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"DIFFUSION APPROXIMATION AND HYPERBOLIC AUTOMORPHISMS OF THE TORUS"

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Diffusion Approximation and Hyperbolic Automorphisms of the Torus.

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Abstract

In this article a diffusion equation is obtained as a limit of a reversible kinetic equation with an ad hoc scaling. The diffusion is produced by the collisions of the particles with the boundary. These particles are assumed be to reflected according to a reversible law having convenient mixing properties. Optimal convergence results are obtained in a very simple manner. This is made possible because the model, based on Arnold's cat map can be handled with Fourier series instead of the symbolic dynamics associated to a Markov partition.

1. — Introduction.

The purpose of this article is the description of a diffusion equation obtained as the limit of a deterministic and reversible linear kinetic equation.

For this purpose the following model is introduced. Beetween two horizontal plates a family of particles evolve as a Knudsen gas i.e. a gas with no interparticles collisions. The vertical velocity of the particles is positive or negative, according to whether they go up or down, and of constant modulus c. Their horizontal velocities $ca(\omega)$ are parametrized by $\mathbf{T}^2 = \mathbf{R}^2/(2\pi\mathbf{Z})^2$. Whenever the particles hit the top or bottom plate, their vertical velocities are changed into their opposite while their horizontal velocities are modified by the right action of an hyperbolic automorphism of $\mathbf{T}^2$ (cf. figure 1).

More precisely the following notations are used : the space position of the particles is $(x,z) \in \mathbf{R}^d \times (0,h)$; the vertical component of the velocity of the particles is $\pm c$; the horizontal component of this velocity is given by $ca(\omega)$, $\omega \in \mathbf{T}^2$ with $a : \mathbf{T}^2 \to \mathbf{R}^d$ denoting a smooth enough mean zero vector field. The nonnegative functions $f_+(t,x,z,\omega)$ (resp. $f_-(t,x,z,\omega)$) represent the density of particles which at time t and point (x,v) have the velocity $(ca(\omega),+c)$ (resp. $(ca(\omega),-c)$).

The following hyperbolic automorphism T of the torus (Arnold's cat map) defined by

$$T\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \quad (\text{mod. } 2\pi). \tag{1}$$

will be the only case treated here; the method developped in the present paper would apply to any hyperbolic automorphism of $\mathbf{T}^n$. The map $T : \mathbf{T}^2 \to \mathbf{T}^2$ is one-to-one and C^∞; it preserves the measure $d\omega_1 d\omega_2 / 4\pi^2$ and its inverse (which also is a C^∞ map) is given by

$$T^{-1}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \quad (\text{mod. } 2\pi)\,. \tag{2}$$

The densities $f^\pm$ satisfy the Liouville equations

$$\partial_t f^\pm + c a(\omega)\cdot\partial_x f^\pm \pm c\partial_z f^\pm = 0\,, \quad x \in \mathbf{R}^d\,,\ 0 < z < h\,,\ \omega \in \mathbf{T}^2\,. \tag{3}$$

with the following boundary conditions on the plates:

$$f^+(t,x,0,\omega) = f^-(t,x,0,T\omega)\,, \quad x \in \mathbf{R}^d\,,\ \omega \in \mathbf{T}^2\,, \tag{4a}$$

$$f^-(t,x,h,\omega) = f^+(t,x,h,T\omega)\,, \quad x \in \mathbf{R}^d\,,\ \omega \in \mathbf{T}^2\,. \tag{4b}$$

and their value at $t = 0$ is given by the following initial condition

$$f^\pm(0,x,z,\omega) = \phi(x)\,, \quad x \in \mathbf{R}^d\,,\ 0 < z < h\,,\ \omega \in \mathbf{T}^2\,; \tag{5}$$

which is compatible with an approximation, as $h \to 0$, by an horizontal diffusion, and which avoids the appearance of an initial layer in the limiting process.

Since the densities $f^\pm$ satisfy the equation (3), they are constant along the characteristic lines of the system; hence

$$f^\pm(t,x,z,\omega) = \phi(x - h\sum_{k=0}^{[ct/h]} a(T^k\omega)) + O(h)\,. \tag{6}$$

The asymptotic limit leading to an horizontal diffusion will be obtained by letting h go to zero and observing the system for large positive times. A small parameter ϵ being introduced, h is changed into ϵh (thus letting the collision frequency going to ∞) and t into t/ϵ. The problem of interest (3)-(4a,b)-(5) becomes

$$\epsilon^2\partial_t f_\epsilon^\pm + c\epsilon a(\omega)\cdot\partial_x f_\epsilon^\pm \pm c\partial_z f_\epsilon^\pm = 0\,, \quad x \in \mathbf{R}^d\,,\ 0 < z < h\,,\ \omega \in \mathbf{T}^2\,. \tag{7}$$

$$f_\epsilon^+(t,x,0,\omega) = f_\epsilon^-(t,x,0,T\omega)\,, \quad x \in \mathbf{R}^d\,,\ \omega \in \mathbf{T}^2\,, \tag{8a}$$

$$f_\epsilon^-(t,x,h,\omega) = f_\epsilon^+(t,x,h,T\omega)\,, \quad x \in \mathbf{R}^d\,,\ \omega \in \mathbf{T}^2\,, \tag{8b}$$

$$f_\epsilon^\pm(0,x,z,\omega) = \phi(x)\,, \quad x \in \mathbf{R}^d\,,\ 0 < z < h\,,\ \omega \in \mathbf{T}^2\,. \tag{9}$$

Its solution is given by (see (6))

$$f_\epsilon^\pm(t,x,z,\omega) = \phi(x - \epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) + O(\epsilon)\,; \tag{10}$$

therefore most of the analysis is reduced to studying the limit, as $\epsilon \to 0$, of the expression

$$\psi_\epsilon(t,x,\omega) = \phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)). \tag{11}$$

2.— Notations and Main Results.

The following notation will be systematically used below:

$$\langle F \rangle = \frac{1}{4\pi^2}\int_{\mathbf{T}^2} F(\omega)d\omega\,.$$

The formula $U_T f = f \circ T$ with the mapping T given by (1) defines an operator U_T in the space $L^2(\mathbf{T}^2)$; this operator is unitary and therefore its adjoint is given by $U_T^* f = U_T^{-1} f = f \circ T^{-1}$.

Definition 1. A coboundary is an element of $\mathrm{Im}(I - U_T)$. Two functions f and g belonging to $L^2(\mathbf{T}^2)$ are said to be cohomologous if and only if $f - g$ is a coboundary and this equivalence relation will be denoted $f \sim g$.

The next proposition describes the elementary properties of what will be, in the limit $\epsilon \to 0$, the diffusion coefficient. It is essentially based on the ergodic and mixing properties of the mapping T.

Proposition 2.
1) Any function $a \in L^2(\mathbf{T}^2)$ which satisfies the relation $a = a \circ T$ is constant and the subspace $\mathrm{Im}(I - U_T)$ is dense in the space of functions $a \in L^2(\mathbf{T}^2)$ such that $< a >= 0$ (notice that this space is invariant under T).

Let $s > 0$ and $a : \mathbf{T}^2 \to \mathbf{R}^d$ in the Sobolev class $H^s(\mathbf{T}^2)$, with mean value $\langle a \rangle = 0$
2) One has:

$$D(a) = \tfrac{1}{2}\langle a^2 \rangle + \sum_{k\geq 1}\langle a \circ T^k \otimes a \rangle = \tfrac{1}{2}\lim_{N\to\infty}\langle \left(\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a \circ T^k\right)^{\otimes 2}\rangle \geq 0, \tag{12}$$

where the series

$$\sum_{k\geq 1}\langle a \circ T^k \otimes a \rangle$$

is normally convergent.
3) Let $\xi \in \mathbf{R}^d$ and $b \in H^s(\mathbf{T}^2;\mathbf{R}^d)$ such that $\langle b \rangle = 0$. If $a\cdot\xi \sim b\cdot\xi$ then $\xi\cdot D(a)\xi = \xi\cdot D(b)\xi$.
4) For any $\xi \in \mathbf{R}^d$, the following properties are equivalent:
   i) $D(a)\xi = 0$;
   ii) $\xi \cdot D(a)\xi = 0$;

iii) The sequence of functions $f_N \cdot \xi$ of $L^2(\mathbf{T}^2)$

$$f_N \cdot \xi = \sum_{k=1}^{N} (a \circ T^k) \cdot \xi$$

is (uniformly with respect to N) bounded in $L^2(\mathbf{T}^2)$;

iv) The function $a \cdot \xi$ is a coboundary.

The main result of this paper is the following

Theorem 3. Let $s > d/2$, $a : \mathbf{T}^2 \to \mathbf{R}^d$ be in the Sobolev class $H^s(\mathbf{T}^2)$ with mean value $\langle a \rangle = 0$ and let $\phi \in C^2(\mathbf{R}^d)$ be an initial data. Denote by $u(t,x)$ the solution of

$$\partial_t u = hc\nabla_x \cdot (D(a)\nabla_x u)\,, \quad u(0,x) = \phi(x)\,. \tag{13}$$

Then the family of functions $f_\epsilon^\pm$ defined by (7), (8) and (9) converges to $u(x,t)$ as $\epsilon \to 0$ in the following sense: for any $\tau > 0$ and any compact $K \subset \mathbf{R}_x^d$

$$\langle f_\epsilon^\pm(t,x,z,\omega)\rangle \to u(t,x)\,,\ C^0([0,\tau] \times K \times \mathbf{T}^2 \times]0,h[))\,, \tag{14a}$$

$$f_\epsilon^\pm(t,x,z,\omega) \to u(t,x)\,,\ C^0([0,\tau], w^* - L^\infty(\mathbf{R}^d \times \mathbf{T}^2)) \tag{14b}$$

Furthermore with $\psi_\epsilon(x,t)$ defined by the formula (11) one has:

$$\|f_\epsilon^\pm - \psi_\epsilon\|_{L^\infty(\mathbf{R}\times\mathbf{R}^d\times(0,h)\times\mathbf{T}^2)} = O(\epsilon)\,. \tag{15}$$

The proof of this theorem is tailored on the proof of the Ito formula (cf. for instance [d]); in particular it will be shown that the average of the different products appearing in a Taylor expansion are, in the limit decorrelated and therefore converge to the product of the corresponding limiting averages. In the original Ito formula, this point is straightforward since (by construction) the Brownian motion has independent increments. At variance in the present paper, the independence is obtained only in the limit $\epsilon \to 0$ and, as will be shown below, is a consequence of the different mixing properties of the map T.

