Master Thesis

Optimal Control and Asymptotic Analysis of the Cattaneo Model

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Abstract

Optimal control of partial differential equations is an important task in applied mathematics where it is used in order to optimize, for example, industrial or medical processes. In this thesis we investigate an optimal control problem with tracking type cost functional for the Cattaneo equation with distributed control, that is, $\tau y_{tt} + y_t - \Delta y = u$. Our focus is on the theoretical and numerical analysis of the limit process $\tau \to 0$ where we prove the convergence of solutions of the Cattaneo equation to solutions of the heat equation.

We start by deriving both the Cattaneo and the classical heat equation as well as introducing our notation and some functional analytic background. Afterwards, we prove the well-posedness of the Cattaneo equation for homogeneous Dirichlet boundary conditions, that is, we show the existence and uniqueness of a weak solution together with its continuous dependence on the data. We need this in the following, where we investigate the optimal control problem for the Cattaneo equation: We show the existence and uniqueness of a global minimizer for an optimal control problem with tracking type cost functional and the Cattaneo equation as a constraint. Subsequently, we do an asymptotic analysis for $\tau \to 0$ for both the forward equation and the aforementioned optimal control problem and show that the solutions of these problems for the Cattaneo equation converge strongly to the ones for the heat equation. Finally, we investigate these problems numerically, where we examine the different behaviour of the models and also consider the limit $\tau \to 0$, suggesting a linear convergence rate.

Contents

1 Introduction

Solving optimization problems subject to constraints given by partial differential equations (PDEs) among other constraints is an important task in applied mathematics (see, for example, [KSLT09], [HPUU08] or [Tro10]). Problems of that kind arise, for example, in the fields of industrial or medical applications. From a mathematical point of view these problems are very interesting since they require a proper modelling, followed by accurate (numerical) simulations and finally their optimization.

In this thesis we investigate the Cattaneo model for heat transfer. It was introduced in [Cat58] in order to "correct" the non-physical behaviour of infinite speed of propagation that the classical heat equation exhibits. The heat equation is given by

$$y_t - \Delta y = u, \tag{1.1}$$

where y denotes the temperature of some medium and u denotes a heat source. Furthermore, the Cattaneo equation, which can be seen as a delayed heat equation, is given by

$$\tau y_{tt} + y_t - \Delta y = u, \tag{1.2}$$

where $\tau > 0$ is a delay parameter (see Chapter 2). In order to simplify our presentation, we do not discuss initial or boundary conditions needed to get a well-posed problem at this point. Formally setting $\tau = 0$ in the Cattaneo equation (1.2) gives the heat equation (1.1). Therefore, we expect that solutions of the Cattaneo equation converge to the ones of the heat equation for $\tau \to 0$, which we prove rigorously in Chapter 5.

Now let us take a look at an example for an optimization problem where the heat equation is a reasonable constraint. Note that this example is a very simplified one, however, it showcases some of the main ideas for the modelling of PDE constrained optimization problems.

For the treatment of liver tumors so-called laser-induced interstitial thermotherapy (LITT) is used in order to destroy tumors due to coagulative effects. In [HLB+17] it is shown that the heat equation is a good model for the temperature distribution in the tissue, where the heat source u is given by the laser's radiation. Therefore, an optimization problem could be the following: We prescribe a desired temperature distribution y_d that we want to achieve during the treatment. This temperature distribution should be chosen such that the tumorous tissue is coagulated. Additionally, most of the healthy tissue should not be heated in such a way that it is damaged and there should also be some safety region, where healthy tissue is coagulated in order to make sure that all of the tumor is destroyed. Therefore, we consider the optimization problem of finding a radiation of the laser so that the resulting temperature distribution is close to the desired one.

This can be modelled mathematically by the following optimization problem:

$$\min \; ||y - y_d||_H^2 + \lambda \, ||u||_U^2 \quad \text{subject to} \quad y_t - \Delta y = u, \tag{1.3}$$

where $\lambda \geq 0$ is a regularization parameter and the spaces H and U have to be chosen appropriately. In case $\lambda > 0$ we also penalize high values of u which is reasonable since, for example, it can be costly to use a large amount of laser power. Therefore, the parameter λ introduces a weighting between the objectives "get close to the desired temperature distribution" and "do not use too much laser power".

In this thesis, we consider similar optimization problems, but as constraint we use the Cattaneo equation, that is, we consider the problem

$$\min \ ||y - y_d||_H^2 + \lambda \, ||u||_U^2 \quad \text{subject to} \quad \tau y_{tt} + y_t - \Delta y = u. \tag{1.4}$$

As before, we observe that formally setting $\tau = 0$ gives the same optimization problem. Therefore, we again expect a convergence of the solution of the optimization problem with the Cattaneo equation towards the solution of the corresponding problem with the heat equation as a constraint for $\tau \to 0$. However, note that the limit process for the optimization problem is more difficult to treat than the one for the forward problems (1.1) and (1.2) since there are two variables, the state y and the control u, that we have to analyze.

Note that solving such problems analytically is not possible in general. In applications the geometry of the computational domain is often rather complicated: For our example of LITT the domain is given by the patient's liver. As we can see from this example, it is, in general, not possible to give an analytical representation of the computational domain. Furthermore, solving a PDE on a complicated domain or solving a nonlinear optimization problem is, generally, also not possible analytically. For these reasons, we have to employ numerical methods in order to solve these problems. This is done in Chapter 6 where we again focus on the asymptotic behaviour of the Cattaneo equation for $\tau \to 0$.

An overview on PDE constrained optimization can be found, for example, in [HPUU08] or [Tro10], where, among others, the optimal control problem for the heat equation is discussed. We use techniques shown there in order to investigate the optimal control problem for the Cattaneo equation in Chapter 4.

2 Basic Concepts

Let us begin with deriving the Cattaneo equation and introducing basic concepts as well as the notation we use throughout this thesis. In particular, we present theoretical results on both functional analysis and PDE constrained optimization.

2.1 Derivation of the Cattaneo Equation

As we have already stated in the previous chapter, the Cattaneo equation can be interpreted as a delayed heat equation. In order to see this, let us now derive both the heat and the Cattaneo equation.

For the derivation of the heat equation we take a look at the thermal energy balance of a medium whose shape is given by a bounded domain Ω with boundary $\partial\Omega$. Furthermore, we assume that there is only heat transfer due to heat conduction and neglect other possible effects such as convection and radiation. For the moment we also assume that there are no heat sources present in the medium. Moreover, we suppose that the medium is at rest and that it is both homogeneous and isotropic, which simplifies the resulting equation. From physics we know that the thermal energy $E(t)$ of the medium at time t is proportional to its temperature that we denote by $y = y(t, x)$, that is,

$$E(t) = \int_\Omega y(t, x) \; \mathrm{d}x,$$

where we have chosen a proportionality constant of one for simplicity and better readability. Due to our assumptions, the change of energy can be expressed as

$$\frac{d}{dt}E(t) = \int_\Omega y_t(t, x) \; \mathrm{d}x.$$

Additionally, the change of energy is proportional to the heat entering or leaving the domain over the boundary. We can express this with the help of the heat flux $q = q(t, x)$ as

$$\frac{d}{dt}E(t) = \int_\Omega y_t(t, x) \; \mathrm{d}x = -\int_{\partial\Omega} q(t, x) \cdot n \; \mathrm{d}s, \tag{2.1}$$

where n denotes the outer unit normal vector on $\partial\Omega$. Note that the sign in front of the boundary integral comes from the fact that we have a heat flux out of the domain in case $q \cdot n$ is positive, since n is pointing outwards, and, therefore, a decrease of energy in that case.

The second law of thermodynamics states that thermal energy flows from a hot medium to a cold one and never the other way around (cf. [Dem15, Chapter 10.3]). This has been modelled most prominently by the so-called Fourier's law for the heat flux. It states that the heat flux is proportional to the temperature gradient, that is,

$$q(t, x) = -\nabla y(t, x). \tag{2.2}$$

Again, we use a proportionality constant of one for the same reasons as before. The minus sign in Fourier's law is due to the fact that the (temperature) gradient is pointing into the direction of steepest (temperature) ascent, so we need the sign in order to be compatible with the second law of thermodynamics. We plug this into (2.1) and observe

$$\begin{aligned} &\int_\Omega y_t(t,x)\ \mathrm{d}x && (2.3)\\ =&\int_{\partial\Omega} -q(t,x)\cdot n\ \mathrm{d}s && \text{(due to (2.1))}\\ =&\int_{\partial\Omega} \nabla y(t,x)\cdot n\ \mathrm{d}s && \text{(due to (2.2))}\\ =&\int_\Omega \nabla\cdot(\nabla y(t,x))\ \mathrm{d}x && \text{(Gauss' divergence theorem)}\\ =&\int_\Omega \Delta y(t,x)\ \mathrm{d}x. \end{aligned}$$

Since our derivation holds true for any domain Ω and any time $t \in (0,T)$, we observe that the integrands in (2.3) have to coincide, which yields the homogeneous heat equation

$$y_t - \Delta y = 0 \quad \text{in } (0,T)\times\Omega,$$

where $T > 0$ denotes the time horizon up to which we investigate the PDEs. For our purposes an inhomogeneous heat equation is of greater interest, in particular, we consider the classical heat equation

$$y_t - \Delta y = u \quad \text{in } (0,T)\times\Omega,$$

where the inhomogeneity $u = u(t,x)$ can be interpreted as a heat source or sink. In order to get a well-posed problem we need initial and boundary conditions for the temperature. As initial condition we prescribe the temperature distribution at $t = 0$, that is, $y(0,x) = y_0(x)$ in Ω. For the boundary condition we could choose, for example, a homogeneous Dirichlet condition, that is, we demand $y = 0$ on $[0,T]\times\partial\Omega$. Note that later on, for the optimal control problems, we use the heat source u as a (distributed) control, that is, we can influence the temperature distribution y by changing the heat source u.

It is well-known that this equation has the property of "infinite speed of propagation" (cf. [Eva10, Chapter 2.3]) which means that changing the temperature locally directly influences the solution on the whole domain at any future point in time. In order to correct this non-physical property, the following idea regarding the heat flux was introduced by Cattaneo in [Cat58]. Instead of requiring that the heat flux at time t is proportional to the temperature gradient at the same time, Cattaneo introduced a delay time τ and assumed that the heat flux coincides with the negative temperature gradient after that delay time has passed, that is,

$$q(t+\tau,x) = -\nabla y(t,x). \tag{2.4}$$

We could now plug this formula into the equation of energy conservation (2.1) in order to derive a delayed heat equation. However, in [DQR09] it is shown that doing so does not yield a well-posed problem. Instead, we do a Taylor expansion of q around t to observe

$$q(t+\tau,x) = q(t,x) + \tau q_t(t,x) + \mathcal{O}(\tau^2).$$

We drop the terms in $\mathcal{O}(\tau^2)$ and use (2.4) to see

$$q(t,x) + \tau q_t(t,x) = -\nabla y(t,x)$$

or, equivalently,

$$q(t,x) = -\left(\nabla y(t,x) + \tau q_t(t,x)\right). \tag{2.5}$$

We observe that it holds

$$\tau \frac{d}{dt} \int_{\partial\Omega} q \cdot n \, \mathrm{d}s = -\tau \frac{d}{dt} \int_{\Omega} y_t \, \mathrm{d}x = -\tau \int_{\Omega} y_{tt} \, \mathrm{d}x, \tag{2.6}$$

thanks to (2.1), and, therefore, we can use the modified Fourier's law in (2.3) to observe

$$\begin{aligned}
&\int_{\Omega} y_t \, \mathrm{d}x \\
={}& \int_{\partial\Omega} -q \cdot n \, \mathrm{d}s && \text{(due to (2.1))} \\
={}& \int_{\partial\Omega} (\nabla y + \tau q_t) \cdot n \, \mathrm{d}s && \text{(due to (2.5))} \\
={}& \int_{\partial\Omega} \nabla y \cdot n \, \mathrm{d}s + \int_{\partial\Omega} \tau q_t \cdot n \, \mathrm{d}s \\
={}& \int_{\Omega} \Delta y \, \mathrm{d}x + \tau \frac{d}{dt} \int_{\partial\Omega} q \cdot n \, \mathrm{d}s && \text{(Gauss' divergence theorem)} \\
={}& \int_{\Omega} \Delta y - \tau y_{tt} \, \mathrm{d}x, && \text{(due to (2.6))}
\end{aligned}$$

where we have suppressed the time and space dependency of the temperature y. Again, we use the argument that this holds true for any domain Ω and any $t \in (0,T)$, and that, therefore, the integrands have to coincide, leading to the homogeneous Cattaneo equation

$$\tau y_{tt} + y_t - \Delta y = 0 \quad \text{in } (0,T).$$

As before, we actually consider the (inhomogeneous) Cattaneo equation

$$\tau y_{tt} + y_t - \Delta y = u \quad \text{in } (0,T),$$

where u can, again, be interpreted as a heat source. Note that, as for the heat equation, we need appropriate initial and boundary conditions in order to obtain a well-posed problem. For example, we can take the same initial and boundary conditions as for the heat equation together with an additional initial velocity $y_t(0,x) = y_1(x)$ in Ω since the Cattaneo equation is second order in time.

Notice that this equation is hyperbolic in the sense of [Str07, Chapter 1.6], whereas the classical heat equation is one of the most prominent examples for a parabolic equation. In fact, the Cattaneo equation (for large τ) shares many similarities with the wave equation. It can either be interpreted as we did in our derivation, or it can be seen as a damped wave equation, the latter being useful for our theoretical results later on. Typically, hyperbolic equations have a finite speed of propagation (cf. [Eva10, Chapter 2.4]) which is the goal of the modification in Fourier's law.

In this thesis we investigate the behaviour of this equation in comparison to the classical heat equation. Of particular interest is the limit $\tau \to 0$ which we investigate both analytically and numerically for the forward equation as well as for the optimal control problem. Note that we often use the term "forward" equation for the Cattaneo and heat equation since we later on also have the corresponding backward equations and want to distinguish between them. It turns out that the solutions of both the forward equation and the optimal control problem for the Cattaneo equation converge to the solutions of the corresponding problems for the heat equation for $\tau \to 0$ which is shown in Chapter 5.

2.2 Background and Notation

In this chapter we give a brief introduction of our notation and recall some useful functional analytic facts that help us with the theory for the Cattaneo equation. We assume that the reader is familiar with basic functional analysis, like Lebesgue and Sobolev spaces, as well as weak (spatial) derivatives. A good introduction to these topics with focus on application in an optimal control context can be found in [HPUU08]. Other recommendable books include, for example, [Alt16], [Eva10], [Tro10], [DL00] and [Tem97], as well as many other books on partial differential equations. Note that this chapter as well as our notation mostly follows [HPUU08, Chapters 1.2 and 1.3].

Notation 2.1
We have the following notations and conventions:

- Throughout the rest of this thesis Ω denotes a bounded Lipschitz domain in $\mathbb{R}^n$ with boundary $\partial\Omega$.
- Let X be a Banach space and H be a Hilbert space. We denote the corresponding norm of X by $||\cdot||_X$ and the scalar product (which we also call inner product) in H is denoted by $\langle \cdot, \cdot \rangle_H$.
- Furthermore, we denote by X^* the dual space of X, i.e.,

$$X^* := \{ \varphi \colon X \to \mathbb{R} \mid \varphi \text{ is linear and bounded} \}.$$

  We denote the application of a linear functional $\varphi \in X^*$ to some $y \in X$ by the duality pairing

$$\langle \varphi, y \rangle_{X^*,X} := \varphi(y).$$

  Additionally, the duality pairing is "symmetric" in the sense that we write

$$\langle \varphi, y \rangle_{X^*,X} = \langle y, \varphi \rangle_{X,X^*}.$$

- More generally, for Banach spaces X and Y we denote by $\mathscr{L}(X,Y)$ the space

$$\mathscr{L}(X,Y) := \{ \varphi \colon X \to Y \mid \varphi \text{ is linear and bounded} \}.$$

The operator norm on $\mathscr{L}(X,Y)$ is given by

$$||\varphi||_{X^*} := \sup_{||y||_X=1} ||\varphi(y)||_Y \,.$$

With this we see that the dual space of X is given as $X^* = \mathscr{L}(X,\mathbb{R})$.

- A special notation is used for the dual space of $H_0^1(\Omega)$, namely we define

$$H^{-1}(\Omega) := (H_0^1(\Omega))^*.$$

For the sake of better readability we also skip the domain when writing duality pairings and, for example, write

$$\langle y,\varphi\rangle_{H^1,H^{-1}} := \langle y,\varphi\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \,.$$

- We say that a sequence $(y_k) \subset X$ converges weakly to some $y \in X$ if it holds that

$$\langle y_k,\varphi\rangle_{X,X^*} \to \langle y,\varphi\rangle_{X,X^*} \qquad \text{for all } \varphi \in X^*.$$

Furthermore, we denote the weak convergence by

$$y_k \rightharpoonup y \text{ in } X.$$

- A functional $\varphi\colon X \to \mathbb{R}$ is called weakly lower semicontinuous if it holds that

$$y_k \rightharpoonup y \text{ in } X \quad \Rightarrow \quad \varphi(y) \le \liminf_{k\to\infty} \varphi(y_k).$$

Every convex and continuous functional is in fact weakly lower semicontinuous (see [HPUU08, Theorem 1.18]). In particular, the norm of a Banach space is weakly lower semicontinuous thanks to the triangle inequality.

- We say that some subset $V \subseteq X$ of a Banach space X is weakly closed if for every sequence $(y_k) \subset V$ with $y_k \rightharpoonup y$ in X we have $y \in V$.

- With C we denote a generic constant $C > 0$ possibly having different values in one and the same equation. We denote by $C = C(\alpha)$ the dependence of the constant on a parameter α.

- We denote by $C_0^\infty(\Omega)$ the space of all infinitely often continuously differentiable functions with values in $\mathbb{R}$ and compact support on Ω, i.e.,

$$C_0^\infty(\Omega) := \{\, y \in C^\infty(\Omega) \mid \operatorname{supp} y \text{ compact} \,\},$$

where the support of a function $y\colon \Omega \to \mathbb{R}$ is defined as

$$\operatorname{supp} y := \overline{\{\, x \in \Omega \mid y(x) \neq 0 \,\}}.$$

Remark 2.2
Our notation of a duality pairing is rather similar to the one we use for a scalar product. In fact, this is intended since we can interpret a scalar product as a duality pairing. The converse holds true if the underlying space is a Hilbert space thanks to the Riesz representation theorem. However, they are not to be confused since the inner product has only one subscript (which is the corresponding Hilbert space), whereas the duality pairing always has both a Banach space and its dual as subscripts.

In the following, we introduce so-called Bochner spaces. These are spaces of vector-valued functions since their elements are functions with values in a Banach space and, therefore, their values are vectors. Throughout the rest of this thesis $T > 0$ denotes the time horizon of the PDEs.

Definition 2.3
Let X be a Banach space, H be a Hilbert space and $1 \leq p < \infty$. We define the spaces

$$L^p(0,T;X) := \left\{ y\colon [0,T] \to X \text{ strongly measurable} \;\middle|\; \int_0^T ||y(t)||_X^p \, \mathrm{d}t < \infty \right\}$$

as well as

$$L^\infty(0,T;X) := \left\{ y\colon [0,T] \to X \text{ strongly measurable} \;\middle|\; \text{ess sup}_{t\in[0,T]} \, ||y(t)||_X < \infty \right\}.$$

These spaces are endowed with the norms

$$||y||_{L^p(0,T;X)} := \left(\int_0^T ||y(t)||_X^p \, \mathrm{d}t \right)^{\frac{1}{p}}$$

and also

$$||y||_{L^\infty(0,T;X)} := \text{ess sup}_{t\in[0,T]} \, ||y(t)||_X .$$

Furthermore, we equip the space $L^2(0,T;H)$ with the inner product

$$\langle y, v \rangle_{L^2(0,T;H)} := \int_0^T \langle y(t), v(t) \rangle_H \, \mathrm{d}t.$$

For these choices the spaces $L^p(0,T;X)$ are Banach spaces and the space $L^2(0,T;H)$ is a Hilbert space. Note that it holds $L^q(0,T;X) \subseteq L^p(0,T;X)$ for $1 \leq p \leq q \leq \infty$ thanks to the Hölder inequality. Therefore, every function $y \in L^p(0,T;X)$ is, in fact, also an element of $L^1(0,T;X)$.

Additionally, it can be shown that the dual space of $L^p(0,T;X)$ for $1 \leq p < \infty$ can be isometrically identified with $L^q(0,T;X^*)$, where $1 < q \leq \infty$ such that $1/p + 1/q = 1$. The duality pairing is then given by

$$\langle y, v \rangle_{L^p(0,T;X),L^q(0,T;X^*)} = \int_0^T \langle y(t), v(t) \rangle_{X,X^*} \, \mathrm{d}t.$$

Furthermore, spaces of continuous and continuously differentiable vector-valued functions are of interest for our investigations.

Definition 2.4
Let $k \in \mathbb{N}$ and X be a Banach space. We define the spaces

$$C^k([0,T];X) := \{\, y\colon [0,T] \to X \mid y \text{ is } k\text{-times continuously differentiable on } [0,T] \,\}.$$

Note that for $k = 0$ we only require that y is continuous on $[0,T]$ and we write

$$C([0,T];X) := C^0([0,T];X).$$

We equip the space $C([0,T];X)$ with the norm

$$||y||_{C([0,T];X)} := \sup_{t\in[0,T]} ||y(t)||_X$$

and the space $C^k([0,T];X)$ with the norm

$$||y||_{C^k([0,T];X)} := \sum_{j=0}^{k} \left|\left| \frac{d^j}{dt^j} y \right|\right|_{C([0,T];X)},$$

which makes them Banach spaces. As before, the space $C_0^\infty((0,T);X)$ is defined as the space of infinitely often differentiable functions with compact support on $(0,T)$ and values in X.

For functions in $L^p(0,T;X)$ we have the following density result available (see [HPUU08, Lemma 1.9]).

Lemma 2.5
Functions of the form

$$\sum_{i=1}^{m} \varphi_i(t) y_i$$

with $\varphi_i \in C_0^\infty((0,T))$ and $y_i \in X$ are dense in $L^p(0,T;X)$ for $1 \leq p < \infty$.

Furthermore, the spaces $C_0^\infty((0,T);X)$ as well as $C^k([0,T];X)$, for $k \in \mathbb{N}$, are dense in $L^p(0,T;X)$ for $1 \leq p < \infty$.

In analogy to weak spatial derivatives of “classical” Sobolev spaces we define weak time derivatives in the following way.

Definition 2.6
Let $y \in L^1(0,T;X)$ and $k \in \mathbb{N}_{>0}$. We say that a function $w \in L^1(0,T;X)$ is the k-th order weak time derivative of y if it holds

$$\int_0^T y(t) \frac{d^k}{dt^k} \varphi(t) \, \mathrm{d}t = (-1)^k \int_0^T w(t)\varphi(t) \, \mathrm{d}t \qquad \text{for all } \varphi \in C_0^\infty((0,T)).$$

In this case we write $\frac{d^k}{dt^k}y := w$. Moreover, for weakly differentiable functions in $L^1(0,T;X)$ we do not distinguish between total and partial time derivative since we interpret them as functions of time with values in a Banach space, i.e., we write

$$\frac{d^k}{dt^k}y = \frac{\partial^k}{\partial t^k}y.$$

In case $k=1$ or $k=2$ we also write

$$\frac{d}{dt}y = y' = y_t \quad \text{and} \quad \frac{d^2}{dt^2}y = y'' = y_{tt}$$

for the weak time derivatives of y.

Note that for k-times continuously differentiable functions the above definition is nothing other than the integration by parts formula. Hence, a strongly differentiable function is also weakly differentiable. The definition of weak time derivatives enables us to define Sobolev spaces of vector-valued functions.

Definition 2.7
Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the space

$$W^{m,p}(0,T;X) := \left\{ y \in L^p(0,T;X) \;\middle|\; \frac{d^k}{dt^k}y \in L^p(0,T;X) \text{ for } k = 1,\dots,m \right\},$$

i.e., the space of all $y \in L^p(0,T;X)$ whose weak time derivatives are also in $L^p(0,T;X)$ up to order m. For $1 \leq p < \infty$ the space $W^{m,p}(0,T;X)$ equipped with the norm

$$||y||_{W^{m,p}(0,T;X)} := \left(\sum_{j=0}^{m} \left|\left| \frac{d^j}{dt^j}y \right|\right|^p_{L^p(0,T;X)} \right)^{\frac{1}{p}}$$

as well as the space $W^{m,\infty}(0,T;X)$ endowed with the norm

$$||y||_{W^{m,\infty}(0,T;X)} := \sum_{j=0}^{m} \left|\left| \frac{d^j}{dt^j}y \right|\right|_{L^\infty(0,T;X)}$$

are Banach spaces. Furthermore, let H be a Hilbert space. Then the space $W^{m,2}(0,T;H)$ equipped with the inner product

$$\langle y, v \rangle_{W^{m,2}(0,T;H)} := \sum_{j=0}^{m} \left\langle \frac{d^j}{dt^j}y, \frac{d^j}{dt^j}v \right\rangle_{L^2(0,T;H)}$$

is a Hilbert space. In this case, we also write $H^m(0,T;H) := W^{m,2}(0,T;H)$.

Note that this definition makes sense: If $y \in L^p(0,T;X) \subseteq L^1(0,T;X)$ is weakly differentiable, then its weak time derivative y' is an element of $L^1(0,T;X)$. For y to be in $W^{1,p}(0,T;X)$ we then additionally require that $y' \in L^p(0,T;X)$. The same argument can then be employed for higher order weak time derivatives. For functions of such a Sobolev space we have the following embedding available.

Theorem 2.8
Let $1 \leq p \leq \infty$. Then, the embedding $W^{1,p}(0,T;X) \hookrightarrow C([0,T];X)$ is continuous, i.e., it holds $W^{1,p}(0,T;X) \subseteq C([0,T];X)$ and there exists a constant $C > 0$ such that

$$||y||_{C([0,T];X)} \leq C\, ||y||_{W^{1,p}(0,T;X)} \qquad \text{for all } y \in W^{1,p}(0,T;X).$$

Proof: The proof can be found in [Eva10, Chapter 5.9, Theorem 2]. □

This theorem tells us that vector-valued functions in $W^{1,p}(0,T;X)$ are, after a suitable modification on a set of measure zero, continuous, in complete analogy to the continuity results for Sobolev functions in one spatial dimension. Moreover, we do not only have results for continuous, but also for compact embeddings available. One well-known result is the so-called Aubin-Lions lemma, which we state in the following.

Theorem 2.9 (Aubin-Lions Lemma)
Let X_0, X and X_1 be Banach spaces where X_0 and X_1 are reflexive. Suppose that X_0 is compactly embedded into X, which is in turn continuously embedded into X_1, i.e.,

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1.$$

Then, for any given p, q with $1 < p, q < \infty$ the space

$$W_0 = \{\, y \in L^p(0,T;X_0) \mid y' \in L^q(0,T;X_1) \,\}$$

equipped with the norm

$$||y||_{W_0} = ||y||_{L^p(0,T;X_0)} + ||y'||_{L^q(0,T;X_1)}$$

is compactly embedded into $L^p(0,T;X)$, i.e., we have

$$W_0 \hookrightarrow\hookrightarrow L^p(0,T;X).$$

Proof: The proof can be found, for example, in [Ruz06, Lemma 3.74]. □

For the theoretical analysis of this thesis the notion of a Gelfand triple is very useful.

Definition 2.10 (Gelfand Triple)
Let V, H be Hilbert spaces and let the embedding $V \hookrightarrow H$ be dense and continuous. Due to the Riesz representation theorem we identify the dual space H^* with H. From this we then get the continuous and dense embeddings

$$V \hookrightarrow H \cong H^* \hookrightarrow V^*,$$

which is called a Gelfand triple. The embedding $H \hookrightarrow V^*$ is given in the following way: We interpret some $y \in H$ as a functional acting on V through the mapping

$$y \in H \mapsto \langle \cdot, y \rangle_H \in H^* \subseteq V^*.$$

Example 2.11
One of the most prominent examples for a Gelfand triple in PDE theory is the one we get for $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$, namely

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega),$$

where each of the embeddings is dense and continuous (cf. [Tro10]).

2.3 Optimization with PDE constraints

In this chapter we give a brief introduction to optimal control problems with PDE constraints. We describe the problem and give useful theoretical results for the existence and uniqueness of an optimal control. Let us first define the optimal control problem mathematically.

Definition 2.12 (Optimal Control Problem)
Let U, Y, Z be Banach spaces, Z^* be the dual space of Z, $U_{\text{ad}} \subseteq U$ be a subset of U and $J\colon Y \times U \to \mathbb{R}$ as well as $e\colon Y \times U \to Z^*$. An optimal control problem is given by

$$\min \; J(y,u)$$

subject to

$$e(y,u) = 0 \quad \text{ in } Z^*$$

and

$$u \in U_{\text{ad}}.$$

Here, J is called cost functional, y is called state (variable) and u is called control (variable). Furthermore, the spaces Y and U are called state and control space, respectively, and the subset U_{ad} is called set of admissible controls. The equation $e(y,u) = 0$ is called state equation and is, in this thesis, a weak formulation of a PDE.

In order to deal with problems of this type, we always assume that the state equation admits a unique solution y for every choice of $u \in U$. This allows us to introduce the solution operator

$$G\colon U \to Y; \quad u \mapsto G(u) = y,$$

that maps the control $u \in U$ to the unique solution of the state equation $y \in Y$, i.e., we have

$$e(G(u), u) = 0 \quad \text{ for all } u \in U.$$

With this available we introduce the so-called reduced cost functional.

Definition 2.13 (Reduced Cost Functional)
Let $J(y,u)$ be a cost functional with state equation $e(y,u) = 0$ and corresponding solution operator $G\colon U \to Y$. The reduced cost functional is given by

$$\hat{J}(u) := J(G(u), u).$$

With this definition we have eliminated the PDE constraint and only have to consider a non-linear optimization problem in a Banach space, namely

$$\min_{u \in U_{\text{ad}}} \hat{J}(u). \tag{2.7}$$

Definition 2.14
A vector $u^* \in U_{\text{ad}}$ is called optimal control for problem (2.7) if we have $\hat{J}(u^*) \leq \hat{J}(u)$ for all $u \in U_{\text{ad}}$, i.e., if u^* is a global minimizer of $\hat{J}$ on U_{ad}. In this case $y^* := G(u^*)$ is called the corresponding optimal state.

Let us also define the notion of coercivity of a functional which turns out to be helpful in the proof of the existence of minimizers.

Definition 2.15
We say that a functional $F\colon U_{\text{ad}} \to \mathbb{R}$ is coercive if it holds that

$$||u||_U \to \infty \quad \Rightarrow \quad F(u) \to \infty.$$

We are now ready to give the following theorem concerning the existence of an optimal control for problem (2.7).

Theorem 2.16
Let $U_{\text{ad}} \subseteq U$ be a non-empty and weakly closed subset of a reflexive Banach space U and let $F\colon U_{\text{ad}} \to \mathbb{R}$ be weakly lower semicontinuous as well as bounded from below. Additionally, we require that either U_{ad} is bounded or that F is coercive. Then, F attains a global minimum on U_{ad}.

Proof: Let $(u_k) \subset U_{\text{ad}}$ be a minimizing sequence of F that exists since F is bounded from below, i.e.,

$$\lim_{k\to\infty} F(u_k) = \inf_{u\in U_{\text{ad}}} F(u).$$

We show that the sequence (u_k) is bounded: First, assume that the set of admissible controls U_{ad} is bounded. In this case we directly get the boundedness of (u_k). In the second case we assume that F is coercive. Since (u_k) is a minimizing sequence we know that $F(u_k)$ is bounded in $\mathbb{R}$ and, therefore, the coercivity of F directly gives the boundedness of the sequence (u_k).

Now, we observe that U_{ad} is weakly compact since it is a weakly closed subset of a reflexive Banach space. This gives us the existence of a subsequence $(u_{k_l}) \subset (u_k)$ and some $u^* \in U_{\text{ad}}$ such that

$$u_{k_l} \rightharpoonup u^* \text{ in } U,$$

since U_{ad} is weakly closed. We now use the weak lower semicontinuity of F to deduce

$$F(u^*) \leq \liminf_{l\to\infty} F(u_{k_l}) = \inf_{u\in U_{\text{ad}}} F(u),$$

which gives $F(u^*) \leq F(u)$ for all $u \in U_{\text{ad}}$ and implies that u^* is indeed a global minimizer of F. $\square$

We apply this theorem in Chapter 4 to deduce the existence of minimizers for the optimal control problem with the Cattaneo equation as constraint.

As in the finite-dimensional case, convex optimization problems have useful properties: We know that if the functional F is convex, then every local minimizer of F is also a global minimizer. If F is even strictly convex, we know that there exists at most one global minimizer, i.e., it is uniquely determined if it exists. These statements can be proved using exactly the same techniques as for the finite-dimensional case. Since we only consider strictly convex cost functionals the above statements always apply for our problems.

We conclude this chapter by stating the first order necessary conditions for a minimizer of our problem.

Theorem 2.17
Let $U_{\mathrm{ad}} \subseteq U$ be a convex subset of a Banach space U and let $\hat{J} \colon U \to \mathbb{R}$ be Fréchet differentiable in an open neighborhood around U_{ad}. If $u^* \in U_{\mathrm{ad}}$ is a solution of the minimization problem (2.7), then it is a stationary point of $\hat{J}$, i.e., it satisfies the variational inequality

$$\left\langle \hat{J}'(u^*), u - u^* \right\rangle_{U^*,U} \geq 0 \quad \text{for all } u \in U_{\mathrm{ad}}.$$

The converse holds true if $\hat{J}$ is additionally convex.

In case $U_{\mathrm{ad}} = U$ the variational inequality reduces to the well-known first order necessary condition

$$\hat{J}'(u^*) = 0.$$

Proof: The proof can be found, for example, in [HPUU08, Theorem 1.46]. □

3 Well-Posedness of the Cattaneo Equation

In this chapter we investigate the well-posedness of the Cattaneo equation. In particular, we show that there exists a unique weak solution that depends continuously on the data. For simplicity we restrict ourselves to homogeneous Dirichlet boundary conditions since other boundary conditions (such as Robin or Neumann boundary conditions) can be treated similarly. This chapter closely follows [Eva10, Chapter 7.2], where the well-posedness of the wave equation is proved. Note that we consider some fixed $\tau > 0$ in this chapter, whereas the limit $\tau \to 0$ is investigated in Chapter 5.

