

Tutorial on Asymptotic Analysis I

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Preface

This text summarizes parts of the exercises of the tutorial on 'Asymptotic Analysis' held in the winter term 1993/94 at the University of Kaiserslautern. The lecture was designed and held by Prof. Dr. H. Neunzert.

The main aspect of the lecture and the tutorial was to investigate the basic techniques used in Asymptotic Analysis. This first part covers the following topics

- 1) Simple Operators in \mathbb{R}
- 2) Asymptotic Expansions
- 3) Asymptotic Expansions of Integrals
- 4) Perturbation Methods in Partial Differential Equations
- 5) Singular Perturbation: Matching

In order to understand the exercises of the sections 4)–6) the reader should be familiar with basic analytical solution techniques for ordinary and partial differential equations.

Each section is provided with a small review on the main topics which were discussed in the lecture. A list of references is given at the beginning of a section.

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1 Simple cases: Operators in \mathbb{R}

References:

Bellman, R.: *Perturbation Techniques in Mathematics, Physics and Engineering*, Holt, Rinehart and Winston, New York, 1975.

Berg, L.: *Asymptotische Darstellungen und Entwicklungen*, VEB Berlin, 1968.

Let us consider an operator $A_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$, where ε is a small parameter.

We are interested in the dependence of the solution of

$$A_\varepsilon(x) = 0 \tag{1}$$

on the small parameter ε . For example, we may study the solution of the equation

$$A_0(x) = 0$$

and on the other hand we may investigate the convergence of the solution x_ε of (1) in the limit when ε goes to zero. We also investigate the case, where A_ε is a differential operator acting on a function space $\mathcal{C}^2([a, b])$.

In order to study the asymptotic behaviour of the solution of (1) we consider an expansion of x_ε in a formal power series

$$x_\varepsilon = \sum_{k \in \mathbb{Z}} a_k \varepsilon^k$$

and try to derive a set of equations for the parameter set $\{a_k\}_{k \in \mathbb{Z}}$.

One problem arises directly: the powers of the expansion are not a priori given by integer numbers. Hence we have to introduce a scaling to derive the correct asymptotic behaviour. Furthermore a scaling is used to derive the asymptotic behaviour of a solution of (1), if the solution is not known explicitly.

Exercise 1.1

Find the sensible scalings of the equation

$$\varepsilon x^3 + x^2 + 2x + \varepsilon = 0$$

We investigate a transformation (scaling) in the form

$$x = \delta(\varepsilon)X$$

with the restriction

$$X = O(1) \quad \frac{1}{X} = O(1) \quad \text{if } \varepsilon \rightarrow 0$$

In order to get the sensible scaling – this means the different orders of magnitude of $\delta(\varepsilon)$ – we consider the various cases.

a) If $\delta(\varepsilon) \sim 1$ we have

$$X^2 + 2X + \text{small} = 0$$

and hence

$$\begin{aligned} X &= -2 + \text{small} \quad \checkmark \\ X &= 0 + \text{small} \quad \text{but } X \neq O(1) \end{aligned}$$

b) If $\delta(\varepsilon) \sim \varepsilon$ we get

$$\varepsilon^2 \delta^3 X^3 + \delta^2 X^2 + 2\delta X + \varepsilon = 0$$

and

$$\begin{aligned} \text{small} + 2X + 1 &= 0 \\ X &= -\frac{1}{2} + \text{small} \end{aligned}$$

c) If $\delta(\varepsilon) \gg 1$ then

$$X(\varepsilon^2 \delta^2 X^2 + \delta X + 2) + \text{small} = 0$$

$X = 0$ is not valid, hence we consider the case

$$\varepsilon^2 \delta^2 X^2 + \delta X + 2 = 0$$

and

$$X = -\frac{1}{2\varepsilon^2 \delta} (1 \pm (1 - 8\varepsilon^2)^{1/2})$$

If we choose $\delta(\varepsilon) \sim \frac{1}{\varepsilon^2}$ then

$$X = 1 + \text{small}$$

Finally we end up with the sensible scalings

$$\delta(\varepsilon) \sim 1 \quad \delta(\varepsilon) \sim \varepsilon \quad \delta(\varepsilon) \sim \frac{1}{\varepsilon^2}$$

and

$$x^{(1)} = -2 + \dots \quad x^{(2)} = -\frac{1}{2} \varepsilon + \dots \quad x^{(3)} = \frac{1}{\varepsilon^2} + \dots$$

Exercise 1.2

Find the first terms of $x(\varepsilon)$ ($\varepsilon = o(1)$), solution of

$$\sqrt{2} \sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{x^2}{2} = -\frac{\varepsilon}{6}$$

We define the curves $K_1(x)$ and $K_2(x)$ by

$$K_1(x) = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right) \quad K_2(x) = 1 + x - \frac{x^2}{2} =$$

The solutions of the unperturbed problem ($\varepsilon = 0$) are given by the intersection points of $K_1(x)$ and $K_2(x)$.

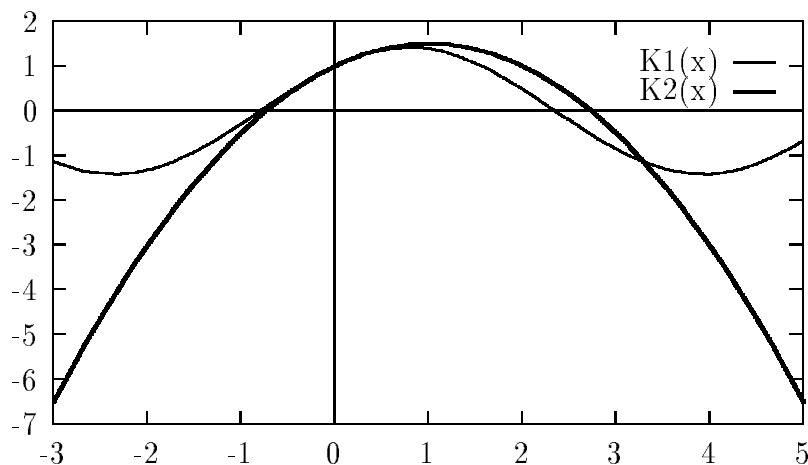


Fig. 1.1: Curves $K_1(x)$ and $K_2(x)$

One intersection is at $x = 0$ and we try to construct an asymptotic expansions for $\varepsilon > 0$ in the neighbourhood of $x = 0$.

The main question is: What asymptotic expansion is useful?

$$x = \varepsilon^{h_0} x_0 + \varepsilon^{h_1} x_1 + \dots$$

What is the correct *Ansatz* for h_0 etc.?

The function $K_1(x)$ can be expanded in a power series around $x = 0$. One gets

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}x - \frac{1}{4}\sqrt{2}x^2 - \frac{1}{12}\sqrt{2}x^3 + \frac{1}{48}\sqrt{2}x^4 + O(x^5)$$

Substituting the expansion into the equation $K_1(x) + K_2(x) = -\frac{\varepsilon}{6}$ we have

$$-\frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 = -\frac{\varepsilon}{6}$$

Hence we get the scaling

$$X\varepsilon^{1/3} = x$$

and

$$\varepsilon X^3 - \frac{1}{4}\varepsilon^{4/3}X^4 - \frac{1}{2}\varepsilon^{5/3}X^5 + \varepsilon + \dots = 0 \quad (2)$$

with an expansion of the form

$$X(\varepsilon) = X_1 + \varepsilon^{1/3}X_2 + \dots$$

Substituting this expansion into (2)

$$\begin{aligned} & \varepsilon(X_1^3 + 3\varepsilon^{1/3}X_1^2X_2 + 3\varepsilon^{2/3}X_1X_2^2 + \varepsilon X_2^3) \\ & - \frac{1}{4}\varepsilon^{4/3}(X_1^4 + 4\varepsilon^{1/3}X_1^3X_2 + 6\varepsilon^{2/3}X_1^2X_2^2 + 4\varepsilon X_1X_2^3 + \varepsilon^{4/3}X_2^4) \\ & - \varepsilon + \dots = 0 \end{aligned}$$

Comparing the powers in ε we get

$$X_1 = 1 \quad X_2 = \frac{1}{12}$$

and

$$X(\varepsilon) = 1 + \frac{1}{12}\varepsilon^{1/3}$$

or

$$x(\varepsilon) = \varepsilon^{1/3} + \frac{1}{12}\varepsilon^{2/3}$$

Exercise 1.3

Find the first two terms for all three roots of

$$a) \quad \varepsilon x^3 + x^2 + (2 + \varepsilon)x + 1 = 0$$

$$b) \quad \varepsilon x^3 + x^2 + (2 - \varepsilon)x + 1 = 0$$

We investigate the sensible scalings

$$x = \delta(\varepsilon)X$$

a) If $\delta(\varepsilon) \sim 1$ we get

$$X^2 + 2X + 1 + \text{small} = 0$$

Hence

$$X = -1 + \text{small}$$

and $X = -1$ is a double root of the unperturbed problem.

b) If $\delta(\varepsilon) \sim \varepsilon$ the equations becomes

$$\varepsilon\delta^3 X^3 + \delta^2 X^2 + (2 \pm \varepsilon)\delta X + 1 = 0$$

or

$$\text{small} + 1 = 0$$

This leads to a contradiction – this scaling gives no reasonable result.

c) If $\delta(\varepsilon) \gg 1$ we have

$$X(\varepsilon\delta^2 X^2 + \delta X + 2 \pm \varepsilon) + \text{small} = 0$$

$X = 0$ is not valid, hence we have to investigate the solution of the quadratic equation

$$X = \frac{1}{2\varepsilon\delta}(-1 \pm (1 - 4(2 \pm \varepsilon)\varepsilon)^{1/2})$$

In the case $\delta = \frac{1}{\varepsilon}$ it follows that

$$X = -1 + \text{small}$$

as sensible scaling.

To evaluate the first terms of the roots we first investigate case a). (In the following we will see that there are strong differences between the two cases a) and b))

Using the scaling $\delta \sim 1$ we have to consider the equation

$$\varepsilon X^3 + X^2 + (2 + \varepsilon)X + 1 = 0 \quad (3)$$

The first term of the asymptotic expansion is given by the solution of the quadratic equation

$$X^2 + 2X + 1 = 0$$

Hence we get the expansion

$$X = -1 + Y(\varepsilon)$$

with $Y(\varepsilon) \rightarrow 0$ if $\varepsilon \rightarrow 0$. In order to compute the suitable ε -power of $Y(\varepsilon)$ we compute

$$\begin{aligned} X^2 &= 1 - 2Y(\varepsilon) + Y^2(\varepsilon) \\ X^3 &= 1 + 3Y(\varepsilon) - 3Y^2(\varepsilon) + Y^3(\varepsilon) \end{aligned}$$

Using equation (3) we get

$$\varepsilon Y^3 + (1 - 3\varepsilon)Y^2 + 2\varepsilon Y = 0$$

If we compare the powers in ε it is obvious that we have to choose an asymptotic expansion in the form

$$X = -1 + \varepsilon^{1/2}X_1 + \dots$$

Furthermore we have the relation

$$X_1^2 - 2 = 0$$

This means that we have two solutions

$$X_1^{(1)} = -\sqrt{2} \quad X_2^{(2)} = \sqrt{2}$$

It follows that the first two terms of the first two roots of equation a) are given by

$$x^{(1)} = -1 - \sqrt{2}\varepsilon^{1/2} + O(\varepsilon) \quad x^{(2)} = -1 + \sqrt{2}\varepsilon^{1/2} + O(\varepsilon)$$

Consider now the scaling $\delta = \frac{1}{\varepsilon}$.

We end up with the equation

$$X^3 + X^2 + (2 + \varepsilon)X + \varepsilon^2 = 0$$

and

$$X = -1 + Y(\varepsilon)$$

The equation for $Y(\varepsilon)$ reads

$$Y^3 + 2Y^2 + (1 + 2\varepsilon)Y - 2\varepsilon = 0$$

If we compare the powers in ε we get

$$X = -1 + 2\varepsilon + O(\varepsilon^2)$$

Therefore the first two terms of the third root are given by

$$x^{(3)} = -\frac{1}{\varepsilon} + 2 + O(\varepsilon^2)$$

Case b):

We investigate again the scaling $\delta \sim 1$, where the corresponding equation is given by

$$\varepsilon X^3 + X^2 + (2 - \varepsilon)X + 1 = 0$$

We know that the expansion has the form

$$X = -1 + Y(\varepsilon)$$

with $Y(\varepsilon) \rightarrow 0$ if $\varepsilon \rightarrow 0$.

