak{q})} t_{\mathfrak{q}}^{\alpha_{\mathfrak{q}}}\right) \\
&= \overline{\sum_{\alpha_{\mathfrak{q}}\in(\Gamma_R)_{\mathfrak{q}}\setminus(\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}} a_{\alpha_{\mathfrak{q}}}^{(\mathfrak{q})} t_{\mathfrak{q}}^{\alpha_{\mathfrak{q}}}}
\end{aligned}$$

for every $\mathfrak{p},\mathfrak{q} \in \operatorname{Min}(R)$. This is equivalent to the condition

$$a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} = \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_{\mathfrak{p}})\, a_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_{\mathfrak{p}})}^{(\mathfrak{q})}$$

for all $\mathfrak{p} \in \operatorname{Min}(R)$, for any $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$, and for every $\alpha_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$. Thus,

$$A = \operatorname{Fib}(\Gamma_R, \overline{w}, \zeta)_{\mathfrak{p}\in\operatorname{Min}(R)},$$

where $\zeta = \left(\left(\zeta_{\mathfrak{p},\mathfrak{q}}\right)_{\mathfrak{q}\in\operatorname{Min}(R)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p}\in\operatorname{Min}(R)}$.

By (3) and the universal property of the fibre product (see Lemma 2.31) there is a unique $\mathbb{C}$-algebra homomorphism

$$\begin{aligned}\Psi\colon R &\to \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta),\\ x &\mapsto (\psi_{\mathfrak{p}}(x))_{\mathfrak{p}\in \mathrm{Min}\,(R)}.\end{aligned}$$

By definition Ψ is the restriction of the isomorphism of Theorem 3.44 (cf. (3) and the proof of (3)). In particular, Ψ is injective.

(5) Let $x \in R$, let $\psi_{\mathfrak{p}}(x) = \sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}} a^{(\mathfrak{p})}_{\alpha_{\mathfrak{p}}} t^{\alpha_{\mathfrak{p}}}$ for any $\mathfrak{p} \in \mathrm{Min}\,(R)$, and for every $d \in \mathbb{Z}$ define $y_d \in \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}\left[\left[t^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$ by

$$(y_d)_{\mathfrak{p}} = \begin{cases} a_{d/\overline{w}_{\mathfrak{p}}} t^{\frac{d}{\overline{w}_{\mathfrak{p}}}} & \text{if there is an } \alpha \in (\Gamma_R)_{\mathfrak{p}} \text{ with } \overline{w}_{\mathfrak{p}}\alpha = d,\\ 0 & \text{else,}\end{cases}$$

for each $\mathfrak{p} \in \mathrm{Min}\,(R)$. Since $\psi_{\mathfrak{p}}$ is homogeneous by (3), for any $\mathfrak{p} \in \mathrm{Min}\,(R)$, Lemma 6.10.(2) yields

$$\psi_{\mathfrak{p}}(x_d) = (\psi_{\mathfrak{p}}(x))_d = (y_d)_{\mathfrak{p}}.$$ □

6.3. Proof of Proposition 6.6

Let R be a quasihomogeneous curve. With the notation of Theorem 6.2, we set

$$A = \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta),$$

where $\zeta = \left((\zeta_{\mathfrak{pq}})_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)}$, i.e. A is the subset of $\prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$ consisting of all elements

$$\left(\sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}} a^{(\mathfrak{p})}_{\alpha_{\mathfrak{p}}} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)} \in \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

with

$$a^{(\mathfrak{p})}_{\alpha_{\mathfrak{p}}} = \zeta_{\mathfrak{pq}}(\alpha_{\mathfrak{p}})\, a^{(\mathfrak{q})}_{\tau_{\mathfrak{pq}}(\alpha_{\mathfrak{p}})}$$

for any $\mathfrak{p} \in \mathrm{Min}\,(R)$, for every $\mathfrak{q} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{p}\}$, and for all $\alpha_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)^{\mathfrak{q}}_{\mathfrak{p}}$, where

$$\begin{aligned}\tau_{\mathfrak{pq}}\colon (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)^{\mathfrak{q}}_{\mathfrak{p}} &\to (\Gamma_R)_{\mathfrak{q}} \setminus (\Gamma_R)^{\mathfrak{p}}_{\mathfrak{q}},\\ \alpha &\mapsto \frac{\overline{w}_{\mathfrak{p}}\alpha}{\overline{w}_{\mathfrak{q}}}.\end{aligned}$$

Lemma 6.13. *In the natural way, A is a $\mathbb{C}$-algebra.*

Proof. Let $\mathcal{C}$ be the category of $\mathbb{C}$-algebras, let $\mathcal{I}$ be a category with $\mathrm{Ob}\,\mathcal{I} = \mathrm{Min}\,(R)$, and let $D\colon \mathcal{I} \to \mathcal{C}$ be a diagram of type $\mathcal{I}$ with $D(\mathfrak{p}) = \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$ for any $\mathfrak{p} \in \mathrm{Min}\,(R)$. Let $\mathcal{J}$ and $F\colon \mathcal{J} \to \mathcal{C}$ be as in Definition 2.29, where for any $(\mathfrak{p}, \mathfrak{q}) \in \mathrm{Ob}\,\mathcal{J}$ with $\mathfrak{p} \neq \mathfrak{q}$ we have

$$F((\mathfrak{p}, \mathfrak{q})) = \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right],$$

$$F((\mathfrak{p}, \mathfrak{p}) \to (\mathfrak{q}, \mathfrak{p})) = \chi_{\mathfrak{p}\mathfrak{q}}\colon \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right] \to \mathbb{C}\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right]$$

and

$$F((\mathfrak{p}, \mathfrak{q}) \to (\mathfrak{q}, \mathfrak{p})) = \sigma_{\mathfrak{p}\mathfrak{q}}.$$

Then Corollary 2.34 yields

$$A = \mathrm{Fib}\,(F).$$

In particular, A is a $\mathbb{C}$-subalgebra of $\prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} \mathbb{C}[[t_{\mathfrak{p}}]]$. $\square$

Lemma 6.14. *The following hold for A.*

1. *We have* $\overline{A} = \overline{\Psi(R)}$. *In particular,* $\dim A = 1$.
2. *There is a bijection*

$$\begin{aligned} \eta\colon\ \mathrm{Min}\,(R) &\to \mathrm{Min}\,(A), \\ \mathfrak{p} &\mapsto \mathfrak{p}A, \\ \mathfrak{q} \cap R &\leftarrow\!\shortmid \mathfrak{q}. \end{aligned}$$

Proof. (1) By Theorem 6.2.(4) and Lemma 6.13 we have $\Psi(R) \subset A \subset Q_{\Psi(R)}$, and hence Lemma A.34 yields $Q_A = Q_{\Psi(R)}$. Since $\Psi(R) \subset A \subset \overline{\Psi(R)}$ by construction (see Theorem 3.44), Proposition B.5 implies $\overline{A} = \overline{\Psi(R)}$. In particular,

$$\dim A = \dim R = 1$$

by Theorem B.14.

(2) This follows from (1) and Theorem A.72. $\square$

Lemma 6.15. *The ring A is local with maximal ideal*

$$\mathfrak{m}_A = \{x \in A \mid \mathrm{ord}_t\,(x) > \mathbf{0}\}.$$

Proof. Assume A is not local, and let $\mathfrak{m}, \mathfrak{n} \in \mathrm{Max}\,(A)$ with $\mathfrak{m} \neq \mathfrak{n}$. Then by Propositions B.3 and B.15 and Theorem B.12 there are $\overline{\mathfrak{m}}, \overline{\mathfrak{n}} \in \mathrm{Max}\left(\overline{A}\right)$ with $\overline{\mathfrak{m}} \cap A = \mathfrak{m}$ and $\overline{\mathfrak{n}} \cap A = \mathfrak{n}$. Since

$$\overline{A} = \overline{\Psi(R)} = \Psi\left(\overline{R}\right) = \prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} \mathbb{C}[[t_{\mathfrak{p}}]]$$

by Theorems 3.44 and 6.2.(4) and Lemma 6.14.(1), there are by Lemma A.6.(2) $\mathfrak{p}_\mathfrak{m}, \mathfrak{p}_\mathfrak{n} \in \mathrm{Min}\,(R)$ such that

$$\begin{aligned}\overline{\mathfrak{m}} &= t_{\mathfrak{p}_\mathfrak{m}}\mathbb{C}[[t_{\mathfrak{p}_\mathfrak{m}}]] \times \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)\setminus(\mathfrak{p}_\mathfrak{m})} \mathbb{C}[[t_\mathfrak{p}]],\\ \overline{\mathfrak{n}} &= t_{\mathfrak{p}_\mathfrak{n}}\mathbb{C}[[t_{\mathfrak{p}_\mathfrak{n}}]] \times \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)\setminus(\mathfrak{p}_\mathfrak{n})} \mathbb{C}[[t_\mathfrak{p}]].\end{aligned}$$

Then for any $x \in \mathfrak{m} \setminus (\mathfrak{n} \cap \mathfrak{m})$ this implies

$$\begin{aligned}\mathrm{pr}_{\mathfrak{p}_\mathfrak{m}}(x) &\in t_{\mathfrak{p}_\mathfrak{m}}\mathbb{C}[[t_{\mathfrak{p}_\mathfrak{m}}]],\\ \mathrm{pr}_{\mathfrak{p}_\mathfrak{n}}(x) &\in \mathbb{C}[[t_{\mathfrak{p}_\mathfrak{n}}]] \setminus t_{\mathfrak{p}_\mathfrak{n}}\mathbb{C}[[t_{\mathfrak{p}_\mathfrak{n}}]],\end{aligned}$$

where for every $\mathfrak{p} \in \mathrm{Min}\,(R)$ we denote by $\mathrm{pr}_\mathfrak{p} \colon \prod_{\mathfrak{q}\in\mathrm{Min}\,(R)} \mathbb{C}[[t_\mathfrak{q}]] \to \mathbb{C}[[t_\mathfrak{p}]]$ the projection. In particular, we obtain $\mathrm{ord}_{t_{\mathfrak{p}_\mathfrak{m}}}(x) > 0$ and $\mathrm{ord}_{t_{\mathfrak{p}_\mathfrak{n}}}(x) = 0$. So writing

$$x = \left(\sum_{\alpha_\mathfrak{p}\in(\Gamma_R)_\mathfrak{p}} a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)$$

we have

$$\begin{aligned}a_0^{(\mathfrak{p}_\mathfrak{m})} &= 0,\\ a_0^{(\mathfrak{p}_\mathfrak{n})} &\neq 0.\end{aligned} \tag{6.17}$$

Since $x \in A$, since Γ_R is local by Theorem 4.9, and since therefore $0 \in (\Gamma_R)_{\mathfrak{p}_\mathfrak{m}} \setminus (\Gamma_R)^{\mathfrak{p}_\mathfrak{n}}_{\mathfrak{p}_\mathfrak{m}}$ by Proposition 4.65, Equation (6.17) and the definition of A yield the contradiction

$$0 = a_0^{(\mathfrak{p}_\mathfrak{m})} = \zeta_{\mathfrak{p}_\mathfrak{m}\mathfrak{p}_\mathfrak{n}}(0)\, a^{(\mathfrak{p}_\mathfrak{n})}_{\tau_{\mathfrak{p}_\mathfrak{m}\mathfrak{p}_\mathfrak{n}}(0)} = \zeta_{\mathfrak{p}_\mathfrak{m}\mathfrak{p}_\mathfrak{n}}(0)\, a_0^{(\mathfrak{p}_\mathfrak{n})} \neq 0,$$

where the last inequality follows as $\zeta_{\mathfrak{p}_\mathfrak{m}\mathfrak{p}_\mathfrak{n}}(0) \neq 0$ by Lemma 6.7.(2). Thus, A is local, and the maximal ideal of A is by Theorems 3.44 and B.12, Propositions B.3 and B.15 and Lemmas A.6.(2) and 6.14

$$\begin{aligned}\mathfrak{m}_A &= \left(\bigcap_{\mathfrak{m}\in\mathrm{Max}\,(\overline{A})} \mathfrak{m}\right) \cap A\\ &= \left(\bigcap_{\mathfrak{p}\in\mathrm{Min}\,(R)} t_\mathfrak{p}\mathbb{C}[[t_\mathfrak{p}]] \times \prod_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}} \mathbb{C}[[t_\mathfrak{q}]]\right) \cap A\\ &= \left(t \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}[[t_\mathfrak{p}]]\right) \cap A\\ &= \{x \in A \mid \mathrm{ord}_t(x) > \mathbf{0}\}.\end{aligned}$$

□

Lemma 6.16. *The ring A is a local complex algebroid curve.*

Proof. By definition the ring R is a local complex algebroid curve, and hence $\Psi(R)$ is a complex algebroid curve. By construction, we have $\Psi(R) \subset A \subset \overline{\Psi(R)}$ (see Theorem 3.44), and hence A is an integral extension of $\Psi(R)$. Thus, A is a complex algebroid curve by Theorem 3.45.(2), and it is local by Lemma 6.15. $\square$

Since A is admissible, we may consider its semigroup of values.

Lemma 6.17. *Let* $\eta\colon \operatorname{Min}(R) \to \operatorname{Min}(A)$ *be the bijection of Lemma 6.14.(2). We have*

$$\mathcal{V}_A = \mathcal{V}_{\Psi(R)} = \{\Psi(V) \mid V \in \mathcal{V}_R\},$$

and for any $\mathfrak{p} \in \operatorname{Min}(A)$ *the corresponding valuation of* Q_A *is* $\operatorname{ord}_{t_{\eta^{-1}(\mathfrak{p})}}$. *Moreover, considered as subsets of* $\mathbb{N}$*, we obtain*

$$(\Gamma_A)_{\mathfrak{p}} = (\Gamma_R)_{\eta^{-1}(\mathfrak{p})},$$

and for every $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$ *we have*

$$(\Gamma_A)_{\mathfrak{p}}^{\mathfrak{q}} = (\Gamma_R)_{\eta^{-1}(\mathfrak{p})}^{\eta^{-1}(\mathfrak{q})}.$$

Proof. Since $\Psi(R)$ and A are admissible (see Definition 6.1, Lemma 6.16, and Proposition 3.41), and since A is by Theorems 3.44 and 6.2.(4) an integral extension of $\Psi(R)$ in $Q_{\Psi(R)}$, Theorem 3.45.(1) yields

$$\mathcal{V}_A = \mathcal{V}_{\Psi(R)} = \{\Psi(V) \mid V \in \mathcal{V}_R\}.$$

This implies

$$\Gamma_R \subset \Gamma_A.$$

Thus, for any $\mathfrak{p} \in \operatorname{Min}(R)$ we obtain

$$(\Gamma_R)_{\mathfrak{p}} \subset (\Gamma_A)_{\eta(\mathfrak{p})}.$$

Moreover, we have by definition

$$\operatorname{pr}_{\mathfrak{p}}(A) \subset \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right],$$

where $\operatorname{pr}_{\mathfrak{p}}\colon \prod_{\mathfrak{q} \in \operatorname{Min}(R)} \mathbb{C}[[t_{\mathfrak{q}}]] \to \mathbb{C}[[t_{\mathfrak{p}}]]$ is the projection. Therefore,

$$(\Gamma_A)_{\eta(\mathfrak{p})} \subset (\Gamma_R)_{\mathfrak{p}}$$

since the valuation corresponding to $\eta(\mathfrak{p})$ is $\operatorname{ord}_{t_{\mathfrak{p}}}$ (see Theorem 6.2.(4)). This yields

$$(\Gamma_A)_{\eta(\mathfrak{p})} = (\Gamma_R)_{\mathfrak{p}}. \tag{6.18}$$

Let now $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$. Since $\Gamma_R \subset \Gamma_A$, we have

$$(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \subset (\Gamma_A)_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})}.$$

Let $\alpha \in (\Gamma_A)^{\eta(\mathfrak{q})}_{\eta(\mathfrak{p})}$. Then by Proposition 4.69 there is an $x \in \eta(\mathfrak{q})$ with $\operatorname{ord}_{t_\mathfrak{p}}(x) = \alpha$. So writing

$$x = \left(\sum_{\alpha_{\mathfrak{p}'} \in (\Gamma_R)_{\mathfrak{p}'}} a^{(\mathfrak{p}')}_{\alpha_{\mathfrak{p}'}} t_{\mathfrak{p}'}^{\alpha_{\mathfrak{p}'}} \right)_{\mathfrak{p}' \in \operatorname{Min}(R)}$$

we have

$$a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} = 0 \text{ for all } \alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \text{ with } \alpha_\mathfrak{p} < \alpha, \tag{6.19}$$

$$a^{(\mathfrak{p})}_{\alpha} \neq 0, \tag{6.20}$$

$$a^{(\mathfrak{q})}_{\alpha_\mathfrak{q}} = 0 \text{ for all } \alpha_\mathfrak{q} \in (\Gamma_R)_\mathfrak{q}. \tag{6.21}$$

By Equation (6.18) we have $\alpha \in (\Gamma_R)_\mathfrak{p}$. Assume $\alpha \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)^\mathfrak{q}_\mathfrak{p}$. Then $\tau_{\mathfrak{p}\mathfrak{q}}(\alpha) \in (\Gamma_R)_\mathfrak{q} \setminus (\Gamma_R)^\mathfrak{p}_\mathfrak{q}$ by Theorem 6.2.(2). Hence, Equation (6.21) and the definition of A yield

$$0 = a^{(\mathfrak{q})}_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha)} = \zeta_{\mathfrak{q}\mathfrak{p}}(\tau_{\mathfrak{p}\mathfrak{q}}(\alpha))\, a^{\mathfrak{p}}_{\alpha}.$$

Since $a^{(\mathfrak{p})}_\alpha \neq 0$ (see Equation (6.20)), this implies $\zeta_{\mathfrak{q}\mathfrak{p}}(\tau_{\mathfrak{p}\mathfrak{q}}(\alpha)) = 0$. However, this is a contradiction to $\zeta_{\mathfrak{q}\mathfrak{p}}(\beta) \neq 0$ for all $\beta \in (\Gamma_R)_\mathfrak{q} \setminus (\Gamma_R)^\mathfrak{p}_\mathfrak{q}$ (see Lemma 6.7.(1)). Thus, $\alpha \in (\Gamma_R)^\mathfrak{q}_\mathfrak{p}$. This yields

$$(\Gamma_A)^{\eta(\mathfrak{q})}_{\eta(\mathfrak{p})} \subset (\Gamma_R)^\mathfrak{q}_\mathfrak{p},$$

and therefore

$$(\Gamma_R)^\mathfrak{q}_\mathfrak{p} = (\Gamma_A)^{\eta(\mathfrak{q})}_{\eta(\mathfrak{p})}. \qquad \square$$

Lemma 6.18. *The $\mathbb{C}$-derivation*

$$\left(\overline{w}_\mathfrak{p} t_\mathfrak{p} \partial_{t_\mathfrak{p}}\right)_{\mathfrak{p} \in \operatorname{Min}(R)}$$

of $\prod_{\mathfrak{p} \in \operatorname{Min}(R)} \mathbb{C}[[t_\mathfrak{p}]]$ restricts to a $\mathbb{C}$-derivation $\mathfrak{d}$ of A.

Proof. Let $x \in A$, i.e.

$$x = \left(\sum_{\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p}} a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} t_\mathfrak{p}^{\alpha_\mathfrak{p}} \right)_{\mathfrak{p} \in \operatorname{Min}(R)} \in \prod_{\mathfrak{p} \in \operatorname{Min}(R)} \mathbb{C}\left[\left[t_\mathfrak{p}^{(\Gamma_R)_\mathfrak{p}}\right]\right]$$

with

$$a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} = \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})\, a^{(\mathfrak{q})}_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})}$$

for any $\mathfrak{p} \in \operatorname{Min}(R)$, for every $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$, and for all $\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)^\mathfrak{q}_\mathfrak{p}$. Then

$$\mathfrak{d}(x) = \left(\sum_{\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \setminus \{0\}} \overline{w}_\mathfrak{p} \alpha_\mathfrak{p} a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} t_\mathfrak{p}^{\alpha_\mathfrak{p}} \right)_{\mathfrak{p} \in \operatorname{Min}(R)}.$$

Now Theorem 6.2.(2) implies for any $\mathfrak{p} \in \operatorname{Min}(R)$, for every $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$, and for all $\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \setminus \left((\Gamma_R)_\mathfrak{p}^\mathfrak{q} \cup \{0\}\right)$

$$\overline{w}_\mathfrak{p}\alpha_\mathfrak{p} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} = \overline{w}_\mathfrak{q}\alpha_\mathfrak{q}\zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})\, a_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})}^{(\mathfrak{q})}.$$

Thus, $\mathfrak{d}(x) \in A$. □

Lemma 6.19. *An element*

$$x = \left(\sum_{\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p}} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p} \in \operatorname{Min}(R)} \in A$$

is an eigenvector of $\mathfrak{d}$ (see Lemma 6.18) if and only if there is $d \in \mathbb{Z}$ such that for any $\mathfrak{p} \in \operatorname{Min}(R)$ we have

$$x_\mathfrak{p} = \begin{cases} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} & \text{if there is an } \alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \text{ such that } \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} = d, \\ 0 & \text{else.} \end{cases}$$

In particular, $\mathfrak{d}$ has only eigenvalues in $\mathbb{N}$.

Proof. Let

$$x = \left(\sum_{\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p}} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p} \in \operatorname{Min}(R)} \in A$$

be an eigenvector of $\mathfrak{d}$, i.e. there is $c \in \mathbb{C}$ such that

$$\begin{aligned} cx &= \mathfrak{d}(x) \\ &= \left(\sum_{\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \setminus \{0\}} \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p} \in \operatorname{Min}(R)}. \end{aligned}$$

This implies

$$x_\mathfrak{p} = \begin{cases} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} & \text{if there is an } \alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \text{ such that } \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} = c, \\ 0 & \text{else.} \end{cases}$$

for any $\mathfrak{p} \in \operatorname{Min}(R)$. In particular, we have $c \in \mathbb{N}$ since $\overline{w} \in \mathbb{N}^{\operatorname{Min}(R)}$ and $\Gamma_R \subset \mathbb{N}^{\operatorname{Min}(R)}$.

Let now $d \in \mathbb{Z}$, and let

$$x = \left(\sum_{\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p}} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p} \in \operatorname{Min}(R)} \in A$$

with

$$x_\mathfrak{p} = \begin{cases} a_{\alpha_\mathfrak{p}}^{(\mathfrak{p})} & \text{if there is an } \alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \text{ such that } \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} = d, \\ 0 & \text{else.} \end{cases}$$

for any $\mathfrak{p} \in \operatorname{Min}(R)$. Then

$$\begin{aligned}\mathfrak{d}(x) &= \left(\sum_{\alpha_\mathfrak{p}\in(\Gamma_R)_\mathfrak{p}\setminus\{0\}} \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p}\in\operatorname{Min}(R)}\\ &= d\left(\sum_{\alpha_\mathfrak{p}\in(\Gamma_R)_\mathfrak{p}\setminus\{0\}} a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p}\in\operatorname{Min}(R)}\\ &= dx.\end{aligned}$$

Note that $x = 0$ if $d < 0$ since $\overline{w} \in \mathbb{N}^{\operatorname{Min}(R)}$ and $\Gamma_R \subset \mathbb{N}^{\operatorname{Min}(R)}$. □

Lemma 6.20. *Let*

$$x = \left(\sum_{\alpha_\mathfrak{p}\in(\Gamma_R)_\mathfrak{p}} a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p}\in\operatorname{Min}(R)} \in A,$$

and let $d \in \mathbb{Z}$. *For any* $\mathfrak{p} \in \operatorname{Min}(R)$, *for every* $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$, *and for all* $\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$ *we define*

$$b^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} = \begin{cases} a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} & \text{if } \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} = d,\\ 0 & \text{else.}\end{cases}$$

Then

$$\left(\sum_{\alpha_\mathfrak{p}\in(\Gamma_R)_\mathfrak{p}} b^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)_{\mathfrak{p}\in\operatorname{Min}(R)} \in A.$$

Proof. Let $d \in \mathbb{Z}$, let $\mathfrak{p} \in \operatorname{Min}(R)$, let $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$, and let $\alpha_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$. First suppose $\overline{w}_\mathfrak{p}\alpha_\mathfrak{p} \neq d$. Then also

$$\overline{w}_\mathfrak{q}\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p}) = \overline{w}_\mathfrak{q}\frac{\overline{w}_\mathfrak{p}\alpha_\mathfrak{p}}{\overline{w}_\mathfrak{q}} = \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} \neq d,$$

and hence

$$b^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} = 0 = b^{(\mathfrak{q})}_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})}.$$

This implies

$$b^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} = \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})\, b^{(\mathfrak{q})}_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})}.$$

Assume now that $\overline{w}_\mathfrak{p}\alpha_\mathfrak{p} = d$. Then also

$$\overline{w}_\mathfrak{q}\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p}) = \overline{w}_\mathfrak{q}\frac{\overline{w}_\mathfrak{p}\alpha_\mathfrak{p}}{\overline{w}_\mathfrak{q}} = \overline{w}_\mathfrak{p}\alpha_\mathfrak{p} = d,$$

Hence,

$$b^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} = a^{(\mathfrak{p})}_{\alpha_\mathfrak{p}} = \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})\, a^{(\mathfrak{q})}_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})} = \zeta_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})\, b^{(\mathfrak{q})}_{\tau_{\mathfrak{p}\mathfrak{q}}(\alpha_\mathfrak{p})}.$$

This implies

$$\left(\sum_{\alpha_{\mathfrak{p}}\in(\Gamma_R)_{\mathfrak{p}}} b^{(\mathfrak{p})}_{\alpha_{\mathfrak{p}}} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)} \in A. \qquad \square$$

Lemma 6.21. *For any $x \in A$ there is a sequence $(x_d)_{d\in\mathbb{Z}} \in A^{\mathbb{Z}}$, where for every $d \in \mathbb{Z}$ either $x_d = 0$ or $\mathfrak{d}(x_d) = dx_d$, such that $x = \sum_{d\in\mathbb{Z}} x_d$.*

Proof. This follows from Lemmas 6.19 and 6.20. □

Lemma 6.22. *The maximal ideal $\mathfrak{m}_A$ of A (see Lemma 6.15) is generated by eigenvectors of $\mathfrak{d}$ with positive eigenvalues.*

Proof. We want to show that $\mathfrak{m}_A$ is generated by the set

$$M = \{x \in A \mid \mathrm{ord}_t\,(x) > \mathbf{0} \text{ and } \mathfrak{d}(x) = d_x x \text{ for some } d_x \in \mathbb{Z}\}.$$

Lemma 6.15 immediately yields $M \subset \mathfrak{m}_A$.

Let $x \in \mathfrak{m}_A$. Then $\mathrm{ord}_t\,(x) > \mathbf{0}$ by Lemma 6.15, and by Lemma 6.21 there is a sequence $(x_d)_{d\in\mathbb{Z}} \in A^{\mathbb{Z}}$ with $x_d = 0$ or $\mathfrak{d}(x_d) = dx_d$ for every $d \in \mathbb{Z}$ such that $x = \sum_{d\in\mathbb{Z}} x_d$. In particular, we have $\mathrm{ord}_t\,(x_d) > \mathbf{0}$ (see Lemma 6.20), and hence $x_d \in \mathfrak{m}_A$ for every $d \in \mathbb{Z}$ by Lemma 6.15.

Pick an $\alpha \in C_{\Gamma_R}$ with $\overline{w}_{\mathfrak{p}}\alpha_{\mathfrak{p}} = \overline{w}_{\mathfrak{q}}\alpha_{\mathfrak{q}}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}\,(R)$. Then $t^\alpha \in \Psi(\mathfrak{C}_R) \subset R \subset A$ by Proposition 4.56 and Theorem 6.2.(4). Moreover,

$$\begin{aligned}\mathfrak{d}(t^\alpha) &= \left(\overline{w}_{\mathfrak{p}}\alpha_{\mathfrak{p}} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)}\\ &= d_\alpha t^\alpha,\end{aligned}$$

where $d_\alpha = \overline{w}_{\mathfrak{p}}\alpha_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Min}\,(R)$, i.e. $t^\alpha \in M$. Then

$$t^{\alpha+\gamma_{\Gamma_R}} \prod_{\mathfrak{p}\in\mathrm{Min}\,(R)} \mathbb{C}[[t_{\mathfrak{p}}]] = t^\alpha\Psi(\mathfrak{C}_R) \subset t^\alpha R \subset t^\alpha A \subset \mathfrak{m}_A \tag{6.22}$$

by Theorems 3.44 and 6.2.(4) and Lemma 6.15, and we can write

$$x = \sum_{\substack{d\in\mathbb{Z}\\ \mathrm{ord}_t\,(x_d)\not\geq\alpha+\gamma_{\Gamma_R}}} x_d + \sum_{\substack{d\in\mathbb{Z}\\ \mathrm{ord}_t\,(x_d)\geq\alpha+\gamma_{\Gamma_R}}} x_d,$$

where

$$\sum_{\substack{d\in\mathbb{Z}\\ \mathrm{ord}_t\,(x_d)\geq\alpha+\gamma_{\Gamma_R}}} x_d \in t^\alpha A$$

by Equation (6.22).

Let now $d \in \mathbb{Z}$ such that $\mathrm{ord}_t\,(x_d) \not\geq \alpha+\gamma_{\Gamma_R}$. Then by Lemma 6.19 there is a $\mathfrak{p} \in \mathrm{Min}\,(R)$ such that $d = \overline{w}_{\mathfrak{p}}\,\mathrm{ord}_{t_{\mathfrak{p}}}\,(x_d) \leq \overline{w}_{\mathfrak{p}}(\alpha + \gamma_{\Gamma_R})_{\mathfrak{p}}$. In particular, we have

$$d \leq \max\left\{\overline{w}_{\mathfrak{p}}(\alpha + \gamma_{\Gamma_R})_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Min}\,(R)\right\}.$$

This implies that

$$\sum_{\substack{d\in\mathbb{Z}\\ \mathrm{ord}_t\,(x_d)\ngeq\alpha+\gamma_{\Gamma_R}}} x_d$$

is finite. Thus,

$$x = \sum_{\substack{d\in\mathbb{Z}\\ \mathrm{ord}_t\,(x_d)\ngeq\alpha+\gamma_{\Gamma_R}}} x_d + t^\alpha y \in \langle M\rangle$$

with some $y \in A$.

Finally, note that by Lemma 6.19 the eigenvalue of every $x \in M$ with respect to $\mathfrak{d}$ is positive. □

Proof of Proposition 6.6. (1) By Lemma 6.16 A is a local complex algebroid curve. By Lemma 6.22 (and since A is Noetherian) there is a generating system $(x_i)_{i=1}^n$ for the maximal ideal $\mathfrak{m}_A$ of A such that $\mathfrak{d}(x_i) = w_i x_i$ for some $w_i \in \mathbb{N}$ with $w_i > 0$ for every $i = 1, \dots, n$. Thus, A is quasihomogeneous.

(2) Lemma 6.14.(2) yields the bijection $\eta\colon \mathrm{Min}\,(R) \to \mathrm{Min}\,(A)$. Since the grading on A is induced by the restriction of the $\mathbb{C}$-derivation

$$\left(\overline{w}_{\mathfrak{p}} t_{\mathfrak{p}} \partial_{t_{\mathfrak{p}}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(R)} = \left(\overline{w}_{\eta^{-1}(\mathfrak{q})} t_{\eta^{-1}(\mathfrak{q})} \partial_{t_{\eta^{-1}(\mathfrak{q})}}\right)_{\mathfrak{q}\in\mathrm{Min}\,(A)}$$

of $\overline{\Psi(R)} = \overline{A}$ (see Lemma 6.14.(1)), and since the valuation of Q_A is ord_t (see Theorem 6.2.(3), Equation (6.6)), A has normal weights $\overline{w}$ (see Definition 6.3).

(3) Lemma 6.17 yields

$$A = \mathrm{Fib}\,(\Gamma_A, \overline{w}, \zeta).$$

Since $\overline{A} = \overline{\Psi(R)}$ by Lemma 6.14.(1), A is a fibre product with connecting maps $\zeta = \left(\left(\zeta_{\eta^{-1}(\mathfrak{p})\eta^{-1}(\mathfrak{q})}\right)_{\mathfrak{q}\in\mathrm{Min}\,(A)\setminus\{\mathfrak{p}\}}\right)_{\mathfrak{p}\in\mathrm{Min}\,(A)}$ (see Definitions 6.3 and 6.4).

(4) Let $x \in (\mathfrak{C}_R)^{\mathrm{reg}}$. Then

$$x\Psi^{-1}(A) \subset x\overline{R} \subset \mathfrak{C}_R \subset R,$$

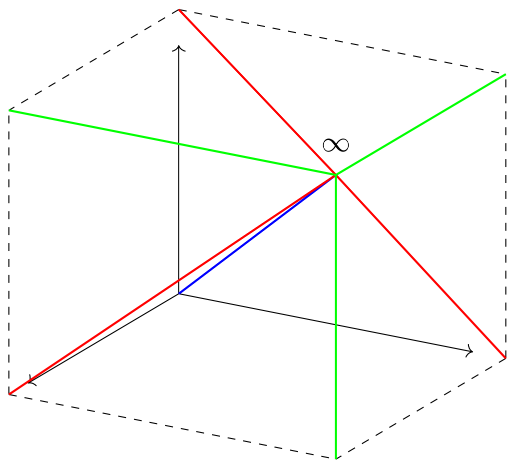
and hence $\Psi^{-1}(A) \in \mathcal{R}_R$ since $\emptyset \neq R^{\mathrm{reg}} \subset \left(\Psi^{-1}(A)\right)^{\mathrm{reg}}$. The rest of the statement follows from Lemma 6.17. □

7. Quasihomogeneous Semigroups

In this chapter we consider two approaches to introduce quasihomogeneity on good semigroups. First we define gradings on good semigroups in analogy to gradings on rings as in Definition 6.1 and E.1 (see Section 7.1, in particular Definitions 7.2 and 7.3). Alternatively, we use properties of the values of homogeneous ring elements to define "homogeneous" semigroup elements (see Section 7.2, in particular Definition 7.14). Then a good semigroup is quasihomogeneous if it is generated by taking sums and infima of these elements. It turns out that both approaches lead to the same concept of quasihomogeneity (see Theorem 7.19). Moreover, the quasihomogeneity on good semigroups is compatible with the quasihomogeneity on algebroid curves under taking values, i.e. the semigroup of values of a quasihomogeneous curve is quasihomogeneous (see Proposition 7.6).

An element of a graded ring can be decomposed as a sum of its homogeneous components (see Proposition E.4). The semigroup operation corresponding to the addition on rings is the infimum. Thus, in a quasihomogeneous semigroup we want to represent any element as an infimum of its homogeneous components.

The values of homogeneous elements of a quasihomogeneous curve lie on lines which are determined by the normal weights of the curve (see Theorem 6.2.(1)), like the blue, red, and green lines in the following illustration.



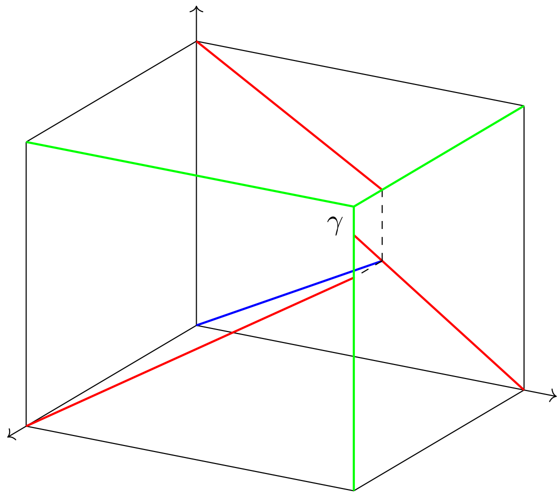
Here the color depends on the number of minimal primes a homogeneous element is contained in.

In a graded ring the element 0 is homogeneous of any degree. The value of zero is ∞ but ∞ is not contained in a good semigroup. However, with Lemma 4.33 we can consider the conductor instead of ∞. This motivates the following definition.

Definition 7.1. Let S be a good semigroup. On S we have an equivalence relation $\sim$ defined by $\alpha \sim \beta$ for $\alpha, \beta \in S$ if for any $i \in I$ we have $\beta_i = \alpha_i$ if $\alpha_i < (\gamma_S)_i$ and $\beta_i \geq (\gamma_S)_i$ if $\alpha_i \geq (\gamma_S)_i$. Then $\widetilde{S} = S/\sim$ denotes the quotient set of S by $\sim$.

7.1. Gradings

Using Definition 7.1 the graded parts of a quasihomogeneous semigroup S will be constructed from equivalence classes in $\widetilde{S}$. In particular, for each class we can choose a representative which is less than or equal to the conductor γ_S of S. Then those representatives defining the graded parts of S have to lie on the colored lines in the following illustration.



In analogy to Definition E.1 we first introduce a general notion of a G-grading on a good semigroup for some additive abelian group G.

Definition 7.2. Let S be a good semigroup, and let G be an additive abelian group.

(1) A *(G-)grading* of S is a system $(\psi_d)_{d\in G}$ of maps $\psi_d \colon S \to \widetilde{S}$ such that the following hold:

   (1) For any $d \in G$ and every $\alpha, \beta \in \bigcup_{\delta\in S} \psi_d(\delta)$ we have
   $$\inf\{\alpha, \beta\} \in \bigcup_{\delta\in S} \psi_d(\delta).$$

   (2) For any $d, d' \in G$ we have
   $$\bigcup_{\delta\in S} \psi_d(\delta) + \bigcup_{\epsilon\in S} \psi_{d'}(\epsilon) \subset \bigcup_{\eta\in S} \psi_{d+d'}(\eta).$$

   (3) For any $\alpha \in S$ there is
   $$\left(\alpha^{(d)}\right)_{d\in G} \in \prod_{d\in G} \psi_d(\alpha)$$

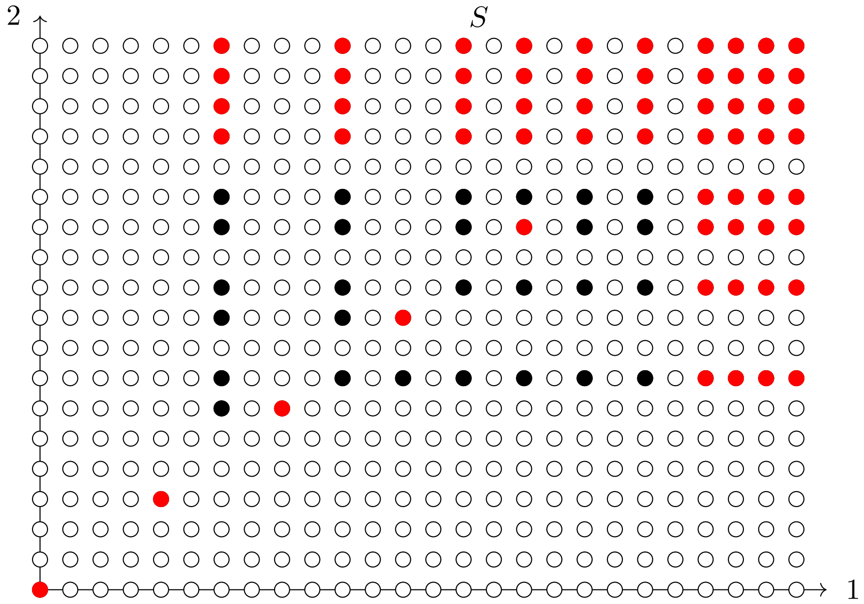


Figure 7.1.: The good semigroup S is quasihomogeneous of type $(3, 4)$, see Example 7.4. Its homogeneous elements are marked red.

with $\alpha^{(d)} \in \psi_d\left(\alpha^{(d)}\right)$ for all $d \in G$ such that

$$\alpha = \inf\left\{\alpha^{(d)} \,\middle|\, d \in G\right\}.$$

If there is a G-grading of S, then S is called *(G-)graded*.

(2) Let S be G-graded, and let $\alpha \in S$. For any $d \in G$ we call every $\beta \in \psi_d(\alpha)$ a *d-th homogeneous component* of α. If $\alpha \in \psi_d(\alpha)$ for some $d \in G$, then α is called *homogeneous*, and d is the *degree* of α. We denote the degree of α by $\deg(\alpha)$.

As discussed above, Theorem 6.2 leads to the following definition of quasihomogeneous semigroups.

Definition 7.3. Let S be a good semigroup, and let $w \in \mathbb{N}^I$ with $w_i > 0$ for all $i \in I$. Then S is called *quasihomogeneous (of type w)* if there is a $\mathbb{Z}$-grading $(\psi_d)_{d\in\mathbb{Z}}$ of S such that for any $d \in \mathbb{Z}$ every $\alpha \in \bigcup_{\beta\in S} \psi_d(\beta)$ satisfies

$$w_i \alpha_i = d$$

for all $i \in I$ with $\alpha_i < (\gamma_S)_i$.

Example 7.4. The good semigroup S depicted in Figure 7.1 is quasihomogeneous of type $(3, 4)$.

Proposition E.4 shows that the decomposition of an element of a quasihomogeneous curve into its homogeneous components is unique. The decomposition on quasihomogeneous semigroups has weaker properties.

Proposition 7.5. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and let $\alpha \in S$. Then for any $d \in \mathbb{Z}$ and for every $\alpha^{(d)} \in \psi_d(\alpha)$ we have*

$$\inf\left\{\alpha^{(d)}, \gamma_S\right\} \geq \inf\left\{\alpha, \gamma_S\right\}.$$

Moreover, for any $i \in I$ with $\alpha_i < (\gamma_S)_i$ we have $\beta_i = \alpha_i$ for all $\beta \in \psi_{w_i \alpha_i}(\alpha)$.

Proof. Since S is quasihomogeneous, there is a family

$$\left(\beta^{(d)}\right)_{d\in\mathbb{Z}} \in \prod_{d\in\mathbb{Z}} \psi_d(\alpha)$$

such that

$$\alpha = \inf\left\{\beta^{(d)} \,\middle|\, d \in \mathbb{Z}\right\}, \tag{7.1}$$

see Definition 7.2.(3). In particular, we have $\beta^{(d)} \geq \alpha$. Let now $\alpha^{(d)} \in \psi_d(\alpha)$. Since $\alpha^{(d)} \sim \beta^{(d)}$ (see Definition 7.2.(1)), this implies for any $i \in I$

$$\left(\alpha^{(d)}\right)_i \geq (\gamma_S)_i$$

if $\left(\beta^{(d)}\right)_i \geq (\gamma_S)_i$, and

$$\left(\alpha^{(d)}\right)_i = \left(\beta^{(d)}\right)_i \geq \alpha_i$$

if $\left(\beta^{(d)}\right)_i < (\gamma_S)_i$. Thus,

$$\inf\left\{\alpha^{(d)}, \gamma_S\right\} \geq \inf\left\{\alpha, \gamma_S\right\}.$$

By Equation 7.1 there is for any $i \in I$ a $d_i \in \mathbb{Z}$ such that $\left(\beta^{(d_i)}\right)_i = \alpha_i$. Suppose that $\alpha_i < (\gamma_S)_i$. Then

$$d_i = w_i\left(\beta^{(d_i)}\right)_i = w_i \alpha_i,$$

see Definition 7.3. Let now

$$\beta \in \psi_{w_i\alpha_i}(\alpha) = \psi_{d_i}(\alpha).$$

Then $\beta \sim \beta^{(d_i)}$, see Definition 7.2.(1). Since $\left(\beta^{(d_i)}\right)_i = \alpha_i < (\gamma_S)_i$, this implies

$$\beta_i = \left(\beta^{(d_i)}\right)_i = \alpha_i. \qquad \square$$

Being constructed in analogy to the quasihomogeneity on algebroid curves we expect the quasihomogeneity on good semigroups to be compatible with its algebraic prototype under taking values. More precisely, we show the following.

Proposition 7.6. *Let R be a quasihomogeneous curve with normal weights $\overline{w}$ (see Definition 6.3). Then Γ_R is quasihomogeneous of type $\overline{w}$.*

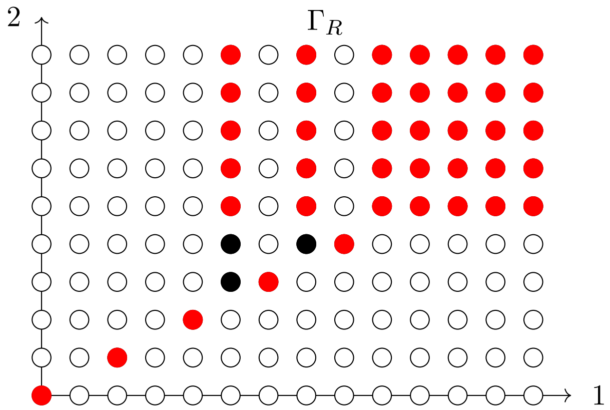


Figure 7.2.: The semigroup of values of the quasihomogeneous curve $R = \mathbb{C}[[X,Y]]/\langle (X^5 - Y^2)Y \rangle$ with normal weights $(1,2)$ is quasihomogeneous of type $(1,2)$, see Example 7.7. The homogeneous elements are marked red.

Proof. See Section 7.7. □

Example 7.7. The algebroid curve

$$\begin{aligned} R &= \mathbb{C}[[X,Y]]/\Big\langle \Big(X^5 - Y^2\Big)Y \Big\rangle \\ &= \mathbb{C}[[X,Y]]/\Big(\Big\langle X^5 - Y^2 \Big\rangle \cap \langle Y \rangle\Big) \\ &\cong \mathbb{C}\Big[\Big[\Big(t_1^2, t_2\Big), \Big(t_1^5, 0\Big)\Big]\Big] \end{aligned}$$

is quasihomogeneous of type $(2,5)$. Since $X \mapsto (t_1^2, t_2)$ and $Y \mapsto (t_1^5, 0)$, Theorem 6.2.(1) implies that the normal weights of R are $(1,2)$ (see Definition 6.3). Then by Proposition 7.6 the semigroup of values Γ_R of R is quasihomogeneous of type $(1,2)$, see Figure 7.2.

Let R be a quasihomogeneous curve with normal weights $\overline{w}$. Then by Theorem 6.2.(2) there is for any $\mathfrak{p}, \mathfrak{q} \in \operatorname{Min}(R)$ with $\mathfrak{p} \neq \mathfrak{q}$ a bijection

$$\begin{aligned} \tau_{\mathfrak{p}\mathfrak{q}} \colon (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} &\to (\Gamma_R)_{\mathfrak{q}} \setminus (\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}, \\ \alpha &\mapsto \frac{\overline{w}_{\mathfrak{p}}\alpha}{\overline{w}_{\mathfrak{q}}}. \end{aligned}$$

Therefore, Proposition 7.6 suggests the following.

Proposition 7.8. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$. For any $i, j \in I$ with $i \neq j$ there is a bijection*

$$\begin{aligned} \tau_{ij} \colon S_i \setminus S_i^j &\to S_j \setminus S_j^i, \\ \alpha &\mapsto \frac{w_i\alpha}{w_j}. \end{aligned}$$

Proof. Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and let $i, j \in I$ with $i \neq j$. Let $\alpha \in S_i \setminus S_i^j$. In particular, we have $\alpha < (\gamma_S)_i$, see Definition 4.60. Pick a $\delta \in S$ with $\delta_i = \alpha$. Since S is quasihomogeneous, there is by Proposition 7.5 a $\beta \in \psi_{w_i\alpha}(\delta)$ with $\beta_i = \alpha$. As $\alpha \in S_i \setminus S_i^j$, we have $\beta_j < (\gamma_S)_j$, see Definition 4.60. Since $\beta \in \bigcup_{\epsilon \in S} \psi_{w_i\alpha}(\epsilon)$, this implies

$$w_j\beta_j = w_i\alpha = w_i\beta_i, \tag{7.2}$$

see Definition 7.3. Hence, $\frac{w_i\alpha}{w_j} = \beta_j \in S_j$. Suppose that $\beta_j \in S_j^i$. Then there is a $\zeta \in S$ with $\zeta_i \geq (\gamma_S)_i$ and $\zeta_j = \beta_j$. So property (E2) applied to β and ζ yields an $\eta \in S$ with $\eta_i = \min\{\beta_i, \zeta_i\} = \beta_i = \alpha$ and $\eta_j > \beta_j = \zeta_j$. Then by Proposition 7.5 there is a $\theta \in \psi_{w_i\alpha}(\eta)$ with $\theta_i = \eta_i = \alpha$ and $\theta_j \geq \eta_j > \beta_j$. Since $\alpha \in S_i \setminus S_i^j$, we have $\theta_j < (\gamma_S)_j$, see Definition 4.60. With Equation (7.2) this yields the contradiction

$$w_i\alpha = w_j\theta_j > w_j\beta_j = w_i\alpha,$$

see Definition 7.3. Thus, there is a map

$$\begin{aligned} \tau_{ij} \colon S_i \setminus S_i^j &\to S_j \setminus S_j^i, \\ \alpha &\mapsto \frac{w_i\alpha}{w_j}. \end{aligned}$$

Since, moreover,

$$\tau_{ji} \circ \tau_{ij}(\alpha) = \frac{w_j \frac{w_i\alpha}{w_j}}{w_i} = \alpha,$$

it follows that τ_{ij} is bijective. $\square$

Example 7.9.

(1) By Proposition 7.8 there is for the quasihomogeneous semigroup S of type $(3, 4)$ of Example 7.4 and Figure 7.1 a bijection

$$\begin{aligned} \tau_{12} \colon S_1 \setminus S_1^2 &\to S_2 \setminus S_2^1, \\ \alpha &\mapsto \frac{3\alpha}{4}, \end{aligned}$$

see Figure 7.3.

(2) Similarly, for the quasihomogeneous semigroup Γ_R of type $(1, 2)$ of Example 7.7 and Figure 7.2 there is a bijection

$$\begin{aligned} \tau_{12} \colon (\Gamma_R)_1 \setminus (\Gamma_R)_1^2 &\to (\Gamma_R)_2 \setminus (\Gamma_R)_2^1, \\ \alpha &\mapsto \frac{1\alpha}{2}, \end{aligned}$$

see Figure 7.4.

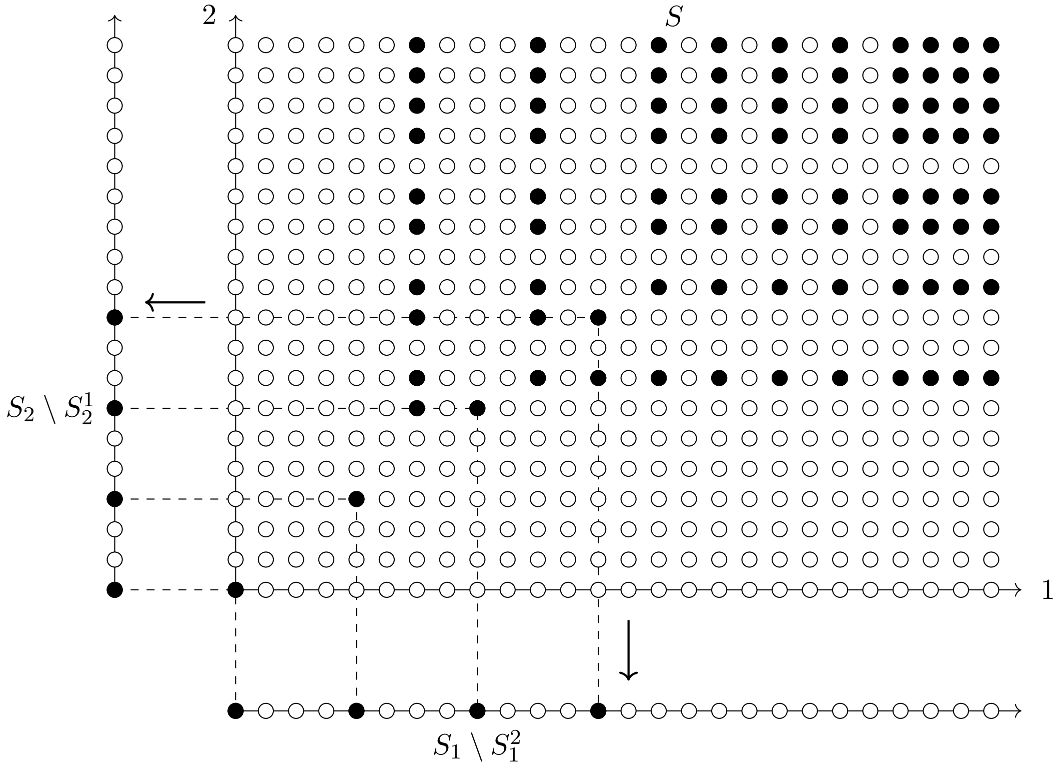


Figure 7.3.: The quasihomogeneous semigroup S of type $(3,4)$ of Example 7.4 and Figure 7.1 with the bijection $\tau_{12}\colon S_1 \setminus S_1^2 \to S_2 \setminus S_2^1$ of Proposition 7.8, see Example 7.9.(1).

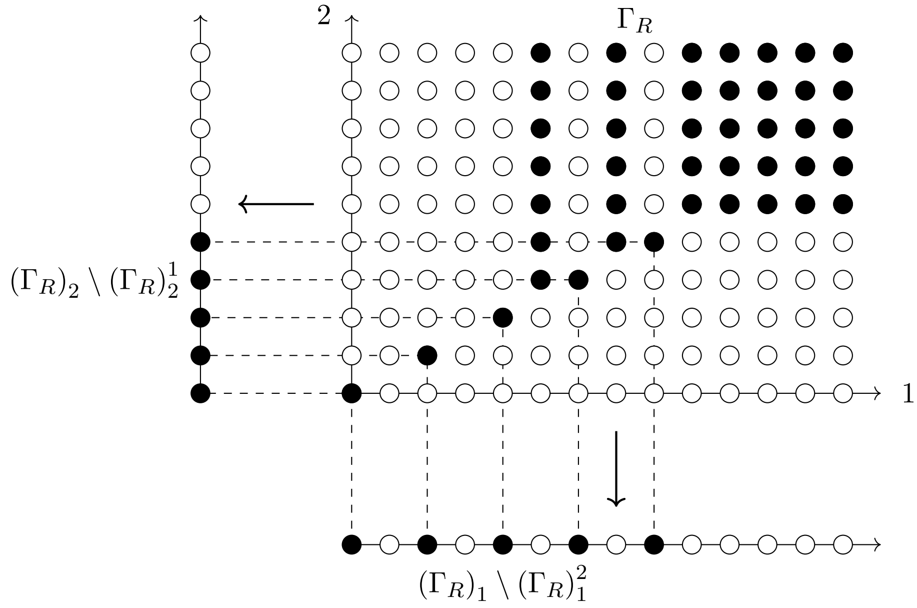


Figure 7.4.: The quasihomogeneous semigroup Γ_R of type $(1, 2)$ of Example 7.7 and Figure 7.2 with the bijection $\tau_{12} \colon (\Gamma_R)_1 \setminus (\Gamma_R)_1^2 \to (\Gamma_R)_2 \setminus (\Gamma_R)_2^1$ of Proposition 7.8, see Example 7.9.(2).

Let R be a quasihomogeneous curve. With the normal weights $\overline{w}$ and the connecting maps ζ we construct the fibre product

$$\mathrm{Fib}\,(R, \overline{w}, \zeta)$$

of the semigroup rings

$$\mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

for $\mathfrak{p} \in \mathrm{Min}\,(R)$ over the semigroup rings

$$\mathbb{C}\left[\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{p}}/(\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}}\right]\right]$$

for $\mathfrak{q} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{p}\}$ (see Theorem 6.2.(4)).

Now the semigroup of values Γ_R of R is quasihomogeneous of type $\overline{w}$ by Proposition 7.6. So we want to extend the construction of $\mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$ to general quasihomogeneous semigroups S, given "connecting maps" $\zeta_{ij} \colon S_i \setminus S_i^j \to \mathbb{C}$ satisfying the properties of Lemma 6.7.

Definition 7.10. Let S be a local quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and for any $i, j \in I$ with $i \neq j$ let $\zeta_{ij} \colon S_i \setminus S_i^j \to \mathbb{C}$ be a map satisfying the following

(1) $\zeta_{ij}(\alpha + \beta) = \zeta_{ij}(\alpha)\,\zeta_{ij}(\beta)$ for all $\alpha, \beta \in S_i \setminus S_i^j$ with $\alpha + \beta \in S_i \setminus S_i^j$,

(2) $\zeta_{ij}(0) = 1$, and

(3) $\zeta_{ij}(\alpha) \neq 0$ for all $\alpha \in S_i \setminus S_i^j$.

Then with $\zeta = \left((\zeta_{ij})_{j \in I \setminus \{i\}} \right)_{i \in I}$ we denote by

$$\mathrm{Fib}\,(S, w, \zeta)$$

the subset of $\prod_{i \in I} \mathbb{C}\left[\left[t_i^{S_i}\right]\right]$ consisting of all elements

$$\left(\sum_{\alpha_i \in S_i} a_{\alpha_i}^{(i)} t_i^{\alpha_i} \right)_{i \in I} \in \prod_{i \in I} \mathbb{C}\left[\left[t_i^{S_i}\right]\right]$$

satisfying

$$a_{\alpha_i}^{(i)} = \zeta_{ij}(\alpha_i)\, a_{\tau_{ij}(\alpha_i)}^{(j)}$$

for all $i \in I$, for any $j \in I \setminus \{i\}$, and for every $\alpha_i \in S_i \setminus S_i^j$ (see Proposition 7.8).

If $\zeta_{ij}(\alpha_i) = 1$ for all $i \in I$, for any $j \in I \setminus \{i\}$, and for every $\alpha_i \in S_i \setminus S_i^j$, we write $\mathrm{Fib}\,(S, w)$ instead of $\mathrm{Fib}\,(S, w, \zeta)$.

The object $\mathrm{Fib}\,(S, w, \zeta)$ constructed in Definition 7.10 is indeed a fibre product.