3.1 Weak Solution Theory

Before we begin with proving the well-posedness of the Cattaneo equation we first have to define a suitable formulation of the equation for which we seek solutions. We have to do this since the strong formulation of the equation is too strict for our purposes (cf. [Tro10]). Let us now motivate an appropriate weak formulation.

The Cattaneo equation with homogeneous Dirichlet boundary conditions is given by

$$\begin{aligned} \tau y_{tt} + y_t - \Delta y &= u && \text{in } (0,T) \times \Omega \\ y &= 0 && \text{on } [0,T] \times \partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega \\ y_t(0,\cdot) &= y_1 && \text{in } \Omega, \end{aligned} \tag{3.1}$$

for some $\tau > 0$. For the remainder of this chapter we assume that

- $u \in L^2(0,T;L^2(\Omega))$,
- $y_0 \in H_0^1(\Omega)$ and
- $y_1 \in L^2(\Omega)$

are given.

In order to motivate our notion of a weak solution, we assume that the solution of the Cattaneo equation is sufficiently smooth for all of the following computations. Let us multiply the equation for a fixed $t \in (0,T)$ with a test function $v \in H_0^1(\Omega)$ and integrate over Ω

$$\int_\Omega \tau y_{tt}\ v + y_t\ v - \Delta y\ v\ \mathrm{d}x = \int_\Omega u\ v\ \mathrm{d}x.$$

Integration by parts (for the spatial variables) then gives

$$\int\limits_\Omega \tau y_{tt}\ v + y_t\ v + \nabla y \cdot \nabla v\ \mathrm{d}x - \underbrace{\int_{\partial\Omega} n \cdot \nabla y\ v\ \mathrm{d}s}_{=\,0} = \int_\Omega u\ v\ \mathrm{d}x, \tag{3.2}$$

due to the fact that $v \in H_0^1(\Omega)$ vanishes on $\partial\Omega$.

From standard elliptic PDE theory we know that it is useful to introduce a bilinear form a that replaces the Laplacian operator in the weak formulation:

$$a\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}; \quad (y,v) \mapsto a[y,v] := \int_\Omega \nabla y \cdot \nabla v \, \mathrm{d}x = \langle \nabla y, \nabla v \rangle_{L^2(\Omega)} . \tag{3.3}$$

Recall that this bilinear form has the following properties:

- It is symmetric, i.e., $a[y,v] = a[v,y]$ for all $y, v \in H_0^1(\Omega)$.
- It is also uniformly coercive, i.e., there exists a constant $\alpha = \alpha(\Omega) > 0$, only depending on the domain Ω, such that
$$a[y,y] \geq \alpha \, ||y||^2_{H_0^1(\Omega)} \tag{3.4}$$
for all $y \in H_0^1(\Omega)$.
- Lastly, it is continuous, i.e., there exists a constant $C > 0$ such that
$$|a[y,v]| \leq C \, ||y||_{H_0^1(\Omega)} \, ||v||_{H_0^1(\Omega)} \tag{3.5}$$
for all $y, v \in H_0^1(\Omega)$.

With the help of the bilinear form a we can rewrite (3.2) as

$$\langle \tau y_{tt}, v \rangle_{L^2(\Omega)} + \langle y_t, v \rangle_{L^2(\Omega)} + a[y,v] = \langle u, v \rangle_{L^2(\Omega)} .$$

This motivates the following definitions for a weak solution of the Cattaneo equation.

Definition 3.1
We define the space $Y(0,T)$ by

$$Y(0,T) := \left\{ y \in L^2(0,T;H_0^1(\Omega)) \;\middle|\; y' \in L^2(0,T;L^2(\Omega)) \text{ and } y'' \in L^2(0,T;H^{-1}(\Omega)) \right\}$$

and equip this space with the inner product

$$\langle y, v \rangle_{Y(0,T)} := \langle y, v \rangle_{L^2(0,T;H_0^1(\Omega))} + \langle y', v' \rangle_{L^2(0,T;L^2(\Omega))} + \langle y'', v'' \rangle_{L^2(0,T;H^{-1}(\Omega))} .$$

Remark 3.2
It is easy to see that the space $Y(0,T)$ equipped with the above scalar product is, in fact, a Hilbert space. This can be shown analogously to the proof of [Emm04, Theorem 8.1.6]. Furthermore, we use this space in the following chapters as the state space for our optimization problem.

Definition 3.3
A function $y \in Y(0,T)$ is called a weak solution of the Cattaneo equation (3.1) if it satisfies the variational equation

$$\begin{aligned} &\tau \langle y''(t), v \rangle_{H^{-1},H^1} + \langle y'(t), v \rangle_{L^2(\Omega)} + a[y(t),v] = \langle u(t), v \rangle_{L^2(\Omega)} \\ &\text{for all } v \in H_0^1(\Omega) \text{ and almost every (a.e.) } t \in [0,T] \end{aligned}$$

and the initial conditions $y(0) = y_0$ and $y'(0) = y_1$.

Remark 3.4
If y is a weak solution of the Cattaneo equation, it holds that $y \in L^2(0,T;H_0^1(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$, $y' \in L^2(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;H^{-1}(\Omega))$ and $y'' \in L^2(0,T;H^{-1}(\Omega))$. Therefore, we have $y \in H^1(0,T;L^2(\Omega))$ as well as $y' \in H^1(0,T;H^{-1}(\Omega))$. Applying Theorem 2.8 yields that $y \in C([0,T];L^2(\Omega))$ and $y' \in C([0,T];H^{-1}(\Omega))$. This, however, ensures that the point evaluation of y and y' is well-defined and that it makes sense to talk about the initial conditions.

Furthermore, we can apply the same argument as in [HPUU08, Theorem 1.33] to find an equivalent description of a weak solution to the one given in Definition 3.3.

Lemma 3.5
A function $y \in Y(0,T)$ is a weak solution of the Cattaneo equation if and only if it satisfies the variational equation

$$\int_0^T \tau \left\langle y''(t), v(t) \right\rangle_{H^{-1},H^1} + \left\langle y'(t), v(t) \right\rangle_{L^2(\Omega)} + a[y(t), v(t)] \,\mathrm{d}t = \int_0^T \left\langle u(t), v(t) \right\rangle_{L^2(\Omega)} \,\mathrm{d}t$$

for all $v \in L^2(0,T;H_0^1(\Omega))$

and the initial conditions $y(0) = y_0$ and $y'(0) = y_1$.

3.2 Existence of a Weak Solution

Let us start proving the well-posedness of the Cattaneo equation by showing the existence of a weak solution. We do so by constructing an approximate solution with the Faedo-Galerkin-Method (cf. [Fae49]) and then show that this solution converges to a weak solution.

First, we choose a countable set of functions $w_k \in H_0^1(\Omega)$, $k \in \mathbb{N}_{>0}$, that are linearly independent such that the linear span of $\{ w_k \mid k \in \mathbb{N}_{>0} \}$ is dense in $H_0^1(\Omega)$. Such a set of functions exists because $H_0^1(\Omega)$ is separable (cf. [Alt16, Chapter 2]). Furthermore, the linear span of $\{ w_k \mid k \in \mathbb{N}_{>0} \}$ is also dense in $L^2(\Omega)$ since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$.

Let $m \in \mathbb{N}_{>0}$ and consider a function y_m of the form

$$y_m(t) = \sum_{k=1}^{m} \varphi_{mk}(t)\, w_k, \quad \varphi_{mk} \in H^2(0,T). \tag{3.6}$$

It is easy to see that $y_m \in Y(0,T)$ and that the weak time derivatives of y_m are given by

$$y_m'(t) = \sum_{k=1}^{m} \varphi_{mk}'(t)\, w_k \quad \text{and} \quad y_m''(t) = \sum_{k=1}^{m} \varphi_{mk}''(t)\, w_k,$$

where $\varphi_{mk}' \in H^1(0,T)$ and $\varphi_{mk}'' \in L^2(0,T)$ are the weak time derivatives of φ_{mk}. Thanks to Theorem 2.8, we immediately observe that it holds $y_m \in C([0,T];H_0^1(\Omega))$ as well as $y_m' \in C([0,T];H_0^1(\Omega))$.

We aim at showing that y_m is an approximate solution of the Cattaneo equation in the space spanned by the first m basis functions, i.e.,

$$\langle \tau y_m'', w_k\rangle_{L^2(\Omega)} + \langle y_m', w_k\rangle_{L^2(\Omega)} + a[y_m, w_k] = \langle u, w_k\rangle_{L^2(\Omega)} \tag{3.7a}$$
$$\varphi_{mk}(0) = \beta_{mk}, \quad \varphi_{mk}'(0) = \gamma_{mk} \tag{3.7b}$$

for $k = 1, \dots, m$ and a.e. $t \in [0, T]$, where the coefficients β_{mk} and γ_{mk} satisfy

$$y_m(0) = \sum_{k=1}^{m} \beta_{mk} w_k \to y_0 \text{ in } H_0^1(\Omega) \text{ as } m \to \infty \quad \text{and} \tag{3.8}$$
$$y_m'(0) = \sum_{k=1}^{m} \gamma_{mk} w_k \to y_1 \text{ in } L^2(\Omega) \text{ as } m \to \infty.$$

Note that, in fact, we choose $y_m(0)$ and $y_m'(0)$ to be the orthogonal projections of y_0 in $H_0^1(\Omega)$ and y_1 in $L^2(\Omega)$, respectively, into the subspace spanned by $w_1, \dots, w_m$. Therefore, we immediately have the estimates

$$||y_m(0)||_{H_0^1(\Omega)} \le ||y_0||_{H_0^1(\Omega)} \quad \text{and} \quad ||y_m'(0)||_{L^2(\Omega)} \le ||y_1||_{L^2(\Omega)}\,. \tag{3.9}$$

In the following lemma we prove the existence of such an approximate solution.

Lemma 3.6
For every $m \in \mathbb{N}_{>0}$ there exists a unique function y_m of the form (3.6) that satisfies (3.7).

Proof: Let y_m have the form given in (3.6). We rewrite (3.7a) as

$$\tau \sum_{j=1}^{m} \langle w_j, w_k\rangle_{L^2(\Omega)} \varphi_{mj}'' + \sum_{j=1}^{m} \langle w_j, w_k\rangle_{L^2(\Omega)} \varphi_{mj}' + \sum_{j=1}^{m} a[w_j, w_k]\varphi_{mj} = \langle u, w_k\rangle_{L^2(\Omega)} \tag{3.10}$$

for all $k = 1, \dots, m$ and a.e. $t \in [0, T]$. Introducing the matrices

$$(A)_{k,j} := \langle w_j, w_k\rangle_{L^2(\Omega)} \quad \text{and} \quad (M)_{k,j} := a[w_j, w_k]$$

as well as the vectors

$$\varphi_m(t) := [\varphi_{m1}(t), \dots, \varphi_{mm}(t)]^T \quad \text{and}$$
$$u_m(t) := [\langle u(t), w_1\rangle_{L^2(\Omega)}, \dots, \langle u(t), w_m\rangle_{L^2(\Omega)}]^T$$

allows us to rewrite (3.10) with the initial conditions (3.7b) as

$$\tau A\varphi_m'' + A\varphi_m' + M\varphi_m = u_m \tag{3.11}$$
$$\varphi_{mk}(0) = \beta_{mk}, \quad \varphi_{mk}'(0) = \gamma_{mk} \tag{3.7b}$$

for $k = 1, \dots, m$ and a.e. $t \in [0, T]$. Note that the (Gramian) matrix A is invertible since the functions w_k are linearly independent. Equations (3.11) and (3.7b), however, are a linear system of m ordinary differential equations (ODEs) for the coefficients φ_m. From ODE theory for measurable functions (cf. [O'R13, Chapter 3]) we get the existence and uniqueness of functions $\varphi_{mk} \in H^2(0, T)$, $k = 1, \dots, m$, that solve the ODE system (3.11) and (3.7b). □

Now that we have shown the existence and uniqueness of an approximate solution we would like to pass to the limit $m \to \infty$ and show that the sequence (y_m) converges to a weak solution. Before we can do so, we have to give some a-priori energy estimates for the approximate solution.

Theorem 3.7

There exists a constant $C = C(\tau) > 0$ which only depends on Ω, τ and T such that for the approximate solution y_m of (3.7) it holds that

$$\begin{aligned} &\max_{t\in[0,T]} \left(||y_m(t)||^2_{H^1_0(\Omega)} + ||y_m'(t)||^2_{L^2(\Omega)}\right) + ||y_m''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H^1_0(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right). \end{aligned} \tag{3.12}$$

In particular, this implies that we have $y_m \in L^\infty(0,T;H^1_0(\Omega))$, $y_m' \in L^\infty(0,T;L^2(\Omega))$ and $y_m'' \in L^2(0,T;H^{-1}(\Omega))$ independently of m.

Proof: Multiplying (3.7a) by φ_{mk}' and summing over $k = 1,\dots,m$ results in

$$\sum_{k=1}^m \varphi_{mk}'\left(\tau\,\langle y_m'', w_k\rangle_{L^2(\Omega)} + \langle y_m', w_k\rangle_{L^2(\Omega)} + a[y_m,w_k]\right) = \sum_{k=1}^m \varphi_{mk}'\,\langle u, w_k\rangle_{L^2(\Omega)}$$

for a.e. $t \in [0,T]$. Using the form of y_m (given in (3.6)) and the bilinearity of both the L^2 scalar product and a we arrive at

$$\tau\,\langle y_m'', y_m'\rangle_{L^2(\Omega)} + \langle y_m', y_m'\rangle_{L^2(\Omega)} + a[y_m,y_m'] = \langle u, y_m'\rangle_{L^2(\Omega)}\,, \tag{3.13}$$

which holds for a.e. $t \in [0,T]$. Note that the following holds for a.e. $t \in [0,T]$:

$$\begin{aligned} \langle y_m'', y_m'\rangle_{L^2(\Omega)} &= \frac{1}{2}\frac{d}{dt}\,||y_m'||^2_{L^2(\Omega)}\,, \\ \langle y_m', y_m'\rangle_{L^2(\Omega)} &= ||y_m'||^2_{L^2(\Omega)} \quad\text{and} \\ a[y_m,y_m'] &= \int_\Omega \nabla y_m\cdot\nabla y_m'\;\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\int_\Omega \nabla y_m\cdot\nabla y_m\;\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\;a[y_m,y_m]. \end{aligned}$$

With these reformulations, (3.13) reads

$$\frac{\tau}{2}\frac{d}{dt}\,||y_m'||^2_{L^2(\Omega)} + ||y_m'||^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\;a[y_m,y_m] = \langle u, y_m'\rangle_{L^2(\Omega)} \tag{3.14}$$

for a.e. $t \in [0,T]$. We use this to make the following estimate

$$\begin{aligned} &\frac{d}{dt}\left(||y_m'||^2_{L^2(\Omega)} + a[y_m,y_m]\right) && (3.15)\\ &\leq C(\tau)\frac{d}{dt}\left(\tau\,||y_m'||^2_{L^2(\Omega)} + a[y_m,y_m]\right) \\ &\leq C(\tau)\left(\tau\frac{d}{dt}\,||y_m'||^2_{L^2(\Omega)} + 2\,||y_m'||^2_{L^2(\Omega)} + \frac{d}{dt}\;a[y_m,y_m]\right) \\ &= C(\tau)\,\langle u, y_m'\rangle_{L^2(\Omega)} && \text{(due to (3.14))}\\ &\leq C(\tau)\;\;||u||_{L^2(\Omega)}\,||y_m'||_{L^2(\Omega)} && \text{(Cauchy-Schwarz inequality)}\\ &\leq C(\tau)\left(||u||^2_{L^2(\Omega)} + ||y_m'||^2_{L^2(\Omega)}\right) && \text{(Young's inequality)}\\ &\leq C(\tau)\left(||u||^2_{L^2(\Omega)} + ||y_m'||^2_{L^2(\Omega)} + a[y_m,y_m]\right) && \text{(due to (3.4))} \end{aligned}$$

for a.e. $t \in [0,T]$, since τ is positive. Now, we introduce

$$\eta(t) := ||y_m'(t)||^2_{L^2(\Omega)} + a[y_m(t), y_m(t)] \tag{3.16}$$

and

$$\xi(t) := ||u(t)||^2_{L^2(\Omega)} \, . \tag{3.17}$$

With this we can rewrite (3.15) as

$$\frac{d}{dt}\,\eta(t) \leq C(\tau)\,(\eta(t) + \ \xi(t))$$

for a.e. $t \in [0,T]$. Applying the Gronwall lemma (cf. [Eva10]) to the above yields

$$\begin{aligned} \eta(t) &\leq \ \exp\left(C(\tau)t\right)\left(\eta(0) + C(\tau)\ \int_0^t \xi(s)\ \mathrm{d}s\right) \\ &= C(\tau)\left(\eta(0) + \int_0^t \xi(s)\ \mathrm{d}s\right) \end{aligned} \tag{3.18}$$

for all $t \in [0,T]$. It holds that

$$\eta(0) = ||y_m'(0)||^2_{L^2(\Omega)} + a[y_m(0), y_m(0)] \leq C\left(||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right), \tag{3.19}$$

due to (3.9) and the continuity of the bilinear form a (cf. (3.5)). We use the coercivity of a and (3.18) to estimate:

$$\begin{aligned} &||y_m'(t)||^2_{L^2(\Omega)} + ||y_m(t)||^2_{H_0^1(\Omega)} \\ &\leq C\left(||y_m'(t)||^2_{L^2(\Omega)} + a[y_m(t), y_m(t)]\right) && \text{(due to (3.4))} \\ &= C\ \eta(t) && \text{(due to (3.16))} \\ &\leq C(\tau)\ \left(\eta(0) + \ \int_0^t \xi(s)\ \mathrm{d}s\right) && \text{(due to (3.18))} \\ &\leq C(\tau)\left(||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} + \int_0^t \xi(s)\ \mathrm{d}s\right) && \text{(due to (3.19))} \\ &= C(\tau)\left(||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} + \int_0^t ||u(s)||^2_{L^2(\Omega)}\ \ \mathrm{d}s\right) && \text{(due to (3.17))} \\ &\leq C(\tau)\left(||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} + ||u||^2_{L^2(0,T;L^2(\Omega))}\right) \end{aligned}$$

for all $t \in [0,T]$. Taking the maximum over all $t \in [0,T]$ we arrive at

$$\begin{aligned} &\max_{t\in[0,T]}\left(||y_m(t)||^2_{H_0^1(\Omega)} + ||y_m'(t)||^2_{L^2(\Omega)}\right) \\ &\leq C(\tau)\left(||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} + ||u||^2_{L^2(0,T;L^2(\Omega))}\right). \end{aligned} \tag{3.20}$$

Now, we choose some $v \in H_0^1(\Omega)$ with $||v||_{H_0^1(\Omega)} \leq 1$ and decompose it as $v = v_1 + v_2$ where $v_1 \in \operatorname{span}\{ w_k \mid k = 1, \dots, m \}$ and $\langle v_2, w_k \rangle_{L^2(\Omega)} = 0$ for all $k = 1, \dots, m$. It is obvious that doing so also yields $||v_1||_{H_0^1(\Omega)} \leq 1$. We have

$$\langle y_m'', v \rangle_{H^{-1},H^1} = \langle y_m'', v \rangle_{L^2(\Omega)} = \langle y_m'', v_1 \rangle_{L^2(\Omega)} = \frac{1}{\tau} \left(\langle u, v_1 \rangle_{L^2(\Omega)} - \langle y_m', v_1 \rangle_{L^2(\Omega)} - a[y_m, v_1] \right)$$

for a.e. $t \in [0, T]$. Therefore, we estimate

$$\begin{aligned}
&\left| \langle y_m'', v \rangle_{H^{-1},H^1} \right| \\
&= \frac{1}{\tau} \left| \langle u, v_1 \rangle_{L^2(\Omega)} - \langle y_m', v_1 \rangle_{L^2(\Omega)} - a[y_m, v_1] \right| \\
&\leq C(\tau) \left(\left| \langle u, v_1 \rangle_{L^2(\Omega)} \right| + \left| \langle y_m', v_1 \rangle_{L^2(\Omega)} \right| + |a[y_m, v_1]| \right) \\
&\leq C(\tau) \left(||u||_{L^2(\Omega)} ||v_1||_{L^2(\Omega)} + ||y_m'||_{L^2(\Omega)} ||v_1||_{L^2(\Omega)} + ||y_m||_{H_0^1(\Omega)} ||v_1||_{H_0^1(\Omega)} \right) \\
&\leq C(\tau) \left(||u||_{L^2(\Omega)} ||v_1||_{H_0^1(\Omega)} + ||y_m'||_{L^2(\Omega)} ||v_1||_{H_0^1(\Omega)} + ||y_m||_{H_0^1(\Omega)} ||v_1||_{H_0^1(\Omega)} \right) \\
&\leq C(\tau) \left(||u||_{L^2(\Omega)} + ||y_m'||_{L^2(\Omega)} + ||y_m||_{H_0^1(\Omega)} \right)
\end{aligned} \tag{3.21}$$

for a.e. $t \in [0, T]$, where the last inequality comes from the fact that $||v_1||_{H_0^1(\Omega)} \leq 1$. We use the definition of the operator norm to deduce

$$\begin{aligned}
||y_m''||_{H^{-1}(\Omega)} &= \sup_{||v||_{H_0^1(\Omega)}=1} \left| \langle y_m'', v \rangle_{H^{-1},H^1} \right| \\
&\leq C(\tau) \left(||u||_{L^2(\Omega)} + ||y_m'||_{L^2(\Omega)} + ||y_m||_{H_0^1(\Omega)} \right)
\end{aligned}$$

for a.e. $t \in [0, T]$ due to (3.21). Integrating this over $[0, T]$ yields:

$$\begin{aligned}
||y_m''||^2_{L^2(0,T;H^{-1}(\Omega))} &= \int_0^T ||y_m''(t)||^2_{H^{-1}(\Omega)} \, \mathrm{d}t \\
&\leq C(\tau) \left(\int_0^T ||u(t)||^2_{L^2(\Omega)} \, \mathrm{d}t + \int_0^T ||y_m'(t)||^2_{L^2(\Omega)} \, \mathrm{d}t + \int_0^T ||y_m(t)||^2_{H_0^1(\Omega)} \, \mathrm{d}t \right) \\
&\leq C(\tau) \left(||u||^2_{L^2(0,T,L^2(\Omega))} + T \max_{t \in [0,T]} ||y_m'(t)||^2_{L^2(\Omega)} + T \max_{t \in [0,T]} ||y_m(t)||^2_{H_0^1(\Omega)} \right) \\
&\leq C(\tau) \left(||u||^2_{L^2(0,T,L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right),
\end{aligned}$$

thanks to (3.20). Putting everything together, we end up with

$$\begin{aligned}
&\max_{t \in [0,T]} \left(||y_m(t)||^2_{H_0^1(\Omega)} + ||y_m'(t)||^2_{L^2(\Omega)} \right) + ||y_m''||^2_{L^2(0,T;H^{-1}(\Omega))} \\
&\leq C(\tau) \left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right).
\end{aligned}$$

□

With the estimates we derived above we are able to show the existence of a weak solution of the Cattaneo equation by investigating the limit of the approximate solutions, which is done in the following theorem.

Theorem 3.8
There exists a weak solution of the Cattaneo equation.

Proof: First note that from Theorem 3.7 it follows that

- (y_m) is bounded in $L^2(0,T;H_0^1(\Omega))$ independently of m,
- (y_m') is bounded in $L^2(0,T;L^2(\Omega))$ independently of m and
- (y_m'') is bounded in $L^2(0,T;H^{-1}(\Omega))$ independently of m,

due to the embedding $L^\infty(0,T;X) \hookrightarrow L^2(0,T;X)$ for a Banach space X (cf. the proof of Lemma 3.10). Since all of these spaces are reflexive Banach spaces (in fact, they are Hilbert spaces), we get the existence of weakly convergent subsequences (y_{m_l}), (y_{m_l}') as well as (y_{m_l}'') such that

$$\begin{aligned} y_{m_l} &\rightharpoonup y && \text{in } L^2(0,T;H_0^1(\Omega)), \\ y_{m_l}' &\rightharpoonup y' && \text{in } L^2(0,T;L^2(\Omega)) \text{ and} \\ y_{m_l}'' &\rightharpoonup y'' && \text{in } L^2(0,T;H^{-1}(\Omega)) \end{aligned} \tag{3.22}$$

as $l \to \infty$. It is easy to see that y' and y'' are indeed the weak time derivatives of y and, therefore, we have $y \in Y(0,T)$.

We show that y is, in fact, a weak solution of the Cattaneo equation. We fix some $N \in \mathbb{N}_{>0}$ and choose v of the form

$$v(t) = \sum_{k=1}^{N} \varphi_k(t)\ w_k, \quad \varphi_k \in C^\infty((0,T)). \tag{3.23}$$

Now we choose $m_l \geq N$ and use the fact that y_{m_l} is an approximate solution to get

$$\tau \left\langle y_{m_l}'', v \right\rangle_{H^{-1},H^1} + \left\langle y_{m_l}', v \right\rangle_{L^2(\Omega)} + a[y_{m_l}, v] = \langle u, v \rangle_{L^2(\Omega)}$$

for a.e. $t \in [0,T]$. Integrating this expression over $[0,T]$ gives

$$\int_0^T \tau \left\langle y_{m_l}'', v \right\rangle_{H^{-1},H^1} + \left\langle y_{m_l}', v \right\rangle_{L^2(\Omega)} + a[y_{m_l}, v]\ \mathrm{d}t = \int_0^T \langle u, v \rangle_{L^2(\Omega)}\ \mathrm{d}t. \tag{3.24}$$

Let us now use the weak convergence and pass to the limit $l \to \infty$. We directly see that this gives

$$\int_0^T \tau \langle y'', v \rangle_{H^{-1},H^1} + \langle y', v \rangle_{L^2(\Omega)} + a[y, v]\ \mathrm{d}t = \int_0^T \langle u, v \rangle_{L^2(\Omega)}\ \mathrm{d}t. \tag{3.25}$$

In fact, this even holds for all $v \in L^2(0,T;H_0^1(\Omega))$ since functions of the form (3.23) are dense in this space (cf. Lemma 2.5). To conclude that y solves the Cattaneo equation we only have to show that the initial conditions are satisfied. As discussed in Remark 3.4

we see that we have $y \in C([0,T];L^2(\Omega))$ and $y' \in C([0,T];H^{-1}(\Omega))$ which allows us to investigate the initial conditions.

In order to do so, we choose $v \in C^2([0,T];H_0^1(\Omega))$ with $v(T) = v'(T) = 0$. Integration by parts in (3.25) reveals

$$\begin{aligned}&\int_0^T \tau \langle y, v'' \rangle_{L^2(\Omega)} + \langle y', v \rangle_{L^2(\Omega)} + a[y,v] \; \mathrm{d}t \\ &= \int_0^T \langle u, v \rangle_{L^2(\Omega)} \; \mathrm{d}t - \tau \langle y(0), v'(0) \rangle_{L^2(\Omega)} + \tau \langle y'(0), v(0) \rangle_{H^{-1},H^1} .\end{aligned} \tag{3.26}$$

We analogously apply integration by parts to (3.24) and obtain

$$\begin{aligned}&\int_0^T \tau \langle y_{m_l}, v'' \rangle_{L^2(\Omega)} + \left\langle y'_{m_l}, v \right\rangle_{L^2(\Omega)} + a[y_{m_l},v] \; \mathrm{d}t \\ &= \int_0^T \langle u, v \rangle_{L^2(\Omega)} \; \mathrm{d}t - \tau \langle y_{m_l}(0), v'(0) \rangle_{L^2(\Omega)} + \tau \left\langle y'_{m_l}(0), v(0) \right\rangle_{L^2(\Omega)} .\end{aligned} \tag{3.27}$$

Recall that y_{m_l} and its derivatives are weakly convergent (cf. (3.22)) and that we also have

$$\begin{aligned}&\lim_{l\to\infty} y_{m_l}(0) = y_0 \text{ in } H_0^1(\Omega) \text{ and} \\ &\lim_{l\to\infty} y'_{m_l}(0) = y_1 \text{ in } L^2(\Omega),\end{aligned}$$

due to (3.8). Passing to the limit $l \to \infty$ in (3.27) yields:

$$\begin{aligned}&\int_0^T \tau \langle y, v'' \rangle_{L^2(\Omega)} + \langle y', v \rangle_{L^2(\Omega)} + a[y,v] \; \mathrm{d}t \\ &= \int_0^T \langle u, v \rangle_{L^2(\Omega)} \; \mathrm{d}t - \tau \langle y_0, v'(0) \rangle_{L^2(\Omega)} + \tau \langle y_1, v(0) \rangle_{L^2(\Omega)} .\end{aligned} \tag{3.28}$$

As in [Eva10], comparing equations (3.26) and (3.28) directly reveals

$$y(0) = y_0 \quad \text{as well as} \quad y'(0) = y_1$$

since $v(0)$ and $v'(0)$ are arbitrary. This together with (3.25) yields that y is indeed a weak solution of the Cattaneo equation, concluding the proof. □

Remark 3.9
In Theorem 3.11 we show that the weak solution of the Cattaneo equation is unique. Therefore, we see that the computations done in the previous theorem also hold true for any subsequence of (y_m) and thanks to the uniqueness of the limit we get the weak convergence of the whole sequence (y_m) to the weak solution of the Cattaneo equation.

3.3 Continuous Dependence on the Data

In order to prove the uniqueness of a weak solution, we use the a-priori estimates to derive an energy estimate for a weak solution. This also gives the continuous dependence on the data we need for the well-posedness of the Cattaneo equation.

Lemma 3.10
A weak solution of the Cattaneo equation depends continuously on the data, i.e., there exists a constant $C = C(\tau) > 0$ such that

$$\begin{aligned} ||y||^2_{Y(0,T)} = {} & ||y||^2_{L^2(0,T;H_0^1(\Omega))} + ||y'||^2_{L^2(0,T;L^2(\Omega))} + ||y''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ & \leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right). \end{aligned} \tag{3.29}$$

Proof: Recall the proof of Theorem 3.7 where we had

$$\begin{aligned} & \max_{t\in[0,T]}\left(||y_m(t)||^2_{H_0^1(\Omega)} + ||y_m'(t)||^2_{L^2(\Omega)}\right) + ||y_m''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ & \leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right), \end{aligned} \tag{3.12}$$

with the constant $C(\tau)$ being independent of m. In Theorem 3.8 we have shown that this inequality allows us to extract a weakly convergent subsequence (y_{m_l}) such that

$$\begin{aligned} y_{m_l} & \rightharpoonup y && \text{in } L^2(0,T;H_0^1(\Omega)), \\ y'_{m_l} & \rightharpoonup y' && \text{in } L^2(0,T;L^2(\Omega)) \text{ and} \\ y''_{m_l} & \rightharpoonup y'' && \text{in } L^2(0,T;H^{-1}(\Omega)) \end{aligned} \tag{3.22}$$

as $l \to \infty$, where y is a weak solution of the Cattaneo equation. Equation (3.12) also holds true for this subsequence, and since $C(\tau)$ is independent of m it is also valid in the limit $l \to \infty$. Therefore, we get

$$\begin{aligned} & \sup_{t\in[0,T]}\left(||y(t)||^2_{H_0^1(\Omega)} + ||y'(t)||^2_{L^2(\Omega)}\right) + ||y''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ & \leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right). \end{aligned} \tag{3.30}$$

Now, we observe that we have

$$||y||^2_{L^2(0,T;H_0^1(\Omega))} = \int_0^T ||y(t)||^2_{H_0^1(\Omega)} \; \mathrm{d}t \leq T \sup_{t\in[0,T]} ||y(t)||^2_{H_0^1(\Omega)}$$

as well as

$$||y'||^2_{L^2(0,T;L^2(\Omega))} = \int_0^T ||y'(t)||^2_{L^2(\Omega)} \; \mathrm{d}t \leq T \sup_{t\in[0,T]} ||y'(t)||^2_{L^2(\Omega)}.$$

Using this together with inequality (3.30) then reveals

$$\begin{aligned} ||y||^2_{Y(0,T)} = {} & ||y||^2_{L^2(0,T;H_0^1(\Omega))} + ||y'||^2_{L^2(0,T;L^2(\Omega))} + ||y''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ & \leq C \sup_{t\in[0,T]}\left(||y(t)||^2_{H_0^1(\Omega)} + ||y'(t)||^2_{L^2(\Omega)}\right) + ||y''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ & \leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right), \end{aligned}$$

completing the proof. □

3.4 Uniqueness of the Weak Solution

Now that we have shown both the existence of a weak solution as well as the continuous dependence on the data, we investigate the uniqueness of a weak solution. This, however, immediately follows from the continuous dependence on the data since the Cattaneo equation is linear as is shown in the following.

Theorem 3.11
A weak solution of the Cattaneo equation is unique.

Proof: Thanks to the linearity of the Cattaneo equation we have the superposition principle available. Therefore, all we have to show is that the Cattaneo equation for $u = 0 = y_0 = y_1$ has only $y = 0$ as a weak solution. In this case it holds that $||u||_{L^2(0,T;L^2(\Omega))} = ||y_0||_{H_0^1(\Omega)} = ||y_1||_{L^2(\Omega)} = 0$. Using Lemma 3.10 we obtain:

$$\begin{aligned} ||y||^2_{Y(0,T)} = &\ ||y||^2_{L^2(0,T;H_0^1(\Omega))} + ||y'||^2_{L^2(0,T;L^2(\Omega))} + ||y''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right) \\ &= 0, \end{aligned}$$

which immediately gives $y = 0$ and, hence, finishes the proof. □

Therefore, we have proved the well-posedness of the Cattaneo equation. Furthermore, we can employ the same techniques as in [Tem97, Chapter 2.4] to observe that we even have $y \in C([0,T]; H_0^1(\Omega))$ as well as $y' \in C([0,T]; L^2(\Omega))$.

Let us summarize the results of this chapter in the following theorem.

Theorem 3.12
The Cattaneo equation (3.1) is well-posed in the following sense: Let $u \in L^2(0,T;L^2(\Omega))$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. Then, the following statements hold:

- There exists a weak solution $y \in Y(0,T)$ of the Cattaneo equation.
- The weak solution of the Cattaneo equation is unique.
- The weak solution of the Cattaneo equation depends continuously on the data, i.e., there exists a constant $C = C(\tau) > 0$ such that

$$\begin{aligned} ||y||^2_{Y(0,T)} = &\ ||y||^2_{L^2(0,T;H_0^1(\Omega))} + ||y'||^2_{L^2(0,T;L^2(\Omega))} + ||y''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right). \end{aligned} \tag{3.29}$$

Proof: This has been proved throughout Chapter 3. □

3.5 The Solution Operator of the Cattaneo Equation

We finish this chapter by taking a look at the solution operator of the Cattaneo equation which we need for the optimal control problems in the following chapter.

