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A Homological Approach to Numerical Godeaux Surfaces

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Abstract

Numerical Godeaux surfaces are minimal surfaces of general type with the smallest possible numerical invariants. It is known that the torsion group of a numerical Godeaux surface is cyclic of order $m \leq 5$. A full classification has been given for the cases $m = 3, 4, 5$ by the work of Reid and Miyaoka. In each case, the corresponding moduli space is 8-dimensional and irreducible. There exist explicit examples of numerical Godeaux surfaces for the orders $m = 1, 2$, but a complete classification for these surfaces is still missing.

In this thesis we present a construction method for numerical Godeaux surfaces which is based on homological algebra and computer algebra and which arises from an experimental approach by Schreyer. The main idea is to consider the canonical ring $R(X)$ of a numerical Godeaux surface X as a module over some graded polynomial ring S . The ring S is chosen so that $R(X)$ is finitely generated as an S -module and a Gorenstein S -algebra of codimension 3. We prove that the canonical ring of any numerical Godeaux surface, considered as an S -module, admits a minimal free resolution of length 3 whose middle map is alternating. Moreover, we show that a partial converse of this statement is true under some additional conditions. Afterwards we use these results to construct (canonical rings of) numerical Godeaux surfaces. Hereby, we restrict our study to surfaces whose bicanonical system has no fixed component but 4 distinct base points, in the following referred to as marked numerical Godeaux surfaces.

The particular interest of this thesis lies on marked numerical Godeaux surfaces whose torsion group is trivial. For these surfaces we will study the fibration of genus 4 over \mathbb{P}^1 induced by the bicanonical system. Catanese and Pignatelli showed that the general fibre is non-hyperelliptic and that the number \tilde{h} of hyperelliptic fibres is bounded by 3. The two explicit constructions of numerical Godeaux surfaces with a trivial torsion group due to Barlow and Craighero-Gattazzo, respectively, satisfy $\tilde{h} = 2$.

With the method from this thesis, we construct an 8-dimensional family of numerical Godeaux surfaces with a trivial torsion group and whose general element satisfy $\tilde{h} = 0$. Furthermore, we establish a criterion for the existence of hyperelliptic fibres in terms of a minimal free resolution of $R(X)$. Using this criterion, we verify experimentally the existence of a numerical Godeaux surface with $\tilde{h} = 1$.

Notation

Let X be a smooth projective complex surface, and let D, D' be divisors on X . We use the following notation:

$D \sim D'$	D and D' are linearly equivalent
$D \equiv D'$	D and D' are numerically equivalent
$\text{Pic } X$	the Picard group of X
$\text{Tors } X$	the subgroup of torsion elements of $\text{Pic } X$
$H^i(X, \mathcal{O}_X(D))$, or simply $H^i(X, D)$	the i th cohomology group of the sheaf $\mathcal{O}_X(D)$
$h^i(X, \mathcal{O}_X(D))$, or simply $h^i(X, D)$	the vector space dimension of $H^i(X, \mathcal{O}_X(D))$
$\chi(\mathcal{O}_X(D))$	the Euler-Poincaré characteristic of $\mathcal{O}_X(D)$, that is $h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) + h^2(X, \mathcal{O}_X(D))$
K_X	a canonical divisor of X
$p_g(X)$	the geometric genus of X , that is $h^0(X, \mathcal{O}_X(K_X))$
$q(X)$	the irregularity of X , that is $h^1(X, \mathcal{O}_X)$
$P_n(X)$	the n th-plurigenus of X , that is $h^0(X, \mathcal{O}_X(nK_X))$ for $n \geq 1$
$\kappa(X)$	the Kodaira dimension of X

If X is clear from the context, we will often write p_g, q, P_n, χ and K^2 instead of $p_g(X), q(X), P_n(X), \chi(\mathcal{O}_X)$ and K_X^2 .

Throughout this thesis, all considered rings will be commutative and unitary. Let A be a Noetherian ring, and let $I \subseteq A$ be an ideal. If M is a finitely generated A -module, we set:

$\text{depth}(I, M)$	the length of any maximal M -sequence in I
$\text{ann}_A M$	the annihilator of M
$\text{projdim}_A M$	the projective dimension of M

For a map $\psi: F \rightarrow G$ between free A -modules, we write:

$\text{rank}(\psi)$	the rank of ψ
$I_j(\psi)$	the j th determinantal ideal of ψ
$I(\psi)$	$I_{\text{rank}(\psi)}(\psi)$

If d is a matrix representing ψ , then $\text{rank}(\psi)$ is the same as the size of the largest non-vanishing minor of d . By abuse of notation, we sometimes write $\text{rank}(d)$ instead of $\text{rank}(\psi)$.

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1 Introduction

In the study of algebraic surfaces throughout the last centuries, complex surfaces with geometric genus $p_g = 0$ and irregularity $q = 0$ have always been of a particular interest. Around 1870, Max Noether posed the question whether every surface satisfying $p_g = q = 0$ is rational. A first negative answer was given by Enriques in 1894. Enriques considered a surface of degree 6 in \mathbb{P}^3 passing doubly through the edges of a tetrahedron. The normalization is then a non-rational surface with the desired invariants. Shortly afterwards, Castelnuovo presented his celebrated rationality criterion stating that a surface is rational if and only if $P_2 = q = 0$, where P_2 denotes the second plurigenus, and another example of a non-rational surface with $p_g = q = 0$ ([Cas96]).

In the 1930s, the first examples of surfaces of general type with $p_g = q = 0$ were constructed independently by Campedelli and Godeaux ([Cam32], [God34]). Campedelli constructed a surface with $p_g = 0$ and $K^2 = 2$ as the minimal model of a double cover of \mathbb{P}^2 branched along a curve of degree 10 having a specific configuration of singularities. Godeaux considered a quintic surface Y in \mathbb{P}^3 on which the cyclic group of order 5 acts without fixed points. Then Y is a surface of general type with $K_Y^2 = 5$, $p_g(Y) = 4$ and $q(Y) = 0$ and the smooth minimal model of the quotient is a surface of general type with $K^2 = 1$ and $p_g = q = 0$. In honor of their work, minimal surfaces of general type with $p_g = 0$, and hence $q = 0$, are nowadays called *numerical Godeaux surfaces* if $K^2 = 1$ and *numerical Campedelli surfaces* if $K^2 = 2$.

In 1977, Gieseker showed the existence of a quasi-projective coarse moduli space $\mathfrak{M}_{a,b}$ parametrizing isomorphism classes of minimal models of surfaces of general type with $K^2 = a$ and $\chi = b$ ([Gie77]). Having a complete description of $\mathfrak{M}_{a,b}$ for as many types of (K^2, χ) is a constant aspiration in algebraic geometry. A first natural question is for which values of (K^2, χ) , there exists a minimal surface of general type with these invariants, and hence a non-empty moduli space. Although there is no general answer to this question, the numerical invariants of a minimal surface of general type X satisfy several well-known inequalities:

- $K_X^2 \geq 1$ and $\chi(\mathcal{O}_X) \geq 1$,
- $K_X^2 \leq 9\chi(\mathcal{O}_X)$ (*Bogomolov-Miyaoka-Yau-inequality*),
- $K_X^2 \geq 2\chi(\mathcal{O}_X) - 6$ (*Noether's inequality*).

If the surface X is irregular, that means $q(X) > 0$, a further restriction is given by Debarre's inequality stating that $K_X^2 \geq 2\chi(\mathcal{O}_X)$ in this case. This shows, in particular, that the numerical Godeaux surfaces are exactly the minimal surfaces of general type satisfying $K^2 = \chi = 1$. Thus it is fair to say that numerical Godeaux surfaces are the surfaces of general type with the smallest possible invariants.

From the beginning of the 1970s, several steps towards a complete classification of these surfaces have been achieved. Miyaoka showed that the group $H^2(X, \mathbb{Z})_{\text{tors}} \cong \text{Tors } X$ is cyclic of order ≤ 5 , where $\text{Tors } X$ is the torsion subgroup of the Picard group of a numerical Godeaux surface X ([Miy76]). For a surface with a non-trivial torsion group of order m , the original construction due to Godeaux generalizes to the following: There exists a finite étale covering $Z \rightarrow X$ of degree m corresponding to $\text{Tors } X$. Construct first the cover surface Z and realize X then as the quotient. Using this method, Reid gave a complete description of the canonical ring of a numerical Godeaux surface X with $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ ([Rei78]). In each case, Reid showed that the moduli space of these surfaces is irreducible and 8-dimensional.

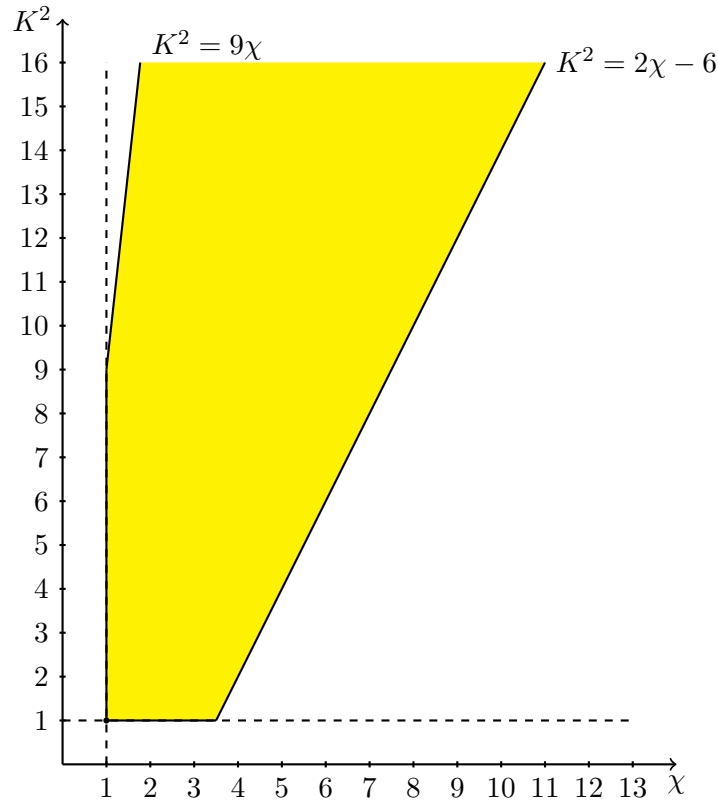


Figure 1.1: The geography of minimal surfaces of general type

In the 1980s, Barlow constructed numerical Godeaux surfaces with $\text{Tors } X = 0$ and $\text{Tors } X = \mathbb{Z}/2\mathbb{Z}$ ([Bar82]). The surfaces with a trivial torsion group were also the first examples of simply connected numerical Godeaux surfaces. Since then there have been further examples of numerical Godeaux surfaces with $\text{Tors } X = \mathbb{Z}/2\mathbb{Z}$ ([Bar85], [Wer94], [Wer97], [Cou09], [KLP10]) and trivial torsion group ([CG94]), but a complete classification for these surfaces is still missing.

The main subject of this thesis is a construction method for numerical Godeaux surfaces based on homological algebra and computer algebra. The construction arises from an experimental approach by Schreyer ([Sch05]). The basic idea of this approach is to study the canonical ring

$$R(X) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nK_X))$$

of a numerical Godeaux surface X as a module over a polynomial ring S . More precisely, let x_0, x_1 be a basis of $H^0(X, \mathcal{O}_X(2K_X))$, and let y_0, \dots, y_3 be a basis of $H^0(X, \mathcal{O}_X(3K_X))$. We will show that $R(X)$ is a finitely generated $S = \mathbb{k}[x_0, x_1, y_0, \dots, y_3]$ -module for any numerical Godeaux surface X . Thus, geometrically, we study the canonical model $X_{can} = \text{Proj}(R(X))$ of X via its image under the finite morphism $X_{can} \rightarrow \text{Proj}(S) = \mathbb{P}(2^2, 3^4)$.

In Chapters 3, 4 and 5 we prove some general results on the canonical ring $R(X)$, considered as an S -module, and its minimal free resolution. These statements provide the theoretical foundation for our construction which we present in Chapters 6, 7 and 8. In doing so, we restrict our study to numerical Godeaux surfaces whose bicanonical system has no fixed component but four distinct base points. We refer to these surfaces as *marked numerical Godeaux surfaces*. Note that the torsion group of any such surface is of odd order. But anyway, our main interest lies in numerical Godeaux surfaces with a trivial torsion group.

Besides the theoretical results, one of the major features of our method is that we can compute explicit examples of numerical Godeaux surfaces. Let us briefly explain this by means of the following diagram:

$$\begin{array}{ccccccc}
 \tilde{X} & \longrightarrow & X & \xrightarrow{\pi} & X_{can} & \hookrightarrow & \mathbb{P}(2^2, 3^4, 4^4, 5^3) \\
 \downarrow f & \searrow & \downarrow g & \searrow & \downarrow \varphi & & \\
 \mathbb{P}^1 & \longleftarrow & \mathbb{P}^1 \times \mathbb{P}^3 & \longleftarrow & \mathbb{P}(2^2, 3^4) & &
 \end{array}$$

X denotes a marked numerical Godeaux surface and $\pi: X \rightarrow X_{can}$ is the birational morphism onto the canonical model of X . The map g is a product rational map induced by the linear systems $|2K_X|$ and $|3K_X|$ and is birational onto its image $W \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ ([CP00], [Pig00]). The bicanonical system induces a fibration $f: \tilde{X} \rightarrow \mathbb{P}^1$ of genus 4, where \tilde{X} is the blow-up of X at the four base points of $|2K_X|$.

We focus on the morphism $\varphi: X_{can} \rightarrow \mathbb{P}(2^2, 3^4)$ and the image $Y \subseteq \mathbb{P}(2^2, 3^4)$. Via our construction we obtain in the first place only the surface Y which is a birational model of the canonical model X_{can} . However, we develop an algorithm for determining the defining relations of $X_{can} \subseteq \mathbb{P}(2^2, 3^4, 4^4, 5^3)$ from a (given) minimal free resolution of $R(X)$ as an S -module. Hence, we obtain the canonical model X_{can} and the different birational models Y and W and we can study the geometry of these surfaces explicitly via computer algebra.

Now let us assume that $\text{Tors } X = 0$. The general fibre of f is non-hyperelliptic and a complete intersection curve of type $(2, 3)$. However, hyperelliptic fibres may occur and Pignatelli showed that the number \tilde{h} of hyperelliptic fibres (counted with multiplicity) is ≤ 3 ([Pig00]). The two explicit constructions of torsion-free numerical Godeaux surfaces due to Barlow and Craighero-Gattazzo, respectively, satisfy $\tilde{h} = 2$. We believe that these surfaces are rather special and that the general elements of the moduli space of torsion-free numerical Godeaux surfaces satisfy $\tilde{h} = 0$. Up to now, the existence of surfaces with $\tilde{h} = 1$ has been unknown.

Throughout this thesis we establish several new relations between Schreyer's original approach and geometric properties of X such as the order of the torsion group or the existence of hyperelliptic curves in the bicanonical system $|2K_X|$. One main result of this thesis is the existence of an 8-dimensional family of torsion-free numerical Godeaux surfaces whose general element has no hyperelliptic bicanonical curves. Furthermore, we develop a criterion for the existence of (smooth) hyperelliptic bicanonical curves which allows us, in particular, to determine the number of these curves from the first syzygy matrix of a minimal free resolution of $R(X)$ as an S -module. Moreover, using this criterion, we compute an explicit example of a numerical Godeaux surface (defined over a field with characteristic $p > 0$) satisfying $\tilde{h} = 1$.

We end this introductory part by giving an outline of the chapters and results of this thesis:

In Chapter 2 we briefly introduce most of the notation which will be used throughout this thesis such as minimal free resolutions, canonical modules, and minimal and canonical surfaces.

Chapter 3 then focuses on numerical Godeaux surfaces. It is known that the canonical ring $R(X)$ of a numerical Godeaux surface X is generated in degree ≤ 6 . We refine this statement by showing that $R(X)$ is generated in degree ≤ 5 and describe a minimal set of algebra generators. Afterwards we consider the canonical ring $R(X)$ as a module over the polynomial ring S introduced above. We show that the morphism $\varphi: X_{can} \rightarrow Y$ is the normalization of Y . At the end of Chapter 3, we will see that a minimal free resolution of $R(X)$ as an S -module has length 3 and determine the Betti numbers of $R(X)$.

In Chapter 4 we present our main theorem on the canonical ring of numerical Godeaux surfaces. We prove that the canonical ring $R(X)$, considered as an S -module, admits a minimal free

resolution whose middle map is alternating. This result can be seen as a generalization of the celebrated structure theorem of Buchsbaum and Eisenbud for Gorenstein ideals in codimension 3 ([BE77]) and the proof of our statement is based on similar ideas.

In Chapter 5 we prove a partial converse of the structure theorem of Chapter 4. We first show that any finitely generated graded S -module having a self-dual resolution as described in Chapter 4 supports, under some additional (ring) condition, the structure of a Gorenstein ring. Afterwards we show that, under some further mild assumptions, the corresponding scheme is the canonical model of a numerical Godeaux surface X . Furthermore, we explain how a complete set of defining relations of $R(X)$ (as a ring) can be obtained from the S -linear relations.

One important ingredient for our construction is that the elements $x_0, x_1 \in S$ form a regular sequence for $R(X)$. In Chapter 6 we show that modulo this regular sequence the minimal free resolution of $R(X)$ splits into a direct sum of three exact sequences. Restricting to marked numerical Godeaux surfaces then, we give an explicit description of these three complexes. Furthermore, we call a minimal free resolution of $R(X)$ *standard* if the middle map is alternating and the complexes are of a particular fixed form modulo x_0, x_1 . We will show that the canonical ring of any marked numerical Godeaux surface admits such a standard resolution.

In Chapter 7 we present and explain the individual steps of our construction. We will see that constructing S -modules R having a standard resolution is basically equivalent to choosing a line in a complete intersection $Q \subseteq \mathbb{P}^{11}$ and a point in some vector space. More precisely, our construction depends on pairs of the form (l, p) , where l is a 2×12 matrix representing a line in Q and p is a point in a vector space $\mathcal{V}(l)$. In particular, we can assign such a pair to any standard resolution of $R(X)$. The variety Q is defined by four quadrics of rank 6 which are the Pfaffians of some skew-symmetric matrices of size 4. We present a method for finding lines in Q and study also some local properties of the corresponding Fano scheme of lines $F_1(Q)$.