Remark 7.11. Let S be a local quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and for any $i, j \in I$ with $i \neq j$ let $\zeta_{ij} \colon S_i \setminus S_i^j \to \mathbb{C}$ be a map satisfying the following

(1) $\zeta_{ij}(\alpha + \beta) = \zeta_{ij}(\alpha)\, \zeta_{ij}(\beta)$ for all $\alpha, \beta \in S_i \setminus S_i^j$ with $\alpha + \beta \in S_i \setminus S_i^j$,

(2) $\zeta_{ij}(0) = 1$, and

(3) $\zeta_{ij}(\alpha) \neq 0$ for all $\alpha \in S_i \setminus S_i^j$.

With $\tau_{ij} \colon S_i \setminus S_i^j \to S_j \setminus S_j^i$ as in Proposition 7.8 we define for any $i, j \in I$ with $i \neq j$ a $\mathbb{C}$-algebra isomorphism

$$\begin{aligned} \sigma_{ij} \colon \mathbb{C}\left[t^{S_i / S_i^j}\right] &\to \mathbb{C}\left[t^{S_j / S_j^i}\right], \\ \overline{t^{\alpha}} &\mapsto \overline{\zeta_{ij}(\alpha)\, t^{\tau_{ij}(\alpha)}}. \end{aligned}$$

Let $\mathcal{C}$ be the category of $\mathbb{C}$-algebras, let $\mathcal{I}$ be a category with $\mathrm{Ob}\,\mathcal{I} = I$, and let $D \colon \mathcal{I} \to \mathcal{C}$ be a diagram of type $\mathcal{I}$ with $D(i) = \mathbb{C}\left[\left[t^{S_i}\right]\right]$ for any $i \in I$. Let $\mathcal{J}$ and $F \colon \mathcal{J} \to \mathcal{C}$ be as in Definition 2.29, where for any $(i, j) \in \mathrm{Ob}\,\mathcal{J}$ with $i \neq j$ we have

$$F((i,j)) = \mathbb{C}\left[t^{S_i / S_i^j}\right],$$

$$F((i,j) \to (j,i)) = \chi_{ij} \colon \mathbb{C}\left[\left[t^{S_i}\right]\right] \to \mathbb{C}\left[t^{S_i / S_i^j}\right]$$

as in Proposition 4.79, and

$$F((i,j) \to (j,i)) = \sigma_{ij}.$$

Then Corollary 2.34 yields

$$\mathrm{Fib}\,(S, w, \zeta) = \mathrm{Fib}\,(F),$$

where $\zeta = \left((\zeta_{ij})_{j \in I \setminus \{i\}} \right)_{i \in I}$. In particular, $\mathrm{Fib}\,(S, w, \zeta)$ is a $\mathbb{C}$-subalgebra of $\prod_{i \in I} \mathbb{C}\left[\left[t_i^{S_i}\right]\right]$.

In analogy to Proposition 6.6 we obtain the following.

Proposition 7.12. *Let S be a local quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and for any $i, j \in I$ with $i \neq j$ let $\zeta_{ij} \colon S_i \setminus S_i^j \to \mathbb{C}$ be a map satisfying the following*

(1) $\zeta_{ij}(\alpha + \beta) = \zeta_{ij}(\alpha)\,\zeta_{ij}(\beta)$ *for all* $\alpha, \beta \in S_i \setminus S_i^j$ *with* $\alpha + \beta \in S_i \setminus S_i^j$,

(2) $\zeta_{ij}(0) = 1$, *and*

(3) $\zeta_{ij}(\alpha) \neq 0$ *for all* $\alpha \in S_i \setminus S_i^j$.

Then $\mathrm{Fib}\,(S, w, \zeta)$ *(with* $\zeta = \left((\zeta_{ij})_{j \in I \setminus \{i\}}\right)_{i \in I}$*) is a quasihomogeneous curve with normal weights w (see Definition 6.3). Moreover,* $\mathrm{Fib}\,(S, w)$ *is a fibre product (see Definition 6.4).*

Proof. See Section 7.8. □

7.2. w-Elements

The second approach to quasihomogeneity on good semigroups is based on the properties of values of homogeneous elements of a quasihomogeneous curve.

Proposition 7.13. *Let R be a quasihomogeneous curve with normal weights $\overline{w} \in \mathbb{N}^{\mathrm{Min}\,(R)}$ (see Definition 6.3), and let $x \in R$ be a homogeneous element. Then for any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}\,(R)$ with $\mathfrak{p} \neq \mathfrak{q}$ the following hold:*

(1) If $\nu_\mathfrak{p}(x) \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$, *then* $\overline{w}_\mathfrak{q}\nu_\mathfrak{q}(x) = \overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x)$.

(2) If $\nu_\mathfrak{p}(x) \in (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$, *then either* $\nu_\mathfrak{q}(x) = \infty$ *(i.e.* $x \in \mathfrak{q}$*) or* $\nu_\mathfrak{q}(x) \in (\Gamma_R)_\mathfrak{q}^\mathfrak{p}$ *with* $\overline{w}_\mathfrak{q}\nu_\mathfrak{q}(x) = \overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x)$.

See Figure 7.5.

Proof. Let $x \in R$ be a homogeneous element, set $d = \deg\,(x)$, and let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}\,(R)$ with $\mathfrak{p} \neq \mathfrak{q}$. We may suppose that $x \notin \mathfrak{p}$. Then

$$d = \overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x) \tag{7.3}$$

by Theorem 6.2.(1).

Let

$$\Psi \colon R \to \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$$

be the injective homomorphism of Theorem 6.2.(4), where $\zeta = \left((\zeta_{\mathfrak{p}\mathfrak{q}})_{\mathfrak{q} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{p}\}}\right)_{\mathfrak{p} \in \mathrm{Min}\,(R)}$ are the connecting maps for R. We write

$$\Psi(x) = \left(\sum_{\alpha_{\mathfrak{p}'} \in (\Gamma_R)_{\mathfrak{p}'}} a^{(\mathfrak{p}')}_{\alpha_{\mathfrak{p}'}} t_{\mathfrak{p}'}^{\alpha_{\mathfrak{p}'}}\right)_{\mathfrak{p}' \in \mathrm{Min}\,(R)}.$$

Then $a^{(\mathfrak{p})}_{d/\overline{w}_\mathfrak{p}} \neq 0$, and

$$(\Psi(x))_\mathfrak{p} = a^{(\mathfrak{p})}_{d/\overline{w}_\mathfrak{p}} t_\mathfrak{p}^{\frac{d}{\overline{w}_\mathfrak{p}}}$$

by Theorem 6.2.(5) and Equation (7.3).

Suppose that $\frac{d}{\overline{w}_\mathfrak{p}} = \nu_\mathfrak{p}(x) \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$. Then

$$\frac{d}{\overline{w}_\mathfrak{q}} = \tau_{\mathfrak{p}\mathfrak{q}}(\nu_\mathfrak{p}(x)) \in (\Gamma_R)_\mathfrak{q} \setminus (\Gamma_R)_\mathfrak{q}^\mathfrak{p}$$

by Theorem 6.2.(2). Therefore, Theorem 6.2.(5) implies

$$(\Psi(x))_\mathfrak{q} = a^{(\mathfrak{q})}_{d/\overline{w}_\mathfrak{q}} t_\mathfrak{q}^{\frac{d}{\overline{w}_\mathfrak{q}}}.$$

Since

$$a^{(\mathfrak{q})}_{d/\overline{w}_\mathfrak{q}} = \zeta_{\mathfrak{q}\mathfrak{p}}\left(\frac{d}{\overline{w}_\mathfrak{q}}\right) a^{(\mathfrak{p})}_{d/\overline{w}_\mathfrak{p}}$$

by Theorem 6.2.(4) (see Equation (6.9)), and hence $a^{(\mathfrak{q})}_{d/\overline{w}_\mathfrak{q}} \neq 0$, this implies by Theorem 6.2.(3)

$$\nu_\mathfrak{q}(x) = \frac{d}{\overline{w}_\mathfrak{q}} = \frac{\overline{w}_\mathfrak{p}}{\overline{w}_\mathfrak{q}} \nu_\mathfrak{p}(x).$$

Now suppose $\nu_\mathfrak{p}(x) \in (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$, and assume that $\nu_\mathfrak{q}(x) \in (\Gamma_R)_\mathfrak{q}$. Then

$$\overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x) = \deg(x) = \overline{w}_\mathfrak{q}\nu_\mathfrak{q}(x)$$

by Theorem 6.2.(1). If $\nu_\mathfrak{q}(x) \in (\Gamma_R)_\mathfrak{q} \setminus (\Gamma_R)_\mathfrak{q}^\mathfrak{p}$, then

$$\nu_\mathfrak{p}(x) = \frac{\overline{w}_\mathfrak{q}\nu_\mathfrak{q}(x)}{\overline{w}_\mathfrak{p}} = \tau_{\mathfrak{q}\mathfrak{p}}(\nu_\mathfrak{q}(x)) \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$$

by Theorem 6.2.(2). But this contradicts the assumption. □

The properties of Proposition 7.13 lead to the following definition.

Definition 7.14. Let S be a good semigroup, and let $w \in \mathbb{N}^I$. An element $\alpha \in \prod_{i\in I} S_i$ is called a *w-element (of S)* if for any $i, j \in I$ with $i \neq j$ the following hold (see Figure 7.6):

(1) If $\alpha_i \in S_i \setminus S_i^j$, then $w_i\alpha_i = w_j\alpha_j$.

(2) If $\alpha_i \in S_i^j \setminus (C_S)_i$, then either $\alpha_j \in (C_S)_j$ or $\alpha_j \in S_j^i$ with $w_i\alpha_i = w_j\alpha_j$.

(3) If $\alpha_i \in (C_S)_i$, then $\alpha_j \in S_j^i$.

Remark 7.15. Let S be a good semigroup, let $w \in \mathbb{N}^I$, let α be a w-element of S, and let $i, j \in I$ with $i \neq j$. If $\alpha_i \in S_i \setminus S_i^j$, then $\alpha_j \in S_j \setminus S_j^i$.

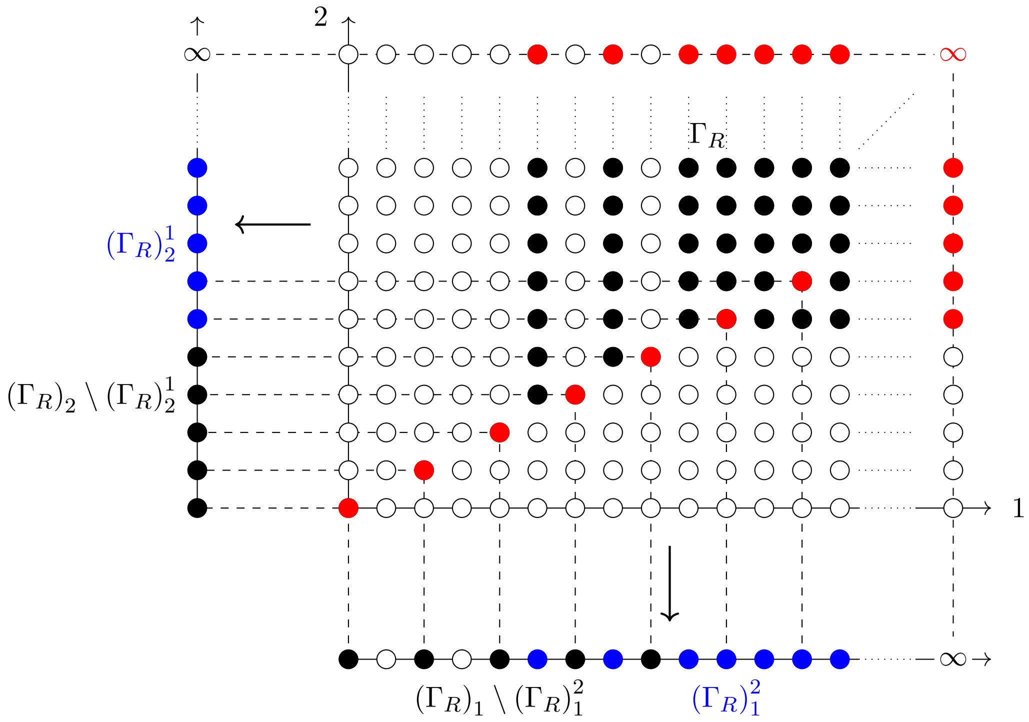


Figure 7.5.: The values of homogeneous elements of the quasihomogeneous curve $R = \mathbb{C}[[X,Y]]/\langle (X^5 - Y^2)Y \rangle$ (red).

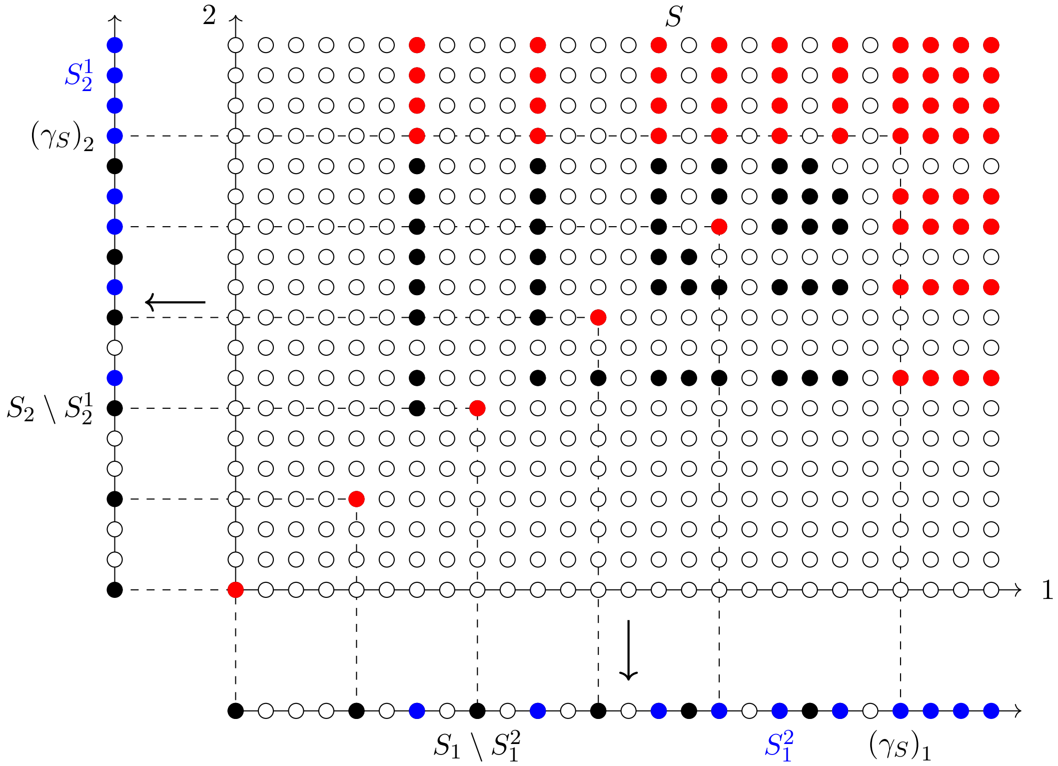


Figure 7.6.: A good semigroup S with its w-elements (red) for $w = (3, 4)$. Note that all w-elements of S are contained in S (also see Proposition 7.25).

Let S be a good semigroup, and let $w \in \mathbb{N}^I$. If S is quasihomogeneous of type w, the w-elements of S contained in S will be exactly the homogeneous elements of S (see Theorem 7.19). First we associate to a w-element a "degree".

Proposition 7.16. *Let S be a good semigroup, and let $w \in \mathbb{N}^I$. If $\alpha \in \prod_{i \in I} S_i$ is a w-element of S, then there is a $d \in \mathbb{Z}$ such that $w_i \alpha_i = d$ for all $i \in I$ with $\alpha_i < (\gamma_S)_i$. Moreover, d is unique if $\alpha \in (\prod_{i \in I} S_i) \setminus C_S$.*

Proof. If $\alpha \in C_S$, the statement is trivial. So suppose $\alpha \in \prod_{i \in I} S_i \setminus C_S$.

Let $i \in I$ with $\alpha_i < (\gamma_S)_i$, and set $d = w_i \alpha_i$. Let $j \in I \setminus \{i\}$. If $\alpha_i \in S_i \setminus S_i^j$, then $w_j \alpha_j = w_i \alpha_i = d$. If $\alpha_i \in S_i^j$, then $\alpha_j \in S_j^i$ with $\alpha_j \geq (\gamma_S)_j$ or $w_j \alpha_j = w_i \alpha_i = d$. So for any w-element $\alpha \in \prod_{i \in I} S_i \setminus C_S$ we obtain a unique $d \in \mathbb{Z}$ such that $w_i \alpha_i = d$ for all $i \in I$ with $\alpha_i < (\gamma_S)_i$. $\square$

Definition 7.17. Let S be a good semigroup, let $w \in \mathbb{N}^I$, and let $\alpha \in \prod_{i \in I} S_i$ be a w-element of S. If $\alpha \in \prod_{i \in I} S_i \setminus C_S$, we define the *$w$-degree* of α as

$$\deg_w(\alpha) = w_i \alpha_i$$

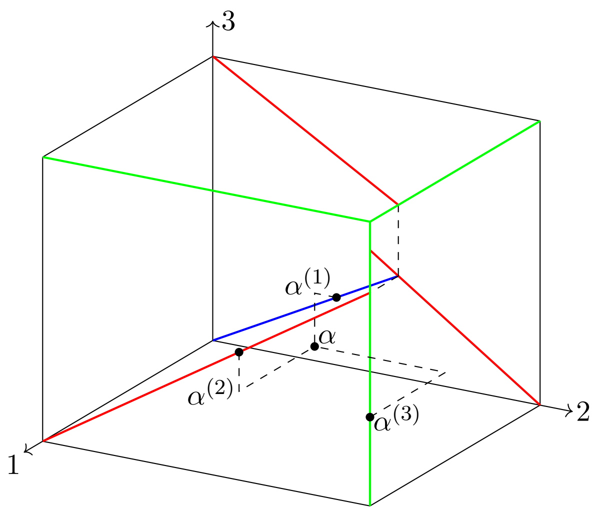
for some $i \in I$ with $\alpha_i < (\gamma_S)_i$ (see Proposition 7.16). If $\alpha \in C_S$, then $\deg_w(\alpha)$ is arbitrary.

Remark 7.18. Let S be a good semigroup, let $w \in \mathbb{N}^I$, let $\alpha \in \prod_{i \in I} \setminus C_S$ be a w-element of S, and let $i \in I$ with $\alpha_i < (\gamma_S)_i$. Then Proposition 7.16 yields $\beta_i \geq \alpha_i$ for any w-element β of S with $\deg_w(\beta) = \deg_w(\alpha)$.

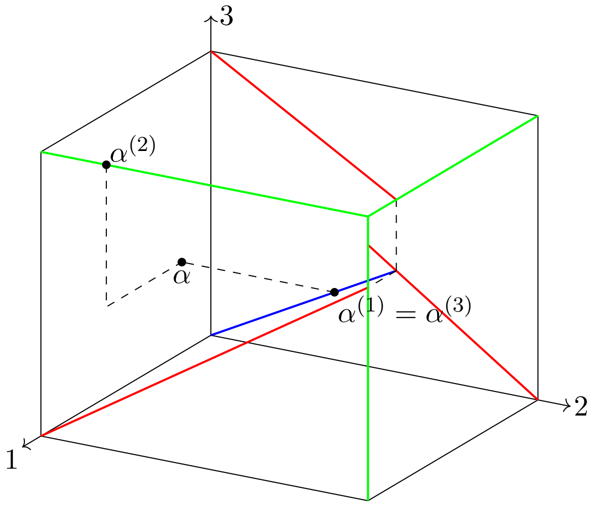
Let S be a good semigroup, and let $w \in \mathbb{N}^I$ with $w_i > 0$ for all $i \in I$. We want to use the w-elements of S to determine quasihomogeneity of S. In a homogeneous ring every element can be decomposed into a sum of its homogeneous components (see Proposition E.4). The semigroup operation corresponding to addition in rings is the infimum. So we would like to call a good semigroup quasihomogeneous of type w if for any element $\alpha \in S$ there is a family $\left(\alpha^{(i)}\right)_{i \in I}$ of w-elements of S with $\alpha^{(i)} \in S$ such that $\left(\alpha^{(i)}\right)_i = \alpha_i$ and

$$\alpha = \inf\left(\alpha^{(i)} \,\middle|\, i \in I\right).$$

This definition can be illustrated by



or



Indeed it turns out that the definition of quasihomogeneity on good semigroups using w-elements yields the same concept as the one introduced in Section 7.1 (see Definition 7.3).

Theorem 7.19. *Let S be a good semigroup, and let $w \in \mathbb{N}^I$. The following are equivalent:*

(a) S is quasihomogeneous of type w.

(b) For any $\alpha \in S$ there is a family

$$\left(\alpha^{(i)}\right)_{i\in I} \in S^I$$

of w-elements such that

$$\left(\alpha^{(i)}\right)_i = \alpha_i$$

for any $i \in I$ and

$$\alpha = \inf\left(\alpha^{(i)} \,\middle|\, i \in I\right).$$

In particular, we have $\alpha^{(i)} \geq \alpha$ for all $i \in I$.

If S is quasihomogeneous of type w, an element $\alpha \in S$ is homogeneous if and only if it is a w-element, and for homogeneous elements $\alpha \in S$ we have $\deg(\alpha) = \deg_w(\alpha)$.

Proof. See Section 7.4. □

Example 7.20.

(1) The good semigroup S in Figure 7.6 is not quasihomogeneous of type $w = (3, 4)$. Indeed, we have for example $(15, 11) \in S$ but there is no w-element $\alpha \in S$ with $\alpha_1 = 15$.

(2) Since the good semigroups in Figures 7.1 and 7.2 are quasihomogeneous, their homogeneous elements are their w-elements which are contained in the respective semigroup. In fact, these are all the w-elements (also see Proposition 7.25).

7.3. Properties

A quasihomogeneous curve R is a fibre product (see Definition 6.4) if its homogeneous elements only satisfy certain relations between any pair of branches of R (cf. Theorem 6.2.(5)) depending on the normal weights and the connecting maps (see Definition 6.3). After taking values these relations correspond to the definition of $\overline{w}$-elements of Γ_R (see Propositions 7.6 and 7.13, Definition 7.14, and Theorem 7.19. So in analogy to being a fibre product we introduce the following closedness property of a quasihomogeneous semigroup.

Definition 7.21. Let S be a good semigroup and let $w \in \mathbb{N}^I$. Then S is called *closed with respect to w* or *w-closed* if $\alpha \in S$ for any w-element $\alpha \in \prod_{i\in I} S_i$ of S.

Let R be a quasihomogeneous curve with normal weights $\overline{w}$ and connecting maps ζ (see Definition 6.3). Recall that the fibre product $\mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$ is by Remark 6.5 the largest quasihomogeneous curve in

$$\prod_{\mathfrak{p}\in \mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

with normal weights $\overline{w}$ and connecting maps ζ. For an analogous construction on quasihomogeneous semigroups we use the property of w-closedness.

Proposition 7.22. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$. There is a unique quasihomogeneous semigroup S^w of type w which is w-closed and satisfies*

$$S \subset S^w \subset \overline{S}$$

and

$$\left(S_i, \left(S_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I} = \left((S^w)_i, \left((S^w)_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I}.$$

*The semigroup S^w is called the w-*closure *of S, and it is generated by the w-elements of S in the following sense: for any element $\alpha \in D_S$ we have $\alpha \in S^w$ if and only if there is a family $\left(\alpha^{(i)}\right)_{i\in I}$ of w-elements of S such that*

$$\alpha = \inf\left(\alpha^{(i)} \;\middle|\; i \in I\right).$$

Proof. See Section 7.6. □

Indeed, if R is a quasihomogeneous curve with normal weights $\overline{w}$ and connecting maps ζ (see Definition 6.3), the fibre product $\mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)$ corresponds to the $\overline{w}$-closure of Γ_R in the following sense.

Theorem 7.23. *Let R be a quasihomogeneous curve with normal weights $\overline{w}$ (see Definition 6.3). Then R is a fibre product (see Definition 6.4) if and only if Γ_R is quasihomogeneous of type $\overline{w}$ and $\overline{w}$-closed. In particular, if R has normal weights $\overline{w}$ and connecting maps ζ (see Definition 6.3), then*

$$\Gamma_{\mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta)} = (\Gamma_R)^{\overline{w}}.$$

Proof. See Section 7.9. □

Theorem 7.24. *Let S be a local quasihomogeneous semigroup of type $w \in \mathbb{N}^I$. Then $S^w = \Gamma_{\mathrm{Fib}\,(S,w)}$. In particular, S is w-closed if and only if $S = \Gamma_{\mathrm{Fib}\,(S,w)}$.*

Proof. See Section 7.9. □

As the following proposition shows, Theorem 7.23 leads to the description of a quasihomogeneous curve with two branches in terms of the semigroups of values of its branches by Kunz and Ruppert (see [9, Satz 4.2]).

Proposition 7.25. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$. If $|I| \leq 2$, then S is w-closed. In particular, if R is a quasihomogeneous curve with $|\mathrm{Min}\,(R)| \leq 2$, then R is a fibre product (see Definition 6.4).*

Proof. If $|I| = 1$, the claim is trivial. So suppose that $I = \{1, 2\}$, and let α be a w-element of S. Assume that $\alpha_1 \in S_1 \setminus S_1^2$. Then $\alpha_2 \in S_2 \setminus S_2^1$ with

$$w_1\alpha_1 = w_2\alpha_2 \tag{7.4}$$

(see Definition 7.14 and Remark 7.15). Moreover, by Theorem 7.19 there is a w-element β of S with $\beta \in S$ and $\beta_1 = \alpha_1$. Then Equation (7.4) implies $\alpha = \beta \in S$.

So assume now that $\alpha_1 \in S_1^2$. Then there is a $\beta \in S$ with $\beta_1 = \alpha_1$ and $\beta_2 \geq (\gamma_S)_2$. By Lemma 4.33 we may assume that $\beta_2 \geq \alpha_2$. Since S is quasihomogeneous, there is by Theorem 7.19 a w-element δ of S with $\delta \in S$, $\delta_1 = \beta_1 = \alpha_1$, and $\delta_2 \geq \beta_2 \geq \alpha_2$. Since $\alpha_1 \in S_1^2$ implies $\alpha_2 \in S_2^1$ (see Definition 7.14), there also is a w-element ϵ of S with $\epsilon \in S$, $\epsilon_2 = \alpha_2$, and $\epsilon_1 \geq \alpha_1$. This implies $\alpha = \inf\{\delta, \epsilon\} \in S$. Thus, S is w-closed.

The particular claim follows then with Proposition 7.6 and Theorem 7.23. □

A quasihomogeneous curve R can be embedded into the fibre product of its branches over their pairwise intersections. Theorem 7.23 gives a criterion on the semigroup of values of R which characterizes this embedding to be an isomorphism. If R is a fibre product, it can be reconstructed from information on its branches. By Theorem 7.24 this implies that any quasihomogeneous semigroup of type $w \in \mathbb{N}^I$ which is w-closed can be reconstructed from data on its branches. In fact, we can extend this statement to arbitrary quasihomogeneous semigroups. In order to make this statement more precise we define the following "stronger" version of w-elements.

Definition 7.26. Let S be a good semigroup, and let $w \in \mathbb{N}^I$. A w-element α of S is called *maximal* if the following hold:

(1) $\alpha \in S$.

(2) If $\alpha \in S \setminus C_S$, then there is $i \in I$ with $\alpha_i < (\gamma_S)_i$ such that $\inf\{\alpha, \gamma_S\} = \inf\{\beta, \gamma_S\}$ for all w-elements β of S with $\beta \in S$, $\beta_i = \alpha_i$, and $\inf\{\alpha, \gamma_S\} \leq \inf\{\beta, \gamma_S\}$.

The set of maximal w-elements of S is denoted by $\mathcal{M}_w(S)$.

Theorem 7.27. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$. Then the following data are equivalent:*

(a) The semigroup S.

(b) The family

$$\left(S_i, \left(S_i^J\right)_{J \subset I \setminus \{i\}}\right)_{i \in I}$$

consisting of the branches S_i of S together with all ideals S_i^J for every $J \subset I \setminus \{i\}$.

(c) The set of maximal w-elements $\mathcal{M}_w(S)$ of S.

The maximal w-elements determine the semigroup S in the following way: For an element $\alpha \in D_S$ we have $\alpha \in S$ if and only if there is a family

$$\left(\alpha^{(i)}\right)_{i \in I} \in \left(\mathcal{M}_w(S)\right)^I$$

with

$$\left(\alpha^{(i)}\right)_i = \alpha_i$$

for every $i \in I$ and

$$\left(\alpha^{(i)}\right)_j \geq \alpha_j$$

for each $j \in I \setminus \{i\}$. In particular,

$$\alpha = \inf\left(\alpha^{(i)} \;\middle|\; i \in I\right).$$

Proof. See Section 7.5. ☐

7.4. Proof of Theorem 7.19

Lemma 7.28. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, let $\alpha \in S$, let $d \in \mathbb{Z}$, and let $\beta \in \psi_d(\alpha)$. Then for any $i \in I$ the following hold:*

(1) If $\alpha_i < (\gamma_S)_i$ or $\beta_i < (\gamma_S)_i$, then $\beta_i \geq \alpha_i$. In particular, $\beta_i < (\gamma_S)_i$ implies $\alpha_i < (\gamma_S)_i$.

(2) If $\alpha_i < (\gamma_S)_i$ and $w_i\alpha_i = d$, then $\beta_i = \alpha_i$.

(3) If $w_i\alpha_i > d$ or $\alpha_i \geq (\gamma_S)_i$, then $\beta_i \geq (\gamma_S)_i$.

Proof. (1) Since there is $\left(\delta^{(g)}\right)_{g \in \mathbb{Z}} \in \prod_{g \in \mathbb{Z}} \psi_g(\alpha)$ such that $\alpha = \inf\left\{\delta^{(g)} \;\middle|\; g \in \mathbb{Z}\right\}$, we have

$$\delta^{(g)} \geq \alpha \tag{7.5}$$

for every $g \in \mathbb{Z}$.

Let $\alpha_i < (\gamma_S)_i$ or $\beta_i < (\gamma_S)_i$, and assume that $\beta_i < \alpha_i$. Then we have $\beta_i < (\gamma_S)_i$ in either case. So suppose that $\beta_i < (\gamma_S)_i$, and assume that $\beta_i < \alpha_i$. Then $\delta^{(d)} \sim \beta$ implies $\left(\delta^{(d)}\right)_i = \beta_i < \alpha_i$ since $\beta, \delta^{(d)} \in \psi_d(\alpha)$ (see Definition 7.1). But this contradicts Equation (7.5).

(2) Suppose that $\alpha_i < (\gamma_S)_i$ and $w_i\alpha_i = d$. Since there is $\left(\delta^{(g)}\right)_{g\in\mathbb{Z}} \in \prod_{g\in\mathbb{Z}} \psi_g(\alpha)$ such that $\alpha = \inf\left\{\delta^{(g)} \;\middle|\; g \in \mathbb{Z}\right\}$, there is an $e \in \mathbb{Z}$ such that $\left(\delta^{(e)}\right)_i = \alpha_i$. This implies $\left(\delta^{(e)}\right)_i < (\gamma_S)_i$, and hence

$$e = \deg\left(\delta^{(e)}\right) = w_i\left(\delta^{(e)}\right)_i = w_i\alpha_i = d.$$

Thus, $\beta, \delta^{(d)} \in \psi_d(\alpha)$, i.e. $\beta \sim \delta^{(d)}$, and we obtain $\beta_i = \left(\delta^{(d)}\right)_i = \alpha_i$.

(3) Suppose $\alpha_i \geq (\gamma_S)_i$, and assume that $\beta_i < (\gamma_S)_i$. Then $\beta_i < \alpha_i$, contradicting (1).

Suppose now that $\alpha_i < (\gamma_S)_i$, and assume that $\beta_i < (\gamma_S)_i$. Then (1) yields $d < w_i\alpha_i \leq w_i\beta_i$. But as $\beta \in \psi_d(\alpha)$ with $\beta_i < (\gamma_S)_i$, this is a contradiction to $w_i\beta_i = \deg(\beta) = d$. □

Lemma 7.29. *Let S be a quasihomogeneous semigroup (of type $w \in \mathbb{N}^I$), and let $\alpha \in S$ be a homogeneous element. If $\alpha_i \in S_i^j$ for some $i, j \in I$ with $i \neq j$, then $\alpha_j \in S_j^i$.*

Proof. Let $\alpha \in S$ with $\alpha_i \in S_i^j$, and assume $\alpha_j \in S_j \setminus S_j^i$. Then $\alpha_i < (\gamma_S)_i$ (see Definition 4.60), and $\alpha_j < (\gamma_S)_j$ by Lemma 4.63. Since α is homogeneous, this implies $w_i\alpha_i = \deg(\alpha) = w_j\alpha_j$.

As $\alpha_i \in S_i^j$, there is a $\beta \in S$ with $\beta_i = \alpha_i$ and $\beta_j > (\gamma_S)_j$ (see Definition 4.60). So property (E2) yields a $\delta \in S$ with $\delta_i > \alpha_i$ and $\delta_j = \min\{\alpha_j, \beta_j\} = \alpha_j$.

Let now $\epsilon \in \psi_{\deg(\alpha)}(\delta)$. Then $\epsilon_j = \delta_j$ by Lemma 7.28.(2) since $\delta_j = \alpha_j < (\gamma_S)_j$ and $w_j\delta_j = w_j\alpha_j = \deg(\alpha)$, and $\epsilon_i \geq (\gamma_S)_i$ by Lemma 7.28.(3) since $w_i\delta_i > w_i\alpha_i = \deg(\alpha)$ (see Definition 7.3). This implies $\alpha_j = \delta_j = \epsilon_j \in S_j^i$, contradicting the assumption. □

Corollary 7.30. *Let S be a quasihomogeneous semigroup (of type $w \in \mathbb{N}^I$). Any homogeneous element $\alpha \in S$ is a w-element of S with* $\deg_w(\alpha) = \deg(\alpha)$.

Proof. Let $\alpha \in S$ be homogeneous, and let $i, j \in I$ with $i \neq j$. First assume $\alpha_i \in S_i \setminus S_i^j$. Then Lemma 7.29 yields $\alpha_j \in S_j \setminus S_j^i$. So, in particular, $\alpha_i < (\gamma_S)_i$ and $\alpha_j < (\gamma_S)_j$ (see Definition 4.60). Since α is homogeneous, this implies $w_i\alpha_i = \deg(\alpha) = w_j\alpha_j$ (see Definition 7.3).

Let now $\alpha_i \in S_i^j \setminus (C_S)_i$. Then Lemma 7.29 yields $\alpha_j \in S_j^i$. So assume $\alpha_j \in S_j^i \setminus (C_S)_j$. Since α is homogeneous, this implies again $w_i\alpha_i = \deg(\alpha) = w_j\alpha_j$.

If $\alpha_i \in (C_S)_i$, then $\alpha_j \in S_j^i$ since $\alpha \in S$. Thus, α is a w-element of S.

Finally, for any $i \in I$ we have $\alpha_i \geq (\gamma_S)_i$ or $\deg(\alpha) = w_i\alpha_i = \deg_w(\alpha)$. □

Lemma 7.31. *Let S be a good semigroup, let $w \in \mathbb{N}^I$ with $w_i > 0$ for every $i \in I$, and let $\alpha, \beta \in S$ be w-elements with* $\deg_w(\alpha) = \deg_w(\beta)$. *Then* $\inf\{\alpha, \beta\} \in S$ *is a w-element with* $\deg_w(\inf\{\alpha, \beta\}) = \deg_w(\alpha) = \deg_w(\beta)$.

Proof. If $\alpha, \beta \in C_S$, then $\inf\{\alpha, \beta\} \in C_S$ by Lemma 4.18 and Definition 4.26. So assume that $\alpha \in S \setminus C_S$ or $\beta \in S \setminus C_S$.

Set $\delta = \inf\{\alpha, \beta\}$, and let $i, j \in I$ with $i \neq j$. First assume that $\delta_i \in S_i \setminus S_i^j$. Then without loss of generality $\alpha_i = \delta_i \in S_i \setminus S_i^j$ and $\beta_i \geq \alpha_i$. Since $\alpha_i < (\gamma_S)_i$ by Lemma 4.63, Proposition 7.16 yields $w_i\alpha_i = w_j\alpha_j$ and $\beta_j \in (C_S)_j$ or $w_j\beta_j = \deg_w(\beta) = \deg_w(\alpha) = w_i\alpha_i$. In particular, since $\alpha_j < (\gamma_S)_j$, we have $\beta_j \geq \alpha_j$. Hence $w_j\delta_j = w_j\alpha_j = w_i\alpha_i = w_i\delta_i$.

Now suppose that $\delta_i \in S_i^j \setminus (C_S)_i$, and again assume without loss of generality that $\delta_i = \alpha_i \leq \beta_i$. Then $\alpha_i = \delta_i \in S_i^j \setminus (C_S)_i$ implies $\alpha_j \in S_j^i$ with $\alpha_j \in (C_S)_j$ or $w_j\alpha_j = w_i\alpha_i = \deg_w(\alpha)$ (see Definition 7.14

Assume that $\beta_i \in S_i \setminus S_i^j$. Then, in particular, $\beta_i < (\gamma_S)_i$ by Lemma 4.63. Therefore, we have by Proposition 7.16 $w_i\alpha_i = \deg_w(\alpha) = \deg_w(\beta) = w_i\beta_i$ but $\alpha_i = \beta_i$ is a contradiction to $\alpha_i \in S_i^j$ and $\beta_i \in S_i \setminus S_i^j$. Thus, we have $\beta_i \in S_i^j$.

This implies $\beta_j \in S_j^i$ (see Definition 4.60). In particular, we have $\beta_j \in (C_S)_j$ or $w_j\beta_j = \deg_w(\beta) = \deg_w(\alpha)$ by Proposition 7.16. Since $\delta_j = \min\{\alpha_j, \beta_j\}$, this yields $\delta_j \in S_i^j$ with $\delta_j \in (C_S)_j$ or $w_j\delta_j = \deg_w(\alpha) = \deg_w(\beta)$. Therefore, $\inf\{\alpha, \beta\}$ is a w-element.

Let $i \in I$ with $\delta_i < (\gamma_S)_i$. Without loss of generality, we may assume $\alpha_i = \delta_i < (\gamma_S)_i$ by Proposition 7.16. Hence, $w_i\delta_i = w_i\alpha_i = \deg_w(\alpha)$. Thus, $\deg_w(\inf\{\alpha, \beta\}) = \deg_w(\alpha) = \deg_w(\beta)$. ☐

Lemma 7.32. *Let S be a good semigroup, let $w \in \mathbb{N}^I$ with $w_i > 0$ for all $i \in I$, and let $\alpha, \beta \in \prod_{i \in I} S_i$.*

(1) If $(\alpha + \beta)_i \in S_i \setminus S_i^j$ for some $i, j \in I$ with $i \neq j$, then $\alpha_i, \beta_i \in S_i \setminus S_i^j$.

(2) Let α and β be w-elements of S, and suppose that for every $i, j \in I$ with $i \neq j$ there is a bijection

$$\tau_{ij} \colon S_i \setminus S_i^j \to S_j \setminus S_j^i, \qquad \delta \mapsto \frac{w_i\delta}{w_j}.$$

Then $\alpha + \beta$ is a w-element of S with $\deg_w(\alpha + \beta) = \deg_w(\alpha) + \deg_w(\beta)$.

Proof. (1) Let $i, j \in I$ with $i \neq j$ such that $(\alpha + \beta)_i \in S_i \setminus S_i^j$, and assume $\alpha_i \in S_i^j$. Since $\alpha, \beta \in \prod_{i \in I} S_i$, there are $\delta, \epsilon \in S$ with $\delta_i = \alpha_i$, $\delta_j \geq (\gamma_S)_j$ and $\epsilon_i = \beta_i$. This yields $\delta + \epsilon \in S$ with $(\delta + \epsilon)_i = (\alpha + \beta)_i$ and $(\delta + \epsilon)_j \geq (\gamma_S)_j$. In particular, we obtain $(\alpha + \beta)_i \in S_i^j$, contradicting the assumption.

(2) Since $\alpha, \beta \in \prod_{i \in I} S_i$, for every $i \in I$ there are $\alpha^{(i)}, \beta^{(i)} \in S$ with $\left(\alpha^{(i)}\right)_i = \alpha_i$ and $\left(\beta^{(i)}\right)_i = \beta_i$. Thus, $(\alpha + \beta)_i = \left(\alpha^{(i)} + \beta^{(i)}\right)_i \in S_i$.

Let $i, j \in I$ with $i \neq j$, and assume that $(\alpha + \beta)_i \in S_i \setminus S_i^j$. Then $\alpha_i, \beta_i \in S_i \setminus S_i^j$ by (1). This implies $w_i\alpha_i = w_j\alpha_j$ and $w_i\beta_i = w_j\beta_j$, see Definition 7.14. Hence,

$$w_i(\alpha + \beta)_i = w_i\alpha_i + w_i\beta_i = w_j\alpha_j + w_j\beta_j = w_j(\alpha + \beta)_j. \tag{7.6}$$

Suppose now that $(\alpha+\beta)_i \in S_i^j$, and assume that $(\alpha+\beta)_j \in S_j \setminus S_j^i$. Then $w_j(\alpha+\beta)_j = w_i(\alpha+\beta)_i$ by Equation (7.6) (with i and j interchanged). This implies

$$(\alpha+\beta)_i = \frac{w_j(\alpha+\beta)_j}{w_i} = \tau_{ij}\Big((\alpha+\beta)_j\Big) \in S_i \setminus S_i^j,$$

contradicting the assumption $(\alpha+\beta)_i \in S_i^j$. Thus, $(\alpha+\beta)_j \in S_j^i$.

Assume that $(\alpha+\beta)_i \in S_i^j \setminus (C_S)_i$. Since $\alpha,\beta \geq \mathbf{0}$, $(\alpha+\beta)_i < (\gamma_S)_i$ implies $\alpha_i, \beta_i < (\gamma_S)_i$, i.e. $\alpha_i, \beta_i \in S_i \setminus (C_S)_i$. If $(\alpha+\beta)_j < (\gamma_S)_j$, then $\alpha_j, \beta_j < (\gamma_S)_j$, and hence

$$w_i(\alpha+\beta)_i = w_i\alpha_i + w_i\beta_i = w_j\alpha_j + w_j\beta_j = w_j(\alpha+\beta)_j.$$

If $\alpha_j \geq (\gamma_S)_j$ or $\beta_j \geq (\gamma_S)_j$, then $(\alpha+\beta)_j \geq (\gamma_S)_j$. Thus, $\alpha+\beta$ is a w-element of S. □

Lemma 7.33. *Let S be a good semigroup, and let $\alpha,\beta \in S$ with $\alpha \sim \beta$. If α is a w-element of S, then β is a w-element of S with* $\deg_w(\alpha) = \deg_w(\beta)$.

Proof. Let $\alpha,\beta \in S$ with $\alpha \sim \beta$, and suppose that α is a w-element of S. Let $i,j \in I$ with $i \neq j$. First assume that $\beta_i \in S_i \setminus S_i^j$. Then $\beta_i < (\gamma_S)_i$ by Lemma 4.63, and hence $\alpha_i = \beta_i \in S_i \setminus S_i^j$ as $\alpha \sim \beta$. Since α is a w-element, this implies $\alpha_j \in S_j \setminus S_j^i$ with $w_i\alpha_i = w_j\alpha_j$ (see Definition 7.14 and Remark 7.15). In particular, we have $\alpha_j < (\gamma_S)_j$ (see Definition 4.60), and hence

$$w_j\beta_j = w_j\alpha_j = w_i\alpha_i = w_i\beta_i$$

since $\alpha \sim \beta$.

Assume now that $\beta_i \in S_i^j \setminus (C_S)_i$. Then $\beta_i < (\gamma_S)_i$, and hence $\alpha_i = \beta_i \in S_i^j \setminus (C_S)_i$. Since α is a w-element, this implies $\alpha_j \in (C_S)_j$ or $\alpha_j \in S_j^i$ with $w_i\alpha_i = w_j\alpha_j$. If $\alpha_j \in (C_S)_j$, then $\beta_j \in (C_S)_j$ since $\alpha \sim \beta$. If $\alpha_j \in S_j^i \setminus (C_S)_j$, then $\beta_j = \alpha_j \in S_j^i \setminus (C_S)_j$ since $\alpha \sim \beta$, and we obtain

$$w_i\beta_i = w_i\alpha_i = w_j\alpha_j = w_j\beta_j$$

since α is a w-element.

If there is an $i \in I$ with $\beta_i < (\gamma_S)_i$, then $\beta_i = \alpha_i < (\gamma_S)_i$ since $\alpha \sim \beta$. This implies $\deg_w(\alpha) = \deg_w(\beta)$. □

Lemma 7.34. *Let S be a good semigroup, let $w \in \mathbb{N}^I$ with $w_i > 0$ for all $i \in I$, and suppose that for any $\alpha \in S$ there is a family $\left(\alpha^{(i)}\right)_{i\in I} \in S^I$ of w-elements such that $\left(\alpha^{(i)}\right)_i = \alpha_i$ for any $i \in I$ and $\alpha = \inf\left(\alpha^{(i)} \,\middle|\, i \in I\right)$. Then for every $i,j \in I$ with $i \neq j$ there is a bijection*

$$\begin{aligned}\tau_{ij} \colon S_i \setminus S_i^j &\to S_j \setminus S_j^i,\\ \alpha &\mapsto \frac{w_i\alpha}{w_j}.\end{aligned}$$

Proof. Let $i, j \in I$ with $i \neq j$, and let $\alpha \in S_i \setminus S_i^j$. Then by assumption there is a w-element $\beta \in S$ of S with $\beta_i = \alpha$. This implies

$$\frac{w_i \alpha}{w_j} = \frac{w_i \beta_i}{w_j} = \beta_j \in S_j$$

(see Definition 7.14). Suppose that $\beta_j \in S_i^j$. Since β is a w-element, this implies $\beta_i \in S_i^j$, contradicting $\beta_i = \alpha \in S_i \setminus S_i^j$. Thus, there is a map

$$\begin{aligned} \tau_{ij} \colon S_i \setminus S_i^j &\to S_j \setminus S_j^i, \\ \alpha &\mapsto \frac{w_i \alpha}{w_j}. \end{aligned}$$

Since, moreover,

$$\tau_{ji} \circ \tau_{ij}(\alpha) = \frac{w_j \frac{w_i \alpha}{w_j}}{w_i} = \alpha,$$

it follows that τ_{ij} is bijective. $\square$

Proof of Theorem 7.19. (a) $\Longrightarrow$ (b) Let S be quasihomogeneous of type w. Then for any $\alpha \in S$ there is a family

$$\left(\alpha^{(d)}\right)_{d \in \mathbb{Z}} \in \prod_{d \in \mathbb{Z}} \psi_d(\alpha)$$

such that

$$\alpha = \inf\left(\alpha^{(d)} \;\middle|\; d \in \mathbb{Z}\right).$$

In particular, this implies that for any $i \in I$ there is an $\alpha^{(d_i)} \in \psi_{d_i}(\alpha)$ with $\left(\alpha^{(d_i)}\right)_i = \alpha_i$ and $\left(\alpha^{(d_i)}\right)_j \geq \alpha_j$ for all $j \in I \setminus \{i\}$. Thus, we have

$$\alpha = \inf\left(\alpha^{(d_i)} \;\middle|\; i \in I\right),$$

and by Corollary 7.30 $\alpha^{(d_i)}$ is a w-element for any $i \in I$.

(b) $\Longrightarrow$ (a) Let $\alpha \in S$, and suppose that there is a family $\left(\alpha^{(i)}\right)_{i \in I} \in S^I$ of w-elements such that for any $i \in I$ we have $\left(\alpha^{(i)}\right)_i = \alpha_i$ and $\left(\alpha^{(i)}\right)_j \geq \alpha_j$. Note that if $\alpha_i \geq (\gamma_S)_i$ for some $i \in I$, then we may assume that $\alpha^{(i)} \geq (\gamma_S)_i$.

For every $d \in \mathbb{Z}$ we set

$$\beta_\alpha^{(d)} = \inf\left(\{\gamma_S\} \cup \left\{\alpha^{(i)} \;\middle|\; i \in I \text{ and } \deg_w\left(\alpha^{(i)}\right) = d\right\}\right). \tag{7.7}$$

Then for any $d \in \mathbb{Z}$ we have

$$\left(\beta_\alpha^{(d)}\right)_i \geq \alpha_i \tag{7.8}$$

for all $i \in I$ with $\alpha_i \leq (\gamma_S)_i$. Moreover, $\beta_\alpha^{(d)}$ is by Lemma 7.31 a w-element with $\deg_w\left(\beta_\alpha^{(d)}\right) = d$ for any $d \in \mathbb{Z}$, and we have $\beta_\alpha^{(d)} \in S$ since S satisfies property (E1). So repeating this construction for all $\delta \in S$ we may define a map

$$\begin{aligned} \psi_d \colon S &\to \widetilde{S}, \\ \delta &\mapsto \overline{\beta_\delta^{(d)}} \end{aligned} \tag{7.9}$$

for every $d \in \mathbb{Z}$. If α is a w-element, then Proposition 7.16 and Equation 7.7 yield for any $d \in \mathbb{Z}$

$$\psi_d(\alpha) = \begin{cases} \overline{\inf\{\alpha, \gamma_S\}} = \overline{\alpha} & \text{if } \deg_w(\alpha) = d, \\ \overline{\gamma_S} & \text{else.} \end{cases}$$

In particular, we have

$$\alpha \in \psi_{\deg_w(\alpha)}(\alpha). \tag{7.10}$$

We now verify that the map ψ satisfies the properties in Definition 7.2.(1).

(1) Let $d \in \mathbb{Z}$, and let $\alpha, \beta \in \bigcup_{\delta \in S} \psi_d(\delta)$. Then α and β are by Lemma 7.33 and Equation (7.9) w-elements of S with $\alpha, \beta \in S$ and $\deg_w(\alpha) = d = \deg_w(\beta)$ since $\beta_\delta^{(d)}$ is a w-element of S with $\deg_w\left(\beta_\delta^{(d)}\right) = d$ for every $\delta \in S$. Therefore, Lemma 7.31 implies that also $\inf\{\alpha, \beta\}$ is a w-element of S with $\deg_w(\inf\{\alpha, \beta\}) = d$. Moreover, we have $\inf\{\alpha, \beta\} \in S$ since S satisfies property (E1). Thus, Equation (7.10) yields

$$\inf\{\alpha, \beta\} \in \psi_d(\inf\{\alpha, \beta\}) \subset \bigcup_{\delta \in S} \psi_d(\delta).$$

(2) Let $d, d' \in \mathbb{Z}$, and let $\alpha \in \bigcup_{\delta \in S} \psi_d(\delta)$ and $\beta \in \bigcup_{\epsilon \in S} \psi_{d'}(\epsilon)$. Then as before α and β are w-elements of S with $\alpha, \beta \in S$ and $\deg_w(\alpha) = d$ and $\deg_w(\beta) = d'$ by Lemma 7.33. Therefore, Lemmas 7.32.(2) and 7.34 imply that $\alpha + \beta$ is a w-element of S with $\deg_w(\alpha + \beta) = d + d'$. Moreover, we have $\alpha + \beta \in S$ since S is a monoid. Thus, Equation (7.10) yields

$$\alpha + \beta \in \psi_{d+d'}(\alpha + \beta) \subset \bigcup_{\delta \in S} \psi_{d+d'}(\delta).$$

(3) Let now $\alpha \in S$ be any element, let $d \in \mathbb{Z}$, and let $\delta^{(d)} \in \psi_d(\alpha)$. Then $\delta^{(d)} \sim \beta_\alpha^{(d)}$ (see Equation (7.9)), where $\beta_\alpha^{(d)}$ is defined as in Equation (7.7). Let $i \in I$, and assume that $\alpha_i < (\gamma_S)_i$. Since $\left(\alpha^{(j)}\right)_i \geq \alpha_i$ for all $j \in I$, Equation (7.7) implies $(\gamma_S)_i \geq \left(\beta_\alpha^{(d)}\right)_i \geq \alpha_i$, and hence

$$\left(\delta^{(d)}\right)_i \geq \left(\beta_\alpha^{(d)}\right)_i \geq \alpha_i \tag{7.11}$$

(see Definition 7.1).

If $d = w_i\alpha_i$, then $\deg_w\left(\alpha^{(i)}\right) = d$ since $\left(\alpha^{(i)}\right)_i = \alpha_i < (\gamma_S)_i$. As $\epsilon_i \geq \alpha_i = \left(\alpha^{(i)}\right)_i$ for all w-elements ϵ of S with $\deg_w(\epsilon) = d$ (see Remark 7.18), this implies

$$\begin{aligned}\left(\beta_\alpha^{(d)}\right)_i &= \min\left(\{(\gamma_S)_i\} \cup \left\{\left(\alpha^{(j)}\right)_i \;\middle|\; j \in I \text{ and } \deg_w\left(\alpha^{(j)}\right) = d\right\}\right)\\ &= \left(\alpha^{(i)}\right)_i\\ &= \alpha_i.\end{aligned}$$

Therefore,

$$\left(\delta^{(d)}\right)_i = \left(\beta_\alpha^{(d)}\right)_i = \alpha_i \tag{7.12}$$

since $\left(\beta_\alpha^{(d)}\right)_i = \alpha_i < (\gamma_S)_i$ and $\beta_\alpha^{(d)} \sim \delta^{(d)}$.

Let now $i \in I$ with $\alpha_i \geq (\gamma_S)_i$. Then $\left(\alpha^{(j)}\right)_i \geq \alpha_i \geq (\gamma_S)_i$ for all $j \in I$. Hence,

$$\left(\beta_\alpha^{(d)}\right)_i = \min\left(\{(\gamma_S)_i\} \cup \left\{\left(\alpha^{(j)}\right)_i \;\middle|\; j \in I \text{ and } \deg_w\left(\alpha^{(j)}\right) = d\right\}\right) = (\gamma_S)_i.$$

Since $\delta^{(d)} \sim \beta_\alpha^{(d)}$, this implies $\left(\delta^{(d)}\right)_i \geq (\gamma_S)_i$. Hence, we may by Lemma 4.33 assume that

$$\left(\delta^{(d)}\right)_i = \alpha_i \tag{7.13}$$

(also see Definition 7.14).

Since $w_i\alpha_i \in \mathbb{Z}$ for any $i \in I$ with $\alpha_i < (\gamma_S)_i$, Equations (7.11), (7.12), and (7.13) imply

$$\alpha = \inf\left\{\delta^{(d)} \;\middle|\; d \in \mathbb{Z}\right\}.$$

Thus, S is $\mathbb{Z}$-graded. In fact, S is quasihomogeneous of type w by Equation (7.9) since $\overline{\beta_\delta^{(d)}}$ is for any $\delta \in S$ a w-element (see Proposition 7.16.

Let S be a quasihomogeneous semigroup of type w. Then a homogeneous element α of S is by Corollary 7.30 a w-element of S with $\deg_w(\alpha) = \deg(\alpha)$. Conversely, a w-element β of S is by Equation (7.10) a homogeneous element of S with $\deg(\beta) = \deg_w(\beta)$. □

7.5. Proof of Theorem 7.27

Lemma 7.35. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$. An element $\alpha \in \prod_{i\in I} S_i$ is a maximal w-element of S if and only if $\alpha \in C_S$ or there is an $i \in I$ such that*

$$\begin{aligned}\alpha_i &< (\gamma_S)_i,\\ \alpha_J &\geq (\gamma_S)_J,\\ w_k\alpha_k &= w_i\alpha_i \text{ for all } k \in I \setminus J,\end{aligned}$$

where $J \subset I \setminus \{i\}$ such that $\alpha_i \in S_i^J$ and $J' = J$ for all subsets J' of $I \setminus \{i\}$ containing J with $\alpha_i \in S_i^{J'}$.

Proof. Let $\alpha \in S \setminus C_S$. First suppose that α is a maximal w-element of S. Then there is by Definition 7.26 an $i \in I$ with $\alpha_i < (\gamma_S)_i$ such that $\inf\{\alpha, \gamma_S\} = \inf\{\beta, \gamma_S\}$ for all w-elements β of S with $\beta \in S$, $\beta_i = \alpha_i$ and $\inf\{\alpha, \gamma_S\} \leq \inf\{\beta, \gamma_S\}$. Set

$$J = \left\{ j \in I \;\middle|\; \alpha_j \geq (\gamma_S)_j \right\}.$$

Then $\alpha_i \in S_i^J$ since $\alpha \in S$ (see Definition 4.60), and Proposition 7.16 and Definition 7.26 yield

$$\begin{aligned} \alpha_i &< (\gamma_S)_i, \\ \alpha_J &\geq (\gamma_S)_J, \\ w_k \alpha_k &= \deg_w(\alpha) = w_i \alpha_i \text{ for all } k \in I \setminus J. \end{aligned}$$

Let now $J' \subset I \setminus \{i\}$ such that $J \subset J'$ and $\alpha_i \in S_i^{J'}$. Then there is $\beta \in S$ with

$$\begin{aligned} \beta_i &= \alpha_i, \\ \beta_{J'} &\geq (\gamma_S)_{J'} \end{aligned}$$

(see Definition 4.60). Since S is quasihomogeneous, Theorem 7.19 yields a w-element $\delta \in S$ with $\delta \geq \beta$ and $\delta_i = \beta_i = \alpha_i$. In particular, if we set

$$J'' = \left\{ j \in I \;\middle|\; \delta_j \geq (\gamma_S)_j \right\},$$

then $J' \subset J''$. Since

$$w_j \alpha_j = w_i \alpha_i = w_i \delta_i = w_j \delta_j$$

for all $j \in I \setminus J''$ by Proposition 7.16, we obtain

$$\begin{aligned} \min\left\{\alpha_j, (\gamma_S)_j\right\} = (\gamma_S)_j &= \min\left\{\delta_j, (\gamma_S)_j\right\} \text{ for all } j \in J, \\ \min\{\alpha_k, (\gamma_S)_k\} = \alpha_k < (\gamma_S)_k &= \min\{\delta_k, (\gamma_S)_k\} \text{ for all } k \in J'' \setminus J, \\ \min\{\alpha_l, (\gamma_S)_l\} = \alpha_l = \delta_l &= \min\{\delta_l, (\gamma_S)_l\} \text{ for all } l \in I \setminus J''. \end{aligned}$$

This implies

$$\inf\{\alpha, \gamma_S\} \leq \inf\{\delta, \gamma_S\}.$$

Therefore, we have $\inf\{\alpha, \gamma_S\} = \inf\{\delta, \gamma_S\}$ since α is maximal. Thus, $J = J' = J''$.