The solution operator G_c of the Cattaneo equation is defined as

$$\begin{aligned} &G_c\colon L^2(0,T;L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega) \to Y(0,T) \\ &(u, y_0, y_1) \mapsto G_c(u, y_0, y_1) = y, \end{aligned}$$

where $y \in Y(0,T)$ is the unique weak solution of the Cattaneo equation. The continuous dependence on the data (cf. (3.29)) immediately reveals that G_c is bounded since

$$\begin{aligned} ||G_c(u, y_0, y_1)||^2_{Y(0,T)} = &\; ||y||^2_{L^2(0,T;H_0^1(\Omega))} + ||y'||^2_{L^2(0,T;L^2(\Omega))} + ||y''||^2_{L^2(0,T;H^{-1}(\Omega))} \\ &\leq C(\tau)\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right). \end{aligned} \tag{3.31}$$

Furthermore, we know that the Cattaneo equation is linear and, therefore, the superposition principle is valid. Hence, we split the solution operator into two terms: First, the solution operator $G_u\colon L^2(0,T;L^2(\Omega)) \to Y(0,T)$ that maps the right-hand side u to the unique weak solution of the Cattaneo equation with zero initial conditions, i.e., $y_0 = y_1 = 0$. We denote the second part of the solution operator by $G_0\colon H_0^1(\Omega) \times L^2(\Omega) \to Y(0,T)$, and G_0 maps the initial conditions y_0 and y_1 to the unique weak solution of the Cattaneo equation with zero right-hand side, i.e., $u = 0$. Thanks to the superposition principle it holds

$$G_c(u, y_0, y_1) = G_u(u) + G_0(y_0, y_1).$$

Due to (3.31) and the linearity of G_c it is obvious that we have

$$G_u \in \mathscr{L}(L^2(0,T;L^2(\Omega)), Y(0,T)).$$

We use this decomposition of the solution operator in the next chapter in order to investigate the optimal control problem.

4 Optimal Control of the Cattaneo Equation

Now that we have proved the well-posedness of the Cattaneo equation in the previous chapter, let us take a look at the corresponding optimal control problem. We prove the existence and uniqueness of a minimizer and derive the adjoint equation as well as the first order optimality conditions for this problem. In order to do so, we use methods stated in [HPUU08] and [Tro10].

Let us begin with stating the optimal control problem we investigate. This consists of a tracking type cost functional with the Cattaneo equation as a constraint:

$$\min \; J(y,u) = \frac{1}{2} \left|\left|E(y) - y_d\right|\right|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \left|\left|u\right|\right|^2_{L^2(0,T;L^2(\Omega))} \tag{4.1}$$

subject to

$$\begin{aligned} \tau y_{tt} + y_t - \Delta y &= u && \text{in } (0,T) \times \Omega \\ y &= 0 && \text{on } [0,T] \times \partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega \\ y_t(0,\cdot) &= y_1 && \text{in } \Omega \end{aligned}$$

and

$$u \in U_{\text{ad}},$$

where $\lambda \geq 0$ is a parameter, $y_d \in L^2(0,T;L^2(\Omega))$ is the desired state and $y_0 \in H^1_0(\Omega)$ and $y_1 \in L^2(\Omega)$ are given initial conditions, as before. As mentioned earlier, we do not consider the Cattaneo equation in the strong, but in the weak form since this is less restrictive for the optimal control problem. Thanks to our results from Chapter 3 we know that it is reasonable to consider the space

$$Y(0,T) = \left\{ \, y \in L^2(0,T;H^1_0(\Omega)) \,\middle|\, y' \in L^2(0,T;L^2(\Omega)) \text{ and } y'' \in L^2(0,T;H^{-1}(\Omega)) \, \right\}$$

as the state space, as well as the control space $U := L^2(0,T;L^2(\Omega))$ for the optimization problem. As usual, we demand that the set of admissible controls $U_{\text{ad}} \subseteq U$ is a weakly closed and convex subset of the control space U which we also assume to be bounded in case $\lambda = 0$. Therefore, we search for $y \in Y(0,T)$ and $u \in U_{\text{ad}} \subseteq U$. Finally, we note that we have the continuous embedding $Y(0,T) \hookrightarrow L^2(0,T;L^2(\Omega))$ which we denote with E, i.e., E maps a function $y \in Y(0,T)$ to the same function $y \in L^2(0,T;L^2(\Omega))$. This we need in order to derive the adjoint equation rigorously later on.

4.1 Existence and Uniqueness of an Optimal Control

In order to investigate the question of existence and uniqueness of an optimal control for problem (4.1) we introduce the reduced cost functional (cf. Chapter 2.3). Recall that for this we need the solution operator G_c introduced in Chapter 3.5. Therefore, the reduced cost functional is given by

$$\hat{J}(u) = J(G_c(u), u) = \frac{1}{2} \|E(G_c(u, y_0, y_1)) - y_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;L^2(\Omega))} .$$

Let us define the operator $S_c\colon L^2(0,T;L^2(\Omega)) \times H^1_0(\Omega) \times L^2(\Omega) \to L^2(0,T;L^2(\Omega))$ by $S_c := E \circ G_c$. Hence, the reduced cost functional reads

$$\hat{J}(u) = \frac{1}{2} \|S_c(u, y_0, y_1) - y_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;L^2(\Omega))} .$$

Now, we split the operator S_c in the same way we decomposed the solution operator G_c in Chapter 3.5: We define the operators $S_u\colon L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ and $S_0\colon H^1_0(\Omega) \times L^2(\Omega) \to L^2(0,T;L^2(\Omega))$ by $S_u := E \circ G_u$ and $S_0 := E \circ G_0$. Thanks to the linearity of the embedding E it holds

$$\begin{aligned} S_c(u, y_0, y_1) = E(G_c(u, y_0, y_1)) &= E(G_u(u) + G_0(y_0, y_1)) \\ &= E(G_u(u)) + E(G_0(y_0, y_1)) = S_u(u) + S_0(y_0, y_1). \end{aligned}$$

It is obvious that $S_u \in \mathscr{L}(L^2(0,T;L^2(\Omega)), L^2(0,T;L^2(\Omega)))$. We use this decomposition to rewrite the reduced cost functional as follows

$$\begin{aligned} \hat{J}(u) &= \frac{1}{2} \|S_c(u, y_0, y_1) - y_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;L^2(\Omega))} \\ &= \frac{1}{2} \|S_u(u) + S_0(y_0, y_1) - y_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;L^2(\Omega))} \\ &= \frac{1}{2} \|S_u(u) - q\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;L^2(\Omega))} , \end{aligned} \tag{4.2}$$

where we define the function $q \in L^2(0,T;L^2(\Omega))$ as

$$q := y_d - S_0(y_0, y_1). \tag{4.3}$$

Due to (4.2), the optimal control problem (4.1) is equivalent to the reduced problem

$$\min_{u \in U_{\text{ad}}} \hat{J}(u) = \frac{1}{2} \|S_u(u) - q\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;L^2(\Omega))} , \tag{4.4}$$

with q given in (4.3). We show the existence and uniqueness of a minimizer for this problem in the following theorem.

Lemma 4.1
Let H and U be Hilbert spaces, $q \in H$ and $\lambda \geq 0$. Additionally, let $S \in \mathscr{L}(U, H)$ and $U_{\text{ad}} \subseteq U$ be a non-empty, weakly closed and convex subset of U which is bounded in case $\lambda = 0$. Then, the problem

$$\min_{u \in U_{\text{ad}}} f(u) := \frac{1}{2} \|S(u) - q\|^2_H + \frac{\lambda}{2} \|u\|^2_U$$

has a minimizer $u^* \in U_{\text{ad}}$. Furthermore, this minimizer is uniquely determined if $\lambda > 0$ or S is injective.

Proof: We check whether the conditions of Theorem 2.16 are satisfied for f. First notice that f is bounded from below since $f \geq 0$. Let us now investigate the weak lower semicontinuity of f: Since f is a composition of norms and a continuous operator S, it is both continuous and convex. Therefore, we know that f is weakly lower semicontinuous. Additionally we know that the space U is a reflexive Banach space since it is, in fact, a Hilbert space. In case $\lambda = 0$ our assumption immediately tells us that U_{ad} is bounded. For $\lambda > 0$ we observe that f is, in fact, coercive, and therefore we apply Theorem 2.16 in order to deduce the existence of a global minimizer $u^* \in U_{\text{ad}}$.

For the uniqueness of the minimizer we note that it is easy to see that f is strictly convex if $\lambda > 0$ or if S is injective. □

Therefore, we get both the existence and uniqueness of an optimal for problem (4.4).

Theorem 4.2
Let U_{ad} be a non-empty, convex and weakly closed subset of $L^2(0,T;L^2(\Omega))$ which is bounded in case $\lambda = 0$. Then, problem (4.1) has a unique optimal control $u^* \in U_{\text{ad}}$.

Proof: We apply Lemma 4.1 with $U = H = L^2(0,T;L^2(\Omega))$, $S = S_u$ and $q = y_d - S_0(y_0, y_1)$. Note that the operator S_u is injective since both the embedding E and the solution operator G_c are injective, where the latter is injective thanks to the unique solvability of the Cattaneo equation. □

4.2 First Order Optimality Conditions

In order to state the first order optimality conditions and to solve problem (4.1) numerically we need the following theorems where we derive the adjoint solution operator of the Cattaneo equation.

Lemma 4.3
Let $z \in L^2(0,T;L^2(\Omega))$. Then, there exists a unique weak solution $p \in Y(0,T)$ of the backward Cattaneo equation

$$\begin{aligned} \tau p_{tt} - p_t - \Delta p &= z && \text{in } (0,T)\times\Omega \\ p &= 0 && \text{on } [0,T]\times\partial\Omega \\ p(T,\cdot) &= 0 && \text{in } \Omega \\ p_t(T,\cdot) &= 0 && \text{in } \Omega, \end{aligned} \tag{4.5}$$

that satisfies the variational formulation

$$\int_0^T \tau \langle p''(t), v(t)\rangle_{H^{-1},H^1} - \langle p'(t), v(t)\rangle_{L^2(\Omega)} + a[p(t), v(t)] \,\mathrm{d}t = \int_0^T \langle z(t), v(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t \tag{4.6}$$

for all $v \in L^2(0,T;H_0^1(\Omega))$,

where a is defined in (3.3), and the terminal conditions $p(T) = 0$ as well as $p'(T) = 0$. Furthermore, there exists a constant $C = C(\tau) > 0$ such that p satisfies

$$||p||^2_{Y(0,T)} = ||p||^2_{L^2(0,T;H_0^1(\Omega))} + ||p'||^2_{L^2(0,T;L^2(\Omega))} + ||p''||^2_{L^2(0,T;H^{-1}(\Omega))} \leq C(\tau)\,||z||^2_{L^2(0,T;L^2(\Omega))}.$$

Proof: We introduce the time transformation $\theta = T - t$ and write $\bar{p}(\theta) := p(T-t)$, $\bar{v}(\theta) := v(T-t)$ and $\bar{z}(\theta) := z(T-t)$. The chain rule of calculus reveals

$$\frac{\partial \bar{p}}{\partial t} = \frac{\partial \bar{p}}{\partial \theta}\frac{\partial \theta}{\partial t} = (-1)\frac{\partial \bar{p}}{\partial \theta},$$

and this immediately gives

$$\frac{\partial^n \bar{p}}{\partial t^n} = (-1)^n \frac{\partial^n \bar{p}}{\partial t^n}.$$

Hence, we see that it holds $\bar{p}(0) = p(T)$ as well as $\bar{p}'(0) = -p'(0)$. For the weak formulation in (4.6) we get with this transformation

$$\int_0^T \tau \left\langle \bar{p}''(t), \bar{v}(t)\right\rangle_{H^{-1},H^1} + \left\langle \bar{p}'(t), \bar{v}(t)\right\rangle_{L^2(\Omega)} + a[\bar{p}(t), \bar{v}(t)] \, \mathrm{d}t = \int_0^T \left\langle \bar{z}(t), \bar{v}(t)\right\rangle_{L^2(\Omega)} \, \mathrm{d}t$$

for all $v \in L^2(0,T;H_0^1(\Omega))$, with initial conditions given by $\bar{p}(0) = 0$ and $\bar{p}'(0) = 0$. Note that this is the weak formulation of the (forward) Cattaneo equation

$$\begin{aligned} \tau \bar{p}_{tt} + \bar{p}_t - \Delta \bar{p} &= \bar{z} && \text{in } (0,T)\times\Omega \\ \bar{p} &= 0 && \text{on } [0,T]\times\partial\Omega \\ \bar{p}(0,\cdot) &= 0 && \text{in } \Omega \\ \bar{p}_t(0,\cdot) &= 0 && \text{in } \Omega. \end{aligned}$$

We apply Theorem 3.12 to this in order to obtain the existence and uniqueness of a weak solution $\bar{p} \in Y(0,T)$ with

$$||\bar{p}||^2_{L^2(0,T;H_0^1(\Omega))} + ||\bar{p}'||^2_{L^2(0,T;L^2(\Omega))} + ||\bar{p}''||^2_{L^2(0,T;H^{-1}(\Omega))} \leq C(\tau)\, ||\bar{z}||^2_{L^2(0,T;L^2(\Omega))}.$$

A backward transformation then completes the proof. □

The following theorem may, at a first glance, not seem very helpful, but we use it in order to compute the adjoint solution operator later on.

Theorem 4.4
Let $y \in Y(0,T)$ be the unique weak solution of the Cattaneo equation

$$\begin{aligned} \tau y_{tt} + y_t - \Delta y &= u && \text{in } (0,T)\times\Omega \\ y &= 0 && \text{on } [0,T]\times\partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega \\ y_t(0,\cdot) &= y_1 && \text{in } \Omega, \end{aligned} \tag{4.7}$$

where $u \in L^2(0,T;L^2(\Omega))$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. Furthermore, let $p \in Y(0,T)$ be the unique weak solution of the backward Cattaneo equation (4.5) with right-hand side $z \in L^2(0,T;L^2(\Omega))$. Then, it holds that

$$\int_0^T \left\langle z, y\right\rangle_{L^2(\Omega)} \, \mathrm{d}t + \tau \left\langle y_0, p'(0)\right\rangle_{L^2(\Omega)} = \int_0^T \left\langle u, p\right\rangle_{L^2(\Omega)} \, \mathrm{d}t + \tau \left\langle y_1, p(0)\right\rangle_{L^2(\Omega)} + \left\langle y_0, p(0)\right\rangle_{L^2(\Omega)}.$$

Proof: Since y is the weak solution of (4.7) it holds that $y(0) = y_0$ and $y'(0) = y_1$ as well as

$$\int_0^T \tau \langle y'', v\rangle_{H^{-1},H^1} + \langle y', v\rangle_{L^2(\Omega)} + a[y, v] \,\mathrm{d}t = \int_0^T \langle u, v\rangle_{L^2(\Omega)} \,\mathrm{d}t \tag{4.8}$$

for all $v \in L^2(0, T; H^1_0(\Omega))$. Thanks to p being the weak solution of (4.5) it holds that $p(T) = 0$ and $p'(T) = 0$ in addition to

$$\int_0^T \tau \langle p'', w\rangle_{H^{-1},H^1} - \langle p', w\rangle_{L^2(\Omega)} + a[p, w] \,\mathrm{d}t = \int_0^T \langle z, w\rangle_{L^2(\Omega)} \,\mathrm{d}t \tag{4.9}$$

for all $w \in L^2(0, T; H^1_0(\Omega))$.

Choosing $v = p$ in (4.8) gives

$$\int_0^T \tau \langle y'', p\rangle_{H^{-1},H^1} + \langle y', p\rangle_{L^2(\Omega)} + a[y, p] \,\mathrm{d}t = \int_0^T \langle u, p\rangle_{L^2(\Omega)} \,\mathrm{d}t, \tag{4.10}$$

and choosing $w = y$ in (4.9) gives

$$\int_0^T \tau \langle p'', y\rangle_{H^{-1},H^1} - \langle p', y\rangle_{L^2(\Omega)} + a[p, y] \,\mathrm{d}t = \int_0^T \langle z, y\rangle_{L^2(\Omega)} \,\mathrm{d}t. \tag{4.11}$$

We apply integration by parts to the first two terms in (4.10) and see

$$\begin{aligned} &\int_0^T -\tau \langle y', p'\rangle_{L^2(\Omega)} - \langle y, p'\rangle_{L^2(\Omega)} + a[y, p] \,\mathrm{d}t \\ &= \int_0^T \langle u, p\rangle_{L^2(\Omega)} \,\mathrm{d}t + \tau \langle y_1, p(0)\rangle_{L^2(\Omega)} + \langle y_0, p(0)\rangle_{L^2(\Omega)}, \end{aligned} \tag{4.12}$$

due to the initial and terminal conditions of y and p, respectively. Similarly, we apply integration by parts to the first term in (4.11) and observe

$$\begin{aligned} &\int_0^T -\tau \langle p', y'\rangle_{L^2(\Omega)} - \langle p', y\rangle_{L^2(\Omega)} + a[p, y] \,\mathrm{d}t \\ &= \int_0^T \langle z, y\rangle_{L^2(\Omega)} \,\mathrm{d}t + \tau \langle p'(0), y_0\rangle_{L^2(\Omega)}, \end{aligned} \tag{4.13}$$

again due to the given initial and terminal conditions. Note that the left-hand side of (4.12) is the same as the left-hand side of (4.13), which implies that the right-hand sides coincide, too, which reveals

$$\int_0^T \langle z, y\rangle_{L^2(\Omega)} \,\mathrm{d}t + \tau \langle p'(0), y_0\rangle_{L^2(\Omega)} = \int_0^T \langle u, p\rangle_{L^2(\Omega)} \,\mathrm{d}t + \tau \langle y_1, p(0)\rangle_{L^2(\Omega)} + \langle y_0, p(0)\rangle_{L^2(\Omega)},$$

and finishes the proof. □

As in Chapter 3.5 we also introduce the solution operator $Q\colon L^2(0,T;L^2(\Omega)) \to Y(0,T)$ that maps the right-hand side $z \in L^2(0,T;L^2(\Omega))$ to the unique weak solution $p \in Y(0,T)$ of the backward Cattaneo equation with terminal conditions $p(T) = p'(T) = 0$, i.e., the weak solution of (4.5).

Analogously, we also define the operator $R\colon L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ with the help of the continuous embedding $Y(0,T) \hookrightarrow L^2(0,T;L^2(\Omega))$, again denoted by E, as $R = E \circ Q$ and observe that $R \in \mathscr{L}(L^2(0,T;L^2(\Omega)), L^2(0,T;L^2(\Omega)))$ thanks to the estimate in Lemma 4.3. We now prove that R is actually the adjoint operator of S_u.

Theorem 4.5
The adjoint solution operator $S_u^* \in \mathscr{L}(L^2(0,T;L^2(\Omega)), L^2(0,T;L^2(\Omega)))$ is given by R.

Proof: We choose $u, z \in L^2(0,T;L^2(\Omega))$ and define $y := S_u(u)$ as well as $p := R(z)$. By the definition of S_u and R, respectively, y is the unique weak solution of the Cattaneo equation with $y_0 = y_1 = 0$ and p is the unique weak solution of the backward Cattaneo equation (4.5). Applying Theorem 4.4 with these particular initial conditions for y reveals

$$\langle z, y\rangle_{L^2(0,T;L^2(\Omega))} = \int_0^T \langle z, y\rangle_{L^2(\Omega)} \, \mathrm{d}t = \int_0^T \langle u, p\rangle_{L^2(\Omega)} \, \mathrm{d}t = \langle u, p\rangle_{L^2(0,T;L^2(\Omega))},$$

since all initial and terminal conditions are zero. We use this to observe

$$\langle z, S_u(u)\rangle_{L^2(0,T;L^2(\Omega))} = \langle z, y\rangle_{L^2(0,T;L^2(\Omega))} = \langle p, u\rangle_{L^2(0,T;L^2(\Omega))} = \langle R(z), u\rangle_{L^2(0,T;L^2(\Omega))}.$$

Since this holds true for all $u, z \in L^2(0,T;L^2(\Omega))$, we conclude that we have $R = S_u^*$. □

We use this information about the adjoint solution operator in order to derive the first order optimality conditions. Recall that our optimization problem is given by

$$\min_{u \in U_{\mathrm{ad}}} \hat{J}(u) = \frac{1}{2} \|S_u(u) - q\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \|u\|^2_{L^2(0,T;L^2(\Omega))}, \tag{4.4}$$

where q is defined in (4.3) as $q = y_d - S_0(y_0, y_1)$.

The first order optimality conditions for this problem given in Theorem 2.17 are

$$\left\langle \hat{J}'(u^*), u - u^* \right\rangle_{U^*,U} = \left\langle \hat{J}'(u^*), u - u^* \right\rangle_{L^2(0,T;L^2(\Omega))} \geq 0 \quad \text{for all } u \in U_{\mathrm{ad}},$$

since $U = L^2(0,T;L^2(\Omega))$ is a Hilbert space, or, in case $U_{\mathrm{ad}} = U$

$$\hat{J}'(u^*) = 0.$$

Therefore, let us compute the gradient $\hat{J}'(u)$. It is easy to see (cf. [Tro10, Chapter 2.6]) that the following holds:

$$\begin{aligned}\left\langle \hat{J}'(u), h \right\rangle_{L^2(0,T;L^2(\Omega))} &= \langle S_u(u) - q, S_u(h)\rangle_{L^2(0,T;L^2(\Omega))} + \lambda \langle u, h\rangle_{L^2(0,T;L^2(\Omega))} \\ &= \langle S_u^*(S_u(u) - q), h\rangle_{L^2(0,T;L^2(\Omega))} + \lambda \langle u, h\rangle_{L^2(0,T;L^2(\Omega))},\end{aligned}$$

and using Riesz' representation theorem we observe that the gradient is given by

$$\hat{J}'(u) = S_u^*(S_u(u) - q) + \lambda u.$$

Recall that $S_u(u) - q = S_c(u, y_0, y_1) - y_d$. Let us now introduce the state $y = S_c(u, y_0, y_1)$, i.e., y is the weak solution of the Cattaneo equation (4.7). Furthermore, we introduce the adjoint state $p = S_u^*(S_u(u) - q) = S_u^*(y - y_d)$, i.e., p is the weak solution of the backward Cattaneo equation

$$\begin{aligned} \tau p_{tt} - p_t - \Delta p &= y - y_d && \text{in } (0,T) \times \Omega \\ p &= 0 && \text{on } [0,T] \times \partial\Omega \\ p(T, \cdot) &= 0 && \text{in } \Omega \\ p'(T, \cdot) &= 0 && \text{in } \Omega, \end{aligned}$$

cf. Theorem 4.5. With these definitions, the gradient of the reduced cost functional can be written as

$$\hat{J}'(u) = p + \lambda u.$$

We summarize the first order optimality conditions in the following theorem.

Theorem 4.6

Let $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$ be given initial conditions. The first order optimality conditions for $u^* \in U_{\text{ad}} \subseteq U = L^2(0,T; L^2(\Omega))$ being an optimal control of problem (4.1) with corresponding optimal state $y^* \in Y(0,T)$ and adjoint optimal state $p^* \in Y(0,T)$ are given by

1. The optimal state y^* is the weak solution of the state equation

$$\begin{aligned} \tau y_{tt}^* + y_t^* - \Delta y^* &= u^* && \text{in } (0,T) \times \Omega \\ y^* &= 0 && \text{on } [0,T] \times \partial\Omega \\ y^*(0, \cdot) &= y_0 && \text{in } \Omega \\ y_t^*(0, \cdot) &= y_1 && \text{in } \Omega. \end{aligned}$$

2. The adjoint optimal state p^* is the weak solution of the adjoint equation

$$\begin{aligned} \tau p_{tt}^* - p_t^* - \Delta p^* &= y^* - y_d && \text{in } (0,T) \times \Omega \\ p^* &= 0 && \text{on } [0,T] \times \partial\Omega \\ p^*(T, \cdot) &= 0 && \text{in } \Omega \\ p_t^*(T, \cdot) &= 0 && \text{in } \Omega. \end{aligned}$$

3. We have the following optimality condition:

$$\left\langle \hat{J}'(u^*), u - u^* \right\rangle_{L^2(0,T;L^2(\Omega))} = \langle p^* + \lambda u^*, u - u^* \rangle_{L^2(0,T;L^2(\Omega))} \geq 0 \quad \text{for all } u \in U_{\text{ad}}.$$

   This again reduces to

$$\hat{J}'(u^*) = p^* + \lambda u^* = 0$$

   in case $U_{\text{ad}} = U$.

Proof: This is done in the computations above. Note that these conditions are also sufficient since our cost functional is convex (cf. Theorem 2.17). □

5 Asymptotic Analysis

In this chapter we perform an asymptotic analysis of the Cattaneo model for $\tau \to 0$: We investigate whether or not solutions of the Cattaneo equation converge to the ones of the heat equation as it is expected from the derivation of the Cattaneo equation in Chapter 2.1. We consider both the forward problem from Chapter 3 and the corresponding optimal control problem from Chapter 4.

Recall that we have proved the well-posedness of the Cattaneo equation for any $\tau > 0$ in Chapter 3. There we needed some a-priori energy estimates (cf. Theorem 3.7) in order to prove the existence of a weak solution. These estimates, however, do not work in the limit $\tau \to 0$ since the constants there are dependent on the choice of τ, in particular, it is easy to see that the constants tend to infinity for $\tau \to 0$. Therefore, we cannot use these for the limit $\tau \to 0$ and have to find stronger ones. Recall that the existence of an approximate weak solution was independent of the choice of $\tau > 0$ (cf. Lemma 3.6):

For every $m \in \mathbb{N}_{>0}$ and every $\tau > 0$ there exists a unique function y_m of the form

$$y_m(t) = \sum_{k=1}^{m} \varphi_{mk}(t)\ w_k, \quad \varphi_{mk} \in H^2(0,T),$$

which is an approximate solution as defined in Chapter 3, i.e.,

$$\langle \tau y_m'', w_k \rangle_{L^2(\Omega)} + \langle y_m', w_k \rangle_{L^2(\Omega)} + a[y_m, w_k] = \langle u, w_k \rangle_{L^2(\Omega)} \tag{3.7a}$$

$$\varphi_{mk}(0) = \beta_{mk}, \quad \varphi_{mk}'(0) = \gamma_{mk} \tag{3.7b}$$

for $k = 1, \dots, m$ and a.e. $t \in [0,T]$, where the coefficients β_{mk} and γ_{mk} satisfy

$$y_m(0) = \sum_{k=1}^{m} \beta_{mk} w_k \to y_0 \text{ in } H_0^1(\Omega) \text{ as } m \to \infty \quad \text{and} \tag{3.8}$$

$$y_m'(0) = \sum_{k=1}^{m} \gamma_{mk} w_k \to y_1 \text{ in } L^2(\Omega) \text{ as } m \to \infty.$$

As in Chapter 3 we choose $y_m(0)$ and $y_m'(0)$ as the orthogonal projection of y_0 in $H_0^1(\Omega)$ and y_1 in $L^2(\Omega)$, respectively, into the subspace spanned by $w_1, \dots, w_m$. Throughout the rest of this chapter we assume that $\tau \leq 1$. This is justified and without loss of generality since we are only interested in the limit $\tau \to 0$ anyway.

5.1 Energy Estimates

In the following we give the energy estimates required to obtain convergence of the weak solution of the Cattaneo equation to the weak solution of the heat equation. Let us begin our investigation by showing that y_m' is bounded in $L^2(0,T;L^2(\Omega))$ independently of τ.

Lemma 5.1
Let y_m be the approximate solution of (3.7a) described above. Then, there exists a constant $C > 0$ independent of τ such that we have

$$||y_m'||^2_{L^2(0,T;L^2(\Omega))} \leq C \left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right). \tag{5.1}$$

Proof: As in Lemma 3.7, we multiply (3.7a) with $\varphi_{mk}'(t)$ and sum over $k = 1, \dots, m$ to obtain

$$\tau \langle y_m'', y_m' \rangle_{L^2(\Omega)} + \langle y_m', y_m' \rangle_{L^2(\Omega)} + a[y_m, y_m'] = \langle u, y_m' \rangle_{L^2(\Omega)} \tag{3.13}$$

for a.e. $t \in [0, T]$. Again, we see that this can be rewritten as

$$\frac{\tau}{2}\frac{d}{dt} ||y_m'(t)||^2_{L^2(\Omega)} + ||y_m'(t)||^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt} a[y_m(t), y_m(t)] = \langle u(t), y_m'(t) \rangle_{L^2(\Omega)}$$

for a.e. $t \in [0, T]$. Next, we integrate this over $[0, T]$ and obtain

$$\begin{aligned}
&\int_0^T \langle u(t), y_m'(t) \rangle_{L^2(\Omega)} \, \mathrm{d}t \qquad (5.2)\\
&= \int_0^T \frac{\tau}{2}\frac{d}{dt} ||y_m'(t)||^2_{L^2(\Omega)} + ||y_m'(t)||^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt} a[y_m(t), y_m(t)] \, \mathrm{d}t \\
&= \frac{\tau}{2} ||y_m'(T)||^2_{L^2(\Omega)} - \frac{\tau}{2} ||y_m'(0)||^2_{L^2(\Omega)} + ||y_m'||^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \frac{1}{2} a[y_m(T), y_m(T)] - \frac{1}{2} a[y_m(0), y_m(0)].
\end{aligned}$$

As discussed in Chapter 3, we have the estimates

$$\begin{aligned}
&||y_m'(0)||^2_{L^2(\Omega)} \leq ||y_1||^2_{L^2(\Omega)} \text{ and} \\
&a[y_m(0), y_m(0)] \leq C\, ||y_m(0)||^2_{H_0^1(\Omega)} \leq C\, ||y_0||^2_{H_0^1(\Omega)},
\end{aligned}$$

where the constant C is independent of τ (cf. (3.5) and (3.9)). We use these to obtain the following estimate

$$\begin{aligned}
&2\, ||y_m'||^2_{L^2(0,T;L^2(\Omega))} && (5.3)\\
&\leq 2\, ||y_m'||^2_{L^2(0,T;L^2(\Omega))} + \tau\, ||y_m'(T)||^2_{L^2(\Omega)} + a[y_m(T), y_m(T)] && \text{(due to (3.4))}\\
&= 2 \int_0^T \langle u(t), y_m'(t) \rangle_{L^2(\Omega)} \, \mathrm{d}t + \tau\, ||y_m'(0)||^2_{L^2(\Omega)} + a[y_m(0), y_m(0)] && \text{(due to (5.2))}\\
&\leq \int_0^T ||u(t)||^2_{L^2(\Omega)} + ||y_m'(t)||^2_{L^2(\Omega)} \, \mathrm{d}t + C\, ||y_0||^2_{H_0^1(\Omega)} + \tau\, ||y_1||^2_{L^2(\Omega)} \\
&\leq ||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_m'||^2_{L^2(0,T;L^2(\Omega))} + C\, ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)},
\end{aligned}$$

since $\tau \leq 1$. Note that we used both the Cauchy-Schwarz and Young's inequality for the second estimate. Subtracting $||y_m'||^2_{L^2(0,T;L^2(\Omega))}$ from (5.3) yields

$$||y_m'||^2_{L^2(0,T;L^2(\Omega))} \leq C\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right),$$

where the constant C is again independent of τ, which also finishes the proof. □

Now, we want to prove a similar result for y_m. In order to do so, we start by proving a Poincaré inequality.

Lemma 5.2
Let $y \in H^1(0,T;L^2(\Omega))$. Then, there exists a constant $C > 0$ only depending on T such that

$$||y||^2_{L^2(0,T;L^2(\Omega))} \leq C\left(||y'||^2_{L^2(0,T;L^2(\Omega))} + ||y(0)||^2_{L^2(\Omega)}\right).$$

Proof: Note that $C^1([0,T];L^2(\Omega))$ is dense in $H^1(0,T;L^2(\Omega))$ (cf. Lemma 2.5), so it suffices to prove the inequality for $y \in C^1([0,T];L^2(\Omega))$. For such a y we observe

$$\begin{aligned}||y(t)-y(0)||^2_{L^2(\Omega)} &= \left|\left|\int_0^t y'(s)\ \mathrm{d}s\right|\right|^2_{L^2(\Omega)} \leq \left(\int_0^t ||y'(s)||_{L^2(\Omega)}\ \mathrm{d}s\right)^2 \\ &\leq t\int_0^t ||y'(s)||^2_{L^2(\Omega)}\ \mathrm{d}s \leq T\int_0^T ||y'(s)||^2_{L^2(\Omega)}\ \mathrm{d}s = T\,||y'||^2_{L^2(0,T;L^2(\Omega))},\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the second estimation. Now, we integrate this over $[0,T]$ to obtain

$$\begin{aligned}||y-y(0)||^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T ||y(t)-y(0)||^2_{L^2(\Omega)}\ \mathrm{d}t \\ &\leq T\int_0^T ||y'||^2_{L^2(0,T;L^2(\Omega))}\ \mathrm{d}t \\ &= T^2\,||y'||^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Taking the square root reveals

$$||y-y(0)||_{L^2(0,T;L^2(\Omega))} \leq T\,||y'||_{L^2(0,T;L^2(\Omega))}. \tag{5.4}$$

Finally, we employ the triangle inequality to find

$$\begin{aligned}||y||_{L^2(0,T;L^2(\Omega))} &= ||y-y(0)+y(0)||_{L^2(0,T;L^2(\Omega))} \\ &\leq ||y-y(0)||_{L^2(0,T;L^2(\Omega))} + ||y(0)||_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{5.5}$$

Note that we have

$$||y(0)||^2_{L^2(0,T;L^2(\Omega))} = \int_0^T ||y(0)||^2_{L^2(\Omega)}\ \mathrm{d}t = T\,||y(0)||^2_{L^2(\Omega)},$$

and, therefore, (5.4) and (5.5) yield

$$||y||_{L^2(0,T;L^2(\Omega))} \leq C(T) \left(||y'||_{L^2(0,T;L^2(\Omega))} + ||y(0)||_{L^2(\Omega)} \right),$$

completing the proof. □

We know that the approximate solution y_m is in $H^2(0,T;H_0^1(\Omega)) \hookrightarrow H^1(0,T;L^2(\Omega))$ and, thus, we apply Lemma 5.1 and Lemma 5.2 to find

$$\begin{aligned} ||y_m||^2_{L^2(0,T;L^2(\Omega))} &\leq C(T) \left(||y'_m||^2_{L^2(0,T;L^2(\Omega))} + ||y_m(0)||^2_{L^2(\Omega)} \right) \\ &\leq C(T) \left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right), \end{aligned} \tag{5.6}$$

with a constant $C(T) > 0$ independent of τ since we have $||y_m(0)||_{L^2(\Omega)} \leq ||y_0||_{H_0^1(\Omega)}$.