Taking

$$\begin{aligned} X^2 &= 1 - 2Y(\varepsilon) + Y^2(\varepsilon) \\ X^3 &= 1 + 3Y(\varepsilon) - 3Y^2(\varepsilon) + Y^3(\varepsilon) \end{aligned}$$

one gets

$$\varepsilon Y^3 + (1 - 3\varepsilon)Y^2 + 2\varepsilon Y = 0$$

for $Y(\varepsilon)$.

In contrast to case a) we now have the asymptotic expansion

$$X = -1 + \varepsilon X_1 + O(\varepsilon^2)$$

where X_1 is the solution of

$$X_1(X_1 + 2) = 0 \tag{4}$$

It follows that the first two terms of the first root are given by

$$x^{(1)} = -1 + 2\varepsilon + O(\varepsilon^2) \tag{5}$$

The second solution of equation (4) says that there is now a perturbation of the order ε . But we already know that it is not possible to have any perturbation of the order ε^h with $h > 1$. Therefore the root $x^{(2)} = -1$ of the unperturbed problem must be also a root of the perturbed equation for all $\varepsilon \in \mathbb{R}$. This may be simply verified by substitution.

We consider the scaling $\delta \sim \frac{1}{\varepsilon}$.

The corresponding equation reads

$$X^3 + X^2 + (2 - \varepsilon)X + \varepsilon^2 = 0$$

With the *Ansatz*

$$X = -1 + Y(\varepsilon)$$

the equation for $Y(\varepsilon)$ gets

$$Y^3 - 2Y^2 + (1 + 2\varepsilon - \varepsilon^2)Y + 2\varepsilon^2 - 2\varepsilon = 0$$

We get the same as in case a), namely

$$x^{(3)} = -\frac{1}{\varepsilon} + 2 + O(\varepsilon)$$

What can be done with the asymptotic expansions?

As an example we consider the expansion of the third root

$$X^{(3)} = -\frac{1}{\varepsilon} + 2 + O(\varepsilon)$$

It follows that

$$\lim_{\varepsilon \rightarrow 0} x^{(3)} = \infty$$

Using the expansion it is easy to estimate the location of $x^{(3)}$.

Example:

Choose $\varepsilon = 0.01$ we have

$$x^{(3)} = -98 + O(0.01)$$

Choose $\varepsilon = 0.001$ we have

$$x^{(3)} = -998 + O(0.001)$$

Figures 1.2 and 1.3 show the curve $K(x) = \varepsilon x^3 + x^2 + (2 + \varepsilon)x + 1$ for the region around the third root given above.

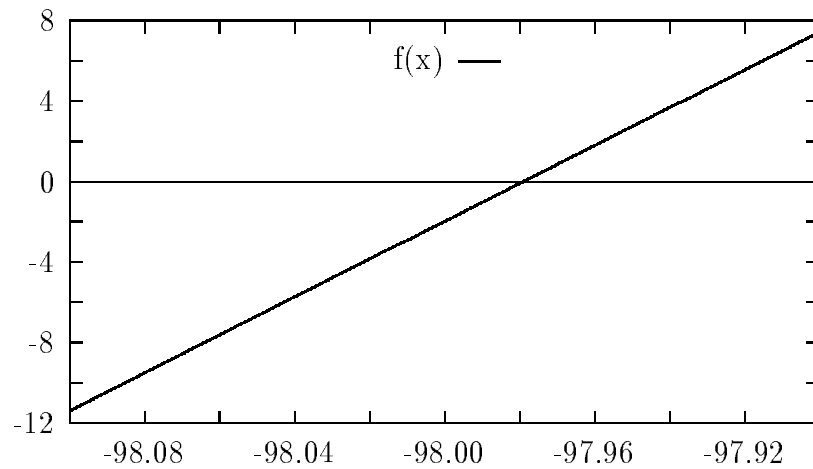


Fig. 1.2: Curve $K(x)$ for $\varepsilon = 0.01$

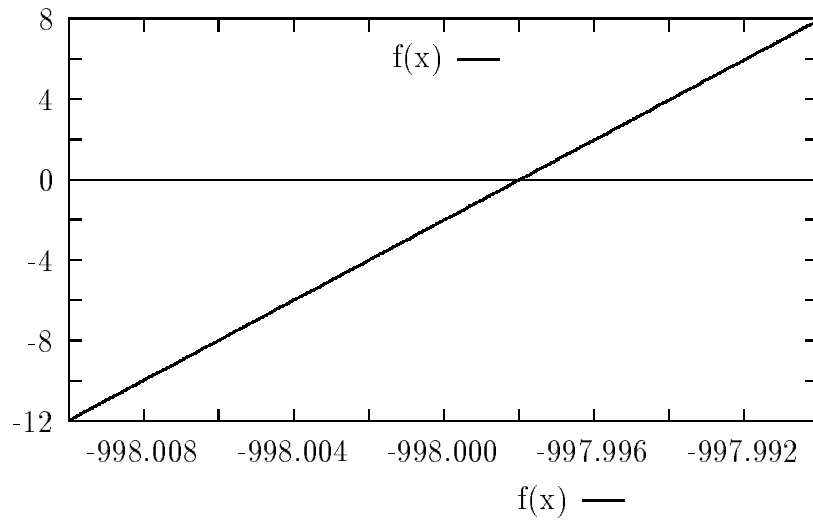


Fig. 1.3: Curve $K(x)$ for $\varepsilon = 0.001$

In the same way we may estimate the location of the first root, which has the expansion

$$x^{(1)} = -1 - 2\varepsilon + O(\varepsilon^2)$$

Figure 1.4 shows the given function with $\varepsilon = 0.001$. One may recognize the stable root at $x = -1$ and the root shifted by the perturbation of order ε . The expansion gives

$$x^{(1)} = -1.002 + O(10^{-6})$$

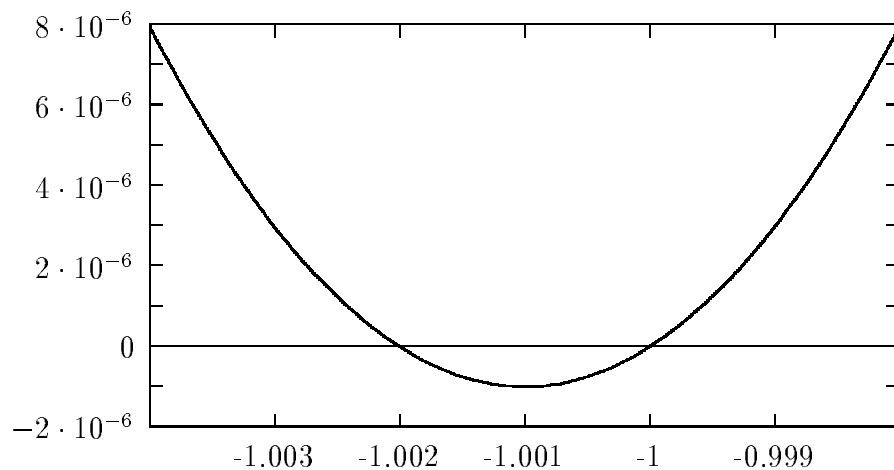


Fig. 1.4: Curve $K(x)$ for $\varepsilon = 0.001$

Exercise 1.4

Find the first order perturbation of the eigenvalues of

$$y'' + \lambda y + \varepsilon y^n = 0 \quad (6)$$

in $0 \leq x \leq \pi$, $y(0) = y(\pi) = 0$, $n = 1, 2$ and 3 .

Consider the unperturbed problem ($\varepsilon = 0$)

$$y'' + \lambda y = 0 \quad (7)$$

With the prescribed boundary condition there exists nontrivial solution only if

$$\lambda_n = n^2 \quad n \in \mathbb{N}$$

This are the eigenvalues of equation (7) together with the eigenfunctions

$$y_n = \left(\frac{2}{\pi}\right)^{1/2} \sin nx$$

The set $\{y_n\}_{n \in \mathbb{N}}$ forms an orthonormal basis on the space $\mathcal{C}([0, 1])$ of functions with boundary conditions $f(0) = f(\pi) = 0$ and we have

$$\begin{aligned} \|y_n\|_2 &= \int_0^\pi y_n^2(x) dx = 1 \\ \langle y_n, y_m \rangle &= \int_0^\pi y_n(x) y_m(x) dx = 0 \\ &\text{falls } n \neq m \end{aligned}$$

If $y(x) \in \mathcal{C}([0, 1])$ we have

$$y(x) = \sum_{k \in \mathbb{N}} b_k y_k \quad b_k = \int_0^\pi y(x) y_k(x) dx$$

We use this Fourier expansion to find the expansion of the eigenvalues λ_n and the eigenfunction y_n

$$\begin{aligned} y(x) &= y_n(x) + \varepsilon \sum_{k \neq n} a_k^{(n)} y_k(x) + O(\varepsilon^2) \\ \lambda &= n^2 + \sum_{n=1}^{\infty} \varepsilon^n \lambda^{(n)} \end{aligned} \quad (8)$$

We may use an expansion of the inhomogeneous part of (6)

$$y^p(x) = \sum_{k \in N} b_k y_k(x)$$

Using (8) we have

$$b_k = b_k^{(0)} + O(\varepsilon^2)$$

where

$$b_k^{(0)} = \int_0^\pi y_n^p(x) y_k(x) dx$$

($b_k^{(0)}$ does not vanish if $k \neq n!$)

If we use the expansions for the differential equation we get

$$\begin{aligned} y_n'' + \varepsilon \sum_{k \neq n} a_k^{(n)} y_k'' + (n^2 + \varepsilon \lambda_1^{(n)})(y_n + \varepsilon \sum_{k \neq n} a_k^{(n)} y_k) \\ + \varepsilon \sum_{k \in N} b_k^{(0)} y_k + O(\varepsilon^2) = 0 \end{aligned}$$

It follows that

$$\begin{aligned} -n^2 y_n - \varepsilon \sum_{k \neq n} k^2 a_k^{(n)} y_k + n^2 y_n + \varepsilon \lambda_1^{(n)} y_n \\ + \varepsilon \sum_{k \neq n} n^2 a_k^{(n)} y_k + \varepsilon \sum_{k \in N} b_k^{(0)} y_k + O(\varepsilon^2) = 0 \end{aligned}$$

Comparing powers in ε one gets

$$\begin{aligned} \varepsilon^0 : \quad -n^2 y_n + n^2 y_n = 0 \\ \varepsilon^1 : \quad \sum_{k \neq n} (n^2 - k^2) a_k^{(n)} y_k + \lambda_1^{(n)} y_n + \sum_{k \in N} b_k^{(0)} y_k = 0 \end{aligned}$$

The zeroth order equation holds. Hence we consider the first order equation:

Because the set of function $\{y_n\}$ form an orthonormal basis the first order equation must hold for all $k \in N$ separately

$$\begin{aligned} k = n \quad \lambda_1^{(n)} + b_n^{(0)} = 0 \\ k \neq n \quad (n^2 - k^2) a_k^{(n)} + b_k^{(0)} = 0 \end{aligned}$$

Hence we get

$$\begin{aligned} \lambda_1^{(n)} &= b_n^{(0)} \\ a_k^{(n)} &= -\frac{b_k^{(0)}}{n^2 - k^2} \end{aligned}$$

or

$$\begin{aligned}\lambda &= n^2 + \varepsilon \lambda_1^{(n)} + O(\varepsilon^2) \\ &= n^2 - \varepsilon b_n^{(0)} + O(\varepsilon^2)\end{aligned}$$

where

$$b_n^{(0)} = \int_0^\pi y_n^p(x) y_n(x) dx$$

and

$$\begin{aligned}y(x) &= y_n + \varepsilon \sum_{k \neq n} a_k^{(n)} y_k(x) + O(\varepsilon^2) \\ &= y_n(x) - \varepsilon \sum_{k \neq n} \frac{b_k^{(0)}}{n^2 - k^2} y_k(x) + O(\varepsilon^2)\end{aligned}$$

Explicit calculation gives:

If $p = 2$

$$b_n^{(0)} = \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{2}{3n} & \text{if } n \text{ odd} \end{cases}$$

If $p = 3$

$$b_n^{(0)} = \frac{3}{8\sqrt{\pi}}$$

2 Asymptotic Expansions

References:

Bleistein, N. and Handelsman, R.A.: *Asymptotic Expansions of Integrals*, Holt, Rinehart and Winston, New York, 1975.