To decide if a pair (l, p) results in the canonical ring of a numerical Godeaux surface X , we use the statements from Chapter 5. If this is the case, we assign to the pair (l, p) the isomorphism class $[X] \in \mathfrak{M}_{1,1}$ and speak of an *admissible* pair. In Chapter 8 we characterize and identify (admissible) pairs which lead to isomorphic surfaces. The identification part is mainly done by introducing appropriate group operations on the parameter space of the matrices l and the vector spaces $\mathcal{V}(l)$. Furthermore, we give some criteria which allow us to determine the order of the torsion group of X from a given standard resolution of $R(X)$. Finally, we use our construction to show the existence of an 8-dimensional family of numerical Godeaux surfaces having a trivial torsion group which Schreyer's experimental results already suggested. The proof of this result is based on the fact that we can construct one such surface with our method which will be verified in Chapter 11.

In Chapter 9 we briefly recall the classification results on numerical Godeaux surfaces with torsion group $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ due to Godeaux, Miyaoka and Reid, respectively. For every member X of the 8-dimensional family of $\mathbb{Z}/5\mathbb{Z}$ -Godeaux surfaces we give an explicit description of the first syzygy matrix of $R(X)$ as an S -module. In particular, we will see that such a surface is a marked numerical Godeaux surface. Furthermore, in both cases, we deduce some properties on the standard resolutions of the corresponding canonical rings.

In Chapter 10 we restrict our attention to torsion-free marked numerical Godeaux surfaces. We show that our finite birational morphism $\varphi: X_{can} \rightarrow Y \subseteq \mathbb{P}(2^2, 3^4)$ is not an isomorphism if there exists a (smooth) hyperelliptic bicanonical curve. Furthermore, we give an explicit characterization of the existence of such curves in terms of a standard resolution of $R(X)$.

The aim of Chapter 11 is to present some explicit computations with MACAULAY2 ([GS]). In particular, we construct a marked numerical Godeaux surface (defined over a finite field extension over \mathbb{Q}) having a trivial torsion group and no hyperelliptic fibres. Furthermore, we verify by an explicit computation the existence of a torsion-free numerical Godeaux surface (defined over a finite field) having exactly one hyperelliptic bicanonical curve.

2 Preliminaries

The aim of this chapter is to introduce the main objects and tools which are used throughout the thesis. We start by describing minimal free resolutions and Betti numbers. Then we introduce the notion of *local rings and *canonical modules. At the end of the chapter, we give a brief introduction to algebraic surfaces, where the main focus lies on surfaces of general type.

Notation 2.0.1. Throughout this thesis, \mathbb{k} denotes an algebraically closed field unless otherwise stated. When dealing with surfaces, we will often assume that $\mathbb{k} = \mathbb{C}$.

2.1 Minimal Free Resolutions and Betti Numbers

In this section we state some well-known results on minimal free resolutions and syzygies which can be found in [Eis05]. Let $S = \mathbb{k}[t_0, \dots, t_r]$ be the graded polynomial ring in $r + 1$ variables with $\deg(t_i) = a_i > 0$ for $i = 0, \dots, r$. If $a_i = 1$ for all i , then we call S *standard graded*. Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded S -module, and let $m_1, \dots, m_n \in M$ be homogeneous generators of M with $\deg(m_j) = b_j$. Sending the canonical basis vectors of the free module $F_0 = \bigoplus_j S(-b_j)$ to the m_j , we get a surjective map of degree 0

$$0 \leftarrow M \leftarrow F_0.$$

The kernel M_1 of this map is again a finitely generated graded S -module. We call the elements of M_1 *syzygies* on the generators m_j or *syzygies* of M . Continuing this procedure with the module M_1 yields an exact sequence

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_i \leftarrow F_{i+1} \leftarrow \dots . \quad (2.1)$$

We call its free part

$$F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_i \leftarrow F_{i+1} \leftarrow \dots$$

a (*graded*) *free resolution* of M . By abuse of notation, we also call every sequence as in (2.1) a free resolution of M .

Theorem 2.1.1 (Hilbert's Syzygy Theorem). *Let M be a finitely generated graded S -module. Then M has a finite graded free resolution*

$$F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_{m-1} \leftarrow F_m \leftarrow 0$$

of length $m \leq r + 1$ with finitely generated free modules F_i .

Proof. See [Eis05], Theorem 1.1 and Exercise 1.1.2. □

A free resolution of a finitely generated S -module M is in general not unique. However, we obtain uniqueness up to isomorphism if we are working with minimal free resolutions:

Definition 2.1.2. Let $\mathfrak{m} = (t_0, \dots, t_r) \subseteq S$. We call a complex of free graded S -modules

$$\dots \leftarrow F_{i-1} \xleftarrow{\delta_i} F_i \leftarrow \dots$$

minimal if for every i the image of δ_i is contained in $\mathfrak{m}F_{i-1}$.

One can show that a free resolution is minimal if and only if a minimal set of module generators is chosen in every step of the construction of the free resolution. In particular, such a resolution exists. Furthermore, a minimal free resolution can be computed from any free resolution. In the following, M will always denote a finitely generated graded S -module. In view of the following result we will often speak of the minimal graded free resolution of M .

Theorem 2.1.3. *If \mathbf{F} and \mathbf{G} are two minimal free graded resolutions of M , then there exists a graded isomorphism of complexes $\mathbf{F} \rightarrow \mathbf{G}$ which induces the identity on M .*

Proof. See [Eis13], Theorem 20.2. □

One of the most important features of this result is that the number of generators of a given degree in each module of a minimal free resolution does only depend on the module M and not on the resolution.

Definition 2.1.4. Let \mathbf{F} be the minimal free resolution of M . We write

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}(M)}.$$

Then the numbers $\beta_{i,j}(M)$ are called the (graded) Betti numbers of M . We simply write $\beta_{i,j}$ for $\beta_{i,j}(M)$ if M is clear from the context.

The Betti numbers of M can be represented in a table, called the *Betti table* of M :

$j \setminus i$	0	1	\cdots	m
\vdots	\vdots	\vdots	\cdots	\vdots
0	$\beta_{0,0}$	$\beta_{1,1}$	\cdots	$\beta_{m,m}$
1	$\beta_{0,1}$	$\beta_{1,2}$	\cdots	$\beta_{m,m+1}$
\vdots	\vdots	\vdots	\cdots	\vdots
s	$\beta_{0,s}$	$\beta_{1,1+s}$	\cdots	$\beta_{m,m+s}$

There exists a characterization of these numbers using the functor Tor . Note that \mathbb{k} is the residue field S/\mathfrak{m} . The module $\text{Tor}_i(M, \mathbb{k})$ is the i th homology module of the complex $\mathbf{F} \otimes_S \mathbb{k}$. Since \mathbf{F} is a minimal free resolution, every map of the complex $\mathbf{F} \otimes_S \mathbb{k}$ is zero and $\text{Tor}_i(M, \mathbb{k}) = F_i \otimes_S \mathbb{k}$. By the Lemma of Nakayama the number of minimal generators of F_i in degree j is then $\dim_{\mathbb{k}} \text{Tor}_i(M, \mathbb{k})_j$.

Proposition 2.1.5. *For M as above we have*

$$\beta_{i,j} = \dim_{\mathbb{k}} \text{Tor}_i(M, \mathbb{k})_j.$$

There exist several connections between the Betti numbers of M and its Hilbert function respectively Hilbert series. Recall that the *Hilbert function* $H_M: \mathbb{Z} \rightarrow \mathbb{Z}$ of M is defined by $d \mapsto \dim_{\mathbb{k}} M_d$. The *Hilbert series* of M is the formal Laurent series

$$\Psi_M(t) = \sum_{d \in \mathbb{Z}} H_M(d)t^d \in \mathbb{Z}[[t, t^{-1}]].$$

First of all we can express the Hilbert function of M in terms of the Betti numbers:

Proposition 2.1.6. *If $\{\beta_{i,j}\}$ are the graded Betti numbers of M , then the alternating sums $B_j = \sum_{i>0} (-1)^i \beta_{i,j}$ (together with the Hilbert function of S) determine the Hilbert function of M via the formula*

$$H_M(d) = \sum_{j \in \mathbb{Z}} B_j H_S(d - j).$$

If S is standard graded this reduces to the formula

$$H_M(d) = \sum_{j \in \mathbb{Z}} B_j \binom{r + d - j}{r}.$$

Moreover, the values of B_j can be deduced inductively from the Hilbert functions H_M and H_S via the formula

$$B_j = H_M(j) - \sum_{k:k < j} B_k H_S(j - k). \quad (2.2)$$

Proof. See [Eis05], Corollary 1.10. □

Note that the Hilbert function of M does in general not determine the Betti numbers of M . The next statement expresses the Hilbert series of M as a rational function in t depending only on the alternating Betti numbers and the weights of the polynomial ring:

Theorem 2.1.7 (Hilbert, [Eis05], Theorem 1.11). *Let M and B_j be as above, and set $\varphi_M(t) = \sum_{j \in \mathbb{Z}} B_j t^j$. The Hilbert series of M is given by the formula*

$$\Psi_M(t) = \frac{\varphi_M(t)}{\prod_i (1 - t^{a_i})}.$$

We end this section with a result relating free resolutions and regular sequences. Regular sequences play an important role in homological algebra and satisfy various nice properties. The one we are mainly interested in is the fact that free resolutions stay exact modulo regular sequences, or more precisely:

Proposition 2.1.8. *Let M be a finitely generated S -module with free resolution*

$$F_\bullet : F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_m \leftarrow 0.$$

Let $s = s_1, \dots, s_k \in S$ be an M -regular sequence. Then $F_\bullet \otimes S/(s)$ is a free resolution of $M \otimes S/(s) = M/(s)M$ as an $S/(s)$ -module.

Proof. See [BH98], Proposition 1.1.5. □

2.2 Canonical Modules and Gorenstein Rings

Later we will see that our basic object of study is a *local ring which is a Gorenstein ring. A *local ring is the graded counterpart of a local ring. We hereby use the notation introduced in [BH98]. For proofs and further details, we refer to Section 1.5 and 3.6 of [BH98]. Throughout this section $A = \bigoplus_{d \in \mathbb{Z}} A_d$ will denote a graded ring.

Definition 2.2.1 ([BH98], Definition 1.5.13). A graded ideal \mathfrak{m} of A is called *maximal, if every graded ideal that properly contains \mathfrak{m} is equal to A . The ring A is called *local, if it has a unique *maximal ideal \mathfrak{m} . A *local ring A with *maximal ideal \mathfrak{m} will be denoted by (A, \mathfrak{m}) . The *dimension of such a ring, denoted by *dim A , is defined as the height of \mathfrak{m} .

Notation 2.2.2. We call a graded ring A a *positively graded \mathbb{k} -algebra* if

- (i) $A = \bigoplus_{d \geq 0} A_d$,
- (ii) $A_0 = \mathbb{k}$.

Throughout this thesis, we tacitly assume that each positively graded \mathbb{k} -algebra satisfies the additional property:

- (iii) A is finitely generated over \mathbb{k} .

Example 2.2.3. Let A be a positively graded \mathbb{k} -algebra, and let $\mathfrak{m} = \bigoplus_{d \geq 1} A_d$. Then \mathfrak{m} is the unique $*$ maximal ideal of A . Hence every positively graded \mathbb{k} -algebra is $*$ local. In particular, the graded polynomial ring S of Section 2.1 is $*$ local.

Notation 2.2.4. Let M and N be graded A -modules. An A -module homomorphism $f: M \rightarrow N$ satisfying $f(M_d) \subseteq N_{d+i}$ for all d will be called *homogeneous of degree i* . In the case $i = 0$, we simply call f *homogeneous*. Let $\mathcal{M}_0(A)$ be the category of graded A -modules, whose objects are graded A -modules and whose morphisms are homogeneous A -module homomorphisms. We denote the group of A -module homomorphisms between M and N which are homogeneous of degree i by $\text{Hom}_i(M, N)$. The module ${}^* \text{Hom}_A(M, N) = \bigoplus_i \text{Hom}_i(M, N)$ is a graded submodule of $\text{Hom}_A(M, N)$. Note that $\text{Hom}_A(M, N) = {}^* \text{Hom}_A(M, N)$ if M is finitely generated. By ${}^* \text{Ext}_A^i(-, N)$ we denote the i th right derived functor of ${}^* \text{Hom}_A(-, N)$ in $\mathcal{M}_0(A)$. If A is Noetherian and M is finitely generated as an A -module, then ${}^* \text{Ext}_A^i(M, N) = \text{Ext}_A^i(M, N)$.

Using the ${}^* \text{Ext}$ -modules we will now introduce the notion of a $*$ canonical module:

Definition 2.2.5 ([BH98], Definition 3.6.8). Let (A, \mathfrak{m}) be a Cohen-Macaulay $*$ local ring of $*$ dimension d . A finitely generated graded A -module C is a *$*$ canonical module* of A if there exist homogeneous isomorphisms

$${}^* \text{Ext}_A^i(A/\mathfrak{m}, C) \cong \begin{cases} 0 & \text{for } i \neq d \\ A/\mathfrak{m} & \text{for } i = d. \end{cases}$$

The natural question arises whether every Cohen-Macaulay $*$ local ring A admits a $*$ canonical module and, if so, whether this is unique. Let (A, \mathfrak{m}) be as in Definition 2.2.5, and let C be a $*$ canonical module of A . If A has no homogeneous units of positive degree, which is equivalent to the fact that \mathfrak{m} is maximal in the usual sense, then we get the following result:

Proposition 2.2.6. *Let (A, \mathfrak{m}) be a Cohen-Macaulay $*$ local ring, and let C be a $*$ canonical module of A . If \mathfrak{m} is a maximal ideal, then C is uniquely determined up to homogeneous isomorphism.*

Proof. See [BH98], Proposition 3.6.9. □

We will see that every Cohen-Macaulay positively graded \mathbb{k} -algebra admits a $*$ canonical module. A first step towards this result is:

Example 2.2.7 ([BH98], Example 3.6.10). Let $A = \mathbb{k}[t_0, \dots, t_r]$ be the polynomial ring over \mathbb{k} with $\deg(t_j) = a_j > 0$, and let $\mathfrak{m} = (t_0, \dots, t_r) \subseteq A$ be the $*$ maximal ideal of A . The minimal free resolution of A/\mathfrak{m} is the Koszul complex $\mathbf{K}(t_0, \dots, t_r)$. From the self-duality of $\mathbf{K}(t_0, \dots, t_r)$ we see that ${}^* \text{Ext}_A^i(A/\mathfrak{m}, A) = 0$ for $i \neq r$ and ${}^* \text{Ext}_A^r(A/\mathfrak{m}, A) = A/\mathfrak{m}(\sum_{j=0}^r a_j)$. Hence

$${}^* \text{Ext}_A^r(A/\mathfrak{m}, A(-\sum_{j=0}^r a_j)) = A/\mathfrak{m}$$

and $A(-\sum_{j=0}^r a_j)$ is the $*$ canonical module of A .

Note that in this example the \ast -canonical module is easy to describe since it is simply a shift of the ring A . Cohen-Macaulay \ast -local rings fulfilling this property have an equivalent important description:

Proposition 2.2.8. *Let (A, \mathfrak{m}) be a Cohen-Macaulay \ast -local ring with \ast -canonical module ω_A . Then the following conditions are equivalent:*

- (i) A is Gorenstein.
- (ii) $\omega_A \cong A(a)$ for some $a \in \mathbb{Z}$.

Proof. See [BH98], Proposition 3.6.11. □

The number a from Proposition 2.2.8 is uniquely determined by A if \mathfrak{m} is maximal (in the usual sense). If A is a positively graded \mathbb{k} -algebra with \ast -canonical module ω_A , we define the a -invariant of A as

$$a(A) = -\inf\{i \mid (\omega_A)_i \neq 0\}.$$

Hence, for a Gorenstein positively graded \mathbb{k} -algebra we have

$$\omega_A \cong A(a(A)).$$

In Example 2.2.7 we have seen that any positively graded polynomial ring over \mathbb{k} has a \ast -canonical module. The following result shows that any Cohen-Macaulay positively graded \mathbb{k} -algebra admits a \ast -canonical module:

Proposition 2.2.9. *Let (A, \mathfrak{m}) be a Cohen-Macaulay \ast -local ring with \ast -canonical module ω_A . Furthermore, let $\varphi : (A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a ring homomorphism of Cohen-Macaulay \ast -local rings satisfying*

- (i) $\varphi(A_d) \subseteq B_d$ for all $d \in \mathbb{Z}$,
- (ii) $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$,
- (iii) B is a finitely generated graded A -module.

Then ω_B exists and

$$\omega_B \cong \ast\mathrm{Ext}_A^t(B, \omega_A),$$

where $t = \ast\mathrm{dim}A - \ast\mathrm{dim}B$.

Proof. See [BH98], Proposition 3.6.12. □

Remark 2.2.10. Note that the isomorphism above is an isomorphism of B -modules. Given any A -module N , the module $\ast\mathrm{Hom}_A(B, N)$ (and hence also $\ast\mathrm{Ext}_A^i(B, N)$ for any i) has a natural structure as a (graded) B -module. Indeed, for $b \in B$ and $f \in \ast\mathrm{Hom}_A(B, N)$, we define $b \cdot f$ as the homomorphism which sends an element $c \in B$ to $f(bc)$.

At the end of this section, we state a result relating the minimal free resolution of a positively graded \mathbb{k} -algebra with the minimal free resolution of its canonical module:

Proposition 2.2.11. *Let $A = \mathbb{k}[t_0, \dots, t_r]$ be as in Example 2.2.7, and let B be a positively graded \mathbb{k} -algebra. Furthermore, let $\varphi : A \rightarrow B$ be a ring homomorphism satisfying the properties of Proposition 2.2.9. Assume that B has a minimal free resolution (as an A -module)*

$$0 \leftarrow B \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_t \leftarrow 0,$$

with $t = \ast\mathrm{dim}A - \ast\mathrm{dim}B$ and $F_i = \bigoplus_j A(-j)^{\beta_{i,j}}$. Let $d_{B,A} = \max\{j \mid \beta_{i,j} \neq 0\}$. Then

$$a(B) = a(A) + d_{B,A}.$$

Proof. First note that B is a Cohen-Macaulay A -module since

$$\text{depth}(\mathfrak{m}, B) = \text{depth}(\mathfrak{n}, B).$$

Hence ${}^* \text{Ext}_A^i(B, \omega_A) = 0$ for $i \neq t$. Using Proposition 2.2.9 this implies that

$$0 \leftarrow \omega_B \leftarrow {}^* \text{Hom}_A(F_m, \omega_A) \leftarrow {}^* \text{Hom}_A(F_{m-1}, \omega_A) \leftarrow \cdots \leftarrow {}^* \text{Hom}_A(F_0, \omega_A) \leftarrow 0$$

is a minimal free resolution of ω_B which shows the claim. \square

2.3 Minimal and Canonical Surfaces

In this section we introduce the main geometrical objects of this thesis: minimal and canonical surfaces. We also introduce the canonical ring of a surface and recall some well-known statements on surfaces of general type. The definitions and results are mainly extracted from [Bea96], [Bom73] and [BHPVdV15].