We use this to show that $y_m \in L^2(0,T;H_0^1(\Omega))$ independently of τ.

Lemma 5.3
There exists a constant $C > 0$ independent of τ such that

$$||y_m||^2_{L^2(0,T;H_0^1(\Omega))} \leq C \left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right).$$

Proof: First, we multiply (3.7a) with $\varphi_{mk}(t)$ and sum over $k = 1, \dots, m$ to obtain

$$\tau \langle y''_m, y_m \rangle_{L^2(\Omega)} + \langle y'_m, y_m \rangle_{L^2(\Omega)} + a[y_m, y_m] = \langle u, y_m \rangle_{L^2(\Omega)} \tag{5.7}$$

for a.e. $t \in [0,T]$. We observe that we have

$$\tau \langle y''_m, y_m \rangle_{L^2(\Omega)} = \tau \frac{d}{dt} \langle y'_m, y_m \rangle_{L^2(\Omega)} - \tau \langle y'_m, y'_m \rangle_{L^2(\Omega)} = \tau \frac{d}{dt} \langle y'_m, y_m \rangle_{L^2(\Omega)} - \tau\, ||y'_m||^2_{L^2(\Omega)}$$

for a.e. $t \in [0,T]$. Using this we rewrite (5.7) as

$$\begin{aligned} &\tau \frac{d}{dt} \langle y'_m(t), y_m(t) \rangle_{L^2(\Omega)} + \frac{1}{2} \frac{d}{dt} ||y_m(t)||^2_{L^2(\Omega)} + a[y_m(t), y_m(t)] \\ = \ &\langle u(t), y_m(t) \rangle_{L^2(\Omega)} + \tau\, ||y'_m(t)||^2_{L^2(\Omega)} \end{aligned}$$

for a.e. $t \in [0,T]$. Integrating this over $[0,T]$ yields

$$\begin{aligned} &\int_0^T \langle u(t), y_m(t) \rangle_{L^2(\Omega)} + \tau\, ||y'_m(t)||^2_{L^2(\Omega)} \ \mathrm{d}t \\ = &\int_0^T \tau \frac{d}{dt} \langle y'_m(t), y_m(t) \rangle_{L^2(\Omega)} + \frac{1}{2} \frac{d}{dt} ||y_m(t)||^2_{L^2(\Omega)} + a[y_m(t), y_m(t)] \ \mathrm{d}t \\ = &\ \tau \langle y'_m(T), y_m(T) \rangle_{L^2(\Omega)} - \tau \langle y'_m(0), y_m(0) \rangle_{L^2(\Omega)} + \frac{1}{2} ||y_m(T)||^2_{L^2(\Omega)} \\ &- \frac{1}{2} ||y_m(0)||^2_{L^2(\Omega)} + \int_0^T a[y_m(t), y_m(t)] \ \mathrm{d}t. \end{aligned} \tag{5.8}$$

The coercivity of a together with the Cauchy-Schwarz and Young's inequality reveals

$$\alpha \, ||y_m||^2_{L^2(0,T;H^1_0(\Omega))} = \alpha \int_0^T ||y_m(t)||^2_{H^1_0(\Omega)} \, \mathrm{d}t \le \int_0^T a[y_m(t), y_m(t)] \, \mathrm{d}t \tag{5.9}$$

$$\le \int_0^T a[y_m(t), y_m(t)] \, \mathrm{d}t + \frac{1}{2} ||y_m(T)||^2_{L^2(\Omega)}$$

$$= \int_0^T \langle u(t), y_m(t)\rangle_{L^2(\Omega)} \, \mathrm{d}t + \tau \, ||y_m'||^2_{L^2(0,T;L^2(\Omega))} - \tau \, \langle y_m'(T), y_m(T)\rangle_{L^2(\Omega)}$$

$$+ \tau \, \langle y_m'(0), y_m(0)\rangle_{L^2(\Omega)} + \frac{1}{2} ||y_m(0)||^2_{L^2(\Omega)} \qquad \text{(due to (5.8))}$$

$$\le \int_0^T ||u(t)||_{L^2(\Omega)} \, ||y_m(t)||_{L^2(\Omega)} \, \mathrm{d}t + \tau \, ||y_m'||^2_{L^2(0,T;L^2(\Omega))} + \tau \, ||y_m'(T)||_{L^2(\Omega)} \, ||y_m(T)||_{L^2(\Omega)}$$

$$+ \tau \, ||y_m'(0)||_{L^2(\Omega)} \, ||y_m(0)||_{L^2(\Omega)} + \frac{1}{2} ||y_m(0)||^2_{L^2(\Omega)}$$

$$\le \frac{1}{2} \int_0^T \left(||u(t)||^2_{L^2(\Omega)} + ||y_m(t)||^2_{L^2(\Omega)} \right) \, \mathrm{d}t + \tau \, ||y_m'||^2_{L^2(0,T;L^2(\Omega))} + \frac{\tau}{2} ||y_m'(T)||^2_{L^2(\Omega)}$$

$$+ \frac{\tau}{2} ||y_m(T)||^2_{L^2(\Omega)} + \frac{\tau}{2} ||y_m'(0)||^2_{L^2(\Omega)} + \frac{\tau}{2} ||y_m(0)||^2_{L^2(\Omega)} + \frac{1}{2} ||y_m(0)||^2_{L^2(\Omega)}$$

$$\le \frac{1}{2} ||u||^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2} ||y_m||^2_{L^2(0,T;L^2(\Omega))} + ||y_m'||^2_{L^2(0,T;L^2(\Omega))} + \frac{\tau}{2} ||y_m'(T)||^2_{L^2(\Omega)}$$

$$+ \frac{\tau}{2} ||y_m(T)||^2_{L^2(\Omega)} + \frac{\tau}{2} ||y_1||^2_{L^2(\Omega)} + \frac{\tau}{2} ||y_0||^2_{H^1_0(\Omega)} + \frac{1}{2} ||y_0||^2_{H^1_0(\Omega)} .$$

Let us take a more thorough look at this equation. On the left-hand side we have the term we want to estimate by the data u, y_0 and y_1. On the right-hand side, these terms appear amongst others, which we discuss in the following.

- We estimate the term $||y_m||_{L^2(0,T;L^2(\Omega))}$ using (5.6)

$$||y_m||^2_{L^2(0,T;L^2(\Omega))} \le C(T) \left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H^1_0(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right) .$$

- For the term $||y_m'||^2_{L^2(0,T;L^2(\Omega))}$ we have Lemma 5.1 available which gives us

$$||y_m'||^2_{L^2(0,T;L^2(\Omega))} \le C \left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H^1_0(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right) .$$

- We estimate the term $\frac{\tau}{2} ||y_m'(T)||^2_{L^2(\Omega)}$ as follows: We know that

$$\langle y_m''(t), y_m'(t)\rangle_{L^2(\Omega)} = \frac{1}{2} \frac{d}{dt} ||y_m'(t)||^2_{L^2(\Omega)} ,$$

holds and, therefore, we have

$$\frac{\tau}{2}\left|\left|y_m'(T)\right|\right|_{L^2(\Omega)}^2 = \frac{\tau}{2}\left|\left|y_m'(0)\right|\right|_{L^2(\Omega)}^2 + \int_0^T \frac{\tau}{2}\frac{d}{dt}\left|\left|y_m'(t)\right|\right|_{L^2(\Omega)}^2 \; \mathrm{d}t \tag{5.10}$$

$$= \frac{\tau}{2}\left|\left|y_m'(0)\right|\right|_{L^2(\Omega)}^2 + \int_0^T \tau \left\langle y_m''(t), y_m'(t)\right\rangle_{L^2(\Omega)} \; \mathrm{d}t.$$

As in the proof of Lemma 5.1 we observe that

$$\tau \left\langle y_m''(t), y_m'(t)\right\rangle_{L^2(\Omega)} = \left\langle u(t), y_m'(t)\right\rangle_{L^2(\Omega)} - \left\langle y_m'(t), y_m'(t)\right\rangle_{L^2(\Omega)} - a[y_m(t), y_m'(t)]$$

for a.e. $t \in [0,T]$. We use this to rewrite the integral in (5.10) as

$$\frac{\tau}{2}\left|\left|y_m'(T)\right|\right|_{L^2(\Omega)}^2 = \frac{\tau}{2}\left|\left|y_m'(0)\right|\right|_{L^2(\Omega)}^2 + \int_0^T \tau \left\langle y_m''(t), y_m'(t)\right\rangle_{L^2(\Omega)} \; \mathrm{d}t \tag{5.11}$$

$$= \frac{\tau}{2}\left|\left|y_m'(0)\right|\right|_{L^2(\Omega)}^2 + \int_0^T \left\langle u(t), y_m'(t)\right\rangle_{L^2(\Omega)} - \left\langle y_m'(t), y_m'(t)\right\rangle_{L^2(\Omega)} - a[y_m(t), y_m'(t)] \; \mathrm{d}t.$$

Let us make some estimates: It holds

$$\frac{\tau}{2}\left|\left|y_m'(0)\right|\right|_{L^2(\Omega)}^2 \leq \left|\left|y_1\right|\right|_{L^2(\Omega)}^2, \tag{5.12}$$

thanks to $\tau \leq 1$ and (3.9). Furthermore, we have

$$\begin{aligned}
&\int_0^T \left\langle u(t), y_m'(t)\right\rangle_{L^2(\Omega)} \; \mathrm{d}t \\
&\leq \frac{1}{2}\left|\left|u\right|\right|_{L^2(0,T;L^2(\Omega))}^2 + \frac{1}{2}\left|\left|y_m'\right|\right|_{L^2(0,T;L^2(\Omega))}^2 \\
&\leq C\left(\left|\left|u\right|\right|_{L^2(0,T;L^2(\Omega))}^2 + \left|\left|y_0\right|\right|_{H_0^1(\Omega)}^2 + \left|\left|y_1\right|\right|_{L^2(\Omega)}^2\right),
\end{aligned}$$

thanks to the Cauchy-Schwarz and Young's inequality as well as Lemma 5.1. Additionally, we get

$$-\int_0^T \left\langle y_m'(t), y_m'(t)\right\rangle_{L^2(\Omega)} \; \mathrm{d}t = -\left|\left|y_m'\right|\right|_{L^2(0,T;L^2(\Omega))}^2 \leq 0.$$

Last, we observe

$$-\int_0^T a[y_m(t), y_m'(t)] \; \mathrm{d}t = -\int_0^T \frac{1}{2}\frac{d}{dt}a[y_m(t), y_m(t)] \; \mathrm{d}t \tag{5.13}$$

$$\begin{aligned}
&= -\frac{1}{2}a[y_m(T), y_m(T)] + \frac{1}{2}a[y_m(0), y_m(0)] \\
&\leq C\left|\left|y_0\right|\right|_{H_0^1(\Omega)}^2,
\end{aligned}$$

thanks to the coercivity and continuity of the bilinear form a (cf. (3.4) and (3.5)). Now, we use the estimates (5.12) to (5.13) together with (5.11) to deduce

$$\frac{\tau}{2}\,||y_m'(T)||^2_{L^2(\Omega)} \leq C\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right).$$

- Finally, we investigate the term $\frac{\tau}{2}\,||y_m(T)||^2_{L^2(\Omega)}$. Similarly to before, we express this as

$$\begin{aligned}\frac{\tau}{2}\,||y_m(T)||^2_{L^2(\Omega)} &= \frac{\tau}{2}\,||y_m(0)||^2_{L^2(\Omega)} + \int_0^T \frac{\tau}{2}\frac{d}{dt}\,||y_m(t)||^2_{L^2(\Omega)}\,\mathrm{d}t \\ &= \frac{\tau}{2}\,||y_m(0)||^2_{L^2(\Omega)} + \int_0^T \tau\,\langle y_m'(t), y_m(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{5.14}$$

Let us estimate the terms in (5.14): We have

$$\frac{\tau}{2}\,||y_m(0)||^2_{L^2(\Omega)} \leq ||y_0||^2_{H_0^1(\Omega)}, \tag{5.15}$$

due to (3.9) and $\tau \leq 1$. For the term in the integral we use the Cauchy-Schwarz and Young's inequality to deduce

$$\begin{aligned}\int_0^T \tau\,\langle y_m'(t), y_m(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t &\leq \int_0^T \tau\,||y_m'(t)||_{L^2(\Omega)}\,||y_m(t)||_{L^2(\Omega)}\,\mathrm{d}t \\ &\leq \int_0^T \frac{\tau}{2}\,||y_m'(t)||^2_{L^2(\Omega)} + \frac{\tau}{2}\,||y_m(t)||^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\leq \frac{\tau}{2}\,||y_m'||^2_{L^2(0,T;L^2(\Omega))} + \frac{\tau}{2}\,||y_m||^2_{L^2(0,T;L^2(\Omega))} \\ &\leq C\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right),\end{aligned} \tag{5.16}$$

thanks to Lemma 5.1 and (5.6). Using the inequalities (5.15) as well as (5.16) in (5.14) yields

$$\frac{\tau}{2}\,||y_m(T)||^2_{L^2(\Omega)} \leq C\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||_{L^2(\Omega)}\right).$$

Using all of the estimates above in (5.9) we end up with

$$||y_m||^2_{L^2(0,T;H_0^1(\Omega))} \leq C\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right),$$

finishing the proof. □

5.2 The Forward Problem

With the help of these energy estimates we are able to investigate the limit process for the (forward) Cattaneo equation as follows.

We see that y_m and y'_m are elements of $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;L^2(\Omega))$, respectively, and that their norm is bounded in these spaces independently of both m and τ. Therefore, we can extract weakly convergent subsequences $(y_{m_l}) \subset (y_m)$ and $(y'_{m_l}) \subset (y'_m)$ such that

$$\begin{aligned} y_{m_l} &\rightharpoonup y \quad &&\text{in } L^2(0,T;H_0^1(\Omega)) \text{ and} \\ y'_{m_l} &\rightharpoonup y' \quad &&\text{in } L^2(0,T;L^2(\Omega)), \end{aligned}$$

where, as before, it holds that y' is the weak time derivative of y. Since the weak limit is uniquely determined it follows that y is, in fact, the unique weak solution of the Cattaneo equation (cf. the proof of Theorem 3.8 and Remark 3.9). Since the above convergence holds for any subsequence of (y_m) and the limit is uniquely determined as the weak solution of the Cattaneo equation, we even get the weak convergence of the whole sequence (y_m) to y as well as (y'_m) to y'. The next lemma tells us that the energy estimates are still valid for the limits y and y'.

Lemma 5.4
For every $\tau > 0$ there exists a unique weak solution $y_\tau \in Y(0,T)$ of the Cattaneo equation. Additionally, there exists a constant $C > 0$ independent of τ such that

$$||y_\tau||^2_{L^2(0,T;H_0^1(\Omega))} + ||y'_\tau||^2_{L^2(0,T;L^2(\Omega))} \le C\left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right).$$

Proof: In Chapter 3 we have already shown the existence and uniqueness of a weak solution of the Cattaneo equation. The energy estimate follows directly from Lemma 5.1 and Lemma 5.3. □

Let us now investigate the limit $\tau \to 0$. We consider an arbitrary sequence $(\tau_i) \subset \mathbb{R}$ with $\tau_i > 0$ for all $i \in \mathbb{N}$ as well as $\lim\limits_{i\to\infty} \tau_i = 0$. As before, we assume, without loss of generality, that $\tau_i \le 1$ for all $i \in \mathbb{N}$. Furthermore, we fix $u \in L^2(0,T;L^2(\Omega))$ as well as $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$.

We denote by y_i the unique weak solution of the Cattaneo equation with $\tau = \tau_i$, i.e.,

$$\int_0^T \tau_i \langle y''_i(t), v(t)\rangle_{H^{-1},H^1} + \langle y'_i(t), v(t)\rangle_{L^2(\Omega)} + a[y_i(t), v(t)]\,\mathrm{d}t = \int_0^T \langle u(t), v(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t \quad (5.17)$$

for all $v \in L^2(0,T;H_0^1(\Omega))$ and

$$y_i(0) = y_0 \quad \text{and} \quad y'_i(0) = y_1,$$

where $a[\cdot,\cdot]$ is defined as before (cf. (3.3)). Thanks to Lemma 5.4 we know that the sequences (y_i) and (y'_i) are bounded independently of τ in $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;L^2(\Omega))$, respectively. This allows us to extract subsequences $(y_{i_j}) \subset (y_i)$ and $(y'_{i_j}) \subset (y'_i)$ such that

$$\begin{aligned} y_{i_j} &\rightharpoonup y \quad \text{in } L^2(0,T;H_0^1(\Omega)) \text{ and} \\ y'_{i_j} &\rightharpoonup y' \quad \text{in } L^2(0,T;L^2(\Omega)) \end{aligned} \tag{5.18}$$

for some $y \in L^2(0,T;H_0^1(\Omega))$ with weak time derivative $y' \in L^2(0,T;L^2(\Omega))$ as $j \to \infty$. In the following we show that y is in fact a weak solution of the heat equation. Therefore, we assume that the test function v lies in $C_0^\infty((0,T);H_0^1(\Omega))$.

For such a v we apply integration by parts in (5.17) and observe

$$\begin{aligned} &\int_0^T -\tau_{i_j} \left\langle y'_{i_j}(t), v'(t) \right\rangle_{L^2(\Omega)} + \left\langle y'_{i_j}(t), v(t) \right\rangle_{L^2(\Omega)} + a[y_{i_j}(t), v(t)] \,\mathrm{d}t \\ =&\int_0^T \langle u(t), v(t) \rangle_{L^2(\Omega)} \,\mathrm{d}t. \end{aligned} \tag{5.19}$$

From Lemma 5.4 we get that

$$\begin{aligned} \left|\left|\tau_{i_j} y'_{i_j}\right|\right|^2_{L^2(0,T;L^2(\Omega))} &= \tau_{i_j}^2 \left|\left|y'_{i_j}\right|\right|^2_{L^2(0,T;L^2(\Omega))} \\ &\leq \tau_{i_j}^2 C \left(||u||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)} \right), \end{aligned}$$

where $C > 0$ is independent of τ. Due to this, it holds

$$\tau_{i_j} \left|\left|y'_{i_j}\right|\right|_{L^2(0,T;L^2(\Omega))} = \left|\left|\tau_{i_j} y'_{i_j}\right|\right|_{L^2(0,T;L^2(\Omega))} \to 0 \tag{5.20}$$

as $j \to \infty$. With the help of the Cauchy-Schwarz inequality we get

$$\begin{aligned} &\left| \int_0^T \tau_{i_j} \left\langle y'_{i_j}(t), v'(t) \right\rangle_{L^2(\Omega)} \,\mathrm{d}t \right| = \left| \tau_{i_j} \left\langle y'_{i_j}, v' \right\rangle_{L^2(0,T;L^2(\Omega))} \right| \\ \leq &\tau_{i_j} \left|\left|y'_{i_j}\right|\right|_{L^2(0,T;L^2(\Omega))} ||v'||_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Together with (5.20) this implies

$$-\int_0^T \tau_{i_j} \left\langle y'_{i_j}(t), v'(t) \right\rangle_{L^2(\Omega)} \,\mathrm{d}t \to 0 \quad \text{as } j \to \infty.$$

For the other terms we directly see

$$\begin{aligned} &\int_0^T \left\langle y'_{i_j}(t), v(t) \right\rangle_{L^2(\Omega)} \,\mathrm{d}t \to \int_0^T \langle y'(t), v(t) \rangle_{L^2(\Omega)} \,\mathrm{d}t \text{ as well as} \\ &\int_0^T a[y_{i_j}(t), v(t)] \,\mathrm{d}t \to \int_0^T a[y(t), v(t)] \,\mathrm{d}t \text{ as } j \to \infty, \end{aligned}$$

due to the weak convergence of y_{i_j} and y'_{i_j} (cf. (5.18)).

Altogether, passing to the limit $j \to \infty$ in (5.19) yields

$$\int_0^T \langle y'(t), v(t)\rangle_{L^2(\Omega)} + a[y(t), v(t)] \,\mathrm{d}t = \int_0^T \langle u(t), v(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t. \tag{5.21}$$

Note that this also holds true for all $v \in L^2(0, T; H_0^1(\Omega))$ since $C_0^\infty((0, T); H_0^1(\Omega))$ is dense in this space (cf. Lemma 2.5). Let us now choose $v \in C^1([0, T]; H_0^1(\Omega))$ with $v(T) = 0$. Applying integration by parts in (5.21) gives

$$\int_0^T - \langle y(t), v'(t)\rangle_{L^2(\Omega)} + a[y(t), v(t)] \,\mathrm{d}t = \int_0^T \langle u(t), v(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t + \langle y(0), v(0)\rangle_{L^2(\Omega)}. \tag{5.22}$$

On the other hand, we also apply integration by parts to (5.17) and observe

$$\begin{aligned}&\int_0^T -\tau_{i_j} \left\langle y'_{i_j}(t), v'(t)\right\rangle_{L^2(\Omega)} - \left\langle y_{i_j}(t), v'(t)\right\rangle_{L^2(\Omega)} + a[y_{i_j}(t), v(t)] \,\mathrm{d}t \\ &= \int_0^T \langle u(t), v(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t + \tau_{i_j} \langle y_1, v(0)\rangle_{L^2(\Omega)} + \langle y_0, v(0)\rangle_{L^2(\Omega)},\end{aligned} \tag{5.23}$$

since we have $y_{i_j}(0, \cdot) = y_0$ for all j. With the same arguments as before we observe that we have

$$\tau_{i_j} \langle y_1, v'(0)\rangle_{L^2(\Omega)} \to 0 \quad \text{as } j \to \infty,$$

since both y_1 and $v(0)$ are fixed. Analogously to before we take the limit $j \to \infty$ in (5.23) and observe

$$\int_0^T - \langle y(t), v'(t)\rangle_{L^2(\Omega)} + a[y(t), v(t)] \,\mathrm{d}t = \int_0^T \langle u(t), v(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t + \langle y_0, v(0)\rangle_{L^2(\Omega)}. \tag{5.24}$$

Comparing equations (5.22) and (5.24) we obtain

$$\langle y_0, v(0)\rangle_{L^2(\Omega)} = \langle y(0), v(0)\rangle_{L^2(\Omega)},$$

and, therefore, directly

$$y(0) = y_0.$$

Hence, we summarize what we found: The function $y \in L^2(0, T; H_0^1(\Omega))$ and its weak time derivative $y' \in L^2(0, T; L^2(\Omega))$ satisfy

$$\int_0^T \langle y'(t), v(t)\rangle_{L^2(\Omega)} + a[y(t), v(t)] \,\mathrm{d}t = \int_0^T \langle u(t), v(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t,$$

for all $v \in L^2(0, T; H_0^1(\Omega))$ as well as

$$y(0) = y_0.$$

In, for example, [Eva10, Chapter 7.1] it is shown that there exists a unique weak solution of the heat equation

$$\begin{aligned} y_t - \Delta y &= u && \text{in } (0,T)\times\Omega \\ y &= 0 && \text{on } [0,T]\times\partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega \end{aligned} \tag{5.25}$$

in the space

$$W(0,T) = \left\{ y \in L^2(0,T;H_0^1(\Omega)) \,\middle|\, y' \in L^2(0,T;H^{-1}(\Omega)) \right\}$$

for every $u \in L^2(0,T;H^{-1}(\Omega))$ and $y_0 \in L^2(\Omega)$. This weak solution satisfies the variational formulation

$$\int_0^T \langle y'(t), v(t)\rangle_{H^{-1},H^1} + a[y(t),v(t)] \,\mathrm{d}t = \int_0^T \langle u(t), v(t)\rangle_{H^{-1},H^1} \,\mathrm{d}t$$
$$\text{for all } v \in L^2(0,T;H_0^1(\Omega))$$

and the initial condition $y(0) = y_0$. We can directly see that it holds $Y(0,T) \hookrightarrow W(0,T)$ and, hence, we get $y_i \in W(0,T)$ for all $i \in \mathbb{N}$. Therefore, the sequence of solutions of the Cattaneo equation (y_i) can actually converge to the weak solution of the heat equation. In order to interpret y as a weak solution to the classical heat equation we note that we have the continuous embeddings

$$L^2(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;H^{-1}(\Omega)) \quad \text{as well as} \quad H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

Thanks to this, we can interpret $u \in L^2(0,T;L^2(\Omega))$ and $y' \in L^2(0,T;L^2(\Omega))$ as elements of $L^2(0,T;H^{-1}(\Omega))$ and $y_0 \in H_0^1(\Omega)$ as an element of $L^2(\Omega)$. With this it is immediate that the limit y is the weak solution of the heat equation (5.25). Furthermore, the regularity results from [Eva10, Chapter 7.1] tell us that, if the right-hand side u is in $L^2(0,T;L^2(\Omega))$ and the initial value y_0 is in $H_0^1(\Omega)$, which is the case for our setting, then we get $y \in L^2(0,T;H_0^1(\Omega))$ and $y' \in L^2(0,T;L^2(\Omega))$. Thus, we introduce the space

$$\hat{W}(0,T) := \left\{ y \in L^2(0,T;H_0^1(\Omega)) \,\middle|\, y' \in L^2(0,T;L^2(\Omega)) \right\}$$

and observe that in our case the weak solution of the heat equation y lies, in fact, in this space.

Note that the above construction also holds for any subsequence of (y_i) and, therefore, we get the weak convergence of the whole sequence due to the uniqueness of the limit, i.e.,

$$\begin{aligned} y_i &\rightharpoonup y && \text{in } L^2(0,T;H_0^1(\Omega)) \text{ and} \\ y_i' &\rightharpoonup y' && \text{in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

We even get the strong convergence thanks to the Aubin-Lions lemma (Theorem 2.9). We apply this with $X_0 = H_0^1(\Omega)$, $X = L^2(\Omega)$, $X_1 = L^2(\Omega)$ and $p = q = 2$. Hence, we have

$$W_0 = \left\{ y \in L^2(0,T;H_0^1(\Omega)) \,\middle|\, y' \in L^2(0,T;L^2(\Omega)) \right\} = \hat{W}(0,T)$$

and we observe that the sequence (y_i) is in W_0. Additionally, it is shown in [Eva10, Chapter 5.7] that the embedding $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ is compact, and it is obvious that the embedding $L^2 \hookrightarrow L^2$ is continuous. Furthermore, all spaces are reflexive since they are Hilbert spaces. Hence, the conditions of the Aubin-Lions lemma are satisfied and we get the compact embedding

$$W_0 \hookrightarrow\hookrightarrow L^2(0,T;L^2(\Omega)). \tag{5.26}$$

Note that (y_i) is bounded in W_0 thanks to Lemma 5.4. Therefore, the compact embedding (5.26) yields the existence of a subsequence $(y_{i_l}) \subset (y_i)$ and some $y^* \in L^2(0,T;L^2(\Omega))$ such that

$$y_{i_l} \to y^* \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } l \to \infty.$$

However, it is easy to see that we have $y^* = y$ due to the uniqueness of the limit. Hence, we get the strong convergence

$$y_{i_l} \to y \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } l \to \infty.$$

As before, the above arguments also hold true for any subsequence of (y_i) and, hence, it holds that

$$y_i \to y \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } i \to \infty. \tag{5.27}$$

With this, we have proved the strong convergence of the sequence (y_i) of solutions of the Cattaneo equation for $\tau \to 0$ to the weak solution y of the heat equation. We summarize our results in the following theorem.

Theorem 5.5
Let $(\tau_i) \subset \mathbb{R}$ be a sequence with $\tau_i > 0$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \tau_i = 0$. Additionally, let $u \in L^2(0,T;L^2(\Omega))$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$ be given. Let $y_i \in Y(0,T)$ be the unique weak solution of the Cattaneo equation

$$\begin{aligned} \tau_i(y_i)_{tt} + (y_i)_t - \Delta y_i &= u && \text{in } (0,T)\times\Omega \\ y_i &= 0 && \text{on } [0,T]\times\partial\Omega \\ y_i(0,\cdot) &= y_0 && \text{in } \Omega \\ (y_i)_t(0,\cdot) &= y_1 && \text{in } \Omega, \end{aligned}$$

and let $y \in \hat{W}(0,T)$ be the unique weak solution of the heat equation

$$\begin{aligned} y_t - \Delta y &= u && \text{in } (0,T)\times\Omega \\ y &= 0 && \text{on } [0,T]\times\partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega. \end{aligned}$$

Then, the sequence (y_i) converges strongly to y in $L^2(0,T;L^2(\Omega))$ and weakly to y in $L^2(0,T;H_0^1(\Omega))$ and the sequence $(y_i)'$ converges weakly to y' in $L^2(0,T;L^2(\Omega))$.

Proof: This is done in the above computations. □

5.3 The Optimal Control Problem

Now that we have proved the convergence of the forward problem we investigate the optimal control problem. Let us begin by stating the optimal control problem for both the Cattaneo equation and the heat equation.

For the Cattaneo equation we consider the same setting as before: $(\tau_i) \subset \mathbb{R}$ is a sequence with $0 < \tau_i \leq 1$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \tau_i = 0$. Furthermore, we use $U = L^2(0,T;L^2(\Omega))$ as the control space as well as the non-empty, weakly closed and convex set $U_{\text{ad}} \subseteq U$ as set of admissible controls, which is arbitrary but fixed. Moreover, we define the cost functional J by

$$J(y,u) = \frac{1}{2} \left|\left|y - y_d\right|\right|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \left|\left|u\right|\right|_{L^2(0,T;L^2(\Omega))},$$

where $y_d \in L^2(0,T;L^2(\Omega))$ and $\lambda > 0$ are fixed. Note that, in contrast to Chapter 4, we treat the embeddings $Y(0,T) \hookrightarrow \hat{W}(0,T) \hookrightarrow L^2(0,T;L^2(\Omega))$ implicitly for better readability.

We consider the following optimal control problem for the Cattaneo equation:

$$\min \; J(y_i,u) = \frac{1}{2} \left|\left|y_i - y_d\right|\right|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \left|\left|u\right|\right|^2_{L^2(0,T;L^2(\Omega))} \tag{5.28}$$

subject to

$$\begin{aligned} \tau_i (y_i)_{tt} + (y_i)_t - \Delta y_i &= u && \text{in } (0,T)\times\Omega \\ y_i &= 0 && \text{on } [0,T]\times\partial\Omega \\ y_i(0,\cdot) &= y_0 && \text{in } \Omega \\ (y_i)_t(0,\cdot) &= y_1 && \text{in } \Omega \end{aligned} \tag{5.29}$$

and

$$u \in U_{\text{ad}}.$$

We denote by G_{c_i} the solution operator of the Cattaneo equation (5.29), i.e.,

$$G_{c_i} \colon L^2(0,T;L^2(\Omega)) \to Y(0,T); \quad u \mapsto G_{c_i}(u) = y_i,$$

where y_i is the unique weak solution of (5.29). Note that Lemma 5.4 gives us the following estimates for the solution operator:

$$\begin{aligned} &\left|\left|G_{c_i}(u)\right|\right|^2_{L^2(0,T;H_0^1(\Omega))} + \left|\left|G_{c_i}(u)'\right|\right|^2_{L^2(0,T;L^2(\Omega))} \\ &\leq C\left(\left|\left|u\right|\right|^2_{L^2(0,T;L^2(\Omega))} + \left|\left|y_0\right|\right|^2_{H_0^1(\Omega)} + \left|\left|y_1\right|\right|^2_{L^2(\Omega)} \right). \end{aligned}$$

Therefore, we consider the reduced problem

$$\min_{u\in U_{\text{ad}}} \hat{J}_i(u),$$

where the reduced cost functional for the Cattaneo equation is defined as

$$\hat{J}_i(u) := J(G_{c_i}(u), u). \tag{5.30}$$

From Chapter 4.1 we know that this problem has a unique minimizer $u_i^* \in U_{\text{ad}}$ for every $i \in \mathbb{N}$, i.e.,

$$\hat{J}_i(u_i^*) \leq \hat{J}_i(u) \text{ for all } u \in U_{\text{ad}}. \tag{5.31}$$

We denote by y_i^* its corresponding optimal state, i.e.,

$$y_i^* := G_{c_i}(u_i^*). \tag{5.32}$$

Analogously, the corresponding optimal control problem for the heat equation is given by

$$\min \; J(y,u) = \frac{1}{2} \left|\left|y - y_d\right|\right|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2} \left|\left|u\right|\right|_{L^2(0,T;L^2(\Omega))} \tag{5.33}$$

subject to

$$\begin{aligned} y_t - \Delta y &= u && \text{in } (0,T) \times \Omega \\ y &= 0 && \text{on } [0,T] \times \partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega \end{aligned} \tag{5.34}$$

and

$$u \in U_{\text{ad}}.$$

For the heat equation (5.34) we have the solution operator

$$G \colon U \to \hat{W}(0,T); \quad u \mapsto G(u) = y,$$

where y is the unique weak solution of (5.34). Again, we consider the reduced problem, namely

$$\min_{u \in U_{\text{ad}}} \hat{J}(u),$$

where the reduced cost functional for the heat equation is given by

$$\hat{J}(u) := J(G(u), u). \tag{5.35}$$

In, for example, [Tro10] or [HPUU08] it is shown that this problem has a unique minimizer $u^* \in U_{\text{ad}}$, i.e.,

$$\hat{J}(u^*) \leq \hat{J}(u) \text{ for all } u \in U_{\text{ad}}. \tag{5.36}$$

We define $y^* := G(u^*)$, i.e., y^* is the optimal state of the heat equation corresponding to the optimal control u^*.