Let now $\alpha \in \prod_{i \in I} S_i$, and suppose that there is an $i \in I$ such that

$$\alpha_i < (\gamma_S)_i, \tag{7.14}$$

$$\alpha_J \geq (\gamma_S)_J, \tag{7.15}$$

$$w_k \alpha_k = w_i \alpha_i \text{ for all } k \in I \setminus J, \tag{7.16}$$

where $J \subset I \setminus \{i\}$ such that $\alpha_i \in S_i^J$ and $J' = J$ for all $J \subset J' \subset I \setminus \{i\}$ with $\alpha_i \in S_i^{J'}$. First we want to show that α is a w-element of S with $\alpha \in S$. Since $\alpha_i \in S_i^J$, there is $\beta \in S$ with

$$\begin{aligned} \beta_i &= \alpha_i, \\ \beta_J &\geq (\gamma_S)_J. \end{aligned}$$

(see Definition 4.60). Since S is quasihomogeneous of type w, there is by Theorem 7.19 a w-element $\delta \in S$ with

$$\delta_i = \beta_i, \tag{7.17}$$
$$\delta_j \geq \beta_j \text{ for all } j \in I \setminus \{i\}. \tag{7.18}$$

In particular, we have

$$\delta_i = \alpha_i < (\gamma_S)_i, \tag{7.19}$$
$$\delta_J \geq (\gamma_S)_J. \tag{7.20}$$

Assume now that there is a $k \in I \setminus (\{i\} \cup J)$ such that $\delta_k \geq (\gamma_S)_k$. Then $J \subsetneq J \cup \{k\}$ and $\alpha_i = \delta_i \in S_i^{J \cup \{k\}}$. But this is a contradiction to the definition of J. Therefore, we have $\delta_k < (\gamma_S)_k$ for all $k \in I \setminus (\{i\} \cup J)$, and Proposition 7.16 yields

$$w_k \delta_k = \deg_w(\delta) = w_i \delta_i \tag{7.21}$$

for all $k \in I \setminus J$. Since $\alpha_i = \delta_i$ by Equation (7.19), combining Equations (7.16) and (7.21) we obtain $\alpha_j = \delta_j$ for all $j \in I \setminus J$. Since $\alpha_k \geq (\gamma_S)_k$ and $\delta_k \geq (\gamma_S)_k$ for all $k \in J$, this yields $\alpha \sim \delta$. Thus, α is a w-element of S by Lemma 7.33 since δ is a w-element.

Let now ϵ be a w-element of S with $\epsilon \in S$, $\epsilon_i = \alpha_i$ and $\inf\{\alpha, \gamma_S\} \leq \inf\{\epsilon, \gamma_S\}$. Thus, if we set

$$J' = \left\{ j \in I \;\middle|\; \epsilon_j \geq (\gamma_S)_j \right\},$$

then $J \subset J' \subset I \setminus \{i\}$. Since, moreover, we have $\alpha_i = \epsilon_i \in S_i^{J'}$ (see Definition 4.60), the definition of J yields $J = J'$. Hence, we obtain $\inf\{\alpha, \gamma_S\} = \inf\{\epsilon, \gamma_S\}$. Therefore, $\alpha \in S$ by Lemma 4.33, and hence α is a maximal w-element of S. □

Lemma 7.36. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and let $i \in I$. For any $\alpha \in S_i$ and for any $J \subset I \setminus \{i\}$ with $\alpha \in S_i^J$ there is a maximal w-element β of S with $\beta_i = \alpha$ and $\beta_J \geq (\gamma_S)_J$.*

Proof. Possibly replacing J by a larger subset of $I \setminus \{i\}$ containing J we may assume that $\alpha \in S_i^J$ and $J = J'$ for all $J \subset J' \subset I \setminus \{i\}$ with $\alpha \in S_i^{J'}$.

Suppose $\alpha \notin (C_S)_i$. Since $\alpha \in S_i^J$, there is a $\delta \in S$ with

$$\delta_i = \alpha,$$
$$\delta_J \geq (\gamma_S)_J.$$

Since S is quasihomogeneous of type w, there is by Theorem 7.19 a w-element $\beta \in S$ with

$$\beta_i = \delta_i,$$
$$\beta_j \geq \delta_j \text{ for all } j \in I \setminus \{i\}.$$

In particular, we have

$$\beta_i = \alpha < (\gamma_S)_i,$$
$$\beta_J \geq (\gamma_S)_J.$$

Then by Proposition 7.16 and Lemma 7.35 β is a maximal w-element of S. □

Lemma 7.37. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, let $\left(\alpha^{(i)}\right)_{i\in I} \in S^I$, and set $\alpha = \inf\left(\alpha^{(i)} \,\middle|\, i \in I\right)$. Then there is a map $\eta\colon I \to I$ (which is not necessarily bijective) such that*

$$\alpha = \inf\left(\alpha^{(\eta(i))} \,\middle|\, i \in I\right)$$

and

$$\left(\alpha^{(\eta(i))}\right)_i = \alpha_i$$

Proof. Since $\alpha = \inf\left(\alpha^{(i)} \,\middle|\, i \in I\right)$, there is for any $j \in I$ an $i_j \in I$ such that

$$\begin{aligned} \left(\alpha^{(i_j)}\right)_j &= \alpha_j, \\ \left(\alpha^{(i_j)}\right)_k &\geq \alpha_k \text{ for all } k \in I \setminus \{j\}. \end{aligned}$$

We define the map

$$\begin{aligned} \eta\colon I &\to I, \\ j &\mapsto i_j. \end{aligned}$$

Then we have for any $i \in I$

$$\begin{aligned} \left(\alpha^{(\eta(i))}\right)_i &= \alpha_i, \\ \left(\alpha^{(\eta(i))}\right)_j &\geq \alpha_j \text{ for all } j \in I \setminus \{i\}. \end{aligned}$$

This implies

$$\inf\left(\alpha^{(\eta(i))} \,\middle|\, i \in I\right) = \alpha. \qquad \square$$

Proof of Theorem 7.27. Obviously, the family $\left(S_i, \left(S_i^J\right)_{J\subset I\setminus\{i\}}\right)_{i\in I}$ is determined by S.

Since γ_S can be computed from $\left(S_i, \left(S_i^J\right)_{J\subset I\setminus\{i\}}\right)_{i\in I}$ by Proposition 4.64, the set $\mathcal{M}_w(S)$ of all maximal w-elements of S is determined by $\left(S_i, \left(S_i^J\right)_{J\subset I\setminus\{i\}}\right)_{i\in I}$ by Lemma 7.35. So we want to show that S can be constructed from $\mathcal{M}_w(S)$ in the following way: for any element $\alpha \in D_S$ we have $\alpha \in S$ if and only if there is a family $\left(\alpha^{(i)}\right)_{i\in I}$ of maximal w-elements of S with

$$\begin{aligned} \left(\alpha^{(i)}\right)_i &= \alpha_i, \\ \left(\alpha^{(i)}\right)_j &\geq \alpha_j \text{ for all } j \in I \setminus \{i\}, \end{aligned}$$

i.e. $\alpha = \inf\left(\alpha^{(i)} \,\middle|\, i \in I\right)$.

By Definition 7.26 we have $\mathcal{M}_w(S) \subset S$. So for any family $\left(\alpha^{(i)}\right)_{i\in I} \in (\mathcal{M}_w(S))^I$ we have $\inf\left(\alpha^{(i)} \;\middle|\; i \in I\right) \in S$ since S satisfies property (E1). Moreover, by Lemma 7.37 there is a family $\beta^{(i)} \in (\mathcal{M}_w(S))^I$ with

$$\begin{aligned}
\left(\beta^{(i)}\right)_i &= \left(\inf\left(\alpha^{(k)} \;\middle|\; k \in I\right)\right)_i, \\
\left(\beta^{(i)}\right)_j &\geq \left(\inf\left(\alpha^{(k)} \;\middle|\; k \in I\right)\right)_j \text{ for all } j \in I \setminus \{i\},
\end{aligned}$$

Let now $\alpha \in S$, and let $i \in I$. If $\alpha_i \geq (\gamma_S)_i$, we choose $\alpha^{(i)} \in C_S$ with

$$\begin{aligned}
\left(\alpha^{(i)}\right)_i &= \alpha_i, \\
\left(\alpha^{(i)}\right)_j &\geq \alpha_j \text{ for all } j \in I \setminus \{i\}.
\end{aligned}$$

Assume now that $\alpha_i < (\gamma_S)_i$. Since S is quasihomogeneous, there is by Theorem 7.19 a w-element $\beta \in S$ with

$$\begin{aligned}
\beta_i &= \alpha_i, \\
\beta_j &\geq \alpha_j \text{ for all } j \in I \setminus \{i\}.
\end{aligned} \tag{7.22}$$

Set $J = \left\{j \in I \;\middle|\; \beta_j \geq (\gamma_S)_j\right\}$. Then $\alpha_i = \beta_i \in S_i^J$, and by Lemmas 4.33 and 7.36 there is a maximal w-element $\alpha^{(i)}$ of S with

$$\begin{aligned}
\left(\alpha^{(i)}\right)_i &= \beta_i, \\
\left(\alpha^{(i)}\right)_J &\geq \beta_J.
\end{aligned} \tag{7.23}$$

Let $j \in I \setminus J$. Then $\beta_j < (\gamma_S)_j$. So if $\left(\alpha^{(i)}\right)_j < (\gamma_S)_j$, then Proposition 7.16 yields

$$w_j\left(\alpha^{(i)}\right)_j = w_i\left(\alpha^{(i)}\right)_i = w_i\beta_i = w_j\beta_j,$$

and hence

$$\left(\alpha^{(i)}\right)_j = \beta_j. \tag{7.24}$$

Equations (7.23) and (7.24) imply $\alpha^{(i)} \geq \beta$ with $\left(\alpha^{(i)}\right)_i = \beta_i$. Thus, for any $i \in I$ there is by Equation (7.22) a w-element $\alpha^{(i)} \in W_S$ with

$$\begin{aligned}
\left(\alpha^{(i)}\right)_i &= \alpha_i, \\
\left(\alpha^{(i)}\right)_j &\geq \alpha_j.
\end{aligned}$$

□

7.6. Proof of Proposition 7.22

Lemma 7.38. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and let*

$$\left(\alpha^{(i)}\right)_{i\in I}, \left(\beta^{(i)}\right)_{i\in I} \in S^I.$$

Then there are maps $\eta_\alpha\colon I \to I$ and $\eta_\beta\colon I \to I$ such that

$$\inf\left(\alpha^{(i)} \;\middle|\; i \in I\right) = \inf\left(\alpha^{(\eta_\alpha(i))} \;\middle|\; i \in I\right),$$
$$\inf\left(\beta^{(i)} \;\middle|\; i \in I\right) = \inf\left(\beta^{(\eta_\beta(i))} \;\middle|\; i \in I\right),$$

and

$$\inf\left(\alpha^{(i)} \;\middle|\; i \in I\right) + \inf\left(\beta^{(i)} \;\middle|\; i \in I\right) = \inf\left(\alpha^{(\eta_\alpha(i))} + \beta^{(\eta_\beta(i))} \;\middle|\; i \in I\right).$$

Proof. Set $\alpha = \inf\left(\alpha^{(i)} \;\middle|\; i \in I\right)$ and $\beta = \inf\left(\beta^{(i)} \;\middle|\; i \in I\right)$. By Lemma 7.37 there are maps $\eta_\alpha\colon I \to I$ and $\eta_\beta\colon I \to I$ such that for any $i \in I$ we have

$$\left(\alpha^{(\eta_\alpha(i))}\right)_i = \alpha_i,$$
$$\left(\alpha^{(\eta_\alpha(i))}\right)_j \geq \alpha_j \text{ for all } j \in I \setminus \{i\},$$

and

$$\left(\beta^{(\eta_\beta(i))}\right)_i = \beta_i,$$
$$\left(\beta^{(\eta_\beta(i))}\right)_j \geq \beta_j \text{ for all } j \in I \setminus \{i\}.$$

Therefore, we have for any $i \in I$

$$\left(\alpha^{(\eta_\alpha(i))} + \beta^{(\eta_\beta(i))}\right)_i = \alpha_i + \beta_i$$
$$\left(\alpha^{(\eta_\alpha(i))} + \beta^{(\eta_\beta(i))}\right)_j \geq \alpha_j + \beta_j \text{ for all } j \in I \setminus \{i\}.$$

This implies

$$\alpha + \beta = \inf\left(\alpha^{(\eta_\alpha(i))} + \beta^{(\eta_\beta(i))} \;\middle|\; i \in I\right).$$ □

Lemma 7.39. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and let α be a w-element of S. If $i, j, k \in I$ pairwise different with $\alpha_i \in S_i \setminus S_i^j$ and $\alpha_j \in S_j \setminus S_j^k$, then $\alpha_i \in S_i \setminus S_i^k$.*

Proof. Assume $\alpha_i \in S_i^k$. Then there is an element $\beta \in S$ with $\beta_i = \alpha_i$ and $\beta_k \geq (\gamma_S)_k$. Since S is quasihomogeneous, there is by Theorem 7.19 a w-element $\delta \in S$ such that $\delta_i = \beta_i = \alpha_i$ and $\delta_k \geq \beta_k \geq (\gamma_S)_k$, hence $\delta_j \in S_j^k$. Moreover, we have $w_j\delta_j = w_i\delta_i = w_i\alpha_i = w_j\alpha_j$ since $\delta_i = \alpha_i \in S_i \setminus S_i^j$, and since α and δ are w-elements of S (see Definition 7.14). This implies $\alpha_j = \delta_j \in S_j^k$, contradicting the assumption. □

Lemma 7.40. *Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$. If S is w-closed, then*

$$\gamma_S = \left(\max\left(\gamma_{S_i^j} \;\middle|\; j \in I \setminus \{i\}\right)\right)_{i\in I}.$$

Proof. Set

$$\gamma = \left(\max\left(\gamma_{S_i^j} \;\middle|\; j \in I \setminus \{i\}\right)\right)_{i\in I}.$$

For any $i \in I$ we have $(\gamma_S)_i \in C_{S_i^j}$ for every $j \in I\setminus\{i\}$ (see Definition 4.60 and Lemma 4.62). This implies

$$\gamma_S \geq \left(\max\left(\gamma_{S_i^j} \;\middle|\; j \in I \setminus \{i\}\right)\right)_{i\in I} = \gamma.$$

Let $\alpha \in \gamma + \overline{S}$. Then

$$\alpha_i \geq \gamma_i = \max\left(\gamma_{S_i^j} \;\middle|\; j \in I \setminus \{i\}\right)$$

for any $i \in I$. In particular, we have $\alpha_i \in S_i^j$ for every $j \in I \setminus \{i\}$. So if for any $i \in I$ we choose an element $\alpha^{(i)} \in \prod_{i\in I} S_i$ with

$$\begin{aligned}\left(\alpha^{(i)}\right)_i &= \alpha_i \in \bigcap_{k\in I\setminus\{i\}} S_i^k,\\ \left(\alpha^{(i)}\right)_j &\geq \max\left\{\alpha_j, (\gamma_S)_j\right\} \in (C_S)_j \text{ for each } j \in I \setminus \{i\},\end{aligned}$$

then $\alpha^{(i)}$ is a w-element of S (see Definition 7.14). Since S is w-closed, we have $\alpha^{(i)} \in S$ for all $i \in I$. Thus,

$$\alpha = \inf\left(\alpha^{(i)} \;\middle|\; i \in I\right) \in S$$

as S satisfies property (E1). This implies $\gamma \geq \gamma_S$, and hence we obtain $\gamma = \gamma_S$. □

Proof of Proposition 7.22. We show that the w-elements of S generate a good semigroup S^w in the following sense: for any element $\alpha \in D_S$ we have $\alpha \in S^w$ if and only if there is a family $\left(\alpha^{(i)}\right)_{i\in I}$ of w-elements of S such that $\alpha = \inf\left(\alpha^{(i)} \;\middle|\; i \in I\right)$.

S^w is a good semigroup. By Proposition 7.8 and Lemmas 7.32.(2) and 7.38 S^w is a partially ordered cancellative commutative monoid with $D_{S^w} = D_S$ and $\alpha \geq \mathbf{0}$ for all $\alpha \in S^w$. Since $S \subset S^w$ and $\overline{S^w} = \overline{S}$, S^w satisfies property (E0). It remains to verify properties (E1) and (E2) for S^w.

(E1) Let $\alpha, \beta \in S^w$. Then by definition of S^w and by Lemma 7.37 there are families $\left(\alpha^{(i)}\right)_{i\in I}$ and $\left(\beta^{(i)}\right)_{i\in I}$ of w-elements of S with $\left(\alpha^{(i)}\right)_i = \alpha_i$ and $\left(\beta^{(i)}\right)_i = \beta_i$ for any $i \in I$ and $\left(\alpha^{(i)}\right)_j \geq \alpha_j$ and $\left(\beta^{(i)}\right)_j \geq \beta_j$ for all $j \in I \setminus \{i\}$. For any $i \in I$ set

$$\delta^{(i)} = \begin{cases}\alpha^{(i)} & \text{if } \alpha_i < \beta_i,\\ \beta^{(i)} & \text{else.}\end{cases}$$

Then $\left(\delta^{(i)}\right)_i = \min\{\alpha_i, \beta_i\}$ and $\left(\delta^{(i)}\right)_j \geq \min\{\alpha_j, \beta_j\}$ for all $j \in I \setminus \{i\}$. This shows that

$$\inf\{\alpha, \beta\} = \inf\left(\delta^{(i)} \;\middle|\; i \in I\right) \in S^w,$$

and hence S^w satisfies property (E1).

(E2) Suppose there is an $i \in I$ such that $\alpha_i = \beta_i$. First assume $\alpha_i \geq (\gamma_S)_i$. Then $\left(\delta^{(j)}\right)_i \geq (\gamma_S)_i$ for all $j \in I$. Thus, for any $j \in I$ there is a w-element $\epsilon^{(j)}$ of S with $\left(\epsilon^{(j)}\right)_i > \alpha_i$ and $\left(\epsilon^{(j)}\right)_k = \left(\delta^{(j)}\right)_k$ for all $k \in I \setminus \{i\}$ (see Definition 7.14). Then

$$\epsilon = \inf\left(\epsilon^{(j)} \;\middle|\; j \in I\right) \in S^w$$

with

$$\begin{aligned} &\epsilon_i > \alpha_i = \beta_i, \\ &\epsilon_j \geq \min\{\alpha_j, \beta_j\} \text{ for all } j \in I, \\ &\epsilon_k = \min\{\alpha_k, \beta_k\} \text{ for all } k \in I \text{ with } \alpha_k \neq \beta_k. \end{aligned}$$

Next we treat the case $\alpha_i < (\gamma_S)_i$. Set

$$J = \{j \in I \mid \alpha_j \neq \beta_j\}. \tag{7.25}$$

We show that for every $j \in J$ there is a w-element $\eta^{(j)}$ of S with $\eta^{(j)} \geq \inf\{\alpha, \beta\}$, $\left(\eta^{(j)}\right)_i > \alpha_i = \beta_i$, and $\left(\eta^{(j)}\right)_j = \min\{\alpha_j, \beta_j\}$. For this we consider the family $\left(\delta^{(k)}\right)_{k\in I}$ of w-elements with $\delta^{(k)} \geq \inf\{\alpha, \beta\}$ and $\left(\delta^{(k)}\right)_k = \min\{\alpha_k, \beta_k\}$ for any $k \in I$ as above.

Let $j \in J$. Then without loss of generality we may suppose that $\alpha_j < \beta_j$. We distinguish the cases $\alpha_i \in S_i \setminus S_i^j$ and $\alpha_i \in S_i^j \setminus (C_S)_i$.

First assume that $\alpha_i \in S_i \setminus S_i^j$. If $\alpha_j \in S_j \setminus S_j^i$, then

$$\begin{aligned} w_i\alpha_i = w_i\beta_i = w_i\left(\beta^{(i)}\right)_i = w_j\left(\beta^{(i)}\right)_j \geq w_j\beta_j \\ > w_j\alpha_j = w_j\left(\alpha^{(j)}\right)_j = w_i\left(\alpha^{(j)}\right)_i \geq w_i\alpha_i \end{aligned}$$

(see Definition 7.14). This is a contradiction, and hence $\alpha_j \in S_j^i$. This implies $\left(\delta^{(j)}\right)_i > \alpha_i$ as otherwise $\left(\delta^{(j)}\right)_i = \alpha_i \in S_i \setminus S_i^j$, and therefore $\alpha_j = \left(\delta^{(j)}\right)_j \in S_j \setminus S_j^i$ by Remark 7.15 since $\delta^{(j)}$ is a w-element of S. So if we set $\eta^{(j)} = \delta^{(j)}$, then $\eta^{(j)} \geq \inf\{\alpha, \beta\}$ with $\left(\eta^{(j)}\right)_i > \alpha_i = \beta_i$ and $\left(\eta^{(j)}\right)_j = \min\{\alpha_j, \beta_j\} = \alpha_j$.

Now assume that $\alpha_i \in S_i^j \setminus (C_S)_i$. If $\alpha_j \in S_j \setminus S_j^i$, then as above $\left(\delta^{(j)}\right)_i > \alpha_i$ by Remark 7.15 since $\delta^{(j)}$ is a w-element. So if we set $\eta^{(j)} = \delta^{(j)}$, then $\eta^{(j)} \geq \inf\{\alpha, \beta\}$ with $\left(\eta^{(j)}\right)_i > \alpha_i = \beta_i$ and $\left(\eta^{(j)}\right)_j = \alpha_j = \min\{\alpha_j, \beta_j\}$.

Let now $\alpha_j \in S_j^i$, and consider an element $\eta^{(j)} \in \prod_{k\in I} S_k$ with

$$\begin{aligned}&\left(\eta^{(j)}\right)_j = \left(\delta^{(j)}\right)_j,\\&\left(\eta^{(j)}\right)_k \geq \max\left\{(\gamma_S)_k, \left(\delta^{(j)}\right)_k\right\} \text{ for all } k \in I \setminus \{j\} \text{ with } \left(\delta^{(j)}\right)_j \in S_j^k,\\&\left(\eta^{(j)}\right)_l = \left(\delta^{(j)}\right)_l \text{ for all } l \in I \setminus \{j\} \text{ with } \left(\delta^{(j)}\right)_j \in S_j \setminus S_j^l.\end{aligned}$$

We show that $\eta^{(j)}$ is a w-element of S.

So let $m, n \in I$ with $m \neq n$ and

$$\left(\eta^{(j)}\right)_j = \left(\delta^{(j)}\right)_j \in S_j^m \cap S_j^n.$$

Then

$$\left(\eta^{(j)}\right)_m \in (C_S)_m \subset S_m^j \cap S_m^n$$

and

$$\left(\eta^{(j)}\right)_n \in (C_S)_n \subset S_n^j \cap S_n^m$$

(see Lemma 4.63).

Let $m, n \in I$ with $m \neq n$ and

$$\left(\eta^{(j)}\right)_j = \left(\delta^{(j)}\right)_j \in \left(S_j \setminus S_j^m\right) \cap \left(S_j \setminus S_j^n\right).$$

Then

$$\left(\eta^{(j)}\right)_m = \left(\delta^{(j)}\right)_m \in S_m \setminus S_m^j$$

and

$$\left(\eta^{(j)}\right)_n = \left(\delta^{(j)}\right)_n \in S_n \setminus S_n^j$$

by Remark 7.15 since $\delta^{(j)}$ is a w-element of S. This implies

$$\left(\eta^{(j)}\right)_m = \left(\delta^{(j)}\right)_m \in S_m \setminus S_m^n$$

and

$$\left(\eta^{(j)}\right)_n = \left(\delta^{(j)}\right)_n \in S_n \setminus S_n^m$$

by Lemma 7.39. Moreover, we have

$$\begin{aligned}w_j\left(\eta^{(j)}\right)_j &= w_j\left(\delta^{(j)}\right)_j = w_m\left(\delta^{(j)}\right)_m = w_m\left(\eta^{(j)}\right)_m,\\w_j\left(\eta^{(j)}\right)_j &= w_j\left(\delta^{(j)}\right)_j = w_n\left(\delta^{(j)}\right)_n = w_n\left(\eta^{(j)}\right)_n,\\w_m\left(\eta^{(j)}\right)_m &= w_m\left(\delta^{(j)}\right)_m = w_n\left(\delta^{(j)}\right)_n = w_n\left(\eta^{(j)}\right)_n\end{aligned}$$

(see Definition 7.14).

Let $m, n \in I$ with $m \neq n$ and

$$\left(\delta^{(j)}\right)_j \in \left(S_j \setminus S_j^m\right) \cap S_j^n.$$

Then

$$\left(\eta^{(j)}\right)_m = \left(\delta^{(j)}\right)_m$$

and

$$\left(\eta^{(j)}\right)_n \geq (\gamma_S)_n.$$

Suppose that $\eta^{(j)}$ is not a w-element of S. Then

$$\left(\delta^{(j)}\right)_m = \left(\eta^{(j)}\right)_m \in S_m \setminus S_m^n$$

(see Definition 7.14). But then Lemma 7.39 yields the contradiction

$$\left(\delta^{(j)}\right)_j \in S_j \setminus S_j^n.$$

Therefore, $\eta^{(j)}$ is a w-element of S.

Thus, for any $j \in J$ (see Equation (7.25)) there is a w-element $\eta^{(j)}$ of S with $\eta^{(j)} \geq \inf\{\alpha, \beta\}$, $\left(\eta^{(j)}\right)_i > \alpha_i = \beta_i$, and $\left(\eta^{(j)}\right)_j = \left(\delta^{(j)}\right)_j = \min\{\alpha_j, \beta_j\}$. So for every $k \in I \setminus J$ choose an element $j_k \in J$ with

$$\left(\eta^{(j_k)}\right)_k = \min\left(\left(\eta^{(j)}\right)_k \;\middle|\; j \in J\right),$$

and set $\eta^{(k)} = \eta^{(j_k)}$. Then

$$\eta = \inf\left\{\eta^{(j)} \;\middle|\; j \in J\right\} = \inf\left(\eta^{(j)} \;\middle|\; j \in I\right) \in S^w$$

with

$$\begin{aligned}
\eta_i &> \alpha_i = \beta_i,\\
\eta_j &= \min\{\alpha_j, \beta_j\} \text{ for all } j \in J,\\
\eta_k &\geq \min\{\alpha_k, \beta_k\} \text{ for all } k \in I.
\end{aligned}$$

Thus, S^w satisfies property (E2), and hence it is a good semigroup.

S^w is quasihomogeneous of type w. First note that $S \subset S^w \subset \overline{S}$ by Theorem 7.19, and hence $\gamma_S \geq \gamma_{S^w}$. So by construction we have

$$S_i = (S^w)_i \tag{7.26}$$

for any $i \in I$ and

$$S_i^j \subset (S^w)_i^j \tag{7.27}$$

for every $j \in I \setminus \{i\}$.

Let now $i,j \in I$ with $i \neq j$, and let $\alpha \in (S^w)_i^j$. Then there is a $\beta \in S^w$ with $\beta_i = \alpha$ and $\beta_j \geq (\gamma_{S^w})_j$ (see Definition 4.60). Since S^w is a good semigroup, Lemma 4.33 yields a $\delta \in S^w$ with $\delta_i = \beta_i = \alpha$ and $\delta_j \geq (\gamma_S)_j$. Then by construction of S^w there is a w-element ϵ of S with $\epsilon_i = \delta_i = \alpha$ and $\epsilon_j \geq \delta_j \geq (\gamma_S)_j$. Hence, $\alpha = \epsilon_i \in S_i^j$ (see Definition 7.14). Thus, with Equations (7.26) and (7.27) we obtain

$$\left(S_i, \left(S_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I} = \left((S^w)_i, \left((S^w)_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I}. \tag{7.28}$$

Since $S \subset S^w$ by Theorem 7.19, we have $\gamma_S \geq \gamma_{S^w}$. Thus, a w-element of S is also a w-element of S^w since $\left(S_i, \left(S_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I} = \left((S^w)_i, \left((S^w)_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I}$ by Equation (7.28). This implies that S^w is quasihomogeneous of type w by construction and by Theorem 7.19.

S^w is w-closed. Let α be a w-element of S^w, and set

$$J = \{i \in I \mid \alpha_i < (\gamma_{S^w})_i\}.$$

Then there is a w-element β of S^w with

$$\begin{aligned}&\beta_i = \alpha_i \text{ for all } i \in J,\\ &\beta_j \geq \max\left\{\alpha_j, (\gamma_S)_j\right\} \text{ for all } j \in I \setminus J\end{aligned}$$

(see Definition 7.14). By Equation (7.28) β is also a w-element of S (see again Definition 7.14). For any $i \in J$ we set $\alpha^{(i)} = \beta$.

Let now $i \in I \setminus J$. Then $\alpha_i \geq (\gamma_{S^w})_i$, and hence $\alpha_i \in (S^w)_i^j = S_i^j$ for all $j \in I \setminus \{i\}$, see Equation (7.28). Thus, any element $\alpha^{(i)} \in \prod_{j\in I} S_j$ with

$$\begin{aligned}&\left(\alpha^{(i)}\right)_i = \alpha_i,\\ &\left(\alpha^{(i)}\right)_j \geq \max\left\{\alpha_j, (\gamma_S)_j\right\} \text{ for every } j \in I \setminus \{i\}\end{aligned}$$

is a w-element of S (see Definition 7.14). In particular, we obtain

$$\alpha = \inf\left(\alpha^{(i)} \,\middle|\, i \in I\right) \in S^w$$

by construction. Therefore, S^w is w-closed.

S^w is the unique w-closure of S. Assume that S' is a quasihomogeneous semigroup of type w which is w-closed and satisfies $S \subset S' \subset \overline{S}$ and

$$\left(S_i, \left(S_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I} = \left((S')_i, \left((S')_i^j\right)_{j\in I\setminus\{i\}}\right)_{i\in I}.$$

Then S and S' have by Equation 7.28 the same w-elements since $\gamma_{S^w} = \gamma_{S'}$ by Lemma 7.40, and these elements have to be contained in S^w as well as in S'. Then Theorem 7.19 and property (E1) yield $S^w = S'$. $\square$

7.7. Proof of Proposition 7.6

Lemma 7.41. *Let R be a quasihomogeneous curve, and let $\overline{w} \in \mathbb{N}^n$ as in Theorem 6.2. If $x \in R$ is homogeneous and $\alpha \in C_{\Gamma_R}$ with $\alpha_\mathfrak{p} \neq \nu_\mathfrak{p}(x)$ for all $\mathfrak{p} \in \operatorname{Min}(R)$, then* $\inf\{\nu(x), \alpha\} \in \Gamma_R$ *is a $\overline{w}$-element of* Γ_R.

Proof. Let $\alpha \in C_{\Gamma_R}$ with $\alpha > \inf\{\nu(x), \gamma_{\Gamma_R}\}$. Then there is $y \in (\mathfrak{C}_R)^{\mathrm{reg}}$ with $\nu(y) = \alpha$. Since $\nu_\mathfrak{p}(x+y) = \min\{\nu_\mathfrak{p}(x), \nu_\mathfrak{p}(y)\} < \infty$ for all $\mathfrak{p} \in \operatorname{Min}(R)$ by Lemma D.22.(5), Lemma 3.4.(2) yields $x + y \in R^{\mathrm{reg}}$, and hence $\inf\{\nu(x), \alpha\} \in \Gamma_R$.

Set

$$\beta = \inf\{\nu(x), \alpha\}, \tag{7.29}$$

and let $\mathfrak{p} \in \operatorname{Min}(R)$ such that $\beta_\mathfrak{p} < (\gamma_{\Gamma_R})_\mathfrak{p}$. Then $\beta_\mathfrak{p} = \nu_\mathfrak{p}(x)$, and hence $x \notin \mathfrak{p}$ by Theorems 3.2.(2) and A.74.(2) and Proposition D.13.(4). Therefore, Theorem 6.2.(1) yields

$$\overline{w}_\mathfrak{p}\beta_\mathfrak{p} = \overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x) = \deg(x). \tag{7.30}$$

Let $\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}$ such that $\beta_\mathfrak{p} \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$. Then $\beta_\mathfrak{p} < (\gamma_S)_\mathfrak{p}$ by Lemma 4.63, and hence $\beta_\mathfrak{p} = \nu_\mathfrak{p}(x)$ by Equation (7.29). Since x is quasihomogeneous, Theorem 6.2.(5) yields an $a \in \mathbb{C}$ such that

$$(\Psi(x))_\mathfrak{p} = a t_\mathfrak{p}^{\nu_\mathfrak{p}(x)}$$

(see Equation (7.30)), where we use the notation of Theorem 6.2. Moreover, $a \neq 0$ since $\nu = \operatorname{ord}_t \circ \Psi$ by Theorem 6.2.(3). Since $\nu_\mathfrak{p}(x) \in (\Gamma_R)_\mathfrak{p} \setminus (\Gamma_R)_\mathfrak{p}^\mathfrak{q}$, Theorem 6.2.(4) and (5) imply

$$(\Psi(x))_\mathfrak{q} = (\zeta_{\mathfrak{p}\mathfrak{q}}(\nu_\mathfrak{p}(x)))^{-1} a t_\mathfrak{q}^{\tau_{\mathfrak{p}\mathfrak{q}}(\nu_\mathfrak{q}(x))}.$$

Since $\zeta_{\mathfrak{p}\mathfrak{q}}(\nu_\mathfrak{p}(x)) \neq 0$ by Lemma 6.7.(1), Theorem 6.2.(2) and (3) yield where

$$\nu_\mathfrak{q}(x) = \tau_{\mathfrak{p}\mathfrak{q}}(\nu_\mathfrak{p}(x)) = \frac{\overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x)}{\overline{w}_\mathfrak{q}} \in (\Gamma_R)_\mathfrak{q} \setminus (\Gamma_R)_\mathfrak{q}^\mathfrak{p}.$$

Since $\nu_\mathfrak{p}(x) < (\gamma_{\Gamma_R})_\mathfrak{p}$ and $\nu_\mathfrak{q}(x) < (\gamma_{\Gamma_R})_\mathfrak{q}$ by Proposition 4.67.(2), Theorem 6.2.(1) yields

$$\overline{w}_\mathfrak{q}\nu_\mathfrak{q}(x) = \overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x).$$

Moreover, we have $\nu_\mathfrak{q}(x) < (\gamma_{\Gamma_R})_\mathfrak{q}$ by Lemma 4.63, and hence

$$\overline{w}_\mathfrak{q}\beta_\mathfrak{q} = \overline{w}_\mathfrak{q}\nu_\mathfrak{q}(x) = \overline{w}_\mathfrak{p}\nu_\mathfrak{p}(x) = \overline{w}_\mathfrak{p}\beta_\mathfrak{p}$$

(see Equation (7.29)). □

Proof of Proposition 7.6. Let $\alpha \in \Gamma_R$. Then there is an $x \in R^{\mathrm{reg}}$ with $\nu(x) = \alpha$. Let now $\mathfrak{p} \in \operatorname{Min}(R)$. Then Theorem 6.2.(5) yields

$$\begin{aligned}
\nu(x_{\overline{w}_\mathfrak{p}\alpha_\mathfrak{p}}) &= \operatorname{ord}_t \circ \Psi(x_{\overline{w}_\mathfrak{p}\alpha_\mathfrak{p}}) \\
&= \operatorname{ord}_t \left((\Psi(x))_{\overline{w}_\mathfrak{p}\alpha_\mathfrak{p}} \right) \\
&\geq \nu(x) \\
&= \alpha
\end{aligned}$$

with

$$\nu_{\mathfrak{p}}(x_{\overline{w}_{\mathfrak{p}}\alpha_{\mathfrak{p}}}) = \nu_{\mathfrak{p}}(x) = \alpha_{\mathfrak{p}}$$

(also see Theorem 6.2.(3)).

Let now $\beta \in C_{\Gamma_R}$ with $\beta_{\mathfrak{q}} > \alpha_{\mathfrak{q}}$ for all $\mathfrak{q} \in \operatorname{Min}(R)$, and set

$$\alpha^{(\mathfrak{p})} = \inf\{\nu(x_{\overline{w}_{\mathfrak{p}}\alpha_{\mathfrak{p}}}), \beta\}.$$

Then $\alpha^{(\mathfrak{p})}$ is a $\overline{w}$-element of Γ_R by Lemma 7.41 with

$$\begin{aligned} \left(\alpha^{(\mathfrak{p})}\right)_{\mathfrak{p}} &= \alpha_{\mathfrak{p}}, \\ \left(\alpha^{(\mathfrak{q})}\right)_{\mathfrak{q}} &\geq \alpha_{\mathfrak{q}} \text{ for all } \mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}. \end{aligned}$$

Therefore, we can find a family $\left(\alpha^{(\mathfrak{p})}\right)_{\mathfrak{p}\in\operatorname{Min}(R)} \in (\Gamma_R)^{\operatorname{Min}(R)}$ of $\overline{w}$-elements such that

$$\alpha = \inf\left(\alpha^{(\mathfrak{p})} \;\middle|\; \mathfrak{p} \in \operatorname{Min}(R)\right),$$

and hence Γ_R is quasihomogeneous of type $\overline{w}$ by Theorem 7.19. □

7.8. Proof of Proposition 7.12

Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^I$, and for any $i, j \in I$ with $i \neq j$ let $\zeta_{ij} \colon S_i \setminus S_i^j \to \mathbb{C}$ be a map satisfying $\zeta_{ij}(\alpha+\beta) = \zeta_{ij}(\alpha)\,\zeta_{ij}(\beta)$ for all $\alpha, \beta \in S_i \setminus S_i^j$ with $\alpha + \beta \in S_i \setminus S_i^j$. We set

$$A = \operatorname{Fib}(S, w, \zeta)$$

with $\zeta = \left(\left(\zeta_{ij}\right)_{j\in I\setminus\{j\}}\right)_{i\in I}$. Note that A is a $\mathbb{C}$-subalgebra of $\prod_{i\in I}\mathbb{C}[[t_i]]$, see Remark 7.11.

The proof of Proposition 7.12 is in parts analogous to that of Proposition 6.6, see Section 6.3.

Lemma 7.42. *We have* $\overline{A} = \prod_{i\in I}\mathbb{C}[[t_i]]$. *In particular,* $\dim A = 1$.

Proof. First note that we have $A \subset \prod_{i\in I}\mathbb{C}[[t_i]]$, and Proposition 4.64 yields

$$t^{\gamma_S}\prod_{i\in I}\mathbb{C}[[t_i]] \subset A.$$

This implies

$$Q_A = \prod_{i\in I}\mathbb{C}[[t_i]]\left[t_i^{-1}\right].$$

Let now $x = \left(\sum_{\alpha_i \in \mathbb{N}} a_i^{(i)} t_i^{\alpha_i}\right)_{i\in I} \in \mathbb{C}[[t_i]]$. Then

$$\begin{aligned} x &= \left(\sum_{\substack{\alpha_i\in\mathbb{N}\\ \alpha_i\le(\gamma_S)_i}} a_{\alpha_i}^{(i)} t_i^{\alpha_i}\right)_{i\in I} + \left(\sum_{\substack{\alpha_i\in\mathbb{N}\\ \alpha_i>(\gamma_S)_i}} a_{\alpha_i}^{(i)} t_i^{\alpha_i}\right)_{i\in I} \\ &= \sum_{i\in I} \sum_{\substack{\alpha_i\in\mathbb{N}\\ \alpha_i\le(\gamma_S)_i}} a_{\alpha_i}^{(i)} (t^{\mathbf{e}_i})^{\alpha_i} + \left(\sum_{\substack{\alpha_i\in\mathbb{N}\\ \alpha_i>(\gamma_S)_i}} a_{\alpha_i}^{(i)} t_i^{\alpha_i}\right)_{i\in I}, \end{aligned}$$

where for any $i \in I$ we denote by $\mathbf{e}_i$ the i-th unit vector in $\mathbb{Z}^I$. Since

$$\left(\sum_{\substack{\alpha_i\in\mathbb{N}\\ \alpha_i>(\gamma_S)_i}} a_{\alpha_i}^{(i)} t_i^{\alpha_i}\right)_{i\in I} \in t^{\gamma_S} \prod_{i\in I} \mathbb{C}[[t_i]] \subset A,$$

this implies that $\prod_{i\in I} \mathbb{C}[[t_i]]$ is generated as an A-algebra by $\{t^{\mathbf{e}_i} \mid i \in I\}$.

Moreover, for any $i \in I$ we have

$$(t^{\mathbf{e}_i})^{(\gamma_S)_i} \in t^{\gamma_S} \prod_{i\in I} \mathbb{C}[[t_i]] \subset A.$$

Hence, $t^{\mathbf{e}_i}$ is integral over A. Therefore, $\prod_{i\in I} \mathbb{C}[[t_i]]$ is an integral extension of A in its total ring of fractions. Proposition B.5 yields

$$\overline{A} = \prod_{i\in I} \mathbb{C}[[t_i]]$$

since $\prod_{i\in I} \mathbb{C}[[t_i]]$ is integrally closed in Q_A. Moreover, Theorem B.14 yields $\dim A = \dim \prod_{i\in I} \mathbb{C}[[t_i]] = 1$. $\square$

Lemma 7.43. *The ring A is local with maximal ideal*

$$\mathfrak{m}_A = \{x \in A \mid \operatorname{ord}_t(x) > \mathbf{0}\}.$$

Proof. See the proof of Lemma 6.15.

Assume A is not local, and let $\mathfrak{m}, \mathfrak{n} \in \operatorname{Max}(A)$ with $\mathfrak{m} \neq \mathfrak{n}$. Then by Propositions B.3 and B.15 and Theorem B.12 there are $\overline{\mathfrak{m}}, \overline{\mathfrak{n}} \in \operatorname{Max}\left(\overline{A}\right)$ with $\overline{\mathfrak{m}} \cap A = \mathfrak{m}$ and $\overline{\mathfrak{n}} \cap A = \mathfrak{n}$. Since

$$\overline{A} = \prod_{i\in I} \mathbb{C}[[t_i]]$$

by Lemma 7.42, there are by Lemma A.6.(2) $i_\mathfrak{m}, i_\mathfrak{n} \in I$ such that

$$\begin{aligned} \overline{\mathfrak{m}} &= t_{i_\mathfrak{m}} \mathbb{C}[[t_{i_\mathfrak{m}}]] \times \prod_{i\in I\setminus(i_\mathfrak{m})} \mathbb{C}[[t_i]], \\ \overline{\mathfrak{n}} &= t_{i_\mathfrak{n}} \mathbb{C}[[t_{i_\mathfrak{n}}]] \times \prod_{i\in I\setminus(i_\mathfrak{n})} \mathbb{C}[[t_i]]. \end{aligned}$$

Then for any $x \in \mathfrak{m} \setminus (\mathfrak{n} \cap \mathfrak{m})$ this implies

$$\begin{aligned}
\mathrm{pr}_{i_\mathfrak{m}}(x) &\in t_{i_\mathfrak{m}}\mathbb{C}[[t_{i_\mathfrak{m}}]],\\
\mathrm{pr}_{i_\mathfrak{n}}(x) &\in \mathbb{C}[[t_{i_\mathfrak{n}}]] \setminus t_{i_\mathfrak{n}}\mathbb{C}[[t_{i_\mathfrak{n}}]],
\end{aligned}$$

where for every $i \in I$ we denote by $\mathrm{pr}_i \colon \prod_{j\in I}\mathbb{C}[[t_j]] \to \mathbb{C}[[t_i]]$ the projection. In particular, we obtain $\mathrm{ord}_{t_{i_\mathfrak{m}}}(x) > 0$ and $\mathrm{ord}_{t_{i_\mathfrak{n}}}(x) = 0$. So writing

$$x = \left(\sum_{\alpha_i \in (\Gamma_R)_i} a^{(i)}_{\alpha_i} t_\mathfrak{p}^{\alpha_\mathfrak{p}}\right)$$

we have

$$\begin{aligned}
a_0^{(i_\mathfrak{m})} &= 0,\\
a_0^{(i_\mathfrak{n})} &\neq 0.
\end{aligned} \tag{7.31}$$

Since $x \in A$, since S is local, and since therefore $0 \in S_{i_\mathfrak{m}} \setminus S_{i_\mathfrak{m}}^{i_\mathfrak{n}}$ by Proposition 4.65, Equation (7.31) and the definition of A yield the contradiction

$$0 = a_0^{(i_\mathfrak{m})} = \zeta_{i_\mathfrak{m} i_\mathfrak{n}}(0)\, a^{(i_\mathfrak{n})}_{\tau_{i_\mathfrak{m} i_\mathfrak{n}}(0)} = \zeta_{i_\mathfrak{m} i_\mathfrak{n}}(0)\, a_0^{(i_\mathfrak{n})} \neq 0,$$

where the last inequality follows as $\zeta_{i_\mathfrak{m} i_\mathfrak{n}}(0) \neq 0$ by assumption. Thus, A is local, and the maximal ideal of A is by Theorem B.12, Proposition B.15, and Lemmas 6.14 and A.6.(2)

$$\begin{aligned}
\mathfrak{m}_A &= \left(\bigcap_{\mathfrak{m}\in \mathrm{Max}\left(\overline{A}\right)} \mathfrak{m}\right) \cap A\\
&= \left(\bigcap_{i\in I} t_i\mathbb{C}[[t_i]] \times \prod_{j \in I\setminus\{i\}} \mathbb{C}[[t_j]]\right) \cap A\\
&= \left(t \prod_{i\in I} \mathbb{C}[[t_i]]\right) \cap A\\
&= \{x \in A \mid \mathrm{ord}_t(x) > \mathbf{0}\}.
\end{aligned}$$

□

Lemma 7.44. *For any* $\mathfrak{p} \in \mathrm{Min}(A)$ *there is an* $i_\mathfrak{p} \in I$ *such that*

$$\mathfrak{p} = \left\{x \in A \,\middle|\, \mathrm{pr}_{i_\mathfrak{p}}(x) = 0\right\},$$

where $\mathrm{pr}_j \colon A \to \mathbb{C}\left[\left[t_j^{S_j}\right]\right]$ *is the projection for any* $j \in I$. *Conversely, for every* $i \in I$ *we have*

$$\mathfrak{p}_i = \{x \in A \mid \mathrm{pr}_i(x) = 0\} \in \mathrm{Min}(A).$$

In particular, there is a bijection

$$\begin{aligned}
\mathrm{Min}(A) &\to I,\\
\mathfrak{p} &\mapsto i_\mathfrak{p},\\
\mathfrak{p}_i &\leftarrow\!\shortmid i.
\end{aligned}$$

Proof. By Lemma 7.42 we have $\overline{A} = \prod_{i\in I} \mathbb{C}[[t_i]]$. Then Lemma A.6.(2) yields

$$\operatorname{Min}\left(\overline{A}\right) = \left\{0 \times \prod_{j\in I\setminus\{i\}} \mathbb{C}[[t_i]] \;\middle|\; i \in I\right\}.$$

Thus, the statement follows from Theorem A.72. □

Lemma 7.45. *For any $i \in I$ there is a numerical subsemigroup S_i' of S_i such that*

$$\operatorname{pr}_i(A) = \mathbb{C}\left[\left[t_i^{S_i'}\right]\right],$$

where $\operatorname{pr}_i\colon \prod_{j\in I} \mathbb{C}[[t_j]] \to \mathbb{C}[[t_i]]$ *is the projection.*

Proof. Let $i \in I$, and set

$$S_i' = \left\{\alpha \in S_i \;\middle|\; \text{there is } \left(\sum_{\alpha_j\in S_j} a^{(j)}_{\alpha_j} t_j^{\alpha_j}\right)_{j\in I} \in A \text{ with } a^{(i)}_{\alpha_i} \neq 0\right\}. \tag{7.32}$$

Let $\alpha \in S_i'$, and let

$$\left(\sum_{\alpha_j\in S_j} a^{(j)}_{\alpha_j} t_j^{\alpha_j}\right)_{j\in I} \in A$$

with $a^{(i)}_{\alpha_i} \neq 0$. Set

$$J = \left\{j \in I\setminus\{i\} \;\middle|\; \alpha \in S_i^j\right\},$$

and for any $j \in I$ let

$$b_j = \begin{cases} 0 & \text{if } j \in J,\\ a^{(i)}_{\alpha} & \text{if } j = i,\\ a^{(j)}_{\tau_{ij}(\alpha)} & \text{else,}\end{cases}$$

and

$$\beta_j = \begin{cases} \infty & \text{if } j \in J,\\ \alpha & \text{if } j = i,\\ \tau_{ij}(\alpha) & \text{else.}\end{cases}$$

Then we have for any $j,k \in I\setminus J$ with $j \neq k$

$$b^{(j)}_{\beta_j} = a^{(j)}_{\beta_j} = \zeta_{jk}(\beta_j)\, a^{(k)}_{\tau_{jk}(\beta_j)} = \zeta_{jk}(\beta_j)\, b^{(k)}_{\tau_{jk}(\beta_j)}.$$

Let $j \in J$. Since $\alpha \in S_i^j$, and since S is quasihomogeneous of type w, there is a w-element $\delta \in S$ with $\delta_i = \alpha$ and $\delta_j \geq (\gamma_S)_j$. Let now $k \in I\setminus J$. Then $\delta_k < (\gamma_S)_k$, and hence $\delta_k = \tau_{ik}(\alpha)$ by Proposition 7.16. Thus, $\tau_{ik}(\alpha) \in S_k^j$. This implies that

$$\left(b_j t_j^{\beta_j}\right)_{j\in I} \in A, \tag{7.33}$$

where we use the convention $t_j^\infty = 0$ for any $j \in I$. Thus, S_i' is a subsemigroup of S_i. Moreover, $0 \in S_i'$ as $\mathbb{C} \subset A$. Since obviously $(C_S)_i \subset S_i'$, S_i' is a numerical semigroup. Therefore, Equations (7.32) and (7.33) yield

$$\mathbb{C}\left[\left[t_i^{S_i'}\right]\right] \subset \mathrm{pr}_i\,(A).$$ □

Lemma 7.46. *The ring A is Noetherian.*

Proof. By Lemma 7.44 there is a bijection

$$\begin{aligned}&I \to \mathrm{Min}\,(A),\\ &i \mapsto \mathfrak{p}_i = \{x \in A \mid \mathrm{pr}_i\,(x) = 0\} \in \mathrm{Min}\,(A),\end{aligned}$$

where $\mathrm{pr}_i\colon \prod_{j\in I}\mathbb{C}[[t_j]] \to \mathbb{C}[[t_i]]$ is the projection for any $i \in I$. This obviously yields

$$\bigcap_{\mathfrak{p}\in\mathrm{Min}\,(A)} \mathfrak{p} = \{0\}.$$

Moreover, for any $i \in I$ we obtain

$$\ker\,(\mathrm{pr}_i) = \mathfrak{p}_i.$$

Thus, the Homomorphism Theorem yields an isomorphism

$$A/\mathfrak{p}_i \cong \mathrm{pr}_i\,(A).$$

Since by Lemma 7.45 there is a numerical subsemigroup S_i' of S_i such that $\mathrm{pr}_i\,(A) = \mathbb{C}\left[\left[t_i^{S_i'}\right]\right]$, $A/\mathfrak{p}_i$ is Noetherian by Corollary 4.83. Therefore, A is Noetherian by Lemma A.3. □

Lemma 7.47. *The ring A is reduced.*

Proof. This follows from the definition of A as a subring of the reduced ring $\prod_{i\in I}\mathbb{C}[[t_i]]$. □

Lemma 7.48. *The ring A is a local complex algebroid curve.*

Proof. By Remark 7.11 and Lemmas 7.43, 7.46 and 7.47 A is a local complete reduced Noetherian $\mathbb{C}$-algebra with maximal ideal $\mathfrak{m}_A = \{x \in A \mid \mathrm{ord}_t\,(x) > \mathbf{0}\}$. Since $\zeta_{ij}(0) = 1$, and since $0 \in S_i \setminus S_i^j$ for any $i, j \in I$ with $i \neq j$, all components of an element $x \in A$ have the same constant term. This implies $A/\mathfrak{m}_A \cong \mathbb{C}$. Hence, A is a local complex algebroid curve. □

Lemma 7.49. *The $\mathbb{C}$-derivation*

$$(w_i t_i \partial_{t_i})_{i\in I}$$

of $\prod_{i\in I}\mathbb{C}[[t_i]]$ restricts to a $\mathbb{C}$-derivation $\mathfrak{d}$ of A.

Proof. Let $x \in A$, i.e.

$$x = \left(\sum_{\alpha_i \in S_i} a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I} \in \prod_{i \in I} \mathbb{C}[[t_i]]$$

with

$$a^{(i)}_{\alpha_i} = \zeta_{ij}(\alpha_i)\, a^{(j)}_{\tau_{ij}(\alpha_i)}$$

for any $i \in I$, for every $j \in I \setminus \{i\}$, and for all $\alpha_i \in S_i \setminus S_i^j$. Then

$$\mathfrak{d}(x) = \left(\sum_{\alpha_i \in S_i \setminus \{0\}} w_i \alpha_i a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I}.$$

Now Proposition 7.8 implies for any $i \in I$, for every $j \in I \setminus \{i\}$, and for all $\alpha_i \in S_i \setminus \left(S_i^j \cup \{0\} \right)$

$$w_i \alpha_i a^{(i)}_{\alpha_i} = w_j \alpha_j \zeta_{ij}(\alpha_i)\, a^{(j)}_{\tau_{ij}(\alpha_i)}.$$

Thus, $\mathfrak{d}(x) \in A$. $\square$

Lemma 7.50. *An element*

$$x = \left(\sum_{\alpha_i \in S_i} a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I} \in A$$

is an eigenvector of $\mathfrak{d}$ (see Lemma 7.49) if and only if there is a $d \in \mathbb{Z}$ such that for any $i \in I$ we have

$$x_i = \begin{cases} a^{(i)}_{\alpha_i} t_i^{\alpha_i} & \text{if there is an } \alpha_i \in S_i \text{ such that } w_i \alpha_i = d, \\ 0 & \text{else.} \end{cases}$$

In particular, $\mathfrak{d}$ has only eigenvalues in $\mathbb{N}$.

Proof. Let $x = \left(\sum_{\alpha_i \in S_i} a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I}$ be an eigenvector of $\mathfrak{d}$, i.e. there is $c \in \mathbb{C}$ such that

$$\begin{aligned} cx &= \mathfrak{d}(x) \\ &= \left(\sum_{\alpha_i \in S_i \setminus \{0\}} w_i \alpha_i a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I}. \end{aligned}$$

This implies

$$x_i = \begin{cases} a^{(i)}_{\alpha_i} t_i^{\alpha_i} & \text{if there is an } \alpha_i \in S_i \text{ such that } w_i \alpha_i = c, \\ 0 & \text{else.} \end{cases}$$

for any $i \in I$. In particular, we have $c \in \mathbb{N}$ since $w \in \mathbb{N}^I$ and $S \subset \mathbb{N}^I$.

Let now $d \in \mathbb{Z}$, and let $x = \left(\sum_{\alpha_i \in S_i} a^{(i)}_{\alpha_i} t_i^{\alpha_i}\right)_{i \in I}$ with

$$x_i = \begin{cases} a^{(i)}_{\alpha_i} t_i^{\alpha_i} & \text{if there is an } \alpha_i \in S_i \text{ such that } w_i \alpha_i = d, \\ 0 & \text{else.} \end{cases}$$

for any $i \in I$. Then

$$\begin{aligned} \mathfrak{d}(x) &= \left(\sum_{\alpha_i \in S_i \setminus \{0\}} w_i \alpha_i a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I} \\ &= d \left(\sum_{\alpha_i \in S_i \setminus \{0\}} a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I} \\ &= dx. \end{aligned}$$

Note that $x = 0$ if $d < 0$ since $w \in \mathbb{N}^I$ and $S \subset \mathbb{N}^I$. $\square$

Lemma 7.51. *Let*

$$x = \left(\sum_{\alpha_i \in S_i} a^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I} \in A,$$

and let $d \in \mathbb{Z}$. For any $i \in I$, for every $j \in I \setminus \{i\}$, and for all $\alpha_i \in S_i \setminus S_i^j$ we define

$$b^{(i)}_{\alpha_i} = \begin{cases} a^{(i)}_{\alpha_i} & \text{if } w_i \alpha_i = d, \\ 0 & \text{else.} \end{cases}$$

Then

$$\left(\sum_{\alpha_i \in S_i} b^{(i)}_{\alpha_i} t_i^{\alpha_i} \right)_{i \in I} \in A.$$

Proof. Let $d \in \mathbb{Z}$, let $i \in I$, let $j \in I \setminus \{i\}$, and let $\alpha_i \in S_i \setminus S_i^j$. First suppose $w_i \alpha_i \neq d$. Then also

$$w_j \tau_{ij}(\alpha_i) = w_j \frac{w_i \alpha_i}{w_j} = w_i \alpha_i \neq d,$$

and hence

$$b^{(i)}_{\alpha_i} = 0 = b^{(j)}_{\tau_{ij}(\alpha_i)}.$$

This implies

$$b^{(i)}_{\alpha_i} = \zeta_{ij}(\alpha_i)\, b^{(j)}_{\tau_{ij}(\alpha_i)}.$$

Assume now that $w_i \alpha_i = d$. Then also

$$w_j \tau_{ij}(\alpha_i) = w_j \frac{w_i \alpha_i}{w_j} = w_i \alpha_i = d,$$

Hence,

$$b^{(i)}_{\alpha_i} = a^{(i)}_{\alpha_i} = \zeta_{ij}(\alpha_i)\, a^{(j)}_{\tau_{ij}(\alpha_i)} = \zeta_{ij}(\alpha_i)\, b^{(j)}_{\tau_{ij}(\alpha_i)}.$$

This implies

$$\left(\sum_{\alpha_i\in S_i} b^{(i)}_{\alpha_i} t_i^{\alpha_i}\right)_{i\in I} \in A. \qquad \square$$

Lemma 7.52. *For any $x \in A$ there is a sequence $(x_d)_{d\in\mathbb{Z}} \in A^{\mathbb{Z}}$, where for every $d \in \mathbb{Z}$ either $x_d = 0$ or $\mathfrak{d}(x_d) = dx_d$, such that $x = \sum_{d\in\mathbb{Z}} x_d$.*

Proof. This follows from Lemmas 7.50 and 7.51. $\square$

Lemma 7.53. *The maximal ideal $\mathfrak{m}_A$ of A (see Lemma 7.43) is generated by eigenvectors of $\mathfrak{d}$ with positive eigenvalues.*

Proof. We want to show that $\mathfrak{m}_A$ is generated by the set

$$M = \{x \in A \mid \operatorname{ord}_t(x) > \mathbf{0} \text{ and } \mathfrak{d}(x) = d_x x \text{ for some } d_x \in \mathbb{Z}\}.$$

Lemma 7.43 immediately yields $M \subset \mathfrak{m}_A$.