Notation 2.3.1. Throughout this section X denotes a smooth projective complex surface.

To begin with, let us introduce the notion of (-1) -curves and minimal surfaces.

Definition 2.3.2. A curve $C \subseteq X$ is called a (-1) -curve if $C \cong \mathbb{P}^1$ and $C^2 = -1$.

Definition 2.3.3. We call X *minimal* if X does not contain any (-1) -curves.

Any smooth surface X' is birational to a minimal surface which we call a *minimal model* of X' . Hence, when studying (smooth) surfaces up to birational equivalence, a first step is to determine all minimal models in a given birational equivalence class. Whether there exists a unique minimal model in a fixed equivalence class depends on the Kodaira dimension of the surfaces in this class:

Definition 2.3.4. Let D be a divisor on X , and let $\phi_{|nD|}: X \dashrightarrow \mathbb{P}^{a_n}$ be the rational map associated to the linear system $|nD|$, $n \geq 1$. The *Kodaira dimension* of D , denoted by $\kappa(D)$, is defined as follows. If $h^0(X, nD) \geq 1$ for some $n \geq 1$, then we define

$$\kappa(D) = \max_n \dim \phi_{|nD|}(X).$$

Otherwise we set $\kappa(D) = -\infty$. For $D = K_X$ we call $\kappa(X) := \kappa(K_X)$ the *Kodaira dimension* of X .

Remark 2.3.5. Note that by definition $\kappa(X) \in \{-\infty, 0, 1, 2\}$. The Kodaira dimension is a birational invariant of X . Hence, while studying surfaces up to birational equivalence, we can subdivide the class of surfaces with respect to their Kodaira dimension. Any surface X with $\kappa(X) \geq 0$ admits a unique minimal model. Thus, for these surfaces it is sufficient to classify minimal surfaces.

Definition 2.3.6. A surface X with $\kappa(X) = \dim X (= 2)$ is called a *surface of general type*. Otherwise we call X a *surface of special type*.

Next we introduce two further birational invariants of a surface - the plurigenera and the canonical ring.

Definition 2.3.7. Let $n \geq 0$. We call the linear system $|nK_X|$ the *n th pluricanonical system* of X . The number $P_n = h^0(X, nK_X)$ is called the *n th plurigenus* of X . We denote the corresponding rational map, if it exists, by

$$\phi_n: X \dashrightarrow \mathbb{P}^{P_n-1}.$$

Definition 2.3.8. The *canonical ring* of X is the graded \mathbb{C} -algebra

$$R(X) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nK_X)).$$

If $\kappa(X) = -\infty$, then $R(X) \cong \mathbb{C}$. In 1962, Mumford proved that for X being of general type, the canonical ring is a finitely generated graded \mathbb{C} -algebra. The main part of his proof is to show that $|nK_X|$ is base-point-free for sufficiently large n . We call any divisor with such a property *semi-ample*.

Theorem 2.3.9. *The canonical ring of X is a finitely generated \mathbb{C} -algebra. For $\kappa(X) \geq 0$, we have*

$$\dim R(X) = \kappa(X) + 1.$$

Proof. See [Mum62]. □

Note that the question whether the equivalent statement holds in higher dimensions has been an open problem for many decades. However, recently it has been proved that the canonical ring of any smooth projective variety of general type is finitely generated (see [BCHM10]).

Now let us restrict our study to surfaces of general type. There exists a sufficient condition for X being of general type in terms of the canonical divisor of X :

Proposition 2.3.10. *If K_X is ample, then X is of general type.*

Proof. As K_X is ample, there exists an integer $n > 0$ such that nK_X is very ample. Therefore, the rational map ϕ_n is a closed embedding and hence $\kappa(X) = 2$. □

The converse of this statement is in general false. For example, if X is a minimal surface of general type containing a curve C with $K_X C = 0$, then there exists no integer n such that the rational map ϕ_n is an embedding. The irreducible curves on X satisfying $K_X C = 0$ are completely characterized:

Definition 2.3.11. A curve $C \subseteq X$ is called a *(-2)-curve* if $C \cong \mathbb{P}^1$ and $C^2 = -2$.

Proposition 2.3.12. *Let X be a minimal surface of general type, and let $C \subseteq X$ be an irreducible curve. Then $K_X C = 0$ if and only if C is a (-2)-curve. Furthermore, these curves form a finite set and are numerically independent.*

Proof. See [Bom73], Proposition 2.1. □

From now on, we restrict our study to surfaces of general type. In this case, the canonical ring defines again a surface:

Definition 2.3.13. Let X be a surface of general type. We call the projective surface $X_{can} = \text{Proj}(R(X))$ the *canonical model* of X .

The canonical model depends only on the birational equivalence class of X . It is a normal surface having (at most) rational double points as singularities. We can obtain the canonical model from the minimal model of X by contracting the finitely many (-2)-curves on the minimal model. In particular, if X is minimal, we obtain a birational morphism

$$\pi: X \rightarrow X_{can}$$

which is a minimal resolution of singularities.

Remark 2.3.14. The singularities of X_{can} are also known as canonical singularities, Du Val singularities or simple surface singularities. All these notions are equivalent in dimension 2.

In the following we want to study the rational maps associated to the pluricanonical systems on X_{can} . To do this, we first have to extend the definition of a canonical divisor to normal projective surfaces:

Definition 2.3.15. Let V be a normal projective surface. Being normal implies that the singular locus $\text{Sing}(V)$ consists of at most finitely many points. Let $V_{reg} = V \setminus \text{Sing}(V)$. Then V_{reg} is a smooth surface and has a canonical divisor $K_{V_{reg}}$ which is a Weil divisor. Since $\text{codim}(\text{Sing}(V), V) \geq 2$, the divisor $K_{V_{reg}}$ corresponds to a (unique) Weil divisor on V , which we call the canonical divisor of V , denoted by K_V .

Let V be as in the previous definition. Since V is projective it admits a dualizing sheaf ω_V (see [Har77], III. Section 7). Then $\omega_V \cong \mathcal{O}_V(K_V)$. Note that in contrast to a smooth surface, the canonical divisor on V may not be Cartier, or equivalently, ω_V may not be invertible.

Definition 2.3.16. Let X be a minimal surface of general type with canonical ring $R := R(X) = \bigoplus_{m \geq 0} R_m$, and let $n \geq 1$. We define the n th Veronese subring of R as

$$R^{(n)} = \bigoplus_{m \geq 0} R_{nm}.$$

Then the inclusion $R^{(n)} \subseteq R$ induces an isomorphism of schemes $\text{Proj}(R) \cong \text{Proj}(R^{(n)})$. Furthermore, the subring

$$R^{[n]} = \bigoplus_{m \geq 0} R_n^m \subseteq R$$

is again a graded Noetherian ring and the corresponding scheme

$$X_{can}^{[n]} := \text{Proj}(R^{[n]})$$

is called the n th canonical image of X . The inclusion $R^{[n]} \subseteq R$ induces a rational map $\kappa_n: X_{can} \dashrightarrow X_{can}^{[n]}$.

For any $n \geq 1$, the rational map $\phi_n: X \dashrightarrow \mathbb{P}^{P_n-1}$ factors through the canonical model X_{can} :

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X_{can} \\ & \searrow \phi_n & \downarrow \kappa_n \\ & & X_{can}^{[n]} \end{array}$$

Furthermore, by Bombieri's famous theorem on pluricanonical maps, we know that the n th canonical image is isomorphic to the canonical model for n large enough:

Theorem 2.3.17. *Let X be a surface of general type with canonical model X_{can} . Then*

$$\kappa_n: X_{can} \rightarrow X_{can}^{[n]}$$

is an isomorphism for all $n \geq 5$.

Proof. See [Bom73], Main Theorem. □

Let us summarize some of the main properties of the canonical model which will be used throughout this thesis:

Theorem 2.3.18. *Let X be a minimal surface of general type with canonical model X_{can} . Then*

- (i) X_{can} is a normal projective surface, birational to X ,
- (ii) $K_{X_{can}}$ is Cartier and ample,
- (iii) X_{can} has only canonical singularities,
- (iv) $H^0(X, \mathcal{O}_X(mK_X)) = H^0(X_{can}, \mathcal{O}_{X_{can}}(mK_{X_{can}}))$ for every $m \geq 0$.

Proof. See [Rei85], Chapter 1 and [Mat13], Chapter 1. □

Next we study the canonical ring of minimal surfaces of general type with $q = 0$. It is known that the canonical ring of these surfaces is Gorenstein:

Theorem 2.3.19. *Let X be a minimal surface of general type. Then $R = R(X)$ is Gorenstein if and only if $q(X) = 0$.*

For the sake of completeness we will prove this result here. The proof relies on local cohomology and some results of Goto and Watanabe. Let us first introduce some notation:

Definition 2.3.20 (see [GW78], Chapter 5). A positively graded \mathbb{k} -algebra R fulfills the condition (\sharp) if there exists $d_0 \geq 0$ such that for all $d \geq d_0$ the d th Veronese subring $R^{(d)}$ is generated by R_d over R_0 .

Remark 2.3.21. Note that if $R = R_{Y, \mathcal{L}} = \bigoplus_n H^0(Y, \mathcal{L}^{\otimes n})$ is the section ring of an ample line bundle \mathcal{L} over a projective variety Y , then R satisfies the condition (\sharp) . In particular, the canonical ring $R(X) \cong R(X_{can})$ of a minimal surface of general type satisfies (\sharp) and $\omega_{X_{can}} \cong \widetilde{R(X)}(1) = \mathcal{O}_{X_{can}}(1)$ (see [GW78], Notation 5.1.7).

Lemma 2.3.22. *Let X and R be as in Theorem 2.3.19. Then R is Cohen-Macaulay if and only if $q = 0$.*

Proof. Assume first that R is a Cohen-Macaulay ring. Then $\text{depth } R = \dim R = 3$ implies that $H_m^i(R) = 0$ for all $i \leq 2$, where $\mathfrak{m} = \bigoplus_{d>0} R_d$. The result follows now from the fact that

$$H^1(X, \mathcal{O}_X) \cong H^1(X_{can}, \mathcal{O}_{X_{can}}) \cong H_{\mathfrak{m}}^2(R)_0 = 0,$$

and hence $q = h^1(X, \mathcal{O}_X) = 0$.

Now we assume that $q = 0$. First notice that $q = 0$ implies that

$$h^1(X_{can}, \mathcal{O}_{X_{can}}(nK_{X_{can}})) = h^1(X, \mathcal{O}_X(nK_X)) = 0 \tag{2.3}$$

for all $n \in \mathbb{Z}$. Indeed, for $n \geq 2$ and $n \leq -1$ this is clear from the proof of Proposition 2.3.26 below. The case $n = 0$ is the assumption and the case $n = 1$ follows from Serre duality. To prove that R is Cohen-Macaulay of depth 3 it is enough to show that $H_m^i(R) = 0$ for $i \leq 2$. The exact sequence relating local and global cohomology

$$0 \rightarrow H_{\mathfrak{m}}^0(R) \rightarrow R \rightarrow \bigoplus_n H^0(X_{can}, \mathcal{O}_{X_{can}}(n)) \rightarrow H_{\mathfrak{m}}^1(R) \rightarrow 0$$

and the fact that $R \cong \bigoplus H^0(X_{can}, \mathcal{O}_{X_{can}}(n))$ yields $H_m^0(R) = H_m^1(R) = 0$. The vanishing of $H_m^2(R)$ follows now from (2.3) and

$$H_m^2(R) \cong \bigoplus_n H^1(X_{can}, \mathcal{O}_{X_{can}}(n)) \cong \bigoplus_n H^1(X_{can}, \mathcal{O}_{X_{can}}(nK_{X_{can}})). \quad \square$$

It remains to show that $R(X)$ is also Gorenstein under this assumption. For this we use the following two statements:

Lemma 2.3.23. *Let $R = R_{Y,\mathcal{L}}$ be as in Remark 2.3.21, and let ω_R be the *canonical module of R . Then $\omega_R \cong \bigoplus_{d \in \mathbb{Z}} H^0(Y, \omega_Y(n))$ and $\omega_Y \cong \widetilde{\omega}_R$, where ω_Y is the dualizing sheaf of Y .*

Proof. See [GW78], 5.1.8. □

Lemma 2.3.24. *Let $R = R_{Y,\mathcal{L}}$ be as in Remark 2.3.21 and assume that R is Cohen-Macaulay. Then $R_{Y,\mathcal{L}}$ is Gorenstein if and only if $\omega_Y \cong \mathcal{L}^{\otimes n}$ for some $n \in \mathbb{Z}$.*

Proof. See [GW78], 5.1.9. □

Note that in the latter case $\omega_R \cong R(n)$. Applying this to $R = R_{X_{can}, \omega_{X_{can}}}$ yields:

Lemma 2.3.25. *Let X and R be as in Theorem 2.3.19. If R is Cohen-Macaulay, then R is Gorenstein with $\omega_R \cong R(1)$.*

Proof. Follows directly from Remark 2.3.21 and Lemma 2.3.24. □

Combining Lemma 2.3.22 and 2.3.25, we see that $R(X)$ is a Gorenstein ring if and only if $q = 0$ which proves Theorem 2.3.19.

We end this section by stating a well-known formula for the plurigenera of a minimal surface of general type:

Proposition 2.3.26. *Let X be a minimal surface of general type. Then*

$$P_n = h^0(X, \mathcal{O}_X(nK_X)) = \begin{cases} 1 & \text{for } n = 0, \\ p_g & \text{for } n = 1, \\ \binom{n}{2} K_X^2 + \chi(\mathcal{O}_X) & \text{for } n \geq 2. \end{cases}$$

The proof of this formula is a direct consequence of the Riemann-Roch theorem for surfaces and Mumford's vanishing theorem:

Theorem 2.3.27 (Mumford's vanishing theorem for surfaces). *Let D be a divisor on a smooth projective surface X . If D is semi-ample with Kodaira dimension $\kappa(D) = 2$, then*

$$h^i(X, \mathcal{O}_X(-D)) = 0 \text{ for all } i < \dim X.$$

Recall that we call a divisor D on X semi-ample if, for some $m \geq 1$, the linear system $|mD|$ is base-point-free. By a further result of Bombieri, we know that the canonical divisor of a minimal surface of general type is semi-ample:

Theorem 2.3.28. *Let X be a minimal surface of general type. Then $|mK_X|$ is base-point-free for all $m \geq 4$.*

Proof. See [Bom73], Theorem 5.2. □

Proof of Proposition 2.3.26. The statement is trivial for $n = 0, 1$. So let us assume that $n \geq 2$. Applying the Riemann-Roch theorem to $\mathcal{O}_X(nK_X)$ yields

$$\chi(\mathcal{O}_X(nK_X)) = \binom{n}{2} K_X^2 + \chi(\mathcal{O}_X).$$

Thus, it remains to show that $\chi(\mathcal{O}_X(nK_X)) = h^0(X, \mathcal{O}_X(nK_X))$. By Serre duality we have $h^2(X, \mathcal{O}_X(nK_X)) = h^0(X, \mathcal{O}_X((1-n)K_X))$, where the right-hand side is zero since the canonical divisor of a minimal surface of general type is nef and big. Furthermore, applying Theorem 2.3.27 to the semi-ample divisor $(n-1)K_X$, we obtain

$$h^1(X, \mathcal{O}_X(nK_X)) = h^1(X, \mathcal{O}_X((1-n)K_X)) = 0. \quad \square$$

3 Numerical Godeaux Surfaces: Basics

In this chapter we study the canonical ring $R(X)$ of a numerical Godeaux surface X and give the basic idea of our construction method. First we will consider $R(X)$ as a \mathbb{C} -algebra and determine the degrees of a minimal set of homogeneous algebra generators. Afterwards we take the 6 generators of lowest degree and study $R(X)$ as a module over the algebra S generated by these elements. We show that $R(X)$ is a finitely generated Cohen-Macaulay S -module and compute the Betti numbers of $R(X)$. Furthermore, we show that the corresponding morphism of projective schemes $\text{Proj}(R(X)) \rightarrow \text{Proj}(S)$ is finite and birational onto its image. Let us first recall the definition of a numerical Godeaux surface:

Definition 3.0.1. A minimal surface X of general type is called a *numerical Godeaux surface* if $K_X^2 = 1$ and $p_g = q = 0$.

Throughout the whole chapter, X will denote a numerical Godeaux surface, and $X_{can} = \text{Proj}(R(X))$ the canonical model of X . Furthermore, we assume that $\mathbb{k} = \mathbb{C}$.

3.1 The Bi- and the Tricanonical System

In our construction we use some classical results on the bi- and the tricanonical system of a numerical Godeaux surface which we will briefly recall here. Let us start with a more detailed study of the bicanonical system. We write

$$|2K_X| = |M| + F,$$

where M denotes a generic member of the moving part and F the fixed part of $|2K_X|$. Note that $|M|$ is a pencil by Proposition 2.3.26.

Proposition 3.1.1. *If M is generically chosen, M is reduced and irreducible. Moreover, M and F satisfy one of the following conditions*

- (i) $F = 0$,
- (ii) $K_X F = 0, F^2 = -2, M^2 = 2, MF = 2$,
- (iii) $K_X F = 0, F^2 = -4, M^2 = 0, MF = 4$.

Proof. See [Miy76], Lemma 6. □

The proposition shows that the fixed part of $|2K_X|$, if non-empty, is supported on the (-2) -curves of X . As these curves are contracted by the morphism $\pi: X \rightarrow X_{can}$, we get the following consequence for the bicanonical system on the canonical model:

Corollary 3.1.2. $|2K_{X_{can}}|$ is free from fixed components and its generic member is irreducible.

Next we focus on the tricanonical system.

Proposition 3.1.3. $|3K_X|$ has no fixed part.

Proof. See [Miy76], Proposition 2. □

Miyaoka also studied the question whether a point on X can be a base point of both the bi- and the tricanonical system.