An analogous result has been proven by [de-ph] for suspensions of finite type subshifts under hölderian maps. It would be theoretically possible to reduce the present analysis to this situation by coding the mapping T with a Markov partition. However our goal is to produce a proof of a diffusion limit as explicit as possible and which in the present case uses only elementary techniques (Fourier series expansions instead of Markov partition as a coding of the system).

In spite of the fact that the initial model given by the equations (6)-(7)-(8) is time reversible (it is given by a one-to-one and onto broken hamiltonian flow which preserves the measure of the phase space $(\delta_c + \delta_{-c})dx dz d\omega$), the limiting equation (14) is well posed only for $t \geq 0$. In particular, the $L^2(dx)$ norm of the solution of the equation (14) is for $t > 0$ strictly decreasing; it is constant only in the special case where the solution is space homogenous. And the diffusion equation (14) obtained as the limit of a reversible system is in some sense the simplest and most basic exemple of an irreversible partial differential equation, in spite of the fact that for some very smooth (holomorphic) data it is possible to invert for a small time the diffusion equation; this is realised in $\mathbf{R}^d$ by the inversion formula, due to Lebeau [l], of the Fourier-Bros-Iagolnitzer transform (cf. also [go-le] for the extension of this formula to analytic real compact Riemannian manifolds). However such inversion formula is not valid for non analytic functions, and this is precisely in this situation, in a topology much weaker than any space of holomorphic functions (namely $C^0([0,\tau], w^* - L^\infty(\mathbf{R}^d \times \mathbf{T}^2)))$ (cf. (14)-(15)) that the approximation of (6)-(7)-(8) by (14) is proven.

The rest of the paper is organized as follows. In section 3, the basic mixing properties of the map T are given (property (H1) of proposition 5); they rely on a diophantine estimate due to Kronecker. The mixing property (H1) is used to prove proposition 2. Then the decorrelation properties of the process are first (property (H2)) proven in proposition 6. This implies property (H3) of corollary 7, used in the estimate of the remainder of the "Ito" formula. Then proposition 8 and corollary 9 provide all the material for the proof of theorem 3 which belongs to section 4. Section 5 contains final remarks and comments on the numerical simulations.

3.— Proofs of the Mixing Properties and of Proposition 2.

As we said above, the proofs rely on the use of Fourier series. Indeed the properties of the dynamical system induced by the map T on the torus $\mathbf{T}^2$ are most conveniently translated into those of the dynamical system defined in $\mathbf{Z}^2$ (the dual group of $\mathbf{T}^2$) by the iteration of the matrix

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$

The matrix M is strictly hyperbolic, ie. has two (distinct) real eigenvalues given by

$$\lambda_+ = 1 + \theta , \quad \lambda_- = \lambda_+^{-1} , \quad \text{with } \theta = \frac{1+\sqrt{5}}{2} \tag{16}$$

and that the corresponding eigenvectors (generating the unstable and stable manifolds) are

$$e_+ = \mu \begin{pmatrix} \theta \\ 1 \end{pmatrix} , \ e_- = \mu \begin{pmatrix} 1 \\ -\theta \end{pmatrix} . \tag{17}$$

with $\mu = (1+\theta^2)^{-1/2}$. The vectors (e_+, e_-) define an orthonormal basis. In this basis M reduces the diagonal form

$$M \sim \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} .$$

In the next lemma we recall the following classical estimate (due to Kronecker).

Lemma 4. Let $\theta = \frac{1+\sqrt{5}}{2}$. Then:

(i) (best rational approximation for θ)

$$\inf_{(p,q)=1} |\theta - \frac{p}{q}| \geq \frac{1}{(1+\sqrt{5})q^2} \tag{18}$$

(ii) (diophantine estimate for θ)

$$\inf_{(p,q)\in \mathbf{Z}^2\setminus\{0\}} |q\theta - p| \geq \frac{1}{(1+\sqrt{5})\sup(|p|,|q|)} \tag{19}$$

Proof. The minimal polynomial of θ over $\mathbf{Q}$ is $P(X) = X^2 - X - 1 = (X-\theta)(X+\theta^{-1})$. Let p and q be two integers $(p,q) = 1$. First $P(\frac{p}{q}) \neq 0$ since P has no rational root, and therefore

$$|P(\tfrac{p}{q})| = \frac{|p^2 - qp - q^2|}{q^2} \geq \frac{1}{q^2} \tag{20}$$

On the other hand

$$|\tfrac{p}{q} - \theta| = \frac{|P(\frac{p}{q})|}{|\frac{p}{q} - \theta + \theta + \theta^{-1}|} = \frac{|P(\frac{p}{q})|}{|\frac{p}{q} - \theta + \sqrt{5}|} \geq \frac{1}{q^2(|\frac{p}{q} - \theta| + \sqrt{5})}.$$

Considering both cases $|\frac{p}{q} - \theta| > 1$ and $|\frac{p}{q} - \theta| < 1$, one obtains for any pair of integers $(p,\, q)$ with $q \neq 0$

$$|\tfrac{p}{q} - \theta| \geq \inf\left(1, \frac{1}{(1+\sqrt{5})q^2}\right) \tag{21}$$

which proves point i). Point ii) follows directly from point i). //

Proposition 5. Let $0 \leq \chi(R)$ be a nonincreasing positive function such that

$$\lim_{R\to\infty} \chi(R) = 0.$$

Consider the class of functions:

$$H_\chi = \{f \in L^2(\mathbf{T}^2) \text{ st. } \sum_{|k_1|,|k_2|>R} |\hat{f}(k)|^2 \leq \chi(R)^2 \|f\|_2^2\}.$$

Then
i) **"Rate of Mixing":** For all $(f,\, g) \in H_\chi$ such that $\langle f \rangle = \langle g \rangle = 0$

$$|\langle f \circ T^n \cdot g \rangle| \leq \tfrac{1}{2\pi^2} \|f\|_2 \|g\|_2 \chi\left(C_0 \left(\tfrac{3+\sqrt{5}}{2}\right)^{n/2}\right), \text{ with } C_0 = \frac{1+\sqrt{5}}{\sqrt{2}}. \tag{22}$$

ii) **Property (H1): "Exponential mixing":** For all $s > 0$ and all $f \in H^s(\mathbf{T}^2)$ such that $\langle f \rangle = 0$, the self-correlation coefficient defined as

$$C_f(n) = \langle f \circ T^n \cdot f \rangle$$

satifies the decay estimate:

$$|C_f(n)| \leq C\|f\|_s^2 e^{-sn\alpha} \text{ with } \alpha = \log\left(\tfrac{3+\sqrt{5}}{2}\right). \tag{23}$$

Proof . With the Plancherel formula one has, for any pair $(f,g) \in L^2(\mathbf{T}^2)$ with mean value $\langle f \rangle = \langle g \rangle = 0$, the formula:

$$\langle f \circ T^n \cdot g \rangle = \tfrac{1}{4\pi^2} \sum_{k \neq 0} \hat{f}(M^{-n}k)\hat{g}(-k) \tag{24}$$

For any $R > 0$, decompose the above sum into two parts corresponding to K_R and K_R^c with K_R given by

$$K_R = \{k \in \mathbf{Z}^2 \text{ s.t. } \sup(|k_1|, |k_2|) \leq R\}.$$

Since g belongs to the class H_χ, the Cauchy-Schwartz inequality yields the estimate:

$$\left|\sum_{k \in K_R^c} \hat{f}(M^{-n}k)\hat{g}(-k)\right| \leq \|f\|_2\|g\|_2\chi(R). \tag{25}$$

For $k \in K_R$, one introduces the decomposition $k = (k \cdot e_+)e_+ + (k \cdot e_-)e_-$; Kronecker's estimate (19) shows that

$$|k \cdot e_-| \geq \theta^{-1}|k|^{-1} \geq (\sqrt{2}R\theta)^{-1}$$

whence

$$|M^{-n}k| \geq \frac{\lambda_+^n}{\sqrt{2}R\theta}.$$

Using that $f \in H_\chi$ and that χ is nonincreasing, this implies the estimate:

$$\left|\sum_{k \in K_R - \{0\}} \hat{f}(M^{-n}k)\hat{g}(-k)\right| \leq \|f\|_2\|g\|_2\chi\left(\frac{\lambda_+^n}{\sqrt{2}R\theta}\right). \tag{26}$$

The relation (22) is obtained by choosing $R = \lambda_+^{n/2}$ in (25) and (26). To obtain the exponential rate of mixing (23), one specializes (22) to the case $f = g$ and uses, the expression $\chi(R) = R^{-s}(\|f\|_s\|f\|_2^{-1})^{1/2}$. //

Proof of Proposition 2. Let $k \in \mathbf{Z}^2 \setminus 0$. Since Ker $(M - \lambda_- I)$ is a line with irrational slope, the orthogonal projection of k on Ker $(M - \lambda_- I)$ is not 0 and therefore $|M^{-n}k| \to +\infty$ as $n \to +\infty$.

Let $a \in L^2(\mathbf{T}^2)$ and consider its Fourier series:

$$a(\omega) = \sum_{k \in \mathbf{Z}^2} \hat{a}(k)e^{ik\omega}$$

The relation $a = a \circ T$ shows that for any $k \in \mathbf{Z}^2$ and any n one has

$$\hat{a}(k) = \hat{a}(M^{-n}k)\,.$$

But $a \in l^2(\mathbf{Z}^2)$ since $a \in L^2(\mathbf{T}^2)$: hence for all $k \neq 0$ in $\mathbf{Z}^2$ $\hat{a}(k) = \hat{a}(M^{-n}k) \to 0$ as $n \to +\infty$. Therefore $\hat{a} \equiv 0$ on $\mathbf{Z}^2 \setminus 0$, which means that a is a constant. This proves the ergodic property $\mathrm{Ker}(I - U_T) = \mathbf{R}$. The same remark applies to the kernel of the adjoint $(I - U_T)^* = U_T^{-1}(U_T - I)$ in $L^2(\mathbf{T}^2)$ which also is reduced to the constants. Therefore the set of coboundaries is dense in the space of functions $a \in L^2(\mathbf{T}^2)$ such that $< a >= 0$.