Let $x \in \mathfrak{m}_A$. Then $\operatorname{ord}_t(x) > \mathbf{0}$ by Lemma 6.15, and by Lemma 7.52 there is a sequence $(x_d)_{d\in\mathbb{Z}} \in A^{\mathbb{Z}}$ with $x_d = 0$ or $\mathfrak{d}(x_d) = dx_d$ for every $d \in \mathbb{Z}$ such that $x = \sum_{d\in\mathbb{Z}} x_d$. In particular, we have $\operatorname{ord}_t(x_d) > \mathbf{0}$ (see Lemma 7.51), and hence $x_d \in \mathfrak{m}_A$ for every $d \in \mathbb{Z}$ by Lemma 7.43.

Pick an $\alpha \in C_S$ with $w_i\alpha_i = w_j\alpha_j$ for all $i, j \in I$. Then $t^\alpha \in A$ by the definition of A since $\alpha_i \in S_i^j$ for any $i, j \in I$ with $i \neq j$, see Lemma 4.63. Moreover,

$$\begin{aligned}\mathfrak{d}(t^\alpha) &= (w_i\alpha_i t_i^{\alpha_i})_{i\in I}\\ &= d_\alpha t^\alpha,\end{aligned}$$

where $d_\alpha = w_i\alpha_i$ for all $i \in I$, i.e. $t^\alpha \in M$. Then

$$t^{\alpha+\gamma_S}\prod_{i\in I}\mathbb{C}[[t_i]]t^\alpha A \subset \mathfrak{m}_A \tag{7.34}$$

by the definition of A since $\alpha_i \in S_i^j$ for any $i, j \in I$ with $i \neq j$ (see Lemma 4.63), and we can write

$$x = \sum_{\substack{d\in\mathbb{Z}\\ \operatorname{ord}_t(x_d)\not\geq\alpha+\gamma_S}} x_d + \sum_{\substack{d\in\mathbb{Z}\\ \operatorname{ord}_t(x_d)\geq\alpha+\gamma_S}} x_d,$$

where

$$\sum_{\substack{d\in\mathbb{Z}\\ \operatorname{ord}_t(x_d)\geq\alpha+\gamma_S}} x_d \in t^\alpha A$$

by Equation (7.34).

Let now $d \in \mathbb{Z}$ such that $\operatorname{ord}_t(x_d) \not\geq \alpha+\gamma_S$. Then by Lemma 7.50 there is an $i \in I$ such that $d = w_i \operatorname{ord}_{t_i}(x_d) \leq w_i(\alpha+\gamma_S)_i$. In particular, we have $d \leq \max\{w_i(\alpha+\gamma_S)_i \mid i \in I\}$. This implies that

$$\sum_{\substack{d\in\mathbb{Z}\\ \operatorname{ord}_t(x_d)\not\geq\alpha+\gamma_S}} x_d$$

is finite. Thus,

$$x = \sum_{\substack{d \in \mathbb{Z} \\ \operatorname{ord}_t (x_d) \not\geq \alpha + \gamma_S}} x_d + t^\alpha y \in \langle M \rangle$$

with some $y \in A$.

Finally, note that by Lemma 7.50 the eigenvalue of every $x \in M$ with respect to $\mathfrak{d}$ is positive. □

Lemma 7.54. *Using the bijection* $\eta\colon \operatorname{Min}(\operatorname{Fib}(S,w)) \to I$ *of Lemma 7.44 to identify* $\mathbb{N}^I = \mathbb{N}^{\operatorname{Min}(\operatorname{Fib}(S,w))}$ *we have*

$$S \subset \Gamma_{\operatorname{Fib}(S,w)} \tag{7.35}$$

as well as equalities equalities

$$\left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}} = S_{\eta(\mathfrak{p})}$$

for any $\mathfrak{p} \in \operatorname{Min}(\operatorname{Fib}(S,w))$ *and*

$$\left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}}^{\mathfrak{q}} = S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})}$$

for every $\mathfrak{q} \in \operatorname{Min}(\operatorname{Fib}(S,w)) \setminus \{\mathfrak{q}\}$.

Proof. Let $\alpha \in S$. Then $t^\alpha \in \operatorname{Fib}(S,w)$ since S is quasihomogeneous. Using η to identify $\mathbb{N}^I = \mathbb{N}^{\operatorname{Min}(\operatorname{Fib}(S,w))}$ this implies

$$S \subset \Gamma_{\operatorname{Fib}(S,w)},$$

and hence $S_{\eta(\mathfrak{p})} \subset \left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Min}(\operatorname{Fib}(S,w))$. Moreover, by Lemma 7.44 and the definition of A we have for any $\mathfrak{p} \in \operatorname{Min}(\operatorname{Fib}(S,w))$

$$\operatorname{Fib}(S,w)/\mathfrak{p} \cong \operatorname{pr}_{\eta(\mathfrak{p})}(\operatorname{Fib}(S,w)) \subset \mathbb{C}\left[\left[t_{\eta(\mathfrak{p})}^{S_{\eta(\mathfrak{p})}}\right]\right].$$

This implies

$$\left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}} = S_{\eta(\mathfrak{p})}. \tag{7.36}$$

Let now $\mathfrak{p} \in \operatorname{Min}(\operatorname{Fib}(S,w))$, and let $\mathfrak{q} \in \operatorname{Min}(\operatorname{Fib}(S,w)) \setminus \{\mathfrak{p}\}$. Since $S \subset \Gamma_{\operatorname{Fib}(S,w)}$ (see Equation (7.35)), we have $S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})} \subset \left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}}^{\mathfrak{q}}$.

Let $\alpha \in (\operatorname{Fib}(S,w))_{\mathfrak{p}}^{\mathfrak{q}}$. Then by Proposition 4.69 there is an $x \in \mathfrak{q}$ with $\operatorname{ord}_{t_{\eta(\mathfrak{p})}}(x) = \alpha$. So writing

$$x = \left(\sum_{\alpha_{\mathfrak{p}'} \in S_{\eta(\mathfrak{p}')}} a_{\alpha_{\mathfrak{p}'}}^{(\mathfrak{p}')} t_{\eta(\mathfrak{p}')}^{\alpha_{\mathfrak{p}'}}\right)_{\mathfrak{p}' \in \operatorname{Min}(\operatorname{Fib}(S,w))}$$

we have by Lemma 7.44

$$a_{\alpha_{\mathfrak{p}}}^{(\mathfrak{p})} = 0 \text{ for all } \alpha_{\mathfrak{p}} \in S_{\eta(\mathfrak{p})} \text{ with } \alpha_{\mathfrak{p}} < \alpha, \tag{7.37}$$

$$a_{\alpha}^{(\mathfrak{p})} \neq 0, \tag{7.38}$$

$$a_{\alpha_{\mathfrak{q}}}^{(\mathfrak{q})} = 0 \text{ for all } \alpha_{\mathfrak{q}} \in S_{\eta(\mathfrak{q})}. \tag{7.39}$$

By Equation (7.36) we have $\alpha \in S_{\eta(\mathfrak{p})}$. Assume $\alpha \in S_{\eta(\mathfrak{p})} \setminus S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})}$. Then $\tau_{\eta(\mathfrak{p})\eta(\mathfrak{q})}(\alpha) \in S_{\eta(\mathfrak{q})} \setminus S_{\eta(\mathfrak{q})}^{\eta(\mathfrak{p})}$ by Proposition 7.8. Hence, Equation (7.39) and the definition of A yield

$$0 = a^{(\mathfrak{q})}_{\tau_{\eta(\mathfrak{p})\eta(\mathfrak{q})}(\alpha)} = \zeta_{\eta(\mathfrak{q})\eta(\mathfrak{p})}\Big(\tau_{\eta(\mathfrak{p})\eta(\mathfrak{q})}(\alpha)\Big)\, a^{\mathfrak{p}}_{\alpha}.$$

Since $a^{(\mathfrak{p})}_{\alpha} \neq 0$ (see Equation (7.38)), this implies $\zeta_{\eta(\mathfrak{q})\eta(\mathfrak{p})}\Big(\tau_{\eta(\mathfrak{q})\eta(\mathfrak{p})}(\alpha)\Big) = 0$, contradicting the assumption. Thus, $\alpha \in S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})}$. This yields $(\Gamma_A)_{\mathfrak{p}}^{\mathfrak{q}} \subset S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})}$, and therefore

$$S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})} = (\Gamma_A)_{\mathfrak{p}}^{\mathfrak{q}}. \qquad \square$$

Proof of Proposition 7.12. By Lemma 7.48 A is a local complex algebroid curve. By Lemma 7.53 (and since A is Noetherian) there is a generating system $(x_i)_{i=1}^n$ for the maximal ideal $\mathfrak{m}_A$ of A such that $\mathfrak{d}(x_i) = w_i' x_i$ for some $w_i' \in \mathbb{N}$ with $w_i' > 0$ for every $i = 1, \ldots, n$. Thus, A is quasihomogeneous. Since the grading on A is induced by the $\mathbb{C}$-derivation

$$(w_i t_i \partial_{t_i})_{i\in I}$$

on $\overline{A} = \prod_{i\in I} \mathbb{C}[[t_i]]$ (see Lemma 7.42), and since the valuation of Q_A is ord_t, A has normal weights w (see Definition 6.3).

Using the bijection $\eta\colon \operatorname{Min}(\operatorname{Fib}(S,w)) \to I$ of Lemma 7.44, we have by Lemma 7.54 equalities

$$\Big(\Gamma_{\operatorname{Fib}(S,w)}\Big)_{\mathfrak{p}} = S_{\eta(\mathfrak{p})}$$

for any $\mathfrak{p} \in \operatorname{Min}(\operatorname{Fib}(S,w))$ and

$$\Big(\Gamma_{\operatorname{Fib}(S,w)}\Big)_{\mathfrak{p}}^{\mathfrak{q}} = S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})}$$

for every $\mathfrak{q} \in \operatorname{Min}(\operatorname{Fib}(S,w)) \setminus \{\mathfrak{q}\}$. This yields commutative diagrams

$$\begin{tikzcd}
\mathbb{C}\Big[\Big[t_{\eta(\mathfrak{p})}^{S_{\eta(\mathfrak{p})}}\Big]\Big] \arrow[r, "\cong"] \arrow[d] & \mathbb{C}\bigg[\bigg[t_{\mathfrak{p}}^{\left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}}}\bigg]\bigg] \arrow[d] \\
\mathbb{C}\bigg[\bigg[t_{\eta(\mathfrak{p})}^{S_{\eta(\mathfrak{p})}\setminus S_{\eta(\mathfrak{p})}^{\eta(\mathfrak{q})}}\bigg]\bigg] \arrow[r, "\cong"] & \mathbb{C}\bigg[\bigg[t_{\mathfrak{p}}^{\left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}}\setminus\left(\Gamma_{\operatorname{Fib}(S,w)}\right)_{\mathfrak{p}}^{\mathfrak{q}}}\bigg]\bigg],
\end{tikzcd}$$

and hence

$$\operatorname{Fib}(S,w) = \operatorname{Fib}\Big(\Gamma_{\operatorname{Fib}(S,w)}, w\Big),$$

is a fibre product (see Definition 6.4). $\square$

7.9. Proof of Theorems 7.23 and 7.24

Lemma 7.55. *Let R be a quasihomogeneous curve with normal weights $\overline{w} \in \mathbb{N}^{\mathrm{Min}\,(R)}$ (see Definition 6.3). Then for any $\overline{w}$-element α of Γ_R with $\alpha \in \Gamma_R$ there is a homogeneous element $x \in R_{\deg_{\overline{w}}(\alpha)}$ with*

$$\nu_{\mathfrak{p}}(x) = \begin{cases} \alpha_{\mathfrak{p}} & \text{if } \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ \infty & \text{else,} \end{cases}$$

for any $\mathfrak{p} \in \mathrm{Min}\,(R)$. In particular, for any $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$ there is an $a^{(\mathfrak{p})} \in \mathbb{C} \setminus \{0\}$ such that

$$(\Psi(x))_{\mathfrak{p}} = \begin{cases} a^{(\mathfrak{p})} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}} & \text{if } \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ 0 & \text{else,} \end{cases}$$

where we use the notations of Theorem 6.2.

Proof. If $\alpha \in C_{\Gamma_R}$, then the statement is trivial. So let $\alpha \in \Gamma_R \setminus C_{\Gamma_R}$, and let $x \in R$ with $\nu(x) = \alpha$. Since α is a $\overline{w}$-element of Γ_R, Proposition 7.16 yields $\overline{w}_{\mathfrak{p}} \alpha_{\mathfrak{p}} = \deg_{\overline{w}}(\alpha)$ for every $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$.

In the notation of Theorem 6.2, there is by Theorem 6.2.(5) for any $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$ an $a^{(\mathfrak{p})} \in \mathbb{C}$ such that

$$\left((\Psi(x))_{\deg_{\overline{w}}(\alpha)}\right)_{\mathfrak{p}} = a_{\mathfrak{p}} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}.$$

Then $a_{\mathfrak{p}} \neq 0$ for every $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$ by Theorem 6.2.(3) since $\nu(x) = \alpha$. Let now $y \in Q_R$ with

$$(\Psi(y))_{\mathfrak{p}} = \begin{cases} \left((\Psi(x))_{\deg_{\overline{w}}(\alpha)}\right)_{\mathfrak{p}} & \text{if } \alpha_{\mathfrak{p}} \geq (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ 0 & \text{else,} \end{cases}$$

for any $\mathfrak{p} \in \mathrm{Min}\,(R)$. Then $\nu(y) \geq \gamma_{\Gamma_R}$, and hence $y \in \mathfrak{C}_R$ by Proposition 4.56. Moreover, $y \in R_{\deg_{\overline{w}}(\alpha)}$ by construction. Thus, $x - y \in R_{\deg_{\overline{w}}(\alpha)}$ with

$$(\Psi(x-y))_{\mathfrak{p}} = \begin{cases} a^{(\mathfrak{p})} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}} & \text{if } \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ 0 & \text{else,} \end{cases}$$

for every $\mathfrak{p} \in \mathrm{Min}\,(R)$. Then the claim follows from Theorem 6.2.(3). □

Lemma 7.56. *Let R be a fibre product with normal weights $\overline{w} \in \mathbb{N}^{\mathrm{Min}\,(R)}$ (see Definitions 6.3 and 6.4), let $\alpha \in \prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} (\Gamma_R)_{\mathfrak{p}} \setminus C_{\Gamma_R}$ be a $\overline{w}$-element of Γ_R, let $\beta \in \Gamma_R$ be a $\overline{w}$-element of Γ_R with $\beta_{\mathfrak{p}} = \alpha_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$ (see Proposition 7.6 and Theorem 7.19), and set*

$$J = \left\{\mathfrak{q} \in \mathrm{Min}\,(R) \;\middle|\; \alpha_{\mathfrak{q}} < \min\left\{\beta_{\mathfrak{q}}, (\gamma_{\Gamma_R})_{\mathfrak{q}}\right\}\right\}$$

If $J = \emptyset$*, then* $\alpha \in \Gamma_R$*, and if* $J \neq \emptyset$*, then there is a subset* J' *of* J *with* $J' \neq J$ *and a* $\overline{w}$*-element* δ *of* Γ_R *with* $\delta \in \Gamma_R$ *such that*

$$\begin{aligned} \delta_{\mathrm{Min}\,(R)\setminus J'} &= \alpha_{\mathrm{Min}\,(R)\setminus J'}, \\ \delta_{\mathfrak{q}} &> \alpha_{\mathfrak{q}} \textit{ for all } \mathfrak{q} \in J' \end{aligned}$$

Proof. By Lemma 4.33 we may replace α by $\inf\{\alpha, \gamma_{\Gamma_R}\}$ and β by $\inf\{\beta, \gamma_{\Gamma_R}\}$ (see Definition 7.14). Then

$$J = \{\mathfrak{q} \in \mathrm{Min}\,(R) \mid \alpha_{\mathfrak{q}} < \beta_{\mathfrak{q}}\}, \tag{7.40}$$

and we set

$$J_1 = \{\mathfrak{p} \in \mathrm{Min}\,(R) \mid \beta_{\mathfrak{q}} < \alpha_{\mathfrak{q}}\}. \tag{7.41}$$

Since there is a $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$ and $\alpha_{\mathfrak{p}} = \beta_{\mathfrak{p}}$, we have

$$\deg_{\overline{w}}(\alpha) = \deg_{\overline{w}}(\beta) \tag{7.42}$$

(see Definition 7.17). The construction of the sets J and J_1 yields with Proposition 7.16 and Equation (7.42)

$$\alpha_{\mathrm{Min}\,(R)\setminus(J\cup J_1)} = \beta_{\mathrm{Min}\,(R)\setminus(J\cup J_1)}, \tag{7.43}$$
$$\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in J, \tag{7.44}$$
$$\alpha_{J_1} = (\gamma_{\Gamma_R})_{J_1}, \tag{7.45}$$
$$\beta_J = (\gamma_{\Gamma_R})_J, \tag{7.46}$$
$$\beta_{\mathfrak{q}} < (\gamma_{\Gamma_R})_{\mathfrak{q}} \text{ for all } \mathfrak{q} \in J_1. \tag{7.47}$$

This implies with Definition 7.14

$$\alpha_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \text{ for any } \mathfrak{p} \in \mathrm{Min}\,(R) \text{ and } \mathfrak{q} \in J_1 \text{ with } \mathfrak{p} \neq \mathfrak{q} \tag{7.48}$$
$$\beta_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \text{ for any } \mathfrak{p} \in \mathrm{Min}\,(R) \text{ and } \mathfrak{q} \in J \text{ with } \mathfrak{p} \neq \mathfrak{q}, \tag{7.49}$$
$$\alpha_{\mathfrak{p}} = \beta_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \text{ for any } \mathfrak{p} \in \mathrm{Min}\,(R) \setminus (J \cup J_1) \text{ and } \mathfrak{q} \in J \cup J_1, \tag{7.50}$$
$$\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \text{ for any } \mathfrak{p} \in J \cup J_1 \text{ and } \mathfrak{q} \in \mathrm{Min}\,(R) \setminus (J \cup J_1), \tag{7.51}$$

where Equation (7.51) follows from Equation (7.50) applying Definition 7.14.

Let

$$\Psi \colon Q_R \to \prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} \mathbb{C}[[t_{\mathfrak{p}}]]\left[t_{\mathfrak{p}}^{-1}\right]$$

be the $\mathbb{C}$-algebra isomorphism of Theorem 3.44 (also see Theorem 6.2.(4)). Then by Lemma 7.55 there is a homogeneous element $x \in (\Psi(R))_{\deg_{\overline{w}}(\beta)}$ such that for any $\mathfrak{p} \in \mathrm{Min}\,(R)$ we have

$$x_{\mathfrak{p}} = \begin{cases} a^{(\mathfrak{p})} t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} & \text{if } \beta_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ 0 & \text{else,} \end{cases} \tag{7.52}$$

where $a^{(\mathfrak{p})} \in \mathbb{C} \setminus \{0\}$ for all $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\beta_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$. Note that, in particular, we have $x_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in J$ by Equation (7.46).

Let now

$$y \in \prod_{\mathfrak{p} \in \operatorname{Min}(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

with

$$y_{\mathfrak{p}} = \begin{cases} x_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \operatorname{Min}(R) \setminus J_1, \\ 0 & \text{else,} \end{cases} \tag{7.53}$$

for any $\mathfrak{p} \in \operatorname{Min}(R)$. Then Equations (7.43), (7.46), and (7.52) yield

$$y_{\mathfrak{p}} = \begin{cases} a^{(\mathfrak{p})} t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} = a^{(\mathfrak{p})} t_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}} & \text{if } \mathfrak{p} \in \operatorname{Min}(R) \setminus (J \cup J_1) \text{ and } \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ 0 & \text{else,} \end{cases} \tag{7.54}$$

Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Min}(R)$ with $\mathfrak{p} \neq \mathfrak{q}$, and suppose that $\nu_{\mathfrak{p}}(\Psi^{-1}(y)) \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$. Then

$$\beta_{\mathfrak{p}} = \nu_{\mathfrak{p}}\left(\Psi^{-1}(y)\right) \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \tag{7.55}$$

by Lemma 3.16, Theorem 6.2.(3) (see Equation (6.6)), and Equation (7.54). In particular, this implies

$$\mathfrak{p} \in \operatorname{Min}(R) \setminus (J \cup J_1) \tag{7.56}$$

(see Equation (7.54)). Moreover, since β is a $\overline{w}$-element of Γ_R, Equation (7.55) yields $\beta_{\mathfrak{q}} \in (\Gamma_R)_{\mathfrak{q}} \setminus (\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}$ by Remark 7.15. Therefore, $\mathfrak{q} \in \operatorname{Min}(R) \setminus (J \cup J_1)$ by Equation (7.51), and hence $\alpha_{\mathfrak{q}} = \beta_{\mathfrak{q}} < (\gamma_{\Gamma_R})_{\mathfrak{q}}$ by Lemma 4.63 and Equation (7.43). This implies $y_{\mathfrak{q}} = a^{(\mathfrak{q})} t_{\mathfrak{q}}^{\beta_{\mathfrak{q}}}$ by Equation (7.54). Since β is a $\overline{w}$-element of Γ_R, we have $\tau_{\mathfrak{p}\mathfrak{q}}(\beta_{\mathfrak{p}}) = \beta_{\mathfrak{q}}$, and since $x \in R$, Theorem 6.2.(4) and Equations (7.52) and (7.53) yield $a^{(\mathfrak{p})} = \zeta_{\mathfrak{p}\mathfrak{q}}(\beta_{\mathfrak{p}})\, a^{(\mathfrak{q})}_{\tau_{\mathfrak{p}\mathfrak{q}}(\beta_{\mathfrak{p}})}$. Thus, $y \in \Psi(R)$ by Theorem 6.2.(4) since R is a fibre product (see Definition 6.4). Hence,

$$y \in (\Psi(R))_{\deg_{\overline{w}}(\beta)} = (\Psi(R))_{\deg_{\overline{w}}(\alpha)} \tag{7.57}$$

by Theorem 6.2.(5), Proposition 7.16, and Equation (7.42).

Let $\mathfrak{p} \in J$. Then by Proposition 7.6 and Theorem 7.19 there is a $\overline{w}$-element $\epsilon \in \Gamma_R$ with $\epsilon_{\mathfrak{p}} = \alpha_{\mathfrak{p}}$. In particular, this implies

$$\deg_{\overline{w}}(\alpha) = \deg_{\overline{w}}(\epsilon) \tag{7.58}$$

by Proposition 7.16 and Equation (7.44). Set

$$J_2 = \{\mathfrak{q} \in J \mid \alpha_{\mathfrak{q}} < \epsilon_{\mathfrak{q}}\}.$$

Then $J_2 \subsetneq J$ if $J \neq \emptyset$ since $\mathfrak{p} \in J$, or $J_2 = \emptyset$ otherwise. Moreover, Proposition 7.16 and Equation (7.44) yield

$$\epsilon_{\mathfrak{q}} = \alpha_{\mathfrak{q}} < (\gamma_{\Gamma_R})_{\mathfrak{q}} \tag{7.59}$$

for all $\mathfrak{q} \in J \setminus J_2$ and

$$\epsilon_{\mathfrak{q}} \geq (\gamma_{\Gamma_R})_{\mathfrak{q}} \tag{7.60}$$

for all $\mathfrak{q} \in J_2$.

By Lemma 7.55 there is for any $\mathfrak{p} \in \mathrm{Min}\,(R)$ with $\epsilon_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$ a $b^{(\mathfrak{p})} \in \mathbb{C} \setminus \{0\}$ such that for the element

$$z \in \prod_{\mathfrak{q} \in \mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_{\mathfrak{q}}^{(\Gamma_R)_{\mathfrak{q}}}\right]\right]$$

defined by

$$z_{\mathfrak{p}} = \begin{cases} b^{(\mathfrak{p})} t^{\epsilon_{\mathfrak{p}}} & \text{if } \epsilon_{\mathfrak{p}} < (\gamma_{\Gamma_R}), \\ 0 & \text{else}, \end{cases} \tag{7.61}$$

we have $z \in (\Psi(R))_{\deg_{\overline{w}}(\epsilon)} = (\Psi(R))_{\deg_{\overline{w}}(\alpha)}$ (see Equation (7.58)).

Let now

$$u \in \prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} \mathbb{C}\left[\left[t_{\mathfrak{p}}^{(\Gamma_R)_{\mathfrak{p}}}\right]\right]$$

with

$$u_{\mathfrak{p}} = \begin{cases} z_{\mathfrak{p}} & \text{if } \mathfrak{p} \in J, \\ 0 & \text{else}, \end{cases} \tag{7.62}$$

for any $\mathfrak{p} \in \mathrm{Min}\,(R)$. Then Equations (7.59), (7.60), and (7.61) yield

$$u_{\mathfrak{p}} = \begin{cases} b^{(\mathfrak{p})} t^{\epsilon_{\mathfrak{p}}} = b^{(\mathfrak{p})} t^{\alpha_{\mathfrak{p}}} & \text{if } \mathfrak{p} \in J \setminus J_2, \\ 0 & \text{else}. \end{cases} \tag{7.63}$$

Let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Min}\,(R)$ with $\mathfrak{p} \neq \mathfrak{q}$, and suppose that $\nu_{\mathfrak{p}}(\Psi^{-1}(u)) \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$. Then

$$\epsilon_{\mathfrak{p}} = \nu_{\mathfrak{p}}(u) \in (\Gamma_R)_{\mathfrak{p}} \setminus (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}} \tag{7.64}$$

by Lemma 3.16, Theorem 6.2.(3) (see Equation (6.6)), and Equation (7.63). In particular, this implies

$$\mathfrak{p} \in J \setminus J_2 \tag{7.65}$$

(see Equation (7.63)). Moreover, Equations (7.51), (7.59), and (7.65) yield $\mathfrak{q} \in J \cup J_1$, and Equations (7.48), (7.59), and (7.65) imply $\mathfrak{q} \notin J_1$ as otherwise $\epsilon_{\mathfrak{p}} = \alpha_{\mathfrak{p}} \in (\Gamma_R)_{\mathfrak{p}}^{\mathfrak{q}}$ in both cases. Thus, we have

$$\mathfrak{q} \in J. \tag{7.66}$$

Since ϵ is a $\overline{w}$-element of S, Remark 7.15 and Equation (7.64) yield $\epsilon_{\mathfrak{q}} \in (\Gamma_R)_{\mathfrak{q}} \setminus (\Gamma_R)_{\mathfrak{q}}^{\mathfrak{p}}$, and hence $\epsilon_{\mathfrak{q}} < (\gamma_{\Gamma_R})_{\mathfrak{q}}$ by Lemma 4.63. Therefore, $\mathfrak{q} \in J \setminus J_2$ by Equations (7.59) and (7.66). Then Equation (7.63) yields $u_{\mathfrak{q}} = b^{(\mathfrak{q})} t_{\mathfrak{q}}^{\epsilon_{\mathfrak{q}}}$. Since ϵ is a $\overline{w}$-element of S, we have $\tau_{\mathfrak{pq}}(\epsilon_{\mathfrak{p}}) = \epsilon_{\mathfrak{q}}$ (see Definition 7.14 and Equation (7.64)), and since $z \in \Psi(R)$, Theorem 6.2.(4) and Equations (7.61) and (7.62) yield $b^{(\mathfrak{p})} = \zeta_{\mathfrak{pq}}(\epsilon_{\mathfrak{p}})\, b^{(\mathfrak{q})}_{\tau_{\mathfrak{pq}}(\epsilon_{\mathfrak{p}})}$. Thus, $u \in \Psi(R)$ since R is a fibre product (see Definition 6.4), and hence

$$u \in (\Psi(R))_{\deg_{\overline{w}}(\epsilon)} = (\Psi(R))_{\deg_{\overline{w}}(\alpha)} \tag{7.67}$$

by Theorem 6.2.(5) and Equation (7.58).

Now Equations (7.57) and (7.67) yield

$$y + u \in (\Psi(R))_{\deg_{\overline{w}}(\alpha)}, \tag{7.68}$$

and by Equations (7.54) and (7.63) we have

$$(y+u)_{\mathfrak{p}} = \begin{cases} y_{\mathfrak{p}} = a^{(\mathfrak{p})}t^{\alpha_{\mathfrak{p}}} & \text{if } \mathfrak{p} \in \operatorname{Min}(R) \setminus (J \cup J_1) \text{ and } \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ u_{\mathfrak{p}} = b^{(\mathfrak{p})}t^{\alpha_{\mathfrak{p}}} & \text{if } \mathfrak{p} \in J \setminus J_2, \\ 0 & \text{else,} \end{cases} \tag{7.69}$$

for any $\mathfrak{p} \in \operatorname{Min}(R)$. Then

$$\delta = \inf\left\{\nu\Big(\Psi^{-1}(y+u)\Big), \gamma_{\Gamma_R}\right\}$$

is by Lemma 7.41 and Equation (7.68) a $\overline{w}$-element of Γ_R with $\delta \in \Gamma_R$. Moreover, for any $\mathfrak{p} \in \operatorname{Min}(R)$ Equations (7.59) and (7.69) yield

$$\delta_{\mathfrak{p}} = \begin{cases} \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}} & \text{if } \mathfrak{p} \in \operatorname{Min}(R) \setminus (J_1 \cup J_2) \text{ with } \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}, \\ (\gamma_{\Gamma_R})_{\mathfrak{p}} & \text{else,} \end{cases}$$

since $\nu\big(\Psi^{-1}(y+u)\big) = \operatorname{ord}_t(y+u)$ by Theorem 6.2.(4). As $\alpha \leq \gamma_{\Gamma_R}$ with $\alpha_{J_1} = (\gamma_{\Gamma_R})_{J_1}$ by Equation (7.45), this implies

$$\begin{aligned} \delta_{\operatorname{Min}(R)\setminus J_2} &= \alpha_{\operatorname{Min}(R)\setminus J_2}, \\ \delta_{J_2} &= (\gamma_{\Gamma_R})_{J_2}. \end{aligned}$$

In particular, if we set

$$J' = \left\{\mathfrak{p} \in J_2 \;\middle|\; \alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}\right\} \subset J_2$$

then

$$\begin{aligned} \delta_{\operatorname{Min}(R)\setminus J'} &= \alpha_{\operatorname{Min}(R)\setminus J'}, \\ \delta_{\mathfrak{q}} &= (\gamma_{\Gamma_R})_{\mathfrak{q}} > \alpha_{\mathfrak{q}} \text{ for all } \mathfrak{q} \in J' \end{aligned}$$

□

Lemma 7.57. *Let R be a quasihomogeneous curve with normal weights $\overline{w} \in \mathbb{N}^{\operatorname{Min}(R)}$ (see Definition 6.3), and let $\alpha \in \prod_{\mathfrak{p}\in\operatorname{Min}(R)} (\Gamma_R)_{\mathfrak{p}}$ be a $\overline{w}$-element of Γ_R. If R is a fibre product (see Definition 6.4), then $\alpha \in \Gamma_R$.*

Proof. If $\alpha \in C_{\Gamma_R}$, the statement is trivial. So suppose that $\alpha \in \prod_{\mathfrak{p}\in\operatorname{Min}(R)} (\Gamma_R)_{\mathfrak{p}} \setminus C_{\Gamma_R}$. Then there is a $\mathfrak{p} \in \operatorname{Min}(R)$ such that $\alpha_{\mathfrak{p}} < (\gamma_{\Gamma_R})_{\mathfrak{p}}$. Since $\alpha_{\mathfrak{p}} \in$
$br\Gamma_{R\mathfrak{p}}$, and since R is quasihomogeneous, there is by Proposition 7.6 and Theorem 7.19 a $\overline{w}$-element β of S with $\beta \in \Gamma_R$ and $\alpha_{\mathfrak{p}} = \beta_{\mathfrak{p}}$. In particular, this implies

$$\deg_{\overline{w}}(\alpha) = \deg_{\overline{w}}(\beta) \tag{7.70}$$

(see Proposition 7.16 and Definition 7.17). Inductively applying Lemma 7.56 yields a chain of subsets $\ldots \subset J_1 \subsetneq J \subset \operatorname{Min}(R)$ such that for any $i \geq 1$ we have $J_i = \emptyset$ or $J_{i+1} \subsetneq J_i$, and there is a $\overline{w}$-element $\beta^{(i)}$ of Γ_R with

$$\left(\beta^{(i)}\right)_{\operatorname{Min}(R)\setminus J_i} = \alpha_{\operatorname{Min}(R)\setminus J_i}.$$

Since $\operatorname{Min}(R)$ is finite by Corollary A.46, we eventually obtain that $J_n = \emptyset$ for some n, and hence $\alpha = \beta^{(n)} \in \Gamma_R$. □

Proof of Theorem 7.23. Let R be a quasihomogeneous curve with normal weights $\overline{w}$ and connecting maps $\zeta = \left((\zeta_{\mathfrak{p}\mathfrak{q}})_{\mathfrak{q} \in \mathrm{Min}\,(R) \setminus \{\mathfrak{p}\}} \right)_{\mathfrak{p} \in \mathrm{Min}\,(R)}$ (see Definition 6.3). Then Γ_R is quasihomogeneous of type $\overline{w}$ by Proposition 7.6.

Let R be a fibre product (see Definition 6.4), and let $\alpha \in \prod_{\mathfrak{p} \in \mathrm{Min}\,(R)} (\Gamma_R)_{\mathfrak{p}}$ be a $\overline{w}$-element of Γ_R. Then $\alpha \in \Gamma_R$ by Lemma 7.57, and hence Γ_R is $\overline{w}$-closed.

Suppose now that Γ_R is $\overline{w}$-closed. Set

$$A = \mathrm{Fib}\,(\Gamma_R, \overline{w}, \zeta).$$

Then A is a quasihomogeneous curve with normal weights $\overline{w}$ by Proposition 6.6.(1) and (2), and it is a fibre product by Proposition 6.6.(3). Therefore, Γ_A is $\overline{w}$-closed. Since $\Psi^{-1}(A) \in \mathcal{R}_R$ by Proposition 6.6.(4), Propositions 6.6.(4) and 7.22 yield

$$\Gamma_R = \Gamma_{\Psi^{-1}(A)}.$$

Thus, $R = A$ is a fibre product by Corollary 4.52 since $R \subset \Psi^{-1}(A)$ by Theorem 6.2.(4).

The particular claim follows with Proposition 6.6. $\square$

Proof of Theorem 7.24. Let S be a quasihomogeneous semigroup of type $w \in \mathbb{N}^s$. Then $\mathrm{Fib}\,(S, w)$ is a quasihomogeneous curve with normal weights w by Proposition 7.12. Since $\mathrm{Fib}\,(S, w)$ is also a fibre product by Proposition 7.12, $\Gamma_{\mathrm{Fib}\,(S,w)}$ is w-closed by Theorem 7.23. Since $S \subset \Gamma_{\mathrm{Fib}\,(S,w)}$ with $S_i = \left(\Gamma_{\mathrm{Fib}\,(S,w)}\right)_i$ for any $i \in I$ and $S_i^j = \left(\Gamma_{\mathrm{Fib}\,(S,w)}\right)_i^j$ for every $j \in I \setminus \{i\}$ by Lemma 7.54, Proposition 7.22 yields $S^w = \Gamma_{\mathrm{Fib}\,(S,w)}$. $\square$

8. Normalization of Arrangements

Endomorphism rings occur in the construction of blow ups [15] or non-commutative resolutions [16, 17]. A non-commutative crepant resolution of a curve can be computed [18] considering the intermediate steps of a normalization algorithm [19] which is based on a characterization of normality in terms of the endomorphism ring of a so-called *test ideal* [20]: a reduced Noetherian ring R is normal if and only if $R = \mathfrak{i} : \mathfrak{i}$ for a test ideal $\mathfrak{i}$ of R. If R is a reduced one-dimensional Noetherian semilocal ring, then the Jacobson radical $\mathfrak{j}_R$ is a test ideal for R (see Definition B.47); if R is local, then the maximal ideal $\mathfrak{m}_R$ is the unique test ideal for R (see Remark B.49).

The above criterion by Grauert and Remmert yields the following algorithm for normalization (see Proposition B.57). Let R be a reduced Noetherian ring. Then for any test ideal $\mathfrak{i}$ of R there is a sequence of integral extensions

$$R = R^{(0)} \subsetneq R^{(1)} \subsetneq \ldots \subset \overline{R},$$

where for any $i \geq 0$ we set

$$R^{(i+1)} = \mathfrak{i}^{(i)} : \mathfrak{i}^{(i)}$$

and

$$\mathfrak{i}^{(i+1)} = \sqrt{\mathfrak{i}^{(i)} R^{(i+1)}}$$

with $\mathfrak{i}^{(0)} = \mathfrak{i}$. If $\overline{R}$ is finite over R, then $R^{(i)}$ is finite over R for every $i \in \mathbb{N}$, and there is an $n \in \mathbb{N}$ such that $R^{(i)} = R^{(n)} = \overline{R}$ for any $i \geq n$. Examples for classes of rings with finite normalization are admissible rings (see Definition 3.18.(4) and Corollary C.15) or reduced excellent rings (see Theorem B.36.(2)). If R is an admissible ring, then $R^{(i)}$ is an admissible ring for every $i \in \mathbb{N}$ by Theorem 3.45.(1), and if R is a reduced excellent ring, then $R^{(i)}$ is a reduced excellent ring for every $i \in \mathbb{N}$ by Lemma A.27 (since $R^{(i)} \subset Q_R$) and Theorem B.34.

In this chapter we apply the Grauert–Remmert algorithm to two kinds of arrangements. Following an idea by Böhm, Decker, and Schulze [21] we use the semigroup of values to determine the intermediate steps explicitly (also see [35]). We start in Section 8.1 with a plane arrangement of smooth curves which pairwise intersect only transversally and only in finitely many points. Then we can determine the number n of steps needed in the Grauert–Remmert algorithm to obtain the normalization in terms of the number of analytic branches in the singular points of the arrangement (see Theorem 8.1). For this, we investigate the arrangement locally, that is, we consider the completion of the local rings at all points (in fact, we only need to consider the singular points). Then we deal with algebroid curves, and as in [21, 35] the semigroup of values helps to compute explicitly the intermediate steps in the Grauert–Remmert algorithm (see Theorem 8.2).

Using Serre's criterion (see Section B.5.1) which allows for checking normality in codimension one, we apply this result in Section 8.2 to hyperplane arrangements. In fact, the Grauert–Remmert algorithm is compatible with localization (see Proposition B.58). Geometrically, after localization in codimension one we look at "transversal slices" of the arrangement. This reduces the problem to plane line arrangements whose cardinalities are the numbers of hyperplanes intersecting the respective slices. Then the number of steps needed to compute the normalization of the hyperplane arrangement equals the maximum over the number of steps needed in each slice. This number can be deduced from the combinatorics of the arrangement (see Theorem 8.14).

8.1. Plane Arrangements of Smooth Curves

Theorem 8.1. *Let C be a reduced plane curve over a field k, and suppose that the analytic branches at the singular points of C are regular and intersect transversally. For a singular point p of C we denote by n_p the number of analytic branches at p. If $|k| \geq \max\{n_p \mid p \in \operatorname{Sing}(C)\}$, then for any $n \in \mathbb{N}$ we have*

$$(\mathcal{O}_C)^{(n)} = \overline{\mathcal{O}_C}$$

if and only if $n \geq \max\{n_p \mid p \in \operatorname{Sing}(C)\} - 1$.

For the proof of Theorem 8.1 we consider the curve locally at the singular points. Then $\mathcal{O}_{C,p}$ is a local reduced excellent ring by Lemma A.27 and Theorem B.34. Since completion factors through localization, taking the completion with respect to the maximal ideal corresponding to p we obtain

$$\widehat{\mathcal{O}_C} = \widehat{\mathcal{O}_{C,p}} \cong k[[X,Y]]/\prod_{i \in I_p} f_i, \tag{8.1}$$

where I_p is the set of branches of C meeting in p, and

$$f_i = a_i X + b_i Y + \text{terms of higher degree}$$

with $(a_i, b_i) \neq (0,0)$ for any $i \in I_p$ (since the analytic branches are smooth) and $(a_i, b_i) \neq (a_j, b_j)$ for all $i, j \in I_p$ with $i \neq j$ (since the branches intersect transversally).

After a coordinate change we may assume that $a_i \neq 0$. Then replacing f_i by $\frac{1}{a_i} f_i$ we may assume that

$$f_i = X + b_i Y_i + \text{terms of higher order}$$

for all $i \in I_p$. Then locally we can describe the normalization process in more detail.

Theorem 8.2. *Let I be a finite set, let k be a field with $|k| \geq |I|$, and let*

$$R = k[[X,Y]]/\left\langle \prod_{i \in I} f_i \right\rangle,$$

where $f_i \in k[[X,Y]]$ is of the form

$$f_i = X + c_i Y + \textit{terms of higher order}$$

for any $i \in I$ with $c_i \neq c_j$ for all $i, j \in I$ with $i \neq j$. Then for $n \in \mathbb{N}$ we have

$$R^{(n)} = \overline{R}$$

if and only if $n \geq |I| - 1$. Recall that the unique test ideal for R is its maximal ideal (see Remark B.49), and hence the test ideal of any ring $R^{(n)}$ is its Jacobson radical (see Theorem A.12).

Moreover, for $n < |I| - 1$ we have

$$R^{(n)} = R^{(n-1)} + \sum_{j=1}^{n} k \cdot z_j^{(n)} = k\Big[\Big[x, y, z_1^{(n)}, \ldots, z_n^{(n)}\Big]\Big],$$

where

$$\begin{aligned} x &= -ct + \text{terms of higher order}, \\ y &= t, \\ z_j^{(n)} &= c^{|I|-j} t^{|I|-n-1} \text{ for every } j = 1, \ldots, n, \end{aligned}$$

with $t = (t_i)_{i \in I}$, $c = (c_i)_{i \in I}$, and $c^k = \left(c_i^k\right)_{i \in I}$ for $k \in \mathbb{N}$. In particular, we have

$$R^{(n)} = k\Big[\Big[X, Y, Z_1^{(n)}, \ldots, Z_n^{(n)}\Big]\Big] / \mathfrak{i}^{(n)}$$

with

$$\mathfrak{i}^{(n)} = \bigcap_{i \in I} \Big\langle f_i, Z_j^{(n)} - c_i^{|I|-j} Y^{|I|-n-1} \;\Big|\; j = 1, \ldots, n \Big\rangle.$$

For any $n \in \mathbb{N}$ the semigroup of values of $R^{(n)}$ is

$$\begin{aligned} \Gamma_{R^{(n)}} &= \langle \mathbf{1} + \mathbb{N}\mathbf{e}_k \mid k \in I \rangle \cup \Big((|I| - 1 - n)_{i \in I} + \mathbb{N}^I \Big) \\ &= \left\{ \mathbf{k} + \sum_{l=1}^{k} \mathbb{N}\mathbf{e}_{i_l^{(k)}} \;\middle|\; i_j^{(k)} \in I, 0 \leq k \leq |I| \right\} \cup \Big((|I| - 1 - n)_{i \in I} + \mathbb{N}^I \Big) \end{aligned}$$

with conductor

$$\gamma_{\Gamma_{R^{(n)}}} = (|I| - 1 - n)_{i \in I}.$$

With Theorem 8.2 we can prove Theorem 8.1.

Proof of Theorem 8.1. There is a $g \in k[X, Y]$ such that $\mathcal{O}_C = k[X, Y]/\langle g \rangle$. Thus, $\mathcal{O}_C$ is excellent by Theorems B.34 and B.36 since it is a finitely generated algebra over a field. Moreover, $\mathcal{O}_C$ is reduced by assumption.

Since k is Cohen–Macaulay (see Remark C.3), also $k[X, Y]$ is Cohen–Macaulay by Corollary C.9. Thus, $\mathcal{O}_C$ is Cohen–Macaulay by Proposition C.10, and therefore it satisfies Serre's condition (S_2) by Corollary C.5.

Let $n \in \mathbb{N}$. Then Lemma B.61.(4) implies that $(\mathcal{O}_C)^{(n)} = \overline{\mathcal{O}_C}$ if and only if

$$\Big((\mathcal{O}_C)_{\mathfrak{p}} \Big)^{(n)} = \overline{(\mathcal{O}_C)_{\mathfrak{p}}}$$

for all $\mathfrak{p} \in \operatorname{Sing}(\mathcal{O}_C)$ with height $\mathfrak{p} = 1$. Recall that by assumption $\operatorname{Sing}(\mathcal{O}_C)$ is finite, and height $\mathfrak{p} = 1$ for all $\mathfrak{p} \in \operatorname{Sing}(\mathcal{O}_C)$. So let $\mathfrak{p} \in \operatorname{Sing}(\mathcal{O}_C)$. Then $(\mathcal{O}_C)_\mathfrak{p}$ is a reduced excellent ring by Lemma A.27 and Theorem B.34. Inductively applying Proposition B.54 implies that $\left((\mathcal{O}_C)_\mathfrak{p}\right)^{(n)} = \overline{(\mathcal{O}_C)_\mathfrak{p}}$ if and only if

$$\left(\widehat{(\mathcal{O}_C)_\mathfrak{p}}\right)^{(n)} = \overline{\widehat{(\mathcal{O}_C)_\mathfrak{p}}}.$$

With the considerations before Equation (8.1) we may assume that $\widehat{(\mathcal{O}_C)_\mathfrak{p}}$ is of the form

$$\widehat{\mathcal{O}_C} = \widehat{\mathcal{O}_{C,p}} \cong k[[X,Y]]/\prod_{i\in I_p} f_i,$$

where I_p is the set of analytic branches of C meeting in the point corresponding to $\mathfrak{p}$, and

$$f_i = X + b_i Y + \text{terms of higher order}$$

for any $i \in I$ with $c_i \neq c_j$ for all $i,j \in I_p$ with $i \neq j$. Then Theorem 8.2 yields the claim. □

For the proof of Theorem 8.2 we need a few preliminary results. For the rest of this section let I be a finite set, let k be a field with $|k| \geq |I|$, and let

$$R = k[[X,Y]]/\left\langle \prod_{i\in I} f_i \right\rangle,$$

where $f_i \in k[[X,Y]]$ is of the form

$$f_i = X + c_i Y + \text{terms of higher order}$$

for any $i \in I$ with $c_i \neq c_j$ for all $i,j \in I$ with $i \neq j$. In the following we identify R with its image in

$$\overline{R} = \prod_{i\in I} k[[t_i]]$$

(see Theorem 3.44), i.e. we write

$$R = k[[x,y]]$$

with

$$x = -ct + \text{ terms of higher order,} \tag{8.2}$$

$$y = t \tag{8.3}$$

(see [32, page 299]).

Remark 8.3. The ring R is an algebroid curve, and hence admissible by Proposition 3.41. Then for every $n \in \mathbb{N}$ also the ring $R^{(n)}$ is admissible by Theorem 3.45.(1) and Proposition B.57.

Lemma 8.4. *The family* $(c^j)_{j=0}^{|I|-1}$ *is linearly independent, where* $c^j = \left(c_i^j\right)_{i\in I}$ *for any* $j \in \mathbb{N}$.

Proof. Assume that $(c^j)_{j=0}^{|I|-1}$ is linearly dependent. Then there is a non-zero family $(a_j)_{j=1}^{|I|-1} \in \prod_{j=1}^{|I|-1} k$ such that $\sum_{j=0}^{|I|-1} a_j c_i^j = 0$ for all $i \in I$, i.e. the coefficients c_i, $i \in I$, are roots of the polynomial $g = \sum_{j=0}^{|I|-1} a_j X^j \in k[X]$. Since g can have at most $\deg g$ different roots, and since $\deg g \leq |I| - 1$, this is a contradiction to $c_i \neq c_j$ for all $i, j \in I$ with $i \neq j$. □

Lemma 8.5. *The value semigroup of* R *is*

$$\begin{aligned}\Gamma_R &= \langle \mathbf{1} + \mathbb{N}\mathbf{e}_i \mid i \in I\rangle \\ &= \left\{ \mathbf{k} + \sum_{j=1}^{k} n_{i_j^{(k)}} \mathbf{e}_{i_j^{(k)}} \;\middle|\; i_j^{(k)} \in I \text{ and } k, n_{i_j^{(k)}} \in \mathbb{N} \right\}\end{aligned}$$

with $\gamma_{\Gamma_R} = (|I| - 1)_{i\in I}$ *and* $\mu_{\Gamma_A} = \mathbf{1}$.

Proof. Set

$$\Gamma = \langle \mathbf{1} + \mathbb{N}\mathbf{e}_i \mid i \in I\rangle$$

and

$$\Gamma' = \left\{ \mathbf{k} + \sum_{j=1}^{k} n_{i_j^{(k)}} \mathbf{e}_{i_j^{(k)}} \;\middle|\; i_j^{(k)} \in I \text{ and } k, n_{i_j^{(k)}} \in \mathbb{N} \right\}. \tag{8.4}$$

Let $\alpha \in \Gamma'$. Then there is a $k \in \mathbb{N}$, and for $j = 1, \ldots, k$ there are $i_j^{(k)} \in I$ and $n_{i_j^{(k)}} \in \mathbb{N}$ such that

$$\alpha = \mathbf{k} + \sum_{j=1}^{k} n_{i_j^{(k)}} \mathbf{e}_{i_j^{(k)}} = \sum_{j=1}^{k} \left((1)_{i\in I} + n_{i_j^{(k)}} \mathbf{e}_{i_j^{(k)}} \right) \in \Gamma.$$

Now let $\beta \in \Gamma$. Then there is a $k \in \mathbb{N}$, and for $j = 1, \ldots, k$ there are $i_j \in I$ and $n_{i_j} \in \mathbb{N}$ such that

$$\beta = \sum_{j=1}^{k} \left(\mathbf{1} + n_{i_j} \mathbf{e}_{i_j} \right) = \mathbf{k} + \sum_{j=1}^{k} n_{i_j} \mathbf{e}_{i_j} \in \Gamma'.$$

Thus, we have indeed

$$\Gamma = \Gamma'. \tag{8.5}$$

Now we want to show that $\Gamma_R = \Gamma$. For any $k \in \mathbb{N}$ and for all $i \in I$ we have

$$k\mathbf{e}_i + \sum_{j\in I\setminus\{i\}} \mathbf{e}_j = \nu\left(x - c_i y + y^k\right) \in \Gamma_A.$$

Thus, $\Gamma \subset \Gamma_R$.

Let $\alpha \in (|I| - 1)_{i\in i} + \mathbb{N}^I$. If there is an $i \in I$ such that $\alpha_i = |I| - 1$, then

$$\alpha \in \sum_{j\in I\setminus\{i\}} (\mathbf{1} + \mathbb{N}\mathbf{e}_j) \subset \Gamma.$$

If $\alpha_i > |I| - 1$ for all $i \in I$, then

$$\alpha \in \sum_{j \in I} (\mathbf{1} + \mathbb{N}\mathbf{e}_j) \subset \Gamma.$$

This implies $(|I|-1)_{i \in I} + \mathbb{N}^I \subset \Gamma$, and hence $\gamma_\Gamma \leq (|I|-1)_{i \in I}$. Moreover, for any $i \in I$ we have

$$(|I|-2)_{k \in I} + \sum_{j \in I \setminus \{i\}} \mathbf{e}_i = \sum_{j \in I \setminus \{i,l\}} (\mathbf{1} + \mathbf{e}_j) + \mathbf{e}_l \notin \Gamma,$$

where $l \in I \setminus \{i\}$. This implies $\gamma_\Gamma \geq (|I|-2)_{k \in I} + \sum_{j \in I \setminus \{i\}} \mathbf{e}_i$ for all $i \in I$, and hence

$$\gamma_\Gamma = (|I|-1)_{k \in I}. \tag{8.6}$$

Let now $z \in R^{\mathrm{reg}}$, and suppose that $\nu(z) \notin \Gamma$. Then $\nu(z) \not\geq \gamma_\Gamma$. Therefore,

$$d = \min \{\mathrm{ord}_{t_i}(z) \mid i \in I\} < |I| - 1 \tag{8.7}$$

by Equation (8.6). Since $z \in R$, for all $m, n \in \mathbb{N}$ there are $a_{m,n} \in k$ such that

$$z = \sum_{m,n \in \mathbb{N}} a_{m,n} x^m y^n.$$

As $\sum_{k=0}^d a_{k,d-k} c^k \neq 0$ by Equation (8.7) and Lemma 8.4, Equations (8.2) and (8.3) yield

$$\begin{aligned} z &= \sum_{m,n \in \mathbb{N}} a_{m,n} \Big(c^m t^{m+n} + \text{terms of higher order} \Big) \\ &= \sum_{k=0}^{d} a_{k,d-k} c^k t^d + \text{terms of higher order}. \end{aligned} \tag{8.8}$$

So if $\nu(z) \notin \Gamma$, then Equations (8.4), (8.5), and (8.7) imply that there is a $J \subset I$ with $|J| > d$ such that $\nu_i(z) > d$ for all $i \in J$. Therefore, we have by Equation (8.8)

$$\sum_{k=0}^{d} a_{k,d-k} c_i^k t_i^d = 0$$

for all $i \in J$, i.e. the coefficients c_i, $i \in J$, are roots of the polynomial

$$g = \sum_{k=0}^{d} a_{k,d-k} X^d.$$

But this yields a contradiction as g has only $\deg g \leq d$ roots but $|J| > d$ and the coefficients c_i, $i \in I$, are pairwise different. Thus, we obtain $\Gamma_R = \Gamma$. □

Proof of Theorem 8.2. We proof the statement by induction on n. Note that with R also $R^{(n)}$ is an algebroid curve by Theorem 3.45.(2) since $R^{(n)}$ is an integral extension of R in Q_R by Proposition B.57. For $n = 0$ the statement is true since $R^{(0)} = R = k[x, y]$ with

$$\begin{aligned} x &= ct + \text{terms of higher order}, \\ y &= t \end{aligned}$$

(see Equations (8.2) and (8.3)), and the semigroup of values of R is

$$\Gamma_R = \langle \mathbf{1} + \mathbb{N}\mathbf{e}_i \mid i \in I \rangle$$

by Lemma 8.5.

Now let $0 < n < |I| - 1$, and suppose the statement is true for $n-1$. Then $R^{(n-1)}$ is local by Theorem 4.9, and we denote the maximal ideal by $\mathfrak{m}_{R^{(n-1)}}$. By Remark 4.8 and Lemma 4.58 the conductor of $\Gamma_{R^{(n)}} = \Gamma_{\mathfrak{m}_{R^{(n-1)}} : \mathfrak{m}_{R^{(n-1)}}}$ is

$$\begin{aligned}
\gamma_{\Gamma_{R^{(n)}}} &= \gamma_{\Gamma_{\mathfrak{m}_{R^{(n-1)}} : \mathfrak{m}_{R^{(n-1)}}}} \\
&= \gamma_{\Gamma_{\mathfrak{m}_{R^{(n-1)}}}} - \mu_{\Gamma_{\mathfrak{m}_{R^{(n-1)}}}} \\
&= \gamma_{M_{\Gamma_{R^{(n-1)}}}} - \mu_{M_{\Gamma_{R^{(n-1)}}}} \\
&= \gamma_{\Gamma_{R^{(n-1)}}} - \mu_{M_{\Gamma_{R^{(n-1)}}}} \\
&= (|I| - n)_{i \in I} - \mathbf{1} \\
&= (|I| - 1 - n)_{i \in I}.
\end{aligned}$$

Set

$$\begin{aligned}
\Gamma &= \Gamma_{\mathfrak{m}_{R^{(n-1)}}} - \Gamma_{\mathfrak{m}_{R^{(n-1)}}} \\
&= M_{\Gamma_{R^{(n-1)}}} - M_{\Gamma_{R^{(n-1)}}} \\
&= \left\{ \alpha \in D_{\Gamma_{R^{(n-1)}}} \;\middle|\; \alpha + M_{\Gamma_{R^{(n-1)}}} \subset M_{\Gamma_{R^{(n-1)}}} \right\}
\end{aligned}$$

(see Remark 4.8) and

$$\Gamma' = \langle \mathbf{1} + \mathbb{N}\mathbf{e}_k \mid k \in I \rangle \cup \Big((|I| - 1 - n)_{i \in I} + \mathbb{N}^I \Big). \tag{8.9}$$

Then

$$\Gamma' = \left\{ \mathbf{k} + \sum_{l=1}^{k} \mathbb{N}\mathbf{e}_{i_l^{(k)}} \;\middle|\; i_j^{(k)} \in I, 0 \le k \le |I| \right\} \cup \Big((|I| - 1 - n)_{i \in I} + \mathbb{N}^I \Big).$$

(see Lemma 8.5). Moreover, we obviously have $\Gamma' \subset \Gamma$, and Lemma 3.23.(1) yields

$$\Gamma_{R^{(n-1)}} \subset \Gamma.$$

Now let $\alpha \in \mathbb{N}^I \setminus \Gamma'$. Then $\alpha \not\geq \gamma_{\Gamma'}$. Thus, there is a $k \in \mathbb{N}$ and a $J \subsetneq I$ with $|J| > k$ such that

$$\alpha = \mathbf{k} + \sum_{j \in J} n_j \mathbf{e}_j$$

for some $n_j \in \mathbb{N}$, $j \in J$. Let $l \in I \setminus J$, and set

$$\beta = \mathbf{1} + \mathbf{e}_l.$$

Then $\beta \in M_{\Gamma_{R^{(n-1)}}}$, and

$$\alpha + \beta = \mathbf{k} + \mathbf{1} + \sum_{j \in J \cup \{l\}} n_j \mathbf{e}_j$$

with $n_l = 1$. Since

$$\mathbf{k} + \mathbf{1} < \gamma_{\Gamma'} + \mathbf{1} = (|I| - n)_{i \in I} = \gamma_{\Gamma_{R^{(n-1)}}},$$

and since $|J \cup \{l\}| = |J| + 1 > k + 1$, we have $\alpha + \beta \notin M_{\Gamma_{R^{(n-1)}}}$, and therefore $\alpha \notin \Gamma$. This implies

$$\Gamma = \Gamma'. \tag{8.10}$$

Set

$$\widetilde{R} = R^{(n-1)} + \sum_{j=1}^{n} k z_j^{(n)}. \tag{8.11}$$

Since for any $j = 1, \ldots, n$ we have

$$\begin{aligned} z_j^{(n)} x &= \left(c^{|I|-j+1} t^{|I|-n} + \text{terms of higher order}\right) \in \mathfrak{C}_{R^{(n-1)}}, \\ z_j^{(n)} y &= c^{|I|-j} t^{|I|-n} \in \mathfrak{C}_{R^{(n-1)}}, \\ z_j^{(n)} z_{j'}^{(n-1)} &= c^{2|I|-j-j'} t^{2(|I|-n-1)} \in \mathfrak{C}_{R^{(n-1)}} \text{ for all } j' = 1, \ldots, n, \end{aligned}$$

and

$$z_j^{(n-1)} = y z_j^{(n)}$$

for all $j = 1, \ldots, n-1$, it follows that

$$\widetilde{R} = k\left[\left[x, y, z_1^{(n)}, \ldots, z_n^{(n)}\right]\right],$$

is a regular $R^{(n-1)}$-submodule of $\overline{R^{(n-1)}}$, and hence $\widetilde{R} \in \mathcal{R}_{R^{(n-1)}}$. Moreover, we have $\widetilde{R}\mathfrak{m}_{R^{(n-1)}} \subset \mathfrak{m}_{R^{(n-1)}}$ as

$$z_j^{(n)} z_k^{(n-1)} = c^{2|I|-j-k} t^{2(|I|-n)-1} \in \mathfrak{C}_{R^{(n-1)}}$$

for all $k = 1, \ldots, n-1$ and

$$\mathfrak{m}_{R^{(n-1)}} = \left\langle x, y, z_k^{(n-1)} \;\middle|\; k = 1, \ldots, n-1 \right\rangle.$$

Now we want to show that $\Gamma_{\widetilde{R}} = \Gamma$. On the one hand, we have $\Gamma_{\widetilde{R}} \subset \Gamma$ since for all $j = 1, \ldots, n$

$$\begin{aligned} \nu\left(z_j^{n)}\right) \geq \gamma_{\Gamma_{A^{(n-1)}}} - \mathbf{1} &= (|I| - n)_{i \in I} - \mathbf{1} \\ &= (|I| - n - 1)_{i \in I} \\ &= \gamma_{\Gamma} \end{aligned}$$

by Equations (8.9) and (8.10).