Proposition 3.1.4. *Let $|M|$ denote the moving part of $|2K_X|$. If M is generic, then M contains no base points of $|3K_X|$.*

Proof. See [Miy76], Proposition 3. □

Corollary 3.1.5. *Let \hat{M} be the generic member of $|2K_{X_{can}}|$. Then $|3K_{X_{can}}|$ has no base points on \hat{M} .*

Proof. See [Miy76], Corollary after Proposition 3. □

Remark 3.1.6. Combining the previous statements, we conclude that no base point of $|3K_X|$ (respectively of $|3K_{X_{can}}|$) is a base point of $|2K_X|$ (respectively of $|2K_{X_{can}}|$). Hence, for a base point P of $|3K_X|$ there exists a unique divisor $D \in |2K_X|$ which contains P . Furthermore, Miyaoka showed that every base point of $|3K_{X_{can}}|$ is simple and that \hat{P} is a base point of $|3K_{X_{can}}|$ if and only if $\hat{P} = \hat{D}_1\hat{D}_2$, where \hat{D}_1, \hat{D}_2 are two distinct effective curves which are numerically equivalent to $K_{X_{can}}$ with $\hat{D}_1 + \hat{D}_2 \in |2K_{X_{can}}|$. The last fact gives indeed a very precise description of the number of base points of $|3K_X|$.

Theorem 3.1.7. *Every base point of the tricanonical system $|3K_X|$ is simple and the number b of base points is given as follows:*

$$b = \frac{\#\{t \in H^2(X, \mathbb{Z})_{\text{tors}} \mid t \neq -t\}}{2}.$$

Proof. See [Miy76], Theorem 2. □

Note that for numerical Godeaux surfaces $H^2(X, \mathbb{Z})_{\text{tors}} = \text{Tors } X = H_1(X, \mathbb{Z})$, where $\text{Tors } X$ is the torsion subgroup of the Picard group of X . Bombieri showed that the order of the torsion group of a numerical Godeaux surface is less than or equal to 6. Miyaoka refined this result in the following way:

Theorem 3.1.8. *The torsion group of X is cyclic of order less than or equal to 5.*

Proof. See [Miy76], Lemma 11 and Remark after Theorem 2'. □

Combining these two statements we obtain the following important result:

Theorem 3.1.9. *As above, let b denote the number of base points of $|3K_X|$. Then*

$$b = \begin{cases} 0 & \text{if } \text{Tors } X \cong 0 \text{ or } \mathbb{Z}/2\mathbb{Z}, \\ 1 & \text{if } \text{Tors } X \cong \mathbb{Z}/3\mathbb{Z} \text{ or } \mathbb{Z}/4\mathbb{Z}, \\ 2 & \text{if } \text{Tors } X \cong \mathbb{Z}/5\mathbb{Z}. \end{cases}$$

Proof. See [Miy76], Theorem 2'. □

Later we will use this characterization and Lemma 3.1.11 below to determine the torsion group of our constructed surfaces.

Lemma 3.1.10. *Assume $\text{Tors } X \neq 0$ and let $\tau \in \text{Tors } X$ be a non-trivial torsion element. Then*

$$h^0(X, K_X + \tau) = 1, \quad h^1(X, K_X + \tau) = 0.$$

Proof. See [Rei78], Lemma 0.3. □

Lemma 3.1.11. *Assume that $|2K_X|$ has no fixed part and has $(2K_X)^2 = 4$ distinct (simple) base points. Then the order of $\text{Tors } X$ is odd. In particular, for the number b of base points of $|3K_X|$, we find that*

- $b = 0$ if and only if $\text{Tors } X \cong 0$,
- $b = 1$ if and only if $\text{Tors } X \cong \mathbb{Z}/3\mathbb{Z}$,
- $b = 2$ if and only if $\text{Tors } X \cong \mathbb{Z}/5\mathbb{Z}$.

Proof. Suppose to the contrary that the order of the torsion group is even, that means $\text{Tors } X \cong \mathbb{Z}/2\mathbb{Z}$ or $\text{Tors } X \cong \mathbb{Z}/4\mathbb{Z}$. Let $\tau \in \text{Tors } X$ be a non-trivial torsion element of order 2. By Lemma 3.1.10 there exists an effective divisor $D \in |K_X + \tau|$. But then $|2K_X|$ contains the double curve $2D$ and thus, cannot have 4 distinct base points. The second part is an immediate consequence of Theorem 3.1.9. \square

3.2 The Canonical Ring $R(X)$

This section is devoted to describe the degrees of a minimal set of generators of the canonical ring of a numerical Godeaux surface X . Recall that by Proposition 2.3.26 the plurigenera of X are given as

$$P_n = h^0(X, nK_X) = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n = 1, \\ \binom{n}{2} + 1 & \text{for } n \geq 2. \end{cases}$$

Let x_0, x_1 be a basis of $H^0(X, 2K_X)$, and let y_0, y_1, y_2, y_3 be a basis of $H^0(X, 3K_X)$.

Lemma 3.2.1. x_0^2, x_0x_1, x_1^2 are linearly independent.

Proof. Suppose that there exist $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{C}$, not all zero, such that $\lambda_0x_0^2 + \lambda_1x_0x_1 + \lambda_2x_1^2 = 0$ in $R(X)$. But every quadratic form in two variables decomposes into a product of two linear forms. Since $R(X)$ is an integral domain, one of the linear factors must be zero. But x_0, x_1 being linearly independent implies that the coefficients of this factor are zero, and hence that all λ_i are zero, contradicting our assumption. \square

Since $H^0(X, 4K_X)$ is 7-dimensional, we can choose $z_0, \dots, z_3 \in H^0(X, 4K_X)$ so that $x_0^2, x_0x_1, x_1^2, z_0, \dots, z_3$ is a basis of $H^0(X, 4K_X)$. Our next task is to give a basis for the vector space $H^0(X, 5K_X)$. To start with we consider the natural multiplication map

$$\mu: H^0(X, 2K_X) \otimes H^0(X, 3K_X) \rightarrow H^0(X, 5K_X).$$

Lemma 3.2.2. *The map μ is injective.*

Proof. From $R(X) \cong R(X_{can})$ we get a commutative diagram

$$\begin{array}{ccc} H^0(X, 2K_X) \otimes H^0(X, 3K_X) & \xrightarrow{\mu} & H^0(X, 5K_X) \\ \cong \downarrow & & \downarrow \cong \\ H^0(X_{can}, 2K_{X_{can}}) \otimes H^0(X_{can}, 3K_{X_{can}}) & \xrightarrow{\tilde{\mu}} & H^0(X_{can}, 5K_{X_{can}}) \end{array}$$

Since the vertical maps are isomorphisms, it is sufficient to prove that $\tilde{\mu}$ is injective. Let \tilde{x}_0, \tilde{x}_1 be a basis of $H^0(X_{can}, 2K_{X_{can}})$. By Corollary 3.1.2 we know that the bicanonical system has no fixed part on the canonical model. Therefore, $V(\tilde{x}_0)$ and $V(\tilde{x}_1)$ do not have a common component and the following sequence is exact

$$0 \rightarrow \mathcal{O}_{X_{can}}(-4K_{X_{can}}) \xrightarrow{\begin{pmatrix} \tilde{x}_1 \\ -\tilde{x}_0 \end{pmatrix}} \begin{array}{c} \mathcal{O}_{X_{can}}(-2K_{X_{can}}) \\ \oplus \\ \mathcal{O}_{X_{can}}(-2K_{X_{can}}) \end{array} \xrightarrow{(\tilde{x}_0, \tilde{x}_1)} \mathcal{O}_{X_{can}} \rightarrow \mathcal{O}_Z \rightarrow 0,$$

where $Z = V(\tilde{x}_0) \cap V(\tilde{x}_1)$. Now tensoring with $\mathcal{O}_{X_{can}}(5K_{X_{can}})$ and taking global sections we get a sequence

$$0 \rightarrow \begin{array}{c} H^0(X_{can}, 3K_{X_{can}}) \\ \oplus \\ H^0(X_{can}, 3K_{X_{can}}) \end{array} \xrightarrow{(\tilde{x}_0, \tilde{x}_1)} H^0(X_{can}, 5K_{X_{can}})$$

which is exact since $h^0(X_{can}, K_{X_{can}}) = h^1(X_{can}, \mathcal{O}_{X_{can}}) = 0$. Now, since \tilde{x}_0, \tilde{x}_1 form a basis for $H^0(X_{can}, 2K_{X_{can}})$, the second map is simply the map $\tilde{\mu}$ which shows the claim. \square

The lemma shows that the global sections $x_i y_j$ for $i = 0, 1$ and $j = 0, \dots, 3$ define an 8-dimensional subspace of $H^0(X, 5K_X)$. Now as $h^0(X, 5K_X) = 11$, we can choose sections $w_0, w_1, w_2 \in H^0(X, 5K_X)$ extending these elements to a basis. Since we will use the same notation for the generators in the rest of this thesis, let us summarize the previous results in one table:

n	$h^0(X, nK_X)$	basis of $H^0(X, nK_X)$
2	2	x_0, x_1
3	4	y_0, \dots, y_3
4	7	$x_0^2, x_0 x_1, x_1^2, z_0, \dots, z_3$
5	11	$x_0 y_0, \dots, x_1 y_3, w_0, w_1, w_2$

From the previous chapter we know that $R(X)$ is a positively graded \mathbb{C} -algebra. To describe the structure of this algebra in more detail we are interested in the highest degree of a minimal homogeneous system of generators and in the number of such generators for each fixed degree. By the previous results, the second problem is settled for generators up to degree 5 (the elements marked in blue in the table above). For the first problem, Ciliberto gave upper bounds for the canonical ring of surfaces of general type depending on p_g and K^2 . For example, he showed that if $p_g \geq 1$ then the canonical ring is generated in degree ≤ 5 . The highest bound applies to surfaces of general type with $p_g = q = 0$:

Theorem 3.2.3. *Let Y be a minimal surface of general type with $p_g = q = 0$. Then $R(Y)$ is generated in degree ≤ 6 .*

Proof. See[Cil83], Theorem 3.5. \square

By some Hilbert series calculations and the results of Miyaoka in the previous chapter, we will refine this statement for numerical Godeaux surfaces:

Proposition 3.2.4. *The canonical ring $R(X)$ of a numerical Godeaux surface is generated in degree ≤ 5 .*

To prove this statement, we consider first the Hilbert series

$$\Psi(t) = \sum_{n \geq 0} P_n t^n = 1 + \sum_{n \geq 2} \binom{n}{2} t^n$$

of $R(X)$. Then $\Psi(t)$ has a representation as a rational function of the form

$$\Psi(t) = \frac{1 - 3t + 5t^2 - 3t^3 + t^4}{(1 - t)^3}.$$

Recall that no base point of the tricanonical system is a base point of the bicanonical system. Hence, there exists a form \tilde{y}_0 of degree 3 such that $\text{Proj}(R(X)/(x_0, x_1, \tilde{y}_0)) = \emptyset$. Hence $R(X)$ is a finitely generated module over $A = \mathbb{k}[x_0, x_1, \tilde{y}_0]$ by Proposition 3.2.6 below. Algebraically this means that x_0, x_1, \tilde{y}_0 is a homogeneous system of parameters for $R(X)$. Furthermore, since $R(X)$ is Cohen-Macaulay, $R(X)$ is a free module over A . Now let us use Theorem 2.1.7 on the alternating Betti numbers to compute the degrees of the free generators:

$$(1 - t^2)^2(1 - t^3)\Psi(t) = 1 + 3t^3 + 4t^4 + 3t^5 + t^8.$$

Hence

$$R(X) \cong A \oplus A(-3)^3 \oplus A(-4)^4 \oplus A(-5)^3 \oplus A(-8). \quad (3.1)$$

as A -modules. Choose $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \in H^0(X, 3K_X)$ so that $\tilde{y}_0, \dots, \tilde{y}_3$ is a basis of $H^0(X, 3K_X)$. Then clearly

$$1, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, z_0, \dots, z_3, w_0, w_1, w_2$$

is a basis of $R(X)$ as an A -module up to degree 5.

Proof of Proposition 3.2.4. By the previous arguments, it remains to show that any element in $R(X)$ is a polynomial expression in the algebra generators of $R(X)$ of degree ≤ 5 . Since we already know that $R(X)$ is generated in degree ≤ 6 , it is enough to show this for any form of degree 6. So let $r \in R(X)$ be an arbitrary form of degree 6. Then we can represent r as an A -linear combination of the module generators up to degree 4 (since there are no linear forms of degree 1 in A). Hence r is a polynomial expression in the elements $x_0, x_1, y_0, \dots, y_3, z_0, \dots, z_3$ which proves the claim. \square

So, if $\hat{S} = \mathbb{k}[x_0, x_1, y_0, \dots, y_3, z_0, \dots, z_3, w_0, w_1, w_2]$ is the weighted polynomial ring with degrees as assigned before, then there exists a surjection of rings $\hat{S} \twoheadrightarrow R(X)$, and hence a closed embedding

$$\varphi: X_{can} = \text{Proj}(R(X)) \hookrightarrow \text{Proj}(\hat{S}) = \mathbb{P}(2^2, 3^4, 4^4, 5^3). \quad (3.2)$$

Therefore we can consider X_{can} as a subvariety of a weighted projective space of dimension 12. Furthermore, using Theorem 2.1.7 and Proposition 2.3.26, we deduce from

$$(1 - t^2)^2(1 - t^3)^4(1 - t^4)^4(1 - t^5)^3\Psi(t) = 1 - 6t^6 - 12t^7 - 18t^8 - 4t^9 + 30t^{10} + 72t^{11} + \dots$$

that $X_{can} \subseteq \mathbb{P}(2^2, 3^4, 4^4, 5^4)$ is defined by at least 40 homogeneous relations. Later we will specify this statement and show that we need exactly 54 defining relations.

Studying this embedding is difficult for various reasons. As a (cyclic) \hat{S} -module, $R(X)$ has a minimal free resolution of length 10. While there is the structure theorem for codimension 3 Gorenstein ideals by Buchsbaum-Eisenbud ([BE77]) or the results on codimension 4 Gorenstein ideals by Reid ([Rei13a]), only little is known for higher codimension. In particular, to describe the minimal free resolution of a codimension 10 Gorenstein ideal in general seems hopeless. Furthermore, from a computational point of view, codimension 10 is not very promising for irreducibility or non-singularity tests. Schreyer's basic construction idea addresses these problems:

Construction 3.2.5 (Schreyer's idea). We do not consider $R(X)$ as an algebra but as a finitely generated S -module, where $S \subseteq \hat{S}$ is a subring chosen appropriately. Geometrically, we study the image of $\text{Proj}(R(X))$ under the projection into the smaller projective space $\text{Proj}(S)$.

So as a first step of the construction we have to define a subring S such that $R(X)$ is finitely generated as an S -module. Note that for the computational side a small codimension of the projected surface is desirable. For the choice of S we use the following geometric characterization:

Proposition 3.2.6. *Let B be a positively graded \mathbb{k} -algebra, and let $f_1, \dots, f_k \in B$ be homogeneous elements of positive degree. Then B is a finitely generated $A = \mathbb{k}[f_1, \dots, f_k]$ -module if and only if f_1, \dots, f_k have an empty vanishing locus in $\text{Proj}(B)$.*

Applying this to $B = R(X)$ we have to choose global sections of the pluricanonical systems on X which have an empty vanishing locus on X_{can} . So let

$$S = \mathbb{k}[x_0, x_1, y_0, y_1, y_2, y_3],$$

be the graded polynomial ring, where the x_i and y_j are as before with $\deg(x_i) = 2$ and $\deg(y_j) = 3$. The natural homomorphism

$$f: S \rightarrow R(X)$$

gives $R(X)$ the structure of a graded S -algebra. In the following we consider $R(X)$ as a graded S -module via the homomorphism f .

Proposition 3.2.7. *$R(X)$ is a finitely generated S -module.*

Proof. Clear from Remark 3.1.6 and Proposition 3.2.6. □

Using the closed embedding in (3.2), we will from now on identify X_{can} with its image in $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$. Now, since $R(X)$ is finitely generated as an S -module, the homomorphism $f: S \rightarrow R(X)$ induces a finite morphism of projective schemes

$$\varphi: X_{can} \rightarrow \mathbb{P}(2^2, 3^4).$$

As $R(X)$ is a ring and an S -module we have $\ker(f) = \text{ann}_S R(X)$. Let $S_Y = S / \text{ann}_S R(X)$ and $Y = \text{Proj}(S_Y) \subseteq \mathbb{P}(2^2, 3^4)$. Then Y is the image of X_{can} under φ .

Proposition 3.2.8. *(X_{can}, φ) is the normalization of Y .*

The proof of this result uses a statement of Miyaoka on the rational map ϕ_3 . Recall that we denote by ϕ_n (respectively κ_n) the rational map associated to the linear system $|nK_X|$ (respectively $|nK_{X_{can}}|$). By Bombieri's Theorem 2.3.17 we know that κ_n is an embedding for $n \geq 5$. Furthermore, Bombieri showed that κ_3 is birational except for finitely many choices of tuples (K^2, p_g) . One exception he stated is the case $(K^2, p_g) = (1, 0)$. However, Miyaoka proved later that the statement is indeed true for numerical Godeaux surfaces:

Theorem 3.2.9. *Let $Z \subseteq \mathbb{P}^3$ be the image of X under ϕ_3 . Then $\phi_3: X \dashrightarrow Z$ is birational.*

Proof. See [Miy76], Theorem 4. □

Now let us prove Proposition 3.2.8:

Proof of Proposition 3.2.8. We already know that $\varphi: X_{can} \rightarrow Y$ is a finite morphism. Furthermore, X_{can} is a normal surface. Hence, by the characterization of the normalization of an integral scheme, it remains to show that $\varphi: X_{can} \rightarrow Y$ is birational. Since every pluricanonical map factors through the canonical model we get a commutative diagram

$$\begin{array}{ccc} X & \overset{\phi_3}{\dashrightarrow} & Z \\ & \searrow \pi & \uparrow \kappa_3 \\ & & X_{can} \end{array}$$

where π and ϕ_3 are birational. Hence, κ_3 is birational and, since $X_{can} \subseteq \mathbb{P}(2^2, 3^4, 4^4, 5^3)$, it is the projection of X_{can} to $\mathbb{P}(3^4) \cong \mathbb{P}^3$. Now together with the projection of X_{can} to $\mathbb{P}(2^2, 3^4)$ we get the following commutative diagram of integral schemes

$$\begin{array}{ccc} X_{can} & \overset{\kappa_3}{\dashrightarrow} & Z \subseteq \mathbb{P}^3 \\ & \searrow \varphi & \nearrow \\ & & Y \end{array}$$

On the level of function fields, this corresponds to a commutative diagram

$$\begin{array}{ccc} K(X_{can}) & \overset{\cong}{\longleftarrow} & K(Z) \\ & \searrow & \swarrow \\ & & K(Y) \end{array}$$

But this implies that $K(X_{can}) \cong K(Y)$. Hence, $\varphi: X_{can} \rightarrow Y \subseteq \mathbb{P}(2^2, 3^4)$ is birational and (X_{can}, φ) is the normalization of Y . \square

3.3 The Minimal Free Resolution of $R(X)$ as an S -module

A first part of the description of $R(X)$ as an S -module is to study its minimal free resolution. The main result of this section is the following:

Theorem 3.3.1. *The minimal free resolution of $R(X)$ as an S -module is of type*

$$\begin{array}{cccccccc} & S & & S(-6)^6 & & S(-9)^8 & & S(-12)^3 \\ & \oplus & & \oplus & & \oplus & & \oplus \\ 0 \leftarrow R(X) \leftarrow & S(-4)^4 & \leftarrow & S(-7)^{12} & \leftarrow & S(-10)^{12} & \leftarrow & S(-13)^4 \leftarrow 0. \\ & \oplus & & \oplus & & \oplus & & \oplus \\ & S(-5)^3 & & S(-8)^8 & & S(-11)^6 & & S(-17) \end{array}$$

For the proof of this result we need the notation and results on *canonical modules from Section 2.2. Recall that both S and $R(X)$ are *local Gorenstein rings with $\omega_S \cong S(-16)$ and $\omega_{R(X)} \cong R(X)(1)$. Since we are only dealing with these two *local rings in the following, we will omit the * if no confusion can arise.