Olver, F.W.J.: *Integrals Asymptotics and Special Functions*, Academic Press, New York, 1974.

An expansion $\sum_{n=0}^{\infty} f_n(z)$ is convergent for fixed z , if

$$\forall \varepsilon > 0 \quad \exists N_0(z, \varepsilon) \quad \left| \sum_{n=N}^M f_n(z) \right| < \varepsilon \quad \forall N, M > N_0(z, \varepsilon)$$

The expansion is called uniformly convergent, if $N_0(z, \varepsilon)$ is independent of z .

Convergent expansions may not be useful to get an accurate approximation for all

values of z . One example is the error function

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

A convergent expansion is

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots \right)$$

but for large x it requires a large number of terms to achieve a certain accuracy. Expansions, which may be divergent, can give for large x a high accuracy if only a small number of terms is taken into account:

Consider

$$\operatorname{Erf}(x) = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{x^2} + \frac{1}{(2x^2)^2} + \dots \right) \quad (9)$$

for $x \rightarrow \infty$.

The expansion diverges for all x , but gives for $x \geq 3$ an accuracy of 10^{-5} using 2 terms of (9).

Exercise 2.1

Let the function $f(x)$ be given by

$$f(x) = e^{-1/x^2} \cos(e^{1/x^2})$$

Find an asymptotic expansion of $f(x)$ if $x \rightarrow 0$. Consider the derivative of the expansion and show that the derivative is not an asymptotic expansion of $f'(x)$.

Obviously it hold that

$$\lim_{x \rightarrow 0} f(x) = 0$$

because we have

$$| e^{-1/x^2} \cos(e^{1/x^2}) | \leq e^{1/x^2} \xrightarrow{x \rightarrow 0} 0$$

Furthermore we have

$$\lim_{x \rightarrow 0} \frac{1}{x^m} f(x) = 0 \quad \forall m \in \mathbb{N}$$

Hence one gets $f(x) \sim 0$ if $x \rightarrow 0$.

Consider the derivative of the function $f(x)$

$$f'(x) = \frac{2}{x^3} \left(e^{-1/x^2} \cos(e^{1/x^2}) + \sin(e^{1/x^2}) \right)$$

The first term of $f'(x)$ has the same asymptotic expansion as $f(x)$. The second term

$$\frac{2}{x^3} \sin(e^{1/x^2})$$

does not converge to 0 if $x \rightarrow 0$. Therefore the relation $f'(x) \sim 0$ if $x \rightarrow 0$ can not be valid.

(Remark:

Let $f(x)$ be a continuous function on \mathbb{R} . The formal power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is called asymptotic expansion of f if $x \rightarrow x_0$ (

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n)$$

if

$$\lim_{x \rightarrow x_0} \frac{f(x) - \sum_{n=0}^N a_n (x - x_0)^n}{(x - x_0)^N} = 0$$

holds for all $N \in \mathbb{N}$).

Exercise 2.2

Find the behaviour of $y(x)$ if $x \rightarrow \infty$ and

$$xy'(x) + y(x) = x^{-3} + \frac{1}{2}y''(x) \quad (10)$$

(Use, that

$$\text{Erf}(x) = 1 - \frac{1}{\sqrt{\pi}} \left(\frac{1}{x} + y(x) \right) e^{-x^2})$$

It is easy to verify, that

$$y(x) = \sqrt{\pi} e^{x^2} (1 - \text{Erf}(x)) - \frac{1}{x}$$

is a solution of the differential equation (10).

We now look for an asymptotic expansion of $y(x)$ if $x \rightarrow \infty$. We use a result presented in the lecture, namely

$$\text{Erf}(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}x} \left(1 - \frac{1}{2x^2} + \frac{3}{4x^4} - \frac{15}{8x^6} + O\left(\frac{1}{x^8}\right) \right) \quad (11)$$

If we use the expansion (11) of the Error function we get

$$y(x) = -\frac{1}{2x^3} \left(1 - \frac{3}{2x^2} + \frac{15}{4x^4} + O\left(\frac{1}{x^6}\right) \right)$$

as asymptotic behaviour if $x \rightarrow \infty$.

3 Asymptotic Expansions of Integrals

References:

Bleistein, N. and Handelsman, R.A.: *Asymptotic Expansions of Integrals*, Holt, Rinehart and Winston, New York, 1975.

Olver, F.W.J.: *Integrals Asymptotics and Special Functions*, Academic Press, New York, 1974.

Let us consider an integral $I(x)$

$$I(x) = \int_a^b f(t, x) dt$$

We want to investigate the asymptotic behaviour of $I(x)$ as $x \rightarrow \overset{\circ}{x}$.

This is a very large field in asymptotic analysis (see for example the book of Bleistein and Handelsman) – the exercises cover the following cases

- 1) Laplace Integrals

$$\mathcal{F}(z) = \int_0^{\infty} e^{-zt} f(t) dt$$

where $z \in C$ and $f \in \mathcal{L}_{loc}^1$.

- 2) Generalized Laplace Integrals

$$\int_0^{\infty} e^{-z\Phi(t)} f(t) dt$$

where $\Phi : [a, b] \rightarrow \mathbb{R}$.

- 3) Complex integrals

$$\int_{C_{a,b}} e^{-z\Phi(t)} f(t) dt$$

where $a, b, t \in C$ and $C_{a,b}$ a curve in C connecting a and b .

- 4) Generalized Fourier Integrals

$$I(x) = \int_a^b f(t) e^{ix\Phi(t)} dt$$

where $f, \Phi : [a, b] \rightarrow \mathbb{R}$

The method used are

- 1) The Watson Lemma for integrals of type 1) and 2)
- 2) The method of the steepest descent for integrals of type 3) (or constant phase curves)
- 3) The method of stationary phase for integrals of type 4)

Exercise 3.1

Evaluate the asymptotic expansions if $x \rightarrow \infty$ for

$$a) \int_0^{\pi/2} e^{-x \tan t} dt$$

$$b) \int_0^{\infty} e^{-x \sinh^2 t} dt$$

$$c) \int_{-\pi/2}^{\pi/2} (t+2)e^{-x \cos t} dt$$

We will use Watson's Lemma to evaluate the asymptotic expansions.

Lemma (G.H. Watson, Proc. London Math. Soc., Series 2, 17, 116–148 (1918))

Consider the integral $I(x)$ with

$$I(x) = \int_0^{\infty} e^{-xt} f(t) dt$$

Let $f(t)$ be locally integrable on $(0, \infty)$, bounded for finite t and

$$f(t) = O(e^{at}) \quad \text{if } t \rightarrow \infty, a \in \mathbb{R}$$

$$f(t) \sim \sum_{m=0}^{\infty} c_m t^{a_m}$$

Then

$$I(x) \sim \sum_{m=0}^{\infty} c_m \int_0^{\infty} e^{-xt} t^{a_m} dt = \sum_{m=0}^{\infty} c_m \frac{\Gamma(a_m + 1)}{x^{a_m + 1}}$$

if $x \rightarrow \infty$.

We consider the case a)

$$I(x) = \int_0^{\pi/2} \int_0^{\pi/2} e^{-x \tan t} dt$$

We choose

$$u = \Phi(t) = \tan t \quad \text{with} \quad \Phi'(t) = \frac{1}{\cos^2 t}$$

and get

$$I(x) = \int_0^{\infty} \int_0^{\infty} e^{-xu} \frac{1}{1+u^2} du$$

So we may define

$$f(u) = \frac{1}{1+u^2}$$

and use Watson's Lemma.

The function $f(u)$ has the (Taylor) expansion

$$f(u) \sim \sum_{m=0}^{\infty} c_m u^{2m} = 1 - u^2 + u^4 - u^6 + u^8 - \dots$$

Therefore the asymptotic expansion of $I(x)$ is given by

$$I(x) \sim \sum_{m=0}^{\infty} \frac{\Gamma(2m+1)}{x^{2m+1}}$$

We consider the case b)

$$\int_0^{\infty} e^{-x \sinh^2 t} dt$$

We choose

$$u = \Phi(t) = \sinh^2 t \quad \text{with} \quad \Phi'(t) = 2 \sinh t \cosh t$$

and get

$$I(x) = \int_0^{\infty} e^{-xu} \frac{1}{(4u(1+u^2))^{1/2}} du$$

The asymptotic expansion of $f(u)$ is

$$\begin{aligned} f(u) &\sim \sum_{m=0}^{\infty} c_m u^{2m} \\ &= \frac{1}{2} u^{-1/2} - \frac{1}{4} u^{3/2} + \frac{3}{16} u^{7/2} - \frac{5}{30} u^{11/2} + \dots \end{aligned}$$

Using Watson's Lemma we get

$$I(x) \sim \frac{1}{2} \frac{\Gamma(1/2)}{x^{1/2}} - \frac{1}{4} \frac{\Gamma(5/2)}{x^{5/2}} - \frac{5}{30} \frac{\Gamma(9/2)}{x^{9/2}} + \dots$$

We consider the case c)

$$\int_{-\pi/2}^{\pi/2} \pi/2(t+2)e^{-x \cos t} dt$$

We divide the complete integral into

$$\int_{-\pi/2}^{\pi/2} \pi/2(t+2)e^{-x \cos t} dt = \int_{-\pi/2}^0 0(t+2)e^{-x \cos t} dt + \int_0^{\pi/2} \pi/2(t+2)e^{-x \cos t} dt$$

Using the transformation $u = \cos t$ we get

$$\int_{-\pi/2}^{\pi/2} \pi/2(t+2)e^{-x \cos t} dt = 4 \int_0^1 e^{-xu} \frac{1}{(1-u^2)^{1/2}} du$$

The function $f(u) = 4 \frac{1}{(1-u^2)^{1/2}}$ can be expanded as

$$f(u) = 4 \frac{1}{(1-u^2)^{1/2}} = 4 + 2x^2 + \frac{3}{2}x^4 + \frac{5}{4}x^6 + O(x^8)$$

and we get

$$I(x) \sim \frac{4}{x} + \frac{4}{x^3} + \frac{36}{x^5} + O\left(\frac{1}{x^7}\right)$$

Exercise 3.2

Consider

$$J_0(x) = \operatorname{Re} \frac{1}{i\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ix \cosh t} dt$$

Show, that there exists two curves $\sigma_1(t)$ and $\sigma_2(t)$ which run from $-\infty \pm i\frac{\pi}{2}$ to $\infty \mp i\frac{\pi}{2}$ through the point $t=0$.

We have to separate the function

$$\Phi(t) = i \cosh t \quad t \text{ complex}$$

into real and imaginary part. We set

$$t = \tau + i\sigma$$

and get

$$\begin{aligned}\Phi(\tau + i\sigma) &= i \cosh(\tau + i\sigma) \\ &= -\sinh \tau \sin \sigma + i \cosh \tau \cos \sigma\end{aligned}$$

and

$$\Phi = \Upsilon(\tau, \sigma) + i\Psi(\tau, \sigma) \quad \Upsilon(\tau, \sigma) = -\sinh \tau \sin \sigma \quad \Psi(\tau, \sigma) = \cosh \tau \cos \sigma$$

The curves of constant phase are given by $\Psi(\tau, \sigma) = \text{const}$ and the curves should run through the origin, hence

$$\cosh \tau \cos \sigma = 1$$

The curves are given by the implicit function $K(\tau, \sigma) = 0$ where

$$K(\tau, \sigma) = \cosh \tau \cos \sigma - 1 \tag{12}$$

Hence there exists a function $\sigma = \sigma(\tau)$ in a neighbourhood of a point (τ_0, σ_0) if

$$\frac{\partial K(\tau, \sigma)}{\partial \sigma} = -\cosh \tau \sin \sigma \neq 0$$

The partial derivative is not equal zero as long as $\sigma \neq n\pi$ with $n \in \mathbb{N}$.