For point 2), start with the formula

$$\langle a \circ T^k \otimes a \circ T^l \rangle = \langle a \circ T^{k-l} \otimes a \rangle\,. \tag{27}$$

which follows from the invariance of the measure $d\omega_1 d\omega_2$ under T. Summing with respect to $m = k - l$ yields

$$\langle \left(\sum_{k=0}^{N-1} a \circ T^k \right)^{\otimes 2} \rangle = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \langle a \circ T^{k-l} \otimes a \rangle = N \langle a^{\otimes 2} \rangle + 2 \sum_{m=1}^{N-1} (N - m) \langle a \circ T^m \otimes a \rangle \tag{28}$$

or in other words

$$\langle a^{\otimes 2} \rangle + 2 \sum_{m=0}^{N-1} \langle a \circ T^m \otimes a \rangle = \langle \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a \circ T^k \right)^{\otimes 2} \rangle + \frac{2}{N} \sum_{m=0}^{N-1} m \langle a \circ T^m \otimes a \rangle\,. \tag{29}$$

The exponential decay estimate (23) implies the absolute convergence of the series

$$\sum_{m=0}^{N-1} \langle a \circ T^m \otimes a \rangle$$

which appears in the left hand side of (12) (point 2) of proposition 2) and shows that

$$\sum_{m=0}^{N-1} m \langle a \circ T^m \otimes a \rangle \to 0\,.$$

Therefore (by Cesaro's Theorem) the last term of the right hand side of (29) goes to zero, leading to the relation

$$D(a) = \tfrac{1}{2} \lim_{N \to \infty} \langle \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a \circ T^k \right)^{\otimes 2} \rangle \tag{30}$$

and completing the proof of the point 2) of the proposition 2.

To prove point 3), write $a \sim b$ as $a - b = \phi - \phi \circ T$ and use the relation:

$$\xi \cdot D(b) \cdot \xi = \lim_{N \to \infty} || \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (b \circ T^k \cdot \xi) ||^2 =$$

$$\lim_{N \to \infty} || \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (a \circ T^k \cdot \xi + \phi \circ T^k \cdot \xi - \phi \circ T^{k+1} \cdot \xi) ||^2$$

$$\lim_{N \to \infty} || \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (a \circ T^k \cdot \xi) ||^2 = \xi \cdot D(a) \cdot \xi \,. \tag{31}$$

Since the matrix $D(a)$ is symmetric and nonnegative, points 4 i) and 4 ii) are equivalent. To prove that 4) ii) implies 4 iii) consider the function

$$r(z) = \tfrac{1}{2\pi} \sum_{k \in \mathbf{Z}} \langle a \circ T^k \cdot \xi, a \cdot \xi \rangle e^{ikz} \quad z \in \mathbf{T}^1, \quad \xi \in \mathbf{R}^d \,. \tag{32}$$

Observe that

$$r(0) = \tfrac{1}{2\pi} \sum_{k \in \mathbf{Z}} \langle a \circ T^k \cdot \xi, a \cdot \xi \rangle = \frac{1}{\pi} \xi \cdot D(a) \cdot \xi \tag{33}$$

and that, due to the decay estimate (23), $r(z)$ is holomorphic in the strip $B_{\alpha s} = \{ z \in \mathbf{C} \text{ s.t. } |\Im z| < \alpha s \}$. Therefore one has:

$$||f_N \cdot \xi||^2_{L^2(\mathbf{T}^2)} = \langle \left(\sum_{k=0}^{N-1} a \circ T^k \xi \right)^2 \rangle = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{r}(k-l) = \int_{-\pi}^{\pi} r(x) \left(\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} e^{-i(k-l)x} \right) dx$$

$$= \int_{-\pi}^{\pi} \frac{r(x) + r(-x) - 2r(0)}{2 \sin^2 \frac{x}{2}} \sin^2 (\tfrac{N}{2} x) dx + 2\pi N r(0) \,. \tag{34}$$

The first integral term in the right hand side of (34) is bounded because $r \in C^2(\mathbf{T}^1)$. Therefore the relation

$$0 = \xi \cdot D(a) \cdot \xi = \pi r(0)$$

implies that the series

$$f_N \cdot \xi = \sum_{k=1}^{N} a \circ T^k \cdot \xi$$

is (uniformly with respect to N) bounded in $L^2(\mathbf{T}^2)$.

To prove that iii) implies iv) observe that the sequence $f_N \cdot \xi$ is, in any case, the "formal inverse" for the equation

$$a \cdot \xi = g^{\xi} - g^{\xi} \circ T \,. \tag{35}$$

Since the sequence

$$f_N \cdot \xi = \sum_{k=1}^{N} a \circ T^k \cdot \xi$$

is bounded in $L^2(\mathbf{T}^2)$, one can consider f, one of its weak $L^2(\mathbf{T}^2)$ limit points and, using lemma 4, one has, for any function $\phi \in C^\infty(\mathbf{T}^2)$

$$\langle (f_N - f_N \circ T)\phi \rangle - \langle \xi \cdot a\phi \rangle = -\langle \xi \cdot a \circ T^{N+1}\, \phi \rangle \to 0\,, \text{ for } N \to \infty \tag{36}$$

which shows that $f - f \circ T = a \cdot \xi$. //

As we said above, the proof of theorem 2 follows in many respects the proof of the Ito formula. Therefore it will be important to decorrelate events occuring in two separate intervals of time, uniformly with respect to the size of these intervals, and under the only assumption that their mutual distance is large enough. This is dealt with in the next proposition. What we prove is a property similar to the "Very Weak Bernouilli property" as introduced by Ornstein-Weiss (cf. [bow]). The main difference with the classical formulation as in [bow] is the use of trigonometric polynomials instead of indicator functions of partitions.

Proposition 6. The transformation T has the following

Property (H2). There exist two constants $\beta_0 > 0$ and β_1 such that for all $l, m \in \mathbf{N}$, $U \subset \{n, ..., n+l\}$, $V \subset \{n, ..., n+m\}$ and for all pair of trigonometric polynomials P, Q, of degree less than R, one has, for all $n \geq \beta_0 \log R + \beta_1$

$$\langle \prod_{k \in U} P \circ T^{-k} \prod_{k \in V} Q \circ T^k \rangle - \langle \prod_{k \in U} P \circ T^{-k} \rangle \langle \prod_{k \in V} Q \circ T^k \rangle = 0 \tag{37}$$

Proof . This proof follows that of proposition 5. The method is similar to the one used by Katznelson [katz], but here a more precise result is needed and proven. Observe that it is enough to study expressions of the following type:

$$\frac{1}{4\pi^2} \int_{\mathbf{T}^2} \exp\left(i\omega \cdot \sum_{k \in U} M^{-k} \xi_k \right) \exp\left(i\omega \cdot \sum_{k \in V} M^k \eta_k \right) d\omega$$

$$- \left(\frac{1}{4\pi^2} \int_{\mathbf{T}^2} \exp\left(i\omega \cdot \sum_{k \in U} M^{-k} \xi_k \right) d\omega \right) \left(\frac{1}{4\pi^2} \int_{\mathbf{T}^2} \exp\left(i\omega \cdot \sum_{k \in V} M^k \eta_k \right) d\omega \right) \tag{38}$$

with $\xi_k \in K_R$, $\forall n \leq k \leq n+l$ and $\eta_k \in K_R$ $\forall n \leq k \leq n+m$ (where, as in the proof of the proposition 5, $K_R = \{k \in \mathbf{Z}^2 \text{ s.t. } \sup(|k_1|, |k_2|) \leq R\}$).

With the notations

$$X_U^- = \sum_{k \in U} M^{-k} \xi_k\,, \quad X_V^+ = \sum_{k \in V} M^k \eta_k\,,$$

the relation (37) is equivalent to the following assertion:

There exist $\beta_0 > 0$ and β_1 such that:

$$X_U^- + X_V^+ = 0\,, \;\Rightarrow\; X_U^- = 0\,, \text{ and } X_V^+ = 0\,, \quad \forall n > \beta_0 \log R + \beta_1\,,\ \forall l, m \in \mathbf{N}\,. \tag{39}$$

Denote by $I = \mathbf{R}e_+$ and $S = \mathbf{R}e_-$ the unstable and stable manifolds of M acting on $\mathbf{R}^2$. Let $\xi \in K_R$; one has

$$|X_U^- \cdot e_+| \le \sum_{k\in U} |M^{-n}\xi_k \cdot e_+| = \sum_{k\in U} |\xi_k \cdot M^{-n} e_+| = \sum_{k\in U} \lambda_-^k |\xi_k \cdot M^{-n} e_+|\,.$$

Proceeding likewise with X_V^+, for all ξ and η in K_R one has

$$|X_U^- \cdot e_+| \le \sqrt{2}R\frac{\lambda_-^n}{1-\lambda_-}\,, \qquad |X_V^+ \cdot e_-| \le \sqrt{2}R\frac{\lambda_-^n}{1-\lambda_-}\,, \tag{40}$$

This implies that X_U^- belongs to a neighborhood S_R^n of S while X_V^+ belongs to a neighborhood I_R^n of I given by the formulas:

$$I_R^n = \{X \in \mathbf{R}^2 \text{ s.t. } |X \cdot e_-| \le \sqrt{2}R\frac{\lambda_-^n}{1-\lambda_-}\}$$

$$S_R^n = \{X \in \mathbf{R}^2 \text{ s.t. } |X \cdot e_+| \le \sqrt{2}R\frac{\lambda_-^n}{1-\lambda_-}\}\,, \tag{41}$$

Since $X_U^- + X_V^+ = 0$, both X_U^- and X_V^+ belong to $K'_{R,n} = S_R^n \cap I_R^n$ (cf. fig 2) which for an n greater than a given value N_0 is contained in $K_{R,n}$. Whenever they are not both equal to zero, the diophantine estimate (19) of lemma 4 can be used to give:

$$|X_U^- \cdot e_+| \ge \frac{1}{(1+\sqrt{5})R}\,, \quad \text{or} \quad |X_V^+ \cdot e_-| \ge \frac{1}{(1+\sqrt{5})R}\,. \tag{42}$$