In the following we show the convergence $u_i^* \rightharpoonup u^*$ in $L^2(0,T;L^2(\Omega))$ as well as $y_i^* \rightharpoonup y^*$ in $L^2(0,T;H_0^1(\Omega))$. First, we note that we have

$$\hat{J}_i(u_i^*) \leq \hat{J}_i(u^*) \text{ for all } i \in \mathbb{N}, \tag{5.37}$$

since u_i^* is the unique minimizer of $\hat{J}_i$ in U_{ad} (cf. (5.31)). We use this to get the following estimate

$$\begin{aligned}
&||u_i^*||^2_{L^2(0,T;L^2(\Omega))} && (5.38)\\
&= C\ \frac{\lambda}{2}\,||u_i^*||^2_{L^2(0,T;L^2(\Omega))}\\
&\le C\left(\frac{1}{2}\,||G_{c_i}(u_i^*) - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u_i^*||^2_{L^2(0,T;L^2(\Omega))}\right)\\
&= C\ \hat{J}_i(u_i^*) && \text{(due to (5.30))}\\
&\le C\ \hat{J}_i(u^*) && \text{(due to (5.37))}\\
&= C\left(\frac{1}{2}\,||G_{c_i}(u^*) - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u^*||^2_{L^2(0,T;L^2(\Omega))}\right). && \text{(due to (5.35))}
\end{aligned}$$

Furthermore, we get

$$\begin{aligned}
||G_{c_i}(u^*)||^2_{L^2(0,T;L^2(\Omega))} &\le ||G_{c_i}(u^*)||^2_{L^2(0,T;H_0^1(\Omega))} \qquad (5.39)\\
&\le C\left(||u^*||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right),
\end{aligned}$$

thanks to the continuous embedding $L^2(0,T;H_0^1(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$ and Lemma 5.4. This estimate also holds for all $i \in \mathbb{N}$ since the constant C is independent of τ. Applying the triangle inequality to (5.38) yields

$$\begin{aligned}
||u_i^*||^2_{L^2(0,T;L^2(\Omega))} &\le C\left(\frac{1}{2}\,||G_{c_i}(u^*) - y_d||^2_{L^2(0,T;L^2(\Omega)} + \frac{\lambda}{2}\,||u^*||^2_{L^2(0,T;L^2(\Omega))}\right) \qquad (5.40)\\
&\le C\left(||G_{c_i}(u^*)||^2_{L^2(0,T;L^2(\Omega))} + ||y_d||^2_{L^2(0,T;L^2(\Omega))} + ||u^*||^2_{L^2(0,T;L^2(\Omega))}\right)\\
&\le C\left(||u^*||^2_{L^2(0,T;L^2(\Omega))} + ||y_d||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right),
\end{aligned}$$

thanks to (5.39). Note that the constant C is still independent of τ, hence this also holds true for all $i \in \mathbb{N}$. Moreover, we get the estimate

$$\begin{aligned}
&||y_i^*||^2_{L^2(0,T;H_0^1(\Omega))} + ||(y_i^*)'||^2_{L^2(0,T;L^2(\Omega))}\\
&\le C\left(||u_i^*||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right) && \text{(due to Lemma 5.4)}\\
&\le C\left(||u^*||^2_{L^2(0,T;L^2(\Omega))} + ||y_d||^2_{L^2(0,T;L^2(\Omega))} + ||y_0||^2_{H_0^1(\Omega)} + ||y_1||^2_{L^2(\Omega)}\right). && \text{(due to (5.40))}
\end{aligned}$$

Again this holds for all $i \in \mathbb{N}$ since C is independent of τ.

Therefore, we have the boundedness of the sequence (u_i^*) in $L^2(0,T;L^2(\Omega))$ as well as the boundedness of both (y_i^*) and $((y_i^*)')$ in $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;L^2(\Omega))$, respectively. This implies that we can find subsequences $(u_{i_j}^*)$, $(y_{i_j}^*)$ and $((y_{i_j}^*)')$ such that

$$\begin{aligned}
u_{i_j}^* &\rightharpoonup \bar{u} && \text{in } L^2(0,T;L^2(\Omega)),\\
y_{i_j}^* &\rightharpoonup \bar{y} && \text{in } L^2(0,T;H_0^1(\Omega)) \text{ and}\\
(y_{i_j}^*)' &\rightharpoonup \bar{y}' && \text{in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

As before, we note that $\bar{y}'$ is indeed the weak time derivative of $\bar{y}$. Now, we aim to show that, in fact, $\bar{y} = y^*$ and $\bar{u} = u^*$ holds. Recall that we had

$$\int_0^T \tau_{i_j} \left\langle (y_{i_j}^*)'', v \right\rangle_{H^{-1},H^1} + \left\langle (y_{i_j}^*)', v \right\rangle_{L^2(\Omega)} + a[y_{i_j}^*, v] \;\mathrm{d}t = \int_0^T \left\langle u_{i_j}^*, v \right\rangle_{L^2(\Omega)} \;\mathrm{d}t \tag{5.41}$$

for all $v \in L^2(0,T; H_0^1(\Omega))$ as well as

$$y_{i_j}^*(0) = y_0 \quad \text{and} \quad (y_{i_j}^*)'(0) = y_1$$

for all j since $y_i^* = G_{c_i}(u_i^*)$. With exactly the same arguments as in Chapter 5.2 we observe that the left-hand side of (5.41) converges to

$$\int_0^T \langle \bar{y}', v \rangle_{L^2(\Omega)} + a[\bar{y}, v] \;\mathrm{d}t$$

and the right-hand side of (5.41) goes to

$$\int_0^T \langle \bar{u}, v \rangle_{L^2(\Omega)} \;\mathrm{d}t.$$

Due to the uniqueness of the limit we see that $\bar{y}$ and $\bar{u}$ satisfy

$$\int_0^T \langle \bar{y}', v \rangle_{L^2(\Omega)} + a[\bar{y}, v] \;\mathrm{d}t = \int_0^T \langle \bar{u}, v \rangle_{L^2(\Omega)}$$

for all $v \in L^2(0,T; H_0^1(\Omega))$. Similarly to Chapter 5.2 we also deduce that

$$\bar{y}(0) = y_0.$$

As in Chapter 5.2, we observe that $\bar{y} \in \hat{W}(0,T)$ is the unique weak solution of the heat equation with right-hand side $\bar{u}$, i.e.,

$$\bar{y} = G(\bar{u}).$$

In particular, we get

$$\hat{J}(\bar{u}) = J(G(\bar{u}), \bar{u}) = J(\bar{y}, \bar{u}). \tag{5.42}$$

Thanks to Theorem 5.5 we get, for a fixed u, the strong convergence

$$G_{c_i}(u) \to G(u) \text{ in } L^2(0,T; L^2(\Omega)) \text{ as } i \to \infty.$$

This implies the convergence of

$$\hat{J}_i(u) \to \hat{J}(u) \text{ as } i \to \infty, \tag{5.43}$$

again for a fixed u. The weak lower semi-continuity of J implies that it holds

$$\begin{aligned} & y_i^* \rightharpoonup \bar{y} \text{ in } L^2(0,T;H_0^1(\Omega)) \quad \text{and} \quad u_i^* \rightharpoonup \bar{u} \text{ in } L^2(0,T;L^2(\Omega)) \\ \Rightarrow \quad & J(\bar{y},\bar{u}) \le \liminf_{i\to\infty} J(y_i^*, u_i^*). \end{aligned} \tag{5.44}$$

We use this for the subsequence i_j to obtain the following estimates:

$$\begin{aligned} \hat{J}(\bar{u}) &= J(G(\bar{u}),\bar{u}) \\ &= J(\bar{y},\bar{u}) && \text{(due to (5.42))} \\ &\le \liminf_{j\to\infty} J(y_{i_j}^*, u_{i_j}^*) && \text{(due to (5.44))} \\ &= \liminf_{j\to\infty} J(G_{c_{i_j}}(u_{i_j}^*), u_{i_j}^*) && \text{(due to (5.32))} \\ &= \liminf_{j\to\infty} \hat{J}_{i_j}(u_{i_j}^*) && \text{(due to (5.30))} \\ &\le \liminf_{j\to\infty} \hat{J}_{i_j}(u^*) && \text{(due to (5.37))} \\ &= \hat{J}(u^*). && \text{(due to (5.43))} \end{aligned}$$

With this, we get the following inequality:

$$\hat{J}(\bar{u}) \le \hat{J}(u^*).$$

Recall that u^* is the unique minimizer of $\hat{J}$ in U_{ad} (cf. (5.36)). This implies that, in fact, we have $\bar{u} = u^*$ and, hence, directly $\bar{y} = G(\bar{u}) = G(u^*) = y^*$, which is what we claimed in the beginning. Due to the uniqueness of the weak time derivative we also get $\bar{y}' = (y^*)'$.

The above computations do also hold true for any subsequence and, therefore, we even get the convergence of the whole sequence due to the uniqueness of the limit, i.e.,

$$\begin{aligned} u_i^* &\rightharpoonup u^* && \text{in } L^2(0,T;L^2(\Omega)), \\ y_i^* &\rightharpoonup y^* && \text{in } L^2(0,T;H_0^1(\Omega)) \text{ and} \\ (y_i^*)' &\rightharpoonup (y^*)' && \text{in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

We apply the Aubin-Lions lemma as we did in Chapter 5.2 and deduce the strong convergence

$$y_i^* \to y^* \text{ in } L^2(0,T;L^2(\Omega)). \tag{5.45}$$

In order to derive the strong convergence of the controls u_i^* to u^*, we do the following computations. Due to the weak lower semicontinuity of the norm we have

$$||u^*||^2_{L^2(0,T;L^2(\Omega))} \le \liminf_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))} \le \limsup_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))}. \tag{5.46}$$

Furthermore, we have the estimate

$$\limsup_{i\to\infty} \hat{J}_i(u_i^*) \le \limsup_{i\to\infty} \hat{J}_i(u^*) = \hat{J}(u^*), \tag{5.47}$$

due to (5.37) and (5.43). We rewrite the left-hand side of (5.47) as

$$\begin{aligned}\limsup_{i\to\infty} \hat{J}_i(u_i^*) &= \limsup_{i\to\infty} \frac{1}{2}\,||G_{c_i}(u_i^*) - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u_i^*||^2_{L^2(0,T;L^2(\Omega))} \\ &= \limsup_{i\to\infty} \frac{1}{2}\,||y_i^* - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u_i^*||^2_{L^2(0,T;L^2(\Omega))} \qquad \text{(due to (5.32))} \\ &= \frac{1}{2}\,||y^* - y_d||^2_{L^2(0,T;L^2(\Omega))} + \limsup_{i\to\infty} \frac{\lambda}{2}\,||u_i^*||^2_{L^2(0,T;L^2(\Omega))}\,. \qquad \text{(due to (5.45))}\end{aligned}$$

The right-hand side of (5.47) is given by

$$\begin{aligned}\hat{J}(u^*) &= \frac{1}{2}\,||G(u^*) - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u^*||^2_{L^2(0,T;L^2(\Omega))} \\ &= \frac{1}{2}\,||y^* - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u^*||^2_{L^2(0,T;L^2(\Omega))}\,.\end{aligned}$$

Hence, the estimate (5.47) can be rewritten as

$$\begin{aligned}\limsup_{i\to\infty} \hat{J}_i(u_i^*) &= \frac{1}{2}\,||y^* - y_d||^2_{L^2(0,T;L^2(\Omega))} + \limsup_{i\to\infty} \frac{\lambda}{2}\,||u_i^*||^2_{L^2(0,T;L^2(\Omega))} \\ \leq \hat{J}(u^*) &= \frac{1}{2}\,||y^* - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u^*||^2_{L^2(0,T;L^2(\Omega))}\,.\end{aligned}$$

Subtracting $\frac{1}{2}\,||y^* - y_d||^2_{L^2(0,T;L^2(\Omega))}$ from this inequality reveals

$$\limsup_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))} \leq ||u^*||^2_{L^2(0,T;L^2(\Omega))}\,,$$

since $\lambda > 0$. With this available, we extend the estimate (5.46) to

$$\begin{aligned}||u^*||^2_{L^2(0,T;L^2(\Omega))} &\leq \liminf_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))} \\ &\leq \limsup_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))} \leq ||u^*||^2_{L^2(0,T;L^2(\Omega))}\,.\end{aligned} \tag{5.48}$$

But this means that all inequalities in (5.48) are actually equalities, in particular, we get

$$\liminf_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))} = \limsup_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))} = ||u^*||^2_{L^2(0,T;L^2(\Omega))}\,.$$

This implies that the sequence of real numbers $\left(||u_i^*||^2_{L^2(0,T;L^2(\Omega))}\right) \subset \mathbb{R}$ converges to $||u^*||^2_{L^2(0,T;L^2(\Omega))}$, i.e.,

$$\lim_{i\to\infty} ||u_i^*||^2_{L^2(0,T;L^2(\Omega))} = ||u^*||^2_{L^2(0,T;L^2(\Omega))}\,. \tag{5.49}$$

Using the weak convergence $u_i^* \rightharpoonup u^*$ in $L^2(0,T;L^2(\Omega))$ together with the convergence of the norms given in (5.49) reveals

$$\begin{aligned}
&\lim_{i\to\infty} ||u_i^* - u^*||^2_{L^2(0,T;L^2(\Omega))} \\
&= \lim_{i\to\infty} \langle u_i^* - u^*, u_i^* - u^* \rangle_{L^2(0,T;L^2(\Omega))} \\
&= \lim_{i\to\infty} \left(\langle u_i^*, u_i^* \rangle_{L^2(0,T;L^2(\Omega))} - 2\, \langle u_i^*, u^* \rangle_{L^2(0,T;L^2(\Omega))} + \langle u^*, u^* \rangle_{L^2(0,T;L^2(\Omega))} \right) \\
&= \lim_{i\to\infty} \left(||u_i^*||^2_{L^2(0,T;L^2(\Omega))} - 2\, \langle u_i^*, u^* \rangle_{L^2(0,T;L^2(\Omega))} + ||u^*||^2_{L^2(0,T;L^2(\Omega))} \right) \\
&= 2\, ||u^*||^2_{L^2(0,T;L^2(\Omega))} - 2\, \langle u^*, u^* \rangle_{L^2(0,T;L^2(\Omega))} \\
&= 0,
\end{aligned}$$

which implies the strong convergence $u_i^* \to u^*$ in $L^2(0,T;L^2(\Omega))$.

Let us conclude the chapter with a summary of our results.

Theorem 5.6
Let $U_{\text{ad}} \subseteq U = L^2(0,T;L^2(\Omega))$ be a non-empty, weakly closed and convex subset of the control space $U = L^2(0,T;L^2(\Omega))$ and let $(\tau_i) \subset \mathbb{R}$ be a sequence with $0 < \tau_i \leq 1$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \tau_i = 0$. Furthermore, using the same notation as before, for $i \in \mathbb{N}$ we denote by $u_i^* \in U_{\text{ad}}$ the unique minimizer of problem (5.28) with corresponding optimal state y_i^* and its weak time derivative $(y_i^*)'$. Additionally, let u^* be the unique minimizer of problem (5.33) with corresponding optimal state y^* and weak time derivative $(y^*)'$.

Then, the sequence (y_i^*) converges strongly to y^* in $L^2(0,T;L^2(\Omega))$ and weakly to y^* in $L^2(0,T;H_0^1(\Omega))$. Moreover, we have that the sequence (u_i^*) converges strongly to u^* in $L^2(0,T;L^2(\Omega))$ and the sequence $((y_i^*)')$ converges weakly to $(y^*)'$ in $L^2(0,T;L^2(\Omega))$.

Proof: This is done in the computations above. □

Therefore, we have proved the (strong) convergence of solutions of the optimal control problem. However, a more thorough analysis investigating, for example, convergence rates would go beyond the scope of this thesis. Therefore, we investigate this convergence behaviour numerically in the next chapter.

6 Numerical Results

Let us now take a look at the simulation of both the Cattaneo and the heat equation. We have implemented our simulations in Python with the help of FEniCS (cf. [LMW+12]). In the following, we briefly present our implementation. Afterwards, we consider the forward simulation of the models and investigate the differences between the Cattaneo and heat equation. In the end we discuss the optimal control problem where we, as for the forward problem, also investigate the limit $\tau \to 0$.

6.1 The Forward Problem

In this chapter, we examine a forward problem for the Cattaneo equation and compare the behaviour of its solution to the one of the corresponding heat equation. We discuss the similarities and differences of the two models and also take a look at the asymptotic behaviour of the Cattaneo equation for $\tau \to 0$.

For the numerical simulation of the Cattaneo equation we use the Newmark beta method (cf. [New59]) with parameters $\beta = \gamma = 1/2$ as semi-discretization in time and solve the resulting sequence of PDEs with the help of FEniCS. For the simulation of the heat equation we use the Crank-Nicolson method for the time discretization and, again, FEniCS in order to solve the resulting sequence of PDEs. Fore a more detailed description of the finite element method we refer to, for example, [Bra13] or [GRS07]. More details on the Newmark method can be found in [New59] and [RR12].

Recall that the Cattaneo equation with homogeneous Dirichlet boundary conditions is given by

$$\begin{aligned} \tau y_{tt} + y_t - \Delta y &= u && \text{in } (0,T) \times \Omega \\ y &= 0 && \text{on } [0,T] \times \partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega \\ y_t(0,\cdot) &= y_1 && \text{in } \Omega, \end{aligned} \tag{6.1}$$

and that the heat equation with the same boundary conditions reads

$$\begin{aligned} y_t - \Delta y &= u && \text{in } (0,T) \times \Omega \\ y &= 0 && \text{on } [0,T] \times \partial\Omega \\ y(0,\cdot) &= y_0 && \text{in } \Omega. \end{aligned} \tag{6.2}$$

We restrict our numerical analysis to the two-dimensional case since this is easier to visualize. In particular, we choose our domain as the unit square, that is, $\Omega = [0,1]^2$, and the end-time of the simulation as $T = 1.0$, for simplicity. For our model problem we assume that we have no heat sources, that is, $u = 0$ in $[0,T] \times \Omega$. With this in mind we

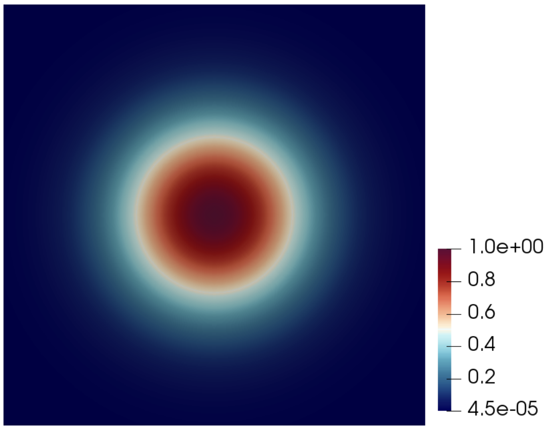


Figure 1: Initial condition for the forward problem.

want to consider the behaviour of the models for a Gaussian pulse as initial condition, that is,

$$y_0(x) = \exp\left(-20\left(\left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 - \frac{1}{2}\right)^2\right)\right).$$

This initial condition corresponds to an (unnormalized) bivariate Gaussian distribution with mean $[1/2, 1/2]^T$ and variance $\sigma_x^2 = \sigma_y^2 = 1/40$. The small variance implies that the Gaussian is rather localized around the center point $[1/2, 1/2]^T$. A plot of this initial condition created with the help of Paraview is shown in Figure 1.

Also note that we have to specify a second initial condition for the Cattaneo equation. In principle, we can choose any function $y_1 \in L^2(\Omega)$. However, we expect that the convergence of the solution is better if we choose an initial condition that is compatible with the heat equation in the following sense: Formally, we could assume that the heat equation is also valid for $t = 0$, that is,

$$y_t(0, x) = \Delta y(0, x) + u(0, x) = \Delta y_0(x) + u(0, x).$$

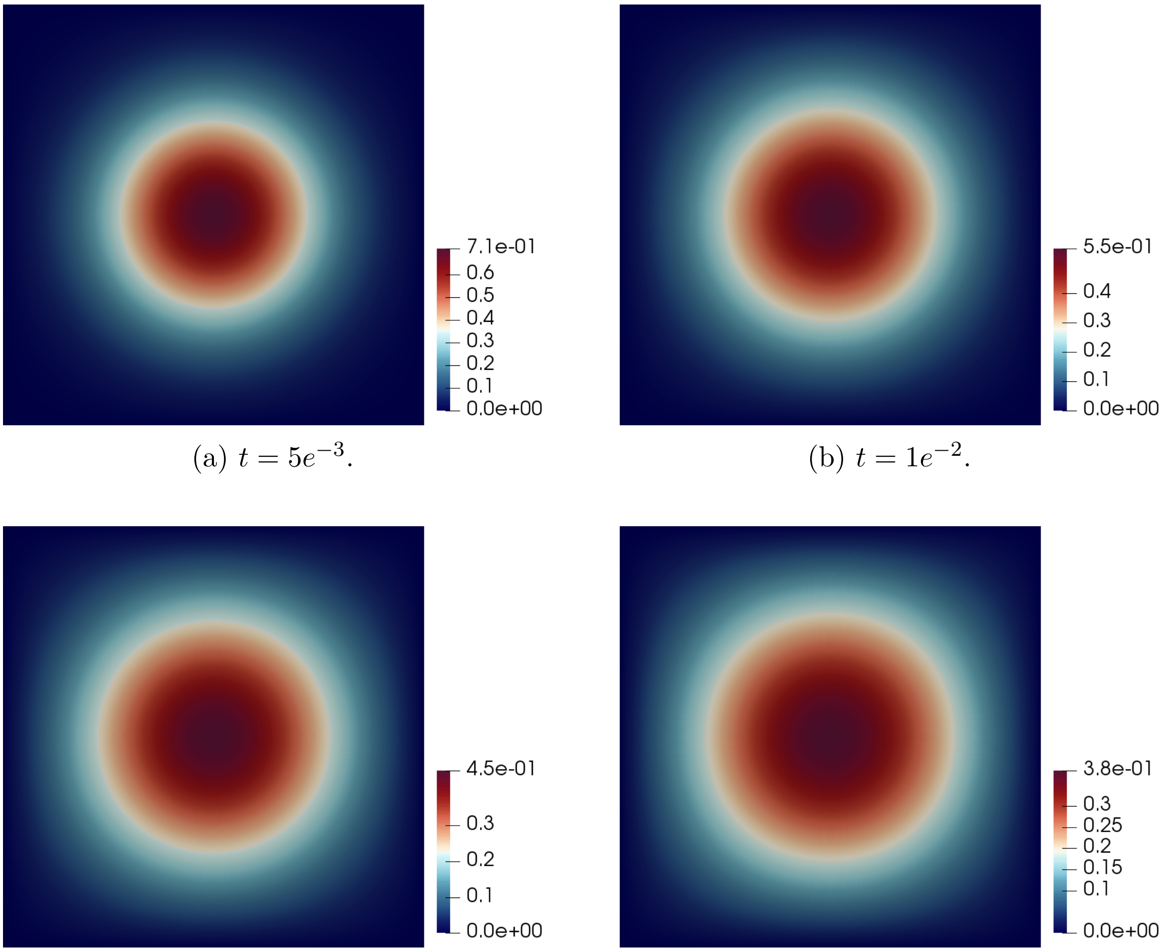
This motivates the choice

$$y_1(x) = y_t(0, x) = \Delta y_0(x) + u(0, x) \tag{6.3}$$

for the second initial condition of the Cattaneo equation. If we choose a different initial velocity we expect an initial layer behaviour of the solution. Therefore, we choose y_1 as motivated in (6.3) which yields a stable numerical scheme.

In order to find an appropriate time discretization we did a convergence analysis which revealed that the numerical solution showed no significant differences for a time step $\Delta t \leq \tau/10$.

Since the smallest τ we consider in this analysis is $\tau = 1e^{-3}$, we choose a step size of $\Delta t = 1e^{-4}$, as suggested by our convergence analysis, in order to simulate both the



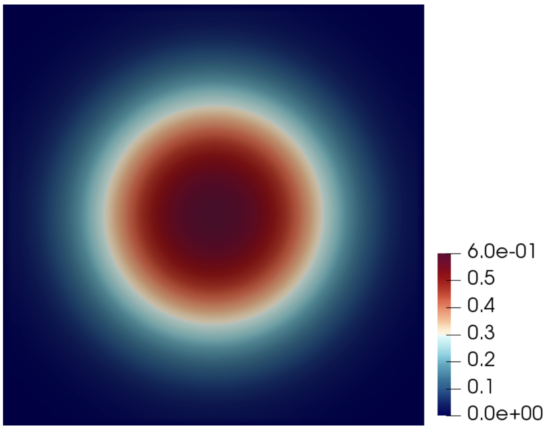
(a) $t = 5e^{-3}$. (b) $t = 1e^{-2}$.

(c) $t = 1.5e^{-2}$. (d) $t = 2e^{-2}$.

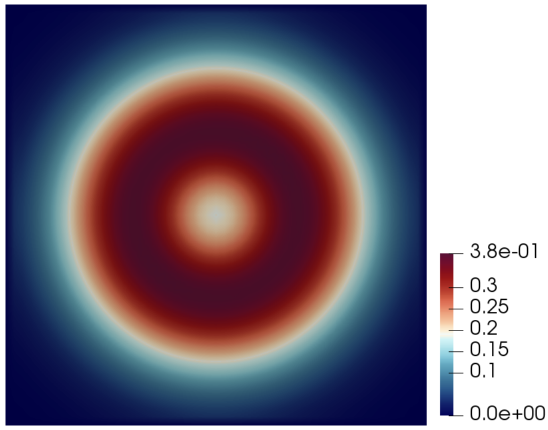
Figure 2: Initial behaviour of the classical heat equation (6.2).

Cattaneo and heat equations. For the spatial discretization we use a mesh consisting of 2601 degrees of freedom in a grid consisting of 5000 triangles. In order to solve the semi-discretized PDEs we use linear Lagrangian finite elements in FEniCS. Furthermore, we note that the spatial discretization can be chosen more or less independently of the choice of τ, that is, we did not obtain noticeably different results when we used more degrees of freedom after having a sufficiently fine spatial discretization.

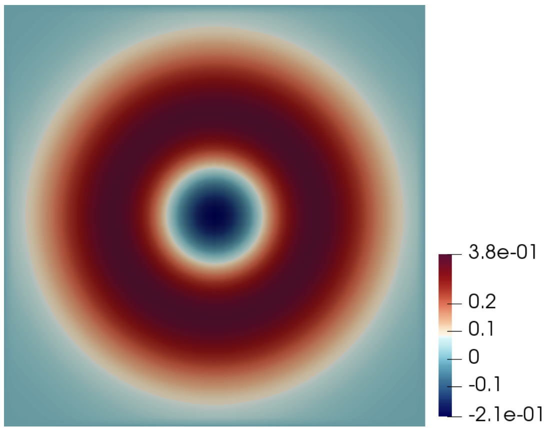
Let us start with discussing the initial behaviour of the models. In order to do so, we investigate the solutions up to $t = 2.0e^{-2}$. Note that due to the rapid decay of the solutions we have rescaled our plots such that the main features of the solutions are well visible. In Figure 2 the solution of the heat equation is shown in this initial phase. We observe that it behaves as one would expect, that is, it takes the initial condition and diffuses it: The Gaussian spreads outwards and decays exponentially, which can be seen even better later on. Therefore, the solution of the heat equation also resembles a decaying Gaussian with increasing variance, a behaviour that is closely related to the fundamental solution of the heat equation (cf. [Eva10, Chapter 2.3]). Last, we note that the maximum of the



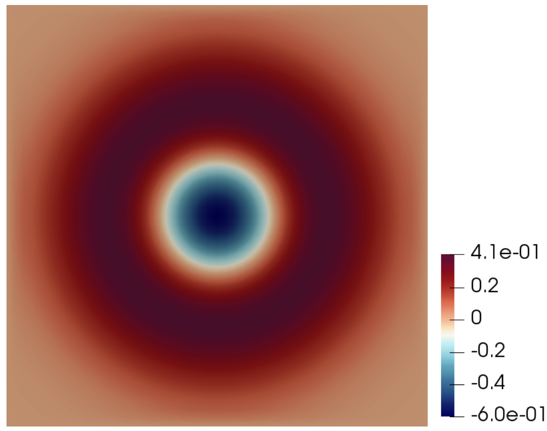
(a) $t = 0.5e^{-2}$.



(b) $t = 1.0e^{-2}$.



(c) $t = 1.5e^{-2}$.



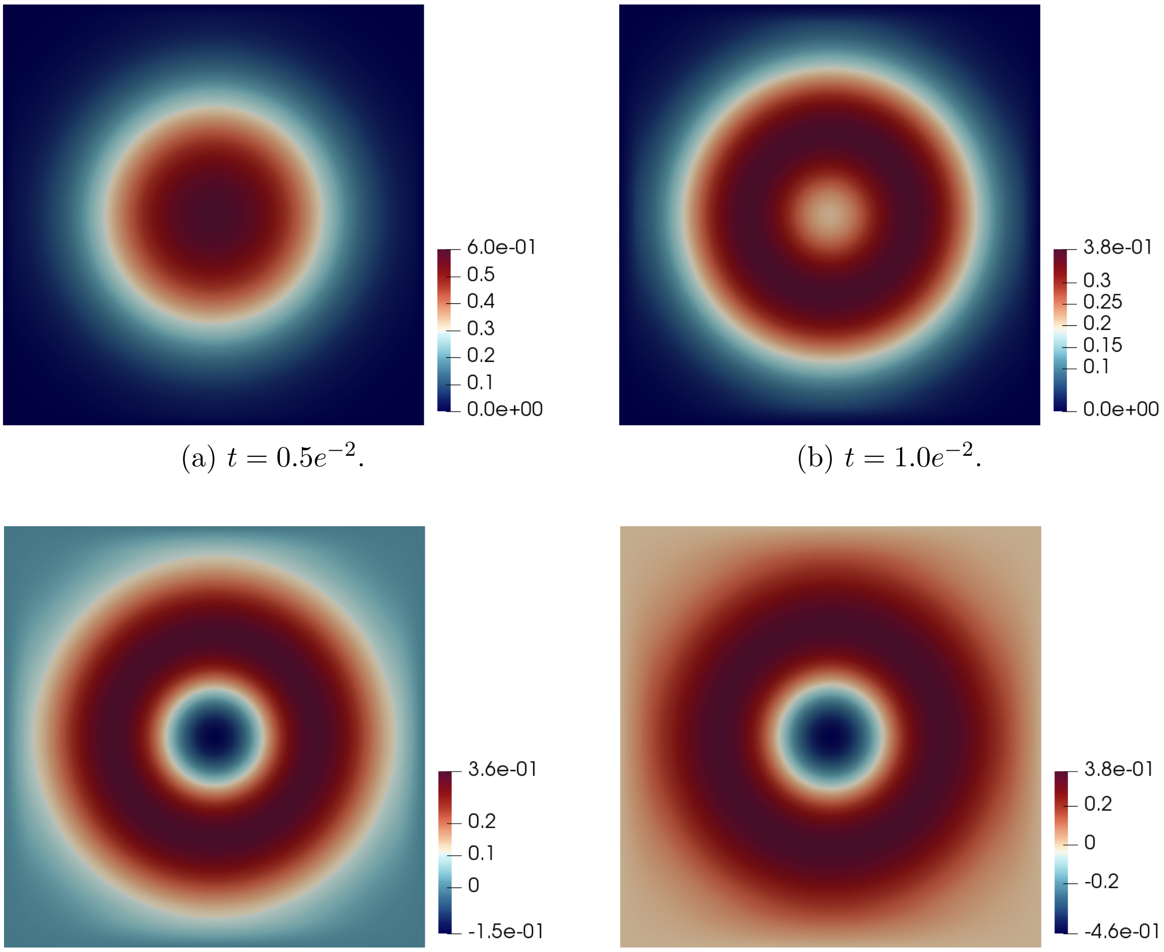
(d) $t = 2.0e^{-2}$.

Figure 3: Initial behaviour of the Cattaneo equation (6.1) for $\tau = 1.0$.

solution is always attained at the center point of the unit square and that the solution is always positive.

This situation changes drastically for the solution of the Cattaneo equation with $\tau = 1.0$ which is shown in Figure 3: Here, we also observe that the solution spreads out in the very beginning. Starting with $t = 1.0e^{-2}$, however, we also observe waves emerging from the center point of the unit square and travelling outwards. Additionally, we see that these waves give rise to oscillations, which are not present in the solution of the heat equation. Furthermore, we see that the solution becomes negative for $t = 1.5e^{-2}$ and $t = 2.0e^{-2}$, due to the aforementioned oscillations. This can be interpreted as some kind of inertia a solution of the (damped) wave equation would also show. Later on we also observe that the behaviour of the oscillations becomes worse. This is due to the fact that the wave-fronts reach the boundary and are reflected there.

A similar behaviour can also be seen for $\tau = 1e^{-1}$. The solution of the Cattaneo equation for this choice of τ is shown in Figure 4. As for $\tau = 1.0$, we see that the Cattaneo



(a) $t = 0.5e^{-2}$.

(b) $t = 1.0e^{-2}$.

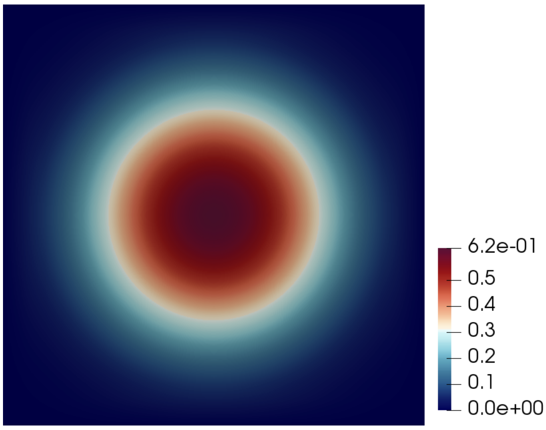
(c) $t = 1.5e^{-2}$.

(d) $t = 2.0e^{-2}$.

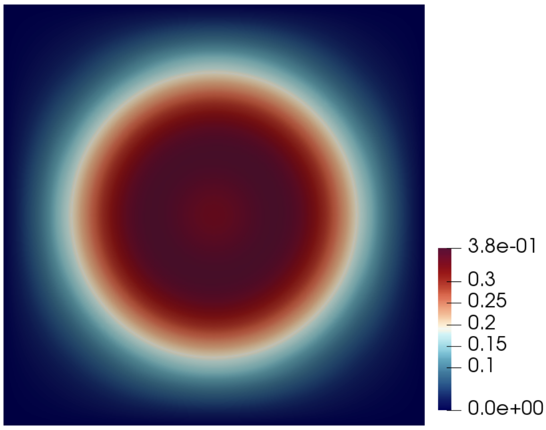
Figure 4: Initial behaviour of the Cattaneo equation (6.1) for $\tau = 1e^{-1}$.

equation gives rise to spherical waves that travel outwards. However, we can observe a slightly stronger damping of the solution, that is, the maximum (minimum) values of the solution for $\tau = 1e^{-1}$ are smaller (larger) than the ones of the solution for $\tau = 1.0$. However, we still observe that the solution of the Cattaneo equation becomes negative at $t = 1.5e^{-2}$ and, as before, the maximum of the solution is not attained at the center point at all times, in contrast to the solution of the heat equation. On the contrary, the solution of the Cattaneo equation has a global minimum at the center point for $t = 1.5e^{-2}$ and $t = 2.0e^{-2}$. Although the solutions of the Cattaneo equation for $\tau = 1.0$ and $\tau = 1e^{-1}$ look very similar initially, we see that the damping is way stronger in the latter case later on.