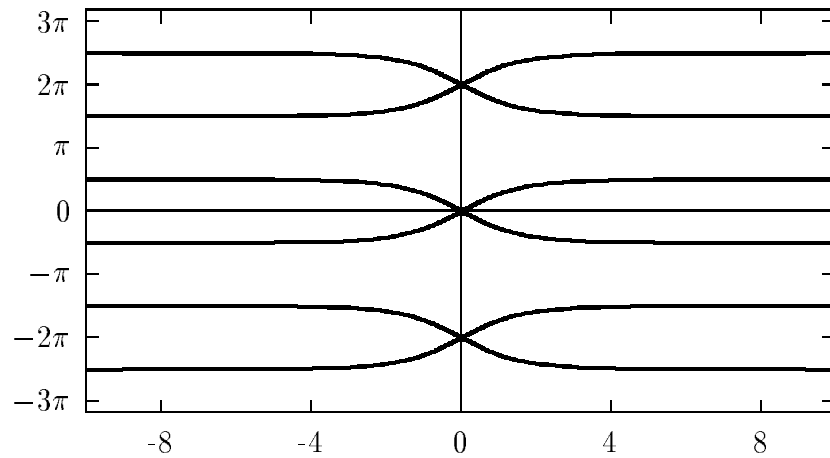


Fig. 3.1: Curves of constant phase

Using the two curves $\sigma_1(\tau)$ and $\sigma_2(\tau)$ we get

$$\int_{-i\pi/2}^{i\pi/2} e^{ix \cosh t} dt = \int_{-i\pi/2}^{-i\pi/2} e^{ix \cosh t} dt + \int_{\sigma_1(t)} e^{ix \cosh t} dt + \int_{i\pi/2}^{\infty+i\pi/2} e^{ix \cosh t} dt + \int_{\sigma_2(t)} e^{ix \cosh t} dt$$

The rest of the exercise is left ot the reader.

Exercise 3.3

Lemma

Consider the generalized Fourier integral

$$I(x) = \int_a^b f(t)e^{ix\Phi(t)} dt$$

Show that, if $f \in \mathcal{L}$, $\Phi \in \mathcal{C}^1([a, b])$ and Φ' has only a finite number of zeroes in $[a, b]$,

$$I(x) \rightarrow 0 \quad \text{if } x \rightarrow \infty$$

Proof:

Denote the zeroes of $\Phi(t)$ in $[a, b]$ by

$$s_1 < s_2 < \dots < s_N$$

Define

$$s_0 = a \quad s_{N+1} = b$$

Now the integral $I(x)$ can be written in the form

$$I(x) = \sum_{n=0}^N \int_{s_n}^{s_{n+1}} f(t)e^{ix\Phi(t)} dt$$

For every intervall (s_n, s_{n+1}) we have

$$\Phi'(t) \neq 0 \quad \forall t \in (s_n, s_{n+1})$$

Therefore the function $\Phi(t)$ is monotone on every intervall. We consider the transformation

$$u = \Phi(t) \quad du = \Phi'(t) dt$$

For the single integrals one gets

$$\int_{s_n}^{s_{n+1}} f(t)e^{ix\Phi(t)} dt = \int_{\Phi(s_n)}^{\Phi(s_{n+1})} \frac{f \circ \Phi(u)}{(\Phi' \circ \Phi^{-1})(u)} e^{ixu} du$$

For every integral we may use Riemann–Lebesgue Lemma, because the function $\frac{f \circ \Phi(u)}{(\Phi' \circ \Phi^{-1})(u)}$ is locally integrable (which is a direct consequence of the local integrability of f). ■

Exercise 3.4

Consider the integral

$$I(x) = \int_a^b f(t)e^{ix\Phi(t)} dt \quad (13)$$

Evaluate the leading term of $I(x)$
if $\Phi'(a) = \dots = \Phi^{(p)}(a) = 0$ and $f(t) \sim A(t-a)^\alpha, \alpha > -1$.

The point $t = a$ is called stationary point if

$$\Phi'(a) = 0$$

Furthermore we consider in this example the case that

$$\Phi'(a) = \dots = \Phi^{(p)}(a) = 0$$

We write (13) in the form

$$I(x) = \int_a^{a+\varepsilon} f(t)e^{ix\Phi(t)} dt + \int_{a+\varepsilon}^b f(t)e^{ix\Phi(t)} dt$$

It holds that

$$\int_{a+\varepsilon}^b f(t)e^{ix\Phi(t)} dt = O(1/x)$$

Consider

$$I_1(x) = \int_a^{a+\varepsilon} f(t)e^{ix\Phi(t)} dt \sim \int_a^{a+\varepsilon} A(t-a)^\alpha e^{ix(\Phi(a)+R_{p+1}(t))} dt$$

where

$$R_{p+1} = \frac{\Phi^{(p+1)}(a)}{(p+1)!}(t-a)^{p+1}$$

We get

$$I_1(x) \sim Ae^{ix\Phi(a)} \int_0^\infty s^\alpha e^{ixCs^{p+1}} ds$$

where

$$C = \frac{1}{(p+1)!}\Phi^{(p+1)}(a)$$

Remark:

Using the transformation we have, at the same time, enlarged the domain of integration on \mathbb{R}_+ . This is valid because we are looking for an asymptotic expansion:

On the interval $[0, \infty]$ the function under the integral has only a vanishing derivative

at the origin. The part which is added by going to the whole half space converges like $1/x$ if x runs to infinity. Because we have a vanishing derivative at the origin one would expect a slower convergence on bounded intervall. Therefore we do not change the asymptotic behaviour by taken the whole half space as domain of integration.

Using the transformation

$$u = \frac{1}{\alpha + 1} s^{\alpha+1}$$

we get

$$I_1(x) \sim A e^{ix\Phi(a)} \int_0^{\infty} e^{ixC'u^\beta} du$$

and

$$\begin{aligned} \beta &= \frac{p+1}{\alpha+1} \\ C' &= (\alpha+1)^\beta \end{aligned}$$

Now we are able to determine the leading term.

Consider the integral

$$J(x) = \int_0^{\infty} e^{ixKs^p} ds$$

and $K > 0$.

In order to evaluate the asymptotic expansion of the integral we try to separate den imaginary part of the integral such that we need only to calculate the real part – if we reduce the problem to a real integral we may use Watson's Lemma.

To separate the imaginary part we use the relation

$$e^{i\phi} = \cos \phi + i \sin \phi$$

Takind a rotation in the complex plane with angle $\frac{\pi}{2p}$, i.e. the transformation

$$s = e^{i\frac{\pi}{2p}\tau}$$

the integral $J(x)$ becomes

$$J(x) = e^{i\frac{\pi}{2p}} \int_0^{\infty} \exp(ixK e^{i\pi/2}\tau^p) d\tau$$

Caution:

This is a coordinate transformation and not a deformation of the path of integration

in the complex plane

It holds that

$$e^{i\pi/2} = \cos \pi/2 + i \sin \pi/2$$

and hence

$$J(x) = e^{i\frac{\pi}{2}\frac{1}{p}} \int_0^{\infty} \exp(-xK\tau^p) d\tau$$

and the integral on the right hand side is real.

If we choose

$$u = K\tau^p \quad \frac{du}{d\tau} = Kp\tau^{p-1}$$

we get

$$J(x) = e^{i\frac{\pi}{2}\frac{1}{p}} \frac{1}{p} K^{1/p} \int_0^{\infty} u^{1-1/p} e^{-xu} du$$

Using Watson's Lemma we get the asymptotic expansion

$$J(x) \sim e^{i\frac{\pi}{2}\frac{1}{p}} \frac{1}{p} K^{1/p} \Gamma(1/p) x^{-1/p}$$

and this leads directly to the asymptotic expansion

$$I_1(x) \sim Ae^{i(x\Phi(a) + \frac{\pi}{2}\frac{1}{p})} \frac{1}{\beta} \Gamma(1/\beta) \left(\frac{1}{C'x}\right)^{1/\beta}$$

Exercise 3.5

Find the leading term of

$$I(x) = \int_0^1 e^{ix(t-\sin t)} dt$$

We may directly apply the result of exercise 3.4.

If we choose

$$\Phi(t) = t - \sin t$$

we get for the first derivatives

$$\begin{aligned} \Phi(0) &= 0 \\ \Phi'(0) &= 1 - \cos 0 = 0 \\ \Phi''(0) &= \sin 0 = 0 \\ \Phi^{(3)}(0) &= \cos 0 = 1 \end{aligned}$$

Comparing with exercise 3.4 we have the parameter

$$p = 2 \quad A = 1 \quad \alpha = 0$$

As leading term we get

$$I(x) \sim \frac{1}{3} e^{i\pi/6} \Gamma(1/3) \left(\frac{1}{6x} \right)^{1/3}$$

4 Perturbation Methods in Differential Equations

References:

Kevorkian, J.: *Partial Differential Equations, Analytical Solution Techniques*, 1990.

Wasov, W.: *Asymptotic Expansions for Ordinary Differential Equations*, John Wiley & Sons, New York, 1965.

A perturbation problem for differential equations is an differential equation including a small parameter ε .

$$P(D, \varepsilon)f = g$$

In the same way we can have the problem

$$P(D)f = g$$

and the small parameter ε is included in the boundary values.

The expansion $f(x, \varepsilon) = \sum \varepsilon^n f_n(x)$ is called a regular perturbation if the solution f_0 of the unperturbed problem (i.e. if $\varepsilon = 0$) is available. The correction f_1, f_2, \dots are computed recursively.

In the following two exercises we consider the case, that the partial differential equation is independent of ε and the small parameter ε is included in the boundary values.

Exercise 4.1

The flux through a slightly corrugated channel is given by the solution $u(x, y; \varepsilon)$ of

$$\begin{aligned} \Delta u &= -1 & \text{in } |y| \leq h(x, \varepsilon) = 1 + \varepsilon \cos kx \\ u &= 0 & \text{on } |y| = h(x, \varepsilon) \end{aligned}$$

and $u(x, y; \varepsilon)$ periodic in x .

Evaluate correct to order ε^2 the average flux per unit width

$$\frac{k}{2\pi} \int_0^{2\pi/k} \left(\int_{-h(x, \varepsilon)}^{h(x, \varepsilon)} u(x, y; \varepsilon) dy \right) dx$$

We consider the unperturbed problem $\varepsilon = 0$

$$\begin{aligned}\Delta u &= -1 & \text{in } |y| \leq 1 \\ u &= 0 & \text{on } |y| = 1 \\ u(0, y) &= u\left(\frac{2\pi}{k}, y\right) & y \in [-1, 1]\end{aligned}$$

For small values of ε we consider the expansion

$$u(x, y; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n u^{(n)}(x, y) = u^{(0)}(x, y) + \varepsilon u^{(1)}(x, y) + \varepsilon^2 u^{(2)} + O(\varepsilon^3) \quad (14)$$

Together with the equation $\Delta u = -1$ we get

$$\sum_{n=0}^{\infty} \varepsilon^n \Delta u^{(n)}(x, y) = -1$$

and compare, as usual, powers in ε

$$\begin{aligned}\varepsilon^0 : & \quad \Delta u^{(0)} = -1 \\ \varepsilon^n : & \quad \Delta u^{(n)} = 0 \quad \forall n \geq 1\end{aligned}$$

The correction functions $u^{(n)}(x, y)$, $n \geq 1$ are harmonic and $u^{(0)}(x, y)$ is the solution of the unperturbed problem.

What are the boundary conditions?