Proposition 3.3.2. *$R(X)$ is a Cohen-Macaulay graded S -module.*

Proof. Let \mathfrak{m} and \mathfrak{n} denote the graded maximal ideals of S and $R(X)$, respectively. Since $R(X)$ is a Cohen-Macaulay ring, we have

$$\text{depth}(\mathfrak{n}, R(X)) = \dim R(X) = 3.$$

Now the natural ring homomorphism $f: S \rightarrow R(X)$ satisfies $f(S_i) \subseteq R(X)_i$ for all i and $f(\mathfrak{m}) \subseteq \mathfrak{n}$. Hence, since $R(X)$ is a finitely generated S -module,

$$\text{depth}(\mathfrak{m}, R(X)) = \text{depth}(\mathfrak{n}, R(X)) = 3.$$

Then the result follows as $\dim_S R(X) = \dim S/\text{ann}_S R(X) = \dim R(X) = 3$, where the second equality follows from the fact that $S/\text{ann}_S R(X) \hookrightarrow R(X)$ is integral. \square

Now, using the fact that $R(X)$ is Gorenstein, we show that the Betti numbers of $R(X)$ as an S -module satisfy the following symmetry condition:

Proposition 3.3.3. *Let*

$$0 \leftarrow R(X) \leftarrow F_0 \leftarrow \dots \leftarrow F_n \leftarrow 0$$

be a minimal free resolution of $R(X)$ as an S -module, where $F_i = \bigoplus_{j \geq 0} S(-j)^{\beta_{i,j}}$.

Then

$$\beta_{i,j} = \beta_{3-i,17-j} \text{ for } 0 \leq i \leq 3, 0 \leq j \leq 17 \text{ and } \beta_{i,j} = 0 \text{ otherwise.}$$

Proof. First note that the previous statement and the Auslander-Buchsbaum formula in the graded case imply that

$$n = \text{projd} \dim R(X) = \text{depth}(\mathfrak{m}, S) - \text{depth}(\mathfrak{m}, R(X)) = 3.$$

So let

$$0 \leftarrow R(X) \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow 0$$

be a minimal free resolution of $R(X)$. As in the proof of Proposition 2.2.11, applying the functor $\text{Hom}_S(-, \omega_S)$ yields a minimal free resolution of $\omega_{R(X)}$:

$$0 \leftarrow \omega_{R(X)} \leftarrow \text{Hom}_S(F_3, \omega_S) \leftarrow \text{Hom}_S(F_2, \omega_S) \leftarrow \text{Hom}_S(F_1, \omega_S) \leftarrow \text{Hom}_S(F_0, \omega_S) \leftarrow 0$$

On the other hand, as $R(X)$ is Gorenstein, $\omega_{R(X)} \cong R(X)(1)$ and, after tensoring with $S(-1)$, we obtain

$$\begin{aligned} 0 \leftarrow R(X) \leftarrow \text{Hom}_S(F_3, S(-17)) \leftarrow \text{Hom}_S(F_2, S(-17)) \\ \leftarrow \text{Hom}_S(F_1, S(-17)) \leftarrow \text{Hom}_S(F_0, S(-17)) \leftarrow 0, \end{aligned}$$

which is another minimal free resolution of $R(X)$. Finally, since the Betti numbers of $R(X)$ are uniquely determined, we obtain the desired equalities

$$\beta_{i,j} = \beta_{3-i,17-j} \text{ for } 0 \leq i \leq 3, 0 \leq j \leq 17 \text{ and } \beta_{i,j} = 0 \text{ otherwise.} \quad \square$$

Notation 3.3.4. If F_i is a free module occurring in a minimal free resolution of $R(X)$, we denote the shift of the dual $\text{Hom}_S(F_i, \omega_S)(-1) \cong \text{Hom}_S(F_i, S(-17))$ from now on by F_i^\vee . Furthermore, if $f: F_i \rightarrow F_{i+1}$ is an S -linear homomorphism given by a matrix α , the dual map $f^\vee: F_{i+1}^\vee \rightarrow F_i^\vee$ is given by the matrix α^{tr} (with an appropriate shift of the grading).

The main idea of the proof of Theorem 3.3.1 is to consider a minimal free resolution of $R(X)$ modulo an $R(X)$ -regular sequence. Recall from Section 2.1 that free resolutions stay exact modulo a regular sequence.

Lemma 3.3.5. $x_0, x_1 \in S$ is a regular sequence for $R(X)$.

Proof. Using again the fact that the bicanonical system on X_{can} has no fixed part, we have $\dim R(X)/(x_0, x_1)R(X) = \dim R(X) - 2$. The result follows now from $R(X)$ being Cohen-Macaulay. \square

So let $\bar{R} = R(X)/(x_0, x_1)R(X)$ and $T = S/(x_0, x_1) \cong \mathbb{k}[y_0, \dots, y_3]$ with $\deg(y_i) = 3$. Then we get the following:

Lemma 3.3.6. As a T -module \bar{R} splits into a direct sum

$$\bar{R} = \bigoplus_{k=0}^2 \bar{R}^{(k)} \quad \text{with} \quad \bar{R}^{(k)} = \bigoplus_{\substack{j \equiv k \\ (\text{mod } 3)}} \bar{R}_j.$$

Moreover, for any minimal free resolution F_\bullet of $R(X)$, the complex $\bar{F}_\bullet = F_\bullet \otimes S/(x_0, x_1)$ decomposes into a direct sum of three T -complexes $\bar{F}_\bullet^{(k)}$ which are minimal free resolutions of $\bar{R}^{(k)}$ as a T -module.

Proof. The statement on the splitting of $R(X)$ and its minimal free resolution is a direct consequence of Proposition 2.1.8 and the fact that the degree of any homogeneous element of T is a multiple of 3. \square

As a last step before proving Theorem 3.3.1 we compute the alternating Betti numbers B_j of $R(X)$ as defined in Section 2.1. Let $\Psi(t)$ again denote the Hilbert series of $R(X)$. Then

$$\begin{aligned} (1-t^2)^2(1-t^3)^4\Psi(t) &= \varphi_{R(X)}(t) = \sum_{j \geq 0} B_j t^j \\ &= 1 + 4t^4 + 3t^5 - 6t^6 - 12t^7 - 8t^8 \\ &\quad + 8t^9 + 12t^{10} + 6t^{11} - 3t^{12} - 4t^{13} - t^{17}. \end{aligned}$$

Note that the polynomial $\varphi_{R(X)}(t)$ is symmetric in the sense that the coefficients of t^k and $-t^{17-k}$ coincide. This is a further consequence of Proposition 3.3.3.

Proof of Theorem 3.3.1. By the results on the numbers and degrees of the generators of $R(X)$ as a \mathbb{k} -algebra, we know that there are no relations between the generators of degree ≤ 5 . Furthermore, using the (known) alternating Betti numbers B_j and the symmetry of the (unknown) Betti numbers, the minimal free resolution of $R(X)$ as an S -module must be of the following form

$$\begin{array}{ccccccc} & S & & & & & S(-10)^{l_2} \\ & \oplus & & S(-6)^{6+l_1} & & S(-8)^{l_3} & \oplus \\ S(-4)^4 & & & \oplus & & \oplus & S(-11)^{l_1} \\ & \oplus & & S(-7)^{12+l_2} & & S(-9)^{8+l_3} & \oplus \\ 0 \leftarrow R(X) \leftarrow & S(-5)^3 & \leftarrow & \oplus & \leftarrow & \oplus & \leftarrow S(-12)^3 \leftarrow 0, \\ & \oplus & & S(-8)^{8+l_3} & & S(-10)^{12+l_2} & \oplus \\ S(-6)^{l_1} & & & \oplus & & \oplus & S(-13)^4 \\ & \oplus & & S(-9)^{l_3} & & S(-11)^{6+l_1} & \oplus \\ & S(-7)^{l_2} & & & & & S(-17) \end{array}$$

where l_1, l_2, l_3 are non-negative integers.

By considering this free resolution modulo the $R(X)$ -regular sequence x_0, x_1 , we show now that $l_1 = l_2 = l_3 = 0$. By Proposition 3.3.6 we know that the sequence \bar{F}_\bullet splits into a direct sum of three T -complexes which are minimal free resolutions of \bar{R}^i for $i = 0, 1, 2$:

$$0 \leftarrow \bar{R}^{(0)} \leftarrow \begin{array}{c} T \\ \oplus \\ T(-6)^{l_1} \end{array} \leftarrow \begin{array}{c} T(-6)^{6+l_1} \\ \oplus \\ T(-9)^{l_3} \end{array} \leftarrow T(-9)^{8+l_3} \leftarrow T(-12)^3 \leftarrow 0 \quad (3.3)$$

$$0 \leftarrow \bar{R}^{(1)} \leftarrow \begin{array}{c} T(-4)^4 \\ \oplus \\ T(-7)^{l_2} \end{array} \leftarrow T(-7)^{12+l_2} \leftarrow T(-10)^{12+l_2} \leftarrow \begin{array}{c} T(-10)^{l_2} \\ \oplus \\ T(-13)^4 \end{array} \leftarrow 0 \quad (3.4)$$

$$0 \leftarrow \bar{R}^{(2)} \leftarrow T(-5)^3 \leftarrow T(-8)^{8+l_3} \leftarrow \begin{array}{c} T(-8)^{l_3} \\ \oplus \\ T(-11)^{6+l_1} \end{array} \leftarrow \begin{array}{c} T(-11)^{l_1} \\ \oplus \\ T(-17) \end{array} \leftarrow 0 \quad (3.5)$$

If $l_2 > 0$, then the minimality of the free resolution (3.4) implies that the last (non-trivial) map of (3.4) has a zero column which is a contradiction. Hence $l_2 = 0$. Using the same argument for the map in the middle of sequence (3.5) we see that l_3 must be zero as well. Then $l_1 = 0$ follows from considering the last map in (3.5). \square

Remark 3.3.7. The theorem shows that $R(X)$ is generated in degree ≤ 5 as an S -module. Hence, this result is an alternative proof of the fact that $R(X)$ is generated in degree ≤ 5 as an algebra.

Corollary 3.3.8. $\text{Proj}(\bar{R}^{(0)})$ is a finite scheme of length 4 in \mathbb{P}^3 .

Proof. We know that $\bar{R}^{(0)} = \bigoplus_{k \geq 0} \bar{R}_{3k}$ is a graded ring. Now the proof of Theorem 3.3.1 implies that

$$0 \leftarrow \bar{R}^{(0)} \leftarrow T \leftarrow T(-2)^6 \leftarrow T(-3)^8 \leftarrow T(-4)^3 \leftarrow 0$$

is a minimal free resolution of $\bar{R}^{(0)}$ as a T -module, where we consider the variables y_j with degree 1. Hence, $\bar{R}^{(0)}$ is a cyclic T -module whose Hilbert polynomial is the constant polynomial 4. Consequently, $\text{Proj}(\bar{R}^{(0)}) \subseteq \mathbb{P}^3$ is a finite scheme of length 4. \square

From Section 3.2 we know that there exists a surjective ring homomorphism

$$\hat{f}: \hat{S} \rightarrow R(X),$$

where $\hat{S} = \mathbb{k}[x_0, x_1, y_0, \dots, y_3, z_0, \dots, z_3, w_0, w_1, w_2]$ is the graded polynomial ring as defined before. In the following, we will determine a minimal generating set of the kernel of \hat{f} . Let $r_0 = 1, r_1 = z_0, \dots, r_4 = z_3, r_5 = w_0, r_6 = w_1, r_7 = w_2$ which generate $R(X)$ as an S -module. Theorem 3.3.1 shows that there are 26 S -linear relations between these module generators:

$$0 = \sum_{k=0}^7 g_{m,k} r_k. \quad (3.6)$$

Furthermore, for the 28 elements $r_i r_j \in R(X)$, $1 \leq i \leq j \leq 7$, there exist elements $s_{i,j,k} \in S$ such that

$$r_i r_j = \sum_{k=0}^7 s_{i,j,k} r_k. \quad (3.7)$$

Note that modulo the relations (3.6) the relations in (3.7) are uniquely determined. Let $I_X \subseteq \hat{S}$ be the ideal generated by the relations in (3.6) and (3.7). Then:

Lemma 3.3.9. $R(X) \cong \hat{S}/I_X$

Proof. Since all generators of I_X define relations in $R(X)$, \hat{f} factors through a surjective homomorphism $\hat{S}/I_X \rightarrow R(X)$. On the other hand, as an S -module, \hat{S}/I_X is also generated by r_0, \dots, r_7 and every relation in (3.6) is also an S -linear relation between the module generators of \hat{S}/I_X . Hence, there exists a surjective S -linear homomorphism $R(X) \rightarrow \hat{S}/I_X$ which shows the claim. \square

In Proposition 3.2.8 we have seen that $\varphi: X_{can} \rightarrow Y$ is the normalization of Y . We end this section by giving a criterion for φ to be an isomorphism. To do so, let us write the first syzygy matrix d_1 of $R(X)$ as

$$\begin{pmatrix} d_1^{(0)} \\ d'_1 \end{pmatrix}: \begin{matrix} S \\ \oplus \\ F'_0 \end{matrix} \leftarrow F_1,$$

and set $M = \text{coker } d'_1$.

Proposition 3.3.10. *The morphism $\varphi: X_{can} \rightarrow Y$ is an isomorphism if and only if $\widetilde{M} = 0$.*

Proof. Let $I_Y = \text{ann}_S R(X) = \text{ann}_S(\text{coker } d_1)$, and let $S_Y = S/I_Y$ as before. We have $\text{ann}_S R(X) = \text{ann}_S 1_{R(X)}$ since $R(X)$ is a ring. Hence, the sequence

$$0 \rightarrow S_Y \rightarrow R(X) \rightarrow M \rightarrow 0$$

is exact, where the non-trivial maps are induced by the ring homomorphism $f: S \rightarrow R(X)$ and the projection of F_0 onto F'_0 , respectively. This yields an exact sequence of coherent $\mathcal{O}_{\mathbb{P}(2^2, 3^4)}$ -modules

$$0 \rightarrow \mathcal{O}_Y \rightarrow \varphi_* \mathcal{O}_{X_{can}} \rightarrow \widetilde{M} \rightarrow 0$$

which proves the claim. \square

The proposition implies that the coherent sheaf \widetilde{M} is supported on the non-normal locus of Y . Let I' be the ideal generated by the 7×7 minors of d'_1 . Then we have

$$\text{Supp}(\widetilde{M}) \subseteq V(I') = \text{Proj}(S/I') \subseteq \mathbb{P}(2^2, 3^4). \quad (3.8)$$

Note that we obtain only an inclusion as we are working over a weighted projective space. Later we will see in explicit examples that this inclusion is often strict.

4 A Structure Theorem for the Canonical Ring

Let X be a numerical Godeaux surface with canonical ring $R(X)$. Furthermore, let $S = \mathbb{k}[x_0, x_1, y_0, \dots, y_3]$ be the graded polynomial ring with $\deg(x_i) = 2$ and $\deg(y_j) = 3$ as in the previous chapter. We have already seen that the resolution of $R(X)$ as an S -module is isomorphic to a twist of its dual. The purpose of this section is to prove a stronger version of this result. More precisely, we will show that there exists a minimal free resolution of $R(X)$ which is equal to a twist of its dual up to the sign (-1) in the middle. Let us first introduce some notation.

Definition 4.0.1 ([BE77], Section 2). Let A be a commutative ring, and let F be a finitely generated free A -module. We call a map $f: F \rightarrow \text{Hom}_A(F, A(s))$, $s \in \mathbb{Z}$, *alternating* if, with respect to some (and hence any) basis and dual basis of F and $\text{Hom}_A(F, A(s))$ respectively, the matrix corresponding to f is skew-symmetric.

The main result of this chapter is the following.

Theorem 4.0.2. *There is a minimal free resolution of $R(X)$ as an S -module of type*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{\eta_1} F_1 \xleftarrow{\eta_2} F_1^\vee \xleftarrow{\eta_1^\vee} F_0^\vee \leftarrow 0,$$

where η_2 is alternating.

This statement can be seen as a modification of the famous structure theorem of Buchsbaum and Eisenbud for codimension 3 Gorenstein ideals, see [BE77]. Note that we cannot apply their results directly since $R(X)$ is not a cyclic S -module. As in [BE77], the central ingredient of our proof is to define a multiplication on the minimal free resolution of $R(X)$ and to show that this multiplication is commutative.

4.1 Preliminaries

In this introductory part, A will denote a graded commutative ring, and $\mathbf{F} = (F_\bullet, d)$ a chain complex of finitely generated free A -modules with $F_i = 0$ for $i < 0$. We will consider $\mathbf{F} = \bigoplus_{i \geq 0} F_i$ as a graded A -module. Note that we have two different gradings on \mathbf{F} . On the one hand, we have the grading coming from homology, that means we say $f \in \mathbf{F}$ is (homological) homogeneous of degree i if $f \in F_i$. We denote the (homological) degree of f by $|f|$. On the other hand, we have the grading coming from A . We denote the degree of any element $f \in \mathbf{F}$ which is homogeneous with respect to the grading of A by $\deg(f)$.