For $n > (2\log R + \log(\sqrt{2}(1+\sqrt{5}))/\log\lambda_+$, one has

$$\frac{1}{(1+\sqrt{5})R} > \sqrt{2}R\frac{\lambda_-^n}{1-\lambda_-}\,. \tag{43}$$

Therefore, if n is greater than

$$\sup\left(n_0, \frac{2\log R + \log(\sqrt{2}(1+\sqrt{5}))}{\log\lambda_+}\right)\,,$$

X_U^- and X_V^+ are both equal to zero. //

Corollary 7. Let $0 \le \chi(R)$ be a nonincreasing positive function going to 0 as R tends to infinity, such that

$$\chi(R) = O((1/\log R)^6) \quad \text{for} \quad R \to +\infty \tag{44}$$

Consider the class of functions defined by:

$$W_\chi = H_\chi \cap \{f \in L^\infty(\mathbf{T}^2) \text{ s.t. } \frac{1}{4\pi^2} \sum_{\sup(|k_1|,|k_2|)>R} |\hat{f}(k)| \le ||f||_\infty \chi(R)\,,\ \forall R > 0\} \tag{45}$$

Then
Property (H3) For all $f \in W_\chi$ such that $\langle f \rangle = 0$

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N} f \circ T^k$$

is uniformly bounded (with respect to N) in $L^4(\mathbf{T}^2)$.
Proof . The proof follows the same line as the one indicated by Ratner [rat] when the mapping T is replaced by any K system. For the sake of being complete, we give a proof based only on property (H2) (which is more restrictive than assuming the K property). First observe that

$$\langle |\sum_{k=0}^{N} f \circ T^k|^4 \rangle = \sum_{0\le k_1,k_2,k_3,k_4 \le N} \langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle$$

$$= 4! \sum_{0\le k_1 \le k_2 \le k_3 \le k_4 \le N} \langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle\,. \tag{46}$$

and introduce the following sets of indices:

$$A = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \le k_1 \le k_2 \le k_3 \le k_4 \le N\,,\ \sup_{2\le i \le 4} |k_i - k_{i-1}| \le N^{1/3}\}\,,$$

$$B = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \le k_1 \le k_2 \le k_3 \le k_4 \le N\,,\ k_2 - k_1 > N^{1/3}\}\,,$$
$$C = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \le k_1 \le k_2 \le k_3 \le k_4 \le N\,,\ k_3 - k_2 > N^{1/3}\}\,,$$
$$D = \{(k_1, k_2, k_3, k_4) \text{ s.t. } 0 \le k_1 \le k_2 \le k_3 \le k_4 \le N\,,\ k_4 - k_3 > N^{1/3}\}\,.$$

One has

$$A \cap B = A \cap C = A \cap D = \emptyset \quad \text{and} \quad 0, 1, ..., N^4 = A \cup B \cup C \cup D \tag{47}$$

with the subsets B, C D having non empty intersection. ¿From the relation (47) one deduce the estimate:

$$\langle |\sum_{k=0}^{N} f \circ T^k|^4 \rangle \le 4! \sum_{(k_1,k_2,k_3,k_4)\in A} |\langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle|$$

$$+4! \sum_{(k_1,k_2,k_3,k_4)\in B} |\langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle|$$

$$+4! \sum_{(k_1,k_2,k_3,k_4)\in C} |\langle f\circ T^{k_1} f\circ T^{k_2} f\circ T^{k_3} f\circ T^{k_4}\rangle|$$

$$+4! \sum_{(k_1,k_2,k_3,k_4)\in D} |\langle f\circ T^{k_1} f\circ T^{k_2} f\circ T^{k_3} f\circ T^{k_4}\rangle|\,. \tag{48}$$

To prove the corollary we show that all the terms which appears in (48) are uniformly bounded with respect to N^2. This will be done in three steps: step 1: terms of (48) with support in A; step 2: terms with support in B or D (the proof being similar in these two cases); and step 3: terms with support in C. For steps 2 and 3 the Fourier expansion of f truncated at degree R will be used; it is denoted by $P_R(f)$ and since $f\in W_\chi$ one has;

$$|f(\omega)-P_R(f)(\omega)|\le \chi(R)||f||_\infty\,. \tag{50}$$

Step 1. Summation with support in A.

Observe that $\sharp A\le 3!N(N^{1/3}+1)^3$; this implies the estimate:

$$\sum_{(k_1,k_2,k_3,k_4)\in A} |\langle f\circ T^{k_1} f\circ T^{k_2} f\circ T^{k_3} f\circ T^{k_4}\rangle| \le 3!N(N^{1/3}+1)^3||f||_\infty^4\,. \tag{51}$$

Step 2. Summation with support in B and D.

As already noticed, the proof is similar for these two terms and therefore only the sum with support in B is considered.

Since f belongs to W_χ, estimate (50) shows that

$$\sum_{(k_1,k_2,k_3,k_4)\in B} |\langle f\circ T^{k_1} f\circ T^{k_2} f\circ T^{k_3} f\circ T^{k_4}\rangle|$$

$$\le \sum_{(k_1,k_2,k_3,k_4)\in B} |\langle P_R(f)\circ T^{k_1} P_R(f)\circ T^{k_2} P_R(f)\circ T^{k_3} P_R(f)\circ T^{k_4}\rangle|$$

$$+4N^4||f||_\infty^4\chi(R)(\sup(\chi(R),1))^3\,. \tag{52}$$

To use property (H2), write:

$$\langle P_R(f)\circ T^{k_1} P_R(f)\circ T^{k_2} P_R(f)\circ T^{k_3} P_R(f)\circ T^{k_4}\rangle =$$

$$\langle P_R(f)\circ T^{k_1-\kappa} P_R(f)\circ T^{k_2-\kappa} P_R(f)\circ T^{k_3-\kappa} P_R(f)\circ T^{k_4-\kappa}\rangle \tag{53}$$

In (53) choose $\kappa=[\frac{k_1+k_2}{2}]$, notice that $P_R(f)$ is of mean value zero (because f is of mean value zero) and apply proposition 6 to the sets

$$U=\{\kappa-k_1\},\quad V=\{k_2-\kappa,k_3-\kappa,k_4-\kappa\}$$

It shows that, for

$$R=\exp(\frac{N^{1/3}-2\beta_1}{2\beta_0})$$

and for any $(k_1, k_2, k_3, k_4) \in B$, one has

$$\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle = 0 \,. \tag{54}$$

Therefore with $\chi(R) = O((1/\log R)^6)$ for $R \to +\infty$, the estimate

$$\sum_{(k_1,k_2,k_3,k_4)\in B} |\langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle| \leq$$

$$4N^4 ||f||_\infty^4 \chi(\exp(\frac{N^{1/3} - 2\beta_1}{2\beta_0})) = O(N^2) \tag{55}$$

is deduced from (52).

Step 3. Summation with support in C. As in step 2 write:

$$\sum_{(k_1,k_2,k_3,k_4)\in C} |\langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle|$$

$$\leq \sum_{(k_1,k_2,k_3,k_4)\in C} |\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle|$$

$$+4N^4 ||f||_\infty^4 \chi(R) (\sup(\chi(R), 1))^3 \,. \tag{56}$$

Then for any $(k_1, k_2, k_3, k_4) \in C$, apply proposition 6 to the sets

$$U = \{\kappa - k_1, \kappa - k_2\}, \qquad \text{and} \qquad V = \{k_3 - \kappa, k_4 - \kappa\}$$

with $\kappa = [\frac{k_2+k_3}{2}]$. It shows that, for

$$R = \exp(\frac{N^{1/3} - 2\beta_1}{2\beta_0})$$

one has:

$$\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle =$$
$$\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} \rangle \langle P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle \,. \tag{57}$$

Therefore

$$\sum_{(k_1,k_2,k_3,k_4)\in C} |\langle f \circ T^{k_1} f \circ T^{k_2} f \circ T^{k_3} f \circ T^{k_4} \rangle|$$

$$\leq \sum_{(k_1,k_2,k_3,k_4)\in C} |\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} \rangle \langle P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle| + 4N^4 ||f||_\infty^4 \chi(R) \,. \tag{58}$$

On the other hand:

$$\sum_{(k_1,k_2,k_3,k_4)\in C} |\langle P_R(f) \circ T^{k_1} P_R(f) \circ T^{k_2} \rangle \langle P_R(f) \circ T^{k_3} P_R(f) \circ T^{k_4} \rangle|$$

$$\leq \sum_{0\leq k_1,k_2,k_3,k_4\leq N} |\langle P_R(f)\circ T^{k_1} P_R(f)\circ T^{k_2}\rangle||\langle P_R(f)\circ T^{k_3} P_R(f)\circ T^{k_4}\rangle|$$

$$= \left(\sum_{0\leq k_1,k_2\leq N} |\langle P_R(f)\circ T^{k_1} P_R(f)\circ T^{k_2}\rangle|\right)^2 . \tag{59}$$

Since f belongs to the class $W_\chi \subset H_\chi$, $P_R(f)$ also belongs to the class H_χ and the estimate (22) of proposition 5 can be used to give

$$|\langle P_R(f)\circ T^{k_1} P_R(f)\circ T^{k_2}\rangle| \leq \tfrac{1}{2\pi^2}||f||_2^2\chi(C_0\lambda_+^{(k_2-k_1)/2}) \tag{60}$$

The sum $\sum_{n\geq 0}\chi(C_0\lambda_+^{n/2})$ converges (by assumption: see (44)) and therefore:

$$\left(\sum_{0\leq k_1,k_2\leq N} |\langle P_R(f)\circ T^{k_1} P_R(f)\circ T^{k_2}\rangle|\right)^2$$

$$\leq (\tfrac{1}{2\pi^2}||f||_2^2)^2 \sum_{0\leq k_1,k_2\leq N} \chi(C_0\lambda_+^{(k_2-k_1)/2}) = O(N^2) \tag{61}$$

With (56), (57), (58) and (61) one obtains, for

$$R = \exp(\frac{N^{1/3}-2\beta_1}{2\beta_0}),$$

the formula

$$\sum_{(k_1,k_2,k_3,k_4)\in C} |\langle f\circ T^{k_1} f\circ T^{k_2} f\circ T^{k_3} f\circ T^{k_4}\rangle|$$

$$\leq 4N^4||f||_\infty^4\chi(\exp(\frac{N^{1/3}-2\beta_1}{2\beta_0})) + O(N^2) \leq CN^2 .$$

which concludes the proof of step 3 and of corollary 7. //

To further extend the decorrelation properties, this section is concluded with proposition 8 involving functions $f\in H^s(\mathbf{R}^d)$ and corollary 9 which allows to consider smooth functions with subquadratic growth at infinity.