We consider the boundary condition on the upper boundary of the channel

$$u(x, h(x, \varepsilon)) = 0$$

Because the function $u(x, y; \varepsilon)$ is given in the form (14) we may consider the Taylor expansion of $u(x, h(x, \varepsilon))$ around $(x, 1)$.

$$u(x, h(x, \varepsilon)) = \sum_{n=0}^{\infty} \frac{\partial^n u}{\partial y^n}(x, 1)(h(x, \varepsilon) - 1)^n = \sum_{n=0}^{\infty} \frac{\partial^n u}{\partial y^n}(x, 1)\varepsilon^n \cos^n kx$$

together with the condition

$$u(x, h(x, \varepsilon)) = 0$$

We again use the expansion (14) to get

$$\begin{aligned}u(x, 1) &= \sum_{n=0}^{\infty} u^{(n)}(x, 1) \\ \frac{\partial^m u(x, 1)}{\partial y^m} &= \sum_{n=0}^{\infty} \frac{\partial^m u^{(n)}}{\partial y^m}(x, 1) \quad \forall m \geq 1\end{aligned}$$

and finally

$$\sum_{n=0}^{\infty} \varepsilon^n u^{(n)}(x, 1) + \sum_{n=0}^{\infty} \frac{\partial u^{(n)}}{\partial y}(x, 1) \varepsilon^{n+1} \cos kx + \sum_{n=0}^{\infty} \frac{\partial^2 u^{(n)}}{\partial y^2}(x, 1) \varepsilon^{n+2} \cos^2 kx + \dots = 0$$

The comparison of ε -powers gives

$$\begin{aligned} \varepsilon^0 : \quad u^{(0)}(x, 1) &= 0 \\ \varepsilon^1 : \quad u^{(1)}(x, 1) &= -\cos kx \frac{\partial u^{(0)}}{\partial y}(x, 1) \\ \varepsilon^2 : \quad u^{(2)}(x, 1) &= -\cos kx \frac{\partial u^{(1)}}{\partial y}(x, 1) - \cos^2 kx \frac{\partial u^{(0)}}{\partial y}(x, 1) \end{aligned}$$

Let us conclude:

In order to evaluate the integral

$$\frac{k}{2\pi} \int_0^{2\pi/k} \left(\int_{-h(x,\varepsilon)}^{h(x,\varepsilon)} u(x, y; \varepsilon) dy \right) dx$$

up to order ε^2 we have to solve, recursively, the following three boundary value problems

1)

$$\begin{aligned} \Delta u^{(0)} &= -1 \\ u^{(0)}(x, |y| = 1) &= 0 \\ u^{(0)}(0, y) &= u^{(0)}\left(\frac{2\pi}{k}, y\right) \end{aligned}$$

2)

$$\begin{aligned} \Delta u^{(1)} &= 0 \\ u^{(1)}(x, 1) &= -\cos kx \frac{\partial u^{(0)}}{\partial y}(x, 1) \\ u^{(1)}(x, -1) &= -\cos kx \frac{\partial u^{(0)}}{\partial y}(x, -1) \\ u^{(1)}(0, y) &= u^{(1)}\left(\frac{2\pi}{k}, y\right) \end{aligned}$$

3)

$$\Delta u^{(2)} = 0$$

$$\begin{aligned}
u^{(2)}(x, 1) &= -\cos kx \frac{\partial u^{(1)}}{\partial y}(x, 1) - \cos^2 kx \frac{\partial u^{(0)}}{\partial y}(x, 1) \\
u^{(2)}(x, -1) &= -\cos kx \frac{\partial u^{(1)}}{\partial y}(x, -1) - \cos^2 kx \frac{\partial u^{(0)}}{\partial y}(x, -1) \\
u^{(2)}(0, y) &= u^{(1)}\left(\frac{2\pi}{k}, y\right)
\end{aligned}$$

The solution of problem 1):

$$\Delta u = -1 \tag{15}$$

on a rectangle R with homogenous boundary conditions.

We get a solution of this problem using the Green's function $G(x, y; \xi, \eta)$:

Let us suppose that the rectangle R is given by $R = [0, a] \times [0, b]$. The Green's function is constructed using a conformal mapping of the circle into the given rectangle. The derivation can be found Courant, Hilbert: Methoden der mathematischen Physik I, Seite 335.

One gets

$$G(x, y; \xi, \eta) = \frac{4}{ab\pi^2} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\sin k \frac{\pi}{a} x \sin m \frac{\pi}{b} y \sin k \frac{\pi}{a} \xi \sin m \frac{\pi}{b} \eta}{\frac{m^2}{a^2} + \frac{k^2}{b^2}}$$

The solution of problem (15) with homogeneous boundary condition is given by

$$u(x, y) = \int_0^a \int_0^b G(x, y; \xi, \eta) d\xi d\eta$$

Solution of problem 2):

We have the equation

$$\Delta u = 0 \tag{16}$$

on a rectangle with given boundary values.

Again we assume that the rectangle is given by $R = [0, a] \times [0, b]$.

Using separation of variables

$$u(x, y) = X(x)Y(y)$$

we get two ordinary differential equations for X and Y .

The general solution of (16) is given by

$$u(x, y) = \sum_{n=1}^{\infty} \left[C_n \sinh \frac{n\pi y}{b} + D_n \cosh \frac{n\pi y}{b} \right] \sin \frac{n\pi x}{a}$$

Using C_n and D_n the boundary conditions have to be fitted. The boundary conditions of u depend on the solution of problem 1).

Solution of problem 3):

Make the same as in case 2)

Exercise 4.2

$\varphi(x, y, ; \varepsilon)$ und $\lambda(\varepsilon)$ are solutions of the eigenvalue problem

$$\varphi_{xx} + \varphi_{yy} + \lambda\varphi = 0 \quad \text{in} \quad 0 \leq x \leq \pi, \varepsilon(\pi - x)x \leq y \leq \pi$$

and $\varphi = 0$ at the boundary.

Find the order ε correction for the unperturbed eigenvalue 2 and the corresponding eigenfunction $\varphi = \sin x \sin y$.

How do we calculate the solution of the unperturbed problem?

Consider the linear eigenvalue problem

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{in} \quad [0, \pi] \times [0, \pi]$$

and $\varphi = 0$ at the boundary.

We use an *Ansatz*

$$\varphi(x, y) = X(x)Y(y)$$

and get the equation

$$X_{xx}Y + XY_{yy} + \lambda XY = 0$$

This equation can be written as

$$\frac{X_{xx}}{X} = -\frac{Y_{yy} + \lambda Y}{Y}$$

Now, the left hand side only depends on the variable x , whereas the right hand side depends only on y . Hence we get for X and Y the ordinary differential equations

$$\begin{aligned} X_{xx} - kX &= 0 \\ Y_{yy} + (\lambda - k)Y &= 0 \end{aligned}$$

with an arbitrary constant $k \in \mathbb{R}$.

The corresponding boundary conditions are

$$X(0) = X(\pi) = 0 \quad \text{und} \quad Y(0) = Y(\pi) = 0$$

We already know that nontrivial solutions together with the prescribed boundary conditions only exists if the constants k and λ satisfy the conditions

$$-k = n^2 \quad \text{und} \quad \lambda - k = m^2$$

with arbitrary integers $n, m \in \mathbb{N}$.

Hence we get

$$\lambda = n^2 + m^2 \quad \text{with} \quad n, m \in \mathbb{N} \quad (17)$$

In the current exercise we investigate the perturbation of the unperturbed eigenvalue $\lambda = 2$. If we set $n = m = 1$ relation (17) is fulfilled!

We now try to calculate the solution of

$$\Delta\varphi + \lambda\varphi = 0 \quad 0 \leq x \leq \pi, \quad \varepsilon x(\pi - x) \leq y \leq \pi$$

and $\varphi = 0$ at the boundary (up to order ε).

As usual, we consider the series expansion with respect to the small parameter ε :

$$\begin{aligned} \varphi(x, y; \varepsilon) &= \sum_{n=0}^{\infty} \varepsilon^n \varphi^{(n)}(x, y) \\ \lambda(\varepsilon) &= \sum_{n=0}^{\infty} \varepsilon^n \lambda^{(n)} \end{aligned}$$

Using the given eigenvalue we get (we are only interested on the order ε)

$$\Delta\varphi^{(0)} + \varepsilon\Delta\varphi^{(1)} + (\lambda^{(0)} + \varepsilon\lambda^{(1)})(\varphi^{(0)} + \varepsilon\varphi^{(1)}) + O(\varepsilon^2) = 0$$

If we compare coefficients in ε -powers we get

$$\begin{aligned} \varepsilon^0 : \quad \Delta\varphi^{(0)} + \lambda^{(0)}\varphi^{(0)} &= 0 \\ \varepsilon^1 : \quad \Delta\varphi^{(1)} + \lambda^{(0)}\varphi^{(1)} + \lambda^{(1)}\varphi^{(0)} &= 0 \end{aligned}$$

Now we have to derive the boundary conditions for both equations at the boundary of $[0, \pi] \times [0, \pi]$ (we use the same technique as in exercise 4.1)

$$\varphi(x, \varepsilon x(\pi - x); \varepsilon) = \sum_{n=0}^{\infty} \frac{\partial^n \varphi}{\partial y^n}(x, 0) \frac{\varepsilon^n}{n!} x^n (\pi - x)^n$$

and

$$\varphi(x, \varepsilon x(\pi - x); \varepsilon) = 0$$

If we use the expansion of $\varphi(x, y; \varepsilon)$ we get the two boundary conditions

$$\begin{aligned} \varphi^{(0)}(x, y) &= 0 \\ \varphi^{(1)}(x, y) &= x(x - \pi) \frac{\partial \varphi^{(0)}}{\partial y}(x, 0) \end{aligned}$$

The function $\varphi^{(0)}$ is exactly the solution of the unperturbed system and we set

$$\lambda^{(0)} = 2 \quad \varphi^{(0)}(x, y) = \sin x \sin y$$

Hence the boundary condition for $\varphi^{(1)}(x, 0)$ is

$$\varphi^{(1)} = x(x - \pi) \sin x$$

We now consider the problem

$$\Delta\varphi^{(1)} + 2\varphi^{(1)} + \lambda^{(1)} \sin x \sin y = 0 \quad (18)$$

with boundary conditions

$$\varphi^{(1)}(0, y) = \varphi^{(1)}(\pi, y) = \varphi^{(1)}(x, \pi) = 0 \quad \varphi^{(1)}(x, 0) = x(x - \pi) \sin x$$

Because of the boundary conditions (especially the condition $\varphi^{(1)}(0, y) = \varphi^{(1)}(\pi, y) = 0$) we assume the solution to be a Fourier series in x

$$\varphi^{(1)}(x, y) = \sum_{n=1}^{\infty} a_n(y) \sin nx$$

where the coefficients $a_n = a_n(y)$, $n \in \mathbb{N}$ are functions of y .

Using equation (18) one gets

$$\sum_{n=1}^{\infty} [(a_n''(y) + (2 - n^2)a_n(y)) \sin nx] + \lambda^{(1)} \sin x \sin y = 0$$

with boundary conditions

$$\begin{aligned} \sum_{n=1}^{\infty} a_n(0) \sin nx &= x(x - \pi) \sin x \\ \sum_{n=1}^{\infty} a_n(\pi) \sin nx &= 0 \end{aligned}$$

The set of functions $\{\sin nx\}_{n \in \mathbb{N}}$ constitutes a basis on the space $\mathcal{C}([0, \pi])$ of functions with boundary conditions $f(0) = f(\pi) = 0$.

Hence we get

$$\begin{aligned} a_1''(y) + a_1(y) + \lambda^{(1)} \sin y &= 0 \\ a_n''(y) + (2 - n^2)a_n(y) &= 0 \quad \forall n > 1 \end{aligned}$$

Now we have to construct the boundary conditions for the given equations.