On the other hand, note that $x^k y^{(|I|-1-n-k)} \in R^{(n-1)}$ for all $k = 0, \ldots, |I| - 1 - n$, and we have
$$x^k y^{|I|-1-n-k} = -c^k t^{|I|-1-n} + \mathcal{O}\left(t^{\gamma_{R^{(n-1)}}}\right).$$
Since also $c^{|I|-j} t^{|I|-1-n} = z_j^{(n)} \in \widetilde{R}$ for $j = 1, \ldots, n$, we have
$$c^k t^{|I|-1-n} \in \widetilde{R}$$
for all $k = 0, \ldots, |I| - 1$. Then for any $i \in I$ and for any $k = 0, \ldots, |I| - 1$ there is an $a_k^{(i)} \in k$ such that
$$t_i^{|I|-1-n} \mathbf{e}_i = \sum_{k=0}^{|I|-1} a_k^{(i)} c^k t^{|I|-1-n} \in \widetilde{R}$$
as the family $\left(c^k\right)_{k=0}^{|I|-1}$ is by Lemma 8.4 linearly independent. This implies that $\Gamma = \Gamma' \subset \Gamma_{\widetilde{R}}$, and hence
$$\Gamma = \Gamma_{\widetilde{R}}.$$

So we have $\widetilde{R} \in \mathcal{R}_{A^{(n-1)}}$ with $\widetilde{R}\mathfrak{m}_{R^{(n-1)}} \subset \mathfrak{m}_{R^{(n-1)}}$ and $\Gamma_{\widetilde{R}} = \Gamma_{\mathfrak{m}_{R^{(n-1)}}} - \Gamma_{\mathfrak{m}_{R^{(n-1)}}}$. Thus, Lemma 4.53 yields
$$\widetilde{R} = \mathfrak{m}_{R^{(n-1)}} : \mathfrak{m}_{R^{(n-1)}} = R^{(n)}.$$

Moreover, we can compute $\mathfrak{i}^{(n)} = \ker \Phi$, where
$$\begin{aligned} \Phi\colon k\left[\left[X, Y, Z_1^{(n)}, \ldots, Z_n^{(n)}\right]\right] &\to k\left[\left[x, y, z_i^{(n)}, \ldots, z_n^{(n)}\right]\right], \\ X &\mapsto x, \\ Y &\mapsto y, \\ Z_j^{(n)} &\mapsto z_j^{(n)} \text{ for any } j = 1, \ldots, n. \end{aligned}$$

Let $n = |I| - 1$. Then Lemma 4.58 yields
$$\begin{aligned} \gamma_{\Gamma_{R^{(|I|-1)}}} &= \gamma_{\Gamma_{\mathfrak{m}_{R^{(|I|-2)}}:\mathfrak{m}_{R^{(|I|-2)}}}} \\ &= \gamma_{\Gamma_{\mathfrak{m}_{R^{(|I|-2)}}}} - \mu_{\Gamma_{\mathfrak{m}_{R^{(|I|-2)}}}} \\ &= \gamma_{M_{\Gamma_{R^{(|I|-2)}}}} - \mu_{M_{\Gamma_{R^{(|I|-2)}}}} \\ &= \gamma_{\Gamma_{R^{(|I|-2)}}} - \mu_{M_{\Gamma_{R^{(|I|-2)}}}} \\ &= (|I| - |I| - 1)_{i \in I} - \mathbf{1} \\ &= \mathbf{0}. \end{aligned}$$
Thus, we have $\gamma_{\Gamma_{R^{(|I|-1)}}} = \mu_{\Gamma_{R^{(|I|-1)}}}$. Then we obtain with Lemma 4.15 and Proposition 4.56 (also see Lemma A.34)
$$\mathfrak{Q}^{\mu_{\Gamma_{R^{(|I|-1)}}}} = \mathfrak{Q}^{\gamma_{\Gamma_{R^{(|I|-1)}}}} = \mathfrak{C}_{R^{(|I|-1)}} \subset R^{(|I|-1)} \subset \mathfrak{Q}^{\mu_{\Gamma_{R^{(|I|-1)}}}},$$
and hence $R^{(|I|-1)} = \mathfrak{C}_{R^{(|I|-1)}}$. This implies $R^{(|I|-1)} = \overline{R^{(|I|-1)}}$. $\square$

8.2. Hyperplane Arrangements

In this section we apply the results of Section 8.1 to determine the number of steps needed to compute the normalization of a hyperplane arrangement using the Grauert–Remmert algorithm of Section B.5.2.

Definition 8.6. Let k be a field, and let V be a k-vector space of dimension n. A *hyperplane* in V is an affine subspace H of V of dimension $n-1$. A *hyperplane arrangement* $(\mathcal{A}, V)$ is given by a finite set $\mathcal{A}$ of hyperplanes in V. A *subarrangement* of $(\mathcal{A}, V)$ is a hyperplane arrangement $(\mathcal{B}, V)$ with $\mathcal{B} \subset \mathcal{A}$.

Remark 8.7. Let k be a field, and let $(\mathcal{A}, k^n)$ be an arrangement of hyperplanes. We describe the arrangement $(\mathcal{A}, k^n)$ by its *defining polynomial*

$$\mathcal{Q}_{\mathcal{A}} = \prod_{H \in \mathcal{A}} f_H,$$

where each factor

$$f_H = \sum_{j=1}^{n} a_j^{(H)} X_j \in k[X_1, \ldots, X_n] \tag{8.12}$$

defines a hyperplane $H \in \mathcal{A}$ (see [37, page 11]). We assume that the hyperplanes in $\mathcal{A}$ are pairwise different, i.e. the family $\left(\left(a_j^{(H)}\right)_{j=1,\ldots,n}\right)_{H \in \mathcal{A}}$ is linearly independent. Then the ring

$$R_{\mathcal{A}} = k[X_1, \ldots, X_n] / \langle \mathcal{Q}_{\mathcal{A}} \rangle$$

describing $(\mathcal{A}, V)$ is reduced. Moreover, since k is excellent by Theorem B.35, and since $R_{\mathcal{A}}$ is a finitely generated k-algebra, it is excellent by Theorem B.34 (see Definition B.33).

In Theorem 8.14 we determine the number of steps needed in the Grauert–Remmert algorithm to compute the normalization of a hyperplane arrangement. We want to deduce this number from the combinatorics of the arrangement.

Definition 8.8. Let k be a field, let V be a k-vector space, and let $(\mathcal{A}, V)$ be an arrangement of hyperplanes. We denote by $L(\mathcal{A})$ the set of all non-empty intersections of elements of $\mathcal{A}$. In particular, $L(\mathcal{A})$ includes V as the intersection of the empty collection of hyperplanes. On $L(\mathcal{A})$ we define a partial order by reverse inclusion, i.e. for $X, Y \in L(\mathcal{A})$ we have $X \leq Y$ if and only if $Y \subset X$.

Definition 8.9. Let k be a field, let V be a k-vector space, and let $(\mathcal{A}, V)$ be an arrangement of hyperplanes. A map

$$\mu_{\mathcal{A}} \colon L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}$$

is called *Möbius function* of the arrangement $(\mathcal{A}, V)$ if for any $X, Y \in L(\mathcal{A})$ we have

$$\mu_{\mathcal{A}}(X, Y) = \begin{cases} 1 & \text{if } X = Y, \\ -\displaystyle\sum_{\substack{Z \in L(\mathcal{A}) \\ X \leq Z < Y}} \mu_{\mathcal{A}}(X, Z) & \text{if } X < Y, \\ 0 & \text{otherwise.} \end{cases} \tag{8.13}$$

Remark 8.10. Let k be a field, let V be a k-vector space, and let $(\mathcal{A}, V)$ be an arrangement of hyperplanes. Then there is a unique map $\mu_{\mathcal{A}} \colon L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}$ satisfying the conditions of Equation (8.13), see [37, page 33].

Definition 8.11. Let k be a field, let V be a k-vector space, and let $(\mathcal{A}, V)$ be an arrangement of hyperplanes. For any $X \in L(\mathcal{A})$ we define a subarrangement $(\mathcal{A}_X, V)$ of $(\mathcal{A}, V)$ by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}.$$

Remark 8.12. Let k be a field, and let $(\mathcal{A}, k^n)$ be an arrangement of hyperplanes with defining polynomial

$$\mathcal{Q}_{\mathcal{A}} = \prod_{H \in \mathcal{A}} f_H.$$

Let $X \in L(\mathcal{A})$. Then the defining polynomial of the hyperplane arrangement $(\mathcal{A}_X, k^n)$ is

$$\mathcal{Q}_{\mathcal{A}_X} = \prod_{\substack{H \in \mathcal{A} \\ X \subset H}} f_H,$$

see Remark 8.7.

Proposition 8.13. *Let be a field, let V be a k-vector space, let $(\mathcal{A}, V)$ be an arrangement of hyperplanes, and let $X \in L(\mathcal{A})$ with* $\operatorname{codim} X = 2$. *Then*

$$\mu(V, X) = |\mathcal{A}_X| - 1.$$

Proof. See [37, page 35]. □

Theorem 8.14. *Let k be an algebraically closed field, and let $(\mathcal{A}, k^n)$ be an arrangement of hyperplanes. We write*

$$R_{\mathcal{A}} = k[X_1 \dots, X_n]/\langle \mathcal{Q}_{\mathcal{A}} \rangle,$$

where $\mathcal{Q}_{\mathcal{A}}$ is the defining polynomial of $\mathcal{A}$. Then $(R_{\mathcal{A}})^{(q)}$ is normal if and only if

$$\begin{aligned} q &\geq \max\{|\mathcal{A}_X| \mid X \in L(\mathcal{A}) \text{ with } \operatorname{codim} X = 2\} - 1 \\ &= \max\{\mu(k^n, X) \mid X \in L(\mathcal{A}) \text{ with } \operatorname{codim} X = 2\}. \end{aligned}$$

In the following let k be an algebraically closed field, let V be an n-dimensional k-vector space, and let $(\mathcal{A}, V)$ be an arrangement of hyperplanes such that $|k| \geq |\mathcal{A}|$. We write

$$R_{\mathcal{A}} = k[X_1 \dots, X_n]/\langle \mathcal{Q}_{\mathcal{A}} \rangle,$$

where $\mathcal{Q}_{\mathcal{A}}$ is the defining polynomial of $\mathcal{A}$.

Remark 8.15. The ring $R_{\mathcal{A}}$ is Cohen–Macaulay by Corollary C.9 and Proposition C.10. Since $R_{\mathcal{A}}$ is also reduced by definition and excellent by Theorem B.35, it is normalization-finite by Theorem B.36.

For the proof of Theorem 8.14 we first show that the Grauert–Remmert algorithm behaves well with respect to field extensions.

Lemma 8.16. *Any extension field L of k is flat over k.*

Proof. See [26, Chapitre IV, § 2, no. 4, Proposition 3]. $\square$

Lemma 8.17. *Let L be an extension field of k, and let R be a k-algebra with $R/\mathfrak{m} \cong k$ for every $\mathfrak{m} \in \operatorname{Max}(R)$. Then for any $\mathfrak{m} \in \operatorname{Max}(R)$ we have $\mathfrak{m} \otimes_k L \in \operatorname{Max}(R \otimes_k L)$.*

Proof. For every $\mathfrak{m} \in \operatorname{Max}(R)$ there is an exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0.$$

Since L is flat over k by Lemma 8.16, this yields the exact sequence

$$0 \to \mathfrak{m} \otimes_k L \to R \otimes_k L \to (R/\mathfrak{m}) \otimes_k L = L \to 0.$$

Moreover, since $\mathfrak{m} \otimes_k L$ is an ideal of $R \otimes_k L$, we have the exact sequence

$$0 \to \mathfrak{m} \otimes_k L \to R \otimes_k L \to (R \otimes_k L)/(\mathfrak{m} \otimes_k L) \to 0.$$

This implies that

$$(R \otimes_k L)/(\mathfrak{m} \otimes_k L) = (R/\mathfrak{m}) \otimes_k L$$

is a field, and hence $\mathfrak{m} \otimes_k L \in \operatorname{Max}(R \otimes_k L)$. $\square$

Lemma 8.18. *Let L be an extension field of k, and let R be a k-algebra. If $\mathfrak{i}$ is an ideal of R, then $\sqrt{\mathfrak{i} \otimes_k L} = \sqrt{\mathfrak{i}} \otimes_k L$.*

Proof. We first show that

$$\mathfrak{i} \otimes_k L \subset \sqrt{\mathfrak{i}} \otimes_k L \subset \sqrt{\mathfrak{i} \otimes_k L}. \tag{8.14}$$

The first inclusion follows from Lemma 8.16. For the second let $\sum_{i=1}^{N} f_i \otimes a_i \in \sqrt{\mathfrak{i}} \otimes_k L$. Then for every $i = 1, \ldots, N$ there is a $d_i > 0$ such that $f_i^{d_i} \in \mathfrak{i}$. Set $d = \max_{i=1,\ldots,N} d_i$. Then $f_i^d \in \mathfrak{i}$ for each $i = 1, \ldots, N$. Moreover, we have

$$\left(\sum_{i=1}^{N} f_i \otimes a_i \right)^{Nd} = \sum_{|\alpha|=Nd} b_\alpha \prod_{i=1}^{N} f_i^{\alpha_i} \otimes \prod_{i=1}^{N} a_i^{\alpha_i}$$

with some coefficients $b_\alpha \in k$. Now $|\alpha| = Nd$ implies that there is a $j \in \{1, \ldots, N\}$ such that $\alpha_j \geq d$, and hence $f_j^{\alpha_j} \in \mathfrak{i}$. Thus, we have

$$\prod_{i=1}^{N} f_i^{\alpha_i} = f_j^{\alpha_j} \prod_{i \in \{1,\ldots,N\} \setminus \{j\}} f_i^{\alpha_i} \in \mathfrak{i},$$

and therefore

$$\left(\sum_{i=1}^{N} f_i \otimes a_i \right)^{Nd} \in \mathfrak{i} \otimes_k L.$$

This implies

$$\sum_{i=1}^{N} f_i \otimes a_i \in \sqrt{\mathfrak{i} \otimes_k L},$$

and hence

$$\sqrt{\mathfrak{i}} \otimes_k L \subset \sqrt{\mathfrak{i} \otimes_k L}.$$

Next we want to show that $\sqrt{\mathfrak{i}} \otimes_k L$ is a radical ideal in $R \otimes_k L$. Then Equation (8.14) implies

$$\sqrt{\mathfrak{i} \otimes_k L} \subset \sqrt{\sqrt{\mathfrak{i}} \otimes_k L} \subset \sqrt{\mathfrak{i}} \otimes_k L,$$

and therefore

$$\sqrt{\mathfrak{i} \otimes_k L} = \sqrt{\sqrt{\mathfrak{i}} \otimes_k L} = \sqrt{\mathfrak{i}} \otimes_k L.$$

Since L is flat over k by Lemma 8.16, the exact sequence

$$0 \to \sqrt{\mathfrak{i}} \to R \to R/\sqrt{\mathfrak{i}} \to 0.$$

yields an exact sequence

$$0 \to \sqrt{\mathfrak{i}} \otimes_k L \to R \otimes_k L \to \left(R/\sqrt{\mathfrak{i}}\right) \otimes_k L \to 0.$$

As $\mathfrak{i} \otimes_k L$ is an ideal of $R \otimes_k L$, the exact sequence

$$0 \to \sqrt{\mathfrak{i}} \otimes_k L \to R \otimes_k L \to (R \otimes_k L)/\left(\sqrt{\mathfrak{i}} \otimes_k L\right) \to 0$$

implies

$$(R \otimes_k L)/\left(\sqrt{\mathfrak{i}} \otimes_k L\right) = \left(R/\sqrt{\mathfrak{i}}\right) \otimes_k L. \tag{8.15}$$

Now $R/\sqrt{\mathfrak{i}}$ is a reduced k-algebra as $\sqrt{\mathfrak{i}}$ is a radical ideal. Since k is a perfect field, and since L is a reduced k-algebra, $(R \otimes_k L)/\left(\sqrt{\mathfrak{i}} \otimes_k L\right)$ is by Equation (8.15) and Theorem A.8 reduced, as well. This implies that $\sqrt{\mathfrak{i}} \otimes_k L$ is a radical ideal in $R \otimes_k L$. $\square$

Lemma 8.19. *Let L be an extension field of k, and let R be a k-algebra. Suppose that R and $R \otimes_k L$ are reduced excellent rings, and let $\mathfrak{i}$ be a test ideal for R (see Definition B.47) such that $\mathfrak{i} \otimes_k L$ is a test ideal for $R \otimes_k L$. Then*

$$(R \otimes_k L)^{(1)} = \operatorname{End}_{R \otimes_k L}(\mathfrak{i} \otimes_k L) = \operatorname{End}_R(\mathfrak{i}) \otimes_k L = R^{(1)} \otimes_k L, \tag{8.16}$$

and the test ideal for $(R \otimes_k L)^{(1)}$ used in the Grauert–Remmert algorithm is

$$\sqrt{(\mathfrak{i} \otimes_k L)\left((R \otimes_k L)^{(1)}\right)} = \sqrt{\mathfrak{i} R^{(1)}} \otimes_k L.$$

Proof. First note that

$$\mathfrak{i} \otimes_k L = \mathfrak{i} \otimes_R R \otimes_k L. \tag{8.17}$$

Since L is flat over k by Lemma 8.16, also $R \otimes_k L$ is flat over R by Lemma A.9. Moreover, $\mathfrak{i}$ is finitely presented as an $R \otimes_k L$-module by Remark A.41 since excellent rings are Noetherian (see Definition B.33). Thus, Equation (8.17) and Proposition A.40 yield

$$\begin{aligned}\operatorname{End}_R(\mathfrak{i}) \otimes_k L &= \operatorname{End}_R(\mathfrak{i}) \otimes_R R \otimes_k L \\ &= \operatorname{End}_{R\otimes_k L}(\mathfrak{i} \otimes_R R \otimes_k L) \\ &= \operatorname{End}_{R\otimes_k L}(\mathfrak{i} \otimes_k L).\end{aligned}$$

Then Equation (8.16) and Lemma 8.18 yield

$$\begin{aligned}\sqrt{(\mathfrak{i} \otimes_k L)\big((R \otimes_k L)'\big)} &= \sqrt{(\mathfrak{i} \otimes_k L)(R' \otimes_k L)} \\ &= \sqrt{\mathfrak{i}R' \otimes_k L} \\ &= \sqrt{\mathfrak{i}R'} \otimes_k L.\end{aligned}$$ □

Remark 8.20. Note that in Lemma 8.19 the reducedness of $R \otimes_k L$ follows from the reducedness of R and L, see Theorem A.8.

Lemma 8.21. *Let L be an extension field of k, and let R be a k-algebra. Suppose that R and $R \otimes_k L$ are reduced excellent rings, and let $\mathfrak{i}$ be a test ideal for R (see Definition B.47) such that $\mathfrak{i} \otimes_k L$ is a test ideal for $R \otimes_k L$. Then $R \otimes_k L$ is normal if and only if R is normal.*

Proof. By Theorem B.48 the ring $R \otimes_k L$ is normal if and only if $R \otimes_k L = \operatorname{End}_{R\otimes_k L}(\mathfrak{i} \otimes_k L)$. By Lemma 8.19 this is equivalent to $R = \operatorname{End}_R(\mathfrak{i})$, and hence to R being normal by Theorem B.48. □

Lemma 8.22. *Let L be an extension field of k, and let R be a k-algebra. Suppose that R and $R \otimes_k L$ are reduced excellent rings, and let $\mathfrak{i}$ be a test ideal for R (see Definition B.47) such that $\mathfrak{i} \otimes_k L$ is a test ideal for $R \otimes_k L$. Then*

$$(R \otimes_k L)^{(q)} = R^{(q)} \otimes_k L$$

and

$$(\mathfrak{i} \otimes_k L)^{(q)} = \mathfrak{i}^{(q)} \otimes_k L$$

for every $q \geq 0$.

Proof. This follows inductively from Lemma 8.19 and Proposition B.57. □

Using Proposition B.58 we want to apply the Grauert–Remmert algorithm locally to $R_{\mathcal{A}}$. By Lemma B.61.(4) we only have to consider $\operatorname{Sing}(R_{\mathcal{A}})$ (see Definition B.38).

Lemma 8.23. *Let $\mathfrak{p}$ be a prime ideal of $R_{\mathcal{A}}$. Then $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ if and only if there are $H, H' \in \mathcal{A}$ with $H \neq H'$ such that $f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle, f_{H'} + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of $R_{\mathcal{A}}$. Since $R_{\mathcal{A}} = k[X_1, \dots, X_n]/\langle \mathcal{Q}_{\mathcal{A}} \rangle$ with $\mathcal{Q}_{\mathcal{A}} = \prod_{H \in \mathcal{A}} f_H$, there is by Proposition A.10.(3) a prime ideal $\mathfrak{q}$ of $k[X_1, \dots, X_n]$ with $\langle \mathcal{Q}_{\mathcal{A}} \rangle \subset \mathfrak{q}$ such that $\mathfrak{q}/\langle \mathcal{Q}_{\mathcal{A}} \rangle = \mathfrak{p}$. Since $\mathcal{Q}_{\mathcal{A}} = \prod_{H \in \mathcal{A}} f_H$, there is at least one $H \in \mathcal{A}$ with $f_H \in \mathfrak{q}$.

Suppose that there is exactly one $H \in \mathcal{A}$ with $f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}$. After a linear coordinate change we may assume that $f_H = X_1$. Then Theorem A.36 yields

$$\begin{aligned}
(R_{\mathcal{A}})_{\mathfrak{p}} &= (k[X_1, \dots, X_n]/\langle \mathcal{Q}_{\mathcal{A}} \rangle)_{\mathfrak{p}} \\
&= (k[X_1, \dots, X_n])_{\mathfrak{q}}/\mathcal{Q}_{\mathcal{A}}(k[X_1, \dots, X_n])_{\mathfrak{q}} \\
&= (k[X_1, \dots, X_n])_{\mathfrak{q}}/X_1(k[X_1, \dots, X_n])_{\mathfrak{q}} \\
&= (k[X_1, \dots, X_n]/X_1)_{\overline{\mathfrak{q}}} \\
&= (k[X_2, \dots, X_n])_{\overline{\mathfrak{q}}},
\end{aligned}$$

where $\overline{\mathfrak{q}}$ is the image of $\mathfrak{q}$ in $k[X_1, \dots, X_n]/X_1 = k[X_2, \dots, X_n]$. The ring $k[X_2, \dots, X_n]$ is regular (see [38, Theorem 2.2.13]. So if $\mathfrak{m}$ is a maximal ideal of $k[X_2, \dots, X_n]$ containing $\overline{\mathfrak{q}}$, then $(k[X_2, \dots, X_n])_{\mathfrak{m}}$ is regular. Since $\overline{\mathfrak{q}}(k[X_2, \dots, X_n])_{\mathfrak{m}}$ is by Proposition A.20.(2) a prime ideal of $(k[X_2, \dots, X_n])_{\mathfrak{m}}$, also the ring

$$(k[X_2, \dots, X_n])_{\overline{\mathfrak{q}}} = ((k[X_2, \dots, X_n])_{\mathfrak{m}})_{\overline{\mathfrak{q}}(k[X_2, \dots, X_n])_{\mathfrak{m}}}$$

(see Corollary A.23 for the equality) is regular (see [38, Corollary 2.2.9]). This implies $\mathfrak{p} \notin \operatorname{Sing}(R_{\mathcal{A}})$.

Now suppose that there are $H, H' \in \mathcal{A}$ with $H \neq H'$ such that $f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle, f_{H'} + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}$. Then Theorem A.36 yields

$$\begin{aligned}
(R_{\mathcal{A}})_{\mathfrak{p}} &= \Big(k[X_1, \dots, X_n]/\langle \mathcal{Q}_{\mathcal{A}} \rangle_{\mathfrak{p}}\Big) \\
&= (k[X_1, \dots, X_n])_{\mathfrak{q}}/\mathcal{Q}_{\mathcal{A}}(k[X_1, \dots, X_n])_{\mathfrak{q}} \\
&= (k[X_1, \dots, X_n])_{\mathfrak{q}}/ \prod_{f_{H''} \in \mathfrak{q}} f_{H''}(k[X_1, \dots, X_n])_{\mathfrak{q}}.
\end{aligned}$$

Since $f_H, f_{H'} \in \mathfrak{q}$, this implies that the images of f_H and $f_{H'}$ are non-zero but zerodivisors in $(R_{\mathcal{A}})_{\mathfrak{p}}$. Thus, $(R_{\mathcal{A}})_{\mathfrak{p}}$ is not regular (see [38, Proposition 2.2.3]), i.e. $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$. $\square$

In fact, since $R_{\mathcal{A}}$ is Cohen–Macaulay by Remark 8.15, it suffices by Proposition C.12 to consider only prime ideals $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ with $\operatorname{height} \mathfrak{p} = 1$.

Lemma 8.24. *Let $\mathfrak{p}$ be a prime ideal of $R_{\mathcal{A}}$. Then $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ with $\operatorname{height} \mathfrak{p} = 1$ if and only if there are $H, H' \in \mathcal{A}$ with $H \neq H'$ such that $\mathfrak{p} = \langle f_H, f_{H'} \rangle R_{\mathcal{A}}$. In particular, for every $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ with $\operatorname{height} \mathfrak{p} = 1$ there is a linear coordinate change $X_i + \langle Q_{\mathcal{A}} \rangle \mapsto y_i$, $i = 1, \dots, n$, such that $\mathfrak{p} = \langle y_1, y_2 \rangle$.*

Proof. By Lemma 8.23 we have $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ if and only if there are $H, H' \in \mathcal{A}$ with $H \neq H'$ such that $f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle, f_{H'} + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}$. After a linear coordinate change we may assume that $f_H = X_1$ and $f_{H'} = X_2$. Then $\langle X_1, X_2 \rangle$ is a prime ideal of $k[X_1, \dots, X_n]$ containing $\mathcal{Q}_{\mathcal{A}}$. Therefore, $\langle X_1, X_2 \rangle R_{\mathcal{A}}$ is by Proposition A.10.(3) a prime ideal of $R_{\mathcal{A}}$. The claim follows since $\operatorname{height} \langle X_1, X_2 \rangle R_{\mathcal{A}} = 1$. $\square$

Lemma 8.25. *There is a bijection*

$$\begin{aligned}\{\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}}) \mid \operatorname{height} \mathfrak{p} = 1\} &\to \{X \in L(\mathcal{A}) \mid \operatorname{codim} X = 2\},\\ \mathfrak{p} &\mapsto X_{\mathfrak{p}} = \bigcap_{\substack{H \in \mathcal{A} \\ f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}}} H,\\ \langle f_H \mid X \subset H \rangle R_{\mathcal{A}} &\hookleftarrow X.\end{aligned}$$

Proof. Let $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ with $\operatorname{height} \mathfrak{p} = 1$. By Lemma 8.24 there are $H, H' \in \mathcal{A}$ with $H \neq H'$ such that $\mathfrak{p} = \langle f_H, f_{H'} \rangle R_{\mathcal{A}}$. Then for any $H'' \in \mathcal{A}$ we have $f_{H''} + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}$ if and only if $f_{H''} \in \langle f_H, f_{H'} \rangle$. This implies

$$X_{\mathfrak{p}} = H \cap H'.$$

In particular, $\operatorname{codim} X_{\mathfrak{p}} = 2$.

Conversely, let $X \in L(\mathcal{A})$ with $\operatorname{codim} X = 2$. Then there are $H, H' \in \mathcal{A}$ with $H \neq H'$ such that $X = H \cap H'$. Thus, for any $H'' \in \mathcal{A}$ with $X \subset H''$ we have $f_{H''} \in \langle f_H, f_{H'} \rangle$. This implies

$$\langle f_{H''} \mid H'' \in \mathcal{A} \text{ with } X \subset H'' \rangle R_{\mathcal{A}} = \langle f_H, f_{H'} \rangle R_{\mathcal{A}}.$$

Moreover, we have by Lemma 8.23 $\langle f_H, f_{H'} \rangle R_{\mathcal{A}} \in \operatorname{Sing}(R_{\mathcal{A}})$ with $\operatorname{height} \langle f_H, f_{H'} \rangle = 1$. $\square$

Lemma 8.26. *Let* $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ *with* $\operatorname{height} \mathfrak{p} = 1$. *Then for any* $q \geq 0$ *the ring* $\left((R_{\mathcal{A}})_{\mathfrak{p}}\right)^{(q)}$ *is normal if and only if*

$$q \geq |\mathcal{A}_{X_{\mathfrak{p}}}| - 1,$$

where

$$X_{\mathfrak{p}} = \bigcap_{\substack{H \in \mathcal{A} \\ f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}}} H \in L(\mathcal{A}).$$

Proof. Let $q \geq 0$, and let $\mathfrak{q}$ be a prime ideal of $(R_{\mathcal{A}})^{(q)}$ with $\mathfrak{q} \cap R_{\mathcal{A}} = \mathfrak{p}$ (see Proposition B.57 and Theorem B.12). Since $\operatorname{height} \mathfrak{p} = 1$, we may by Lemma 8.24 assume that $\mathfrak{p} = \langle x_1, x_2 \rangle$, where for any $j = 1, \ldots, n$ we set $x_j = X_j + \langle \mathcal{Q}_{\mathcal{A}} \rangle$. Moreover, Lemma B.18 yields $\operatorname{height} \mathfrak{q} = 1$.

Let $H \in \mathcal{A}$. If $\overline{f_H} = f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}$, then there is a $g \in k[X_1, \ldots, X_n]$ such that

$$f_H \left(1 + g \prod_{H' \in \mathcal{A} \setminus \{H\}} f_{H'} \right) = f_H + g \mathcal{Q}_{\mathcal{A}} \in \langle X_1, X_2 \rangle.$$

This implies

$$f_H \in \langle X_1, X_2 \rangle$$

since $\langle X_1, X_2 \rangle$ is a prime ideal and $1 + g \prod_{H' \in \mathcal{A} \setminus \{H\}} f_{H'} \notin \langle X_1, X_2 \rangle$. If $\overline{f_H} \notin \mathfrak{p}$, then $f_H \in \left(k[X_1, \ldots, X_n]_{\langle X_1, X_2 \rangle}\right)^*$, and hence

$$\mathcal{Q}_{\mathcal{A}} k[X_1, \ldots, X_n]_{\langle X_1, X_2 \rangle} = f k[X_1, \ldots, X_n]_{\langle X_1, X_2 \rangle}$$

with
$$f = \prod_{\substack{H\in\mathcal{A} \\ f_H\in\langle X_1,X_2\rangle}} f_H = \prod_{\substack{H\in\mathcal{A} \\ f_H+\langle \mathcal{Q}_\mathcal{A}\rangle\in\mathfrak{p}}}. \tag{8.18}$$

Therefore, Theorem A.36 yields

$$\begin{aligned}
(R_\mathcal{A})_\mathfrak{p} &= (k[X_1,\dots,X_n]/\langle \mathcal{Q}_\mathcal{A}\rangle)_{\langle x_1,x_2\rangle} \\
&= k[X_1,\dots,X_n]_{\langle X_1,X_2\rangle}/\mathcal{Q}_\mathcal{A}k[X_1,\dots,X_n]_{\langle X_1,X_2\rangle} \\
&= k[X_1,\dots,X_n]_{\langle X_1,X_2\rangle}/fk[X_1,\dots,X_n]_{\langle X_1,X_2\rangle} \\
&= \Big(k[X_1,X_2]_{\langle X_1,X_2\rangle}\otimes_k L\Big)/\Big(fk[X_1,X_2]_{\langle X_1,X_2\rangle}\otimes_k L\Big) \\
&= R\otimes_k L
\end{aligned}$$

with
$$L = k(X_3,\dots,X_n)$$
and
$$R = (k[X_1,X_2]/\langle f\rangle)_{\langle x_1,x_2\rangle},$$
where by abuse of notation we consider $f\in k[X_1,X_2]$, and we write $x_j = X_j+\langle f\rangle$ for $j=1,2$.

Since $(R_\mathcal{A})_\mathfrak{p}$ and R are local rings, Lemma 8.17 yields $\mathfrak{p}(R_\mathcal{A})_\mathfrak{p} = \mathfrak{m}\otimes_k L$, where $\mathfrak{m}$ is the maximal ideal of R. Moreover, $(R_\mathcal{A})_\mathfrak{p}$ and R are by Theorems B.34 and B.35 reduced excellent rings. Hence, by Remark B.49 $\mathfrak{p}(R_\mathcal{A})_\mathfrak{p}$ is the unique test ideal for $(R_\mathcal{A})_\mathfrak{p}$, and $\mathfrak{m}$ is the unique test ideal for R. Then Lemma 8.22 yields for any $q\geq 0$
$$\Big((R_\mathcal{A})_\mathfrak{p}\Big)^{(q)} = R^{(q)}\otimes_k L.$$

Thus, $\Big((R_\mathcal{A})_\mathfrak{p}\Big)^{(q)}$ is by Lemmas 8.21 and 8.22 normal if and only if $R^{(q)}$ is normal.

By Theorem B.36.(3) $R^{(q)}$ is normal if and only if $\widehat{R^{(q)}}$ is normal, and Proposition B.54 yields $\widehat{R^{(q)}} = \widehat{R}^{(q)}$. Moreover, with
$$\widehat{R} = k[[X_1,X_2]]/\langle f\rangle$$
Equation (8.18), Theorem 8.2, Remark 8.12, and Lemma 8.25 imply that $\widehat{R}^{(q)}$ is normal if and only if
$$\begin{aligned}
q &\geq |\{H\in\mathcal{A}\mid f_H+\langle\mathcal{Q}_\mathcal{A}\rangle\in\mathfrak{p}\}|-1 \\
&= |\mathcal{A}_{X_\mathfrak{p}}|-1,
\end{aligned}$$
where
$$X_\mathfrak{p} = \bigcap_{\substack{H\in\mathcal{A} \\ f_H+\langle\mathcal{Q}_\mathcal{A}\rangle\in\mathfrak{p}}} H\in L(\mathcal{A}). \qquad\square$$

Proof of Theorem 8.14. Since $R_{\mathcal{A}}$ is Cohen–Macaulay by Remark 8.15, it satisfies Serre's condition (S_2) by Corollary C.5. Then for any $q \geq 0$ the ring $(R_{\mathcal{A}})^{(q)}$ satisfies Serre's condition (S_2) by Lemma A.27, Theorem B.34, and Propositions B.46.(1) and B.57. Hence, $(R_{\mathcal{A}})^{(q)}$ is by Lemma B.61.(1) normal if and only if it satisfies Serre's condition (R_1). By Lemma B.61.(4) and Proposition C.12 this is equivalent to $\left((R_{\mathcal{A}})_{\mathfrak{p}}\right)^{(q)}$ being normal for every $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ with height $\mathfrak{p} = 1$.

Let $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ with height $\mathfrak{p} = 1$. Then $\left((R_{\mathcal{A}})_{\mathfrak{p}}\right)^{(q)}$ is by Lemma 8.26 normal if and only if

$$q \geq |\mathcal{A}_{X_{\mathfrak{p}}}| - 1,$$

where

$$X_{\mathfrak{p}} = \bigcap_{\substack{H \in \mathcal{A} \\ f_H + \langle \mathcal{Q}_{\mathcal{A}} \rangle \in \mathfrak{p}}} H \in L(\mathcal{A}).$$

By Lemma 8.25 this implies that $\left((R_{\mathcal{A}})_{\mathfrak{p}}\right)^{(q)}$ is normal for every $\mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}})$ with height $\mathfrak{p} = 1$ if and only if

$$\begin{aligned} q &\geq \max\left\{|\mathcal{A}_{X_{\mathfrak{p}}}| \mid \mathfrak{p} \in \operatorname{Sing}(R_{\mathcal{A}}) \text{ with } \operatorname{height}\mathfrak{p} = 1\right\} - 1 \\ &= \max\{|\mathcal{A}_X| \mid X \in L(\mathcal{A}) \text{ with } \operatorname{codim} X = 2\} - 1 \\ &= \max\{\mu(k^n, X) \mid X \in L(\mathcal{A}) \text{ with } \operatorname{codim} X = 2\}, \end{aligned}$$

where the last equality follows from Lemma 8.13. $\square$

A. Commutative Algebra

Theorem A.1. *Any homomorphic image of a Noetherian ring is Noetherian. Furthermore, if R is a Noetherian ring, and A is a finitely generated algebra over R, then A is Noetherian.*

Proof. See [39, Corollary 1.3]. $\square$

Theorem A.2 (Prime Avoidance)**.** *Let $\mathfrak{i}_1, \dots, \mathfrak{i}_n, \mathfrak{j}$ be ideals of a ring R, and suppose that $\mathfrak{j} \subset \bigcup_{i=1}^n \mathfrak{i}_i$.If at most two of the ideals $\mathfrak{i}_i$, $\mathfrak{i} = 1, \dots, n$ are not prime, then $\mathfrak{j}$ is contained in one of the $\mathfrak{i}_i$.*

Proof. See [39, Lemma 3.3]. $\square$

Lemma A.3. *Let R be a ring, and let $\mathfrak{i}_1, \dots, \mathfrak{i}_n$ be ideals of R such that $\bigcap_{i=1}^n \mathfrak{i}_i = 0$. If $R/\mathfrak{i}_i$ is Noetherian for all $i = 1, \dots, n$, then R is Noetherian.*

Proof. Since $\bigcap_{i=1}^n \mathfrak{i}_i = 0$, the canonical map

$$\begin{aligned} \phi \colon R &\to \prod_{i=1}^n R/\mathfrak{i}_i, \\ x &\mapsto (x + \mathfrak{i}_i)_{i=1,\dots,n}, \end{aligned}$$

is injective.

Let $\mathfrak{j}_1 \subset \mathfrak{j}_2 \subset \dots$ be an ascending chain of ideals in R. Then for any $i = 1, \dots, n$ there is an ascending chain $\mathfrak{j}_1 + \mathfrak{i}_i \subset \mathfrak{j}_2 + \mathfrak{i}_i \subset \dots$ of ideals of $R/\mathfrak{i}_i$. Since $R/\mathfrak{i}_i$ is Noetherian, there is n_i such that $\mathfrak{j}_{n_i} + \mathfrak{i}_i = \mathfrak{j}_{n_i+1} + \mathfrak{i}_i = \dots$. This implies

$$\phi\Big(\mathfrak{j}_{\max_{i=1,\dots,n} n_i}\Big) = \phi\Big(\mathfrak{j}_{(\max_{i=1,\dots,n} n_i)+1}\Big) = \dots.$$

Since ϕ is injective, this implies that R is Noetherian. $\square$

Theorem A.4 (Krull Intersection Theorem)**.** *Let R be a Noetherian ring, and let $\mathfrak{i}$ be an ideal of R. If M is a finitely generated R-module, then there is an element $r \in \mathfrak{i}$ such that*

$$(1 - r)\left(\bigcap_{i=i}^{\infty} \mathfrak{i}^i M\right) = 0.$$

If R is a domain or a local ring, then

$$\bigcap_{i=1}^{\infty} \mathfrak{i}^i = 0.$$

Proof. See [39, Corollary 5.4]. $\square$

Corollary A.5. *Let R be a Noetherian ring, and let $\mathfrak{i}$ be a proper ideal of R. Then*

$$\bigcap_{i=1}^{\infty} \mathfrak{i}^i \subset R \setminus R^{\mathrm{reg}}.$$

Proof. Since R is a finitely generated R-module, Theorem A.4 yields an $r \in \mathfrak{i}$ such that

$$(1-r)\left(\bigcap_{i=i}^{\infty} \mathfrak{i}^i\right) = 0.$$

Since $\mathfrak{i}$ is a proper ideal of R, we have $r \neq 1$, and hence $1 - r \neq 0$. Thus, every element in $\left(\bigcap_{i=i}^{\infty} \mathfrak{i}^i\right)$ is a zerodivisor. ☐

Lemma A.6. *Let $(R_i)_{i\in I}$ be a finite family of rings, and let $R = \prod_{i\in I} R_i$. For $j \in I$, we denote by*

$$\mathrm{pr}_j\colon R = \prod_{i\in I} R_i \to R_j$$

the projection.

(1) Let $\mathfrak{q}$ be a prime ideal of R. Then there is a $j \in I$ such that

$$\mathrm{pr}_j(\mathfrak{q}) \in \mathrm{Spec}(R_j)$$

and

$$\mathrm{pr}_i(\mathfrak{q}) = R_i$$

for all $i \in I \setminus \{j\}$.

(2) There is a bijection

$$\mathrm{Spec}(R) \to \bigsqcup_{i\in I} \mathrm{Spec}(R_i)$$

which is induced by the bijections

$$\begin{aligned} \Big\{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathrm{pr}_j(\mathfrak{q}) \in \mathrm{Spec}(R_j)\Big\} &\to \mathrm{Spec}(R_j), \\ \mathfrak{q} &\mapsto \mathrm{pr}_j(\mathfrak{q}), \\ \mathfrak{p} \times \prod_{i\in I\setminus\{j\}} R_i &\leftarrow\!\shortmid \mathfrak{p} \end{aligned}$$

for $j \in I$. In particular, for any $i \in I$ we have

$$\mathrm{height}\, \mathfrak{q} = \mathrm{height}\, \mathrm{pr}_i(\mathfrak{q}).$$

Proof. (1) Let $\mathfrak{q}$ be a prime ideal of R. Then

$$R/\mathfrak{q} = \prod_{i\in I} (R_i/\mathrm{pr}_i(\mathfrak{q}))$$

is a domain. This implies that there is $j \in I$ such that

$$R_i / \operatorname{pr}_i(\mathfrak{q}) = 0$$

for all $i \in I \setminus \{j\}$, and hence

$$\operatorname{pr}_i(\mathfrak{q}) = R_i$$

for all $i \in I \setminus \{j\}$. This implies

$$R/\mathfrak{q} \cong R_j / \operatorname{pr}_j(\mathfrak{q}),$$

and hence $R_j / \operatorname{pr}_j(\mathfrak{q})$ is a domain. Thus, $\operatorname{pr}_j(\mathfrak{q})$ is a prime ideal of R_j.

(2) The statement follows from (1) since for any prime ideal $\mathfrak{p}$ of R_i we obviously have

$$\mathfrak{p} \times \prod_{i \in I \setminus \{j\}} R_i \in \operatorname{Spec}(R). \qquad \square$$

Lemma A.7. *Let R and A be rings, let $\phi\colon R \to A$ be a ring homomorphism. If A is flat as an R-module (with respect to ϕ), then $\phi(R^{\mathrm{reg}}) \subset A^{\mathrm{reg}}$.*

Proof. An element $x \in R$ is regular if and only if multiplication by x defines an injective ring homomorphism $\phi_x\colon R \to R$. If A is flat, tensoring by A yields an injective ring homomorphism $\phi_x \otimes 1\colon R \otimes_R A \to R \otimes_R A$, see [40, Proposition 2.19]. The claim follows since we can identify $R \otimes_R A$ with A and $\phi_x \otimes 1$ with multiplication by $\alpha(x)$. $\square$

Theorem A.8. *Let k be a perfect field, and let R and S be two reduced k-algebras. Then $R \otimes_k S$ is reduced.*

Proof. See [41, Chapitre V, §15, no. 5, Theorem 3.(d)]. $\square$

Lemma A.9. *Let R be a ring, and let A be an R-algebra, and let M be a flat R-module. Then $M \otimes_R A$ is a flat A-module.*

Proof. See [42, Chapter 2, (3.C)]. $\square$

Proposition A.10. *Let R and A be rings, and let $\phi\colon R \to A$ be a ring homomorphism.*

(1) There is an injective map from the set of ideals of R into the set of ideals of A given by

$$\mathfrak{i} \mapsto \phi(\mathfrak{i})A.$$

(2) There is a surjective map from the set of ideals of A onto the set of ideals of R given by

$$\mathfrak{j} \mapsto \phi^{-1}(\mathfrak{j}),$$

and sending prime ideals to prime ideals.

(3) If ϕ is surjective, then the maps given in (1) and (2) yield mutually inverse bijections between the set of ideals of R containing $\ker\phi$ and the set of ideals of A, where prime ideals correspond to prime ideals.

Proof. See [40, page 9]. □

Lemma A.11. *Let A be a ring, let I be a finite set, and let $(R_i)_{i\in I}$ be a family of subrings of A. Then $R = \bigcap_{i\in I} R_i$ is a subring of A with $R^* = \bigcap_{i\in I} (R_i)^*$.*

Proof. We obviously have $0, 1 \in R$. Let $x, y \in R$. Then $x, y \in R_i$ for all $i \in I$, and hence $x + y, xy \in \bigcap_{i\in I} R_i = R$. For any $i \in I$ there is an $x_i \in R_i$ with $x + x_i = 0$. Since these equations also hold in A, we have $x_i = -x \in \bigcap_{j\in I} R_j = R$ for every $i \in I$ by uniqueness of inverse elements. Thus, R is a subring of A.

Let now $x \in R^*$. Then there is a $z \in R = \bigcap_{i\in I} R_i$ with $xz = 1$. Thus, $x \in \bigcap_{i\in I} (R_i)^*$.

Conversely, let $x \in \bigcup_{i\in I} (R_i)^*$. Then for any $i \in I$ there is a $z_i \in R_i$ with $xz_i = 1$. This implies $x \in A^*$, and hence $z_i = x^{-1}$ for all $i \in I$. In particular, $x^{-1} \in R^*$. □

Theorem A.12. *Let R be a semilocal ring with Jacobson radical $\mathfrak{j}_R$, and let A be a finite R-algebra containing R. Then A is semilocal with Jacobson radical $\mathfrak{j}_A = \sqrt{\mathfrak{j}_R A}$.*

Proof. See [43, § 6, Theorem 15]. □

Lemma A.13. *Let R be a ring, let $\mathfrak{i}$ be an ideal of R, let $x \in \mathfrak{i}$, and let $y \in R \setminus \mathfrak{i}$. Then $x + y \in R \setminus \mathfrak{i}$.*

Proof. Since $x \in \mathfrak{i}$, also $-x \in \mathfrak{i}$. So if $x + y \in \mathfrak{i}$, then $y = x + y - x \in \mathfrak{i}$, contradicting the assumption. □

A.1. Large Jacobson Radical

Proposition A.14. *Let R be a ring with Jacobson radical $\mathfrak{j}_R$, and let $x \in R$. Then $x \in \mathfrak{j}_R$ if and only if $1 + xy$ is a unit in R for all $y \in R$.*

Proof. See [30, Section 7, page 422]. □

Proposition A.15. *Let R be a ring with Jacobson radical $\mathfrak{j}_R$. Then the following are equivalent:*

(a) Any prime ideal of R containing $\mathfrak{j}_R$ is maximal.

(b) For each $x \in R$ there is $y \in R$ such that for all $z \in A$ and for all units $r \in R^*$ both $x + ry$ and $1 + zxy$ are units in R.

(c) For each $x \in R$ there is a $y \in R$ such that $x + y$ is a unit in R and $xy \in \mathfrak{j}_R$.

Proof. See [30, Proposition 19]. □

Definition A.16. A ring R is said to have a *large Jacobson radical* if it satisfies the equivalent conditions of Proposition A.15.

Remark A.17. Let R be a ring.

(1) If every prime ideal of R is maximal, or if R is (quasi)semilocal, then R has a large Jacobson radical, see [30, Section 7, page 423].

(2) If Q_R has a large Jacobson radical, then R is a Marot ring, see [23, Chapter I, Proposition 1.12].

Proposition A.18. *Let R be a Noetherian ring. Then R has a large Jacobson radical if and only if it is semilocal.*

Proof. See [30, Section 7, page 423]. □

A.2. Localization

Let R be a ring, and let U be a multiplicatively closed subset of R. We will always assume that $1 \in U$. The *localization* of R at U is the R-algebra $U^{-1}R$ satisfying the following universal property: the homomorphism $\alpha\colon R \to U^{-1}R$ satisfies $\alpha(R^*) \subset \left(U^{-1}R\right)^*$, and if A is a ring such that there is a ring homomorphism $\beta\colon R \to A$ with $\beta(R^*) \subset A^*$, then there is a unique ring homomorphism $\phi\colon U^{-1}R \to A$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & U^{-1}R \arrow[d, dotted, "\phi"] \\ & A \end{tikzcd} \tag{A.1}$$

commutes.

Let M be an R-module. The *localization* $U^{-1}M$ of M at U is the set of equivalence classes $\frac{x}{u}$ with $x \in M$ and $u \in U$, where $\frac{x}{u} = \frac{x}{v}$ if there is an element $s \in U$ such that $s(xv - yu) = 0$. If $M = R$, then $U^{-1}R$ is a ring with the operations

$$\frac{x}{u} + \frac{y}{v} = \frac{xv + yu}{uv}$$

and

$$\frac{x}{u} \cdot \frac{y}{v} = \frac{xy}{uv}$$

for all $\frac{x}{u}, \frac{y}{v} \in U^{-1}R$, and $U^{-1}R$ is an R-algebra with the natural homomorphism $\alpha\colon R \to U^{-1}R$, $x \mapsto \frac{x}{1}$. For an R-module M, the localization $U^{-1}M$ is both an R- and a $U^{-1}R$-module with the obvious operations.

The localization $Q_R = (R^{\mathrm{reg}})^{-1}$ is called the *total ring of fractions* of R. If $\mathfrak{p}$ is a prime ideal of R, then the *localization of R at $\mathfrak{p}$* is $(R \setminus \mathfrak{p})^{-1}R$.

Let M, N be R-modules, and let $\psi\colon M \to N$ be an R-module homomorphism. Then there is an $U^{-1}R$-module homomorphism

$$\begin{aligned} U^{-1}\phi\colon U^{-1}M &\to U^{-1}N, \\ \frac{m}{u} &\mapsto \frac{\phi(m)}{u}, \end{aligned}$$

the *localization* of ϕ, see [39, Chapter 2] and [40, Chapter 3].

Remark A.19. Let R be a ring.

(1) Let U be a multiplicatively closed subset of R^*. Then $U^{-1}R = R$.

(2) The total ring of fractions of R is the "largest" localization of R at a multiplicatively closed set $U \subset R$ such that the natural map $\alpha\colon R \to U^{-1}R$ is an injection, see [39, Chapter 2, page 60]. In particular, we may consider R as a subring of Q_R.

(3) $Q_R^{\mathrm{reg}} = Q_R^*$.

(4) If R is a domain, i.e. if $R^{\mathrm{reg}} = R \setminus \{0\}$, then Q_R is a field.

Proposition A.20. *Let R be a ring, let U be a multiplicatively closed subset of R, and let $\alpha\colon R \to U^{-1}R$ be the natural map $x \mapsto \frac{x}{1}$. Then the following hold:*

(1) For any ideal $\mathfrak{i} \subset U^{-1}R$ we have $\mathfrak{i} = \alpha^{-1}(\mathfrak{i})\, U^{-1}R$. Thus, the map $\mathfrak{i} \mapsto \alpha^{-1}(\mathfrak{i})$ is an injection of the set of ideals of $U^{-1}R$ into the set of ideals of R. It preserves inclusions and intersections, and it takes prime ideals to prime ideals.

(2) An ideal $\mathfrak{j} \subset R$ is of the form $\alpha^{-1}(\mathfrak{i})$ for some ideal $\mathfrak{i} \subset U^{-1}R$ if and only if $\mathfrak{j} = \alpha^{-1}(\mathfrak{j}U^{-1}R)$. This is the case if and only if for each $u \in U$, $xu \in \mathfrak{j}$ implies $x \in \mathfrak{j}$ for any $x \in R$. In particular, the correspondence $\mathfrak{i} \mapsto \alpha^{-1}(\mathfrak{i})$ is a bijection between set of the prime ideals of $U^{-1}R$ and the set of prime ideals of R not meeting U.

Proof. See [39, Proposition 2.2]. □

Corollary A.21. *A localization of a Noetherian ring is Noetherian.*

Proof. See [39, Corollary 2.3]. □

Theorem A.22. *Let R and A be rings with a ring homomorphism $\psi\colon R \to A$, and let U be a multiplicatively closed subset of R. Then $U^{-1}A = (\psi(U))^{-1}A$, and the localized map $U^{-1}\psi\colon U^{-1} \to U^{-1}A$ is a ring homomorphism.*

Proof. See [44, Theorem 4.3]. □

Corollary A.23. *Let R be a ring, let U be a multiplicatively closed subset of R, and let $\mathfrak{p}$ be a prime ideal of R with $\mathfrak{p} \cap U = \emptyset$. Then*

$$\left(U^{-1}R\right)_{\mathfrak{p}U^{-1}R} = R_{\mathfrak{p}}.$$

In particular, if $\mathfrak{q}$ is a prime ideal of R with $\mathfrak{p} \subset \mathfrak{q}$, then

$$(R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}} = R_{\mathfrak{p}}.$$

Proof. See [44, Corollary 4 of Theorem 4.3]. □

Proposition A.24. *Let R be a ring, and let U be a multiplicatively closed subset of R. Then the operation U^{-1} is exact, i.e. if*

$$M' \xrightarrow{\phi} M \xrightarrow{\psi} M'$$

is a sequence of R-modules which is exact at M, then

$$U^{-1}M' \xrightarrow{U^{-1}\phi} U^{-1}M \xrightarrow{U^{-1}\psi} U^{-1}M'$$

is a sequence of $U^{-1}R$-modules which is exact at $U^{-1}M$.

Proof. See [40, Proposition 3.3]. □

Corollary A.25. *Let R be a ring, let U be a multiplicatively closed subset of R, and let $x \in R^{\mathrm{reg}}$. Then $\frac{x}{y} \in \left(U^{-1}R\right)^{\mathrm{reg}}$ for any $y \in U$.*

Proof. This follows from Lemma A.7 and Proposition A.24. □

Remark A.26. The statement of Corollary A.25 can also be proved directly. In fact, let $\frac{v}{w} \in U^{-1}R$ such that $\frac{x}{y}\frac{v}{w} = 0$. Then there is a $u \in U$ such that $uxv = 0$. Since x is regular, this implies $uv = 0$, and hence $\frac{v}{w} = 0$. Therefore, $\frac{x}{y} \in \left(U^{-1}R\right)^{\mathrm{reg}}$.

Lemma A.27. *Let R be a ring, and let U be a multiplicatively closed subset of R. Then R is reduced if and only if $U^{-1}R$ is reduced.*

Proof. If R is not reduced, there is $x \in R$ with $x^n = 0$. This implies $\left(\frac{x}{1}\right)^n = \frac{x^n}{1} = 0$ in $U^{-1}R$. Hence, also $U^{-1}R$ is not reduced.

Suppose that R is reduced, and assume that $U^{-1}R$ is not reduced. Then there is $\frac{a}{b} \in U^{-1}R$ such that $\left(\frac{a}{b}\right)^n = 0$ for some $n \in \mathbb{N}$. Thus, there is an $s \in U$ such that $sa^n = 0$. This yields

$$(sa)^n = s^{n-1}(sa^n) = s^{n-1}0 = 0.$$

Hence, sa is a nilpotent element of R. But this is a contradiction as R is reduced. □

Lemma A.28. *Let R be a ring, let U be a multiplicatively closed set, and let A be a ring. If there is a map $\psi\colon R \to A$ with $\psi(U) \subset A^*$, then the unique ring homomorphism $\phi\colon U^{-1}R \to A$ making Diagram* (A.1) *commutative is given by*

$$\frac{x}{y} \mapsto \psi(x)(\psi(y))^{-1}$$

for any $\frac{x}{y} \in U^{-1}R$.