Proposition 8. Assume that the vector field a is smooth enough (say $a\in C^3(\mathbf{T}^2,\mathbf{R}^2)$) and satisifes $< a >= 0$. Then for any pair of functions f and $g\in H^s(\mathbf{R}^d)$ with $d/2 < s$ one has:

$$|\langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]} a\circ T^k))g(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a\circ T^l\rangle - \langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]} a\circ T^k))\rangle\langle g(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a\circ T^l\rangle|$$

$$\leq ||f||_s||g||_s\left(C\frac{\epsilon^{s-d/2}}{s-d/2}+\epsilon^{-d}H(K,\delta,\epsilon)\right) \tag{62}$$

with $H(K,\delta,\epsilon)$ given by:

$$H(K,\delta,\epsilon)=\frac{C"}{\epsilon^2}\exp\left(-\frac{C'\delta}{\epsilon^2}\right)\exp\left(\frac{C"}{\epsilon^2}\exp\left(-\frac{C'\delta}{\epsilon^2}\right)\right) \tag{63}$$

In (62) and (63) C, C' and $C"$ denote some constants independent of ϵ, δ and t, $t+\tau$ in the bounded interval $[0,K]$.

Proof . First represent f and g in terms of their Fourier transforms:

$$f(x)=\int_{\mathbf{R}^d}e^{i\xi\cdot x}\hat{f}(\xi)\tfrac{d\xi}{(2\pi)^d}\,,\quad g(x)=\int_{\mathbf{R}^d}e^{i\eta\cdot x}\hat{g}(\eta)\tfrac{d\eta}{(2\pi)^d} \tag{64}$$

and observe that since f and g belong to $H^s(\mathbf{R}^d)$ the above integrals can, in the sequel, be replaced by

$$f_\epsilon(x)=\int_{\mathbf{R}^d\cap\{|\epsilon\xi|\leq 1\}}e^{i\xi\cdot x}\hat{f}(\xi)\tfrac{d\xi}{(2\pi)^d}\,,\quad g_\epsilon(x)=\int_{\mathbf{R}^d\cap\{|\epsilon\xi|\leq 1\}}e^{i\eta\cdot x}\hat{g}(\eta)\tfrac{d\eta}{(2\pi)^d}\,. \tag{65}$$

This truncation introduces in the proof an error of the order of $||f||_s||g||_s\frac{\epsilon^{s-d/2}}{s-d/2}$ which correspond to the first term in the right hand side of (62) and reduces the proof of the proposition 8 to estimating the expression:

$$\epsilon^{-d}\sup_{|\xi|\leq 1,|\eta|\leq 1}\Big|\langle\prod_{k=0}^{[ct/\epsilon^2h]}\exp(i\xi\cdot\epsilon ha\circ T^k)\prod_{l=[c(t+\delta)/\epsilon^2h]}^{[c(t+\tau)/\epsilon^2h]}\exp(i\eta\cdot\epsilon ha\circ T^l)\rangle$$

$$-\langle\prod_{k=0}^{[ct/\epsilon^2h]}\exp(i\xi\cdot\epsilon ha\circ T^k)\rangle\langle\prod_{l=[c(t+\delta)/\epsilon^2h]}^{[c(t+\tau)/\epsilon^2h]}\exp(i\eta\cdot\epsilon ha\circ T^l)\rangle\Big|\,. \tag{66}$$

However proposition 6 (property (H2) or relation (37)) cannot be directly applied to the above formula because the functions $\exp(i\xi\cdot\epsilon ha)$ and $\exp(i\eta\cdot\epsilon ha)$ are not trigonometric polynomials in the variable ω. Therefore these functions have to be approximated by their truncated Fourier series. In order to do so it is convenient to introduce the following notations:

$$\theta_\xi(\omega)=\exp(i\xi\cdot ha(\omega)),\ P_R(\theta_\xi)=\sum_{\sup(|k_1|,|k_2|<R)}\hat{\theta}_\xi(k)\exp(ik\cdot\omega)$$

$$\text{and } A_{\xi,R}=\sum_{\sup(|k_1|,|k_2|\geq R)}|\hat{\theta}_\xi(k)|\,. \tag{67}$$

Assuming that $a \in C^3(\mathbf{T}^2, \mathbf{R}^2)$, the quantities $A_{\xi,R}$ are finite and one has:

$$||P_R(\theta_\xi) - \theta_\xi||_\infty \leq \sum_{\sup(|k_1|,|k_2| \geq R} |\hat{\theta}_\xi(k)| = A_{\xi,R},$$

$$|||P_R(\theta_\xi)| - 1||_\infty \leq ||P_R(\theta_\xi) - \theta_\xi||_\infty \leq A_{\xi,R}, \quad ||P_R(\theta_\xi)||_\infty \leq (1 + A_{\xi,R}). \tag{68}$$

and

$$\forall \sigma, \quad 0 < \sigma < 1 \qquad A_R = \sup_{|\xi| \leq 1} A_{\xi,R} \leq \frac{C}{(1-\sigma)R^{1-\sigma}} \tag{69}$$

Going back to (66), one has the inequality:

$$|\langle \prod_{k=0}^{[ct/\epsilon^2 h]} \theta_{\epsilon\xi} \circ T^k \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} \theta_{\epsilon\eta} \circ T^k \rangle - \langle \prod_{k=0}^{[ct/\epsilon^2 h]} \theta_{\epsilon\xi} \circ T^k \rangle \langle \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} \theta_{\epsilon\eta} \circ T^k \rangle|$$

$$\leq |\langle \prod_{k=0}^{[ct/\epsilon^2 h]} P_R(\theta_{\epsilon\xi}) \circ T^k \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} P_R(\theta_{\epsilon\eta}) \circ T^k \rangle$$

$$- \langle \prod_{k=0}^{[ct/\epsilon^2 h]} P_R(\theta_{\epsilon\xi}) \circ T^k \rangle \langle \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} P_R(\theta_{\epsilon\eta}) \circ T^k \rangle|$$

$$+ |\langle \prod_{k=0}^{[ct/\epsilon^2 h]} \theta_{\epsilon\xi} \circ T^k \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} \theta_{\epsilon\eta} \circ T^k \rangle$$

$$- \langle \prod_{k=0}^{[ct/\epsilon^2 h]} P_R(\theta_{\epsilon\xi}) \circ T^k \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} P_R(\theta_{\epsilon\eta}) \circ T^k \rangle|$$

$$+ |\langle \prod_{k=0}^{[ct/\epsilon^2 h]} \theta_{\epsilon\xi} \circ T^k \rangle \langle \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} \theta_{\epsilon\eta} \circ T^k \rangle$$

$$- \langle \prod_{k=0}^{[ct/\epsilon^2 h]} P_R(\theta_{\epsilon\xi}) \circ T^k \rangle \langle \prod_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} P_R(\theta_{\epsilon\eta}) \circ T^k \rangle|. \tag{70}$$

The property (H3) can be used for the first term of the right hand side of (70) which turns out to be zero for

$$R = \exp(\frac{1}{\beta_0}([c(t+\delta)/\epsilon^2 h] - [ct/\epsilon^2 h] - \beta_1)). \tag{71}$$

The estimates (68) and (69) can be used to control the the second and the third term of (70), with $|\epsilon\xi| \leq 1$ and $|\epsilon\eta| \leq 1$ their sum is bounded by:

$$2A_R \left[\frac{c(t+\tau)}{\epsilon^2 h}\right] (1 + A_R)^{[c(t+\tau)/\epsilon^2 h]} \leq 2A_R \left[\frac{c(t+\tau)}{\epsilon^2 h}\right] \exp(A_R \left[\frac{c(t+\tau)}{\epsilon^2 h}\right]). \tag{72}$$

With (69) and the choice of R given by (71), the right hand side of (72) satisfies the following estimate:

$$A_R\left[\frac{c(t+\tau)}{\epsilon^2 h}\right]\exp(A_R\left[\frac{c(t+\tau)}{\epsilon^2 h}\right])\leq \frac{C"}{\epsilon^2}\exp\left(-\frac{C'\delta}{\epsilon^2}\right)\exp\left(\frac{C"}{\epsilon^2}\exp\left(-\frac{C'\delta}{\epsilon^2}\right)\right) \quad (73)$$

which leads to the function $H(K,\delta,\epsilon)$ and completes the proof. //

Corollary 9. Assume that the vector field a satisfies the assumptions of proposition 8. Let $f\in C_0^\infty(\mathbf{R}^d)$ and $g\in C^\infty(\mathbf{R}^d)$ with subquadratic growth at infinity as follows

$$|g(x)|+\sum_{1\leq l\leq[d/2+1]}|\nabla_x g(x)|\leq C_g(1+|x|^2) \quad (74)$$

Then for any positive constant M

$$|\langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]}a\circ T^k)g(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]}a\circ T^l)\rangle-\langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]}a\circ T^k)\rangle\langle g(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]}a\circ T^l)\rangle|$$

$$\leq 2||f||_\infty C_g\sup_{N>0}||(\frac{1}{\sqrt{N}}\sum_{k=0}^{N}a\circ T^k)||_4^3\left(\frac{\sqrt{\tau-\delta+2\epsilon^2}}{M}\right)^2$$

$$+C_g||f||_s M^{\frac{d+2}{2}}\left(C_1\frac{\epsilon^{s-d/2}}{s-d/2}+\epsilon^{-d}H(K,\delta,\epsilon)\right) \quad (75)$$

where s is chosen equal to $[d/2+1]$, $||f||_s$ represent the H^s Sobolev norm of f and $H(K,\delta,\epsilon)$ is the function defined by (63). As before, the constants are independent of t,τ,ϵ and δ.