Definition 4.1.1. By $\mathbf{F} \otimes \mathbf{F}$ we denote the chain complex whose degree n component is

$$(\mathbf{F} \otimes \mathbf{F})_n = \bigoplus_{i+j=n} F_i \otimes F_j$$

with differentials

$$\delta(f \otimes g) = d(f) \otimes g + (-1)^{|f|} f \otimes d(g)$$

for $f, g \in \mathbf{F}$ homogeneous. Let $\mu: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F}$ be a map of chain complexes. We say that the map μ is *homotopy-associative* if the chain maps

$$\mu \circ (\mu \otimes \text{id}_{\mathbf{F}}): \mathbf{F}^{\otimes 3} \rightarrow \mathbf{F} \text{ and } \mu \circ (\text{id}_{\mathbf{F}} \otimes \mu): \mathbf{F}^{\otimes 3} \rightarrow \mathbf{F}$$

are homotopic. Moreover, we define a map of chain complexes $\alpha: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F} \otimes \mathbf{F}$ by

$$f \otimes g \mapsto f \otimes g - (-1)^{|f| \cdot |g|} g \otimes f$$

for $f, g \in \mathbf{F}$ homogeneous. We say that μ is *homotopy-commutative* if the chain map $\mu \circ \alpha: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F}$ is homotopic to zero.

4.1.1 The Module $\text{Hom}_{S_Y}(R(X), R(X))$

One ingredient in the proof of Theorem 4.0.2 is that any non-zero homogeneous S_Y -linear homomorphism $R(X) \rightarrow R(X)$ is already an isomorphism, where $S_Y = S/\text{ann}_S R$ as before. The aim of this subsection is to prove the following result which implies this statement:

Theorem 4.1.2. $R(X) \cong \text{Hom}_{S_Y}(R(X), R(X))$ as S_Y -modules.

Since we will use similar ideas again, we show this in a more general setting.

Lemma 4.1.3. *Let $A \subseteq B$ be an inclusion of integral domains with $Q(A) = Q(B)$. If B is a finitely generated A -module, then B is a fractional ideal of A , that means there exists $0 \neq d \in A$ such that $dB \subseteq A$.*

Proof. Let b_1, \dots, b_m be module generators of B as an A -module. Then B , considered as an A -submodule of $Q(A)$, is generated by $\frac{b_1}{1}, \dots, \frac{b_m}{1}$. But $Q(A) = Q(B)$ implies that for each i there exist $a_i, d_i \in A$ with $d_i \neq 0$ such that $\frac{b_i}{1} = \frac{a_i}{d_i}$. Then $d = d_1 \cdot \dots \cdot d_m$ satisfies $dB \subseteq A$. \square

The next step is to show that any A -linear homomorphism $B \rightarrow B$ for domains $A \subseteq B$ as in the lemma above is the multiplication by an element of B . The proof relies on two well-known results:

Proposition 4.1.4. *Let A be a reduced ring with total ring of fractions L . If L is a direct product of finitely many fields and I, J are A -submodules of L , then every A -linear homomorphism $I \rightarrow J$ is the multiplication by an element of L .*

Proof. See [HS06], Lemma 2.4.1. \square

Proposition 4.1.5. *Let A, L, I, J be as in Proposition 4.1.4. Then the natural map*

$$J :_L I \rightarrow \text{Hom}_A(I, J)$$

is a surjective A -module homomorphism with kernel $0 :_L I$.

Proof. See [HS06], Lemma 2.4.2. \square

Proposition 4.1.6. *Let A and B be as in Lemma 4.1.3. Then $B \cong \text{Hom}_A(B, B)$ as A -modules.*

Proof. Let $L = Q(A) = Q(B)$. From Propositions 4.1.4 and 4.1.5 we see that

$$\tau: B :_L B \rightarrow \text{Hom}_A(B, B)$$

is surjective with kernel $0 :_L B$. We claim that $0 :_L B = 0$. Let $\frac{a}{d} \in 0 :_L B$. Then $\frac{a}{d}b = 0$ for all $b \in B$. But this implies that $ab = 0$ for all $b \in B$. Hence $a = 0$ since B is a faithful A -module. Thus, τ is an isomorphism. It remains to show that $B :_L B = B$. Clearly $B \subseteq B :_L B$. Now let $z \in B :_L B \subseteq L$. But then $z = z \cdot 1 \in B$. \square

To apply this statement in our setting we have to verify that $B = R(X)$ and $A = S_Y$ fulfill the condition assumed above. So let us suppose now that A and B have additionally the structure of positively graded \mathbb{k} -algebras and that B is a finitely generated graded A -module.

Remark 4.1.7. Since B is a finitely generated A -module, $\text{Hom}_A(B, B) = {}^*\text{Hom}_A(B, B)$ (see Notation 2.2.4). Furthermore, the isomorphism τ is compatible with the grading of B and ${}^*\text{Hom}_A(B, B)$. Indeed, given $b \in B_n$, the homomorphism $\tau(b)$ is homogeneous of degree n and hence $\tau(b) \in ({}^*\text{Hom}_A(B, B))_n = \text{Hom}_n(B, B)$. Thus τ is an isomorphism of graded A -modules.

Let q be a homogeneous prime ideal of B , and let T be the set of homogeneous elements in $B \setminus q$. We denote the graded ring $T^{-1}B$ by B_q , and the subring of degree zero elements by $B_{(q)}$. With the same notation for A , the proof of Theorem 4.1.2 uses the following result on the homomorphisms of local rings induced by the inclusion $A \subseteq B$:

Lemma 4.1.8. *Let q be a homogeneous prime ideal of B , and let $p = q \cap A$ be the corresponding (homogeneous) prime ideal of A . If the homomorphism*

$$A_{(p)} \rightarrow B_{(q)}$$

induced by the inclusion $A \subseteq B$ is an isomorphism and if A_p contains an invertible element of degree 1, then the homomorphism

$$A_p \rightarrow B_q$$

is an isomorphism as well.

Proof of Theorem 4.1.2. Identifying S_Y with its image in $R(X)$ via the injective homomorphism $S_Y \hookrightarrow R(X)$, we have to show that $Q(S_Y) = Q(R(X))$. First, the inclusion $S_Y \subseteq R(X)$ implies that $Q(S_Y) \subseteq Q(R(X))$. Let $q = (0)$ be the generic point of $\text{Proj}(R(X))$ and $p = (0) = \varphi(q)$ the generic point of Y . Since $\varphi: X_{\text{can}} \rightarrow Y$ is birational, the ring homomorphism

$$\varphi_q^\#: (S_Y)_{(p)} \rightarrow R(X)_{(q)}$$

is an isomorphism. Furthermore, $(S_Y)_p$ contains an invertible element of degree 1, for example the element $\frac{y_0}{x_1}$. The result follows now from the previous lemma and the fact that $R(X)_q = Q(R(X))$ (and similarly for $(S_Y)_p$). Consequently, the assumptions of Proposition 4.1.6 are satisfied and $R(X) \cong \text{Hom}_{S_Y}(R(X), R(X))$ as graded S_Y -modules. \square

4.2 A Multiplicative Structure on the Minimal Free Resolution

From now on $\mathbf{F} = (F_\bullet, d)$ will denote a (fixed) minimal free resolution of $R(X)$ as an S -module:

$$0 \leftarrow R(X) \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} F_3 \leftarrow 0.$$

The aim of this section is to define a multiplication on \mathbf{F} satisfying various properties needed for the proof of Theorem 4.0.2.

Theorem 4.2.1. *There exists a chain map $\mu: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F}$ such that, writing ab for $\mu(a \otimes b)$, the following holds:*

- (i) μ respects the grading of S , that is for $a, b \in \mathbf{F}$ homogeneous with respect to the grading of S we have $\deg(ab) = \deg(a) + \deg(b)$,
- (ii) $d(fg) = d(f)g + (-1)^{|f|}fd(g)$ for any $f, g \in \mathbf{F}$ homogeneous,

- (iii) μ is homotopy-associative,
- (iv) there exists an element $e_0 \in \mathbf{F}$ of degree 0 which acts as a unit for μ on \mathbf{F} , that means $e_0 g = g = g e_0$ for any $g \in \mathbf{F}$,
- (v) $fg = (-1)^{|f|\cdot|g|}gf$ for any $f, g \in \mathbf{F}$ homogeneous (that means μ is commutative).

We will prove this statement in several steps. Throughout the rest of this chapter, we use the following notation.

Notation 4.2.2. By e_0, \dots, e_7 we denote the canonical basis of the finitely generated free module F_0 , where e_0 corresponds to the summand S in F_0 . Let $m_i := d_0(e_i) \in R(X)$. Then m_0, \dots, m_7 is a generating set of $R(X)$ as an S -module. By $\tilde{\mu}$ we denote the (S -linear) multiplication map from $R(X) \otimes R(X)$ to $R(X)$.

Let us first consider the following diagram:

$$\begin{array}{ccccccccccc}
 0 & \longleftarrow & R(X) \otimes R(X) & \xleftarrow{d_0 \otimes d_0} & (\mathbf{F} \otimes \mathbf{F})_0 & \xleftarrow{\delta_1} & (\mathbf{F} \otimes \mathbf{F})_1 & \xleftarrow{\delta_2} & (\mathbf{F} \otimes \mathbf{F})_2 & \xleftarrow{\delta_3} & (\mathbf{F} \otimes \mathbf{F})_3 & \longleftarrow & \dots \\
 & & \tilde{\mu} \downarrow & & & & & & & & & & & \\
 0 & \longleftarrow & R(X) & \xleftarrow{d_0} & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 & \xleftarrow{d_3} & F_3 & \longleftarrow & 0
 \end{array}$$

where the first row is a complex and the second row is exact. By the comparison theorem for complexes of projective modules we can lift the map $\tilde{\mu}: R(X) \otimes R(X) \rightarrow R(X)$ to a map of complexes $\mu: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F}$ inducing $\tilde{\mu}$. Furthermore, any two lifts of $\tilde{\mu}$ are homotopic.

Proposition 4.2.3. Any chain map $\mu: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F}$ which is a lift of $\tilde{\mu}$ satisfies properties (i)-(iii) of Theorem 4.2.1.

Proof. The three properties are immediate from the definition of μ . Property (i) holds since δ_i and d_i are homogeneous maps of degree 0 with respect to the grading by S . Property (ii) is just the commutativity of the diagram above after adding the chain map μ . To verify the last property we have to show that the chain map

$$\rho := \mu \circ (\mu \otimes \text{id}_{\mathbf{F}}) - \mu \circ (\text{id}_{\mathbf{F}} \otimes \mu): \mathbf{F}^{\otimes 3} \rightarrow \mathbf{F}$$

is homotopic to zero. But ρ is a lift of the map $\tilde{\mu} \circ (\tilde{\mu} \otimes \text{id}_{R(X)}) - \tilde{\mu} \circ (\text{id}_{R(X)} \otimes \tilde{\mu}): R(X)^{\otimes 3} \rightarrow R(X)$ which is the zero map since $R(X)$ is associative. Thus ρ is homotopic to zero. \square

Remark 4.2.4. Similarly one can check that the map μ from Proposition 4.2.3 is homotopy-commutative.

So it remains to show that there is a lift μ of $\tilde{\mu}$ which satisfies also properties (iv) and (v). To do this, we will first introduce the symmetric square of the complex \mathbf{F} as in [BE77]. Let M be the graded submodule of $\mathbf{F} \otimes \mathbf{F}$ generated by

$$\{f \otimes g - (-1)^{|f|\cdot|g|}g \otimes f \mid f, g \in \mathbf{F} \text{ homogeneous}\}.$$

Since $\delta(M) \subseteq M$, the module

$$S_2(\mathbf{F}) = (\mathbf{F} \otimes \mathbf{F})/M$$

inherits the structure of a complex of S -modules (with differentials $\bar{\delta}$).

Lemma 4.2.5. Let $n \geq 0$ and set $V = \bigoplus_{i+j=n, i < j} F_i \otimes F_j$. Then

$$S_2(\mathbf{F})_n \cong \begin{cases} V & \text{if } n \text{ is odd,} \\ V \oplus \wedge^2(F_{n/2}) & \text{if } n \equiv 2 \pmod{4}, \\ V \oplus S_2(F_{n/2}) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

In particular, $S_2(\mathbf{F})$ is a complex of free S -modules.

Proof. See [FSWT11], Theorem 2.9. □

From the definition of $S_2(\mathbf{F})$ we see that any lift $\mu: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F}$ of $\tilde{\mu}$ which factors through the complex $S_2(\mathbf{F})$ satisfies property (v). Let $\pi: \mathbf{F} \otimes \mathbf{F} \rightarrow S_2(\mathbf{F})$ be the map of chain complexes, where each π_i is the canonical projection from $(\mathbf{F} \otimes \mathbf{F})_i$ to $S_2(\mathbf{F})_i$.

Proof of Theorem 4.2.1. To begin with, let us consider the following diagram

$$\begin{array}{ccc} R(X) \otimes R(X) & \xleftarrow{d_0 \otimes d_0} & F_0 \otimes F_0 \\ \downarrow \tilde{\mu} & \swarrow \alpha_0 & \\ 0 & \xleftarrow{d_0} & R(X) \xleftarrow{d_0} F_0 \end{array}$$

where $\alpha_0 := \tilde{\mu} \circ (d_0 \otimes d_0): F_0 \otimes F_0 \rightarrow R(X)$. For $e_i \otimes e_j - e_j \otimes e_i \in M \subseteq F_0 \otimes F_0$ we have

$$\alpha_0(e_i \otimes e_j - e_j \otimes e_i) = m_i m_j - m_j m_i = 0$$

since $R(X)$ is a commutative ring. Thus, α_0 factors through $S_2(\mathbf{F})_0 \cong S_2(F_0)$ which gives the following commutative diagram

$$\begin{array}{ccc} R(X) \otimes R(X) & \xleftarrow{d_0 \otimes d_0} & F_0 \otimes F_0 \\ \downarrow \tilde{\mu} & \swarrow \alpha_0 & \searrow \pi_0 \\ 0 & \xleftarrow{d_0} & R(X) \xleftarrow{d_0} F_0 \\ & & \nearrow \gamma_0 \\ & & S_2(\mathbf{F})_0 \end{array}$$

Since $S_2(\mathbf{F})_0$ is free, there is a map $\beta_0: S_2(\mathbf{F})_0 \rightarrow F_0$ such that $\gamma_0 = d_0 \circ \beta_0$. From this, setting $\mu_0 = \beta_0 \circ \pi_0$, we obtain a new commutative diagram

$$\begin{array}{ccc}
R(X) \otimes R(X) & \xleftarrow{d_0 \otimes d_0} & F_0 \otimes F_0 \\
\downarrow \tilde{\mu} & \searrow \alpha_0 & \downarrow \mu_0 \\
0 & \xleftarrow{d_0} & R(X) \xleftarrow{\gamma_0} S_2(\mathbf{F})_0 \xrightarrow{\beta_0} F_0
\end{array}$$

π_0 (red arrow from $F_0 \otimes F_0$ to $S_2(\mathbf{F})_0$)
 β_0 (red arrow from $S_2(\mathbf{F})_0$ to F_0)

Note that for any $1 \leq i \leq 7$ we have

$$\gamma_0(\pi_0(e_0 \otimes e_i)) = \gamma_0(\pi_0(e_i \otimes e_0)) = \alpha_0(e_i \otimes e_0) = m_i = d_0(e_i).$$

Hence, we can define β_0 in such a way that $\beta_0(\pi_0(e_0 \otimes e_i)) = \beta_0(\pi_0(e_i \otimes e_0)) = e_i$, and thus,

$$\mu_0(e_0 \otimes e_i) = \mu_0(e_i \otimes e_0) = e_i$$

for all $i = 0, \dots, 7$. By linearity this yields

$$\mu_0(f \otimes e_0) = \mu_0(e_0 \otimes f) = f$$

for any $f \in F_0$. Next we show that

$$0 \leftarrow R(X) \xleftarrow{\gamma_0} S_2(\mathbf{F})_0 \xleftarrow{\bar{\delta}_1} S_2(\mathbf{F})_1 \xleftarrow{\bar{\delta}_2} \dots$$

is a complex. Using the fact that $(S_2(\mathbf{F}), \bar{\delta})$ is a complex of free S -modules we have only to show that $\text{im}(\bar{\delta}_1) \subseteq \ker(\gamma_0)$. Let $h \in \text{im}(\bar{\delta}_1)$. Using the commutative diagram

$$\begin{array}{ccc}
(\mathbf{F} \otimes \mathbf{F})_0 & \xleftarrow{\delta_1} & (\mathbf{F} \otimes \mathbf{F})_1 \\
\pi_0 \downarrow & & \pi_1 \downarrow \\
S_2(\mathbf{F})_0 & \xleftarrow{\bar{\delta}_1} & S_2(\mathbf{F})_1 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

we find an element $g \in (\mathbf{F} \otimes \mathbf{F})_1$ such that $h = \pi_0(\delta_1(g))$. Then

$$\gamma_0(h) = \gamma_0(\pi_0(\delta_1(g))) = \alpha_0(\delta_1(g)) = \tilde{\mu}((d_0 \otimes d_0)(\delta_1(g))) = 0$$

since $(d_0 \otimes d_0) \circ \delta_1 = 0$. Thus, $\text{im}(\bar{\delta}_1) \subseteq \ker(\gamma_0)$.

Now, combining this with the previous diagram, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longleftarrow & R(X) & \xleftarrow{\gamma_0} & S_2(\mathbf{F})_0 & \xleftarrow{\bar{\delta}_1} & S_2(\mathbf{F})_1 & \xleftarrow{\bar{\delta}_2} & S_2(\mathbf{F})_2 & \longleftarrow & \dots \\
& & \downarrow \text{id} & & \downarrow \beta_0 & & & & & & \\
0 & \longleftarrow & R(X) & \xleftarrow{d_0} & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 & \longleftarrow & \dots
\end{array}$$

where the first row is a complex and the second row is exact. Using the exactness of the second row, we can extend β_0 to a map of chain complexes $\beta: S_2(\mathbf{F}) \rightarrow \mathbf{F}$:

$$\begin{array}{ccccccc}
0 & \longleftarrow & R(X) & \xleftarrow{\gamma_0} & S_2(\mathbf{F})_0 & \xleftarrow{\bar{\delta}_1} & S_2(\mathbf{F})_1 & \xleftarrow{\bar{\delta}_2} & S_2(\mathbf{F})_2 & \longleftarrow & \dots \\
& & \downarrow \text{id} & & \downarrow \beta_0 & & \downarrow \beta_1 & & \downarrow \beta_2 & & \\
0 & \longleftarrow & R(X) & \xleftarrow{d_0} & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 & \longleftarrow & \dots
\end{array}$$

As above, we can choose the maps β_i successively so that

$$\beta_i(\pi_i(e_0 \otimes g)) = \beta_i(\pi_i(g \otimes e_0)) = g$$

for any $g \in F_i$: If $e_0 \otimes g \in F_0 \otimes F_1$, then

$$\beta_0(\bar{\delta}_1(\pi_1(g \otimes e_0))) = \beta_0(\bar{\delta}_1(\pi_1(e_0 \otimes g))) = \beta_0(\pi_0(\delta_1(e_0 \otimes g))) = \beta_0(\pi_0(e_0 \otimes d_1(g))) = d_1(g).$$

Hence, we can choose β_1 so that $\beta_1(\pi_1(e_0 \otimes g)) = \beta_1(\pi_1(g \otimes e_0)) = g$ and we proceed similarly for $i \geq 2$.