Proof. Let $\frac{x}{y} \in U^{-1}R$, and let $\alpha\colon R \to U^{-1}R$ be the map $x \mapsto \frac{x}{1}$ for $x \in R$. Since $y \in U$, Diagram (A.1) yields

$$\beta(x) = \phi \circ \alpha(x) = \phi\left(\frac{x}{1}\right) = \phi\left(\frac{x}{y}\frac{y}{1}\right) = \phi\left(\frac{x}{y}\right)\phi\left(\frac{y}{1}\right) = \phi\left(\frac{x}{y}\right)\beta(y).$$

Since $\beta(U) \subset A^*$, the statement follows. □

Lemma A.29. *Let R and A be rings, let U, respectively V, be a multiplicatively closed subset of R, respectively A, and let $\alpha\colon R \to A$ with $\alpha(U) \subset V$. Then there is a natural ring homomorphism*

$$\begin{aligned} \epsilon\colon U^{-1}R &\to V^{-1}A, \\ \frac{x}{y} &\mapsto \frac{\alpha(x)}{\alpha(y)}, \end{aligned}$$

such that the diagram

$$\begin{tikzcd} R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & U^{-1}R \arrow[d, dotted, "\epsilon"] \\ A \arrow[r, "\gamma"] & V^{-1}A \end{tikzcd}$$

commutes, where $\beta\colon R \to U^{-1}R$ and $\gamma\colon A \to V^{-1}A$ are the localization maps.

Proof. The localization maps $\beta\colon R \to U^{-1}R$ and $\gamma\colon A \to V^{-1}A$ fit into a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & U^{-1}R \\ A \arrow[r, "\gamma"'] & V^{-1}A, \end{tikzcd}$$

where $\delta = \gamma \circ \alpha$. Since

$$\begin{aligned} \delta(U) &= \gamma \circ \alpha(U) \\ &= \gamma(\alpha(U)) \\ &\subset \gamma(V) \\ &\subset \left(V^{-1}A\right)^* \end{aligned}$$

by assumption, the universal property of localization yields a unique homomorphism $\epsilon\colon U^{-1}R \to V^{-1}A$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & Q_R \arrow[d, dotted, "\epsilon"] \\ A \arrow[r, "\gamma"'] & Q_A \end{tikzcd}$$

commutes. The explicit representation of ϵ follows from Lemma A.28. $\square$

Lemma A.30. *Let R and A be rings, let $\alpha\colon R \to A$ be an injective ring homomorphism, and let U be a multiplicatively closed subset of R such that $\alpha(U) \subset A^{\mathrm{reg}}$. Then there is an injective ring homomorphism*

$$\begin{aligned} \phi\colon U^{-1}R &\to Q_A, \\ \frac{x}{y} &\to \frac{\alpha(x)}{\alpha(y)}. \end{aligned}$$

Proof. Lemma A.29 yields a ring homomorphism

$$\begin{aligned}\phi\colon U^{-1}R &\to Q_A,\\ \frac{x}{y} &\to \frac{\alpha(x)}{\alpha(y)}.\end{aligned}$$

Let $\frac{x}{y}, \frac{x'}{y'} \in U^{-1}R$ such that $\phi\left(\frac{x}{y}\right) = \phi\left(\frac{x'}{y'}\right)$. Then $\frac{\alpha(x)}{\alpha(y)} = \frac{\alpha(x')}{\alpha(y')}$, i.e.

$$\alpha(xy') = \alpha(x)\alpha(y') = \alpha(x')\alpha(y) = \alpha(x'y).$$

Since α is injective, this implies $xy' = x'y$. Hence $\frac{x}{y} = \frac{x'}{y'}$, and therefore ϕ is injective. $\square$

Lemma A.31. *Let R be a ring, and let U be a multiplicatively closed subset of R. Then there are natural ring homomorphisms*

$$\begin{aligned}\epsilon\colon Q_R &\to Q_{U^{-1}R},\\ \frac{x}{y} &\mapsto \frac{x/1}{y/1},\end{aligned}$$

and

$$\begin{aligned}\eta\colon U^{-1}Q_R &\to Q_{U^{-1}R},\\ \frac{x/y}{z} &\mapsto \frac{x/z}{y/1}.\end{aligned}$$

These homomorphisms fit into a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Q_R \arrow[d, dotted, "\epsilon"] \arrow[dr, "\zeta"] & \\
U^{-1}R \arrow[r, "\gamma"] \arrow[rr, "\theta"'] & Q_{U^{-1}R} & U^{-1}Q_R, \arrow[l, dotted, "\eta"']
\end{tikzcd}$$

where $\alpha\colon R \to U^{-1}R$, $\beta\colon R \to Q_R$ and $\gamma\colon U^{-1}R \to Q_{U^{-1}R}$ are the localization maps, and $\theta = U^{-1}\alpha\colon U^{-1}R \to U^{-1}Q_R$.

Proof. By Corollary A.25 and Lemma A.29 there is a natural ring homomorphism

$$\begin{aligned}\epsilon\colon Q_R &\to Q_{U^{-1}R},\\ \frac{x}{y} &\mapsto \frac{x/1}{y/1},\end{aligned}$$

such that the diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Q_R \arrow[d, dotted, "\epsilon"] \\
U^{-1}R \arrow[r, "\gamma"] & Q_{U^{-1}R}
\end{tikzcd}$$

commutes. With the localization map $\zeta\colon Q_R \to U^{-1}Q_R$ this leads to a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & Q_R \arrow[d, dotted, "\epsilon"] \arrow[ddr, "\zeta"] & \\
U^{-1}R \arrow[r, "\gamma"'] & Q_{U^{-1}R} & \\
& & U^{-1}Q_R.
\end{tikzcd}$$

Now we have

$$\epsilon \circ \beta(U) = \gamma \circ \alpha(U) \subset \gamma\Big(\big(U^{-1}R\big)^*\Big) \subset (Q_{U^{-1}R})^*.$$

Since $U^{-1}Q_R = (\beta(U))^{-1}Q_R$ by Theorem A.22, the universal property of localization yields a unique homomorphism $\eta\colon U^{-1}Q_R \to Q_{U^{-1}R}$ such that the diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & Q_R \arrow[d, dotted, "\epsilon"] \arrow[ddr, "\zeta"] & \\
U^{-1}R \arrow[r, "\gamma"'] & Q_{U^{-1}R} & \\
& & U^{-1}Q_R \arrow[ul, dotted, "\eta"']
\end{tikzcd}$$

commutes.

Now consider the localized map $\theta = U^{-1}\beta\colon U^{-1}R \to U^{-1}Q_R$ (see Theorem A.22). For any $\frac{x}{y} \in U^{-1}R$ we have

$$\begin{aligned}
\eta \circ \theta\Big(\frac{x}{y}\Big) &= \eta\Big(\frac{\beta(x)}{y}\Big) \\
&= \eta\Big(\beta(x)(\beta(y))^{-1}\Big) \\
&= \epsilon(\beta(x))(\epsilon(\beta(y)))^{-1} \\
&= \delta(x)(\delta(y))^{-1} \\
&= \gamma \circ \alpha(x)(\gamma \circ \alpha(y))^{-1} \\
&= \gamma(\alpha(x))\gamma\Big((\alpha(y))^{-1}\Big) \\
&= \gamma\Big(\frac{x}{y}\Big).
\end{aligned}$$

Therefore, we obtain a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & Q_R \arrow[d, dotted, "\epsilon"] \arrow[ddr, "\zeta"] & \\
U^{-1}R \arrow[r, "\gamma"] \arrow[drr, "\theta"'] & Q_{U^{-1}R} & \\
 & & U^{-1}Q_R. \arrow[ul, dotted, "\eta"']
\end{tikzcd}$$

□

Lemma A.32. *Let R be a ring, and let U be a multiplicatively closed subset of R. If $\frac{x}{y} \in (U^{-1}R)^{\mathrm{reg}}$ implies $x \in R^{\mathrm{reg}}$ or $x \in U$ for any $\frac{x}{y} \in Q_R$, then $Q_{U^{-1}R} = U^{-1}Q_R$.*

Proof. With the notation of Lemma A.31 we have a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & Q_R \arrow[d, dotted, "\epsilon"] \arrow[ddr, "\zeta"] & \\
U^{-1}R \arrow[r, "\gamma"] \arrow[drr, "\theta"'] & Q_{U^{-1}R} & \\
 & & U^{-1}Q_R. \arrow[ul, dotted, "\eta"']
\end{tikzcd}$$

Let now $\frac{x}{y} \in (U^{-1}R)^{\mathrm{reg}}$. Then $x \in R^{\mathrm{reg}}$ or $x \in U$ by assumption. If $x \in R^{\mathrm{reg}}$, then $\beta(x) \in (Q_R)^*$, and hence $\theta\left(\frac{x}{y}\right) = \frac{\beta(x)}{y} \in (U^{-1}Q_R)^*$ with inverse $\frac{(\beta(x))^{-1}y}{1}$. If $x \in U$, then $\theta\left(\frac{x}{y}\right) = \frac{\beta(x)}{y} \in (U^{-1}Q_R)^*$ with inverse $\frac{\beta(y)}{x}$. Therefore, the universal property of localization yields a unique homomorphism $\iota\colon Q_{U^{-1}R} \to U^{-1}Q_R$ such that the diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] \arrow[dr, "\delta"] & Q_R \arrow[d, dotted, "\epsilon"] \arrow[ddr, bend left, "\zeta"] & \\
U^{-1}R \arrow[r, "\gamma"] \arrow[drr, bend right, "\theta"'] & Q_{U^{-1}R} \arrow[dr, dotted, shift right, "\iota"'] & \\
 & & U^{-1}Q_R. \arrow[ul, dotted, shift right, "\eta"']
\end{tikzcd}$$

commutes. Using the universal property of the localization $U^{-1}Q_R$ of Q_R, respectively the localization $Q_{U^{-1}R}$ of $U^{-1}R$, we obtain $Q_{U^{-1}R} = U^{-1}Q_R$. □

Remark A.33. Note that Lemma A.32 holds, in particular, if R is a domain.

Lemma A.34. *Let R and A be rings such that $R \subset A \subset Q_R$. Then $Q_A = Q_R$.*

Proof. Since $A \subset Q_R$, we have $A^{\mathrm{reg}} \subset Q_R^{\mathrm{reg}} = Q_R^*$. Therefore, the universal property of localization (see Diagram (A.1)) yields a unique homomorphism $Q_A \to Q_R$.

Since $R \subset A$, we have $R^{\mathrm{reg}} \subset A^{\mathrm{reg}} \subset Q_A^*$. So again the universal property of localization yields a unique homomorphism $Q_R \to Q_A$. Thus, we obtain a commutative diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[drr] & A \arrow[r, hook] \arrow[dr] & Q_R \arrow[d, dotted, shift right] \\ & & Q_A. \arrow[u, dotted, shift right] \end{tikzcd}$$

Using the universal property of the localization Q_R over R, respectively the localization Q_A over A, this implies $Q_A = Q_R$. □

Lemma A.35. *Let R be a ring, let U be a multiplicatively closed subset of R, and let M be an R-module. We denote by $\alpha\colon M \to U^{-1}M$ the localization map.*

(1) For any R-submodule N of M we have

$$\alpha^{-1}\Big(U^{-1}N\Big) = \{x \in M \mid ux \in N \text{ for some } u \in U\}.$$

(2) For any $U^{-1}R$-submodule P of $U^{-1}M$ we have

$$U^{-1}\Big(\alpha^{-1}(P)\Big) = P.$$

Proof. (1) Write $N' = \{x \in M \mid ux \in N \text{ for some } u \in U\}$. If $x \in N'$, then there is a $u \in U$ such that $xu \in N$. Thus,

$$\alpha(x) = \frac{x}{1} = \frac{ux}{u} \in U^{-1}N.$$

Let now $y \in \alpha^{-1}(U^{-1}N)$. Then there is an $n \in N$ and a $u \in U$ such that $\frac{y}{1} = \alpha(y) = \frac{n}{u}$. Thus, there is a $v \in U$ such that $vuy = nu$. In particular, we have $vu \in U$ since U is multiplicatively closed, and $nu \in N$ since $U \subset R$ and N is an R-module. This yields $y \in N'$.

(2) Obviously, we have $P \subset U^{-1}(\alpha^{-1}(P))$. So let $\frac{x}{y} \in U^{-1}(\alpha^{-1}(P))$. Then $\frac{x}{1} = \alpha(x) \in (P)$ and $y \in U$. Since $\frac{1}{y} \in U^{-1}R$, this implies $\frac{x}{y} \in P$ as P is a $U^{-1}R$-module. □

Theorem A.36. *Let R be ring, let $\mathfrak{i}$ be an ideal of R, let $\pi\colon R \to R/\mathfrak{i}$ be the canonical surjection, and let U be a multiplicatively closed subset of R. Then*

$$U^{-1}R/\mathfrak{i}U^{-1}R = (\pi\,(U))^{-1}(R/\mathfrak{i}).$$

Proof. See [44, Theorem 4.2]. □

Lemma A.37. *Let R and A be rings such that $R \subset A \subset Q_R$. If $\mathfrak{i}$ is an ideal of R, then $\mathfrak{i}A$ is an ideal of A. Moreover, for any $x \in \mathfrak{i}A$ there are $x_1 \in \mathfrak{i}$ and $x_2 \in R^{\mathrm{reg}}$ such that $x = \frac{x_1}{x_2}$ (considered in Q_R, see Remark A.19.(2)).*

Proof. By Proposition A.10.(1) $\mathfrak{i}A$ is an ideal of A. Also note that for any $y \in A$ there are $y_1 \in R$ and $y_2 \in R^{\mathrm{reg}}$ such that $y = \frac{y_1}{y_2}$ in Q_R. Let $z \in \mathfrak{i}$. Then

$$zy = z\frac{y_1}{y_2} = \frac{zy_1}{y_2},$$

where $zy_1 \in \mathfrak{i}$ and $y_2 \in R^{\mathrm{reg}}$.

So if $x \in \mathfrak{i}A$, there is a finite set I, and for all $i \in I$ there are $x_1^{(i)} \in \mathfrak{i}$ and $x_2^{(i)} \in R^{\mathrm{reg}}$ such that

$$x = \sum_{i\in I} \frac{x_1^{(i)}}{x_2^{(i)}}.$$

Since $Q_R = (R^{\mathrm{reg}})^{-1}R$, we obtain

$$x = \sum_{i\in I} \frac{x_1^{(i)}}{x_2^{(i)}} = \sum_{i\in I} \frac{x_1^{(i)} \prod_{j\in I\setminus\{i\}} x_2^{(j)}}{\prod_{k\in I} x_2^{(k)}} = \frac{\sum_{i\in I} x_1^{(i)} \prod_{j\in I\setminus\{i\}} x_2^{(j)}}{\prod_{k\in I} x_2^{(k)}},$$

where

$$\sum_{i\in I} x_1^{(i)} \prod_{j\in I\setminus\{i\}} x_2^{(j)} \in \mathfrak{i}$$

and

$$\prod_{i\in I} x_2^{(i)} \in R^{\mathrm{reg}}. \qquad \square$$

Proposition A.38. *Let R be a ring, let U be a multiplicatively closed subset of R, and let M be an R-module. The natural map $M \otimes_R U^{-1}R \to U^{-1}M$ sending $\frac{x}{u} \otimes m$ to $\frac{xm}{u}$ is an isomorphism.*

Proof. See [39, Lemma 2.4]. $\square$

Proposition A.39. *Let R be a ring, and let U be a multiplicatively closed subset of R. Then the ring $U^{-1}R$ is flat as an R-module.*

Proof. See [39, Proposition 2.5]. $\square$

Proposition A.40. *Let R be a ring, and let A be an R-algebra. If M and N are R-modules, then there is a unique A-module homomorphism*

$$\alpha\colon \operatorname{Hom}_R(M,N) \otimes_R A \to \operatorname{Hom}_A(M \otimes_R A, N \otimes_R A)$$

that takes an element $\phi \otimes 1 \in \operatorname{Hom}_R(M,N) \otimes_R A$ to the A-module homomorphism $\phi \otimes_R 1\colon M \otimes_R A \to N \otimes A$ in $\operatorname{Hom}_A(M \otimes_R A, N \otimes_R A)$. If A is flat over R and M is finitely presented, then α is an isomorphism. In particular, if M is finitely presented, then $\operatorname{Hom}_R(M,N)$ localizes in the sense that the map α provides a natural isomorphism

$$\operatorname{Hom}_{U^{-1}R}\left(U^{-1}M, U^{-1}N\right) \cong U^{-1}\operatorname{Hom}_R(M,N)$$

for any multiplicatively closed subset U of R.

Proof. See [39, Proposition 2.10]. □

Remark A.41. If R is a Noetherian ring, an R-module M is finitely presented if and only if M is finitely generated, see [39, p. 68]. In particular, every ideal of a Noetherian ring R is finitely presented as an A-module.

Lemma A.42. *Let R be a semilocal ring, and let M be an R-module such that $M_\mathfrak{m}$ is a finite $R_\mathfrak{m}$-module for every $\mathfrak{m} \in \operatorname{Max}(R)$. Then M is a finite R-module.*

Proof. For any $\mathfrak{m} \in \operatorname{Max}(R)$ there is a finite subset $X_\mathfrak{m}$ of $M_\mathfrak{m}$ such that $M_\mathfrak{m}$ is generated as an $R_\mathfrak{m}$-module by the elements in $X_\mathfrak{m}$. By clearing denominators of the elements in $X_\mathfrak{m}$ we find a finite subset $U_\mathfrak{m}$ of M such that $M_\mathfrak{m} = \left\langle \frac{x}{1} \mid x \in U_\mathfrak{m} \right\rangle_{R_\mathfrak{m}}$.

Set $U = \bigcup_{\mathfrak{m} \in \operatorname{Max}(R)} U_\mathfrak{m}$, and let $N = \langle U \rangle_R$ be the R-submodule of M generated by the elements in U. Then we have $N_\mathfrak{m} = M_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$. This yields $M = N$ since equality is a local property. In particular, M is finitely generated as an R-module by the elements in U. □

A.3. Associated and Minimal Prime Ideals

Definition A.43. Let R be a ring. A prime ideal $\mathfrak{p}$ of R is called *minimal* if $\operatorname{height}\mathfrak{p} = 0$. The set of minimal prime ideals of R is denoted by $\operatorname{Min}(R)$.

Definition A.44. Let R be a Noetherian ring, and let M be an R-module.

(1) The *annihilator* of M is

$$\operatorname{Ann}(M) = \{x \in R \mid x\mathfrak{M} = 0\}.$$

(2) A prime ideal $\mathfrak{p}$ of R is *associated* to M if $\mathfrak{p}$ is the annihilator of an element of M. The set of prime ideals associated to M is denoted by $\operatorname{Ass}(M)$.

Theorem A.45. *Let R be a Noetherian ring, and let M be a finite non-zero R-module.*

(1) $\operatorname{Ass}(M)$ is a finite, non-empty set of prime ideals of R, each containing $\operatorname{Ann}(M)$. The set $\operatorname{Ass}(M)$ includes all the prime ideals which are minimal among the prime ideals containing $\operatorname{Ann}(M)$.

(2) The union of the associated prime ideals of M consists of 0 and the set of zerodivisors on M.

(3) The formation of the set $\operatorname{Ass}(M)$ commutes with localization at an arbitrary multiplicatively closed set $U \subset R$, in the sense that

$$\operatorname{Ass}\left(U^{-1}M\right) = \left\{\mathfrak{p}U^{-1}R \;\middle|\; \mathfrak{p} \in \operatorname{Ass}(M) \text{ and } \mathfrak{p} \cap U = \emptyset\right\}.$$

Proof. See [39, Theorem 3.1]. □

Corollary A.46. *Let R be a Noetherian ring. Then* $\operatorname{Min}(R) \subset \operatorname{Ass}(R)$*. In particular,* $\operatorname{Min}(R)$ *is finite.*

Proof. Since $\operatorname{Ann}(R) = \langle 0 \rangle$, and since any ideal of R contains 0, this follows from Theorem A.45.(1). □

Proposition A.47. *Let R be a reduced Noetherian ring. Then*

$$\operatorname{Ass}(R) = \operatorname{Min}(R).$$

Moreover,

$$\bigcup_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p} = R \setminus R^{\mathrm{reg}}.$$

Proof. By Corollary A.46 we have $\operatorname{Min}(R) \subset \operatorname{Ass}(R)$. So assume there are $\mathfrak{p}, \mathfrak{q} \in \operatorname{Ass}(R)$ with $\mathfrak{p} \subsetneq \mathfrak{q}$. Then there is $x \in R$ such that $x\mathfrak{q} = \operatorname{Ann}(R) = \langle 0 \rangle$, and hence $x\mathfrak{p} = \langle 0 \rangle$. Let $y \in \mathfrak{q} \setminus \mathfrak{p}$. Then $xy = 0 \in \mathfrak{p}$ implies $x \in \mathfrak{p}$. But then $x^2 \subset x\mathfrak{p} = \langle 0 \rangle$ which contradicts the reducedness of R. This implies

$$\operatorname{Ass}(R) = \operatorname{Min}(R),$$

and with Theorem A.45.(2) we obtain

$$R \setminus R^{\mathrm{reg}} = \bigcup_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p}.$$ □

Lemma A.48. *Let R be a reduced Noetherian ring. Then*

$$\bigcap_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p} = \langle 0 \rangle,$$

and

$$\langle 0 \rangle \subsetneq \bigcap_{\mathfrak{q} \in \operatorname{Min}(R) \setminus \{\mathfrak{p}\}} \mathfrak{q}$$

for any $\mathfrak{p} \in \operatorname{Min}(R)$*.*

Proof. This follows from Proposition A.47 and Primary Decomposition, see [39, Theorem 3.10]. □

Lemma A.49. *Let R be a reduced Noetherian ring, let* $\mathfrak{p} \in \operatorname{Min}(R)$*, and let* $x \in R \setminus \mathfrak{p}$*. Then* $(x + \mathfrak{p}) \cap R^{\mathrm{reg}} \neq \emptyset$*.*

Proof. If $x \in R^{\mathrm{reg}}$, the statement follows since $0 \in \mathfrak{p}$. So suppose that $x \in R \setminus R^{\mathrm{reg}} = \bigcup_{\mathfrak{q} \in \operatorname{Min}(R)} \mathfrak{q}$, see Proposition A.47. Then the subset

$$I = \{\mathfrak{q} \in \operatorname{Min}(R) \mid x \in \mathfrak{q}\}$$

of $\operatorname{Min}(R)$ is non-empty, and

$$x \in \bigcap_{\mathfrak{q} \in I} \setminus \bigcup_{\mathfrak{q}' \in \operatorname{Min}(R) \setminus I} \mathfrak{q}'. \tag{A.2}$$

Assume now that

$$\bigcap_{\mathfrak{q}\in \operatorname{Min}(R)\setminus I} \mathfrak{q} \subset \bigcup_{\mathfrak{q}'\in I} \mathfrak{q}'.$$

Then there is by Theorem A.2 a $\mathfrak{p}' \in I$ such that $\bigcap_{\mathfrak{q}\in \operatorname{Min}(R)\setminus I} \mathfrak{q} \subset \mathfrak{p}'$. This implies $\bigcap_{\mathfrak{q}\in \operatorname{Min}(R)} \mathfrak{q} = \bigcap_{\mathfrak{q}'\in \operatorname{Min}(R)\setminus\{\mathfrak{p}'\}} \mathfrak{q}'$, contradicting Lemma A.48. Thus, there is a

$$y \in \bigcap_{\mathfrak{q}\in \operatorname{Min}(R)\setminus I} \mathfrak{q} \setminus \bigcup_{\mathfrak{q}'\in I} \mathfrak{q}'. \tag{A.3}$$

Then Lemma A.13, Proposition A.47, and Equations (A.2) and (A.3) yield

$$\begin{aligned} x + y &\in R \setminus \left(\left(\bigcup_{\mathfrak{q}\in I} \mathfrak{q} \right) \cup \left(\bigcup_{\mathfrak{q}'\in \operatorname{Min}(R)\setminus I} \mathfrak{q}' \right) \right) \\ &= R \setminus \bigcup_{\mathfrak{q}\in \operatorname{Min}(R)} \mathfrak{q} \\ &= R^{\mathrm{reg}}. \end{aligned}$$

The claim follows since $\mathfrak{p} \in \operatorname{Min}(R) \setminus I$ by assumption, and hence $y \in \mathfrak{p}$. □

Lemma A.50. *Let R be a reduced Noetherian ring, and let $\mathfrak{p} \in \operatorname{Min}(R)$. Then*

$$Q_{R/\mathfrak{p}} = Q_R/\mathfrak{p}Q_R.$$

Proof. Since $\mathfrak{p} \cap R^{\mathrm{reg}} = \emptyset$ by Proposition A.47, and since $\mathfrak{p}$ is a prime ideal, we have $\operatorname{pr}(R^{\mathrm{reg}}) \subset (R/\mathfrak{p})^{\mathrm{reg}} = (R/\mathfrak{p}) \setminus \{0\}$, where $\pi_{\mathfrak{p}}\colon R \to R/\mathfrak{p}$ is the canonical surjection. Thus, Lemma A.49 implies $\pi_{\mathfrak{p}}(R^{\mathrm{reg}}) = (R/\mathfrak{p})^{\mathrm{reg}}$, and hence the statement follows from Theorem A.36. □

Proposition A.51. *Let R be a reduced Noetherian ring. Then*

$$\operatorname{Min}(Q_R) = \operatorname{Max}(Q_R).$$

Proof. Let $\mathfrak{i}$ be a proper ideal of Q_R. Then $\mathfrak{i} \subset Q_R \setminus Q_R^*$. Since Q_R is reduced by Lemma A.27, and since $Q_R^* = Q_R^{\mathrm{reg}}$ (see Remark A.19.(3)), Proposition A.47 yields

$$\mathfrak{i} \subset Q_R \setminus Q_R^{\mathrm{reg}} = \bigcup_{\mathfrak{p}\in \operatorname{Min}(Q_R)} \mathfrak{p}.$$

Then Theorem A.2 implies that there is a $\mathfrak{p} \in \operatorname{Min}(R)$ containing $\mathfrak{i}$. The claim follows. □

A.4. Completion

For the definition of ideal-adic topologies and completions of rings and modules see for example [45, Chapter II] or [42, Chapter 9].

Theorem A.52. *Let R and A be semilocal rings such that A is a finite R-module. Then the topology of A as a semilocal ring coincides with the topology of A as a finite R-module.*

Proof. See [45, Theorem 16.8]. □

Theorem A.53. *Let R be a Noetherian ring, and let $\mathfrak{i}$ be an ideal of R. Then the $\mathfrak{i}$-adic completion $\widehat{R}$ of R is Noetherian.*

Proof. See [40, Theorem 10.26]. □

Theorem A.54. *Let R be a ring, and let $\mathfrak{i}$ be an ideal of R. Then the $\mathfrak{i}$-adic completion $\widehat{R}$ of R is flat over R.*

Proof. See [44, Theorem 8.8]. □

Theorem A.55. *Let R be a Noetherian ring, let $\mathfrak{i}$ be an ideal of R, and let M be a finite R-module. Writing $\widehat{M}$ and $\widehat{R}$ for the $\mathfrak{i}$-adic completions of M and R we have*

$$\widehat{M} = M \otimes \widehat{R},$$

and the topology of $\widehat{M}$ as the completion of M coincides with its topology as a finite $\widehat{R}$-module.

In particular, if R is complete, so is M.

Proof. See [44, Theorem 8.7]. □

Theorem A.56. *Let R be a ring, let $\mathfrak{i}$ be an ideal of R, and let $\widehat{R}$ be the $\mathfrak{i}$-adic completion of R. Then there is a bijection between the set of regular ideals of R and the set of regular ideals of $\widehat{R}$ given by*

$$\begin{aligned} \mathfrak{I} &\mapsto \mathfrak{I}\widehat{R}, \\ \mathfrak{J} \cap R &\leftarrow\!\shortmid \mathfrak{J}. \end{aligned}$$

Proof. See [29, Theorem 2.8]. □

Definition A.57. A *Zariski ring* is a Noetherian ring R whose topology is defined by an ideal $\mathfrak{i} \subset \mathfrak{j}_R$, where $\mathfrak{j}_R$ is the Jacobson radical of R.

Remark A.58. A Noetherian semilocal ring is a Zariski ring, see [42, 24.B].

Theorem A.59. *Let R be a Zariski ring (with respect to an ideal $\mathfrak{i} \subset \mathfrak{j}_R$), and let $\widehat{R}$ be the $\mathfrak{i}$-adic completion of R.*

(1) R is a subring of $\widehat{R}$.

(2) There is a bijection

$$\begin{aligned} \operatorname{Max}(R) &\to \operatorname{Max}\left(\widehat{R}\right), \\ \mathfrak{m} &\mapsto \mathfrak{m}\widehat{R}, \\ \mathfrak{n} \cap R &\leftarrow\!\shortmid \mathfrak{n}. \end{aligned}$$

In particular, if R is local, then $\widehat{R}$ is local.

(3) *For any* $\mathfrak{m} \in \operatorname{Max}(R)$ *we have* $R/\mathfrak{m} \cong \widehat{R}/\mathfrak{m}\widehat{R}$.

Proof. See [42, Corollary of Theorem 56]. □

Theorem A.60. *Let* R *be a Noetherian ring, and let* $\mathfrak{i}$ *be an ideal of* R*. Then the* $\mathfrak{i}$*-adic completion* $\widehat{R}$ *of* R *is faithfully flat over* R *if and only if* $\mathfrak{i} \subset \mathfrak{j}_R$*, i.e.* R *is a Zariski ring.*

Proof. See [44, Theorem 8.14]. □

Theorem A.61. *Let* R *be a semilocal ring. Then*

$$\widehat{R} = \prod_{\mathfrak{m} \in \operatorname{Max}(R)} \widehat{R_{\mathfrak{m}}}.$$

Proof. See [45, Theorem 17.7]. □

Corollary A.62. *Let* R *be a semilocal Noetherian ring. Then* $\widehat{R_{\mathfrak{m}}} = \widehat{R}_{\mathfrak{m}\widehat{R}}$ *for any* $\mathfrak{m} \in \operatorname{Max}(R)$*. In particular,*

$$\widehat{R} = \prod_{\mathfrak{n} \in \operatorname{Max}\left(\widehat{R}\right)} \widehat{R}_{\mathfrak{n}}.$$

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$. Then $R_{\mathfrak{m}}$ is a Zariski ring, see Corollary A.21 and Remark A.58. Thus, $\widehat{R_{\mathfrak{m}}}$ is by Theorem A.59.(2) a local ring with maximal ideal $\mathfrak{m}R_{\mathfrak{m}}\widehat{R_{\mathfrak{m}}} = \mathfrak{m}\widehat{R_{\mathfrak{m}}}$, where the equality follows from Theorem A.59.(1).

By Theorem A.59.(2) we have $\mathfrak{m}\widehat{R} \in \operatorname{Max}\left(\widehat{R}\right)$, and Theorems A.61 and A.59.(1) yield

$$\begin{aligned}
\mathfrak{m}\widehat{R} &= \mathfrak{m} \prod_{\mathfrak{n} \in \operatorname{Max}(R)} \widehat{R_{\mathfrak{n}}} \\
&= \prod_{\mathfrak{n} \in \operatorname{Max}(R)} \mathfrak{m}\widehat{R_{\mathfrak{n}}} \\
&= \mathfrak{m}\widehat{R_{\mathfrak{m}}} \times \prod_{\mathfrak{n} \in \operatorname{Max}(R) \setminus \{\mathfrak{m}\}} \mathfrak{m}R_{\mathfrak{n}}\widehat{R_{\mathfrak{n}}} \\
&= \mathfrak{m}\widehat{R_{\mathfrak{m}}} \times \prod_{\mathfrak{n} \in \operatorname{Max}(R) \setminus \{\mathfrak{m}\}} R_{\mathfrak{n}}\widehat{R_{\mathfrak{n}}} \\
&= \mathfrak{m}\widehat{R_{\mathfrak{m}}} \times \prod_{\mathfrak{n} \in \operatorname{Max}(R) \setminus \{\mathfrak{m}\}} \widehat{R_{\mathfrak{n}}}.
\end{aligned}$$

Using again Theorem A.61 this implies $\widehat{R}_{\mathfrak{m}\widehat{R}} = \widehat{R_{\mathfrak{m}}}$. The particular claim follows from Theorem A.59.(2). □

Theorem A.63. *Let* R *be a Noetherian local ring, and let* M *be a finite* R*-module. Then* $\dim_R M = \dim_{\widehat{R}} \widehat{M}$.

Proof. See [38, Corollary 2.1.8]. □

Corollary A.64. *Let* R *be a Noetherian semilocal ring. Then* $\dim R = \dim \widehat{R}$.

Proof. Theorems A.59.(2) and A.63 and Corollary A.62 yield

$$\begin{aligned}\dim \widehat{R} &= \max_{\mathfrak{n}\in \operatorname{Max}\left(\widehat{R}\right)} \dim \widehat{R}_{\mathfrak{n}} \\ &= \max_{\mathfrak{m}\in \operatorname{Max}(R)} \dim \widehat{R_{\mathfrak{m}}} \\ &= \max_{\mathfrak{m}\in \operatorname{Max}(R)} \dim R \\ &= \dim R.\end{aligned}$$ □

Corollary A.65. *Let R be a Noetherian semilocal ring. Then* $\operatorname{height}\mathfrak{m} = \operatorname{height}\mathfrak{m}\widehat{R}$ *for every* $\mathfrak{m} \in \operatorname{Max}(R)$*.*

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$. Then Corollaries A.62 and A.64 yield

$$\operatorname{height}\mathfrak{m} = \dim R_{\mathfrak{m}} = \dim \widehat{R_{\mathfrak{m}}} = \dim \widehat{R}_{\mathfrak{m}\widehat{R}} = \operatorname{height}\mathfrak{m}\widehat{R}.$$ □

Definition A.66. Let A be a local ring with maximal ideal $\mathfrak{m}$. A subfield $k \subset A$ is called a *coefficient field* of A if $k \cong A/\mathfrak{m}$ under the canonical surjection $A \twoheadrightarrow A/\mathfrak{m}$.

Theorem A.67 (Cohen Structure Theorem)**.** *Let R be a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and let $\{x_1, \ldots, x_n\}$ be a set of generators for the maximal ideal $\mathfrak{m}$. If R contains a field, then there is a surjective k-algebra homomorphism*

$$\begin{aligned}\pi\colon k[[X_1, \ldots, X_n]] &\twoheadrightarrow R, \\ X_i &\mapsto x_i \text{ for all } i = 1, \ldots, n.\end{aligned}$$

In particular,

$$R \cong k\left[[X_1, \ldots, X_n]\right]/\mathfrak{i},$$

where $\mathfrak{i} = \ker \pi \subset k[[X_1, \ldots, X_n]]$*, and R contains a coefficient field.*

Proof. See [39, Theorem 7.7]. □

Lemma A.68. *Let R be a semilocal ring. Then R is analytically reduced if and only if $R_{\mathfrak{m}}$ is analytically reduced for every* $\mathfrak{m} \in \operatorname{Max}(R)$*.*

Proof. Let R be analytically reduced, i.e. its completion $\widehat{R}$ is reduced. By Lemma A.27 this is equivalent to $\widehat{R}_{\widehat{\mathfrak{m}}}$ being reduced for every $\widehat{\mathfrak{m}} \in \operatorname{Max}\left(\widehat{R}\right)$. Since $\widehat{R}_{\widehat{\mathfrak{m}}} = \widehat{R_{\widehat{\mathfrak{m}}\cap R}}$ for every $\widehat{\mathfrak{m}} \in \operatorname{Max}\left(\widehat{R}\right)$ by Theorem A.59.(2) and Corollary A.62, the statement follows with Theorem A.59.(2). □

A.5. Branches of Rings

Definition A.69. Let R be a ring. A *branch* of R is a quotient ring $R/\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Min}(R)$.

Lemma A.70. *Let R be a reduced Noetherian ring, and let A be a ring with $R \subset A \subset Q_R$. If $\mathfrak{p} \in \mathrm{Min}(R)$, then $\mathfrak{p}A \in \mathrm{Spec}(A)$.*

Proof.

By Proposition A.10.(1) $\mathfrak{p}A$ is an ideal of A. So let $x, y \in A$ such that $xy \in \mathfrak{p}A$. By Lemma A.37, and since $A \subset Q_R$, there are $z_1 \in \mathfrak{p}$, $x_1, y_1 \in R$ and $z_2, x_2, y_2 \in R^{\mathrm{reg}}$ such that

$$xy = \frac{x_1 y_1}{x_2 y_2} = \frac{z_1}{z_2}.$$

This implies

$$x_1 y_1 z_2 = x_2 y_2 z_1 \in \mathfrak{p}.$$

Since $\mathfrak{p}$ is a prime ideal of R, this implies $x_1 y_1 \in \mathfrak{p}$ as otherwise $z_2 \in \mathfrak{p} \cap R^{\mathrm{reg}} = \emptyset$ (see Proposition A.47). Hence, we have $x_1 \in \mathfrak{p}$ or $y_1 \in \mathfrak{p}$. This yields $x \in \mathfrak{p}A$ or $y \in \mathfrak{p}A$, and thus $\mathfrak{p}A$ is a prime ideal of A. □

Lemma A.71. *Let R be a reduced Noetherian ring, and let A be a ring with $R \subset A \subset Q_R$.*

(1) Let $\mathfrak{p} \in \mathrm{Min}(R)$. Then $\mathfrak{p}A \in \mathrm{Min}(A)$.

(2) Let $\mathfrak{q} \in \mathrm{Min}(A)$. Then $\mathfrak{q} \cap R \in \mathrm{Min}(R)$.

Proof. (1) Let $\mathfrak{p} \in \mathrm{Min}(R)$. Then $\mathfrak{p}A$ is by Lemma A.70 a prime ideal of A. Assume that $\mathfrak{p} \notin \mathrm{Min}(A)$. Then there is a prime ideal $\mathfrak{q}$ of A such that

$$\mathfrak{q} \subsetneq \mathfrak{p}A. \tag{A.4}$$

By Proposition A.10.(2) $\mathfrak{p}' = \mathfrak{q} \cap R$ is a prime ideal of R with $\mathfrak{p}' \subset \mathfrak{p}$. Since $\mathfrak{p} \in \mathrm{Min}(R)$, this implies $\mathfrak{p}' = \mathfrak{p}$. However, this yields with Equation (A.4) the contradiction

$$\mathfrak{p}A = \mathfrak{p}'A \subset \mathfrak{q}A = \mathfrak{q} \subsetneq \mathfrak{p}A.$$

Therefore, $\mathfrak{p}A \in \mathrm{Min}(A)$.

(2) Let $\mathfrak{q} \in \mathrm{Min}(A)$. Then $\mathfrak{q} \cap R$ by Proposition A.10.(2) a prime ideal of R. Assume that $\mathfrak{q} \cap R \notin \mathrm{Min}(R)$. Then there is a $\mathfrak{p} \in \mathrm{Min}(R)$ such that

$$\mathfrak{p} \subsetneq \mathfrak{q} \cap R. \tag{A.5}$$

By Proposition A.10.(2) and Lemma A.70 $\mathfrak{p}A$ and $(\mathfrak{q} \cap R)A$ are prime ideals of A, and

$$\mathfrak{p}A \subset (\mathfrak{q} \cap R)A \subset \mathfrak{q}.$$

Since $\mathfrak{q} \in \mathrm{Min}(A)$, this implies

$$\mathfrak{p}A = (\mathfrak{q} \cap R)A = \mathfrak{q}.$$

However, Equation (A.5) then yields the contradiction

$$\mathfrak{q} \cap R = \mathfrak{p}A \cap R = \mathfrak{p} \subsetneq \mathfrak{q} \cap R.$$

Hence, $\mathfrak{q} \cap R \in \mathrm{Min}(R)$. □

Theorem A.72. *Let R be a reduced Noetherian ring, and let A be a ring with $R \subset A \subset Q_R$. Then there is a bijection*

$$\begin{aligned} \operatorname{Min}(R) &\to \operatorname{Min}(A), \\ \mathfrak{p} &\mapsto \mathfrak{p}A, \\ \mathfrak{q} \cap R &\leftarrow\!\shortmid \mathfrak{q}. \end{aligned}$$

Proof. By Lemma A.71 there are maps

$$\begin{aligned} \operatorname{Min}(R) &\to \operatorname{Min}(A), \\ \mathfrak{p} &\mapsto \mathfrak{p}A \end{aligned}$$

and

$$\begin{aligned} \operatorname{Min}(A) &\to \operatorname{Min}(R), \\ \mathfrak{q} &\mapsto \mathfrak{q} \cap R. \end{aligned}$$

Moreover, Proposition A.10.(1) yields $\mathfrak{p}A \cap R = \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Min}(A)$.

Let now $\mathfrak{q} \in \operatorname{Min}(A)$. Then $\mathfrak{q} \cap R \in \operatorname{Min}(R)$ by Lemma A.71.(2). This implies by Lemma A.70 that $(\mathfrak{q} \cap R)A$ is a prime ideal of A, and

$$(\mathfrak{q} \cap R)A \subset \mathfrak{q}A = \mathfrak{q}$$

as $\mathfrak{q}$ is an ideal of A. Since $\mathfrak{q} \in \operatorname{Min}(A)$, this yields

$$(\mathfrak{q} \cap R)A = \mathfrak{q}.$$

Thus, we obtain the statement. □

Corollary A.73. *Let R be a reduced Noetherian ring, and let A be a ring with $R \subset A \subset Q_R$. If R is a domain, then A is a domain.* □

Theorem A.74. *Let R be a reduced Noetherian ring.*

(1) There is a bijection

$$\begin{aligned} \operatorname{Min}(R) &\to \operatorname{Max}(Q_R), \\ \mathfrak{p} &\mapsto \mathfrak{p}Q_R, \\ \mathfrak{m} \cap R &\leftarrow\!\shortmid \mathfrak{m} \end{aligned}$$

such that

$$Q_R/\mathfrak{p}Q_R = Q_{R/\mathfrak{p}}.$$

for any $\mathfrak{p} \in \operatorname{Min}(R)$.

(2) We have

$$Q_R = \prod_{\mathfrak{p} \in \operatorname{Min}(R)} Q_{R/\mathfrak{p}} = \prod_{\mathfrak{m} \in \operatorname{Max}(Q_R)} Q_R/\mathfrak{m}.$$

Proof. Also see [1, Proposition 1.4.27 and Theorem 1.5.20].

(1) This follows from Lemma A.50, Proposition A.51, and Theorem A.72.

(2) The Chinese Remainder Theorem implies

$$Q_R/\bigcap_{\mathfrak{m}\in\mathrm{Max}\,(Q_R)}\mathfrak{m}=\prod_{\mathfrak{m}\in\mathrm{Max}\,(R)}Q_R/\mathfrak{m}=\prod_{\mathfrak{p}\in\mathrm{Min}\,(R)}Q_{R/\mathfrak{p}},$$

where the second equality follows from (1). Since Q_R is by Corollary A.21 and Lemma A.27 Noetherian and reduced, Lemma A.48 and Proposition A.51 yield

$$\bigcap_{\mathfrak{m}\in\mathrm{Max}\,(Q_R)}\mathfrak{m}=\bigcap_{\mathfrak{p}\in\mathrm{Min}\,(Q_R)}\mathfrak{p}=\langle 0\rangle.$$

□

Corollary A.75. *Let R be a reduced Noetherian ring, and let $\mathfrak{m}\in\mathrm{Max}\,(Q_R)$. Then $\mathfrak{p}=\mathfrak{m}\cap R\in\mathrm{Min}\,(R)$, and*

$$\mathfrak{m}=0\times\prod_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}Q_{R/\mathfrak{q}}.$$

Proof. By Theorem A.74.(2),

$$Q_R=\prod_{\mathfrak{q}\in\mathrm{Min}\,(R)}Q_{R/\mathfrak{q}}$$

is a finite product of fields. Hence, there is by Lemma A.6.(2) a $\mathfrak{p}\in\mathrm{Min}\,(R)$ such that

$$\mathfrak{m}=0\times\prod_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}Q_{R/\mathfrak{q}}.$$

Moreover, since $\mathfrak{m}\cap R\in\mathrm{Min}\,(R)$ by Theorem A.74.(2), we obtain

$$\mathfrak{m}\cap R=\left(0\times\prod_{\mathfrak{q}\in\mathrm{Min}\,(R)\setminus\{\mathfrak{p}\}}Q_{R/\mathfrak{q}}\right)\cap R=\mathfrak{p}.$$

□

Lemma A.76. *Let R be a reduced Noetherian ring, let U be a multiplicatively closed subset of R, and let*

$$\begin{aligned}\phi\colon R&\to U^{-1}R,\\ x&\mapsto\frac{x}{1}.\end{aligned}$$

Then for any $\mathfrak{p}\in\mathrm{Spec}\,(R)$ with $\mathfrak{p}\cap U=\emptyset$ we have $Q_{U^{-1}R/\phi(\mathfrak{p})}=Q_{R/\mathfrak{p}}$. Moreover,

$$Q_{U^{-1}R}=\prod_{\substack{\mathfrak{p}\in\mathrm{Min}\,(R)\\ \mathfrak{p}\cap U=\emptyset}}Q_{R/\mathfrak{p}}.$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \cap U = \emptyset$. By Theorem A.36 we have

$$U^{-1}R/\phi(\mathfrak{p}) = (\pi(U))^{-1}(R/\mathfrak{p}),$$

where $\pi\colon R \to R/\mathfrak{p}$ is the canonical surjection. Since $R/\mathfrak{p}$ is a domain, and since $\pi(U)$ is a multiplicatively closed subset of $R/\mathfrak{p}$, Remark A.19.(1) and Lemma A.32 yield

$$Q_{U^{-1}R/\phi(\mathfrak{p})} = Q_{(\pi(U))^{-1}(R/\mathfrak{p})} = Q_{R/\mathfrak{p}}.$$

The second part of the claim follows from Proposition A.20.(2) and Theorem A.74.(2). □

Proposition A.77 (See [25], Section 3.2)**.** *Let R be a reduced Noetherian ring, and let U be a multiplicatively closed subset of R. Then*

$$Q_{U^{-1}R} = U^{-1}Q_R.$$

Proof. By Proposition A.38 and Theorem A.74.(2) we have

$$\begin{aligned} U^{-1}Q_R &= U^{-1}R \otimes_R Q_R \\ &= U^{-1}R \otimes_R \prod_{\mathfrak{p}\in \operatorname{Min}(R)} Q_{R/\mathfrak{p}} \\ &= \prod_{\mathfrak{p}\in \operatorname{Min}(R)} U^{-1}R \otimes_R Q_{R/\mathfrak{p}}. \end{aligned}$$

Let $\mathfrak{p} \in \operatorname{Min}(R)$. First suppose that $\mathfrak{p} \cap U \neq \emptyset$, i.e. there is $s \in \mathfrak{p} \cap U$. Since any element of $U^{-1}R \otimes_R Q_{R/\mathfrak{p}}$ is of the form $\frac{u}{v} \otimes \frac{x+\mathfrak{p}}{y+\mathfrak{p}}$ with $u, x \in R$, $v \in U$, and $y \in R \setminus \mathfrak{p}$, we obtain

$$\begin{aligned} \frac{u}{v} \otimes \frac{x+\mathfrak{p}}{y+\mathfrak{p}} &= \frac{su}{sv} \otimes \frac{x+\mathfrak{p}}{y+\mathfrak{p}} \\ &= \frac{u}{sv} \otimes \frac{s(x+\mathfrak{p})}{y+\mathfrak{p}} \\ &= \frac{u}{sv} \otimes \frac{sx+\mathfrak{p}}{y+\mathfrak{p}} \\ &= \frac{u}{sv} \otimes \frac{\mathfrak{p}}{y+\mathfrak{p}} \\ &= 0. \end{aligned}$$

Hence, $U^{-1}R \otimes_R Q_{R/\mathfrak{p}} = 0$.

Now suppose that $\mathfrak{p} \cap U = \emptyset$. Then $v \notin \mathfrak{p}$, and hence

$$\begin{aligned} \frac{u}{v} \otimes \frac{x+\mathfrak{p}}{y+\mathfrak{p}} &= \frac{1}{v} \otimes \frac{uvx+\mathfrak{p}}{vy+\mathfrak{p}} \\ &= \frac{v}{v} \otimes \frac{ux+\mathfrak{p}}{vy+\mathfrak{p}} \\ &= 1 \otimes \frac{ux+\mathfrak{p}}{vy+\mathfrak{p}}. \end{aligned}$$

This implies $U^{-1}R \otimes_R Q_{R/\mathfrak{p}} = Q_{R/\mathfrak{p}}$. Therefore, Lemma A.76 yields

$$U^{-1}Q_R = \prod_{\substack{\mathfrak{p}\in \mathrm{Min}\,(R)\\ \mathfrak{p}\cap U=\emptyset}} Q_{R/\mathfrak{p}} = Q_{U^{-1}R}. \qquad \square$$

Lemma A.78. *Let R be a complete Noetherian semilocal ring. Then for any $\mathfrak{p} \in \mathrm{Min}\,(R)$ there is a unique $\mathfrak{m} \in \mathrm{Max}\,(R)$ with $\mathfrak{p} \subset \mathfrak{m}$.*

Proof. Let $\mathfrak{p} \in \mathrm{Min}\,(R)$. Since

$$R = \prod_{\mathfrak{m}\in \mathrm{Max}\,(R)} R_{\mathfrak{m}}.$$

by Corollary A.62, there is by Lemma A.6.(1) an $\mathfrak{m} \in \mathrm{Max}\,(R)$ such that

$$\mathfrak{p} = \mathfrak{p}R_{\mathfrak{m}} \times \prod_{\mathfrak{n}\in \mathrm{Max}\,(R)\setminus\{\mathfrak{m}\}} R_{\mathfrak{n}}.$$

This implies $\mathfrak{p} \subset \mathfrak{m}$ and $\mathfrak{p} \not\subset \mathfrak{n}$ for every $\mathfrak{n} \in \mathrm{Max}\,(R) \setminus \{\mathfrak{m}\}$. $\square$

B. Integral Extensions and Normalization

B.1. Integral Extensions

Definition B.1. Let R be a ring, and let A be an R-algebra.

(1) An element $a \in A$ is called *integral over* R if there is a monic polynomial $p \in R[x]$ such that $p(a) = 0$.

(2) The *integral closure* of R in A is the ring of all elements of A which are integral over R.

(3) Suppose that A contains a copy of R as $R \cdot 1$. Then A is called an *integral extension* of R if every element of A is integral over R.

(4) We denote the integral closure of R in its total ring of fractions Q_R by $\overline{R}$.

(5) If R is reduced, then $\overline{R}$ is called the *normalization* of R. The ring R is said to be *normal* if $\overline{R} = R$.

Lemma B.2. *Let R be a ring, and let A be an integral extension of R. Then $R^* = A^* \cap R$.*

Proof. Let $x \in A \cap R$. Then there are $n \in \mathbb{N}$ and $a_0, \ldots, a_{n-1} \in R$ such that

$$x^{-n} + a_{n-1}x^{-(n-1)} + \ldots + a_0 = 0.$$

Multiplying by x^n, we obtain

$$1 + a_{n-1}x + \ldots + a_0 x^n = 0,$$

and hence

$$1 = x\left(a_{n-1} + \ldots + a_0 x^{n-1}\right).$$

This implies $x \in R^*$, since $x, a_0, \ldots, a_{n-1} \in R$, and hence $\left(a_{n-1} + \ldots + a_0 x^{n-1}\right) \in R$. □

Proposition B.3. *Let R and A be rings such that $R \subset A$. Then the integral closure of R in A is a subring of A containing R. In particular, it is an integral extension of R.*

Proof. See [46, Corollary 2.1.11]. □

Corollary B.4. *Let R be a ring. Then $R^* = \overline{R}^* \cap R$.*

Proof. This follows from Lemma B.2 and Proposition B.3. □

Proposition B.5. *Let R, S and T be rings such that $R \subset S \subset T$. Then S is integral over R and T is integral over S if and only if T is integral over R. In particular, the integral closure of R in an overring is integrally closed.*

Proof. See [46, Corollary 2.1.12]. $\square$

Proposition B.6. *Let R be a ring, and let A be an integral extension of R.*

(1) *For any ideal $\mathfrak{i}$ of A the ring $A/\mathfrak{i}$ is an integral extension of $R/(\mathfrak{i} \cap R)$.*

(2) *For any multiplicatively closed subset U of R the ring $U^{-1}A$ is an integral extension of $U^{-1}R$.*

Proof. See [40, Proposition 5.6]. $\square$

Proposition B.7. *Let R, S and T be rings with $R \subset S \subset T$. Then the following are equivalent:*

(a) *S is the integral closure of R in T.*

(b) *$U^{-1}S$ is the integral closure of $U^{-1}R$ in $U^{-1}T$ for every multiplicatively closed subset U of R.*

(c) *$(R \setminus \mathfrak{p})^{-1}S$ is the integral closure of $R_{\mathfrak{p}}$ in $(R \setminus \mathfrak{p})^{-1}T$ for every prime ideal $\mathfrak{p}$ of R.*

(d) *$(R \setminus \mathfrak{m})^{-1}S$ is the integral closure of $R_{\mathfrak{m}}$ in $(R \setminus \mathfrak{m})^{-1}T$ for every $\mathfrak{m} \in \operatorname{Max}(R)$.*

Proof. See [46, Proposition 2.1.6]. $\square$

Corollary B.8. *Let R be a reduced ring, and let U be a multiplicatively closed subset of R. Then*

$$\overline{U^{-1}R} = U^{-1}\overline{R}.$$

Proof. This follows from Propositions A.77 and B.7. $\square$

Corollary B.9. *Let R be a reduced ring. Then the following are equivalent:*

(a) *R is normal.*

(b) *$U^{-1}R$ is normal for every multiplicatively closed subset U of R.*

(c) *$R_{\mathfrak{p}}$ is normal for every prime ideal $\mathfrak{p}$ of R.*

(d) *$R_{\mathfrak{m}}$ is normal for every $\mathfrak{m} \in \operatorname{Max}(R)$.*

Proof. First note that R is by Lemma A.27 reduced if and only if $U^{-1}R$ is reduced for every multiplicatively closed subset U of R. Then the claim follows from Propositions A.77 and B.7. $\square$

Theorem B.10. *Let R be a ring, and let A be an R-algebra. The set of all elements of A which are integral over R is a subalgebra of A. In particular, if A is generated by elements integral over R, then A is integral over R.*

Proof. See [39, Theorem 4.2]. □

Theorem B.11. *Let R be a ring, and let A be an R-algebra. Then A is finite over R if and only if A is generated as an R-algebra by finitely many integral elements.*

Proof. See [39, Corollary 4.5]. □

Theorem B.12 (Lying Over)**.** *Let R be a ring, and let A be an integral extension of R. Then for any prime ideal $\mathfrak{p}$ of R there is a prime ideal $\mathfrak{q}$ of A such that $\mathfrak{q} \cap R = \mathfrak{p}$.*

Proof. See [46, Theorem 2.2.2]. □

Theorem B.13 (Incomparability)**.** *Let R be a ring, let A be an integral extension of R, and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A with $\mathfrak{p} \subset \mathfrak{q}$. If $\mathfrak{p} \cap R = \mathfrak{q} \cap R$, then $\mathfrak{p} = \mathfrak{q}$.*

Proof. See [46, Theorem 2.2.3]. □

Theorem B.14. *Let R be a ring, and let A be an integral extension of R. Then* $\dim R = \dim A$.

Proof. See [46, Theorem 2.2.5]. □

Proposition B.15. *Let R be a ring, let A be an integral extension of R, and let $\mathfrak{q}$ be a prime ideal of A. Then $\mathfrak{q}$ is a maximal ideal of A if and only if $\mathfrak{q} \cap R$ is a maximal ideal of R.*

Proof. See [39, Corollary 4.17]. □

Proposition B.16. *Let R be a ring, let $\mathfrak{p}$ be a prime ideal of R, and let A be a finitely generated integral extension of R. Then there are only finitely many prime ideals of A lying over $\mathfrak{p}$.*

Proof. By Proposition B.6.(2) $(R \setminus \mathfrak{p})^{-1}A$ is a finitely generated integral extension of $R_\mathfrak{p}$, and hence $(R \setminus \mathfrak{p})^{-1}A/\mathfrak{p}(R \setminus \mathfrak{p})^{-1}A$ is a finitely generated integral extension of $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then $(R \setminus \mathfrak{p})^{-1}A/\mathfrak{p}(R \setminus \mathfrak{p})^{-1}A$ is Noetherian by Theorem A.1 since $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a field. Moreover, $\dim (R \setminus \mathfrak{p})^{-1}A/\mathfrak{p}(R \setminus \mathfrak{p})^{-1}A = \dim R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = 0$ by Theorem B.14. This implies that $(R \setminus \mathfrak{p})^{-1}A/\mathfrak{p}(R \setminus \mathfrak{p})^{-1}A$ is Artinian (see [40, Theorem 8.5]). Then $(R \setminus \mathfrak{p})^{-1}A/\mathfrak{p}(R \setminus \mathfrak{p})^{-1}A$ is a product of local rings (see [40, Theorem 8.7]), and hence it is semilocal by Lemma A.6.(1). Proposition A.10.(3) implies that there are finitely many maximal ideals of $(R \setminus \mathfrak{p})^{-1}A$ containing $\mathfrak{p}(R \setminus \mathfrak{p})^{-1}A$. Then by Proposition B.15 there are only finitely many maximal ideals of $(R \setminus \mathfrak{p})^{-1}A$ which all are all lying over $\mathfrak{p}R_\mathfrak{p}$. By Proposition A.20.(2) this implies that there are only finitely many prime ideals of A lying over $\mathfrak{p}$. Also see [44, Exercise 9.3]. □

Corollary B.17. *Let R be a semilocal ring, and let A be a finitely generated integral extension of R. Then A is semilocal, as well.*

Proof. This follows from Proposition B.15 and B.16. □

Lemma B.18. *Let R be a reduced Noetherian ring, let A be an integral extension of R with $R \subset A \subset Q_R$, and let $\mathfrak{q}$ be a prime ideal of A. If* $\operatorname{height} \mathfrak{q} \cap R = 1$, *then* $\operatorname{height} \mathfrak{q} = 1$.