Proof . Introduce a cutoff function $\chi_M\in C^\infty(\mathbf{R}^2)$ with the following properties:

$$\forall x,\ 0\leq\chi_M\leq 1,\ \chi_M(x)=1 \text{ if } |x|\leq M,\ \chi_M(x)=0 \text{ for } |x|\geq 2M \quad (76)$$

and denote by g_M the function $g\chi_M$. ¿From the formula $|g-g_M|\leq|g|\mathbf{1}_{|x|\geq M}$ one deduces the estimate:

$$|\langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]}a\circ T^k)g(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]}a\circ T^l)\rangle-\langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]}a\circ T^k)\rangle\langle g(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]}a\circ T^l)\rangle|$$

$$\leq|\langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]}a\circ T^k)g_M(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]}a\circ T^l)\rangle$$

$$-\langle f(\epsilon h\sum_{k=0}^{[ct/\epsilon^2 h]}a\circ T^k)\rangle\langle g_M(\epsilon h\sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]}a\circ T^l)\rangle|$$

$$+2||f||_\infty \int_{|\epsilon h \sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a\circ T^l(\omega)|\geq M} |g(\epsilon h \sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a \circ T^l(\omega))| d\omega \tag{77}$$

Corollary 7 shows that

$$\text{mes}\, \{\omega \text{ s.t. } |\epsilon h \sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a \circ T^l(\omega)| \geq M\}$$

$$\leq \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_4^4 \left(\frac{\sqrt{\tau - \delta + 2\epsilon^2}}{M}\right)^4 . \tag{78}$$

Using the Cauchy-Schwarz inequality and the subquadratic growth of g (74), this implies

$$\int_{|\epsilon h \sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a\circ T^l(\omega)|\geq M} |g(\epsilon h \sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a \circ T^l(\omega))| d\omega$$

$$\leq \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_4^3 \left(\frac{\sqrt{\tau - \delta + 2\epsilon^2}}{M}\right)^2 . \tag{79}$$

Therefore, with estimate (62), one concludes that

$$|\langle f(\epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a\circ T^k) g(\epsilon h \sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a\circ T^l)\rangle - \langle f(\epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a\circ T^k)\rangle \langle g \epsilon h \sum_{l=[c(t+\delta)/\epsilon^2 h]}^{[c(t+\tau)/\epsilon^2 h]} a\circ T^l)\rangle|$$

$$\leq 2||f||_\infty C_g \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_4^3 \left(\frac{\sqrt{\tau - \delta + 2\epsilon^2}}{M}\right)^2$$

$$+||f||_s ||g_M||_s \left(C \frac{\epsilon^{s-d/2}}{s-d/2} + \epsilon^{-d} H(K, \delta, \epsilon)\right) \tag{80}$$

Since (74) shows that $||g_M||_s$ is bounded by $C_g M^{\frac{d+2}{2}}$, (80) completes the proof of corollary 9. //

4.— Proof of Theorem 2.

As it was said in the introduction the proof of theorem 2 is inspired by the proof the Ito formula for the Brownian motion. Therefore the starting point is the Taylor formula at order three for the increment.

$$\langle \psi_\epsilon(t+\tau, x, \cdot, \cdot)\rangle - \langle \psi_\epsilon(t, x, \cdot, \cdot)\rangle$$

$$= \langle \nabla\phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) \cdot \epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega)\rangle$$

$$+\tfrac{1}{2}\langle \nabla^2\phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) : \left(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega)\right)^{\otimes 2} \rangle$$

$$+O(\langle \left|(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))\right|^3 \rangle). \tag{81}$$

The analysis of the limit for $\epsilon \to 0$ in the above expression will be done in 3 steps and C will be used to denote any constant independant of ϵ and τ.

Step 1. Estimate of the remainder. One has

$$\langle |(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))|^3\rangle \leq \langle |(\epsilon h \sum_{k=0}^{[c\tau)/\epsilon^2 h]} a(T^k\omega))|^3\rangle + O((\epsilon h)^3) \tag{82}$$

or, using Hölder's inequality with $N = [c\tau/\epsilon^2 h]$

$$\langle |(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))|^3\rangle$$

$$\leq \sqrt{2\pi}(\epsilon h\sqrt{N})^3(\langle \left|\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a(T^k\omega))\right|^4 \rangle)^{3/4} + O((\epsilon h)^3) \tag{83}$$

Using corollary 7, the following bound is deduced from (83)

$$\limsup_{\epsilon\to 0} \langle \left|(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))\right|^3 \rangle \leq C\tau^{3/2}(\sqrt{ch})^{3/2} \tag{84}$$

Step 2. Decorrelation in (81). Since the treatment of the linear term is simpler than the treatment of the quadratic term (but follows the same lines) , only the latter, will be considered in detail. A "small" positive time δ is introduced and one has:

$$|\langle \nabla_x^2\phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) : (\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))^{\otimes 2}\rangle$$

$$-\langle \nabla_x^2\phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) : (\epsilon h \sum_{k=[c(t+\delta)/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))^{\otimes 2}\rangle|$$

$$\leq \langle |\nabla_x^2 \phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega))| \, |(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\delta)/\epsilon^2 h]} a(T^k\omega))|^2 \rangle$$

$$+2\langle |\nabla_x^2 \phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega))| \, |(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\delta)/\epsilon^2 h]} a(T^k\omega))|^2 |(\epsilon h \sum_{k=[c(t+\delta)/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))|^2 \rangle \tag{85}$$

The first term of the right hand side of (85) is bounded by:

$$||\nabla_x^2 \phi||_\infty ||(\epsilon h \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\delta)/\epsilon^2 h]} a \circ T^k)||_2^2$$

$$\leq ||\nabla_x^2 \phi||_\infty \, 2\pi h \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_4^2 (\delta + 2\epsilon^2). \tag{86}$$

Similarly the second term is bounded by:

$$||\nabla_x^2 \phi||_\infty \, 2\pi h \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_4^2 \sqrt{(\tau + 2\epsilon^2)(\delta + 2\epsilon^2)}. \tag{87}$$

The only remaining term is

$$\langle \nabla_x^2 \phi(x - \frac{\epsilon h}{c} \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) : (\frac{\epsilon h}{c} \sum_{k=[c(t+\delta)/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))^{\otimes 2} \rangle$$

Here corollary 9 is used with $g(x) = x^{\otimes 2}$ leading to the following estimate:

$$|\langle \nabla_x^2 \phi(x - \frac{\epsilon h}{c} \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) : (\frac{\epsilon h}{c} \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))^{\otimes 2} \rangle$$

$$-\langle \nabla_x^2 \phi(x - \frac{\epsilon h}{c} \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)) \rangle : \langle (\frac{\epsilon h}{c} \sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))^{\otimes 2} \rangle |$$

$$\leq ||\nabla_x^2 \phi||_\infty \, 2\pi h \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_4^2 (\delta + 2\epsilon^2)$$

$$+||\nabla_x^2 \phi||_\infty \, 2\pi h \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_4^2 \sqrt{(\tau + 2\epsilon^2)(\delta + 2\epsilon^2)}$$

$$+||\nabla_x^2 \phi||_\infty \sup_{N>0} ||(\frac{1}{\sqrt{N}} \sum_{k=0}^{N} a \circ T^k)||_2^2 \, \delta$$

$$+2\|\nabla_x^2\phi\|_\infty \sup_{N>0} \|(\frac{1}{\sqrt{N}}\sum_{k=0}^{N} a\circ T^k)\|_4^3 \left(\frac{\sqrt{\tau-\delta+2\epsilon^2}}{M}\right)^2$$

$$+C\|\nabla_x^2\phi\|_s M^{\frac{d+2}{2}}(C_1\frac{\epsilon^{s-d/2}}{s-d/2}+\epsilon^{-d}H(K,\delta,\epsilon)). \tag{88}$$

For a given $\tau\in(0,1)$, and $0<\epsilon<\tau$ one uses the special form of the function $H(K,\delta,\epsilon)$; it shows that, by choosing

$$\delta=\epsilon \text{ and } M=\epsilon^{-\frac{(2s-d)}{4(d+2)}}$$

one has

$$\limsup_{\epsilon\to 0}\{|\langle\nabla_x^2\phi(x-\frac{\epsilon h}{c}\sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega)):(\frac{\epsilon h}{c}\sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))^{\otimes 2}\rangle$$

$$-\langle\nabla_x^2\phi(x-\frac{\epsilon h}{c}\sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega))\rangle:\langle(\frac{\epsilon h}{c}\sum_{k=[ct/\epsilon^2 h]+1}^{[c(t+\tau)/\epsilon^2 h]} a(T^k\omega))^{\otimes 2}\rangle|\}=0. \tag{89}$$

Step 3. Weak and strong limits. Denote by $u_\epsilon(t,x)$ the family of functions $\langle\psi_\epsilon(t,x,\cdot)\rangle$. Starting with (81), using the estimate for the remainder (84), the L^∞ bounds on $\nabla\phi$ and $\nabla^2\phi$, the inequality

$$\#\{k \text{ s.t. } [ct/\epsilon^2 h]+1\le k\le [c(t+\tau)/\epsilon^2 h]\}\le [c\tau/\epsilon^2 h], \tag{90}$$

and property (H3) from corollary 7, we arrive at the (uniform in $\tau\in[0,1]$ and $\epsilon\in[0,1]$) bound:

$$|u_\epsilon(t+\tau,x)-u_\epsilon(t,x,\cdot)|\le C\sqrt{\tau}. \tag{91}$$

It follows from Ascoli's theorem that the family u_ϵ is relatively compact in $C^0([0,\tau];w^*-$ $-L^\infty(\mathbf{R}^d))$. Let u be a limit point of this family and rename as usual, $f_\epsilon^\pm$, ψ_ϵ and u_ϵ the corresponding subfamilies, with u_ϵ converging to u. Starting from the "Ito" formula (82), using (85) and (89) one obtains:

$$u(t+\tau,x)-u(t,x)=\lim_{\epsilon\to 0}\{u_\epsilon(t+\tau,x)-u_\epsilon(t,x)\}$$

$$=\lim_{\epsilon\to 0}\{\tfrac{1}{2}\nabla_x^2 u_\epsilon(t,x):\lim_{\epsilon\to 0}\langle(\frac{\epsilon h}{c}\sum_{k=0}^{[c(t+\tau)/\epsilon^2 h]-[ct/\epsilon^2 h]-1} a(T^k\omega))^{\otimes 2}\rangle\}+O(\tau)^{4/3} \tag{92}$$

or with (12) (cf. point 2 of proposition 2.)

$$u(t+\tau,x)-u(t,x)=\tfrac{1}{2}h\tau D(a):\nabla_x^2 u(t,x)+O(\tau)^{4/3}. \tag{93}$$

Dividing (93) by τ and letting τ go to zero, one sees that u solves the initial value problem for the diffusion equation:

$$\frac{\partial u}{\partial t}-\tfrac{1}{2}D(a):\nabla_x^2 u,\ \ u(x,0)=\phi(x) \tag{94}$$

The solution of (94) being unique, it follows that the whole families

$$\langle f_\epsilon^{\pm}(t,x,\omega)\rangle \text{ or } u_\epsilon(x,t) = \langle \psi_\epsilon(t,x,\omega)\rangle = \langle \phi(x - \epsilon h \sum_{k=0}^{[ct/\epsilon^2 h]} a(T^k\omega))\rangle$$

converge in $C^0([0,\tau]; w^* - -L^\infty(\mathbf{R}^d))$ to $u(t,x)$.