From the relation

$$\sum_{n=1}^{\infty} a_n(\pi) \sin nx = 0$$

it follows directly

$$a_n(\pi) = 0 \quad \forall n \geq 1$$

In order to calculate the boundary condition at $y = 0$ we have to compute the Fourier coefficient of the function $x(x - \pi) \sin x$:

We know that

$$\sum_{n=1}^{\infty} a_n(0) \sin nx = x(x - \pi) \sin x$$

and hence

$$a_n(0) = \frac{2}{\pi} \int_0^{\pi} x(x - \pi) \sin x \sin nx \, dx$$

One calculates

$$a_n(0) = -4 \frac{(-1)^n (n^3 - n) - (n^3 - n)}{(n^2 - 1)^3}$$

and

$$a_n(0) = \begin{cases} -\frac{1}{6}(\pi^2 + 3) & \text{if } n = 1 \\ 0 & \text{if } n > 1 \text{ gerade} \\ -\frac{8n}{(n^2 - 1)^2} & \text{if } n > 1 \text{ ungerade} \end{cases}$$

We first consider the solution if $n > 1$ and even. According to (19) we have to solve the ordinary differential equation

$$a_n''(y) + (2 - n^2)a_n(y) = 0$$

with boundary conditions

$$a_n(0) = a_n(\pi) = 0$$

Now it holds that

$$2 - n^2 < 0$$

We already know that nontrivial solution only exists, if

$$2 - n^2 = m^2 \quad \text{with} \quad m \in \mathbb{N}$$

Hence

$$a_n(y) = 0$$

If $n > 1$ and n odd, we have to solve the differential equation

$$a_n''(y) = (n^2 - 2)a_n(y)$$

with boundary conditions

$$a_n(0) = -\frac{8n}{(n^2 - 1)^2} \quad a_n(\pi) = 0$$

It holds that

$$2 - n^2 < 0$$

We get the solution

$$a_n(y) = c_n \sinh \sqrt{n^2 - 2} (\pi - y)$$

with

$$c_n = - \frac{1}{\sinh \pi \sqrt{n^2 - 2}} \frac{8n}{(n^2 - 1)^2}$$

Finally it remains equation (19) for the Fourier coefficient $a_1(y)$

$$a_1''(y) + a_1(y) + \lambda^{(1)} \sin y = 0$$

with

$$a_1(0) = -\frac{1}{6}(\pi^2 + 3) \quad a_1(\pi) = 0$$

The homogeneous equation

$$a_1''(y) + a_1(y) = 0$$

has the general solution

$$w(y) = a \sin y + b \cos y$$

(We can directly assume that $a = 0$ because the function $\sin x \sin y$ is the solution of the unperturbed system)

In order to construct a special solution we take the *Ansatz*

$$z(y) = (a + bx) \sin y + (c + dx) \cos y$$

and get the equation

$$z''(y) + z(y) + \lambda^{(1)} \sin y = (2b - a - by) \cos y + (a + bx - 2d + \lambda^{(1)}) \sin y = 0$$

It follows that

$$\begin{aligned} 2b - a - bx &= 0 \\ a + bx - 2d + \lambda^{(1)} &= 0 \end{aligned}$$

and the solution is

$$a = b = 0 \quad \lambda^{(1)} = -\frac{d}{2}$$

The special solution is given by

$$z(y) = -\frac{\lambda^{(1)}}{2} x \cos x$$

and the corresponding solution $a_1(y)$ is

$$a_1(y) = b \cos y - \frac{\lambda^{(1)}}{2} y \cos y$$

Now we have to fulfill the two boundary conditions

$$a_1(0) = \frac{1}{6}(\pi^2 + 3) \quad a_1(\pi) = 0$$

From the first condition we directly conclude that

$$b = \frac{1}{6}(\pi^2 + 3)$$

In order to fulfill the second condition we end up with the following restriction on $\lambda^{(1)}$

$$\lambda^{(1)} = \frac{1}{3\pi}(\pi^2 + 3)$$

Now we are able to give the expressions for the order ε correction of the unperturbed system

$$\begin{aligned} \lambda(\varepsilon) &= \lambda^{(0)} + \varepsilon\lambda^{(1)} + O(\varepsilon^2) \\ &= 2 + \varepsilon\frac{1}{3\pi}(\pi^2 + 3) + O(\varepsilon^2) \\ \varphi(x, y; \varepsilon) &= \varphi^{(0)}(x, y) + \varepsilon\varphi^{(1)}(x, y) + O(\varepsilon^2) \\ &= \sin x \sin y + \varepsilon\frac{1}{6}(\pi^2 + 3)\left(1 - \frac{2}{\pi}y\right) \sin x \cos y \\ &\quad + \sum_{k=0}^{\infty} a_{2k+1}(y) \sin(2k+1)x + O(\varepsilon^2) \end{aligned}$$

where

$$\begin{aligned} a_{2k+1}(y) &= c_{2k+1} \sinh[(4k^2 + 4k - 1)^{1/2}(\pi - y)] \\ c_{2k+1} &= \frac{1}{\sinh[(4k^2 + 4k - 1)^{1/2}\pi]} \frac{8(2k+1)}{(4k^2 + 4k - 1)^2} \end{aligned}$$

5 Singular Perturbation: Matching

References:

- Kevorkian, J.: *Partial Differential Equations, Analytical Solution Techniques*, 1990.
 Chang, K.W. and Howes, F.A.: *Nonlinear Singular Perturbation Phenomena: Theory and Applications*, Springer Verlag, New York, 1984.
 Wasov, W.: *Singular Perturbation: Asymptotic Expansions for Ordinary Differential Equations*, John Wiley & Sons, New York, 1976.

van Dyke, M.: *Perturbation Methods in Fluid Mechanics*, The Parabolic Press, Stanford, 1975.

Consider the perturbation problem

$$L_\varepsilon f = 0$$

A perturbation problem is called singular if the operator L_ε is defined for $\varepsilon > 0$ and L_0 has a different character.

For example we have an ordinary differential equation

$$L_\varepsilon y = \varepsilon y'' + y' + y = 0 \quad \text{in } [0, 1]$$

If $\varepsilon > 0$ we have to prescribe two boundary conditions, for example we set

$$y(0) = y_0 \quad y(1) = y_1 \quad (19)$$

If we consider the unperturbed problem

$$L_0 y = y' + y = 0 \quad \text{in } [0, 1] \quad (20)$$

only one boundary condition is needed and the problem (20) together with boundary conditions (19) may have no solution.

Such problems often lead to infinitesimal layers for example near to a boundary:

The change in the equation from $\varepsilon > 0$ to $\varepsilon = 0$ is a small effect which can only be seen near to a boundary. To get the correct behaviour inside the layer it is useful to introduce a scaling in order to enlarge the layer.

Now we can consider two asymptotic expansions

- 1) an outer expansion which should be valid outside the layer
- 2) an inner expansion which should be valid inside the layer

Furthermore one may try to find 'matching conditions' such that both expansion coincide within an overlapping domain. One may also investigate where the layer is located and what is the correct scaling for the layer.

Exercise 5.1

Consider the equation

$$\varepsilon y'' + (1 + \varepsilon)y' + y = 0 \quad \text{in } [0, 1], \quad y(0) = 0, \quad y(1) = \frac{1}{e} \quad (21)$$

Find 2 terms of the outer expansion, 2 terms of the inner expansion and match.

In order to compute the explicit solution of (21) we compute the roots of the quadratic polynomial

$$P(z) = \varepsilon z^2 + (1 + \varepsilon)z + 1 \quad (22)$$

The zeroes are given by

$$\begin{aligned} \lambda^{(1)} &= -1 \\ \lambda^{(2)} &= -\frac{1}{\varepsilon} \end{aligned}$$

and hence the general solution of (21) is given by

$$y(x) = b_1 e^{-x} + b_2 e^{-x/\varepsilon}$$

Using the boundary condition one gets

$$\begin{aligned} y(0) &= b_1 + b_2 = 0 \\ y(1) &= b_1 \frac{1}{e} + b_2 e^{-1/\varepsilon} = \frac{1}{e} \end{aligned}$$

It follows that the explicit solution is given by

$$\begin{aligned} y(x) &= C \left(e^{-x} - e^{-x/\varepsilon} \right) \\ C &= \frac{e^{-1}}{e^{-1} - e^{-1/\varepsilon}} \end{aligned}$$

Figure 5.1 illustrates the explicit solution for various values of ε .

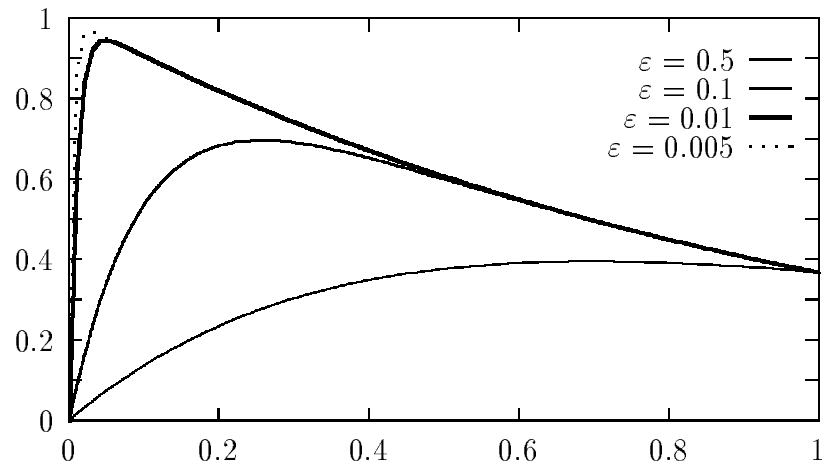


Fig. 5.1: Explicit solution of (21) for various ε

The outer expansion

We want to construct the outer expansion of the solution of (21)

$$y(x; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n f_n(x)$$

and

$$\sum_{n=0}^{\infty} \left(\varepsilon^{n+1} f_n''(x) + (1 + \varepsilon) \varepsilon^n f_n'(x) + \varepsilon^n f_n(x) \right) = 0$$

Now we have

$$\begin{aligned} \varepsilon^0 : \quad f_0' + f_0 &= 0 & f_0(1) &= \frac{1}{e} \\ \varepsilon^n : \quad f_{n-1}'' + f_{n-1}' + f_n' + f_n &= 0 & f_n(1) &= 0 \quad \forall n \geq 1 \end{aligned}$$

The solution is given by

$$\begin{aligned} f_0(x) &= e^{-x} \\ f_n(x) &= 0 \quad \forall n \geq 1 \end{aligned}$$

The outer expansion terminates with f_0 .

The inner expansion

We apply the scaling

$$\xi = \frac{x}{\varepsilon} \quad \hat{y}(\xi; \varepsilon) = y(\varepsilon\xi; \varepsilon)$$

with

$$\hat{y}_\xi = \varepsilon y_x \quad \hat{y}_{\xi\xi} = \varepsilon^2 y_{xx}$$

and get the equation

$$\frac{1}{\varepsilon} \hat{y}_{\xi\xi} + \left(1 + \frac{1}{\varepsilon}\right) \hat{y}_\xi + \hat{y} = 0$$

We take an expansion of the form

$$\hat{y}(\xi; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n g_n(\xi)$$

and

$$\sum_{n=0}^{\infty} \left(\varepsilon^{n-1} g_n'' + \left(1 + \frac{1}{\varepsilon}\right) \varepsilon^n g_n' + \varepsilon^n g_n \right) = 0 \quad (23)$$

and the boundary condition

$$\sum_{n=0}^{\infty} \varepsilon^n g_n(0) = 0$$

(23) leads to the equations

$$\begin{aligned}\varepsilon^{-1} : \quad g_0'' + g_0' &= 0 & g_0(0) &= 0 \\ \varepsilon^n : \quad g_{n-1}' + g_{n-1} + g_n'' + g_n' &= 0 & g_n(0) &= 0 \quad \forall n \geq 0\end{aligned}$$

The solution of the first equation is

$$g_0(\xi) = A_0(1 - e^{-\xi})$$

Hence we get for g_1

$$g_1'' + g_1' + A_0 = 0$$

and

$$g_1(\xi) = A_1(1 - e^{-\xi}) - A_0\xi$$

The equation for g_2 reads

$$g_1' + g_1 + g_2'' + g_2' = 0$$

or

$$g_2'' + g_2' - A_0\xi + A_1 - A_0 = 0$$

and

$$g_2(\xi) = A_2(1 - e^{-\xi}) - A_1\xi + A_0\frac{\xi^2}{2}$$

Now we match the inner and outer expansions.

Choose an intermediate variable η

$$\eta = \frac{x}{\varepsilon^\alpha} = \xi\varepsilon^{1-\alpha}$$

for some $0 < \alpha < 1$, such that if we keep η fixed and let $\varepsilon \rightarrow 0$, then $x \rightarrow 0$ and $\xi \rightarrow \infty$.