Setting $\mu_i = \beta_i \circ \pi_i$ for $i \geq 1$, this yields a chain map $\mu: \mathbf{F} \otimes \mathbf{F} \rightarrow \mathbf{F}$ which factors through $\beta: S_2(\mathbf{F}) \rightarrow \mathbf{F}$:

$$\begin{array}{ccccccc}
& & (\mathbf{F} \otimes \mathbf{F})_0 & \xleftarrow{\delta_1} & (\mathbf{F} \otimes \mathbf{F})_1 & \xleftarrow{\dots} & \dots \\
& \swarrow \pi_0 & & & \swarrow \pi_1 & & \\
S_2(\mathbf{F})_0 & \xleftarrow{\bar{\delta}_1} & S_2(\mathbf{F})_1 & \xleftarrow{\dots} & & & \\
\downarrow \beta_0 & \swarrow \mu_0 & \downarrow \beta_1 & \swarrow \mu_1 & & & \\
F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{\dots} & & &
\end{array}$$

Let $f, g \in \mathbf{F}$. In the following, we will write fg for $\beta(\pi(f \otimes g)) = \mu(f \otimes g)$. For $f = e_0$ we get

$$e_0g = \mu(e_0 \otimes g) = \beta(\pi(e_0 \otimes g)) = g = \beta(\pi(g \otimes e_0)) = \mu(g \otimes e_0) = ge_0.$$

Hence, e_0 acts as a unit element for the multiplication. Finally, by the definition of μ_0 , the chain map μ is a lift of the map $\tilde{\mu}: R(X) \otimes R(X) \rightarrow R(X)$:

$$\begin{array}{ccccccc}
& & R(X) \otimes R(X) & \xleftarrow{d_0 \otimes d_0} & (\mathbf{F} \otimes \mathbf{F})_0 & \xleftarrow{\delta_1} & (\mathbf{F} \otimes \mathbf{F})_1 & \xleftarrow{\quad} & \dots \\
& & \nearrow \tilde{\mu} & & \nearrow & & \nearrow & & \\
& & & & S_2(\mathbf{F})_0 & \xleftarrow{\bar{\delta}_1} & S_2(\mathbf{F})_1 & \xleftarrow{\quad} & \dots \\
& & \nearrow \gamma_0 & & \downarrow \beta_0 & & \downarrow \beta_1 & & \\
0 & \leftarrow & R(X) & \xleftarrow{d_0} & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{\quad} & \dots
\end{array}$$

This completes the proof. \square

4.3 The Induced Chain Map Between the Resolution and its Dual

By the proof of Proposition 3.3.3 we know that

$$0 \leftarrow \omega_{R(X)}(-1) \leftarrow F_3^\vee \xleftarrow{d_3^\vee} F_2^\vee \xleftarrow{d_2^\vee} F_1^\vee \xleftarrow{d_1^\vee} F_0^\vee \leftarrow 0$$

is a minimal free resolution of $\omega_{R(X)}(-1)$ which is isomorphic to $R(X)$. In this section, we first construct a chain map between the given resolution and its dual which is induced by the multiplication map μ defined above. Afterwards we show that this map is an isomorphism of chain complexes. Using this isomorphism, we can finally prove Theorem 4.0.2.

Let $\sigma: F_3 \rightarrow S(-17)$ be the canonical projection map. For any $1 \leq i \leq 3$ we define a map $h_i: F_i \otimes F_{3-i} \rightarrow S(-17)$ by sending $a \otimes b$ to $\sigma(ab)$. Moreover, for each i , this induces a map

$$\begin{aligned}
s_i: F_i &\rightarrow \text{Hom}_S(F_{3-i}, S(-17)) = F_{3-i}^\vee \\
a &\mapsto l_a: F_{3-i} \rightarrow S(-17) \\
b &\mapsto h_i(a \otimes b) = \sigma(ab).
\end{aligned}$$

Proposition 4.3.1. *Define the maps*

$$t_i = \begin{cases} s_i & \text{if } i = 0, 1, \\ -s_i & \text{if } i = 2, 3. \end{cases}$$

Then

$$\begin{array}{ccccccc}
F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 & \xleftarrow{d_3} & F_3 & \xleftarrow{\quad} & 0 \\
\downarrow t_0 & & \downarrow t_1 & & \downarrow t_2 & & \downarrow t_3 & & \\
F_3^\vee & \xleftarrow{d_3^\vee} & F_2^\vee & \xleftarrow{d_2^\vee} & F_1^\vee & \xleftarrow{d_1^\vee} & F_0^\vee & \xleftarrow{\quad} & 0
\end{array}$$

is a commutative diagram, or equivalently, $t: \mathbf{F} \rightarrow \mathbf{F}^\vee$ is a chain map.

Proof. We have to show that for any $0 \leq i \leq 2$ we have

$$t_i \circ d_{i+1} = d_{3-i}^\vee \circ t_{i+1}.$$

So let $f \in F_{i+1}$ and $g \in F_{3-i}$. Then

$$\begin{aligned} d_{3-i}^\vee(t_{i+1}(f))(g) &= t_{i+1}(f)(d_{3-i}(g)) \\ &= \theta_i \sigma(f d_{3-i}(g)) \\ &= \theta_i \sigma(d_{3-i}(g) f), \end{aligned}$$

with

$$\theta_i = \begin{cases} 1 & \text{if } i = 0, \\ -1 & \text{if } i = 1, 2, \end{cases}$$

and where the last equality holds since $f d_{3-i}(g) = (-1)^{(i+1)(2-i)} d_{3-i}(g) f = d_{3-i}(g) f$. On the other hand, since $gf = 0$,

$$0 = d_{3-i}(g) f + (-1)^{3-i} g d_{i+1}(f).$$

Hence

$$\begin{aligned} d_{3-i}^\vee(t_{i+1}(f))(g) &= \theta_i \sigma(d_{3-i}(g) f) = (-1)^i \theta_i \sigma(g d_{i+1}(f)) \\ &= (-1)^i \theta_i \sigma(d_{i+1}(f) g) \\ &= (-1)^i \theta_i s_i(d_{i+1}(f))(g). \end{aligned}$$

Applying this to the three possible values for i , we see that the last term is always equal to $t_i(d_{i+1}(f))(g)$ which proves the claim. \square

Our next task is to show that the maps s_i are dual to each other with respect to some chosen bases. Let us first fix some notation.

Notation 4.3.2. If $\epsilon_0, \dots, \epsilon_n$ is any basis of F_i , we denote by ϵ_i^\vee the map sending ϵ_j to 0 if $j \neq i$ and to 1 otherwise. Clearly, $\epsilon_0^\vee, \dots, \epsilon_n^\vee$ is then a basis of F_i^\vee . We denote by g_0, \dots, g_{25} , v_0, \dots, v_{25} and h_0, \dots, h_7 the canonical bases of F_1 , F_2 and F_3 , respectively.

Then, considering the maps with respect to these bases and its duals, yields the following:

Proposition 4.3.3. For $n = 0, \dots, 3$ we have

$$s_n^\vee = s_{3-n}.$$

Or, in terms of the maps t ,

$$t_n^\vee = -t_{3-n}.$$

Proof. It is enough to prove the claim for $i = 0, 1$. Let us start with the case $i = 0$. Using the bases given above, we get a representation

$$s_0(e_k) = \sum_{i=0}^7 \sigma(e_k h_i) h_i^\vee$$

for any basis element e_k of F_0 . Indeed, we have

$$\left(\sum_{i=0}^7 \sigma(e_k h_i) h_i^\vee \right) (h_j) = \sum_{i=0}^7 \sigma(e_k h_i) h_i^\vee(h_j) = \sigma(e_k h_j) = (s_0(e_k))(h_j).$$

Thus, with respect to the bases e_0, \dots, e_7 and $h_0^\vee, \dots, h_7^\vee$ we can represent s_0 by the matrix

$$\begin{pmatrix} \sigma(e_0 h_0) & \cdots & \sigma(e_7 h_0) \\ \vdots & \ddots & \vdots \\ \sigma(e_0 h_7) & \cdots & \sigma(e_7 h_7) \end{pmatrix}$$

Similarly, for s_3 we have

$$s_3(h_k) = \sum_{i=0}^7 \sigma(h_k e_i) e_i^\vee$$

for any basis element $h_k \in F_3$. Hence, we obtain a representation of s_3 with respect to the bases h_0, \dots, h_7 and $e_0^\vee, \dots, e_7^\vee$ by the matrix

$$\begin{pmatrix} \sigma(h_0 e_0) & \cdots & \sigma(h_7 e_0) \\ \vdots & \ddots & \vdots \\ \sigma(h_0 e_7) & \cdots & \sigma(h_7 e_7) \end{pmatrix}.$$

But

$$e_i h_j = h_j e_i$$

for any $i, j = 0, \dots, 7$ since $\deg(e_i) = 0$ for all i . This implies that the first matrix is the transposed of the second. Hence, identifying $h_i^{\vee\vee}$ with h_i via the isomorphism $F_i \xrightarrow{\cong} F_i^{\vee\vee}$, we get

$$s_0^\vee = s_3.$$

Analogously, for s_1 and s_2 we get representations

$$\begin{array}{c|ccc} s_1 & g_0 & \cdots & g_{25} \\ \hline v_0^\vee & \sigma(g_0 v_0) & \cdots & \sigma(g_{25} v_0) \\ \vdots & \vdots & \ddots & \vdots \\ v_{25}^\vee & \sigma(g_0 v_{25}) & \cdots & \sigma(g_{25} v_{25}) \end{array} \quad \text{and} \quad \begin{array}{c|ccc} s_2 & v_0 & \cdots & v_{25} \\ \hline g_0^\vee & \sigma(v_0 g_0) & \cdots & \sigma(v_{25} g_0) \\ \vdots & \vdots & \ddots & \vdots \\ g_{25}^\vee & \sigma(v_0 g_{25}) & \cdots & \sigma(v_{25} g_{25}) \end{array},$$

respectively. Again, $g_i v_j = v_j g_i$ for any $i, j = 0, \dots, 25$ since $\deg(v_j) = 2$ for all j . Hence,

$$s_1^\vee = s_2. \quad \square$$

4.4 Proof of Theorem 4.0.2

Let us consider the middle square of the commutative diagram from Proposition 4.3.1:

$$\begin{array}{ccc} F_1 & \xleftarrow{d_2} & F_2 \\ t_1 = -t_2^\vee \downarrow & & \downarrow t_2 \\ F_2^\vee & \xleftarrow{d_2^\vee} & F_1^\vee \end{array} \quad -t_2^\vee \circ d_2 = d_2^\vee \circ t_2$$

We claim that the following holds:

Proposition 4.4.1. *For each $i = 0, \dots, 3$, the map t_i is an isomorphism.*

Before proving this proposition, let us first see how the main theorem follows from this statement.

Proof of Theorem 4.0.2. By Proposition 4.4.1, t_2 (or equivalently t_1) is an isomorphism. Then $\tilde{d}_2 := d_2 \circ t_2^{-1}$ is a homomorphism from F_1^\vee to F_1 . But since

$$(t_2^{-1})^\vee = (t_2^\vee)^{-1}$$

we have

$$\begin{aligned} \tilde{d}_2^\vee &= (d_2 \circ t_2^{-1})^\vee \\ &= (t_2^{-1})^\vee \circ d_2^\vee \\ &= -d_2 \circ t_2^{-1} \\ &= -\tilde{d}_2. \end{aligned}$$

Consequently,

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{\tilde{d}_2} F_1^\vee \xleftarrow{d_1^\vee} F_0^\vee \leftarrow 0$$

is a minimal free resolution of $R(X)$ with alternating middle map as desired. \square

It remains to prove that every t_i is an isomorphism. Note that in the setting of [BE77] $F_0 = S$. This implies directly that t_0 is the identity map (lifting the identity on the corresponding rings) and thus, that each t_k is an isomorphism. To show this in our setting some additional work is needed. Again we consider the commutative diagram from Proposition 4.3.1. Let $\tilde{t}: R(X) \rightarrow \omega_{R(X)}(-1)$ be the induced S -linear map:

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & R(X) & \xleftarrow{d_0} & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 & \xleftarrow{d_3} & F_3 & \longleftarrow & 0 \\ & & \tilde{t} \downarrow & & t_0 \downarrow & & t_1 \downarrow & & t_2 \downarrow & & t_3 \downarrow & & \\ 0 & \longleftarrow & \omega_{R(X)}(-1) & \xleftarrow{u_0} & F_3^\vee & \xleftarrow{d_3^\vee} & F_2^\vee & \xleftarrow{d_2^\vee} & F_1^\vee & \xleftarrow{d_1^\vee} & F_0^\vee & \longleftarrow & 0 \end{array}$$

Since both complexes are minimal free resolutions, it is enough to show the following:

Proposition 4.4.2. $\tilde{t}: R(X) \rightarrow \omega_{R(X)}(-1)$ is an isomorphism.

We prove this statement in several steps. First of all we know that $R(X)$ and $\omega_{R(X)}(-1)$ are isomorphic $R(X)$ -modules. Let $\alpha: \omega_{R(X)}(-1) \rightarrow R(X)$ be such an isomorphism. Then for $\epsilon := \alpha^{-1}(1_{R(X)})$ we have

$$\omega_{R(X)}(-1) = \{a\epsilon \mid a \in R(X)\}.$$

Furthermore, since u_0 is surjective and homogeneous of degree 0 we know that $u_0(h_7^\vee) = \lambda\epsilon$ for some $\lambda \in \mathbb{k}^*$. Using this and the following lemma we can show that \tilde{t} is not the zero-map.

Lemma 4.4.3. $s_0(e_0) = t_0(e_0) = h_7^\vee$.

Proof. Recall that s_0 is represented by the matrix

$$\begin{array}{c|ccc} s_0 & e_0 & \cdots & e_7 \\ \hline h_0^\vee & \sigma(e_0 h_0) & \cdots & \sigma(e_7 h_0) \\ \vdots & \vdots & \ddots & \vdots \\ h_7^\vee & \sigma(e_0 h_7) & \cdots & \sigma(e_7 h_7) \end{array}$$

Using the fact that e_0 is an identity element for the multiplication the first column is nothing but $(\sigma(h_0), \dots, \sigma(h_7))^{tr}$. By the choice of the basis h_0, \dots, h_7 , we have $\sigma(h_i) = 1$ for $i = 7$ and 0 otherwise. \square

Lemma 4.4.4. \tilde{t} is a non-zero S -linear homomorphism.

Proof. Clear since $\tilde{t}(1_{R(X)}) = u_0(t_0(e_0)) = u_0(h_7^\vee) = \lambda\epsilon \neq 0$. \square

Recall from Theorem 4.1.2 that every homogeneous S_Y -linear homomorphism from $R(X)$ to $R(X)$ is the multiplication by a homogeneous element of $R(X)$. We want to apply this result to the composition $\alpha \circ \tilde{t}$.

Proof of Proposition 4.4.2. First note that, since α is also S -linear, we have

$$I_Y = \text{ann}_S(R(X)) = \text{ann}_S(\omega_{R(X)}(-1)).$$

Hence \tilde{t} and α are both S_Y -linear homomorphisms. But then $\alpha \circ \tilde{t}$ is a non-zero S_Y -linear homomorphism from $R(X)$ to $R(X)$, and hence the multiplication by a non-zero homogeneous element of $R(X)$. As \tilde{t} and α are both homogeneous (of degree 0), this implies that $\alpha \circ \tilde{t}$ is the multiplication by some $\theta \in \mathbb{k}^*$, and hence an isomorphism. Consequently, \tilde{t} is an isomorphism. \square

5 From Free Resolutions to Numerical Godeaux Surfaces

Let $S = \mathbb{k}[x_0, x_1, y_0, \dots, y_3]$ be the graded polynomial ring as considered in the previous chapters. In this chapter we will consider S -modules R satisfying the following condition:

(*) R is a finitely generated graded S -module with a minimal free resolution of type

$$\begin{array}{cccccccc}
 & & S & & S(-6)^6 & & S(-9)^8 & & S(-12)^3 & & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 0 \leftarrow R \leftarrow & S(-4)^4 & \leftarrow & S(-7)^{12} & \leftarrow & S(-10)^{12} & \leftarrow & S(-13)^4 & \leftarrow & 0 & \\
 & \oplus & & \oplus & & \oplus & & \oplus & & & \\
 & S(-5)^3 & & S(-8)^8 & & S(-11)^6 & & S(-17) & & &
 \end{array}$$

Notation 5.0.1. Let R be an S -module satisfying (*) with minimal free resolution

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} F_3 \leftarrow 0.$$

By e we denote the element of R corresponding to the generator 1 of S in F_0 . Let d'_1 be the submatrix of d_1 obtained by erasing the first row. By $I' \subseteq S$ we denote the zeroth Fitting ideal of $M = \text{coker } d'_1$, that is the ideal generated by the maximal minors of d'_1 . By the properties of Fitting ideals, the ideal I' is independent of the choice of a presentation matrix of M , and hence independent of the choice of a minimal free resolution of R as an S -module. Furthermore, we set $I_Y = \text{ann}_S R$ and $S_Y = S/I_Y$. Finally, we assume that $\mathbb{k} = \mathbb{C}$ throughout this chapter.

Our aim is to prove a partial converse of the structure theorem for numerical Godeaux surfaces presented in the last chapter. More precisely, we will show:

Theorem 5.0.2. *Let R be an S -module satisfying (*) which has a minimal free resolution of the form*

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^\vee} F_0^\vee \leftarrow 0,$$

where $F_i^\vee = \text{Hom}_S(F_i, S(-17))$ and d_2 is skew-symmetric. If

$$(\diamond) \quad \text{depth}(I', S) \geq 5,$$

then R is a Gorenstein ring. Under this condition, let $Y = \text{Proj}(S_Y)$ and $X = \text{Proj}(R)$. Suppose further that

- (i) $x_0, x_1 \in S$ is a regular sequence for R ,
- (ii) $\text{Proj}(S_Y/(y_0, \dots, y_3))$ is empty or 0-dimensional,
- (iii) I_Y is prime,
- (iv) X has only Du Val singularities.

Then X is the canonical model of a numerical Godeaux surface. Furthermore, X is the normalization of Y .