Observing that the problem (7)-(10) is translation invariant in the variable $x \in \mathbf{R}^d$ and using the regularity of $\phi \in C^2(\mathbf{R}^d)$, one can see that the family of functions $u_\epsilon(t,x)$ satisfies the (uniform in ϵ) estimate:

$$\|u_\epsilon(t,x)\|_{L^\infty(\mathbf{R}_t^+, C^2(\mathbf{R}_x^d))} \leq C \tag{95}$$

Acoli's theorem, (91) and (95) show that for any $\tau > 0$ and any compact $K \subset \mathbf{R}^d$, the sequence $u_\epsilon(t,x)$ converges to $u(t,x)$ in $C([0,\tau] \times K)$. This argument completes the proof of the strong convergence of the averages (14a).

Finally let f be in the w^* closure in $L^\infty(\mathbf{R}_t^+ \times \mathbf{R}_x^d \times \mathbf{T}^2)$ of the family $\psi_\epsilon(t,x,\omega)$. One deduces from (11) that f is invariant under the action of T

$$f(t,x,\omega) = f(t,x,T\omega); \tag{96}$$

The ergodicity property (proposition 2 point 1)) implies that f is independent of ω and therefore coincide with the function $u(t,x)$: this demonstrates the convergence (14b) and-concludes the proof of theorem 3. //

5.— Final Remarks and Numerical Experiments..

The model (3), (4a), (4b) and (5) defines a global broken hamiltonian flow which induces an isometry on $L^p(\mathbf{R}^d \times \mathbf{T}^2)$ for all $1 \leq p \leq \infty$. In particular the quantity $\|f_\epsilon^+(t,\cdot,\cdot)\|_2 + \|f_\epsilon^-(t,\cdot,\cdot)\|_2$ is conserved for any $\epsilon > 0$ at variance with the quantity $\|u(t,\cdot)\|_2$. Therefore the type of convergence which is given in the theorem 3 (strong for the average and weak for the solution itself) is optimal with the exception of the trivial case where $D(a) = 0$ which correspond to no diffusion. In this situation the solution exhibits an initial layer near $t = 0$ which is taken care of by the time scaling and the solution $f_\epsilon^{\pm}(x,t,z\omega)$ converges strongly to its initial value $\phi(x)$

The paradox of deriving a well posed irreversible problem for $t > 0$ from a reversible problem can be explained from the following facts.

1) The scaling has been done with the a priori choice of considering the solution for large positive times;

2) It gives the correct approximation at the order ϵ of local averages of the solution in term of the local averages of its initial data; the dependence on ω is lost in the approximation. In some sense, this decay of information can be estimated by the decay of the L^2 norm of u which, for the diffusion equation, is the linearized version of the classical entropy.

The relevance of the above remarks depends of course on the analysis of the strict positivity of the diffusion matrix $D(a)$. In fact it results from proposition 2 that $D(a)=0$ for a varying in a dense subset of $L^2(\mathbf{T}^2)/\mathbf{R}$ (isomorphic to the space of mean zero functions in $L^2(\mathbf{T}^2)$). Furthermore the diffusion is degenerate in the directions ξ for which the "excursion length":

$$f_N \cdot \xi = \sum_{k=1}^{N} (a \circ T^k) \cdot \xi$$

is (uniformly with respect to N) bounded in $L^2(\mathbf{T}^2)$, an observation which turns out to be in agreement with the intuition.

Although the space of coboundaries is dense in the subspace of $L^2(\mathbf{T}^2)$ consisting of mean zero functions, it is not L^2 closed. In other words, it is possible to find smooth funtions of $L^2(\mathbf{T}^2)$ not being coboundaries. For example, observe that for $d=1$ and $a(\omega_1,\omega_2)=\cos\omega_1$ one has $D(a)=\frac{1}{4}$.

More generally it would be extremely useful to have an explicit expression of $D(a)$. For $d=1$ it can be easily obtained in term of the Fibonnaci sequence F_n

$$F_0=0,\ F_1=1,\ F_n=F_{n-1}+F_{n-2} \tag{97}$$

Observe that the minimal polynomial of the matrix M is X^2-3X+1 and introduce the matrix $P-I$ which has for minimal polynomial X^2-X-1, then on one hand one has

$$M=P+1=P^2 \tag{98}$$

and on the other hand

$$P^n=P^{n-1}+P^{n-2} \tag{99}$$

and this implies the formula:

$$M^n=\begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}$$

Therefore, with the diffusion matrix given by (12) one has, for any function $f:\mathbf{T}^2\to\mathbf{R}$ in H^s with $s>0$:

$$D(f)-\tfrac{1}{2}\langle f^2\rangle=\sum_{n\geq 1}\langle f\circ T^n\cdot f\rangle=\frac{1}{4\pi^2}\sum_{n\geq 1}\sum_{k\in\mathbf{Z}^2-\{0\}}\hat{f}(M^n k)\hat{f}(-k)$$

$$=\frac{1}{4\pi^2}\sum_{k\in\mathbf{Z}^2-\{0\}}\sum_{n\geq 1}\hat{f}(F_{2n+2}k_1+F_{2n+1}k_2,F_{2n+1}k_1+F_{2n-1}k_2)\hat{f}(-k_1,-k_2) \tag{100}$$

In particular, since the Fibonnaci sequence is rapidly increasing this will provide and exact formula (involving a small number of non zero terms) for $D(f)$ whenever f is a trigonometric polynomial.

The numerical experiment were done by the third author. They intend to illustrate the difference beetween the diffusive and the non diffusive case. In two space variables the trajectories of 128 particles over 1000 interactions with the upper and lower boundary has been obtained. The diffusive case (see fig. 3) corresponds to horizontal velocity field:

$$a(\omega) = (a_1(\omega),\ a_2(\omega)) = (\cos\omega_1,\ \cos\omega_2) \tag{101}$$

As shown on fig. 4 the trajectory of a single particule is in general ergodic, however notice that even in this case some exceptionnal trajectories are not ergodic. This will be the case for any particle starting with a velocity $a(\omega_0)$ with ω_0 any periodic point for the mapping T. For instance on fig. 4 is plotted the path of a single particle driven by the flow given by (101) with initial velocity:

$$a(\omega_0), \quad \omega_0 = (0,\ \pi/2) \tag{102}$$

Observe the relation $\omega_0 = T^3\omega_0$ which correspond to the behaviour of the particle.

Fig. 5 is devoted to the simulation (128 particles and 1000 collisions) of the non diffusive case with a vector field given by the formula:

$$a'(\omega) = a(T\omega) - a(\omega)$$

First there is an initial layer then the process stabilises to a stationary state.

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References

[bow] R. Bowen: "Equilibrium states and the ergodic theory of Anosov diffeomorphisms" Lectures Notes in Math. 470, Springer Verlag (1975).
[b-s] L. Bunimovich, Ya. Sinai: Commun. in Math. Phys 78, (1980), 479–497.
[cr-car] J. Crawford, J. Cary: Physica D 6, (1982-83), 223-232.
[d] R. Dautray: "Méthodes probabilistes pour les équations de la physique", Masson (1989)
[de-ph] M. Denker, W. Philipp: Ergodic Theory and Dyn. Syst. 4, (1984), 541–552.
[g,le] F. Golse, E. Leichtnam: preprint.
[katz] Y. Katznelson: Israel J. of Math. 10, (1971), 186–195.
[l] G. Lebeau: Ann. Inst. Fourier 35, (1985), 145–216.
[rat] M. Ratner: Israel J. of Math. 16, (1973), 181–197.