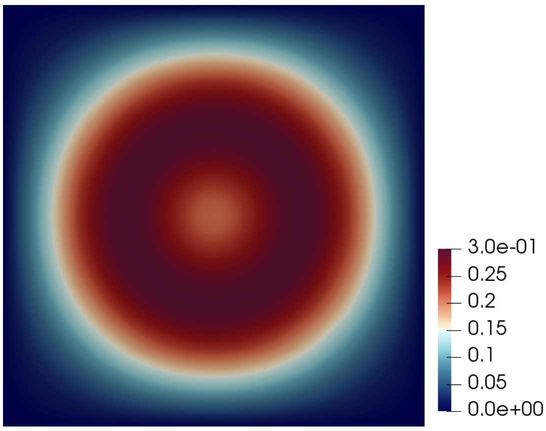
The solution of the Cattaneo equation for $\tau = 1e^{-2}$ is depicted in Figure 5. We directly observe that its behaviour is closer to the one of the heat equation. As before, the solution decays while moving outwards, this time in a more diffusive manner. Also, it does not take on negative values anymore. However, we see that the maximum of the solution is, again, not always attained at the center: Starting with $t = 1.0e^{-2}$ we observe that the solution



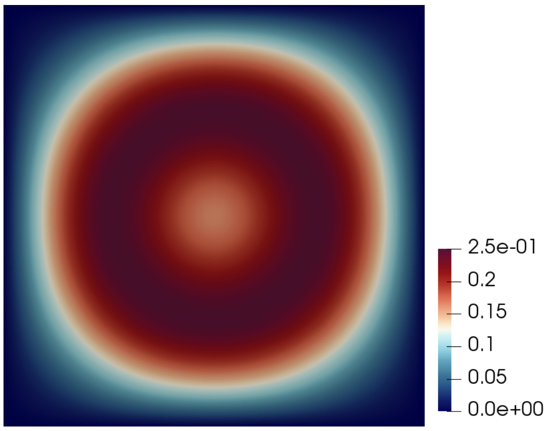
(a) $t = 0.5e^{-2}$.



(b) $t = 1.0e^{-2}$.



(c) $t = 1.5e^{-2}$.

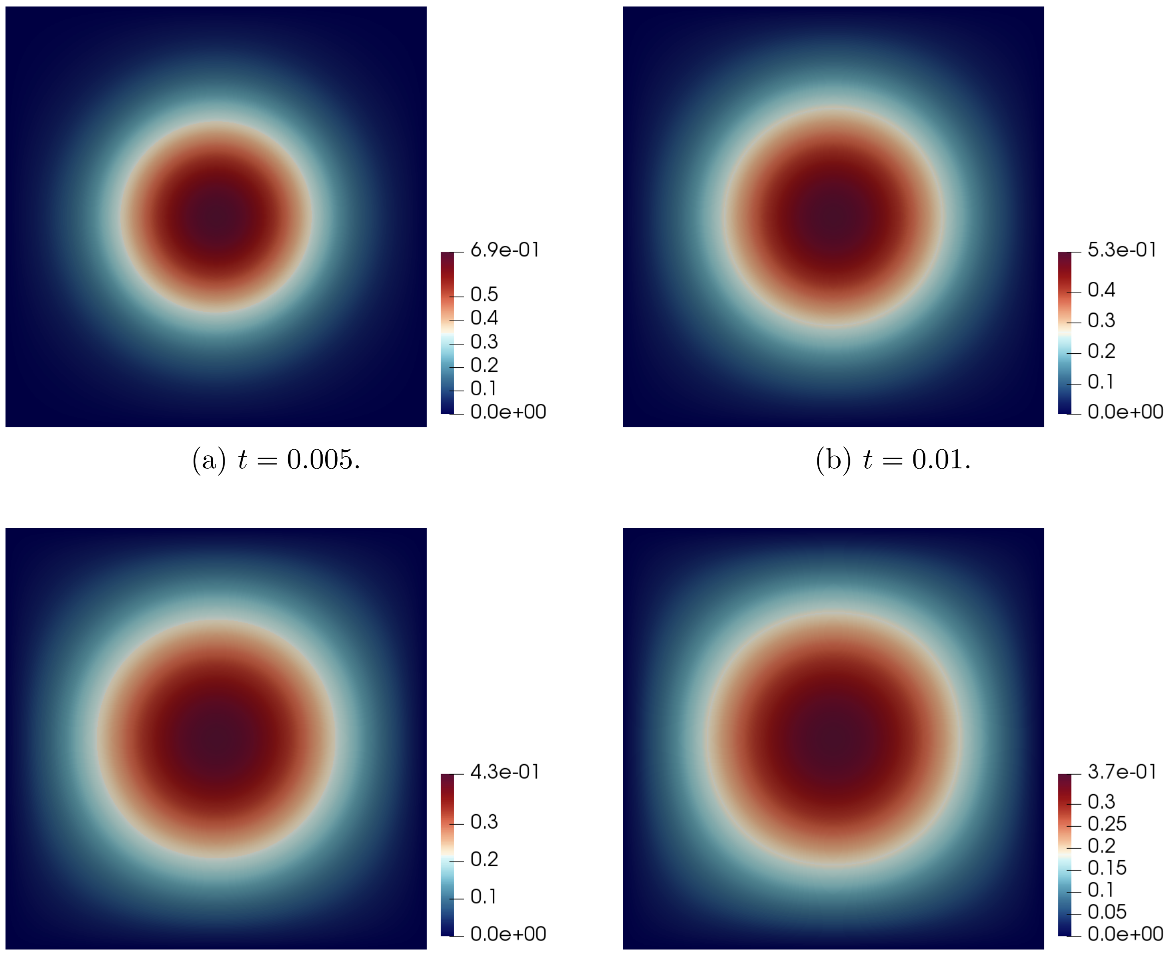


(d) $t = 2.0e^{-2}$.

Figure 5: Initial behaviour of the Cattaneo equation (6.1) for $\tau = 1e^{-2}$.

has a local minimum at this point, in contrast to the solution of the heat equation, that attains a global maximum there. Also note that the decay of this solution in the initial phase is even stronger than the decay of the heat equation: Here, the maximum value at $t = 2.0e^{-2}$ is given by 0.25, whereas we the global maximum of the solution of the heat equation at the same time has a value of 0.38. Furthermore, note that the shape of the peak at $t = 2.0e^{-2}$ is closer to a square than to a circle, which it was for the heat equation. This comes from the fact that the solution approaches the boundary of Ω more quickly than before, allowing it to "move" into the corners of our domain.

Finally, let us consider the Cattaneo equation for $\tau = 1e^{-3}$. Its solution in the initial phase is shown in Figure 6. We directly observe that this solution is the closest to the one of the heat equation: The initial condition is diffused outwards, the solution is always positive and it attains its global maximum at the center of the unit square. Apart from the slightly different numerical values of the maxima, the solution of the Cattaneo equation for $\tau = 1e^{-3}$ and the solution of the heat equation look very similar initially and can barely be distinguished qualitatively.



(a) $t = 0.005$.

(b) $t = 0.01$.

(c) $t = 0.015$.

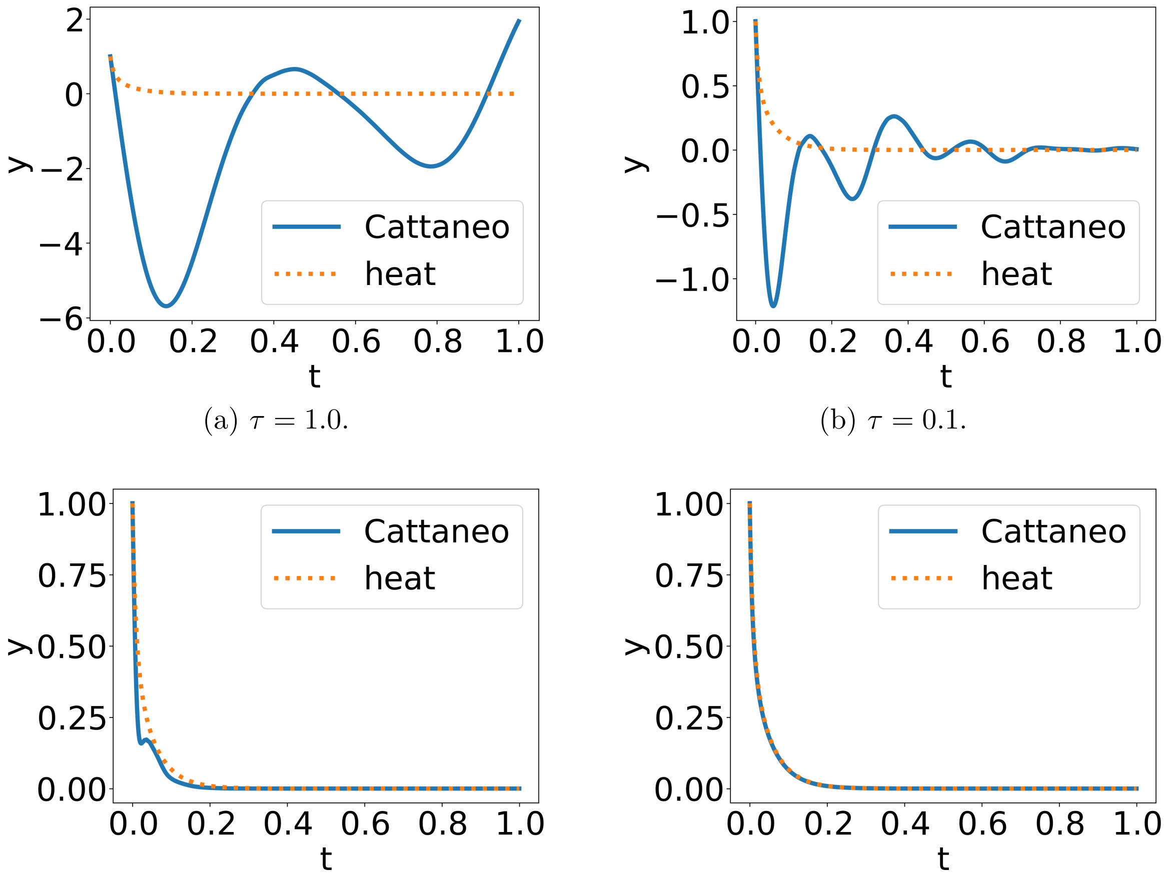
(d) $t = 0.02$.

Figure 6: Initial behaviour of the Cattaneo equation (6.1) for $\tau = 1e^{-3}$.

Now that we have investigated the initial behaviour of the solutions we take a look at their global behaviour over time. We do so by investigating the solutions at a fixed point over the whole time interval.

Let us begin with the center point $[1/2, 1/2]^T$. The solutions of the Cattaneo equation for different choices of τ compared to the solution of the heat equation at this point are shown in Figure 7. Note that the solution of the heat equation at the center point only shows an exponential decay starting immediately.

The solution of the Cattaneo equation for $\tau = 1.0$ (cf. Figure 7a) behaves completely different since it shows oscillations, which we have already seen in our previous investigation. These oscillations seem to be damped, which could, in fact, also be seen for a larger time horizon. Note that the largest absolute value of the solution of the Cattaneo equation at the center point is about 6, which is way larger than the maximum absolute value of the initial condition, namely 1. This comes from the aforementioned choice of the initial velocity for the Cattaneo equation (cf. (6.3)), which produces this “overshoot”.



(a) $\tau = 1.0$. (b) $\tau = 0.1$.

(c) $\tau = 1e^{-2}$. (d) $\tau = 1e^{-3}$.

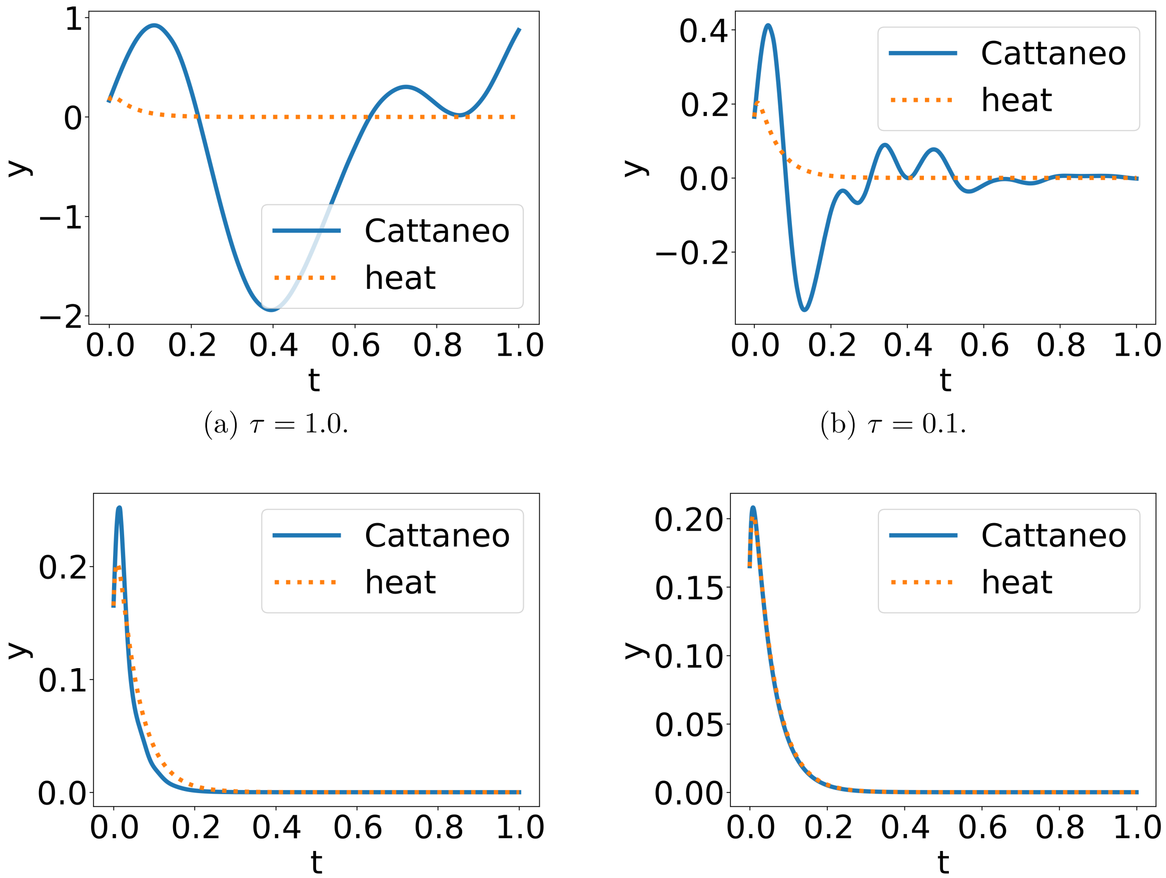
Figure 7: Solutions of Cattaneo and heat equation at $[1/2, 1/2]^T$.

Also note that the solution gets some additional oscillations due to the interaction with the boundary: There, the wave-fronts are reflected and then superimposed with the ones coming from the center. These oscillations also lead to the solution taking on negative values, in contrast to the solution of the heat equation which is always positive.

The damping of the solution can be seen way better for $\tau = 1e^{-1}$, which is shown in Figure 7b. As before, the different heights of the maxima and minima are due to the initial velocity and the reflections from the boundary. Additionally, the long term behaviour of this solution indicates that it also decays to zero after the initial oscillations are damped out.

For $\tau = 1e^{-2}$ and $\tau = 1e^{-3}$ the solutions at the center are shown in Figure 7c and Figure 7d, respectively. The solutions for these values of τ agree pretty well with the one obtained for the heat equation. The only (visible) difference takes place at $t \approx 2e^{-2}$ in Figure 7c in the form of a small oscillation, and there is no visible difference in the solutions for $\tau = 1e^{-3}$.

Let us also investigate a second point, namely $[4/5, 1/2]^T$. This point is rather close to the boundary such that we can investigate the transport effects of the models. The solutions



(a) $\tau = 1.0$. (b) $\tau = 0.1$. (c) $\tau = 1e^{-2}$. (d) $\tau = 1e^{-3}$.

Figure 8: Solutions of Cattaneo and heat equation at $[4/5, 1/2]^T$.

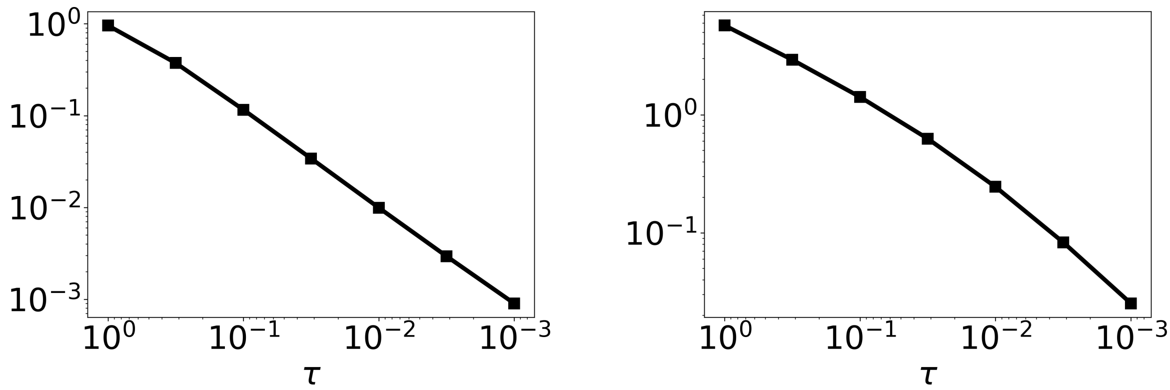
at this point are shown in Figure 8. Here, we see that the solution of the heat equation increases slightly in the initial phase before it, again, starts to decay exponentially.

As for the center point, the solution of the Cattaneo equation for $\tau = 1.0$, depicted in Figure 8a, shows large oscillations together with "overshoots" that come from the initial velocity and the reflections of the solution at the boundary. Investigating the case $\tau = 1e^{-1}$ we observe that the oscillations are damped stronger and that the solution decays to zero for $t \to 1$. For these first two choices of τ we also observe that the first maximum of the solution is shifted to the right compared to the (only) maximum of the solution of the heat equation. This can be seen as one of the delay effects the Cattaneo equation shows (cf. Chapter 2).

For $\tau = 1e^{-2}$ the qualitative behaviour of the Cattaneo model is the same as the one of the heat equation: It increases in the very beginning and then the exponential decay starts. The height of the maximum and the slightly different rate of decay are the only visible differences. As before, the solutions of the Cattaneo equation get closer to the one of the heat equation and there is no visible difference for $\tau = 1e^{-3}$ anymore.

τ	$\|\|y_c - y_h\|\|_{L^2}$	$\|\|y_c-y_h\|\|_{L^2} / \|\|y_h\|\|_{L^2}$	$\|\|y_c - y_h\|\|_{L^\infty}$	$\|\|y_c-y_h\|\|_{L^\infty} / \|\|y_h\|\|_{L^\infty}$
1.0	$9.57e^{-1}$	2384.6 %	5.72	572.4 %
$1e^{-1/2}$	$3.74e^{-1}$	931.3 %	2.93	292.7 %
$1e^{-1}$	$1.16e^{-1}$	288.8 %	1.42	141.7 %
$1e^{-3/2}$	$3.41e^{-2}$	85.0 %	$6.28e^{-1}$	62.8 %
$1e^{-2}$	$9.93e^{-3}$	24.74 %	$2.45e^{-1}$	24.54 %
$1e^{-5/2}$	$2.95e^{-3}$	7.34 %	$8.31e^{-2}$	8.31 %
$1e^{-3}$	$9.02e^{-4}$	2.25 %	$2.52e^{-2}$	2.52 %

Table 1: Absolute and relative difference of the solutions in the L^2 and L^∞ norms.



(a) $||y_c - y_h||_{L^2}$. (b) $||y_c - y_h||_{L^\infty}$.

Figure 9: Convergence rates for the Cattaneo equation.

Let us now take a look at both the absolute and relative difference of the solutions of the Cattaneo equation and the heat equation in the $L^2(0,T;L^2(\Omega))$ and the $L^\infty(0,T;L^\infty(\Omega))$ norm which are given in Table 1 and Figure 9. Table 1 shows both the absolute and relative differences computed in the L^2 and L^∞ norms, and Figure 9 shows the absolute difference of the solutions in both norms in a log-log-plot. Note that we denote with y_h and y_c the numerical solutions of the heat and Cattaneo equation, respectively.

The results depicted there show a similar picture to our previous investigation. The solution of the Cattaneo equation is "far away" from the one of the heat equation for large τ, for example, $\tau = 1.0$ and $\tau = 1e^{-1}$. However, for small τ we get the convergence of the solutions in both norms. Additionally, the plots in Figure 9 together with the numerical values computed in Table 1 suggest that we have, in fact, a linear convergence rate in the $L^2(0,T;L^2(\Omega))$ norm for this example.

6.2 The Optimal Control Problem

Now that we have investigated the convergence of the Cattaneo model to the classical heat equation for the forward problem, we take a look at the simulation of the optimal control problem introduced in Chapter 4 for the Cattaneo equation as well as its asymptotic behaviour.

Recall that the optimal control problem for the Cattaneo equation is given by

$$\min J(y,u) = \frac{1}{2}\,||y - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u||^2_{L^2(0,T;L^2(\Omega))} \tag{6.4}$$

subject to

$$\begin{aligned} \tau y_{tt} + y_t - \Delta y &= u && \text{in } (0,T)\times\Omega \\ y &= 0 && \text{on } [0,T]\times\partial\Omega \\ y(0,x) &= y_0(x) && \text{in } \Omega \\ y_t(0,x) &= y_1(x) && \text{in } \Omega \end{aligned}$$

and

$$u \in U_{\text{ad}},$$

and that the corresponding optimization problem with the heat equation reads

$$\min J(y,u) = \frac{1}{2}\,||y - y_d||^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\,||u||^2_{L^2(0,T;L^2(\Omega))} \tag{6.5}$$

subject to

$$\begin{aligned} y_t - \Delta y &= u && \text{in } (0,T)\times\Omega \\ y &= 0 && \text{on } [0,T]\times\partial\Omega \\ y(0,x) &= y_0(x) && \text{in } \Omega \end{aligned}$$

and

$$u \in U_{\text{ad}},$$

where we again treat the embedding implicitly. As for the forward problem, we consider the domain $\Omega = [0,1]^2$ and the time horizon $T = 1.0$ for simplicity. Additionally, we choose the initial conditions $y_0 = y_1 = 0$. Furthermore, we choose the set of admissible controls $U_{\text{ad}} = U = L^2(0,T;L^2(\Omega))$, that is, we have no control constraints. As desired state y_d we choose the function

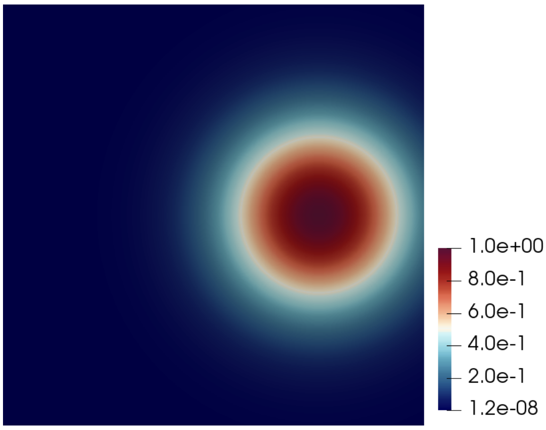
$$y_d(t,x) = \exp\left(-20\left((x_1 - \delta_1(t))^2 + (x_2 - \delta_2(t))^2\right)\right),$$

where

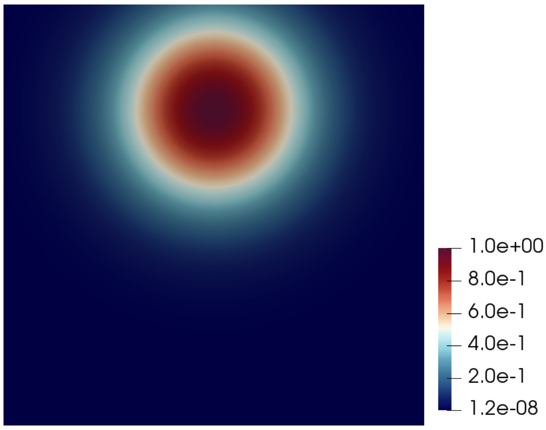
$$\begin{bmatrix}\delta_1(t)\\ \delta_2(t)\end{bmatrix} = \begin{bmatrix}1/2 + 1/4\cos(2\pi t)\\ 1/2 + 1/4\sin(2\pi t)\end{bmatrix}. \tag{6.6}$$

Note that these equations describe a Gaussian pulse with center $[\delta_1(t), \delta_2(t)]^T$ that moves counterclockwise along a circle with midpoint $[1/2, 1/2]^T$ and radius $1/4$, starting at $[3/4, 1/2]^T$ for $t = 0$. A plot of the desired state showing the above mentioned properties is shown in Figure 10. Note that this optimal control problem is actually not as easy as it looks at first glance: First, we prescribe the initial condition $y(0) = y_0 = 0$, which is not compatible with the desired state since $y_d(0) \neq y_0 = 0$. This means that we expect that the solution takes some time until it can reach the desired state. Secondly, we use homogeneous Dirichlet boundary conditions for the PDEs. The desired state y_d, however, does not fulfill this condition. It is close to zero at the parts of the boundary that are distant from the Gaussian, but it is not zero where the Gaussian is close to the boundary. Again, this means that we expect the optimal state to behave differently than the desired state due to the boundary conditions.

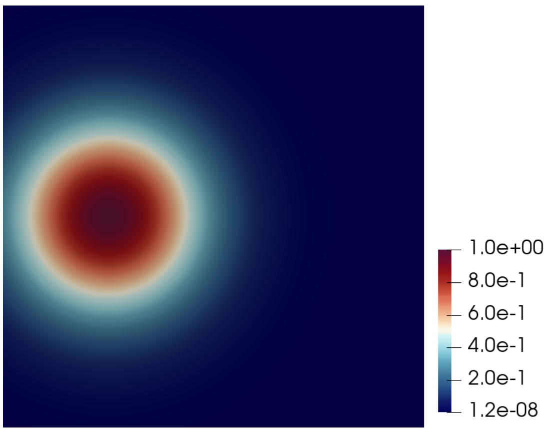
In order to solve this optimization problem numerically, we use the gradient descent method with exact line search since we have no control constraints for our model problem (cf. [HPUU08] or [SHFP17]). Note that this speeds up our computations since we only



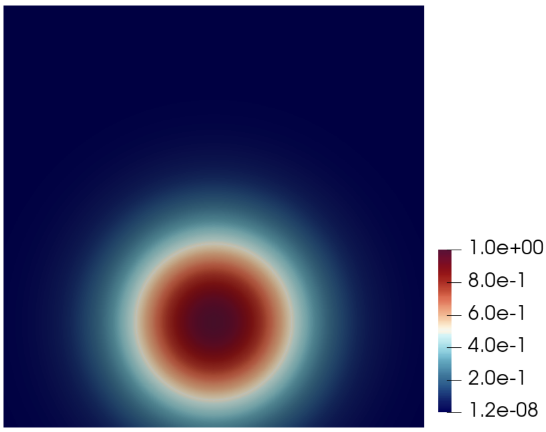
(a) $t = 0$ and $t = 1$.



(b) $t = 0.25$.



(c) $t = 0.5$.



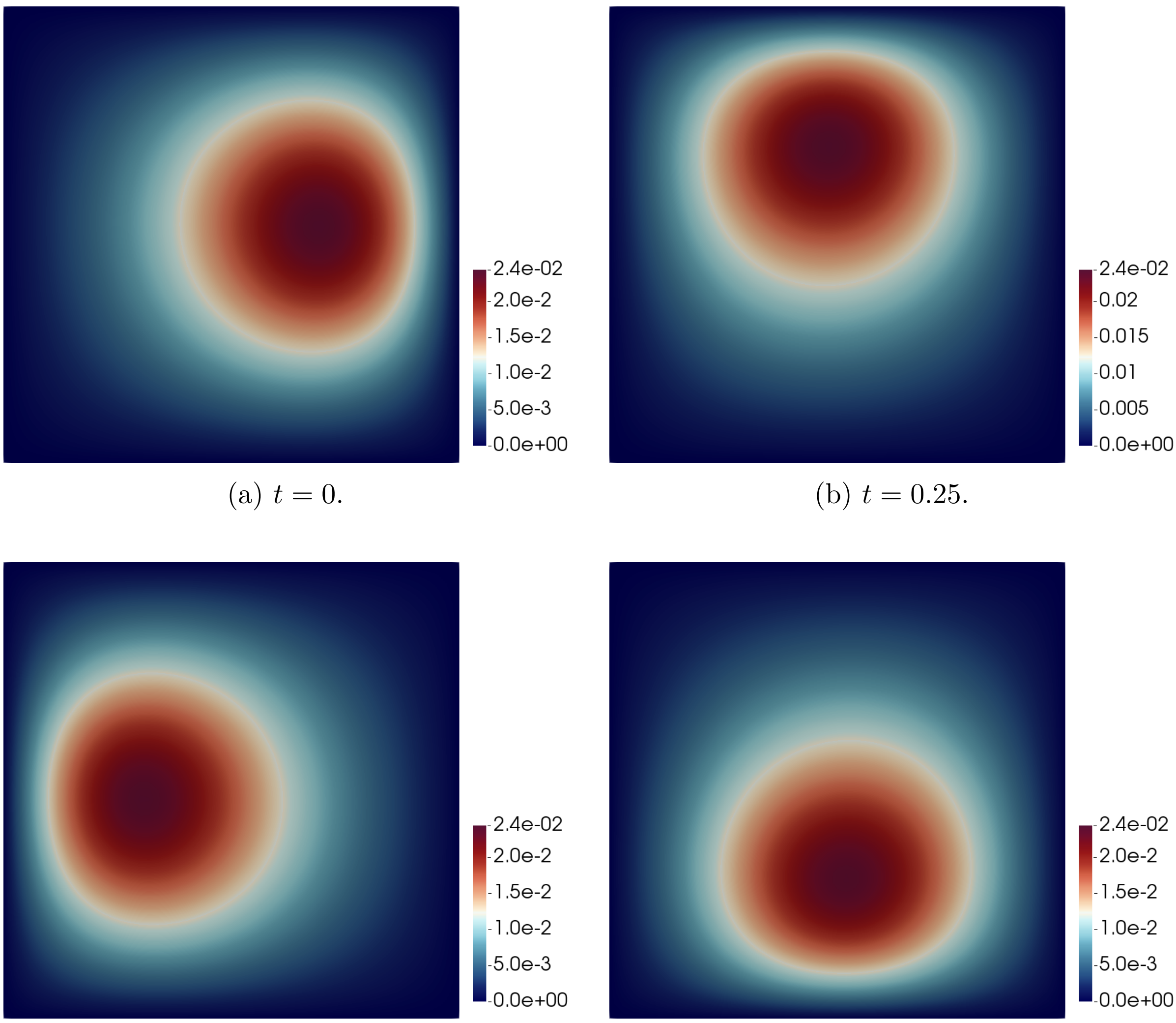
(d) $t = 0.75$.

Figure 10: The desired state y_d at several points in time.

have to solve the forward equation once more (with a different right-hand side) in order to compute the step size into the descent direction, whereas we usually have to do many more forward solves in order to compute a fitting step size when using, for example, the Armijo rule. Furthermore, we use the initial guess $u^0 = 0$ in $[0, T] \times \Omega$ as well as the relative stopping criterion

$$\left|\left|\hat{J}'(u^k)\right|\right|_U \leq \text{TOL} \left|\left|\hat{J}'(u^0)\right|\right|_U$$

for the gradient descent method, where the reduced cost functional $\hat{J}$ is defined as in Chapters 4 and 5. In particular, we chose a tolerance of $\text{TOL} = 1e^{-4}$ for our computations in this chapter. Note that we compute the gradient of the reduced cost functional as discussed in Chapter 4, that is, we solve first the state equation and then the adjoint equation in order to compute the gradient. More details on the numerical treatment of PDE constrained optimization problems can be found, for example, in [Tro10] or [HPUU08], and a



(a) $t = 0$.

(b) $t = 0.25$.

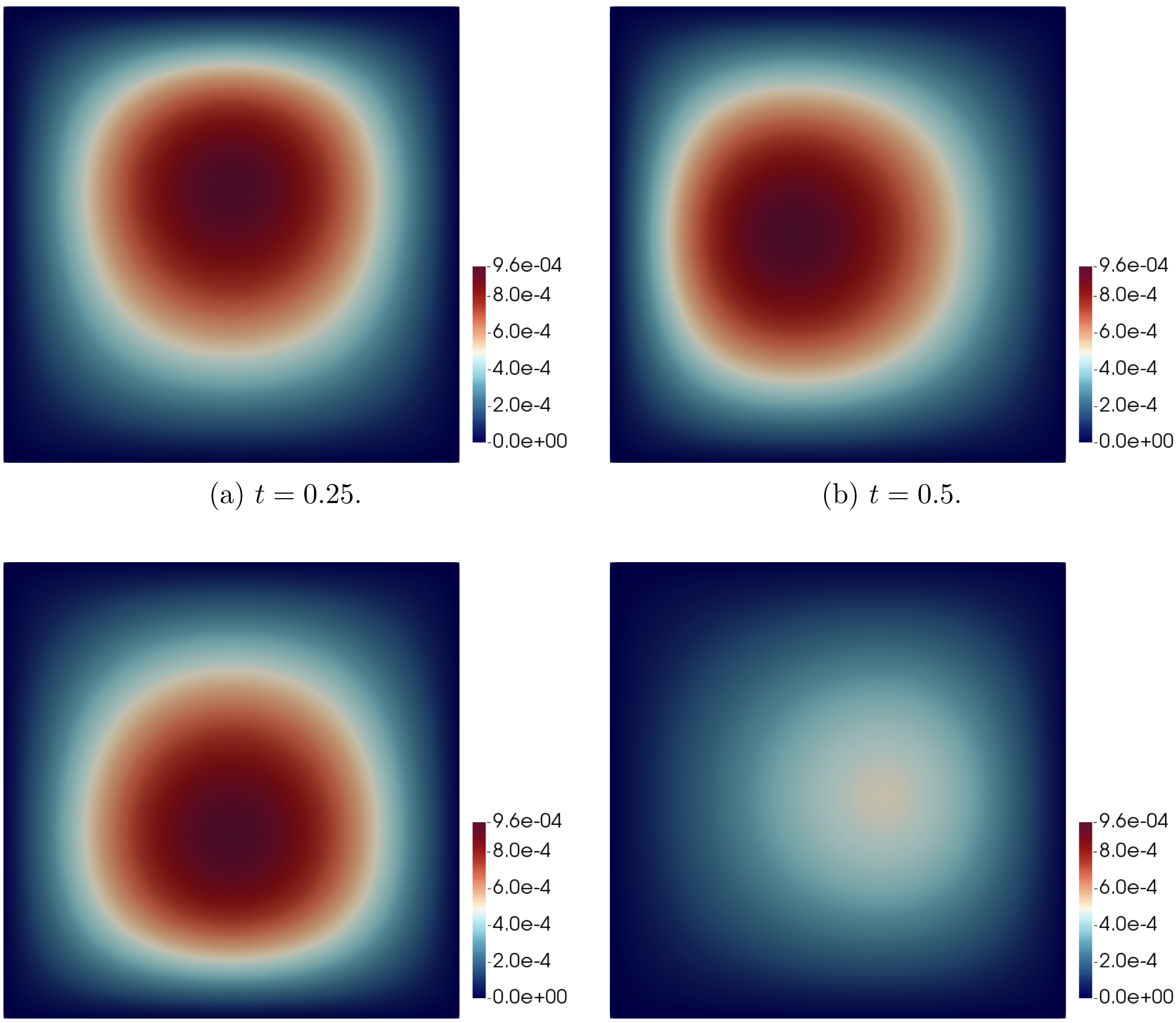
(c) $t = 0.5$.