The outer expansion gets

$$E_1y = e^{-x} = e^{-\varepsilon^\alpha\eta} = 1 - \varepsilon^\alpha\eta + \frac{\varepsilon^{2\alpha}}{2}\eta^2 - \frac{\varepsilon^{3\alpha}}{6}\eta^3 + O(\varepsilon^{4\alpha}\eta^4) \quad (24)$$

and the inner expansion

$$\begin{aligned}H_1y &= A_0 + \text{E.S.T} + \varepsilon(A_1 + \text{E.S.T} - \varepsilon^{\alpha-1}\eta A_0) \\ &\quad + \varepsilon^2(A_2 + \text{E.S.T.} - \varepsilon^{\alpha-1}\eta A_1 + \varepsilon^{2\alpha-2}\eta^2\frac{A_0}{2}) + \dots \\ &= A_0 + \varepsilon^\alpha\eta A_0 + \frac{\varepsilon^{2\alpha}}{2}\eta^2 A_1 + \varepsilon A_1 - \varepsilon x A_1 + \varepsilon^2 A_2 + \text{E.S.T.} + \dots\end{aligned} \quad (25)$$

We get the equations

$$\begin{aligned} A_0 &= 1 \\ A_1 &= 0 \\ A_2 &= 0 \end{aligned}$$

Hence the inner expansion is

$$H_1 y = 1 - e^{-\xi} - \varepsilon \xi + \frac{1}{2} \varepsilon^2 \xi^2$$

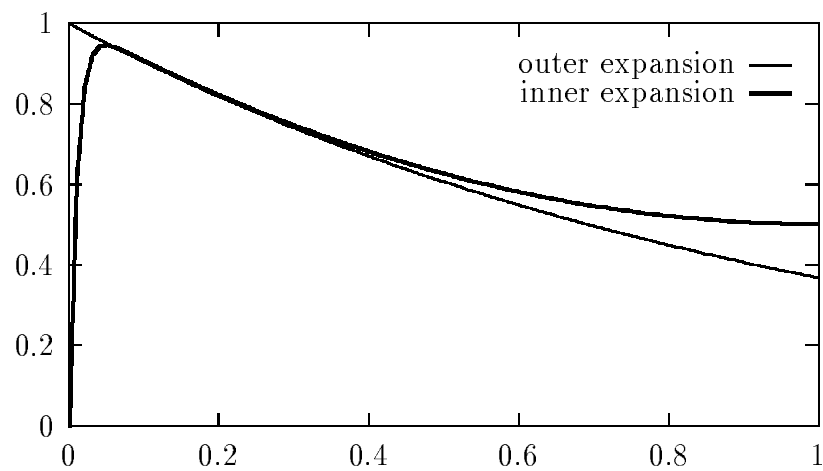


Fig. 5.2: Matched outer and inner expansion for $\varepsilon = 0.01$

One may recognize the overlapping region where the inner and outer expansion coincide and compare the solution with the exact solution shown in figure 5.1.

Exercise 5.2

Consider the equation

$$\varepsilon \Phi_{xx} + \phi_x = h_x \quad \text{in } [0, 1] \quad \Phi(0) = 0, \quad \Phi(1) = 1 \quad (26)$$

Do van Dyke's matching rule for $P = Q = 2$ and $P = 0, Q = 1$. Show for the given example that

$$E_P H_Q \Phi = H_Q E_P \Phi$$

can always be satisfied, i.e. for all P and Q .

For problem (26) the outer expansion with $P + 1$ terms is given by

$$E_P \Phi = 1 + \sum_{n=0}^P (-\varepsilon)^n [h^{(n)}(x) - h^{(n)}(1)]$$

the inner expansion with $Q + 1$ terms by

$$H_Q \Phi = (1 - e^\xi) \sum_{n=0}^Q A_n \varepsilon^n + \sum_{n=1}^Q (-\varepsilon)^n h^{(n)}(0) \sum_{k=1}^n \frac{(-\xi)^k}{k!}$$

and $\xi = \frac{x}{\varepsilon}$.

van Dyke's matching rule for $P = Q = 2$

We compute the two terms expansion $E_2 \Phi$ and $H_2 \Phi$

$$E_2 \Phi = 1 + h(x) - h(1) - \varepsilon(h^{(1)}(x) - h^{(1)}(1)) + \varepsilon^2(h^{(2)}(x) - h^{(2)}(1))$$

$$H_2 \Phi = A_0(1 - e^{-\xi}) + \varepsilon A_1(1 - e^{-\xi}) + \varepsilon^2 A_2(1 - e^{-\xi}) + \varepsilon h^{(1)}(0)\xi + \varepsilon^2 h^{(2)}(0)\left(\frac{\xi^2}{2} - \xi\right)$$

and hence we get

$$\begin{aligned} E_2 H_2 \Phi &= E_2 [A_0(1 - e^{-\xi}) + \varepsilon A_1(1 - e^{-\xi}) + \varepsilon^2 A_2(1 - e^{-\xi}) \\ &\quad + \varepsilon h^{(1)}(0)\xi + \varepsilon^2 h^{(2)}(0)\left(\frac{\xi^2}{2} - \xi\right)] \\ &= E_2 [A_0(1 - e^{-x/\varepsilon}) + \varepsilon A_1(1 - e^{-x/\varepsilon}) + \varepsilon^2 A_2(1 - e^{-x/\varepsilon}) \\ &\quad + x h^{(1)}(0) + \varepsilon^2 h^{(2)}(0)\left(\frac{x^2}{2\varepsilon^2} - \frac{x}{\varepsilon}\right)] \\ &= A_0 + x h^{(1)}(0) + \frac{x^2}{2} h^{(2)}(0) \varepsilon A_1 - \varepsilon x h^{(2)}(0) + \varepsilon^2 A_2 \end{aligned}$$

and

$$\begin{aligned} H_2 E_2 \Phi &= H_2 [1 + h(x) - h(1) - \varepsilon(h^{(1)}(x) - h^{(1)}(1)) + \varepsilon^2(h^{(2)}(x) - h^{(2)}(1))] \\ &= H_2 [1 + h(\varepsilon\xi) - h(1) - \varepsilon(h^{(1)}(\varepsilon\xi) - h^{(1)}(1)) + \varepsilon^2(h^{(2)}(\varepsilon\xi) - h^{(2)}(1))] \\ &= 1 + h(0) + \varepsilon\xi h^{(1)}(0) + \varepsilon^2 \frac{\xi^2}{2} h^{(2)}(0) - h(1) \\ &\quad - \varepsilon h^{(1)}(0) - \varepsilon^2 \xi h^{(2)}(0) + \varepsilon h^{(1)}(1) + \varepsilon^2 h^{(2)}(0) - \varepsilon^2 h^{(2)}(1) \end{aligned}$$

Taking

$$E_2 H_2 \Phi = H_2 E_2 \Phi$$

and using

$$\begin{aligned}\varepsilon\xi h^{(1)}(0) &= xh^{(1)}(0) \\ \varepsilon^2\frac{\xi^2}{2}h^{(2)}(0) &= \frac{x^2}{2}h^{(2)}(0) \\ \varepsilon^2\xi h^{(2)}(0) &= \varepsilon xh^{(2)}(0)\end{aligned}$$

we get

$$\begin{aligned}A_0 &= 1 + h(0) - h(1) \\ A_1 &= h^{(1)}(1) - h^{(1)}(0) \\ A_2 &= h^{(2)}(0) - h^{(2)}(1)\end{aligned}$$

van Dyke's matching rule for $P = 0$, $Q = 1$

We compute $E_0\Phi$ and $H_1\Phi$:

$$\begin{aligned}E_0\Phi &= 1 + h(x) - h(1) \\ H_1\Phi &= A_0(1 - e^{-\xi}) + \varepsilon A_1(1 - e^{-\xi}) + \varepsilon h^{(1)}(0)\xi\end{aligned}$$

Now

$$\begin{aligned}E_0H_1\Phi &= E_0 [A_0(1 - e^{-\xi}) + \varepsilon A_1(1 - e^{-\xi}) + \varepsilon h^{(1)}(0)\xi] \\ &= E_0 [A_0(1 - e^{-x/\varepsilon}) + \varepsilon A_1(1 - e^{-x/\varepsilon}) + h^{(1)}(0)x] \\ &= A_0 + h^{(1)}(0)x\end{aligned}$$

and

$$\begin{aligned}H_1E_0\Phi &= H_1 [1 + h(x) - h(1)] \\ &= H_1 [1 + h(\varepsilon\xi) - h(1)] \\ &= 1 + h(0) - h(1) + \varepsilon\xi h^{(1)}(0)\end{aligned}$$

Using again

$$h^{(1)}(0)x = \varepsilon\xi h^{(1)}(0)$$

we have

$$A_0 = 1 + h(0) - h(1)$$

It is clear, that we only get the constant A_0 of the zeroth order.

van Dyke's matching rule for general P and Q

Take again

$$E_P \Phi = 1 + \sum_{n=0}^P (-\varepsilon)^n [h^{(n)}(x) - h^{(n)}(1)]$$

$$H_Q \Phi = (1 - e^\xi) \sum_{n=0}^Q A_n \varepsilon^n + \sum_{n=1}^Q (-\varepsilon)^n h^{(n)}(0) \sum_{k=1}^n \frac{(-\xi)^k}{k!}$$

We first evaluate $H_Q E_P \Phi$:

$$\begin{aligned} H_Q E_P \Phi &= H_Q \left[1 + \sum_{n=0}^P (-\varepsilon)^n [h^{(n)}(x) - h^{(n)}(1)] \right] \\ &= H_Q \left[1 + \sum_{n=0}^P (-\varepsilon)^n [h^{(n)}(\varepsilon\xi) - h^{(n)}(1)] \right] \\ &= 1 + H_Q \left[\sum_{n=0}^P (-\varepsilon)^n [h^{(n)}(\varepsilon\xi) - h^{(n)}(1)] \right] \\ &= 1 + \sum_{n=0}^{\min(P,Q)} (-\varepsilon)^n \left[\sum_{k=0}^{Q-n} \varepsilon^k \frac{\xi^k}{k!} h^{(n+k)}(0) - h^{(n)}(1) \right] \end{aligned}$$

Now we compute the opposite expansion $E_P H_Q \Phi$:

$$\begin{aligned} E_P H_Q \Phi &= E_P \left[(1 - e^\xi) \sum_{n=0}^Q A_n \varepsilon^n + \sum_{n=1}^Q (-\varepsilon)^n h^{(n)}(0) \sum_{k=1}^n \frac{(-\xi)^k}{k!} \right] \\ &= E_P \left[(1 - e^{x/\varepsilon}) \sum_{n=0}^Q A_n \varepsilon^n + \sum_{n=1}^Q (-\varepsilon)^n h^{(n)}(0) \sum_{k=1}^n \frac{(-x)^k}{k! \varepsilon^k} \right] \\ &= \sum_{n=0}^{\min(P,Q)} A_n \varepsilon^n + E_P \left[\sum_{n=1}^Q (-\varepsilon)^n h^{(n)}(0) \sum_{k=1}^n \frac{(-x)^k}{k! \varepsilon^k} \right] \end{aligned}$$

The rest of the exercise is left to the reader.

Exercise 5.3

Find the interesting scalings for

$$\varepsilon x^m y' + y = 1 \quad \text{in} \quad [0, 1], \quad y(0) = 0 \quad (27)$$

if $0 < m < 1$ and if $m = 1$.

We take as outer expansion

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n f_n(x)$$

and get the equation

$$\sum_{n=0}^{\infty} \left(\varepsilon^{n+1} x^m f'_n(x) + \varepsilon^n f_n(x) \right) = 1$$

hence

$$\begin{aligned} \varepsilon^0 : \quad & f_0(x) = 1 \\ \varepsilon^n : \quad & x^m f'_{n-1}(x) + f_n(x) = 0 \quad \forall n \geq 1 \end{aligned}$$

and

$$\begin{aligned} f_0(x) &= 1 \\ f_n(x) &= 0 \quad \forall n \geq 1 \end{aligned}$$

The outer expansion terminates with f_0 .

Consider the case where $0 < m < 1$:

We try a scaling in the form

$$\xi = \frac{x}{\varepsilon^\alpha}$$

and get

$$\hat{y}(\xi) = y(\varepsilon^\alpha \xi) \quad \hat{y}_\xi(\xi) = \varepsilon^\alpha y_x(\varepsilon^\alpha \xi)$$

Equation (27) becomes

$$\varepsilon^{1+m\alpha-\alpha} \xi^m \hat{y}_\xi + \hat{y} = 1 \tag{28}$$

We get the interesting scaling if (28) is independent of ε :

$$\varepsilon^{1+m\alpha-\alpha} = 1$$

or

$$1 + m\alpha - \alpha = 0 \quad \alpha = \frac{1}{1-m}$$

The first term of the inner solution is calculated by considering the equation

$$\xi^m \hat{y}_\xi + \hat{y} = 1 \quad \hat{y}(0) = 0$$

with solution

$$g_0(\xi) = 1 - \exp\left(\frac{\xi^{1-m}}{m-1}\right)$$

Using the scaling $\xi \varepsilon^{1/(1-m)} = x$ we get

$$\tilde{y}(x) \sim 1 - \exp\left(\frac{x^{1-m}}{\varepsilon(m-1)}\right)$$

which is the exact solution of the problem if $0 < m < 1$.