Proof. By Theorem B.13 we have height $\mathfrak{q} \cap R \geq$ height $\mathfrak{q} = 1$, and Theorem A.72 implies height $\mathfrak{q} > 0$. $\square$

Lemma B.19. *Let R be a ring, let A be an integral extension of R, and let $\mathfrak{q}$ be a prime ideal of A. Set $\mathfrak{p} = \mathfrak{q} \cap A$, $U = A \setminus \mathfrak{p}$ and $B = U^{-1}A$. Then:*

(1) $R \setminus \mathfrak{p} \subset A \setminus \mathfrak{q}$.

(2) $\mathfrak{q}B$ *is a prime ideal of* B*, and* height $\mathfrak{q}B =$ height $\mathfrak{q}$.

(3) $A_{\mathfrak{q}} = B_{\mathfrak{q}B}$.

(4) If B *is local and* $\dim B =$ height $\mathfrak{q}$*, then* $A_{\mathfrak{q}} = B$.

Proof. (1) We have $R \setminus \mathfrak{p} = (A \cap R) \setminus (\mathfrak{q} \cap R) = (A \setminus \mathfrak{q}) \cap R \subset A \setminus \mathfrak{q}$.

(2) By (1) $\mathfrak{q}$ is a prime ideal of A not meeting $R \setminus \mathfrak{p}$. Thus, $\mathfrak{q}B$ is a prime ideal of B by Proposition A.20.(2). Since any prime ideal $\mathfrak{q}'$ of R with $\mathfrak{q}' \subset \mathfrak{q}$ is not meeting $R \setminus \mathfrak{p}$, as well, there is by Proposition A.20.(2) a bijection between chains of prime ideals of A contained in $\mathfrak{q}$ and chains of prime ideals of B contained in $\mathfrak{q}B$. This yields height $\mathfrak{q} =$ height $\mathfrak{q}B$.

(3) We have the following natural homomorphisms

$$\begin{aligned}
\alpha\colon A \to B, \qquad & a \mapsto \frac{a}{1} \\
\gamma\colon B \to B_{\mathfrak{q}B}, \quad & b \mapsto \frac{b}{1} \\
\delta\colon A \to A_{\mathfrak{q}}, \qquad & a \mapsto \frac{a}{1} \\
\epsilon\colon B \to A_{\mathfrak{q}}, \qquad & \frac{a}{p} \mapsto \frac{a}{p}.
\end{aligned}$$

Then $\alpha(A \setminus \mathfrak{q}) \subset B \setminus \mathfrak{q}B$. Otherwise, there are by Proposition A.20.(2) $a \in A \setminus \mathfrak{q}$, $a' \in \mathfrak{q}$ and $b \in R \setminus \mathfrak{p}$ such that $b(a - a') = 0$. This implies $R \setminus \mathfrak{q} \ni ba = ba' \in \mathfrak{q}$ since by (1) $R \setminus \mathfrak{p} \subset A \setminus \mathfrak{q}$, and $A \setminus \mathfrak{q}$ is multiplicatively closed. Thus, setting $\beta = \gamma \circ \alpha\colon A \to B_{\mathfrak{q}B}$ we have $\beta(A \setminus \mathfrak{q}) \subset (B_{\mathfrak{q}B})^*$. Hence, the universal property of $A_{\mathfrak{q}}$ implies that there is a unique homomorphism $f\colon A_{\mathfrak{q}} \to B_{\mathfrak{q}B}$ such that $\beta = f \circ \gamma$.

By definition we have $\epsilon(\mathfrak{q}B) \subset \mathfrak{q}A_{\mathfrak{q}}$, and hence $\epsilon(B \setminus \mathfrak{q}B) \subset (A_{\mathfrak{q}})^*$. Thus, we get by the universal property of $B_{\mathfrak{q}B}$ a unique homomorphism $g\colon B_{\mathfrak{q}B} \to A_{\mathfrak{q}}$ such that $\epsilon = g \circ \gamma$.

These considerations yield the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[rr, "\alpha"] \arrow[dl, "\delta"'] \arrow[dr, "\beta"] & & B \arrow[dl, "\gamma"'] \arrow[dr, "\epsilon"] & \\
A_{\mathfrak{q}} \arrow[rr, dotted, "f"] & & B_{\mathfrak{q}B} \arrow[rr, "g"] & & A_{\mathfrak{q}}.
\end{tikzcd}$$

Since, moreover,

$$\epsilon \circ \alpha\,(a) = \epsilon\left(\frac{a}{1}\right) = \frac{a}{1} = \delta\,(a)\,.$$

for all $a \in A$, we obtain

$$g \circ f \circ \delta = \epsilon \circ \alpha = \delta.$$

Hence, the universal property of $A_{\mathfrak{q}}$ implies $g \circ f = \mathrm{id}_{A_{\mathfrak{q}}}$.

Since $R \setminus \mathfrak{p} \subset A \setminus \mathfrak{q}$ by (1), we have $\beta\,(R \setminus \mathfrak{p}) \subset \beta\,(A \setminus \mathfrak{q}) \subset (B_{\mathfrak{q}B})^*$ (see above). Then the universal property of $B = (R \setminus \mathfrak{p})^{-1} A$ yields a unique homomorphism $h\colon B \to B_{\mathfrak{q}B}$ such that $\beta = h \circ \alpha$. Since $\gamma\colon B \to B_{\mathfrak{q}B}$ such that $\beta = \gamma \circ \alpha$, this implies $h = \gamma$. Hence, we have the following commutative diagram

$$\begin{tikzcd}
& B \arrow[dl, "\gamma"'] \arrow[dr, "\epsilon"] \arrow[drrr, "\gamma"] & & A \arrow[ll, "\alpha"'] \arrow[dr, "\beta"] & \\
B_{\mathfrak{q}B} \arrow[rr, dotted, "g"'] & & (A)_{\mathfrak{q}} \arrow[rr, dotted, "f"'] & & B_{\mathfrak{q}B}.
\end{tikzcd}$$

Then $f \circ g \circ \gamma = \gamma$ yields $f \circ g = \mathrm{id}_{B_{\mathfrak{q}B}}$ by the universal property of $B_{\mathfrak{q}B}$, and hence $A_{\mathfrak{q}} = B_{\mathfrak{q}B}$.

(4) If $\dim B = \operatorname{height} \mathfrak{q}$, then $\mathfrak{q}B$ is a maximal ideal of B by (2). If B is local, then $\mathfrak{q}B$ is the unique maximal ideal of B, and hence $B_{\mathfrak{q}B} = B$. This implies $B = B_{\mathfrak{q}B} = (A')_{\mathfrak{q}}$ by (3). ☐

Definition B.20. A ring R is called *residually rational* if for any $\mathfrak{m} \in \operatorname{Max}(R)$ we have $R/\mathfrak{m} \cong \overline{R}/\mathfrak{n}$ for every $\mathfrak{n} \in \operatorname{Max}\left(\overline{R}\right)$ with $\mathfrak{n} \cap R = \mathfrak{m}$.

Lemma B.21. *Let R be a ring whose residue fields are algebraically closed, and let $\mathfrak{m} \in \operatorname{Max}(R)$. If A is an integral extension of R, then $R/\mathfrak{m} = A/\mathfrak{n}$ for any $\mathfrak{n} \in \operatorname{Max}(R)$ with $\mathfrak{n} \cap R = \mathfrak{m}$. In particular, R is residually rational.*

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$, and let $\mathfrak{n} \in \operatorname{Max}(A)$ with $\mathfrak{n} \cap R = \mathfrak{m}$ (see Theorem B.12 and Proposition B.15). Let $x \in A/\mathfrak{n}$. Then by Proposition B.6.(1) there are $a_0, \ldots, a_{n-1} \in R/\mathfrak{m}$ such that $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$. Thus, $A/\mathfrak{n}$ is an algebraic extension field of $R/\mathfrak{m}$. Since $R/\mathfrak{m}$ is algebraically closed by assumption, this implies $R/\mathfrak{m} = A/\mathfrak{n}$. ☐

B.2. Conductor

Definition B.22. Let R be a ring, and let $\mathfrak{I}$ be an R-submodule of Q_R. The *conductor* of $\mathfrak{I}$ is $\mathfrak{C}_{\mathfrak{I}} = \mathfrak{I} : \overline{R}$.

Remark B.23. Let R be a ring, and let $\mathfrak{I}$ be a regular R-submodule of Q_R.

(1) The conductor $\mathfrak{C}_{\mathfrak{I}}$ is a regular R-submodule of $\mathfrak{I}$. If $\mathfrak{I}, \overline{R} \in \mathcal{R}_R$, then $\mathfrak{C}_{\mathfrak{I}} \in \mathcal{R}_R$, see Proposition 2.7.(2).

(2) $\overline{R}\mathfrak{C}_R\mathfrak{I} \subset R\mathfrak{I} \subset \mathfrak{I}$ implies $\mathfrak{C}_R\mathfrak{I} \subset \mathfrak{C}_{\mathfrak{I}}$.

(3) $\mathfrak{C}_{\mathfrak{I}}$ is both an R- and an $\overline{R}$-submodule of Q_R, and $\mathfrak{C}_{\mathfrak{I}}$ is the largest R-submodule of $\mathfrak{I}$ with this property.

Proposition B.24. *Let R be a ring, and let $\mathfrak{I}$ and $\mathfrak{J}$ be R-submodules of Q_R. Then*

$$\mathfrak{C}_{\mathfrak{I}:\mathfrak{J}} = \mathfrak{C}_{\mathfrak{I}} : \mathfrak{J}.$$

Proof. By Lemma 2.3.(1) and Definition B.22 we have

$$\mathfrak{C}_{\mathfrak{I}:\mathfrak{J}} = (\mathfrak{I} : \mathfrak{J}) : \overline{R} = \left(\mathfrak{I} : \overline{R}\right) : \mathfrak{J} = \mathfrak{C}_{\mathfrak{I}} : \mathfrak{J}.$$ □

Proposition B.25. *Let R be a ring, and let $\mathfrak{I}$ and $\mathfrak{J}$ be R-submodules of Q_R. Then*

$$\mathfrak{C}_{\mathfrak{I}} (\mathfrak{J} : \mathfrak{I}) \subset \mathfrak{C}_{\mathfrak{J}}.$$

Proof. Let $c \in \mathfrak{C}_{\mathfrak{I}} (\mathfrak{J} : \mathfrak{I})$. Then there are $a_i \in \mathfrak{C}_{\mathfrak{I}}$ and $b_i \in \mathfrak{J} : \mathfrak{I}$, $i = 1, \dots, n$ for some $n \geq 0$ such that $c = \sum_{i=1}^n a_i b_i$. Since $a_i \overline{R} \subset \mathfrak{I}$ and $b_i \mathfrak{I} \subset \mathfrak{J}$ for all $i = 1, \dots, n$, we obtain

$$c\overline{R} = \sum_{i=1}^{n} a_i b_i \overline{R} \subset \sum_{i=1}^{n} b_i \mathfrak{I} \subset \mathfrak{J}.$$

This implies $c \in \mathfrak{J} : \overline{R} = \mathfrak{C}_{\mathfrak{J}}$. □

B.3. Equidimensionality

Definition B.26. A ring R is called *equidimensional* if $\dim R/\mathfrak{p} = \dim R$ for all $\mathfrak{p} \in \operatorname{Min}(R)$.

Proposition B.27. *A one-dimensional ring R is equidimensional if and only if* $\operatorname{height} \mathfrak{m} = 1$ *for all* $\mathfrak{m} \in \operatorname{Max}(R)$.

Proof. Let R be a one-dimensional ring, and suppose that R is equidimensional. Let $\mathfrak{m} \in \operatorname{Max}(R)$, and suppose that $\operatorname{height} \mathfrak{m} < 1$, i.e. $\operatorname{height} \mathfrak{m} = 0$. Then $\mathfrak{m} \in \operatorname{Min}(R)$. Since R is equidimensional, we have $\dim R/\mathfrak{m} = \dim R = 1$, and hence there is by Proposition A.10.(3) an $\mathfrak{n} \in \operatorname{Max}(R)$ with $\mathfrak{m} \subset \mathfrak{n}$ and $\operatorname{height} \mathfrak{n} = 1$. This contradicts the maximality of $\mathfrak{m}$. Thus, $\operatorname{height} \mathfrak{m} = 1$.

Suppose now that $\operatorname{height} \mathfrak{m} = 1$ for all $\mathfrak{m} \in \operatorname{Max}(R)$. Let $\mathfrak{p} \in \operatorname{Min}(R)$. Then $\operatorname{height} \mathfrak{p} = 0$, and hence $\mathfrak{p} \notin \operatorname{Max}(R)$. Thus, there is $\mathfrak{m} \in \operatorname{Max}(R)$ with $\operatorname{height} \mathfrak{m} = 1$ and $\mathfrak{p} \subset \mathfrak{m}$. This implies $\dim R/\mathfrak{p} = 1$. Therefore, R is equidimensional. □

Lemma B.28. *Let R be a complete equidimensional Noetherian semilocal ring. Then* $\operatorname{height} \mathfrak{m} = \dim R$ *for all* $\mathfrak{m} \in \operatorname{Max}(R)$.

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$, and let $\mathfrak{q} \in \operatorname{Min}(R_{\mathfrak{m}})$ with $\dim \mathfrak{q} = \dim R_{\mathfrak{m}} = \operatorname{height} \mathfrak{m}$. Then by Proposition A.20.(2) there is a prime ideal $\mathfrak{p}$ of R with $\mathfrak{q} = \mathfrak{p} R_{\mathfrak{m}}$. In particular, this means $\mathfrak{p} \subset \mathfrak{m}$. Since $\dim \mathfrak{p} R_{\mathfrak{m}} = \dim \mathfrak{q} = \operatorname{height} \mathfrak{m}$, Proposition A.20.(2) implies $\mathfrak{p} \in \operatorname{Min}(R)$. Since $\mathfrak{p} \subset \mathfrak{m}$, Proposition A.10.(3) and Lemma A.78 yield $\operatorname{height} \mathfrak{m} = \dim \mathfrak{p} = \dim R$ as R is equidimensional. □

Definition B.29. A ring R is called *formally equidimensional* if its completion $\widehat{R}$ (at the Jacobson radical $\mathfrak{j}_R$ of R) is equidimensional.

Lemma B.30. *Let R be a formally equidimensional Noetherian semilocal ring. Then* $\operatorname{height}\mathfrak{m} = \dim R$ *for all* $\mathfrak{m} \in \operatorname{Max}(R)$.

Proof. Let $\mathfrak{m} \in \operatorname{Max}(R)$. Then Theorem A.59.(2) yields $\mathfrak{m}\widehat{R} \in \operatorname{Max}\left(\widehat{R}\right)$ with

$$\operatorname{height}\mathfrak{m} = \operatorname{height}\mathfrak{m}\widehat{R} = \dim\widehat{R} = \dim R$$

by Corollaries A.64 and A.65 and Lemma B.28. □

Lemma B.31. *Let R be a ring, and let A be an integral extension of R with $R \subset A \subset Q_R$. If R is equidimensional, then so is A.*

Proof. Let $\mathfrak{q} \in \operatorname{Min}(A)$. Then $\mathfrak{q} \cap R \in \operatorname{Min}(R)$ by Theorem A.72, and by Proposition B.6.(1) $A/\mathfrak{q}$ is an integral extension of $R/(\mathfrak{q} \cap R)$. Therefore, Theorem B.14 yields

$$\dim A/\mathfrak{q} = \dim R/(\mathfrak{q} \cap R) = \dim R = \dim A.$$ □

Lemma B.32. *Let R be a formally equidimensional Noetherian semilocal ring, and let A be a finite integral extension of R with $R \subset A \subset Q_R$. Then A is formally equidimensional, as well. In particular,*

$$\operatorname{height}\mathfrak{m} = \operatorname{height}\mathfrak{m} \cap R = \dim R$$

for every $\mathfrak{m} \in \operatorname{Max}(R)$.

Proof. By Theorems A.52 and A.55 $\widehat{A} = A \otimes_R \widehat{R}$ is finite over $\widehat{R}$, and hence an integral extension of $\widehat{R}$ by Theorem B.11. Since $A \otimes \widehat{R} \subset Q_R \otimes \widehat{R} \subset Q_{\widehat{R}}$ by Lemma 2.16.(1) and Theorem A.59.(1), Lemma B.31 implies that $\widehat{A}$ is equidimensional, and hence A is formally equidimensional.

Since A is semilocal by Corollary B.17, and since it is Noetherian by Theorem A.1, Lemma B.30 and Theorem B.14 yield for every $\mathfrak{m} \in \operatorname{Max}(A)$

$$\operatorname{height}\mathfrak{m} = \dim A = \dim R = \operatorname{height}\mathfrak{m} \cap R,$$

where the last equality follows from Proposition B.15. □

B.4. Excellent Rings

Definition B.33. A Noetherian ring R is called *excellent* if it satisfies the following conditions.

(1) R is universally catenary,

(2) For all prime ideals $\mathfrak{p}$ of R, all prime ideals $\mathfrak{q}$ of $R_\mathfrak{p}$, and all finite field extensions L of $k(\mathfrak{q})$ the ring $\widehat{R_\mathfrak{p}} \otimes_{k(\mathfrak{q})} L$ is regular.

(3) For every finitely generated R-algebra A the singular locus $\operatorname{Sing}(A)$ is closed in $\operatorname{Spec}(A)$.

Theorem B.34. *Let R be an excellent ring. Then all localizations of R and all finitely generated R-algebras are excellent.*

Proof. See [42, (34.A)]. ☐

Theorem B.35. *Complete semilocal Noetherian rings are excellent. In particular, any field K, and hence any localization of any finitely generated K-algebra are excellent.*

Proof. See [42, (34.B)]. ☐

The next theorem lists important properties of the normalization of reduced excellent and reduced complete rings.

Theorem B.36. *Let R be a reduced excellent ring.*

(1) For any ideal $\mathfrak{i}$ of R the $\mathfrak{i}$-adic completion $\widehat{R}$ of R is reduced. If R is normal, then $\widehat{R}$ is normal.

(2) The normalization $\overline{R}$ of R is a finite R-module.

(3) If R is semilocal, then $\overline{\widehat{R}} = \widehat{\overline{R}}$. In particular,

  (1) $\overline{\widehat{R}} = \widehat{\overline{R}} = \overline{R} \otimes_R \widehat{R}$ is a finite $\widehat{R}$-module, and

  (2) if R is complete, then $\overline{R}$ is complete.

Proof. See [21, Theorem 1.18]. ☐

B.5. Normalization

Definition B.37. The *non-normal locus* of a reduced ring R is

$$N(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid R_{\mathfrak{p}} \text{ is not normal}\}.$$

Definition B.38. The *singular locus* of a ring R is

$$\operatorname{Sing}(R) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid R_{\mathfrak{p}} \text{ is not regular}\}.$$

Theorem B.39. *A regular local ring is a normal domain. A regular ring is the direct product of regular domains.*

Proof. See [38, Corollary 2.2.20]. ☐

Corollary B.40. *A regular ring is normal.*

Proof. Let R be a regular ring. Then R is by Theorem B.39 the direct product of regular domains. Hence, R is reduced.

Let $\mathfrak{m} \in \operatorname{Max}(R)$. Then $R_{\mathfrak{m}}$ is by definition a regular local ring. Thus, $R_{\mathfrak{m}}$ is by Theorem B.39 a normal domain. Therefore, Corollary B.9 implies that R is normal. ☐

Remark B.41. For any ring R Theorem B.39 implies $\operatorname{Sing}(R) \subset N(R)$.

Theorem B.42 (Splitting of Normalization). *Let R be a reduced Noetherian ring. Then*

$$\overline{R} = \prod_{\mathfrak{p} \in \operatorname{Min}(R)} \overline{R/\mathfrak{p}}.$$

Proof. See Theorem A.74.(2) and [46, Corollary 2.1.13]. ☐

Theorem B.43. *Let R be a reduced Noetherian local ring of dimension one. Then R is normal if and only if it is regular.*

Proof. See [1, Theorem 4.4.9]. ☐

B.5.1. Criteria for Normality

Serre's Conditions

Definition B.44. Let R be a ring, and let $i \geq 0$ be an integer.

(1) Then R satisfies *Serre's condition* $(\mathrm{R_i})$ if for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq i$, $R_{\mathfrak{p}}$ is a regular local ring.

(2) We say that R satisfies *Serre's condition* $(\mathrm{S_i})$ if for all $\mathfrak{p} \in \operatorname{Spec}(R)$ we have $\operatorname{depth} R_{\mathfrak{p}} \geq \min\{i, \dim R_{\mathfrak{p}}\}$.

Theorem B.45. *Let R be a ring.*

(1) The ring R is reduced if and only if it satisfies Serre's conditions $(\mathrm{R_0})$ *and* $(\mathrm{S_1})$.

(2) The ring R is normal if and only if it satisfies Serre's conditions $(\mathrm{R_1})$ *and* $(\mathrm{S_2})$.

Proof. See [38, page 71 and Theorem 2.2.22]. ☐

Proposition B.46. *Let R be a reduced excellent ring, and suppose that R satisfies Serre's condition* $(\mathrm{S_2})$*. Then for any regular radical ideal $\mathfrak{i}$ of R the following hold:*

(1) The ring $\mathfrak{i} : \mathfrak{i}$ satisfies Serre's condition $(\mathrm{S_2})$.

(2) If $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{i})$ with $\operatorname{height} \mathfrak{p} = 1$*, then $\mathfrak{i} : \mathfrak{i} = R$.*

Proof. See [21, Lemma 3.6] and Lemma 2.13. ☐

Grauert and Remmert Criterion

Definition B.47. Let R be a reduced Noetherian ring. A regular radical ideal $\mathfrak{i}$ of R is called a *test ideal* for R if

$$N(R) \subset V(\mathfrak{i}).$$

Theorem B.48 (Grauert and Remmert Criterion). *Let R be a reduced Noetherian ring, and let $\mathfrak{i}$ be a test ideal for R. Then R is normal if and only if*

$$R = \mathfrak{i} : \mathfrak{i}.$$

Proof. See [20, Anhang §3.3, Satz 7], [27, Proposition 3.6.5], and Lemma 2.13. □

Remark B.49. Let R be a one-dimensional reduced Noetherian local ring. Then the maximal ideal $\mathfrak{m}$ of R is the unique test ideal for R, see [21, Remark 4.1].

Remark B.50. Let R be a reduced Noetherian ring. If $\overline{R}$ is finite over R (e.g. if R is excellent, see Theorem B.36.(2)), then

$$V(\mathfrak{C}_R) = N(R),$$

see [21, Remark 2.2].

Definition B.51. Let R be a reduced excellent ring, and let W be a subset of $\operatorname{Spec}(R)$. Then a regular radical ideal $\mathfrak{i}$ is called a *test ideal* at W if

$$V(\mathfrak{C}_{R_\mathfrak{p}}) \subset V(\mathfrak{i}R_\mathfrak{p})$$

for all $\mathfrak{p} \in W$.

Proposition B.52. *Let R be a reduced excellent ring, let $\mathfrak{i}$ be an ideal of R, and let $W \subset \operatorname{Spec}(R)$.*

(1) If $\mathfrak{i}$ is a test ideal at W, then $\mathfrak{i}R_\mathfrak{p}$ is a test ideal for $R_\mathfrak{p}$ for any $\mathfrak{p} \in W$.

(2) If $N(R) \subset W$, then $\mathfrak{i}$ is a test ideal for R if and only if it is a test ideal at W.

Proof. See [21, Lemma 2.4]. □

Proposition B.53. *Let R be a ring, and let $\mathfrak{i}$ be an ideal of A. Then $R = \mathfrak{i} : \mathfrak{i}$ if and only if $R_\mathfrak{p} = \mathfrak{i}R_\mathfrak{p} : \mathfrak{i}R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R.*

Proof. Since equality is a local property, we have $R = \mathfrak{i} : \mathfrak{i}$ if and only if $R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}(\mathfrak{i} : \mathfrak{i})$ for all $\mathfrak{p} \in \operatorname{Min}(R)$. The claim follows since Lemma 2.16.(3) and Proposition A.39 yield $(R \setminus \mathfrak{p})^{-1}(\mathfrak{i} : \mathfrak{i}) = \mathfrak{i}R_\mathfrak{p} : \mathfrak{i}R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. Also see [21, Corollary 2.6] and Lemma 2.13. □

Proposition B.54. *Let R be a ring, let $\mathfrak{j}$ be an ideal of R such that $\mathfrak{j}$ is contained in the Jacobson radical of R, and denote by $\widehat{R}$ the $\mathfrak{j}$-adic completion of R. Then for any ideal $\mathfrak{i}$ of R we have $R = \mathfrak{i} : \mathfrak{i}$ if and only if $\widehat{R} = \mathfrak{i}\widehat{R} : \mathfrak{i}\widehat{R}$.*

Proof. By Lemma 2.16.(4) and Theorem A.60 we have $R = \mathfrak{i} : \mathfrak{i}$ if and only if $R\widehat{R} = (\mathfrak{i} : \mathfrak{i})\widehat{R}$. Then the claim follows since $R\widehat{R} = \widehat{R}$ by Theorem A.59.(1) and $(\mathfrak{i} : \mathfrak{i})\widehat{R} = \mathfrak{i}\widehat{R} : \widehat{R}$ by Lemma 2.16.(3). Also see [21, Corollary 2.7] and Lemma 2.13. □

Proposition B.55. *Let R be a reduced semilocal excellent ring, and let $\mathfrak{i}$ be a test ideal for R. Then $\widehat{\mathfrak{i}}$ is a test ideal for $\widehat{R}$.*

Proof. See [21, Lemma 2.5]. □

B.5.2. Grauert and Remmert Algorithm for Normalization

Proposition B.56. *Let R be a reduced Noetherian ring, and let A be a finite extension ring of R. If $\mathfrak{i}$ is a test ideal for R, then $\sqrt{\mathfrak{i}A}$ is a test ideal for A.*

Proof. See [19, Proposition 3.2]. □

Proposition B.57. *Let R be a reduced Noetherian ring, and suppose that $\overline{R}$ is a finite R-module (e.g. if R is excellent, see Theorem B.36.(2), or if R is a one-dimensional analytically reduced semilocal Cohen–Macaulay ring, see Corollary C.15). Then for any test ideal $\mathfrak{i}$ of R there is a finite sequence of finite integral extensions*

$$R = R^{(0)} \subsetneq R^{(1)} \subsetneq \ldots \subsetneq R^{(n)} = \overline{R},$$

where for any $i \geq 0$ we set

$$R^{(i+1)} = \mathfrak{i}^{(i)} : \mathfrak{i}^{(i)}$$

and

$$\mathfrak{i}^{(i+1)} = \sqrt{\mathfrak{i}^{(i)} R^{(i+1)}}$$

with $\mathfrak{i}^{(0)} = \mathfrak{i}$. Moreover, $R^{(i)} = R^{(n)}$ for any $i \geq n$.

Proof. By Theorem B.48 we have $R = \overline{R}$ if and only if $R = \mathfrak{i} : \mathfrak{i}$. Suppose that R is not normal. Then $R \subsetneq R^{(1)} = \mathfrak{i} : \mathfrak{i} \subset \overline{R}$, and $R^{(1)}$ is by Remark 2.6.(1) and Proposition 2.7.(2) finite over R. Thus, $R^{(1)}$ is by Theorem B.34 excellent. Moreover, since Q_R is reduced by Lemma A.27, and since $R^{(1)} \subset Q_R$, also $R^{(1)}$ is reduced. Therefore, $\mathfrak{i}^{(1)} = \sqrt{\mathfrak{i} R^{(1)}}$ is by Proposition B.56 a test ideal for $R^{(1)}$. So by induction (using Lemma A.34 and Proposition B.5) we obtain a sequence of integral extensions

$$R = R^{(0)} \subsetneq R^{(1)} \subsetneq \ldots \subset \overline{R},$$

and for any $\mathfrak{i} \geq 0$ the ideal $\mathfrak{i}^{(i)}$ is a test ideal for $R^{(i)}$. Since $\overline{R}$ is a finite R-module, there is an n such that $R^{(n)} = R^{(n+1)} = \mathfrak{i}^{(n)} : \mathfrak{i}^{(n)}$, and hence $R^{(n)} = \overline{R^{(n)}} = \overline{R}$ by Proposition B.5 and Theorem B.48. □

Proposition B.58. *Let R be a reduced excellent ring, let $\mathfrak{p}$ be a prime ideal of R, and let $\mathfrak{i}$ be a test ideal of R. Then $\mathfrak{i}R_{\mathfrak{p}}$ is a test ideal of $R_{\mathfrak{p}}$, and*

$$(R \setminus \mathfrak{p})^{-1} R^{(1)} = (R \setminus \mathfrak{p})^{-1} (\mathfrak{i} : \mathfrak{i}) = \mathfrak{i}R_{\mathfrak{p}} : \mathfrak{i}R_{\mathfrak{p}} = (R_{\mathfrak{p}})^{(1)}$$

Proof. This follows from Lemma 2.16.(3) and Propositions A.39 and B.52.(1). □

Proposition B.59. *Let R be a reduced excellent ring, let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus N(R)$, and let $\mathfrak{q} \in \operatorname{Spec}\left(R^{(1)}\right)$ with $\mathfrak{p} = \mathfrak{q} \cap R$ (see Theorem B.12). Then $\left(R^{(1)}\right)_{\mathfrak{q}} = R_{\mathfrak{p}}$.*

Proof. If $\mathfrak{p} \in \operatorname{Spec}(R) \setminus N(R)$, then $R_{\mathfrak{p}}$ is normal by definition. Thus, we have $R_{\mathfrak{p}} = (R_{\mathfrak{p}})^{(1)}$ by Theorem B.48 (recall that by Lemma A.27, Theorem B.34, and Remark B.49 $\mathfrak{p}R_{\mathfrak{p}}$ is the unique test ideal for $R_{\mathfrak{p}}$). Moreover, Proposition B.53 yields

$$R_{\mathfrak{p}} = (R_{\mathfrak{p}})^{(1)} = (R \setminus \mathfrak{p})^{-1} R^{(1)}. \tag{B.1}$$

Let $\mathfrak{q}'$ be a prime ideal of $R^{(1)}$ with $\mathfrak{q} \subset \mathfrak{q}'$ and $\mathfrak{q}' \cap (R \setminus \mathfrak{p}) = \emptyset$. Then

$$\mathfrak{p} = \mathfrak{q} \cap R \subset \mathfrak{q}' \cap R \subset \mathfrak{p}$$

implies $\mathfrak{q}' \cap R = \mathfrak{p}$, and hence $\mathfrak{q} = \mathfrak{q}'$ by Theorem B.13. Thus,

$$\mathfrak{q}\Big((R \setminus \mathfrak{p})^{-1} R^{(1)}\Big) = \mathfrak{p} R_{\mathfrak{p}}$$

is the maximal ideal of the local ring $(R \setminus \mathfrak{p})^{-1} R^{(1)} = R_{\mathfrak{p}}$ (see Equation (B.1)). Then Proposition A.20.(2) yields

$$\operatorname{height} \mathfrak{p} = \operatorname{height} \mathfrak{p} R_{\mathfrak{p}} = \operatorname{height} \mathfrak{q}\Big((R \setminus \mathfrak{p})^{-1} R^{(1)}\Big) = \operatorname{height} \mathfrak{q}.$$

Thus, we obtain by Lemma B.19.(4)

$$R_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1} R^{(1)} = \left(R^{(1)}\right)_{\mathfrak{q}}. \qquad \square$$

Proposition B.60. *Let R be a local ring with maximal ideal $\mathfrak{m}_R$. Then*

$$\operatorname{End}_R(\mathfrak{m}_R) = \begin{cases} R & \text{if } R \text{ is regular,} \\ \operatorname{Hom}_R(\mathfrak{m}_R, R) & \text{otherwise.} \end{cases}$$

In particular, if $\mathfrak{m}_R$ is regular, then

$$\mathfrak{m}_R : \mathfrak{m}_R = \begin{cases} R & \text{if } R \text{ is regular,} \\ R : \mathfrak{m}_R & \text{otherwise.} \end{cases}$$

Proof. See [21, Lemma 3.5]. The particular claim follows with Lemma 2.13. $\square$

Lemma B.61. *Let R be a reduced excellent ring, let $n \in \mathbb{N}$, and set*

$$\mathcal{S}_n = \left\{ \mathfrak{p} \in \operatorname{Sing}(R) \;\middle|\; \text{there is a prime ideal } \mathfrak{q} \text{ of } R^{(n)} \text{ with } \operatorname{height} \mathfrak{q} = 1 \text{ and } \mathfrak{q} \cap R = \mathfrak{p} \right\}.$$

(1) If R satisfies Serre's criterion (S_2), then $R^{(n)}$ is normal if and only if it satisfies (R_1).

(2) The ring $R^{(n)}$ satisfies Serre's condition (R_1) if and only if $\left(R^{(n)}\right)_{\mathfrak{q}}$ is a regular local ring for all prime ideals $\mathfrak{q}$ of $R^{(n)}$ with $\operatorname{height} \mathfrak{q} = 1$ and $\mathfrak{q} \cap R \in \operatorname{Sing}(R)$.

(3) The ring $R^{(n)}$ satisfies Serre's condition (R_1) if and only if $(R_{\mathfrak{p}})^{(n)}$ is normal for all $\mathfrak{p} \in \mathcal{S}_n$.

(4) If R satisfies Serre's criterion (S_2), then $R^{(n)}$ is normal if and only if $(R_{\mathfrak{p}})^{(n)}$ is regular, equivalently normal, for all $\mathfrak{p} \in \mathcal{S}_n$.

Proof. (1) By Theorem B.45.(2) $R^{(n)}$ is normal if and only if it satisfies Serre's criteria (R_1) and (S_2). If $\dim R = 0$, then $\dim R^{(n)} = \dim R = 0$ by Theorem B.14. Thus, $R^{(n)}$ satisfies (S_2). If $\dim R > 0$, then the Jacobson radical $\mathfrak{j}_R$ is regular. Hence, inductively applying Proposition B.46.(1) implies that $R^{(n)}$ satisfies (S_2). So $R^{(n)}$ is normal if and only if it satisfies (R_1).

(2) Since R is reduced, also Q_R is reduced by Lemma A.27, and hence $R^{(n)} \subset Q_R$ is reduced. Hence, $R^{(n)}$ satisfies (R_0) by Theorem B.45.(1). This implies that $R^{(n)}$ satisfies (R_1) if and only if $\left(R^{(n)}\right)_{\mathfrak{q}}$ is a regular local ring for all prime ideals $\mathfrak{q}$ of $R^{(n)}$ with height $\mathfrak{q} = 1$.

Let $\mathfrak{q}$ be a prime ideal of $R^{(n)}$ with height $\mathfrak{q} = 1$, and set $\mathfrak{p} = \mathfrak{q} \cap R$. If $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Sing}(R)$, then $R_{\mathfrak{p}}$ is regular by definition. Hence $R_{\mathfrak{p}}$ is normal by Theorem B.39. So inductively applying Proposition B.59 implies that $\left(R^{(n)}\right)_{\mathfrak{q}} = R_{\mathfrak{p}}$ is regular.

(3) By (2) $R^{(n)}$ satisfies (R_1) if and only if $\left(R^{(n)}\right)_{\mathfrak{q}}$ is a regular local ring for all prime ideals $\mathfrak{q}$ of $R^{(n)}$ with height $\mathfrak{q} = 1$ and $\mathfrak{q} \cap R \in \operatorname{Sing}(R)$. Now let $\mathfrak{q}$ be a prime ideal of $R^{(n)}$ with height $\mathfrak{q} = 1$ and $\mathfrak{q} \cap R \in \operatorname{Sing}(R)$, and set $\mathfrak{p} = \mathfrak{q} \cap R$. By inductively applying Proposition B.58 we obtain

$$(R_{\mathfrak{p}})^{(n)} = (R \setminus \mathfrak{p})^{-1} R^{(n)}.$$

If we set $B = (R \setminus \mathfrak{p})^{-1} R^{(n)}$, then $\mathfrak{q}B$ is by Lemma B.19.(2) a prime ideal of B. Moreover, Lemma B.19.(3) yields

$$B_{\mathfrak{q}B} = \left(R^{(n)}\right)_{\mathfrak{q}}.$$

So if B is normal, then $B_{\mathfrak{q}B}$ is normal by Proposition B.7, and therefore $\left(R^{(n)}\right)_{\mathfrak{q}}$ is normal. Since height $\mathfrak{q} = 1$, this implies by Theorem B.43 that $\left(R^{(n)}\right)_{\mathfrak{q}}$ is regular. Thus, if $(R_{\mathfrak{p}})^{(n)}$ is normal for all $\mathfrak{p} \in \mathcal{S}_n$, then $R^{(n)}$ satisfies Serre's condition (R_1).

Assume now that B is not normal. Then there is a prime ideal $\mathfrak{i}$ of B such that $B_{\mathfrak{i}}$ is not regular. By Proposition A.20.(2) there is a prime ideal $\mathfrak{q}'$ of $R^{(n)}$ with $\mathfrak{i} = \mathfrak{q}'B$. In particular, we have height $\mathfrak{i} =$ height $\mathfrak{q}'B \leq 1$ by Proposition A.20.(1), and $\left(R^{(n)}\right)_{\mathfrak{q}'} = B_{\mathfrak{q}'B} = B_{\mathfrak{i}}$ (see Lemma B.19.(3)) is by Corollary B.9 not normal. Therefore, $\left(R^{(n)}\right)_{\mathfrak{q}'}$, and hence $R^{(n)}$, is not regular by Corollary B.40. So if there is $\mathfrak{p} \in \mathcal{S}_n$ such that $(R_{\mathfrak{p}})^{(n)}$ is not normal, then $R^{(n)}$ does not satisfy (R_1).

(4) This follows from (1) and (3). $\square$

C. Cohen–Macaulay Rings

Proposition C.1. *Let R be a local Noetherian ring, and let M be a finite non-zero R-module. Then* $\operatorname{depth} M \leq \dim M$.

Proof. See [38, Proposition 1.2.12]. $\square$

Definition C.2. Let R be a Noetherian local ring. A finite R-module $M \neq 0$ is a *Cohen-Macaulay module* if $\operatorname{depth} M = \dim M$. If R is a Cohen-Macaulay module over itself, then it is called a *Cohen-Macaulay ring*. A *maximal Cohen-Macaulay module* is a Cohen-Macaulay module M such that $\dim M = \dim R$.

If R is an arbitrary Noetherian ring, then M is a *Cohen-Macaulay module* if $M_{\mathfrak{m}}$ is a Cohen-Macaulay module for all maximal ideals $\mathfrak{m} \in \operatorname{Supp}(M)$. For M to be a *maximal Cohen-Macaulay module* we also require that $M_{\mathfrak{m}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ of R. As in the local case, R is a *Cohen-Macaulay ring* if it is a Cohen-Macaulay module over itself.

Remark C.3. Any zero-dimensional ring is Cohen–Macaulay.

Theorem C.4. *Let R be a Noetherian ring, and let M be a Cohen–Macaulay R-module. Then for any multiplicatively closed subset U of R the localization $U^{-1}M$ is also Cohen–Macaulay. In particular, $M_{\mathfrak{p}}$ is Cohen–Macaulay for every $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. See [38, Theorem 2.1.3.(b)]. $\square$

Corollary C.5. *A Cohen–Macaulay ring satisfies Serre's condition* $(\mathrm{S_k})$ *for any $k \in \mathbb{N}$.* $\square$

Theorem C.6. *Let R be a Noetherian local ring, and let M be a finite R-module. Then M is Cohen–Macaulay if and only if its completion $\widehat{M}$ is Cohen–Macaulay.*

Proof. See [38, Corollary 2.1.8]. $\square$

Corollary C.7. *Let R be a Noetherian semilocal ring. Then R is Cohen–Macaulay if and only if $\widehat{R}$ is Cohen–Macaulay.*

Proof. By definition R is Cohen–Macaulay if and only if $R_{\mathfrak{m}}$ is Cohen–Macaulay for each $\mathfrak{m} \in \operatorname{Max}(R)$. By Theorem C.6 this is equivalent to $\widehat{R_{\mathfrak{m}}}$ being Cohen–Macaulay for every $\mathfrak{m} \in \operatorname{Max}(R)$. This is by Theorem A.59.(2) and Corollary A.62 the case if and only if $\left(\widehat{R}\right)_{\widehat{\mathfrak{m}}}$ is Cohen–Macaulay for every $\widehat{\mathfrak{m}} \in \operatorname{Max}\left(\widehat{R}\right)$. By definition this is equivalent to $\widehat{R}$ being Cohen–Macaulay. $\square$

Proposition C.8. *A Noetherian ring R is Cohen-Macaulay if and only if the polynomial ring $R[x]$ is Cohen-Macaulay.*

Proof. See [39, Proposition 18.9]. ☐

Corollary C.9. *A Noetherian ring R is Cohen-Macaulay if and only if the polynomial ring $R[x_1, \ldots, x_n]$ is Cohen-Macaulay.*

Proof. Apply Proposition C.8 inductively to $R[x_1, \ldots, x_n] = R[x_1, \ldots, x_{n-1}][x_n]$. ☐

Proposition C.10. *Let R be a Cohen–Macaulay ring, and let $\mathfrak{i}$ be an ideal of R. If $\mathfrak{i}$ is generated by* $\operatorname{height} \mathfrak{i}$ *elements, then $R/\mathfrak{i}$ is a Cohen–Macaulay ring.*

Proof. See [39, Proposition 18.13]. ☐

Proposition C.11. *A local Cohen-Macaulay ring is equidimensional.*

Proof. See [39, Corollary 18.11]. ☐

Proposition C.12. *Let R be a reduced Cohen–Macaulay ring, and let A be a finite integral extension of R with $R \subset A \subset Q_R$. Then*

$$\operatorname{height} \mathfrak{p} = \operatorname{height} \mathfrak{p} \cap R$$

for every prime ideal $\mathfrak{p}$ of A.

Proof. Let $\mathfrak{p}$ be a prime ideal of A, and set $\mathfrak{q} = \mathfrak{p} \cap R$. By Proposition B.6.(2) the ring $A' = (R \setminus \mathfrak{q})^{-1} A$ is an integral extension of R. Since A is finite over R, also $A' = A \otimes_R R_{\mathfrak{q}}$ is by Proposition A.38 finite over $R_{\mathfrak{q}}$. As R is reduced, Propositions A.24 and A.77 yield $R_{\mathfrak{p}} \subset (R \setminus \mathfrak{p})^{-1} A \subset (R \setminus \mathfrak{p})^{-1} Q_R = Q_{R_{\mathfrak{p}}}$. Moreover, Proposition A.20.(2) implies $\mathfrak{p}A' \in \operatorname{Max}(A')$.

By Theorem C.4 the ring $R_{\mathfrak{q}}$ is Cohen–Macaulay. Since the Cohen–Macaulay property commutes with completion by Theorem C.6, $\widehat{R_{\mathfrak{p}}}$ is by Theorem A.59.(2) and Proposition C.11 equidimensional, i.e. $R_{\mathfrak{q}}$ is formally equidimensional.

Then Lemma B.32 yields

$$\operatorname{height} \mathfrak{p}A' = \operatorname{height} \mathfrak{q}R_{\mathfrak{q}}.$$

Thus, we obtain

$$\operatorname{height} \mathfrak{p} = \operatorname{height} \mathfrak{q}$$

by Proposition A.20.(2). Also see [47, Proposition 8.7]. ☐

C.1. One-dimensional Cohen–Macaulay Rings

Proposition C.13. *A one-dimensional reduced Noetherian ring is a Cohen–Macaulay ring.*

Proof. Let R be a one-dimensional reduced Noetherian ring, and let $\mathfrak{m} \in \operatorname{Max}(R)$. If $\operatorname{height} \mathfrak{m} = 0$, then $R_{\mathfrak{m}}$ is a Cohen–Macaulay ring (see Remark C.3). So let $\operatorname{height} \mathfrak{m} = 1$. Then $R_{\mathfrak{m}}$ is a one-dimensional local reduced Noetherian ring by Corollary A.21 and Lemma A.27. So in the following let R be a one-dimensional local reduced Noetherian ring with maximal ideal $\mathfrak{m}$. We have to show that $\mathfrak{m}^{\mathrm{reg}} \neq \emptyset$.

So assume $\mathfrak{m}^{\mathrm{reg}} = \emptyset$. Then any $x \in \mathfrak{m}$ is a zerodivisor. Thus,

$$\mathfrak{m} \subset R \setminus R^{\mathrm{reg}} = \bigcup_{\mathfrak{p} \in \mathrm{Min}\,(R)} \mathfrak{p}$$

by Proposition A.47, and hence $\mathfrak{m} \in \mathrm{Min}\,(R)$ by Theorem A.2, and since $\mathfrak{m}$ is prime. This implies $\operatorname{height} \mathfrak{m} = 0$, contradicting the assumption.

Hence, there is an $x \in \mathfrak{m}^{\mathrm{reg}}$, and (x) is a maximal regular sequence in R since

$$\operatorname{depth} R \leq \dim R = \operatorname{height} \mathfrak{m} = 1$$

by Proposition C.1. In particular, we have $\operatorname{depth} R = \dim R$, and hence R is Cohen–Macaulay. $\square$

Theorem C.14. *Let R be a one-dimensional local Cohen–Macaulay ring. Then R is analytically reduced if and only if $\overline{R}$ is a finitely generated R-module.*

Proof. See [23, Chapter II, Theorem 3.22]. $\square$

Corollary C.15. *Let R be a reduced one-dimensional semilocal Cohen–Macaulay ring. Then R is analytically reduced if and only if $\overline{R}$ is a finite R-module.*

Proof. By Lemma A.68 R is analytically reduced if and only if $R_\mathfrak{m}$ is analytically reduced for every $\mathfrak{m} \in \mathrm{Max}\,(R)$. This is by Proposition A.20.(2) and Theorem C.14 equivalent to $\overline{R_\mathfrak{m}} = \overline{R}_\mathfrak{m}$ (see Corollary B.8) being a finite $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \mathrm{Max}\,(R)$ since $R_\mathfrak{m}$ is by Lemma A.27 reduced for every $\mathfrak{m} \in \mathrm{Max}\,(R)$. By Lemma A.42 this is the case if and only if $\overline{R}$ is a finite R-module. $\square$

Corollary C.16. *Let R be a one-dimensional semilocal Cohen–Macaulay ring. If R is analytically reduced, then $\mathfrak{C}_\mathfrak{I} \in \mathcal{R}_{\overline{R}} \subset \mathcal{R}_R$ for any $\mathfrak{I} \in \mathcal{R}_R$.*

Proof. Since R is analytically reduced, we have $\overline{R} \in \mathcal{R}_R$ by Remark 2.6.(1) and Corollary C.15. Hence, the statement follows from Lemmas 2.11 and 2.12. $\square$

C.2. Canonical Module

Definition C.17. Let R be a ring, and let M be an R-module. The *injective dimension* of M, denoted by $\operatorname{inj}\dim M$ or $\operatorname{inj}\dim_R M$, is the smallest integer n for which there exists an injective resolution $I^\bullet$ of M with $I^m = 0$ for all $m > n$. If there is no such n, the injective dimension of M is infinite.

Definition C.18. Let R be a local ring with maximal ideal $\mathfrak{m}_R$, and M be a finite non-zero R-module. Then the number

$$r(M) = \dim_{R/\mathfrak{m}_R} \operatorname{Ext}_R^{\operatorname{depth} M}(R/\mathfrak{m}_R, M)$$

is called the *type* of M.

Definition C.19. Let R be a local Cohen–Macaulay ring. A maximal Cohen–Macaulay module ω_R of type 1 and of finite injective dimension is called a *canonical module* of R.

Let R is an arbitrary Cohen–Macaulay ring. A finite R-module ω_R is a *canonical module* of R if $(\omega_R)_{\mathfrak{m}}$ is a canonical module of $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

Theorem C.20. *Let R be a local Cohen–Macaulay ring, and let ω_R and ω'_R be canonical modules of R.*

(1) The canonical modules ω_R and ω'_R are isomorphic.

(2) We have $\operatorname{Hom}_R(\omega_R, \omega'_R) \cong R$, and any generator ϕ of $\operatorname{Hom}_R(\omega_R, \omega'_R) \cong R$ is an isomorphism.

(3) The canonical homomorphism $R \to \operatorname{End}_R(\omega_R)$ is an isomorphism.

Proof. See [38, Theorem 3.3.4]. □

Theorem C.21. *Let R and A be local Cohen–Macaulay rings, and let $\phi\colon R \to A$ be a local homomorphism such that A is a finite R-module. If the canonical ω_R of R exists, then the canonical module ω_A of A exists, and*

$$\omega_A \cong \operatorname{Ext}_R^{\dim R - \dim A}(A, \omega_R).$$

Proof. See [38, Theorem 3.3.7.(b)]. □

Theorem C.22. *Let R be a local Cohen–Macaulay ring, and let ω_R be a finite R-module. Then the following conditions are equivalent:*

(a) ω_R is the canonical module of R.

(b) For any Cohen–Macaulay modules M of R we have

(1) $\operatorname{Ext}_R^{\dim R - \dim M}(M, \omega_R)$ is a Cohen–Macaulay R-module of dimension $\dim M$,

(2) $\operatorname{Ext}_R^{i}(M, \omega_R) = 0$ for all $i \neq \dim R - \dim M$, and

(3) there is an isomorphism

$$M \to \operatorname{Ext}_R^{\dim R - \dim M}\left(\operatorname{Ext}_R^{\dim R - \dim M}(M, \omega_R), \omega_R\right)$$

which in the case $\dim M = \dim R$ is the natural homomorphism from M into its bidual with respect to ω_R.

(c) For any maximal Cohen–Macaulay modules M of R we have

(1) $\operatorname{Hom}_R(M, \omega_R)$ is a maximal Cohen–Macaulay R-module,

(2) $\operatorname{Ext}_R^{i}(M, \omega_R) = 0$ for $i > 0$, and

(3) the natural homomorphism

$$M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$$

is an isomorphism.

Proof. See [38, Theorem 3.3.10]. □

Proposition C.23. *Let R be a Cohen–Macaulay ring, and let ω_R be a canonical module of R. If R is generically Gorenstein, i.e. if $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Min}(R)$, then ω_R can be identified with an ideal of R.*

C.3. Gorenstein Rings

Definition C.24. A Noetherian local ring R is a *Gorenstein ring* if $\operatorname{inj}\dim_R R < \infty$. A Noetherian ring is a *Gorenstein ring* if $R_{\mathfrak{m}}$ is a Gorenstein ring for every $\mathfrak{m} \in \operatorname{Max}(R)$.

Proposition C.25. *A Gorenstein ring is Cohen–Macaulay.*

Proof. Let R be a Gorenstein ring. Then by definition $R_{\mathfrak{m}}$ is Gorenstein for every $\mathfrak{m} \in \operatorname{Max}(R)$. Hence, $R_{\mathfrak{m}}$ is Cohen–Macaulay for every $\mathfrak{m} \in \operatorname{Max}(R)$ by [38, Proposition 3.1.20]. This implies that R is Cohen–Macaulay (see Definition C.2). □

Theorem C.26. *Let R be a local Cohen–Macaulay ring. Then the following conditions are equivalent:*

(a) R is Gorenstein.

(b) The canonical module ω_R of R exists, and it is isomorphic to R.

Proof. See [38, Theorem 3.3.7.(a)]. □

D. Valuations

D.1. Valuation Rings

Definition D.1. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical.

(1) A *valuation ring* of Q is a subring V of Q with $V \neq Q$ such that the set $Q \setminus V$ is multiplicatively closed.

(2) Let V be a valuation ring of Q. Then for any subring R of V with $Q_R = Q$ we call V a *valuation ring over* R.

(3) If R is a subring of Q with $Q_R = Q$, the set of valuation rings of Q over R is denoted by $\mathcal{V}_R$.

Lemma D.2. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical. A valuation ring V of Q is integrally closed in Q, and $Q = Q_V$ is the total ring of fractions of V.*

Proof. See [23, Chapter I, Lemma 2.1]. □

Theorem D.3. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a subring of Q with $V \neq Q$ and $Q_V = Q$. Then the following statements are equivalent:*

(a) V is a valuation ring of Q.

(b) For any $x \in Q^{\mathrm{reg}}$ we have either $x \in V$ or $x^{-1} \in V$.

(c) The set of regular principal fractional ideals of V is totally ordered by inclusion.

(d) The set $\mathcal{R}_V$ is totally ordered by inclusion.

(e) For any subring $V \subsetneq A \subset Q$ there is a prime ideal $\mathfrak{p} \in \operatorname{Spec}(V)$ such that $\mathfrak{p}A = A$.

Proof. See [23, Chapter I, Theorem 2.2] and Lemma D.2. □

Remark D.4. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q.

(1) Every finitely generated regular fractional ideal of V is principal, see [23, Chapter I, Proposition 2.4.(2)].

(2) Recall that every invertible fractional ideal of any ring is regular and finitely generated, see [23, Chapter II, Remark 2.1.(3) and Proposition 2.2.(2)]. Hence, $\mathcal{R}_V^*$ consists by (1) of the regular principal fractional ideals of V.

(3) The set $\mathcal{R}_V^*$ is by (2) and Theorem D.3.(c) totally ordered by inclusion.

Remark D.5. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q. Then V has a unique regular maximal ideal, denoted by $\mathfrak{m}_V$. In particular, we have $V^{\mathrm{reg}} \setminus V^* \subset \mathfrak{m}_V$. In fact, V has by [23, Chapter I, Theorem 2.2] a unique maximal ideal $\mathfrak{m}_V$ containing all regular non-units of V. Moreover, V is a Marot ring, i.e. any regular ideal $\mathfrak{i}$ of V is generated by its regular elements, and hence $\mathfrak{i} \subset \mathfrak{m}_V$.

The *infinite prime ideal* of V is

$$I_V = V : Q \in \operatorname{Spec}(V) \cap \operatorname{Spec}(Q),$$

see [23, Chapter I, Proposition 2.2.(3a)].

Remark D.6. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q. If V is a domain, then $\mathfrak{m}_V$ is the unique maximal ideal of V, and hence V is local.

Definition D.7. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q. We include $\mathcal{R}_V^*$ into the totally ordered monoid $\mathcal{R}_{V,\infty}^* = \mathcal{R}_V^* \cup \{I_V\}$, where $\mathfrak{I}\mathfrak{J} = I_V$ if $\{\mathfrak{I}, \mathfrak{J}\} \not\subset \mathcal{R}_V^*$, and the order is given by $\mathfrak{I} < I_V$ for all $\mathfrak{I} \in \mathcal{R}_V^*$ and $\mathfrak{I} < \mathfrak{J}$ if $\mathfrak{J} \subset \mathfrak{I}$ for $\mathfrak{I}, \mathfrak{J} \in \mathcal{R}_V^*$, cf. Remark D.4.(3).

Proposition D.8. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q.*

(1) We have

$$I_V = \bigcap_{\mathfrak{I} \in \mathcal{R}_V} \mathfrak{I} = \bigcap_{\mathfrak{I} \in \mathcal{R}_V^*} \mathfrak{I}.$$

(2) For any $x \in Q$ we have

$$\bigcap_{\substack{\mathfrak{I} \in \mathcal{R}_V \\ x \in \mathfrak{I}}} \mathfrak{I} \in \mathcal{R}_{V,\infty}^*$$

with $\bigcap_{x \in \mathfrak{I} \in \mathcal{R}_V} \mathfrak{I} = I_V$ if and only if $x \in I_V$.

(3) Let $x \in Q \setminus I_V$, and let $y \in Q^{\mathrm{reg}}$. Then

$$\bigcap_{\substack{\mathfrak{I} \in \mathcal{R}_V \\ x \in \mathfrak{I}}} \mathfrak{I} = yV$$

if and only if $xy^{-1} \in V \setminus \mathfrak{m}_V$.

Proof. (1) Let $x \in I_V$, and let $y \in Q^{\mathrm{reg}}$. Then $y^{-1} \in Q^{\mathrm{reg}}$, and $xy^{-1} \in V$ by definition of I_V. Therefore, $x \in yV$. Since Q is Marot, and hence any $\mathfrak{I} \in \mathcal{R}_V$ is generated by $\mathfrak{I}^{\mathrm{reg}} \subset Q^{\mathrm{reg}}$, this implies $I_V \subset \bigcap_{\mathfrak{I} \in \mathcal{R}_V} \mathfrak{I}$. Thus, $I_V \subset \bigcap_{\mathfrak{I} \in \mathcal{R}_V} \mathfrak{I}$.

Since $\mathcal{R}_V^* \subset \mathcal{R}_V$, we have $\bigcap_{\mathfrak{I} \in \mathcal{R}_V} \mathfrak{I} \subset \bigcap_{\mathfrak{I} \in \mathcal{R}_V^*} \mathfrak{I}$. Hence, there is an $x \in \bigcap_{\mathfrak{I} \in \mathcal{R}_V^*} \mathfrak{I}$. Let now $y \in Q$. Then there are $a \in V$ and $b \in V^{\mathrm{reg}}$ such that $y = ab^{-1}$. Moreover, we have $x \in bV$ since $bV \in \mathcal{R}_V^*$. Thus, $xy = xab^{-1} \in V$, and therefore $x \in V : Q = I_V$. So the chain of inclusions

$$I_V \subset \bigcap_{\mathfrak{I} \in \mathcal{R}_V} \mathfrak{I} \subset \bigcap_{\mathfrak{I} \in \mathcal{R}_V^*} \mathfrak{I} \subset I_V$$

yields the claim.

(2) If $x \in Q \setminus I_V$, then $\bigcap_{x \in \mathfrak{I} \in \mathcal{R}_V} \mathfrak{I}$ is a regular principal fractional ideal of V, see [23, Chapter I, Proposition 2.4(3b)]. Therefore, $\bigcap_{x \in \mathfrak{I} \in \mathcal{R}_V} \mathfrak{I} \in \mathcal{R}_V^*$ by Remark D.4.(2). The second part of the claim follows immediately from (1).