(d) $t = 0, 75$.

Figure 11: The optimal control for problem (6.5) for $\lambda = 1.0$.

good overview on methods for nonlinear optimization problems (in the finite-dimensional case) can be found in [Kel99].

Recall that the adjoint equations for both the Cattaneo and the heat equation are given by the respective backward equations. Therefore, we use the same numerical methods in order to solve both the forward and adjoint equations, that is, we use the Newmark scheme with $\gamma = \beta = 1/2$ for the forward and adjoint Cattaneo equations as well as the Crank-Nicolson method for the forward and adjoint heat equations. As for the forward problem we numerically found the same step size restriction, that is $\Delta t \leq \tau/10$, for accurate results. Since the smallest value of τ we consider is, again, $\tau = 1e^{-3}$, we choose a time step of $\Delta t = 1e^{-4}$ which is sufficiently small to resolve the (temporal) behaviour of the Cattaneo equation. Furthermore, we also choose the same spatial discretization as before, that is,



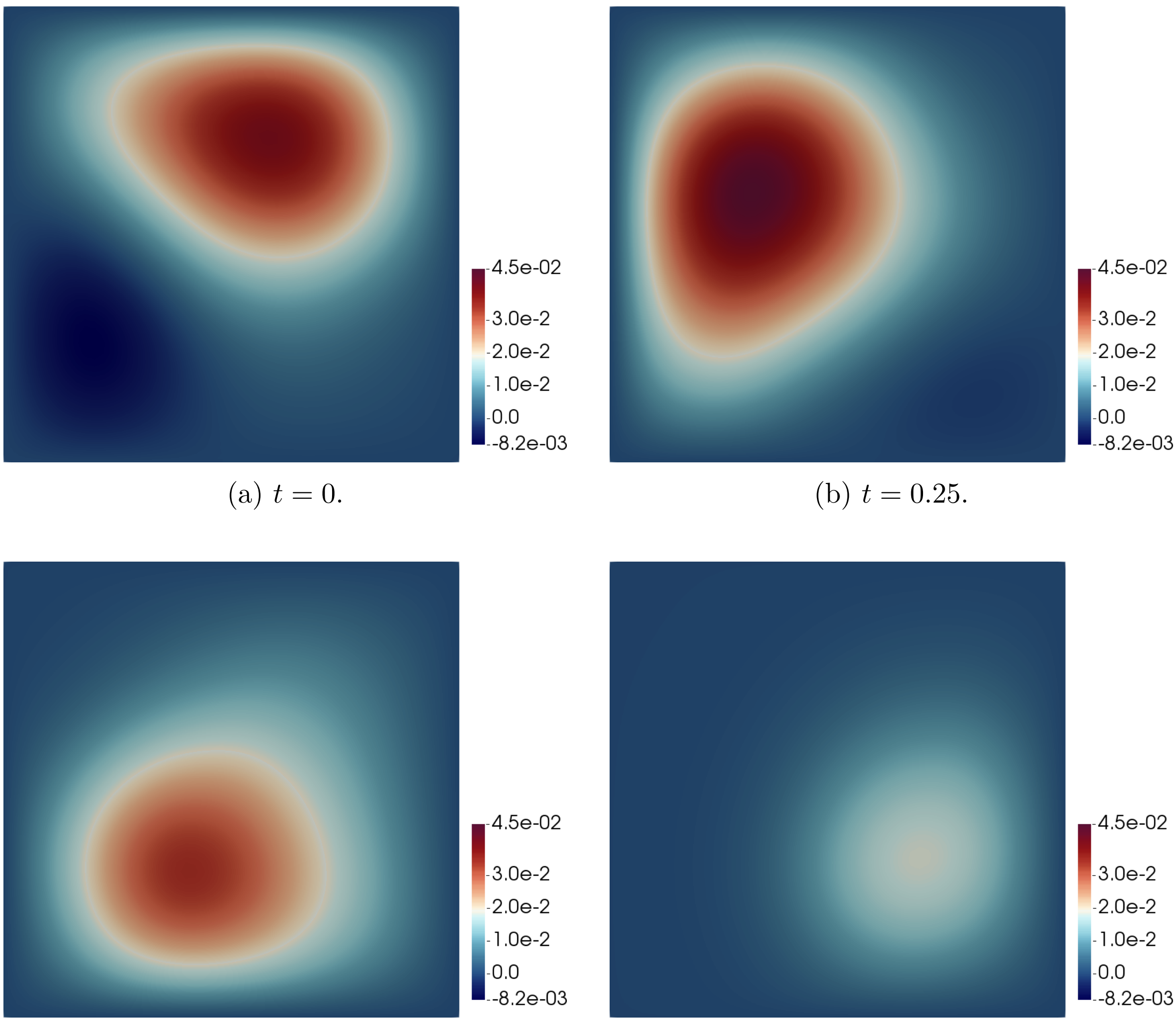
(a) $t = 0.25$. (b) $t = 0.5$. (c) $t = 0.75$. (d) $t = 1$.

Figure 12: The optimal state for problem (6.5) for $\lambda = 1.0$.

we discretize the unit square with a mesh consisting of 2601 nodes in 5000 triangles and use linear Lagrangian elements for FEniCS.

Let us first take a look at the solution we get for the optimal control problem with the heat equation as a constraint. The computed optimal control and state are shown in Figure 11 and Figure 12. Note that for the following plots we do not plot the initial value of the optimal state since this is given by $y^*(0) = y_0 = 0$. Furthermore, we also do not show the optimal control u^* at $t = 1$ since it is easy to see that $u^*(1) = 0$ holds, which is also the result we obtained numerically. Therefore, the plots of the optimal control and state are arranged differently.

The optimal control u^* for the heat equation shown in Figure 11 shows a behaviour which is expected if we want to achieve a temperature distribution like the desired one (depicted in Figure 10). We see that it heats the area that should be heated in that

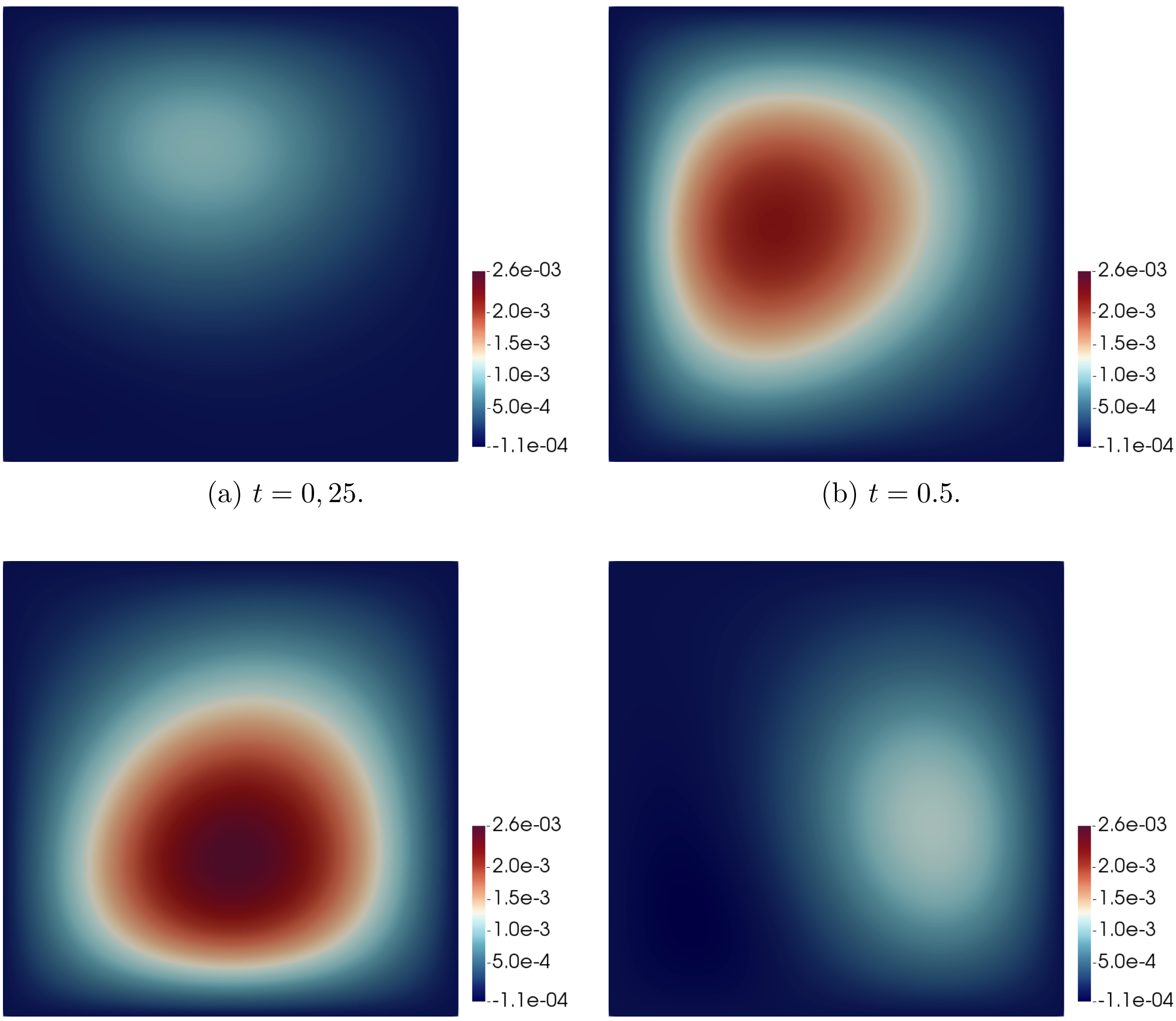


(a) $t = 0$. (b) $t = 0.25$. (c) $t = 0.5$. (d) $t = 0,75$.

Figure 13: The optimal control for problem (6.4) for $\tau = 1.0$ and $\lambda = 1.0$.

particular time step, that is, the control also has a kind of Gaussian shape that rotates with the same velocity as the desired state: We observe that the initial shape of the control is just transported along the circle given by (6.6) which we used to prescribe the desired state. This profile vanishes for $t \to 1$, which can only be seen from the plots for the corresponding optimal state. Also note that we have $u^* \geq 0$ everywhere which implies that we can achieve the optimal state by only heating the medium. However, we immediately see that the heating profile is more smoothed out compared to the rather "focused" Gaussian profile of the desired state. This has a major influence on the resulting optimal temperature profile since we expect that the solution of the heat equation diffuses the heat induced by the control even more.

Figure 12 depicts the optimal state for the heat equation. There, we observe this behaviour immediately: The optimal state is much more "smeared out" compared to the optimal



(a) $t = 0, 25$.

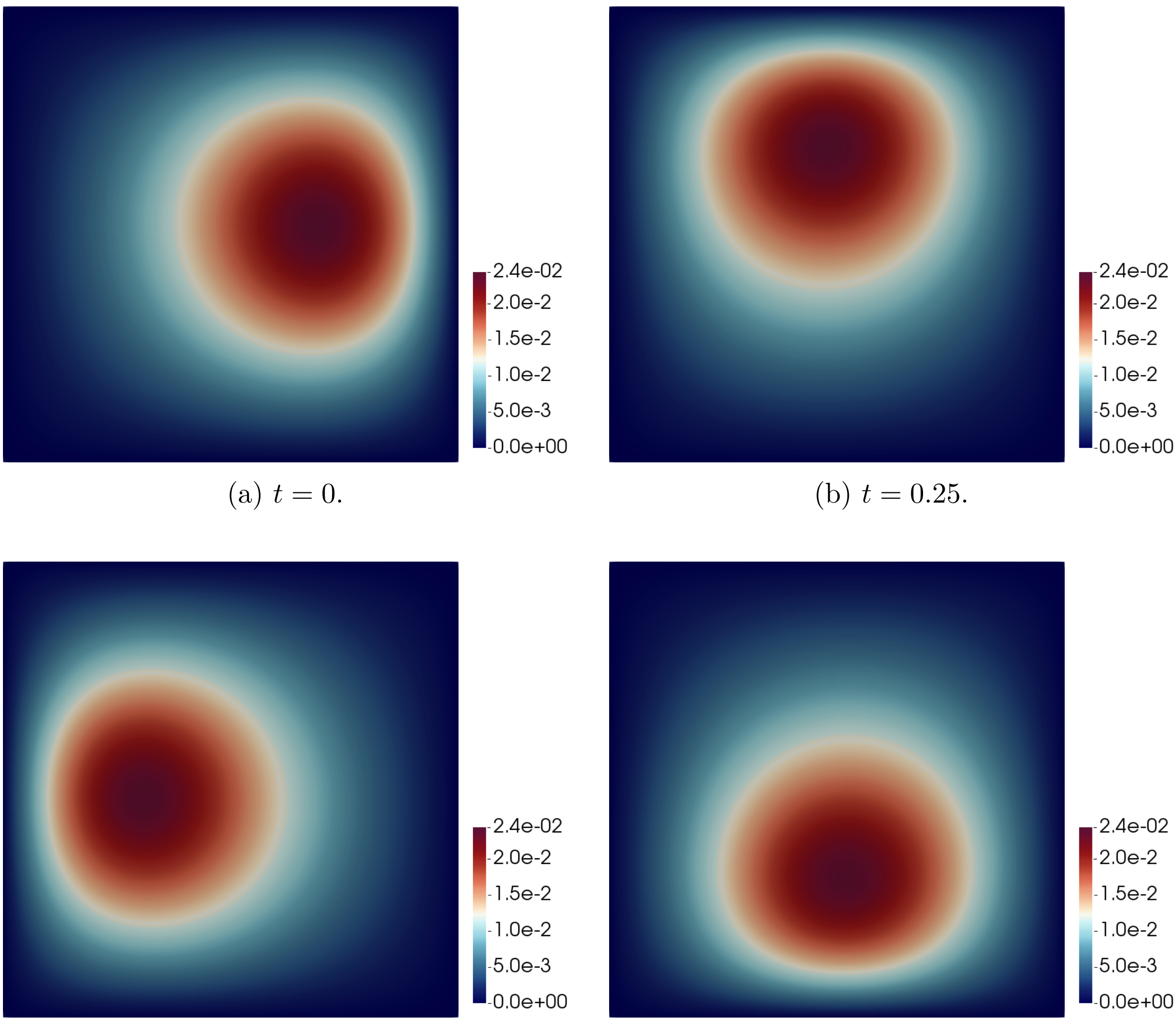
(b) $t = 0.5$.

(c) $t = 0.75$.

(d) $t = 1$.

Figure 14: The optimal state for problem (6.4) for $\tau = 1.0$ and $\lambda = 1.0$.

control. Still, we observe the rotating motion that was given in the desired state. We also directly observe that the optimal state also decays near $t = 1$ which happens due the fact that the optimal control also vanishes there. In order to "fix" this behaviour we could add the term $||y(T) - y_d(T)||^2_{L^2(\Omega)}$ to the cost functional which would lead to a different behaviour of the optimal control for $t \to T$. Furthermore, we see that the "height" of the optimal state is about 1000 times smaller than the "height" of the desired state which comes from the regularizing term in the cost functional: Since we have chosen $\lambda = 1.0$ we value the difference between the state and the desired state as much as the norm of the control. This naturally results in a smaller norm of u, that is, we do not heat as much. However, this also implies that the temperatures obtained for that control cannot be that high. Finally, we note that, as for the optimal control, we also have $y^* \geq 0$ in $[0, T] \times \Omega$ which is not necessarily the case anymore for the Cattaneo equation.



(a) $t = 0$.

(b) $t = 0.25$.

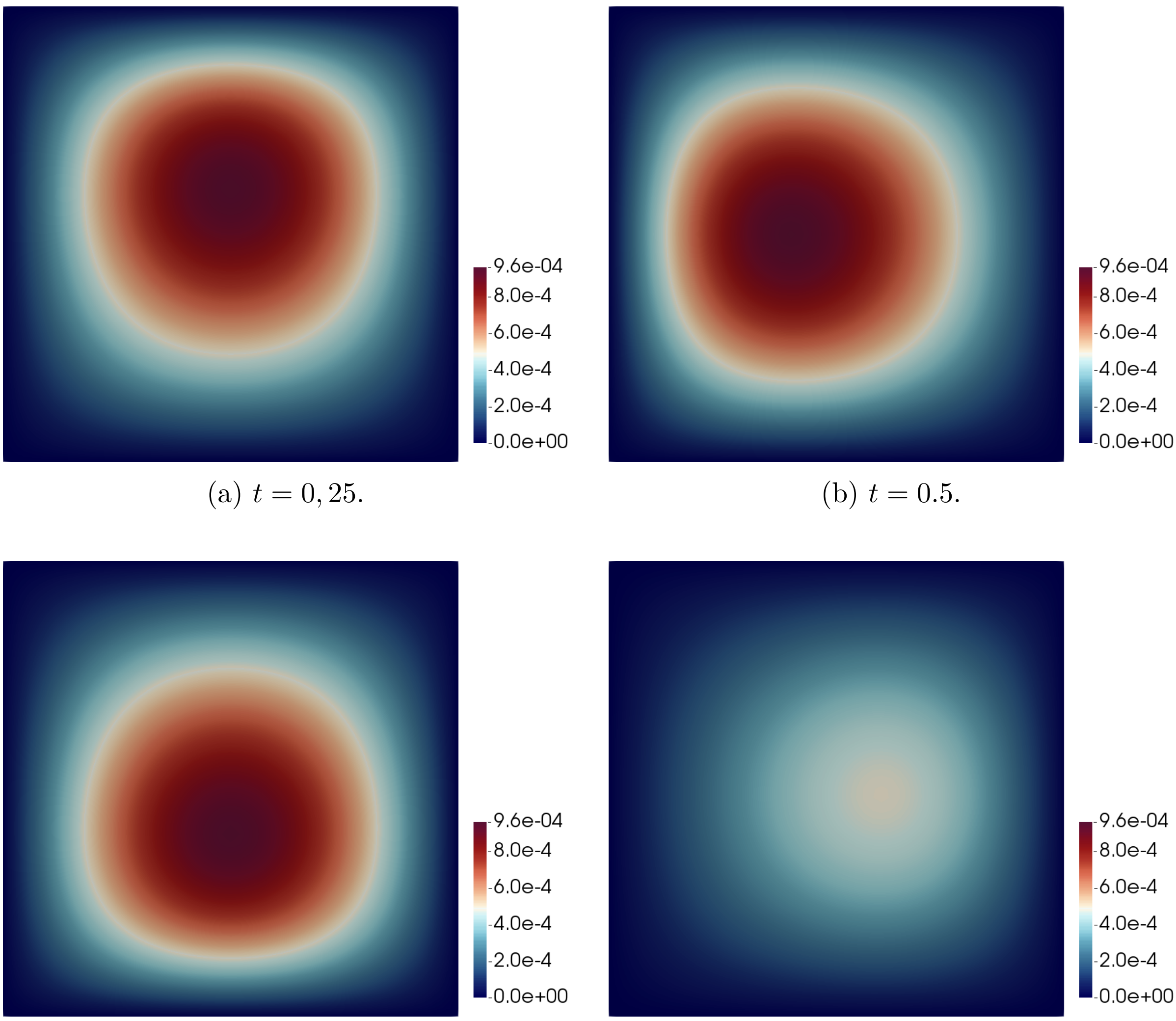
(c) $t = 0.5$.

(d) $t = 0, 75$.

Figure 15: The optimal control for problem (6.4) for $\tau = 1e^{-3}$ and $\lambda = 1.0$.

Figure 13 and Figure 14 show the optimal control and state of the optimization problem with the Cattaneo equation for $\tau = 1.0$ as constraint, again for the same regularizer $\lambda = 1.0$. We observe that both the optimal control and state behave completely different from the ones we discussed for the heat equation. The most significant deviation between the controls is the fact that the one for the Cattaneo equation shows a delayed behaviour: We observe that the optimal control for the Cattaneo equation at $t = 0$ heats the top middle area of the domain where the peaks of both the optimal state and the desired state are going to be later on (at approximately $t = 0.25$).

In contrast, the optimal control of the heat equation heats the right area of the unit square for $t = 0$, which is also where the Gaussian of the desired and optimal state is located at that exact time. An analogous behaviour can also be seen for later times which suggests that the optimal state of the Cattaneo equation "reacts" in a delayed way to the



(a) $t = 0, 25$. (b) $t = 0.5$.

(c) $t = 0.75$. (d) $t = 1$.

Figure 16: The optimal state for problem (6.4) for $\tau = 1e^{-3}$ and $\lambda = 1.0$.

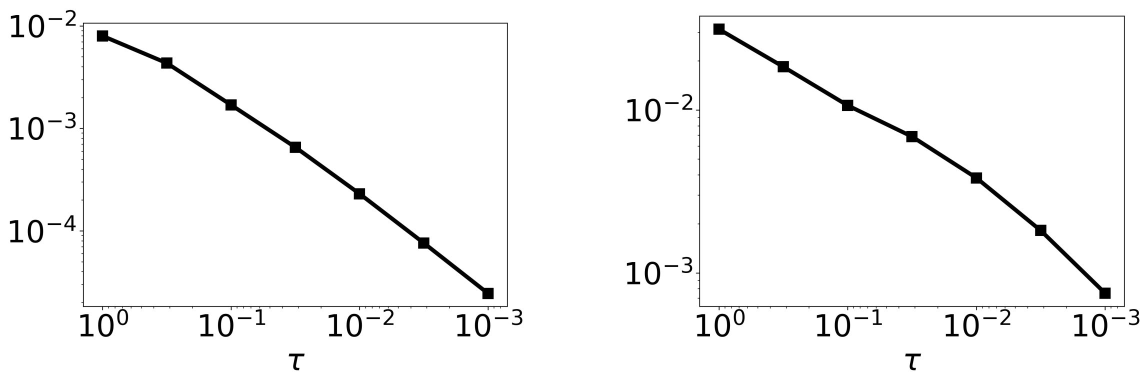
control. This again is a good indication that the Cattaneo equation models a delayed heat transfer. Also note that the optimal control does not only transport the initial heating profile, but, in fact, it is also transformed, in contrast to the heat equation, where the initial condition is only transported most of the time.

Furthermore, we also observe that the optimal control for the Cattaneo equation becomes negative in the beginning, that is, it cools some parts of the domain while it heats others, which is also fundamentally different to the behaviour of the optimal control for the heat equation. Last, we remark that the decay of the optimal control for $t \to 1$ can already be seen for the Cattaneo equation (at $t = 0.75$).

Investigating the optimal state depicted in Figure 14 reveals that we get a heating profile which follows the one given by the desired state, albeit the "height" of the peak is again much lower than the one of y_d. We also observe that this peak is more smoothed out than

τ	$\|\|u_c - u_h\|\|_{L^2}$	$\|\|u_c-u_h\|\|_{L^2}/\|\|u_h\|\|_{L^2}$	$\|\|u_c - u_h\|\|_{L^\infty}$	$\|\|u_c-u_h\|\|_{L^\infty}/\|\|u_h\|\|_{L^\infty}$
1.0	$7.99e^{-3}$	84.88 %	$3.13e^{-2}$	128.81 %
$1e^{-1/2}$	$4.34e^{-3}$	46.08 %	$1.84e^{-2}$	75.77 %
$1e^{-1}$	$1.69e^{-3}$	18.0 %	$1.07e^{-2}$	43.87 %
$1e^{-3/2}$	$6.55e^{-4}$	6.96 %	$6.86e^{-3}$	28.24 %
$1e^{-2}$	$2.3e^{-4}$	2.44 %	$3.83e^{-3}$	15.75 %
$1e^{-5/2}$	$7.6e^{-5}$	0.81 %	$1.82e^{-3}$	7.49 %
$1e^{-3}$	$2.45e^{-5}$	0.26 %	$7.51e^{-4}$	3.09 %

Table 2: Absolute and relative difference of the optimal controls for $\lambda = 1.0$.



(a) $||u_c - u_h||_{L^2}$.

(b) $||u_c - u_h||_{L^\infty}$.

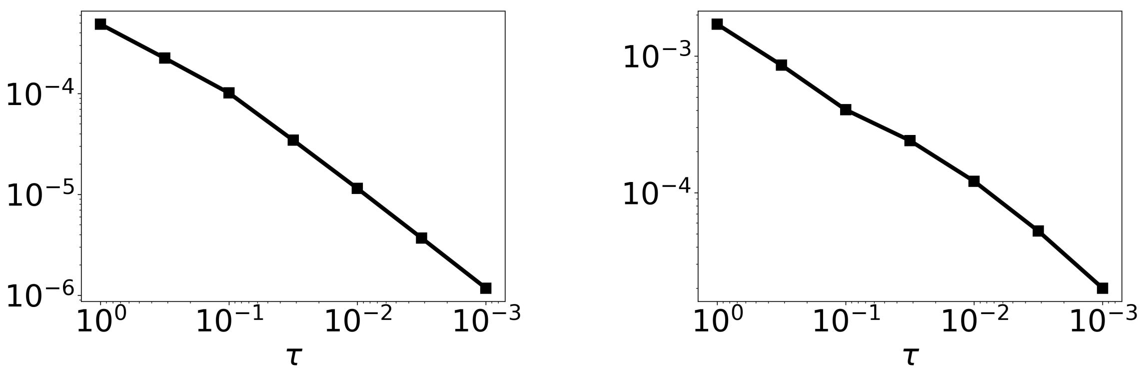
Figure 17: Convergence rates for the optimal controls of problem (6.4) for $\lambda = 1.0$.

the Gaussian given as desired state. However, the optimal state of the Cattaneo equation is a bit “sharper” and more localized than the one of the heat equation. Additionally, we also see that the height of the peak is approximately 2.5-times larger than the peak we got for the optimal state of the heat equation which is due to the stronger heating of the optimal control. This suggests that the Cattaneo equation might be able to approximate a desired state of this form more easily than the heat equation due to the transport character of the former.

In order to again investigate what happens for $\tau \to 0$, we give some plots of the optimal control and optimal state of the Cattaneo equation for $\tau = 1e^{-3}$ in Figure 15 and Figure 16. Note that there is no more visible difference between these and the optimal control and state for the heat equation: Both show the same kind of peaks, with similar heights and positions.

τ	$\|\|y_c - y_h\|\|_{L^2}$	$\|\|y_c-y_h\|\|_{L^2}/\|\|y_h\|\|_{L^2}$	$\|\|y_c - y_h\|\|_{L^\infty}$	$\|\|y_c-y_h\|\|_{L^\infty}/\|\|y_h\|\|_{L^\infty}$
1.0	$4.87e^{-4}$	112.56 %	$1.71e^{-3}$	178.15 %
$1e^{-1/2}$	$2.24e^{-4}$	51.92 %	$8.58e^{-4}$	89.35 %
$1e^{-1}$	$1.01e^{-4}$	23.43 %	$4.05e^{-4}$	42.16 %
$1e^{-3/2}$	$3.45e^{-5}$	7.98 %	$2.41e^{-4}$	25.07 %
$1e^{-2}$	$1.15e^{-5}$	2.65 %	$1.21e^{-4}$	12.63 %
$1e^{-5/2}$	$3.69e^{-6}$	0.85 %	$5.23e^{-5}$	5.45 %
$1e^{-3}$	$1.18e^{-6}$	0.27 %	$2.0e^{-5}$	2.08 %

Table 3: Absolute and relative difference of the optimal states for $\lambda = 1.0$.

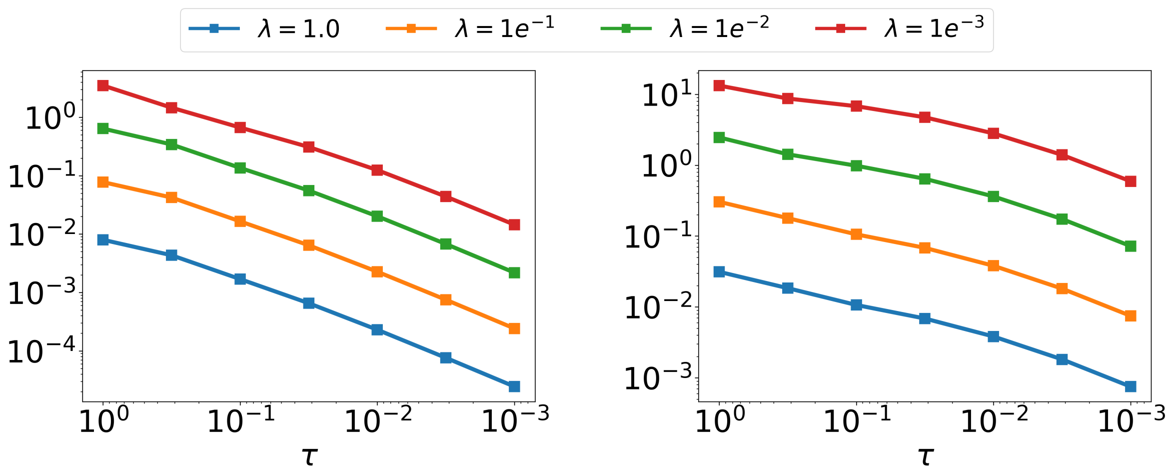


(a) $||y_c - y_h||_{L^2}$. (b) $||y_c - y_h||_{L^\infty}$.

Figure 18: Convergence rates for the optimal states of problem (6.4) for $\lambda = 1.0$.

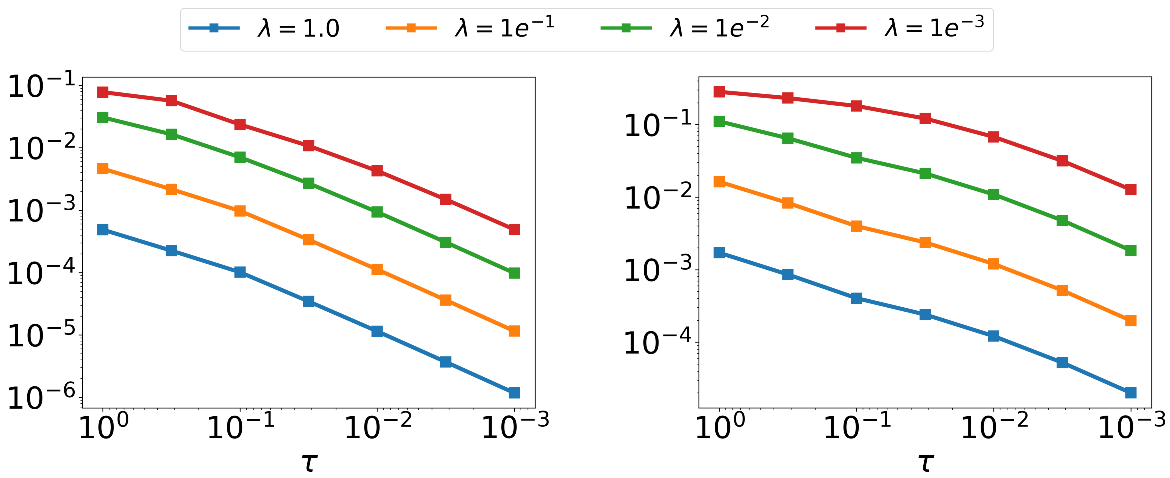
In the following we show the convergence results in the form of tables and figures for both the optimal controls and states. Note that u_h and y_h denote the optimal control and state of the heat equation, respectively, and that u_c and y_c denote the optimal control and state of the Cattaneo equation. The results for the optimal control and $\lambda = 1.0$ are shown in Table 2 and Figure 17, whereas the results for the optimal state for this choice of λ are depicted in Table 3 and Figure 18.

The numerical results displayed there show that we indeed have convergence of both the optimal states and the optimal controls, which we have proved theoretically in Chapter 5.3. The results even suggest that the convergence in the $L^2(0,T;L^2(\Omega))$ norm is a linear one, whereas they indicate a sublinear convergence in the $L^\infty(0,T;L^\infty)$ norm. However, it seems that the convergence rate behaves qualitatively the same for the optimal state and



(a) $||u_c - u_h||_{L^2}$. (b) $||u_c - u_h||_{L^\infty}$.

Figure 19: Convergence rates for the optimal controls of the Cattaneo equation.



(a) $||y_c - y_h||_{L^2}$. (b) $||y_c - y_h||_{L^\infty}$.

Figure 20: Convergence rates for the optimal states of the Cattaneo equation.

the optimal control, that is, they show similar slopes in the convergence plots for the respective norms.

For the sake of brevity we only shortly discuss what happens for smaller values of λ. As is expected due to the structure of the cost functional, the optimal states of both equations get closer to the desired state as $\lambda \to 0$ since the cost functional weights the difference $y - y_d$ more. This means that the optimal state is less "smeared out" and the height of the peaks for both the optimal control and state becomes higher. The corresponding plots for $\lambda = 1e^{-3}$ are shown in the Appendix. For this we observe that the characteristic features discussed above are still similar: We see that the optimal states decay for $t \to 1$, since the optimal control vanishes there. Additionally, we observe that the optimal state of the Cattaneo equation for $\tau = 1.0$ shows the same delayed behaviour as it did for $\lambda = 1.0$ which gives us another indication to interpret the Cattaneo equation as delayed heat equation. And again, as for $\lambda = 1.0$, we see no visible difference in the optimal state and control between heat equation and Cattaneo equation for $\tau = 1e^{-3}$.