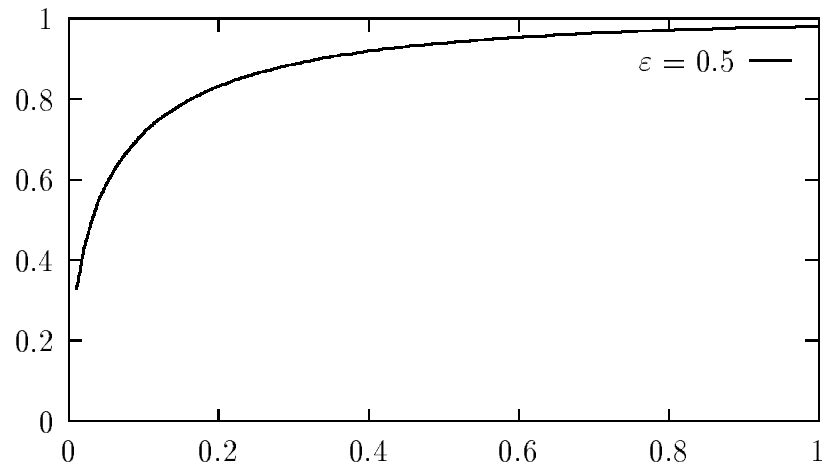


Fig. 5.3: Solution for $m = 1/2$ and $\varepsilon = 0.5$

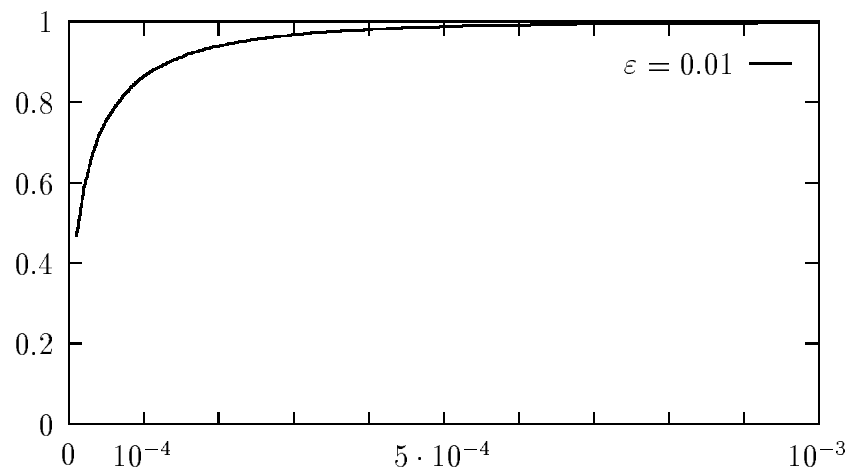


Fig. 5.4: Solution for $m = 1/2$ and $\varepsilon = 0.01$

Now consider the case if $m = 1$:

We try the scaling

$$\xi = \frac{x}{\delta(\varepsilon)} \tag{29}$$

with

$$\hat{y}(\xi) = y(\delta(\varepsilon)\xi) \quad \hat{y}_\xi(\xi) = \delta(\varepsilon)y_x(\delta(\varepsilon)\xi)$$

and equation

$$\varepsilon\xi\hat{y}_\xi + \hat{y} = 1$$

\implies if $m = 1$, the equation is invariant under a scaling of the form (29). We need a scaling which is not of the form (29). Consider

$$\xi = x^{1/\varepsilon}$$

For fixed x we have $\xi \rightarrow 0$, if $\varepsilon \rightarrow 0$ and the boundary layer at $x = 0$ shrinks.

$$\begin{aligned}\hat{y}(\xi) &= y(\xi^\varepsilon) \\ \hat{y}_\xi(\xi) &= \varepsilon \xi^{\varepsilon-1} y'(\xi^\varepsilon)\end{aligned}$$

and the equation gets

$$\begin{aligned}\varepsilon \xi^\varepsilon \hat{y}_\xi \varepsilon^{-1} \xi^{1-\varepsilon} + \hat{y} &= 1 \\ \implies \xi \hat{y}_\xi + \hat{y} &= 1\end{aligned}$$

with boundary condition $\hat{y}(0) = 0$.

The equation can be transformed into

$$(\xi \hat{y})_\xi = 1$$

and by integration we get

$$\xi \hat{y} = \xi + c$$

with an arbitrary constant c .

It follows that

$$\hat{y}(\xi) = 1 + \frac{c}{\xi}$$

but obviously the solution diverges if $\xi \rightarrow 0$.

This means that the equation (27) with the given boundary condition has no solution for $\varepsilon > 0$ and $m = 1$.

Exercise 5.4

Consider

$$\varepsilon y'' + x^{1/2} y' + y = 0 \quad \text{in } [0, 1], \quad y(0) = 0, \quad y(1) = 1 \quad (30)$$

Find the appropriate scaling for a boundary layer at $x = 0$ and construct the inner solution, find the outer solution and match!

We consider the scaling

$$\xi = \frac{x}{\varepsilon^\alpha}$$

and the equation

$$\varepsilon^{1-2\alpha} \hat{y}_{\xi\xi} + \varepsilon^{-\alpha/2} \xi^{1/2} \hat{y}_\xi + \hat{y} = 0 \quad (31)$$

We get the appropriate scaling for the boundary layer at $x = 0$ if we have a balance between the first and second term of (31):

$$\begin{aligned} 1 - 2\alpha &= -\frac{\alpha}{2} \\ \implies \alpha &= \frac{2}{3} \end{aligned}$$

Equation (31) becomes

$$\varepsilon^{-1/3} \hat{y}_{\xi\xi} + \varepsilon^{-1/3} \xi^{1/2} \hat{y}_{\xi} + \hat{y} = 0 \quad (32)$$

To calculate the outer expansion we take

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n f_n(x)$$

and

$$\sum_{n=0}^{\infty} \left(\varepsilon^{n+1} f_n''(x) + \varepsilon^n (x^{1/2} f_n'(x) + f_n(x)) \right) = 0$$

with boundary condition $\sum \varepsilon^n f_n(1) = 1$.

$$\begin{aligned} \varepsilon^0 : \quad & x^{1/2} f_0'(x) + f_0(x) = 0 \quad f_0(1) = 1 \\ \varepsilon^n : \quad & f_{n-1}''(x) + x^{1/2} f_n'(x) + f_n(x) = 0 \quad f_n(1) = 1 \quad n \geq 1 \end{aligned}$$

The solutions are given by

$$\begin{aligned} f_0(x) &= e^{-2x^{1/2}+2} \\ f_1(x) &= -\frac{1}{2}(5 - 4x^{-1/2} - x^{-1})e^{-2x^{1/2}+2} \end{aligned}$$

For the inner expansion we take

$$\hat{y}(\xi) = \sum_{n=0}^{\infty} \varepsilon^{n/3} g_n(\xi)$$

and get

$$\sum_{n=0}^{\infty} \left(\varepsilon^{(n-1)/3} (g_n''(\xi) + \xi^{1/2} g_n'(\xi)) + \varepsilon^{n/3} g_n(\xi) \right) = 0$$

with boundary condition $\sum \varepsilon^{n/3} g_n(0) = 0$.

$$\begin{aligned} \varepsilon^{-1/3} : \quad & g_0'' + \xi^{1/2} g_0' = 0 \quad g_0(0) = 0 \\ \varepsilon^0 : \quad & g_1'' + \xi^{1/2} g_1' + g_0 = 0 \quad g_1(0) = 0 \end{aligned}$$

The solutions are

$$\begin{aligned}g_0(\xi) &= A_0(1 - e^{-\xi}) \\g_1(\xi) &= A_0(1 - \xi - e^{-\xi}(1 + \xi)) + A_1(1 - \exp(-\xi))\end{aligned}$$

As an exercise, the matching of the two expansions

$$\begin{aligned}y(x) &\sim e^{-2x^{1/2}+2} - \varepsilon \frac{1}{2}(5 - 4x^{-1/2} - x^{-1})e^{-2x^{1/2}+2} \\ \hat{y}(\xi) &\sim A_0(1 - e^{-\xi}) + \varepsilon^{1/3} (A_0(1 - \xi - e^{-\xi}(1 + \xi)) + A_1(1 - \exp(-\xi)))\end{aligned}$$

is left to the reader.

Exercise 5.5

Consider the Burger's equation

$$u_t + uu_x = \varepsilon u_{xx} \tag{33}$$

with

$$u(x, 0; \varepsilon) = \begin{cases} -1 & \text{for } x < 0 \\ +1 & \text{for } x \geq 0 \end{cases}$$

Introduce a corner layer by considering

$$\xi = \frac{x \mp t}{\delta(\varepsilon)}$$

and choose $\delta(\varepsilon)$.

(see also Kevorkian, Chapter 8.3, p. 500–501)

The outer expansion

The outer expansion is given by

$$u_0(t, x) = \begin{cases} 1 & x \geq t \\ \frac{x}{t} & -x \leq t \leq x \\ -1 & x \leq -t \end{cases}$$

and terminates with u_0 .

Near the point $x = \pm t$ we need to introduce a corner layer to get a smooth transition between the two sides. We introduce the scaling

$$\xi = \frac{x \mp t}{\delta(\varepsilon)}$$

and

$$\tilde{u}(t, \xi; \varepsilon) = u(t, \delta(\varepsilon)\xi \pm t) \quad (34)$$

We compute

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial \tilde{u}}{\partial t} \mp \frac{1}{\delta(\varepsilon)} \frac{\partial \tilde{u}}{\partial \xi} \\ \frac{\partial u}{\partial x} &= \frac{1}{\delta(\varepsilon)} \frac{\partial \tilde{u}}{\partial \xi} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{1}{\delta(\varepsilon)^2} \frac{\partial^2 \tilde{u}}{\partial \xi^2} \end{aligned}$$

Substituting (34) into (33) we get

$$\frac{\partial \tilde{u}}{\partial t} \mp \frac{1}{\delta(\varepsilon)} \frac{\partial \tilde{u}}{\partial \xi} + \tilde{u} \frac{1}{\delta(\varepsilon)} \frac{\partial \tilde{u}}{\partial \xi} = \frac{\varepsilon}{\delta(\varepsilon)^2} \frac{\partial^2 \tilde{u}}{\partial \xi^2} \quad (35)$$

Since we are looking for a corner layer at $x = \pm t$ we expand the function $u(t, \xi; \varepsilon)$ around the values ± 1

$$\tilde{u}(t, \xi; \varepsilon) = \pm 1 + \gamma(\varepsilon)\tilde{u}_c(t, \xi; \varepsilon) + o(\gamma) \quad (36)$$

Using (36) equation (35) reads

$$\gamma \frac{\partial \tilde{u}_c}{\partial t} \mp \frac{\gamma}{\delta} \frac{\partial \tilde{u}_c}{\partial \xi} + (\pm 1 + \gamma \tilde{u}_c) \frac{\gamma}{\delta} \frac{\partial \tilde{u}_c}{\partial \xi} = \frac{\varepsilon \gamma}{\delta^2} \frac{\partial^2 \tilde{u}_c}{\partial \xi^2} + o(\gamma)$$

This equation can be simplified

$$\gamma \frac{\partial \tilde{u}_c}{\partial t} + \frac{\gamma^2}{\delta} \tilde{u}_c \frac{\partial \tilde{u}_c}{\partial \xi} = \frac{\varepsilon \gamma}{\delta^2} \frac{\partial^2 \tilde{u}_c}{\partial \xi^2} + o(\gamma)$$

To get an equation independent of the scaling parameters we have to set

$$\gamma = \frac{\gamma^2}{\delta} = \frac{\varepsilon \gamma}{\delta^2}$$

or

$$\gamma = \delta = \varepsilon^{1/2}$$

Then the full equation is

$$\frac{\partial \tilde{u}_c}{\partial t} + \tilde{u}_c \frac{\partial \tilde{u}_c}{\partial \xi} = \frac{\partial^2 \tilde{u}_c}{\partial \xi^2}$$

Hence the scaling for the corner layer is given by

$$\delta(\varepsilon) = \varepsilon^{1/2}$$