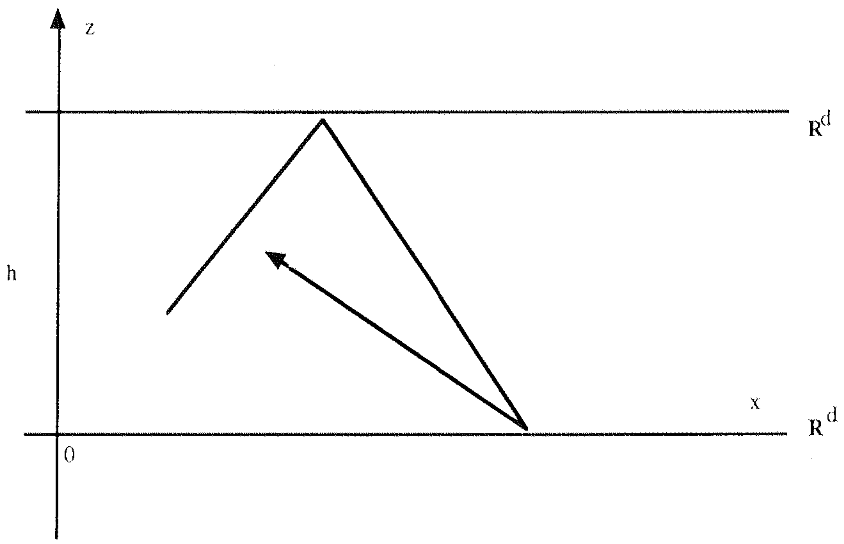

Fig. 1

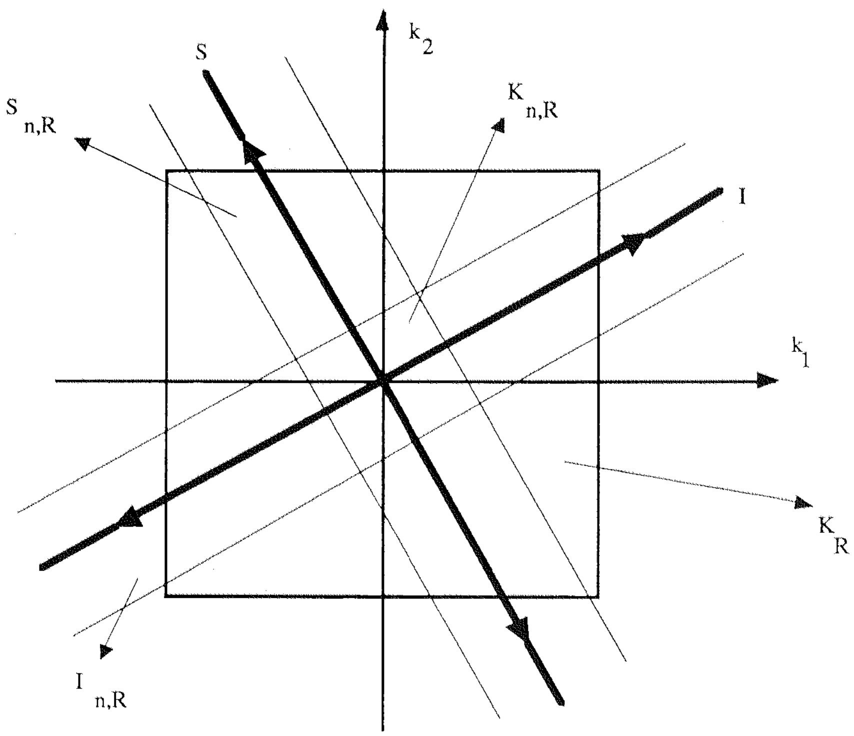

Fig. 2

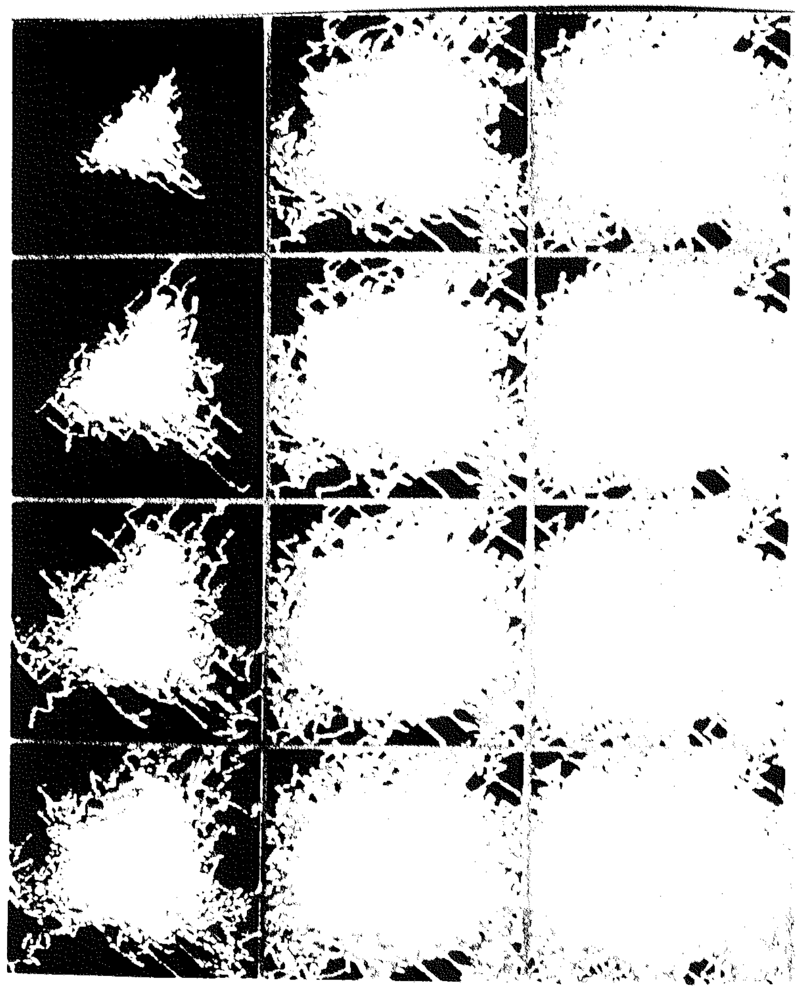

Fig. 3 Evolution of the Diffusive Case
From the top to the bottom and from the left to the right

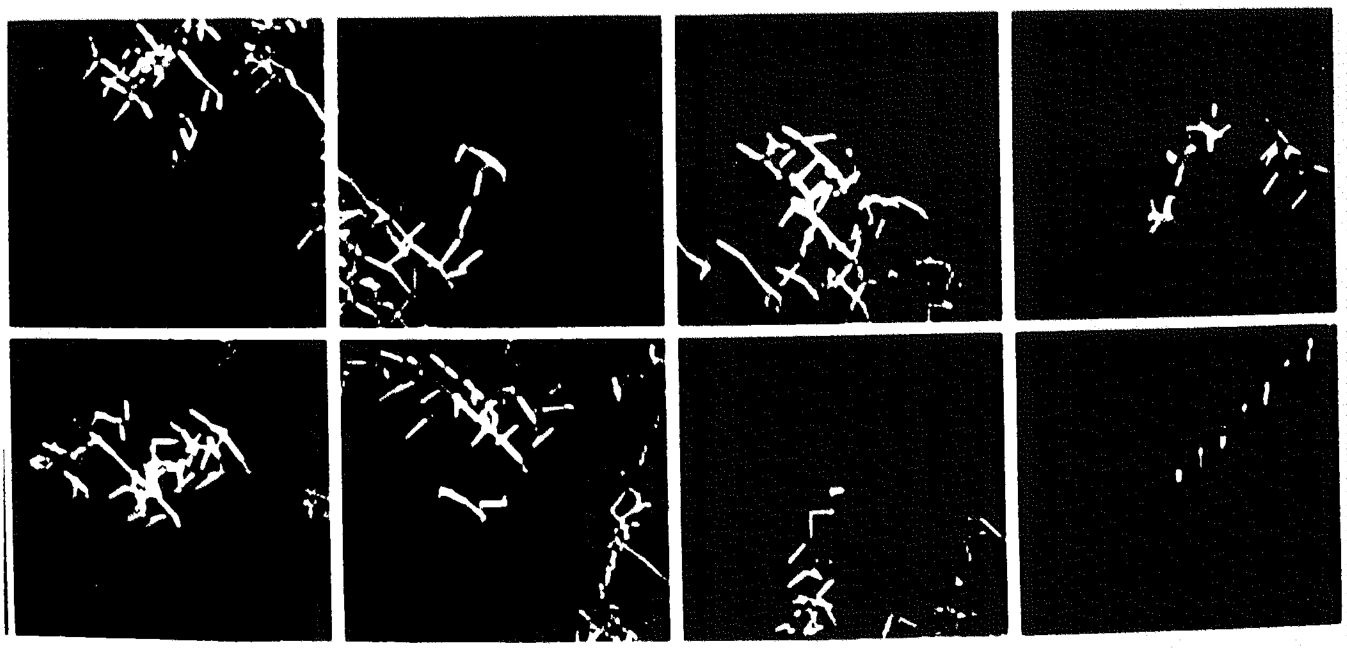

Fig. 4 Ergodic and non ergodic trajectories

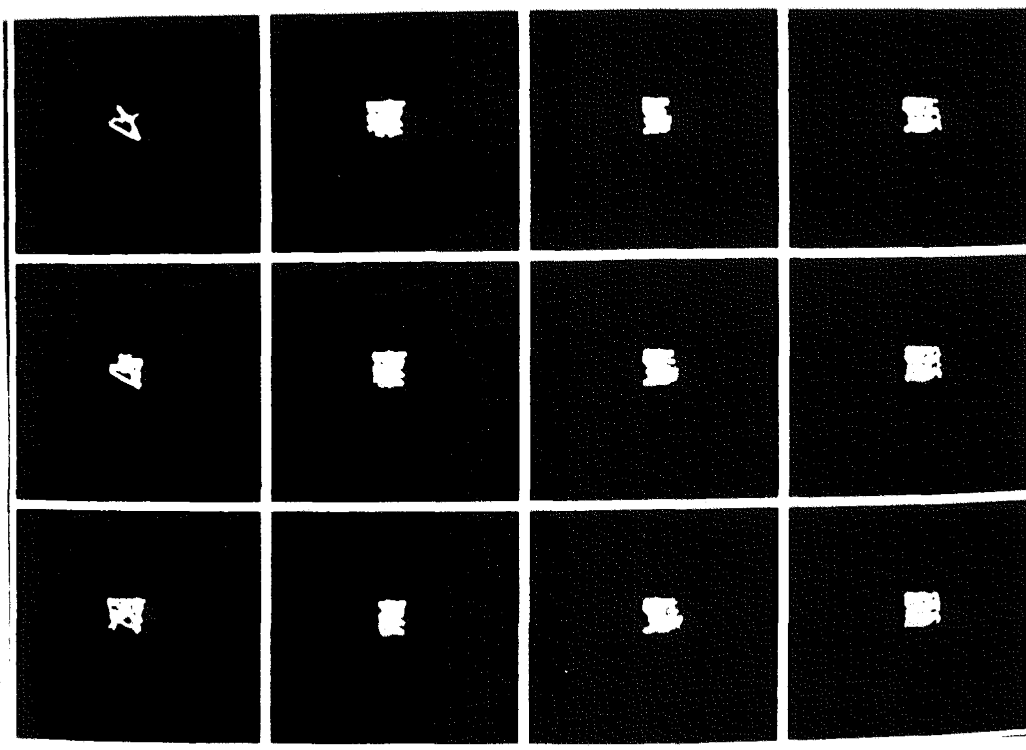

Fig. 5 Evolution of the Non Diffusive Case
From the top to the bottom and from the left to the right