(3) See [23, Chapter I, Proposition 2.4.(3b)]. □

Remark D.9. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical. If V is a valuation ring of Q, then I_V is already the intersection of all regular ideals of V, see [23, Chapter I, Proposition 2.4].

Definition D.10. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q. Considering Proposition D.8 we define the *valuation* of V as the map

$$\begin{aligned} \mu_V \colon Q &\to \mathcal{R}_{V,\infty}^* \\ x &\mapsto \mu_V(x) = \bigcap_{\substack{\mathfrak{I} \in \mathcal{R}_V \\ x \in \mathfrak{I}}} \mathfrak{I}. \end{aligned}$$

Proposition D.11. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q. For any $x, y \in Q$ the valuation μ_V of V satisfies*

(V1) $\mu_V(xy) = \mu_V(x)\,\mu_V(y)$ *and*

(V2) $\mu_V(x+y) \geq \min\{\mu_V(x), \mu_V(y)\}$.

Proof. See [23, Chapter I, Proposition 2.13]. □

Lemma D.12. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, let V be a valuation ring of Q, and let $x \in Q^{\mathrm{reg}}$. Then*

(1) $\mu_V(x) = xV$, *and*

(2) $\mu_V(x^{-1}) = (\mu_V(x))^{-1}$.

Proof. (1) For any $x \in Q$ we have $xV \subset \mu_V(x)$ by Definition D.10. If $x \in Q^{\mathrm{reg}}$, then $xV \in \mathcal{R}_V^*$ by Remark D.4.(2). Since $x \in xV$, Definition D.10 yields $\mu_V(x) \subset xV$.

(2) We have

$$V = \mu_V(1) = \mu_V\left(xx^{-1}\right) = \mu_V(x)\,\mu_V\left(x^{-1}\right).$$

Since $\mu_V(x) \in \mathcal{R}_V^*$ by Proposition D.8.(2), this implies $\mu_V(x^{-1}) = \mu_V(x) : V = (\mu_V(x))^{-1}$, see Section 2.1. □

Proposition D.13. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q. Then*

(1) $V = \{x \in Q \mid \mu_V(x) \geq V\}$,

(2) $V^* = \{x \in Q^{\mathrm{reg}} \mid \mu_V(x) = V\}$,

(3) $\mathfrak{m}_V = \{x \in Q \mid \mu_V(x) > V\}$, *and*

(4) $I_V = \{x \in I_V \mid \mu_V(x) = I_V\}$.

In particular, $V^* = (V \setminus \mathfrak{m}_V)^{\mathrm{reg}}$ *and* $I_V \subset \mathfrak{m}_V$.

Proof. (1) If $x \in V$, then obviously $\mu_V(x) \geq V$. Conversely, if $x \in Q$ with $\mu_V(x) \geq V$, then $x \in \mu_V(x) \subset V$. Hence, $V = \{x \in Q \mid \mu_V(x) \geq V\}$.

(2) Let $x \in V^*$. Then $x \in V^{\mathrm{reg}} \subset Q^{\mathrm{reg}}$ and $x^{-1} \in V$, and hence $\mu_V(x), \mu_V(x^{-1}) \geq V$ by (1). Since $\mu_V(x^{-1}) = (\mu_V(x))^{-1}$ by Lemma D.12.(2), this implies $\mu_V(x) = \mu_V(x^{-1}) = V$.

Let now $x \in Q^{\mathrm{reg}}$ with $\mu_V(x) = V$, i.e. $xV = V$ by Lemma D.12.(1). Then $x \in V$ by (1), and there is a $y \in V$ such that $xy = 1$, i.e. $x \in V^*$.

(3) Set $\mathfrak{m} = \{x \in Q \mid \mu_V(x) > V\}$. If $x, y \in \mathfrak{m}$, then $x + y \in \mathfrak{m}$ by Proposition D.11 (see Equation (V2)). For $z \in V$ we have $\mu_V(z) \geq V$, and hence $xz \in \mathfrak{m}$ by Proposition D.11 (see Equation (V1)). Thus, $\mathfrak{m}$ is an ideal of V, and by (1) and (2) it contains all regular non-units of V. Thus, $\mathfrak{m} = \mathfrak{m}_V$, see Remark D.5.

(4) This follows from Proposition D.8.(2). □

Remark D.14. Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q.

(1) Let $x, y \in Q$ with $\mu_V(x) \neq \mu_V(y)$. Since $\mathcal{R}^*_{V,\infty}$ is totally ordered by reverse inclusion, we have $\mu_V(x + y) = \min\{\mu_V(x), \mu_V(y)\}$.

(2) If Q is a field, then Proposition D.8.(2) and Lemma D.12.(1) yield $I_V = \langle 0 \rangle$.

Proposition D.15. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q, and let $x, y \in Q$ with $x - y \in I_V$. Then $\mu_V(x) = \mu_V(y)$.*

Proof. Recall that by Proposition D.8.(1) and Definition D.10

$$I_V = \bigcap_{\mathfrak{I} \in \mathcal{R}^*_V} \mathfrak{I} \subset \bigcap_{\substack{\mathfrak{I} \in \mathcal{R}^*_V \\ x \in \mathfrak{I}}} \mathfrak{I} = \mu_V(x).$$

This implies $y - x \in \mu_V(x)$, and hence $y = x + y - x \in \mu_V(x)$. Therefore, we obtain $\mu_V(y) \subset \mu_V(x)$.

Interchanging x and y also yields $\mu_V(x) \subset \mu_V(y)$, and thus $\mu_V(x) = \mu_V(y)$. □

Proposition D.16. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let V be a valuation ring of Q such that $I_V \in \mathrm{Max}\,(Q)$. Then V/I_V is a valuation ring of Q/I_V, and there is an order preserving isomorphism of monoids $\phi\colon \mathcal{R}^*_{V,\infty} \to \mathcal{R}^*_{V/I_V,\infty}$ such that the diagram*

$$\begin{tikzcd} Q \arrow[r, "\pi"] \arrow[d, "\mu_V"'] & Q/I_V \arrow[d, "\mu_{V/I_V}"] \\ \mathcal{R}^*_{V,\infty} \arrow[r, "\cong", "\phi"'] & \mathcal{R}^*_{V/I_V,\infty} \end{tikzcd}$$

commutes, where $\pi\colon Q \to Q/I_V$ is the canonical surjection.

Proof. Let $\overline{x}, \overline{y} \in (Q/I_V)\backslash(V/I_V)$, and assume that $\overline{xy} \in V/I_V$. Then there are $x, y \in Q\backslash V$ and $z \in V$ such that $\pi\,(x) = \overline{x}$, $\pi\,(y) = \overline{y}$ and $\pi\,(z) = \overline{xy}$. Since $\pi\,(xy - z) = \pi\,(x)\,\pi\,(y) - \pi\,(z) = 0$, we have $xy - z \in I_V \subset V$, and hence $xy \in V$. But this is a contradiction as $x, y \in Q \setminus V$, and V is a valuation ring of Q, i.e. $Q \setminus V$ is multiplicatively closed. Therefore, since Q/I_V is the field of fractions of V/I_V, V/I_V is a valuation ring of Q/I_V.

Obviously, the map

$$\begin{aligned} \phi\colon \mathcal{R}^*_{V,\infty} &\to \mathcal{R}^*_{V/I_V,\infty} \\ \mathfrak{J} &\mapsto \pi\,(\mathfrak{J}) \end{aligned}$$

is an inclusion preserving homomorphism of monoids.

Let $x \in Q$. Then

$$\begin{aligned} \phi\,(\mu_V\,(x)) &= \pi\left(\bigcap_{x\in\mathfrak{J}\in\mathcal{R}^*_V} \mathfrak{J}\right) \\ &\subset \bigcap_{x\in\mathfrak{J}\in\mathcal{R}^*_V} \pi\,(\mathfrak{J}) \\ &\subset \bigcap_{\pi(x)\in\mathfrak{J}\in\mathcal{R}^*_{V/I_V}} \mathfrak{J} \\ &= \mu_{V/I_V}\,(\pi\,(x))\,. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \phi^{-1}\left(\mu_{V/I_V}\,(\pi\,(x))\right) &= \pi^{-1}\left(\bigcap_{\pi(x)\in\mathfrak{J}\in\mathcal{R}^*_{V/I_V}} \mathfrak{J}\right) \\ &= \bigcap_{\pi(x)\in\mathfrak{J}\in\mathcal{R}^*_{V/I_V}} \pi^{-1}\,(\mathfrak{J}) \\ &\subset \bigcap_{x\in\mathfrak{J}\in\mathcal{R}^*_V} \mathfrak{J} \\ &= \mu_V\,(x)\,. \end{aligned}$$

This implies $\phi \circ \mu_V\,(x) = \mu_{V/I_V} \circ \pi\,(x)$. □

Proposition D.17. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, let $\mathfrak{p} \in \operatorname{Max}(Q)$, and let V be a valuation ring of $Q/\mathfrak{p}$. If $\pi\colon Q \to Q/\mathfrak{p}$ is the canonical surjection, then $\pi^{-1}(V)$ is a valuation ring of Q with infinite prime ideal $I_{\pi^{-1}(V)} = \mathfrak{p}$, and there is an order preserving isomorphism of monoids $\phi\colon \mathcal{R}^*_{\pi^{-1}(V),\infty} \to \mathcal{R}^*_{V,\infty}$ such that the diagram*

$$\begin{tikzcd} Q \arrow[r, "\pi"] \arrow[d, "\mu_{\pi^{-1}(V)}"'] & Q/\mathfrak{p} \arrow[d, "\mu_V"] \\ \mathcal{R}^*_{\pi^{-1}(V),\infty} \arrow[r, "\cong", "\phi"'] & \mathcal{R}^*_{V,\infty} \end{tikzcd}$$

commutes.

Proof. Let $x, y \in Q \setminus \pi^{-1}(V)$, and assume that $xy \in \pi^{-1}(V)$. Then we have $\pi(x), \pi(y) \in V$ and $\pi(x)\pi(y) = \pi(xy) \in V$. But this is a contradiction since V is a valuation ring of $Q/\mathfrak{p}$. Therefore, $\pi^{-1}(V)$ is a valuation ring of Q.

For any $x \in Q$ we have $xQ \subset \pi^{-1}(V)$ if and only if $\pi(xQ) = \pi(x)\pi(Q) \subset V$. Thus, $x \in I_{\pi^{-1}(V)}$ if and only if $\pi(x) \in I_V = 0$ (see Remark D.14.(2)), and this is the case if and only if $x \in \mathfrak{p}$. This implies $I_{\pi^{-1}(V)} = \mathfrak{p}$.

The remaining part of the statement follows now from Proposition D.16. $\square$

Corollary D.18. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, and let $\mathfrak{p} \in \operatorname{Max}(Q)$. There is a one-to-one correspondence between the valuation rings of Q with infinite prime ideal $\mathfrak{p}$ and valuation rings of $Q/\mathfrak{p}$ with infinite prime ideals $\langle 0\rangle_{Q/\mathfrak{p}}$. Moreover, if V and $\overline{V}$ are corresponding valuation rings of Q and $Q/\mathfrak{p}$, respectively, then there is an order preserving isomorphism of monoids $\phi\colon \mathcal{R}^*_{V,\infty} \to \mathcal{R}^*_{\overline{V},\infty}$ such that the diagram*

$$\begin{tikzcd} Q \arrow[r, "\pi"] \arrow[d, "\mu_V"'] & Q/\mathfrak{p} \arrow[d, "\mu_{\overline{V}}"] \\ \mathcal{R}^*_{V,\infty} \arrow[r, "\cong", "\phi"'] & \mathcal{R}^*_{\overline{V},\infty} \end{tikzcd}$$

commutes, where $\pi\colon Q \to Q/\mathfrak{p}$ is the canonical surjection.

Proof. This follows from Propositions D.16 and D.17. Also see [23, Chapter I, Proposition 2.17]. $\square$

Lemma D.19. *Let Q be a ring with $Q^{\mathrm{reg}} = Q^*$ having a large Jacobson radical, let V be a valuation ring of Q, and let U be a multiplicatively closed subset of V such that $\mu_V(u) = V$ for all $u \in U$. We denote by $\alpha\colon Q \to U^{-1}Q$ the localization map.*

(1) We have $\alpha^{-1}(U^{-1}V) = V$. In particular, $U^{-1}V \subsetneq U^{-1}Q$.

(2) The set $U^{-1}Q \setminus U^{-1}V$ is multiplicatively closed. In particular, if $\left(U^{-1}Q\right)^{\mathrm{reg}} = \left(U^{-1}Q\right)^*$ and $U^{-1}Q$ has a large Jacobson radical, then $U^{-1}V$ is a valuation ring of $U^{-1}Q$ (see Definition D.1).

Proof. (1) Since $\mu_V(u) = V$ for all $u \in U$, Lemma A.35.(1) yields

$$\begin{aligned}\alpha^{-1}\Big(U^{-1}V\Big) &= \{x \in Q_R \mid ux \in V \text{ for some } u \in R \setminus \mathfrak{m}\} \\ &\subset \{x \in Q_R \mid ux \in V \text{ for some } u \in V \setminus \mathfrak{m}_V\} \\ &= \{x \in Q_R \mid ux \in V \text{ for some } u \in Q_R \text{ with } \mu_V(u) = V\},\end{aligned}$$

where the last equality follows from Proposition D.13.(1) and (3). So with Proposition D.13.(1) there is for any $x \in \pi^{-1}(U^{-1}V)$ a $u \in Q_R$ with $\mu_V(u) = V$ such that

$$V \leq \mu_V(ux) = \mu_V(u)\mu_V(x) = \mu_V(x)$$

by Proposition D.11, and hence $x \in V$. Therefore,

$$\alpha^{-1}\Big(U^{-1}V\Big) = V,$$

and, in particular, $U^{-1}V \subsetneq U^{-1}Q$.

(2) Let $\frac{a}{b}, \frac{c}{d} \in U^{-1}Q \setminus U^{-1}V$, i.e. $a, c \in Q \setminus V$, and suppose that $\frac{a}{b}\frac{c}{d} \in U^{-1}V$. Then there is an $e \in V$ and an $f \in U$ such that

$$\frac{a}{b}\frac{c}{d} = \frac{ac}{cd} = \frac{e}{f},$$

i.e. there is a $u \in U$ such that

$$u(ace - cdf) = 0.$$

Since $b, d, u \in I$, and since U is multiplicatively closed, we have $bdu \in U$. By assumption this implies $\mu_V(bdu) = V$, and hence $bdu \in V$ by Proposition D.13.(1). So $e \in V$ implies

$$uacf = ubde \in V.$$

Since $u, f \in U$, we have $uf \in U$, and hence $\mu_V(uf) = V$ by assumption. Thus, Propositions D.11 and D.13.(1) yield the contradiction

$$V \leq \mu_V(uacf) = \mu_V(uf)\mu_V(ac) = \mu_V(ac) < V$$

since $ac \in Q \setminus V$ as V is a valuation ring of Q, i.e. $Q \setminus V$ is multiplicatively closed, and $a, c \in Q \setminus V$. □

D.2. Valuations

Definition D.20. Let G be an additive abelian totally ordered group. We include G into the totally ordered commutative monoid $G_\infty = G \cup \{\infty\}$, where ∞ is a symbol such that $x + \infty = \infty$, $\infty + \infty = \infty$, and $\infty > x$ for all $x \in G$.

Definition D.21. Let A be a ring. A *valuation* of A is a map ν from A onto $G_\infty^\nu = (G^\nu)_\infty$, where G^ν is an additive abelian totally ordered group, satisfying

$$\nu(xy) = \nu(x) + \nu(y) \tag{D.1}$$

and

$$\nu(x+y) \geq \min\{\nu(x), \nu(y)\} \tag{D.2}$$

for all $x, y \in A$.

Let $\nu\colon A \to G_\infty^\nu$ a valuation of A.

(1) For $x \in A$ the element $\nu(x) \in G_\infty$ is called the *value* of x in the valuation.

(2) The group G^ν is called the *value group* of the valuation.

(3) The valuation ν is said to be *trivial* if its value group G^ν is trivial, i.e. $G^\nu = \{0\}$. Otherwise, ν is called *non-trivial.*

(4) If a subfield $k \subset A$ is specified as ground field, then ν is said to be a *valuation over* k if ν is trivial on k, i.e. if $\nu(c) = 0$ for all $c \in k$.

In the following, we collect some properties of valuations which follow immediately from the definition.

Lemma D.22. *Let A be a ring, and let ν be a valuation of A.*

(1) We have $\nu(1) = 0$. Moreover, if ν is non-trivial, then $\nu(0) = \infty$.

(2) For any $x \in A^*$ we have $\nu(x^{-1}) = -\nu(x)$. In particular, $\nu(x) < \infty$.

(3) Let $x \in A$. If there is $n \in \mathbb{N}$ such that $x^n = 1$, then $\nu(x) = 0$. In particular, $\nu(-1) = 0$.

(4) For any $x \in A$ we have $\nu(-x) = \nu(x)$.

(5) If $x, y \in A$ such that $\nu(x) \neq \nu(y)$, then

$$\nu(x+y) = \min\{\nu(x), \nu(y)\}.$$

Proof. (1) For any $x \in A$ we have

$$\nu(x) = \nu(x \cdot 1) = \nu(x) + \nu(1),$$

and hence $\nu(1) = 0$.

If ν is non-trivial, then there is an $x \in A$ with $0 \neq \nu(x) \in G^\nu$, and hence

$$\nu(0) = \nu(0x) = \nu(0) + \nu(x)$$

implies $\nu(0) = \infty$ since otherwise $\nu(x) = 0$ yields a contradiction to the assumption.

(2) If $x \in A^*$, then (1) yields

$$0 = \nu(1) = \nu\left(x \cdot x^{-1}\right) = \nu(x) + \nu\left(x^{-1}\right).$$

(3) Assume $\nu(x) \neq 0$, i.e. $\nu(x) > 0$ or $\nu(x) < 0$ since G^ν is totally ordered. Then we have by (1)

$$0 = \nu(1) = \nu(x^n) = \nu(x) + \nu\left(x^{n-1}\right) \lesseqgtr \nu\left(x^{n-1}\right) \lesseqgtr \ldots \lesseqgtr \nu(x) \lesseqgtr 0$$

which is a contradiction.

(4) By (3) we have

$$\nu(-x) = \nu(-1) + \nu(x) = \nu(x).$$

(5) Since $\nu(x) \neq \nu(y)$, we may assume that $\nu(x) > \nu(y)$. Then we have by the definition of a valuation

$$\nu(x+y) \geq \min(\nu(x), \nu(y)) = \nu(y).$$

Moreover, also using Lemma D.22.(4) yields

$$\nu(x+y-x) \geq \min(\nu(x+y), \nu(-x)) = \min(\nu(x+y), \nu(x)).$$

Now assume $\nu(x) \leq \nu(x+y)$. Then

$$\nu(y) = \nu(x+y-x) \geq \min(\nu(x+y), \nu(x)) = \nu(x).$$

But this is a contradiction to the assumption $\nu(x) > \nu(y)$. Thus, we have $\nu(x) > \nu(x+y)$. This implies

$$\begin{aligned} \nu(y) &= \nu(x+y-x) \\ &\geq \min(\nu(x+y), \nu(x)) \\ &= \nu(x+y) \\ &\geq \min(\nu(x), \nu(y)) \\ &= \nu(y), \end{aligned}$$

and hence $\nu(x+y) = \nu(y) = \min(\nu(x), \nu(y))$. □

Definition D.23. Let A be a ring, and let ν be a valuation of A. The *valuation ring* of ν is

$$V_\nu = \{x \in A \mid \nu(x) \geq 0\} \subset A.$$

Moreover, we denote by

$$\mathfrak{m}_\nu = \{x \in A \mid \nu(x) > 0\} \subset V_\nu$$

the *prime ideal* of the valuation, and

$$I_\nu = \nu^{-1}(\infty) = \{x \in A \mid \nu(x) = \infty\}$$

is called the *infinite prime ideal* of ν.

Remark D.24. Let A be a ring, and let ν be a valuation of A. Note that in fact V_ν is a subring of A. If ν is non-trivial, then V_ν is not equal to A since ν is surjective. Moreover, $\mathfrak{m}_\nu$ is a prime ideal of V_ν, and the infinite prime ideal I_ν is a prime ideal of both V_ν and A.

Also note that Lemma D.22.(2) implies

$$V_\nu^* = \{x \in A^* \mid \nu(x) = 0\}.$$

Remark D.25. Let Q be a ring with large Jacobson radical such that $Q^{\mathrm{reg}} = Q^*$.

(1) If ν is a non-trivial valuation of Q, then V_ν is a valuation ring of Q as in Definition D.1.

(2) If V is a valuation ring of Q, then the map μ_V is by Proposition D.11 a valuation of Q.

The following proposition characterizes which subrings of a ring A are rings of valuations of A.

Proposition D.26. *Let A be a ring, let V be a subring of A, and let $\mathfrak{p}$ be a prime ideal of V. Then the following are equivalent.*

(a) For each subring R of A and any ideal $\mathfrak{q}$ of R with $V \subset R$ and $\mathfrak{q} \cap R = \mathfrak{p}$ we have $V = R$.

(b) For any $x \in A \setminus V$ there is an $y \in \mathfrak{p}$ such that $xy \in V \setminus \mathfrak{p}$.

(c) There is a valuation ν of A with $V = V_\nu$ and $\mathfrak{p} = \mathfrak{m}_\nu$.

Proof. See [48, Proposition 1]. □

Proposition D.27. *Let A be a ring, and let ν be a non-trivial valuation of A. Then the infinite prime ideal of ν is*

$$I_\nu = V_\nu : A.$$

Proof. Let $x \in I_\nu$. Then we have for all $y \in A$

$$\nu(xy) = \nu(x) + \nu(y) = \infty + \nu(y) = \infty > 0.$$

This yields $xy \in V_\nu$, and hence $x \in V_\nu : A$.

Now let G^ν be the value group of ν, and assume there is $x \in V_\nu : A$ such that $\nu(x) < \infty$. Since G^ν is a group, we have $-\nu(x) \in G^\nu$, and there is $y \in A$ with $\nu(y) = -\nu(x)$ as ν is surjective. Moreover, there is $z \in A$ with $\nu(z) < 0$ since ν is non-trivial.

Thus, we have $yz \in A$, and therefore $xyz \in V_\nu$ since $x \in V_\nu : A$. Moreover, we have

$$\nu(xyz) = \nu(x) + \nu(y) + \nu(z) = \nu(z) < 0.$$

But this is a contradiction to $xyz \in V_\nu = \{a \in A \mid \nu(a) \geq 0\}$. Hence, we have $\nu(x) = \infty$, and thus $x \in I_\nu$.

Then $I_\nu \subset V_\nu : A$ and $V_\nu : A \subset I_\nu$ yield $I_\nu = V_\nu : A$. □

Definition D.28. Let A be a ring, and let ν and ν' be valuations of A with value groups G_∞^ν and $G_\infty^{\nu'}$, respectively. Then ν and ν' are called *equivalent* if there is an order preserving isomorphism ϕ from G_∞^ν onto $G_\infty^{\nu'}$ such that

$$\nu'(x) = \phi \circ \nu(x)$$

for all $x \in A \setminus I_\nu$. We will identify equivalent valuations.

Proposition D.29. *Let A be a ring, and let ν and ν' be valuations of A. Then ν and ν' are equivalent if and only if $V_\nu = V_{\nu'}$ and $\mathfrak{m}_\nu = \mathfrak{m}_{\nu'}$.*

Proof. See [48, Proposition 2]. □

Proposition D.30. *Let Q be a ring having a large Jacobson radical such that $Q^* = Q^{\mathrm{reg}}$, and let ν be a valuation of Q. Then $\mathfrak{m}_\nu = \mathfrak{m}_{V_\nu}$ and $I_\nu = I_{V_\nu}$.*

Proof. See [23, Chapter I, Proposition 2.12]. □

Corollary D.31. *Let Q be a ring having a large Jacobson radical such that $Q^* = Q^{\mathrm{reg}}$, and let ν and ν' be valuations of Q. Then ν and ν' are equivalent if and only if $V_\nu = V_{\nu'}$.*

Proof. This follows from Remark D.5 and Propositions D.29 and D.30. □

Corollary D.32. *Let Q be a ring having a large Jacobson radical such that $Q^* = Q^{\mathrm{reg}}$. Then there is a bijection*

$$\begin{aligned} V &\mapsto \mu_V, \\ V_\nu &\leftarrow\!\shortmid \nu \end{aligned}$$

between the valuation rings and the valuations of Q.

In particular, we have for any valuation ring V of Q

$$V = V_{\mu_V},$$

and for any valuation ν of Q we obtain

$$\nu = \mu_{V_\nu}$$

(modulo equivalence of valuations).

Proof. This follows from Remark D.25 and Corollary D.31. Also see [23, Chapter I, Propositions 2.12 and 2.13]. □

Proposition D.33. *Let A be a ring, let ν be a valuation of A, and let $x, y \in A$ with $x - y \in I_\nu$. Then $\nu(x) = \nu(y)$.*

Proof. Let $x, y \in A$ such that $x - y \in I_\nu$, and assume $\nu(x) \neq \nu(y)$. Since $\nu(y) = \nu(-y)$ by Lemma D.22.(4), Lemma D.22.(5) yields

$$\infty = \nu(x - y) = \min(\nu(x), \nu(-y)) = \min(\nu(x), \nu(y)).$$

This implies $\nu(x) \geq \infty$ and $\nu(y) \geq \infty$, and hence

$$\nu(x) = \infty = \nu(y)$$

since $\nu(x), \nu(y) \in G_\infty^\nu$. However, this is a contradiction to the assumption $\nu(x) \neq \nu(y)$. □

Proposition D.34. *Let A be a ring, and let ν be a valuation of A. Then there is a valuation μ of the domain A/I_ν such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "\nu"'] & A/I_\nu \arrow[dl, "\mu"] \\ G_\infty^\nu & \end{tikzcd}$$

commutes, where $\pi\colon A \to A/I_\nu$ is the canonical surjection. Moreover, we have

$$V_\mu = V_\nu / I_\nu$$

and

$$I_\mu = \langle 0 \rangle_{A/I_\nu}.$$

Proof. Since by Proposition D.33 $\nu(x) = \nu(y)$ for all $x, y \in A$ with $x - y \in I_\nu$, the map

$$\begin{aligned} \mu\colon A/I_\nu &\to G_\infty^\nu \\ x + I_\nu &\mapsto \nu(x) \end{aligned}$$

is well-defined, and it is clearly a valuation of A/I_ν. The ring of μ is

$$V_\mu = \{x + I_\nu \in A/I_\nu \mid \mu(x + I_\nu) = \nu(x) \geq 0\} = V_\nu / I_\nu.$$ □

Proposition D.35. *Let A be a ring, let $\mathfrak{p}$ be a prime ideal of A, and let μ be a valuation of $A/\mathfrak{p}$ with $I_\mu = \langle 0 \rangle_{A/\mathfrak{p}}$. Then there is a valuation ν of A with $I_\nu = \mathfrak{p}$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "\nu"'] & A/I_\nu \arrow[d, "\mu"] \\ & G_\infty^\mu \end{tikzcd}$$

commutes, where $\pi\colon A \to A/\mathfrak{p}$ is the canonical surjection. Moreover,

$$V_\nu / \mathfrak{p} = V_\mu.$$

Proof. Obviously, the map

$$\begin{aligned} \nu \colon A &\to G^{\mu}_{\infty} \\ x &\mapsto \mu\left(x + \mathfrak{p}\right) \end{aligned}$$

is a valuation of A.

For $x \in \mathfrak{p}$ we have

$$\nu\left(x\right) = \mu\left(x + \mathfrak{p}\right) = \mu\left(0 + \mathfrak{p}\right) = \infty,$$

and hence $x \in I_\nu$. This implies $\mathfrak{p} \subset I_\nu$.

Let now $x \in I_\nu$. Then

$$\infty = \nu\left(x\right) = \mu\left(x + \mathfrak{p}\right).$$

This implies $x + \mathfrak{p} \in I_\mu = \langle 0 \rangle_{A/\mathfrak{p}}$, i.e. $x + \mathfrak{p} \subset \mathfrak{p}$. This implies $x \in \mathfrak{p}$, and hence $I_\nu \subset \mathfrak{p}$.

The remaining part of the statement follows now from Proposition D.34 ☐

Corollary D.36. *Let A be a ring, and let $\mathfrak{p}$ be a prime ideal of A. There is a one-to-one correspondence between valuations of A with infinite prime ideal $\mathfrak{p}$ and valuations of $A/\mathfrak{p}$ with infinite prime ideal $\langle 0 \rangle_{A/\mathfrak{p}}$. Moreover, if ν and $\overline{\nu}$ are corresponding valuations of A and $A/\mathfrak{p}$, respectively, then there is an additive abelian totally ordered group G such that the diagram*

$$\begin{tikzcd} A \arrow[rr, "\pi"] \arrow[dr, "\nu"'] & & A/\mathfrak{p} \arrow[dl, "\overline{\nu}"] \\ & G_\infty & \end{tikzcd}$$

commutes, where $\pi \colon A \to A/\mathfrak{p}$ is the canonical surjection. Also see [23, Chapter I, Proposition 2.17]. ☐

E. Gradings and Derivations

E.1. Gradings

Definition E.1. Let R be a ring, and let G be an additive abelian group.

(1) A *finite (G-)grading* of R is a system

$$\left(\pi_p^R\right)_{p\in G} \in \left(\mathrm{End}\left(R\right)\right)^G$$

of group endomorphisms such that

$$\left(\pi_p^R(R)\right)\left(\pi_q^R(R)\right) \subset \pi_{p+q}^R(R)$$

for all $p,q \in G$ and

$$R \cong \bigoplus_{p\in G} \pi_p^R(R).$$

If there is a finite G-grading of R, then R is called *finitely (G-)graded.*

(2) Let $\left(\pi_p^R\right)_{p\in G}$ R be a finite G-grading of R, and let M be an R-module. A *finite (G-)grading* of M is a system

$$\left(\pi_p^M\right)_{p\in G} \in \left(\mathrm{End}_R\left(M\right)\right)^G$$

of group endomorphisms such that

$$\left(\pi_p^R(R)\right)\left(\pi_q^R(M)\right) \subset \pi_{p+q}^R(M)$$

for all $p,q \in G$ and

$$M \cong \bigoplus_{p\in G} \pi_p^R(M).$$

If there is a finite G-grading of M, then M is called *finitely (G-)graded.*

Definition E.2. Let R be a Zariski ring (see Definition A.57), and let G be an additive abelian group.

(1) A system of group endomorphisms

$$\left(\pi_p^R\right)_{p\in G} \in \left(\mathrm{End}\left(R\right)\right)^G$$

is called a *(G-)grading* of R if for any $n \in \mathbb{N}$ it induces a finite G-grading

$$\left(\pi_p^{R/\mathfrak{n}_R^n}\right)_{p\in G} = \left(\overline{\pi_p^R}\right)_{p\in G}$$

of the ring $R/\mathfrak{n}_R^n$.

If there is a G-grading of R, then R is called *(G-)graded*.

(2) Let R be G-graded, and let M be a finite R-module. A system of group endomorphisms

$$\left(\pi_p^M\right)_{p\in G} \in \left(\operatorname{End}_R(M)\right)^G$$

is called a *(G-)grading* of M if for any $n \in \mathbb{N}$ it induces a finite G-grading

$$\left(\pi_p^{M/\mathfrak{n}_R^n M}\right)_{p\in G} = \left(\overline{\pi_p^M}\right)_{p\in G}$$

of the $R/\mathfrak{n}_R^n$-module $M/\mathfrak{n}_R^n M$, where $R/\mathfrak{n}_R^n$ is graded by the induced grading.

If there is a G-grading of M, then M is called *(G)-graded*.

In the following, let R be a Zariski ring (see Definition A.57), and let G be an additive abelian group such that there is a G-grading $\left(\pi_p^R\right)_{p\in G}$ of R.

Definition E.3. Let M be a G-graded R-module, and let $(x_p)_{p\in G} \in M^G$. The sum $\sum_{p\in G}$ is called *convergent* if there is an $x \in M$ such that for any $n \in \mathbb{N}$ there is a finite subset $E_n \subset G$ such that for all finite subsets $E \subset G$ with $E_n \subset E$ we have

$$x - \sum_{p\in E} x_p \in \mathfrak{n}_R^n M.$$

We also say that $\sum_{p\in G} x_p$ *converges* to $x \in M$, and we write

$$x = \sum_{p\in G} x_p.$$

Proposition E.4. *Let M be a G-graded R-module. Then*

$$x = \sum_{p\in G} \pi_p^R(x)$$

for any $x \in M$. Conversely, if $x = \sum_{p\in G} x_p$ with $x_p \in \pi_p^R(M)$ for all $p \in G$, then $x_p = \pi_p^R(x)$ for all $p \in G$.

Proof. See [49, (1.1)]. □

Definition E.5. Let M be a G-graded R-module.

(1) Let $x \in M$. For any $p \in G$ we call $x_p = \pi_p^M(x)$ the *p-th homogeneous component* of x. If $\pi_p^M(x) = x$ for some $p \in G$, then x is called *homogeneous*, and p is the *degree* of x. We write $\deg(x)$ for the degree of x.

(2) For any $p \in G$ we set

$$M_p = \{x \in M \mid x \text{ homogeneous with } \deg(x) = p\}.$$

(3) An R-submodule N of M is called *homogeneous* if

$$\pi_p^M(N) \subset N$$

for all $p \in G$.

Proposition E.6. *Let M be a G-graded R-module, and let N be an R-submodule of M.*

(1) N is homogeneous if and only if it is generated by homogeneous elements.

(2) Let N be homogeneous. Then

$$\left(\pi_p^N\right)_{p\in G} = \left(\pi_p^M\Big|_N\right)_{p\in G}$$

is a G-grading of N.

(3) Let N be homogeneous. Then the G-grading of M induces a G-grading of the R-module M/N.

Proof. See [49, (1.3), (1.4) and (1.5)]. □

Lemma E.7. *Let M be a G-graded R-module, and let N be a homogeneous R-submodule of M. For any $p \in G$ we have*

$$(M/N)_p = \pi(M_p)$$

(with respect to the induced grading on M/N, see Proposition E.6.(3)), where $\pi\colon M \to M/N$ is the canonical surjection.

Proof. Since we consider the induced grading on M/N, there is for any $p \in G$ a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\pi_p^M"] \arrow[d, two heads, "\pi"'] & M \arrow[d, two heads, "\pi"] \\ M/N \arrow[r, "\pi_p^{M/N}"'] & M/N. \end{tikzcd}$$

This implies

$$\pi(M_p) = \pi \circ \pi_p^M(M) = \pi_p^{M/N} \circ \pi(M) = \pi_p^{M/N}(M/N) = (M/N)_p. \qquad \square$$

Definition E.8. Let M and N be G-graded R-modules. A homomorphism $\phi\colon M \to N$ is called *homogeneous* (of type $q \in G$) if

$$\phi\Big(\pi_p^M(M)\Big) \subset \pi_{p+q}^N(N)$$

or all $p \in G$.

Proposition E.9. *Let M and N be G-graded R-modules, and let $\phi\colon M \to N$ be homogeneous of type $q \in G$. Then*

$$\phi\Big(\pi_p^M(x)\Big) = \pi_{p+q}^N(\phi(x))$$

for all $x \in M$.

Proof. See [49, page 165]. □

E.2. Derivations

Definition E.10. Let k be a valued field. An *analytic k-algebra* is is a complete local Noetherian ring with coefficient field k.

Theorem E.11. *Let k be a valued field, and let R be an analytic k-algebra.*

(1) Let $\mathfrak{d}$ be a k-derivation of R such that $\mathfrak{m}_R$ is generated by eigenvalues of $\mathfrak{d}$. Then there is exactly one k_+-grading $\left(\pi_p^R\right)_{p\in k}$ of R such that $\pi_p^R(R)$ consists of p-eigenvectors of $\mathfrak{d}$ for any $p \in k$.

(2) Let $\left(\pi_p^R\right)_{p\in G}$ be a k_+-grading of R. Then there is exactly one k-derivation $\mathfrak{d}$ of R such that $\pi_p^R(R)$ consists of p-eigenvectors of $\mathfrak{d}$ for any $p \in k$.

Proof. See [49, Satz (2.2) and (2.3)]. □

Definition E.12. Let k be a valued field, and let R be an analytic k-algebra. A k-derivation $\mathfrak{d}$ of R is called *diagonalizable* if $\mathfrak{m}_R$ is generated by eigenvectors of $\mathfrak{d}$.

Theorem E.13. *Let k be a field, let $A = k[[X_1, \ldots, X_n]]$, let $\mathfrak{i}$ be an ideal of A, and let $R = A/\mathfrak{i}$. We denote by $\pi\colon A \to R$ the canonical surjection, and we write $x_i = \pi(X_i)$ for all $i = 1, \ldots, n$. Then for any $w \in k^n$ the following are equivalent:*

(a) R is k_+-graded, and x_i is homogeneous with $\deg(x_i) = w_i$ for any $i = 1, \ldots, n$.

(b) There is a diagonalizable k-derivation $\mathfrak{d}_R$ of R such that $\mathfrak{d}_R(x_i) = w_i x_i$ for all $i = 1, \ldots, n$.

(c) There is a diagonalizable k-derivation $\mathfrak{d}_A$ of A such that $\mathfrak{d}_A(X_i) = w_i X_i$ for all $i = 1, \ldots, n$ and $\mathfrak{i}$ is invariant under $\mathfrak{d}_A$.

(d) The ideal $\mathfrak{i}$ is homogeneous with respect to weighted polynomial degree with weights w.

If these equivalent conditions hold, then there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathfrak{d}_A} & A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ R & \xrightarrow[\mathfrak{d}_R]{} & R. \end{array} \tag{E.1}$$

Moreover, the grading on R is induced by the grading on A corresponding to $\mathfrak{d}_A$.

For the proof of Theorem E.13 we need the following Lemmas.

Lemma E.14. *Let k be a field, let $A = k[[X_1, \ldots, X_n]]$, let $\mathfrak{i} \in A$ be an ideal, and let $R = A/\mathfrak{i}$. We write $\pi \colon A \to R$ for the canonical surjection, and $x_i = \pi(X_i)$ for $i = 1, \ldots, n$. Let $\mathfrak{d}_R$ be a k-derivation of R. For any*

$$(y_i)_{i=1,\ldots,n} \in \prod_{i=1}^{n} \pi^{-1}\left(\mathfrak{d}_R(x_i)\right)$$

there is a k-derivation $\mathfrak{d}_A$ of A such that the diagram

$$\begin{tikzcd} A \arrow[r, "\mathfrak{d}_A"] \arrow[d, "\pi"'] & A \arrow[d, "\pi"] \\ R \arrow[r, "\mathfrak{d}_R"'] & R \end{tikzcd}$$

commutes, and $\mathfrak{d}_A X_i = y_i$ for all $i = 1, \ldots, n$. If there is $(w_i)_{i=1,\ldots,n} \in k^n$ such that $\mathfrak{d}_R x_i = w_i x_i$ for all $i = 1, \ldots, n$, then we may have $\mathfrak{d}_A X_i = w_i X_i$.

In particular, for any diagonalizable k-derivation $\mathfrak{d}_R$ of R there is a diagonalizable k-derivation $\mathfrak{d}_A$ of A with $\mathfrak{d}_R \circ \pi = \pi \circ \mathfrak{d}_A$.

Proof. See [49, (2.1)]. □

Lemma E.15. *Let k be a field, let R be a k_+-graded analytic k-algebra, and let $\mathfrak{d}$ be the k-derivation of R corresponding to the grading (see Theorem E.11.(2)). Then an ideal $\mathfrak{i}$ of R is homogeneous if and only if it is invariant under $\mathfrak{d}$.*

Proof. See [49, (2.4)]. □

Lemma E.16. *Let k be a field, let A be a k_+-graded analytic k-algebra, and let $\mathfrak{d}$ be the k-derivation of A corresponding to the grading (see Theorem E.11.(2)). If $\mathfrak{i}$ is a homogeneous ideal, then the induced grading on $R = A/\mathfrak{i}$ (see Proposition E.6.(3)) corresponds to the derivation*

$$\begin{aligned} \overline{\mathfrak{d}} \colon R &\to R, \\ x + \mathfrak{i} &\mapsto \mathfrak{d}(x) + \mathfrak{i} \end{aligned}$$

(see Theorem E.11.(2)).

Proof. First note that $\overline{\mathfrak{d}}$ is well-defined since $\mathfrak{i}$ is homogeneous, and hence $\mathfrak{d}(\mathfrak{i}) \subset \mathfrak{i}$ by Lemma E.15.

Let now $p \in k_+$, and let $x \in R_p$. Then by Lemma E.7 there is an element $X \in A_p$ such that $x = X + \mathfrak{i}$. Theorem E.11.(2) yields $\delta(X) = pX$. Thus, we obtain

$$\overline{\mathfrak{d}}(x) = \mathfrak{d}(X) + \mathfrak{i} = pX + \mathfrak{i} = px.$$

This implies that $\pi_p^R(R)$ contains of p-eigenvectors of $\overline{\mathfrak{d}}$, and hence $\overline{\mathfrak{d}}$ is by Theorem E.11.(2) the k-derivation of R corresponding to the induced grading on R. □

Proof of Theorem E.13. (a) $\Longrightarrow$ (b) This follows from Theorem E.11.(2).

(b) $\Longrightarrow$ (c) Assume (b) holds. Then by Lemma E.14 there is a k-derivation $\mathfrak{d}_A$ of A such that $\mathfrak{d}_A(X_i) = w_i X_i$ for all $i = 1, \ldots, n$ and $\mathfrak{d}_R \circ \pi = \pi \circ \mathfrak{d}_A$. In particular, this implies

$$\pi \circ \mathfrak{d}_A(\mathfrak{i}) = \mathfrak{d}_R \circ \pi\,(\mathfrak{i}) = \mathfrak{d}_R(0) = 0.$$

Thus,

$$\mathfrak{d}_A(\mathfrak{i}) \subset \ker(\pi) = \mathfrak{i},$$

i.e. $\mathfrak{i}$ is invariant under $\mathfrak{d}_A$.

(c) $\Longrightarrow$ (d) This follows from Theorem E.11 and Lemma E.15.

(d) $\Longrightarrow$ (a) Since $\mathfrak{i}$ is homogeneous, and $\deg(X_i) = w_i$ for all $i = 1, \ldots, n$, this follows from Proposition E.6.(3) as $R = A/\mathfrak{i}$.

If the equivalent conditions hold, the commutativity of Diagram (E.1) follows from Lemma E.14.

Let $y \in R$, and let $Y \in A$ such that $\pi(Y) = y$. Then

$$\begin{aligned} \mathfrak{d}_R(y) &= \mathfrak{d}_R \circ \pi\,(Y) \\ &= \pi \circ \mathfrak{d}_A(Y) \\ &= \mathfrak{d}_A(Y) + \mathfrak{i}. \end{aligned}$$

Thus, the grading on R is by Lemma E.16 induced by the grading on A corresponding to $\mathfrak{d}_A$. □

Proposition E.17. *Let k be a field of characteristic 0, and let R be an analytic k-algebra. Let $\mathfrak{d}$ be a k-derivation of R, and let $\mathfrak{i}$ be an ideal of R with $\mathfrak{d}(\mathfrak{i}) \subset \mathfrak{i}$. Then $\mathfrak{d}(\mathfrak{p}) \subset \mathfrak{p}$ for any associated prime ideal $\mathfrak{p}$ of $\mathfrak{i}$.*

Proof. See [49, (2.5)]. □

Bibliography

[1] Theo de Jong and Gerhard Pfister. *Local analytic geometry*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2000. Basic theory and applications.

[2] Antonio Campillo. *Algebroid curves in positive characteristic*, volume 813 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[3] Rolf Waldi. *Wertehalbgruppe und Singularität einer ebenen algebroiden Kurve*. Dissertation, Universität Regensburg, 1972.

[4] Rolf Waldi. On the equivalence of plane curve singularities. *Comm. Algebra*, 28(9):4389–4401, 2000.

[5] A. Hefez, M. E. Hernandes, and M. E. Rodrigues Hernandes. The analytic classification of plane curves with two branches. *Math. Z.*, 279(1-2):509–520, 2015.

[6] Ernst Kunz. The value-semigroup of a one-dimensional Gorenstein ring. *Proc. Amer. Math. Soc.*, 25:748–751, 1970.

[7] Félix Delgado de la Mata. Gorenstein curves and symmetry of the semigroup of values. *Manuscripta Math.*, 61(3):285–296, 1988.

[8] Marco D'Anna. The canonical module of a one-dimensional reduced local ring. *Comm. Algebra*, 25(9):2939–2965, 1997.

[9] Ernst Kunz and Walter Ruppert. Quasihomogene Singularitäten algebraischer Kurven. *Manuscripta Math.*, 22(1):47–61, 1977.

[10] V. Barucci, M. D'Anna, and R. Fröberg. Analytically unramified one-dimensional semilocal rings and their value semigroups. *J. Pure Appl. Algebra*, 147(3):215–254, 2000.

[11] Joachim Jäger. Längenberechnung und kanonische Ideale in eindimensionalen Ringen. *Arch. Math. (Basel)*, 29(5):504–512, 1977.

[12] Félix Delgado de la Mata. The semigroup of values of a curve singularity with several branches. *Manuscripta Math.*, 59(3):347–374, 1987.

[13] A. Campillo, F. Delgado, and K. Kiyek. Gorenstein property and symmetry for one-dimensional local Cohen-Macaulay rings. *Manuscripta Math.*, 83(3-4):405–423, 1994.

[14] Delphine Pol. *Singularités libres, formes et résidus logarithmiques.* Thése de doctorat, Université d'Angers, 2016.

[15] Joseph Lipman. Stable ideals and Arf rings. *Amer. J. Math.*, 93:649–685, 1971.

[16] Michel van den Bergh. Non-commutative crepant resolutions. In *The legacy of Niels Henrik Abel*, pages 749–770. Springer, Berlin, 2004.

[17] Graham J. Leuschke. Non-commutative crepant resolutions: scenes from categorical geometry. In *Progress in commutative algebra 1*, pages 293–361. de Gruyter, Berlin, 2012.

[18] Graham J. Leuschke. Endomorphism rings of finite global dimension. *Canad. J. Math.*, 59(2):332–342, 2007.

[19] Gert-Martin Greuel, Santiago Laplagne, and Frank Seelisch. Normalization of rings. *J. Symbolic Comput.*, 45(9):887–901, 2010.

[20] H. Grauert and R. Remmert. *Analytische Stellenalgebren.* Springer-Verlag, Berlin-New York, 1971. Unter Mitarbeit von O. Riemenschneider, Die Grundlehren der mathematischen Wissenschaften, Band 176.

[21] Janko Böhm, Wolfram Decker, and Mathias Schulze. Local analysis of Grauert-Remmert-type normalization algorithms. *Internat. J. Algebra Comput.*, 24(1):69–94, 2014.

[22] V. I. Arnol′d, S. M. Guseĭn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. I*, volume 82 of *Monographs in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1985. The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds.

[23] K. Kiyek and J. L. Vicente. *Resolution of curve and surface singularities*, volume 4 of *Algebras and Applications.* Kluwer Academic Publishers, Dordrecht, 2004. In characteristic zero.

[24] Jürgen Herzog and Ernst Kunz, editors. *Der kanonische Modul eines Cohen-Macaulay-Rings.* Lecture Notes in Mathematics, Vol. 238. Springer-Verlag, Berlin-New York, 1971. Seminar über die lokale Kohomologietheorie von Grothendieck, Universität Regensburg, Wintersemester 1970/1971.

[25] Philipp Korell, Mathias Schulze, and Laura Tozzo. Duality on value semigroups. *J. Commut. Algebra*, 2018. Advance publication.

[26] N. Bourbaki. *Éléments de mathématique. Fascicule XXVII. Algèbre commutative. Chapitre 1: Modules plats. Chapitre 2: Localisation.* Actualités Scientifiques et Industrielles, No. 1290. Herman, Paris, 1961.

[27] Gert-Martin Greuel and Gerhard Pfister. *A **Singular** introduction to commutative algebra.* Springer, Berlin, extended edition, 2008. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh and UNIX).

[28] R. G. Swan. *Algebraic K-theory.* Lecture Notes in Mathematics, No. 76. Springer-Verlag, Berlin-New York, 1968.

[29] Eben Matlis. 1*-dimensional Cohen-Macaulay rings.* Lecture Notes in Mathematics, Vol. 327. Springer-Verlag, Berlin-New York, 1973.

[30] Malcolm Griffin. Valuations and Prüfer rings. *Canad. J. Math.*, 26:412–429, 1974.

[31] Ernst Kunz and Rolf Waldi. Über den Derivationenmodul und das Jacobi-Ideal von Kurvensingularitäten. *Math. Z.*, 187(1):105–123, 1984.

[32] G.-M. Greuel, C. Lossen, and E. Shustin. *Introduction to singularities and deformations.* Springer Monographs in Mathematics. Springer, Berlin, 2007.

[33] Delphine Pol. Logarithmic residues along plane curves. *C. R. Math. Acad. Sci. Paris*, 353(4):345–349, 2015.

[34] V. I. Arnol′d. Local normal forms of functions. *Invent. Math.*, 35:87–109, 1976.

[35] Philipp Korell. Normalization Steps for Singularities from Arnold's List. Diplomarbeit, TU Kaiserslautern, March 2014.

[36] Günter Scheja and Hartmut Wiebe. Zur Chevalley-Zerlegung von Derivationen. *Manuscripta Math.*, 33(2):159–176, 1980/81.

[37] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1992.

[38] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1993.

[39] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[40] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[41] Nicolas Bourbaki. *Éléments de mathématique.* Masson, Paris, 1981. Algèbre. Chapitres 4 à 7. [Algebra. Chapters 4–7].

[42] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series.* Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[43] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II.* Springer-Verlag, New York-Heidelberg, 1975. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29.

[44] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[45] Masayoshi Nagata. *Local rings.* Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[46] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[47] Corinna Birghila and Mathias Schulze. Blowup of conductors. *arXiv.org*, 1610:04525, 2016.

[48] Merle E. Manis. Valuations on a commutative ring. *Proc. Amer. Math. Soc.*, 20:193–198, 1969.

[49] Günter Scheja and Hartmut Wiebe. Über Derivationen von lokalen analytischen Algebren. In *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)*, pages 161–192. Academic Press, London, 1973.

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Lebenslauf

Bildung

seit 2014	Doktorand am Fachbereich Mathematik der Technischen Universität Kaiserslautern, Betreuer: Prof. Dr. Mathias Schulze
März 2014	Diplom in Mathematik an der Technischen Universität Kaiserslautern, *Normalization Steps for Singularities from Arnold's List*, Vertiefung: Algebraische Geometrie und Computeralgebra, Betreuer: Prof. Dr. Wolfram Decker und Prof. Dr. Mathias Schulze
September 2012	Diplom in Physik an der Technischen Universität Kaiserslautern, *Phasendiagramm und Spektralfunktion monoatomarer Ketten*, Vertiefung: Theoretische Physik, Betreuer: Jun.-Prof. Dr. Jesko Sirker
März 2007	Abitur am Albert-Schweitzer-Gymnasium Kaiserslautern

Beschäftigung

seit März 2017	Wissenschaftlicher Mitarbeiter am Fachbereich Mathematik der Technischen Universität Kaiserslautern
Oktober 2015 – Februar 2017	Doktorand im EU Marie Curie Career Integration Grant "Codimension-one properties of singularities" bei Prof. Dr. Mathias Schulze am Fachbereich Mathematik der Technischen Universität Kaiserslautern

Veröffentlichungen

2016	Philipp Korell, Mathias Schulze, and Laura Tozzo. *Duality on Value Semigroups.* to appear on J. Comm. Alg., arXiv.org:1510.04072
2013	Nicholas Sedlmayr, Philipp Korell, and Jesko Sirker. *Two-band Luttinger liquid with spin-orbit coupling: Applications to monatomic chains on surfaces.* Phys. Rev. B 88, 195113

Vorträge und Workshops

September 2017	Imaginary-Workshop, Schülerinnentag 2017, Technische Universität Kaiserslautern
Juni 2016	Invited Talk *Quasihomogeneous Semigroups*, Meeting of the Catalan, Swedish, Spansih Math. Societies, Umeå (Schweden)
2015 – 2017	Doktorandenseminar *Algebraic Geometry*, Technische Universität Kaiserslautern
September 2016	Imaginary-Workshop, Schülerinnentag 2016, Technische Universität Kaiserslautern
Juli 2017	Invited Talk *Duality on Value Semigroups*, International Meeting on numerical semigroups with applications, Levico Terme (Italien)
Juli 2016	Imaginary-Workshop, Tag der Mathematik 2016, Technische Universität Kaiserslautern
Juni 2016	Invited Talk *Duality on Value Semigroups*, Second joint Conference of the Belgian, Royal Spanish and Luxembourg Mathematical Societies, Logroño (Spanien)

Lehre

Wintersemester 2017/18	Mathematik für Informatiker: Algebraische Strukturen, *Assistent*
	Mathematik für Informatiker: Kombinatorik und Analysis, *Assistent und Übungsleiter*
Sommersemester 2017	Mathematik für Informatiker: Algebraische Strukturen, *Assistent und Übungsleiter*
	Mathematik für Informatiker: Kombinatorik und Analysis, *Assistent*
Wintersemester 2016/17	Mathematik für Informatiker: Algebraische Strukturen, *Übungsleiter*
Sommersemester 2016	Einführung in das symbolische Rechnen, *Assistent und praktische Übungen in Singular*
Wintersemester 2015/16	Seminar Hyperebenenarrangements, *Assistent*
Sommersemester 2015	Grundlagen der Mathematik II, *Übungsleiter*
Wintersemester 2013/14	Vielteilchentheorie wechselwirkender Quantengase und magnetischer Systeme, *Assistent und Übungsleiter*
Sommersemester 2013	Theoretische Grundlagen der klassischen Physik, *Assistent und Übungsleiter*

Stipendien

2014 – 2015	Promotionsstipendium des Fachbereichs Mathematik der Technischen Universität Kaiserslautern

Curriculum Vitae

Education

June 2014 – present	Doctoral researcher, Department of Mathematics, Technische Universität Kaiserslautern, Advisor: Prof. Dr. Mathias Schulze
March 2014	Diploma in Mathematics, Technische Universität Kaiserslautern, *Normalization Steps for Singularities from Arnold's List*, Specialization: Algebrac Geometry and Computer Algebra, Advisors: Prof. Dr. Wolfram Decker and Prof. Dr. Mathias Schulze
September 2012	Diploma in Physics, Technische Universität Kaiserslautern, *Phasendiagramm und Spektralfunktion monoatomarer Ketten*, Specialization: Theoretical Physics, Advisor: Jun.-Prof. Dr. Jesko Sirker
March 2007	Abitur, Albert-Schweitzer-Gymnasium Kaiserslautern

Employment

March 2017 – present	Scientific Assistant, Department of Mathematics, Technische Universität Kaiserslautern
October 2015 – February 2017	Doctoral Researcher in the EU Marie Curie Career Integration Grant "Codimension-one properties of singularities" with Prof. Dr. Mathias Schulze, Department of Mathematics, Technische Universität Kaiserslautern

Publications

2016	Philipp Korell, Mathias Schulze, and Laura Tozzo. *Duality on Value Semigroups.* to appear on J. Comm. Alg., arXiv.org:1510.04072
2013	Nicholas Sedlmayr, Philipp Korell, and Jesko Sirker. *Two-band Luttinger liquid with spin-orbit coupling: Applications to monatomic chains on surfaces.* Phys. Rev. B 88, 195113

Talks and Workshops

September 2017	Imaginary-Workshop, Schülerinnentag 2017, Technische Universität Kaiserslautern
June 2016	Invited Talk *Quasihomogeneous Semigroups*, Meeting of the Catalan, Swedish, Spansih Math. Societies, Umeå (Sweden)
2015 – 2017	PhD Seminar *Algebraic Geometry*, Technische Universität Kaiserslautern
September 2016	Imaginary-Workshop, Schülerinnentag 2016, Technische Universität Kaiserslautern
July 2017	Invited Talk *Duality on Value Semigroups*, International Meeting on numerical semigroups with applications, Levico Terme (Italy)
July 2016	Imaginary-Workshop, Tag der Mathematik 2016, Technische Universität Kaiserslautern
June 2016	Invited Talk *Duality on Value Semigroups*, Second joint Conference of the Belgian, Royal Spanish and Luxembourg Mathematical Societies, Logroño (Spain)

Teaching

Winter 2017/18	Mathematik für Informatiker: Algebraische Strukturen (Mathematics for Computer Sciences: Algebraic Structures), *teaching assistant* Mathematik für Informatiker: Kombinatorik und Analysis (Mathematics for Computer Sciences: Combinatorics and Analysis), *teaching assistant and exercise class*
Summer 2017	Mathematik für Informatiker: Algebraische Strukturen (Mathematics for Computer Sciences: Algebraic Structures), *teaching assistant and exercise class* Mathematik für Informatiker: Kombinatorik und Analysis (Mathematics for Computer Sciences: Combinatorics and Analysis), *teaching assistant*
Winter 2016/17	Mathematik für Informatiker: Algebraische Strukturen (Mathematics for Computer Sciences: Algebraic Structures), *exercise class*
Summer 2016	Introduction to symbolic computation, *teaching assistant and Singular courses*
Winter 2015/16	Seminar Hyperplane Arrangements, *teaching assistant*
Summer 2015	Grundlagen der Mathematik II (Linear Algebra and Analysis II), *exercise class*
Winter 2013/14	Vielteilchentheorie wechselwirkender Quantengase und magnetischer Systeme (Many-body theory of interacting quantum gases and magnetic systems), *teaching assistant and exercise class*
Summer 2013	Theoretische Grundlagen der klassischen Physik (Theoretical foundations of classical physics), *teaching assistant and exercise class*

Scholarships

2014 – 2015 PhD scholarship of the Department of Mathematics at the Technische Universität Kaiserslautern