Last, we take a look at the convergence behaviour for $\lambda \to 0$. The convergence rates are depicted in Figure 19 and Figure 20, and the corresponding tables are shown in the Appendix. We directly observe that the absolute error becomes worse for $\lambda \to 0$. However, it can be seen in the convergence tables in the Appendix that the relative error stays about the same. This is due to the following: As λ decreases to 0 the weight of the cost functional shifts to term $||y - y_d||^2$ and away from $||u||^2$. In our previous investigation we already found out that the former term is rather large for $\lambda = 1.0$. Therefore, the optimal states for smaller values of λ also increase in height by several orders of magnitude (see, for example, Figures 21 to 26 in the Appendix for $\lambda = 1e^{-3}$) and this means that, even though the absolute error is increasing, the relative error stays the same.

From the figures above we conclude that our numerical scheme behaves well for $\lambda \to 0$ since we observe the same qualitative convergence rates for all considered values of λ. Again, the results suggest a linear convergence rate for both the optimal state and control in the $L^2(0, T; L^2(\Omega))$ norm for all $\lambda > 0$. Additionally, we observe that the convergence rates are very similar for all values of τ and λ except for $\lambda = 1e^{-3}$ where there are some differences for large τ. These observations conclude this chapter about the numerical investigation of the Cattaneo equation.

7 Conclusion and Outlook

In this thesis we investigated both the (forward) Cattaneo equation as well as the optimal control problem with the Cattaneo equation as constraint. Our main focus was the asymptotic analysis of the limit $\tau \to 0$.

Therefore, we first derived the Cattaneo equation and introduced PDE constrained optimization problems. Subsequently, we proved the well-posedness of the Cattaneo equation for homogeneous Dirichlet boundary conditions, which we chose in order to simplify our notation and computations.

Following this, we investigated the optimal control problem for the Cattaneo equation: We proved the existence and uniqueness of an optimal control and derived the adjoint Cattaneo equation which we used to compute the gradient for the numerical solution of the optimal control problem.

With these results available we examined the asymptotic behaviour of both the forward and the optimal control problem for $\tau \to 0$ analytically. We proved the strong convergence of the solution of the Cattaneo equation to the one of the heat equation for the forward problem. For the optimization problem we proved the strong convergence of both optimal state and optimal control of the Cattaneo equation to the ones for the heat equation.

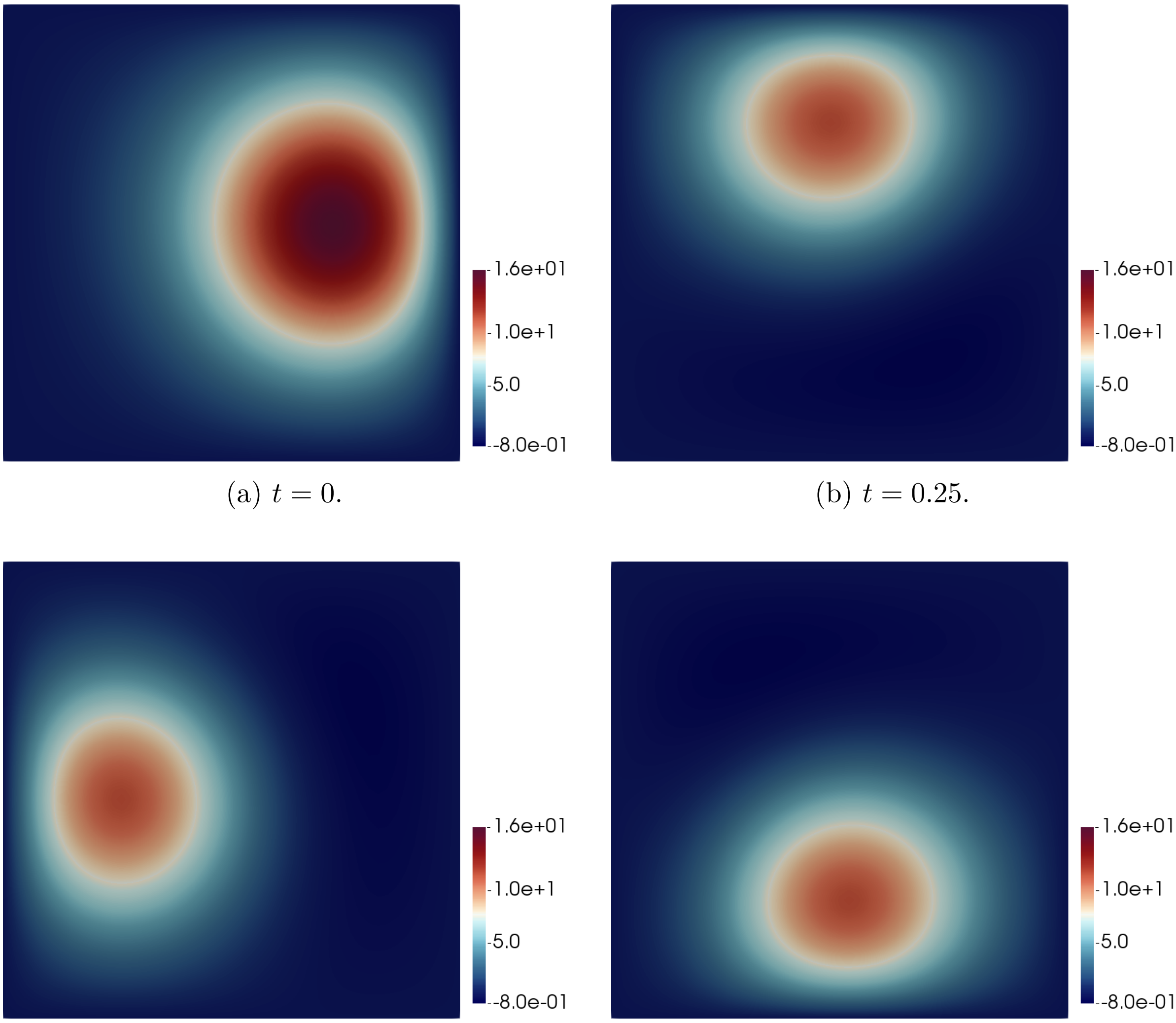
Finally, we also investigated the Cattaneo equation numerically for some academic problems. Our results indicate that it behaves rather differently compared to the heat equation for large τ but shows a linear convergence behaviour in $\tau \to 0$ in the $L^2(0,T;L^2(\Omega))$ norm. The numerical results for the optimal control problem showed similar features, where we again obtained a linear convergence for the optimal control and state in the same norm.

Further work on this topic could be done both theoretically and numerically. As a theoretical topic one could investigate the behaviour of the equation for other boundary conditions such as Neumann or Robin boundary conditions. One could also consider other types of controls, for example, boundary or initial controls. Furthermore, one could investigate, whether or not there is a convergence rate for the forward and optimal control problem in $\tau \to 0$, where our numerical simulations suggested that we do have in fact a linear convergence.

An investigation of a Cattaneo equation with non-linear terms would be of both theoretical and practical interest. This could be done, for example, for radiative heat transfer with the Rosseland approximation. Additionally, one might also investigate a coupled system of PDEs with the Cattaneo equation. However, examining these additional topics would have gone beyond the scope of this thesis.

Altogether, we have successfully investigated the optimal control problem constrained by the Cattaneo equation and proved the convergence of solutions for the Cattaneo problems to the solutions for the corresponding problems constrained by the heat equation theoretically and also did a numerical investigation which suggests a linear convergence rate.

Appendix: Numerical Results



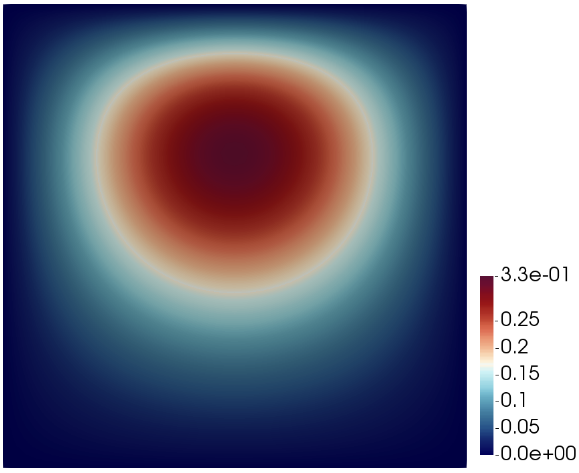
(a) $t = 0$.

(b) $t = 0.25$.

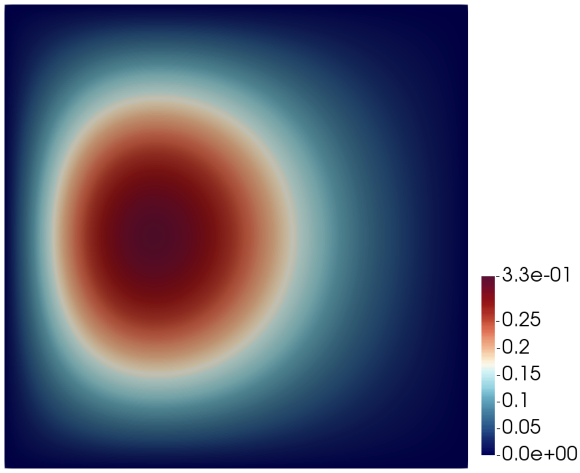
(c) $t = 0.5$.

(d) $t = 0,75$.

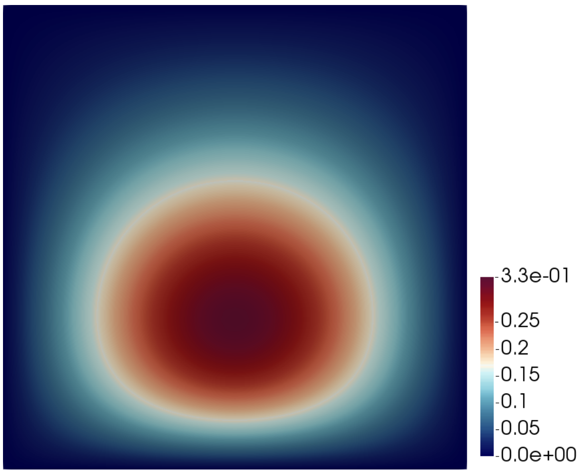
Figure 21: The optimal control for problem (6.5) for $\lambda = 1e^{-3}$.



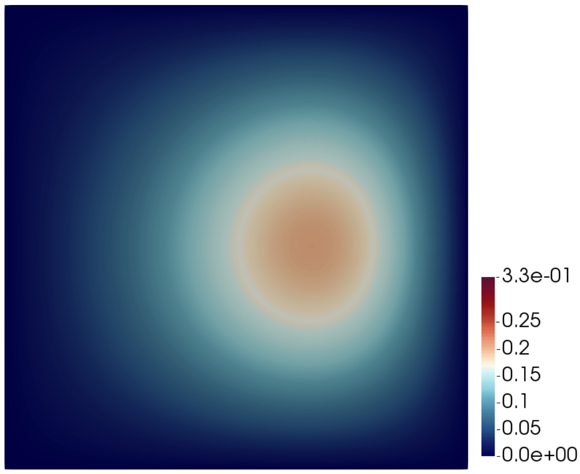
(a) $t = 0.25$.



(b) $t = 0.5$.

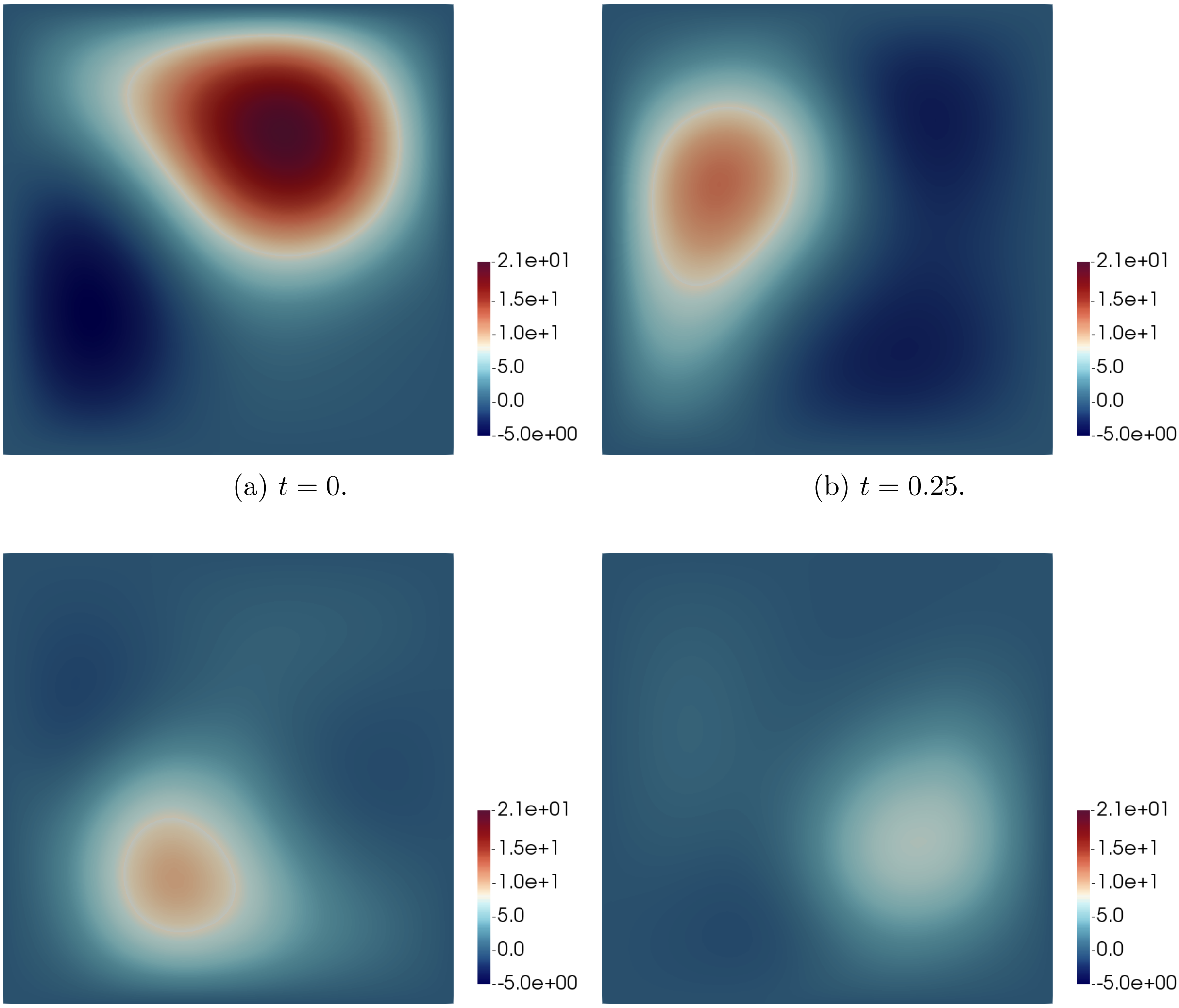


(c) $t = 0.75$.



(d) $t = 1$.

Figure 22: The optimal state for problem (6.5) for $\lambda = 1.e^{-3}$.



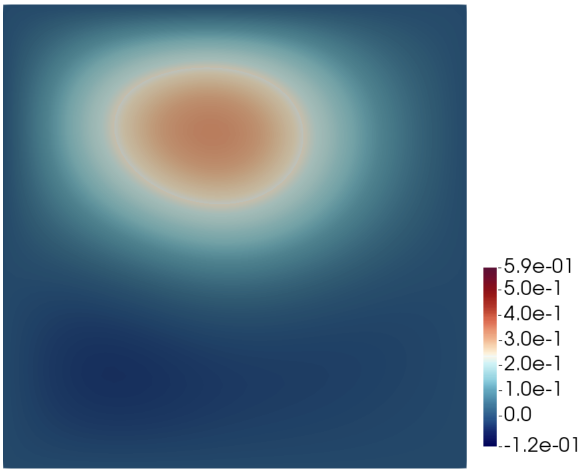
(a) $t = 0$.

(b) $t = 0.25$.

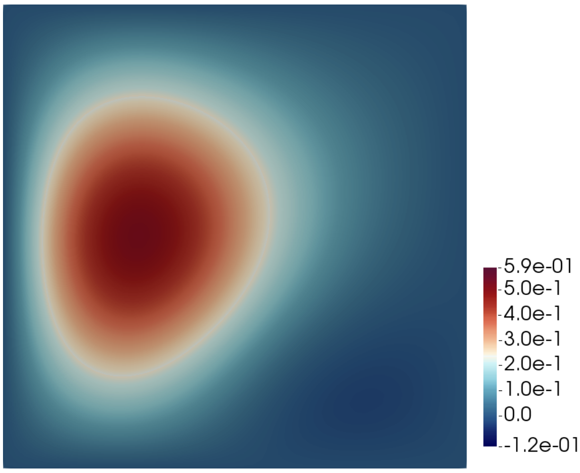
(c) $t = 0.5$.

(d) $t = 0,75$.

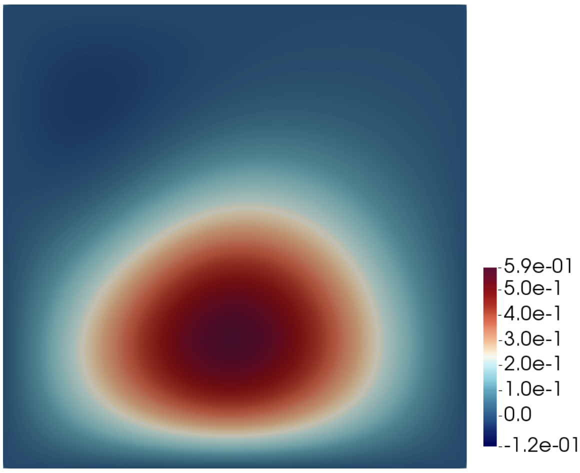
Figure 23: The optimal control for problem (6.4) for $\tau = 1.0$ and $\lambda = 1e^{-3}$.



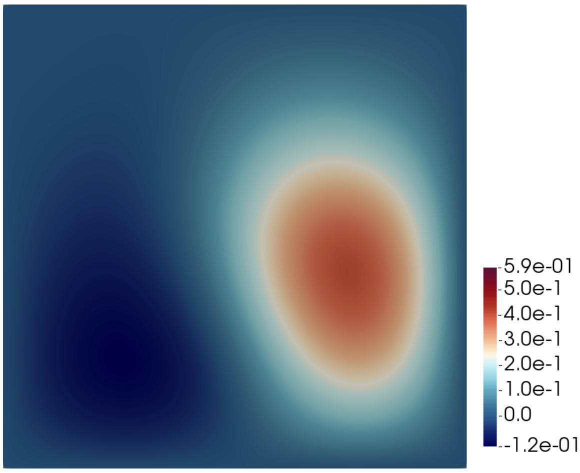
(a) $t = 0, 25$.



(b) $t = 0.5$.

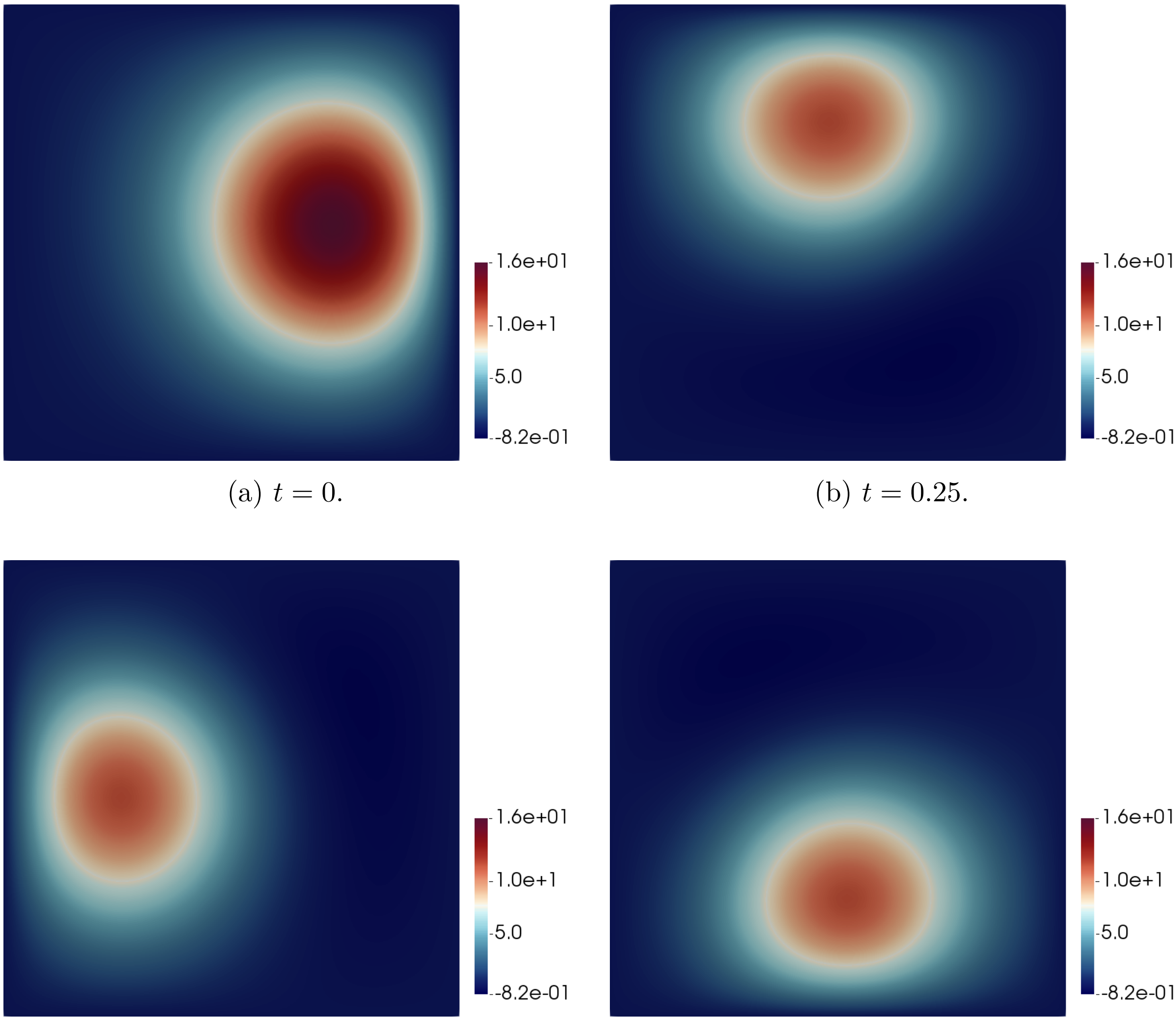


(c) $t = 0.75$.



(d) $t = 1$.

Figure 24: The optimal state for problem (6.4) for $\tau = 1.0$ and $\lambda = 1e^{-3}$.



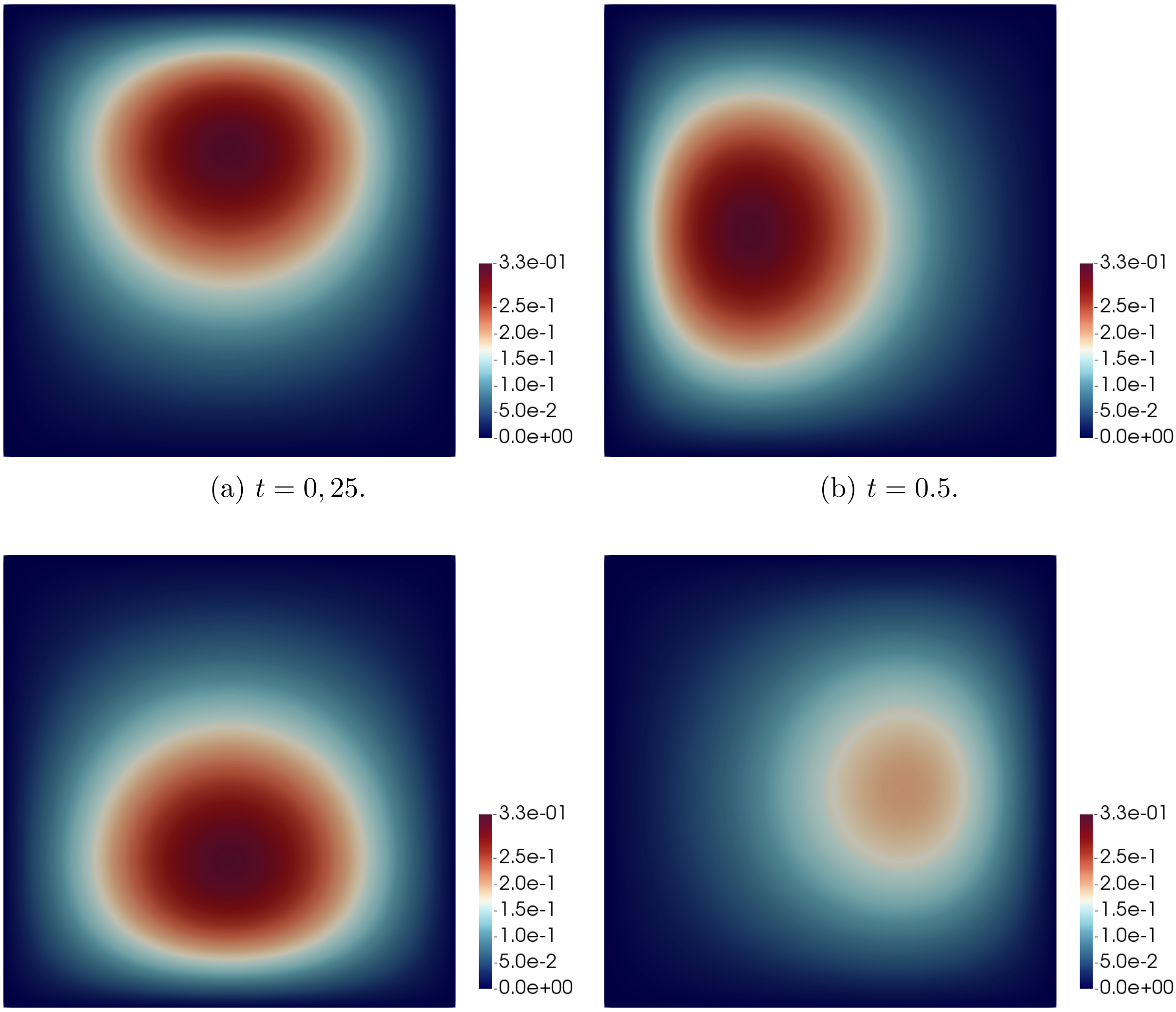
(a) $t = 0$.

(b) $t = 0.25$.

(c) $t = 0.5$.

(d) $t = 0, 75$.

Figure 25: The optimal control for problem (6.4) for $\tau = 1e^{-3}$ and $\lambda = 1e^{-3}$.



(a) $t = 0, 25$.

(b) $t = 0.5$.

(c) $t = 0.75$.

(d) $t = 1$.

Figure 26: The optimal state for problem (6.4) for $\tau = 1e^{-3}$ and $\lambda = 1e^{-3}$.

τ	$\|\|u_c - u_h\|\|_{L^2}$	$\|\|u_c-u_h\|\|_{L^2}/\|\|u_h\|\|_{L^2}$	$\|\|u_c - u_h\|\|_{L^\infty}$	$\|\|u_c-u_h\|\|_{L^\infty}/\|\|u_h\|\|_{L^\infty}$
1.0	$7.76e^{-2}$	84.07 %	$3.04e^{-1}$	127.01 %
$1e^{-1/2}$	$4.22e^{-2}$	45.72 %	$1.79e^{-1}$	74.67 %
$1e^{-1}$	$1.65e^{-2}$	17.87 %	$1.06e^{-1}$	44.1 %
$1e^{-3/2}$	$6.43e^{-3}$	6.96 %	$6.82e^{-2}$	28.43 %
$1e^{-2}$	$2.27e^{-3}$	2.45 %	$3.81e^{-2}$	15.88 %
$1e^{-5/2}$	$7.5e^{-4}$	0.81 %	$1.81e^{-2}$	7.56 %
$1e^{-3}$	$2.42e^{-4}$	0.26 %	$7.48e^{-3}$	3.12 %

Table 4: Absolute and relative difference of the optimal control for $\lambda = 1e^{-1}$.

τ	$\|\|y_c - y_h\|\|_{L^2}$	$\|\|y_c-y_h\|\|_{L^2}/\|\|y_h\|\|_{L^2}$	$\|\|y_c - y_h\|\|_{L^\infty}$	$\|\|y_c-y_h\|\|_{L^\infty}/\|\|y_h\|\|_{L^\infty}$
1.0	$4.61e^{-3}$	109.05 %	$1.63e^{-2}$	173.1 %
$1e^{-1/2}$	$2.16e^{-3}$	51.13 %	$8.3e^{-3}$	88.3 %
$1e^{-1}$	$9.72e^{-4}$	22.96 %	$3.98e^{-3}$	42.38 %
$1e^{-3/2}$	$3.36e^{-4}$	7.94 %	$2.38e^{-3}$	25.28 %
$1e^{-2}$	$1.12e^{-4}$	2.65 %	$1.2e^{-3}$	12.75 %
$1e^{-5/2}$	$3.62e^{-5}$	0.85 %	$5.18e^{-4}$	5.51 %
$1e^{-3}$	$1.15e^{-5}$	0.27 %	$1.98e^{-4}$	2.11 %

Table 5: Absolute and relative difference of the optimal state for $\lambda = 1e^{-1}$.

τ	$\|\|u_c - u_h\|\|_{L^2}$	$\|\|u_c-u_h\|\|_{L^2}/\|\|u_h\|\|_{L^2}$	$\|\|u_c - u_h\|\|_{L^\infty}$	$\|\|u_c-u_h\|\|_{L^\infty}/\|\|u_h\|\|_{L^\infty}$
1.0	$6.41e^{-1}$	81.68 %	2.47	111.11 %
$1e^{-1/2}$	$3.43e^{-1}$	43.65 %	1.42	64.17 %
$1e^{-1}$	$1.36e^{-1}$	17.33 %	$9.82e^{-1}$	44.27 %
$1e^{-3/2}$	$5.57e^{-2}$	7.1 %	$6.42e^{-1}$	28.94 %
$1e^{-2}$	$2.02e^{-2}$	2.57 %	$3.62e^{-1}$	16.33 %
$1e^{-5/2}$	$6.76e^{-3}$	0.86 %	$1.74e^{-1}$	7.83 %
$1e^{-3}$	$2.19e^{-3}$	0.28 %	$7.21e^{-2}$	3.25 %

Table 6: Absolute and relative difference of the optimal control for $\lambda = 1e^{-2}$.

τ	$\|\|y_c - y_h\|\|_{L^2}$	$\|\|y_c-y_h\|\|_{L^2}/\|\|y_h\|\|_{L^2}$	$\|\|y_c - y_h\|\|_{L^\infty}$	$\|\|y_c-y_h\|\|_{L^\infty}/\|\|y_h\|\|_{L^\infty}$
1.0	$3.06e^{-2}$	87.49 %	$1.11e^{-1}$	141.21 %
$1e^{-1/2}$	$1.64e^{-2}$	46.94 %	$6.49e^{-2}$	82.74 %
$1e^{-1}$	$7.04e^{-3}$	20.11 %	$3.48e^{-2}$	44.42 %
$1e^{-3/2}$	$2.69e^{-3}$	7.7 %	$2.12e^{-2}$	27.08 %
$1e^{-2}$	$9.34e^{-4}$	2.67 %	$1.1e^{-2}$	13.89 %
$1e^{-5/2}$	$3.06e^{-4}$	0.87 %	$4.77e^{-3}$	6.08 %
$1e^{-3}$	$9.79e^{-5}$	0.28 %	$1.84e^{-3}$	2.35 %

Table 7: Absolute and relative difference of the optimal state for $\lambda = 1e^{-2}$.

τ	$\|\|u_c - u_h\|\|_{L^2}$	$\|\|u_c-u_h\|\|_{L^2}/\|\|u_h\|\|_{L^2}$	$\|\|u_c - u_h\|\|_{L^\infty}$	$\|\|u_c-u_h\|\|_{L^\infty}/\|\|u_h\|\|_{L^\infty}$
1.0	3.5	93.24 %	$1.33e^{1}$	82.95 %
$1e^{-1/2}$	1.46	38.88 %	8.72	54.47 %
$1e^{-1}$	$6.67e^{-1}$	17.77 %	6.8	42.47 %
$1e^{-3/2}$	$3.1e^{-1}$	8.26 %	4.74	29.59 %
$1e^{-2}$	$1.25e^{-1}$	3.32 %	2.8	17.53 %
$1e^{-5/2}$	$4.39e^{-2}$	1.17 %	$1.4e$	8.73 %
$1e^{-3}$	$1.45e^{-2}$	0.39 %	$5.93e^{-1}$	3.7 %

Table 8: Absolute and relative difference of the optimal control for $\lambda = 1e^{-3}$.

τ	$\|\|y_c - y_h\|\|_{L^2}$	$\|\|y_c-y_h\|\|_{L^2}/\|\|y_h\|\|_{L^2}$	$\|\|y_c - y_h\|\|_{L^\infty}$	$\|\|y_c-y_h\|\|_{L^\infty}/\|\|y_h\|\|_{L^\infty}$
1.0	$7.78e^{-2}$	57.79 %	$2.83e^{-1}$	84.61 %
$1e^{-1/2}$	$5.67e^{-2}$	42.1 %	$2.32e^{-1}$	69.43 %
$1e^{-1}$	$2.36e^{-2}$	17.51 %	$1.81e^{-1}$	54.04 %
$1e^{-3/2}$	$1.07e^{-2}$	7.98 %	$1.21e^{-1}$	36.28 %
$1e^{-2}$	$4.26e^{-3}$	3.16 %	$6.77e^{-2}$	20.26 %
$1e^{-5/2}$	$1.49e^{-3}$	1.1 %	$3.16e^{-2}$	9.45 %
$1e^{-3}$	$4.89e^{-4}$	0.36 %	$1.27e^{-2}$	3.8 %

Table 9: Absolute and relative difference of the optimal state for $\lambda = 1e^{-3}$.

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Statement of Authorship

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

Sebastian Blauth Kaiserslautern, September 3, 2018