To prove this theorem, we proceed in several steps. First we show that the condition (\diamond) implies that R carries a unique structure as an S_Y -algebra with identity element e . Then, we use the additional properties (i) - (iv) to prove that R is the canonical ring of a surface of general type. As a last step, we use the Hilbert series of R to deduce that this surface is a numerical Godeaux surface.

5.1 Preliminaries

Let B be a faithful A -module with distinguished element $e \in B$. In [EU97], Eisenbud and Ulrich studied conditions under which B supports the structure of an A -algebra with identity element e . In case of a finite birational A -module, there is at most one such structure:

Definition 5.1.1 ([EU97]). Let A be a commutative ring, and let B be a faithful A -module with a distinguished element $e \in B$. Then B is called a *finite birational A -module* if there is an element $d \in A$ which is a non-zerodivisor on B such that

$$dB \subseteq Ae \subseteq B.$$

If B is a finite birational A -module and carries the structure of an A -algebra with identity element e , then the algebra structure of B is uniquely determined by the fact that B is a subalgebra of $B[d^{-1}] = A[d^{-1}]$.

Example 5.1.2. Suppose that $A \subseteq B$ are integral domains with $Q(A) = Q(B)$ such that B is finitely generated as an A -module. Then B is a birational A -module by Lemma 4.1.3.

Lemma 5.1.3. Let $A \hookrightarrow B$ be an injective homomorphism of rings, where B is a finite birational A -module (with identity element e). Furthermore, let N be a B -module. Then every A -module isomorphism $\varphi: B \rightarrow N$ is also B -linear.

Proof. Let $d \in A$ and $e \in B$ be as in Definition 5.1.1, that is $dB \subseteq Ae \subseteq B$. Note that since φ is an A -module isomorphism, the element d is also a non-zerodivisor on N . Now let $g, r \in B$ be arbitrary. By assumption, there is an element $a \in A$ such that $dg = a$. Then

$$d\varphi(gr) = \varphi(dgr) = \varphi(ar) = a\varphi(r) = dg\varphi(r),$$

and therefore

$$\varphi(gr) = g\varphi(r)$$

since d is a non-zerodivisor on N . □

5.2 The Induced Algebra Structure

We will show that the condition (\diamond) implies that R carries the structure of a (commutative) S -algebra. This extends a result of Böhning ([Böh05], Theorem 1.1) on Gorenstein algebras from codimension 2 to codimension 3:

Theorem 5.2.1. Let R be an S -module satisfying $(*)$ with minimal free resolution

$$0 \leftarrow R \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} F_3 \leftarrow 0,$$

and let e , I' and S_Y be as in Notation 5.0.1. Furthermore, assume that $\text{depth}(\text{ann}_S R, S) = \text{projdim}_S(R) = 3$. If

$$(\diamond) \text{depth}(I', S) \geq 5,$$

then R carries a unique structure of a (commutative) S_Y -algebra with identity element e .

Proof. The first step is to show that R is a Cohen-Macaulay S -module. We have

$$\begin{aligned} \dim_S R &= \dim S / \text{ann}_S R = \dim S - \text{codim ann}_S R \\ &= \dim S - \text{depth}(\text{ann}_S R, S) = 3, \end{aligned}$$

where the second and third equality follow from the fact that S is Cohen-Macaulay and the last one holds by assumption. Furthermore, using the graded version of the Auslander-Buchsbaum formula, we see that

$$\text{depth } R = \text{depth } S - \text{projdim}_S(R) = \dim S - 3 = 3.$$

Thus, $\dim_S R = \text{depth } R$, and hence R is Cohen-Macaulay.

Next we show that R is a finite birational S_Y -module with distinguished element e . The matrix d'_1 is a presentation matrix (over S) of the module $R/S_Y e$. This implies that $I' \subseteq \text{ann}_S(R/S_Y e)$, and hence that $(I' S_Y)R \subseteq S_Y e$. Now since R is a Cohen-Macaulay S -module and $\text{depth}(I', S) \geq 5$, we have $\text{depth}(I', R) = \dim R - \dim R/I'R \geq 2$. Hence, there exists an element $d \in I' S_Y$ which is a non-zerodivisor on R . Consequently,

$$dR \subseteq S_Y e \subseteq R$$

as required. In particular, $\text{ann}_S e = \text{ann}_S R$.

In the following, $X = \text{Spec}(S)$ is the affine cone over $\text{Proj}(S)$ and

- $Y = V(\text{ann}_S R) \subseteq X$ is the closed subscheme of X of codimension 3,
- $Z \subseteq Y$ is the closed subscheme of Y defined by $(I' + \text{ann}_S R)$ with complement $U = Y \setminus Z$, and
- \mathcal{R} is the coherent sheaf associated to R as an S_Y -module.

From the above, we know that the map $S_Y \rightarrow R$ sending s to se is an isomorphism onto its image $S_Y e$. Then, as d'_1 is a presentation matrix of $R/S_Y e$, we have $\mathcal{R}|_U \cong \mathcal{O}_Y|_U$.

Now consider the standard exact sequence of local and global cohomology ([Gro67], Proposition 2.2)

$$0 \rightarrow H_Z^0(Y, \mathcal{R}) \rightarrow H^0(Y, \mathcal{R}) \rightarrow H^0(U, \mathcal{R}) \rightarrow H_Z^1(Y, \mathcal{R}) \rightarrow 0.$$

Suppose that $H_Z^0(Y, \mathcal{R}) = H_Z^1(Y, \mathcal{R}) = 0$. Then $H^0(Y, \mathcal{R}) \cong H^0(U, \mathcal{R})$ and thus $R \cong H^0(U, \mathcal{O}_Y|_U)$ as S_Y -modules. Using this isomorphism, R inherits the structure of a commutative S_Y -algebra with identity element e . Since the homomorphisms in the exact sequence above preserve the grading, R is a graded S_Y -algebra, and hence a graded \mathbb{k} -algebra. Furthermore, the S_Y -algebra structure is uniquely determined since R is a finite birational S_Y -module.

So it remains to show that $H_Z^0(Y, \mathcal{R})$ and $H_Z^1(Y, \mathcal{R})$ vanish. By [Gro67], Theorem 3.8, it is enough to show that $\text{depth}(Z, \mathcal{R}) \geq 2$. But, since Y is affine,

$$\text{depth}(Z, \mathcal{R}) = \text{depth}(I' S_Y, R) \geq 2$$

which proves the claim. □

Definition 5.2.2 ([Gra96], Definition 3.1). Let A be a positively graded polynomial ring, and let B be an A -algebra. Let $c = \dim A - \dim_A B$ denote the codimension of B . Then B is called a *Gorenstein algebra* of codimension c (and twist $s \in \mathbb{Z}$) if

$$B \cong \text{Ext}_A^c(B, A(s))$$

as B -modules.

Now, if the S -module R in Theorem 5.2.1 admits a minimal free resolution which is self-dual up to a sign, then R is a Gorenstein S -algebra:

Theorem 5.2.3. *Let R be an S -module satisfying $(*)$ which has a minimal free resolution of the form*

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^\vee} F_0^\vee \leftarrow 0,$$

where $F_i^\vee = \text{Hom}_S(F_i, S(-17))$ and d_2 is skew-symmetric. If $\text{depth}(I', S) \geq 5$, then R is a Gorenstein S -algebra of codimension 3.

Proof. To use Theorem 5.2.1 we first have to show that $\text{depth}(\text{ann}_S R, S) = 3$. By [BE73], as the above complex is exact, we know that $\text{depth}(I(d_1^\vee), S) \geq 3$, where $I(d_1^\vee)$ is the ideal generated by the minors of size $\text{rank}(d_1^\vee)$ of a matrix representing d_1^\vee . Hence,

$$\text{depth}(I(d_1), S) = \text{depth}(I(d_1^\vee), S) \geq 3.$$

Furthermore, $\text{depth}(I(d_1), S) = \text{depth}(\text{ann}_S R, S)$ as the two ideals have the same radical ideal. The result follows then from the fact that

$$\text{depth}(\text{ann}_S R, S) \leq \text{projdim}_S(R) = 3.$$

So, all assumptions of Theorem 5.2.1 are satisfied and hence, R is a finite birational S_Y -module and an S -algebra of codimension 3 (with identity element e). Furthermore, we can identify S_Y with its image in R via the injective homomorphism $S_Y \rightarrow R$ (sending an element s to se).

Applying the functor $\text{Hom}_S(-, S(-17))$ to the resolution above, we get a complex

$$0 \rightarrow F_0^\vee \xrightarrow{d_1^\vee} F_1^\vee \xrightarrow{d_2^\vee = -d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow \text{Ext}_S^3(R, S(-17)) \rightarrow 0$$

which is exact since $\text{depth}(\text{ann}_S R, S) = 3$ implies $\text{Ext}_S^i(R, S) = 0$ for $i < 3$. Comparing these two complexes, we can construct a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_0^\vee & \xrightarrow{d_1^\vee} & F_1^\vee & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow -\text{id}_{F_0^\vee} & & \downarrow -\text{id}_{F_1^\vee} & & \downarrow \text{id}_{F_1} & & \downarrow \text{id}_{F_0} & & \downarrow u & & \\ 0 & \longrightarrow & F_0^\vee & \xrightarrow{d_1^\vee} & F_1^\vee & \xrightarrow{-d_2} & F_1 & \xrightarrow{d_1} & F_0 & \longrightarrow & \text{Ext}_S^3(R, S(-17)) & \longrightarrow & 0 \end{array}$$

The graded isomorphism of complexes in the diagram above induces an S -linear isomorphism $u : R \rightarrow \text{Ext}_S^3(R, S(-17))$. Now, since u is also S_Y -linear, Lemma 5.1.3 implies that u is an R -linear homomorphism. Hence,

$$R \cong \text{Ext}_S^3(R, S(-17)), \tag{5.1}$$

showing that R is a Gorenstein S -algebra. \square

Remark 5.2.4. We call the condition (\diamond) the *ring condition*. Now let R_1 and R_2 be two S -modules as in Theorem 5.2.3 satisfying the ring condition. Then, if R_1 and R_2 are isomorphic as S -modules (and hence as S_Y -modules), the S_Y -algebras R_1 and R_2 are isomorphic as rings by Theorem 5.2.1.

5.3 Proof of Theorem 5.0.2

Before proving the main theorem of this chapter we need a further preliminary result concerning the dualizing sheaf for a scheme of type $\text{Proj}(R)$, where R is a Cohen-Macaulay graded ring. But let us first state a result on dualizing sheaves and finite morphisms.

Let $f: X \rightarrow Y$ be a finite morphism of Noetherian schemes. Recall that f_* gives an equivalence from the category of quasi-coherent \mathcal{O}_X -modules to the category of quasi-coherent $f_*\mathcal{O}_X$ -modules. Let us denote the inverse functor by $\tilde{}$. Then given any quasi-coherent \mathcal{O}_Y -module \mathcal{G} , the sheaf $f^!\mathcal{G}$ is the quasi-coherent \mathcal{O}_X -module $\mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})\tilde{}$.

Lemma 5.3.1 (See [Har77], III Exercise 7.2). *Let $f: X \rightarrow Y$ be a finite morphism of projective schemes of the same dimension over \mathbb{k} , and let ω_Y be a dualizing sheaf for Y . Then $f^!\omega_Y$ is a dualizing sheaf for X .*

With the help of this lemma we show in the next statement how the canonical module of a Cohen-Macaulay ring and the dualizing sheaf for the corresponding projective scheme are related.

Proposition 5.3.2. *Let R be a positively graded \mathbb{k} -algebra of dimension $n + 1 \geq 2$, and let $X = \text{Proj}(R)$ be the corresponding projective scheme over \mathbb{k} with dualizing sheaf ω_X . Moreover, assume that R is Cohen-Macaulay and let ω_R be the canonical module of R . Then*

$$\widetilde{\omega}_R \cong \omega_X.$$

Proof. Since R is Cohen-Macaulay there exists a homogeneous system of parameters t_0, \dots, t_n in R such that R is a finitely generated free $A = \mathbb{k}[t_0, \dots, t_n]$ -module. Furthermore, by replacing the t_i with suitable powers, we may assume that all elements have the same degree. The natural ring homomorphism $A \rightarrow R$ induces then a finite morphism

$$\psi: X \rightarrow \mathbb{P}^n.$$

Then Lemma 5.3.1 implies that $\omega_X \cong \psi^!\omega_{\mathbb{P}^n}$ and thus

$$\psi_*\omega_X \cong \mathcal{H}om_{\mathbb{P}^n}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}^n}). \quad (5.2)$$

On the other hand, since both A and R are Cohen-Macaulay and R is a finite A -module, we get

$$\omega_R \cong \text{Hom}_A(R, \omega_A)$$

by Proposition 2.2.9. Considering the associated coherent sheaves and combining with the isomorphism in (5.2) yields

$$\psi_*(\widetilde{\omega}_R) \cong \widetilde{A\omega_R} \cong \mathcal{H}om_{\mathbb{P}^n}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}^n}) \cong \psi_*\omega_X,$$

where $A\omega_R$ means considering ω_R as an A -module. Now applying the functor $\tilde{}$ which is inverse to ψ_* we obtain the claimed isomorphism

$$\widetilde{\omega}_R \cong \omega_X. \quad \square$$

We have now all the ingredients for proving the main theorem of this chapter:

Proof of Theorem 5.0.2. By Theorem 5.2.3 we already know that R is a Gorenstein algebra of codimension 3. The scheme $Y = \text{Proj}(S_Y)$ is a closed subscheme of $\mathbb{P} = \mathbb{P}(2^2, 3^4)$ with support $V(I_Y)$. Since $\dim S_Y = \dim S - \text{codim}(\text{ann}_S R) = \dim S - 3 = 3$, Y is a projective surface. Using the inclusion of rings $S_Y \subseteq R$ yields a surjective morphism of surfaces

$$\varphi : X \rightarrow Y,$$

where $X = \text{Proj}(R)$. This morphism is finite since R is a finitely generated S_Y -module. Hence $\varphi_*\mathcal{O}_X$ is a coherent sheaf of \mathcal{O}_Y -algebras and

$$\text{Proj}(R) = X \cong \text{Spec}(\varphi_*\mathcal{O}_X).$$

Furthermore, by the proof of Theorem 5.2.1, there exists a non-zerodivisor $d \in S_Y$ such that

$$S_Y[d^{-1}] = R[d^{-1}]. \quad (5.3)$$

Hence, φ is birational, and since X has at most Du Val singularities $\varphi : X \rightarrow Y$ is the normalization of Y . Furthermore, (5.3) and (iii) imply that R is an integral domain. Hence X is an integral normal projective scheme.

Since R is a finite S_Y -module and $S_Y \subseteq R$, we know that $\dim R = 3$. Hence R is a Cohen-Macaulay ring. Applying Proposition 2.2.9 to S and R , we obtain

$$\omega_R \cong \text{Ext}_S^3(R, \omega_S) = \text{Ext}_S^3(R, S(-16))$$

as R -modules. Combining this with the isomorphism from (5.1) yields

$$\omega_R \cong \text{Ext}_S^3(R, S(-16)) \cong R(1).$$

Thus, R is a Gorenstein ring. Sheafifying gives then an isomorphism of \mathcal{O}_X -modules

$$\widetilde{\omega}_R \cong \mathcal{O}_X(1).$$

By assumption, all singularities of X are rational double points, and hence Gorenstein of index 1. In particular, X is Gorenstein and admits an invertible dualizing sheaf ω_X . Furthermore, the Weil divisor K_X on the normal scheme X (as defined in Section 2.3) is Cartier and $\omega_X \cong \mathcal{O}_X(K_X)$. On the other hand, $\omega_X \cong \widetilde{\omega}_R$ by Proposition 5.3.2. Hence

$$\omega_X \cong \mathcal{O}_X(1), \quad (5.4)$$

and $\mathcal{O}_X(1)$ is invertible. Next we want to show that

$$\mathcal{O}_X(1)^{\otimes n} \cong \mathcal{O}_X(n) \quad (5.5)$$

for any $n \geq 0$. Note that, as $\mathcal{O}_X(1)^{\otimes n}$ is an invertible sheaf and $\mathcal{O}_X(n)$ is a reflexive sheaf on the normal integral scheme X , it is enough to show that the two sheaves coincide outside a closed subset of codimension 2. By assumption we know that $Y_b := Y \setminus (V(x_0, x_1) \cup V(y_0, \dots, y_3))$ is finite. Then $V = \varphi^{-1}(Y_b)$ is a codimension 2 closed subset of X . Let $U = X \setminus V$. Then, for any $n \geq 0$

$$\mathcal{O}_X(1)^{\otimes n}|_U \cong \mathcal{O}_X(n)|_U.$$

Indeed, let $p \in U$ be a prime ideal. Then by the definition of the set U , there exist integers $i \in \{0, 1\}$ and $j \in \{0, \dots, 3\}$ such that $x_i, y_j \notin p$. But since $R(1)_{(p)} \cong R_{(p)}$ as $R_{(p)}$ -modules,

the morphism

$$R_{(p)} \cong R(1)_{(p)} \otimes \cdots \otimes R(1)_{(p)} \rightarrow R(n)_{(p)}$$

is an isomorphism of $R_{(p)}$ -modules with inverse given by multiplication with $\frac{x_i^n}{y_j^n}$.

Let $\mathfrak{n} = \bigoplus_{n \geq 0} R_n$. Now $\text{depth}(R) = 3$ implies $H_n^i(R) = 0$ for $i \leq 2$. Using the exact sequence relating local and global cohomology this results to

$$R \cong \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(n)).$$

Using the isomorphisms in (5.4) and (5.5) we obtain

$$R_n = H^0(X, \mathcal{O}_X(n)) \cong H^0(X, \mathcal{O}_X(1)^{\otimes n}) \cong H^0(X, \omega_X^{\otimes n}). \quad (5.6)$$

Now let $\pi: \tilde{X} \rightarrow X$ be a minimal resolution of singularities. Since all singularities are canonical, we have $\pi_* \omega_{\tilde{X}} \cong \omega_X$ and

$$H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes n}) \cong H^0(X, \omega_X^{\otimes n})$$

for all $n \geq 0$. In particular,

$$R \cong \bigoplus_{n \geq 0} H^0(X, \omega_X^{\otimes n}) \cong \bigoplus_{n \geq 0} H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes n}).$$

Consequently, R is the canonical ring of the smooth surface \tilde{X} and, since $\dim R = 3$, \tilde{X} is a surface of general type. Furthermore, $H_n^2(R) = 0$ implies that

$$0 = \bigoplus_{n \geq 0} H^1(X, \mathcal{O}_X(n)) \cong \bigoplus_{n \geq 0} H^1(X, \omega_X^{\otimes n}) \cong \bigoplus_{n \geq 0} H^1(\tilde{X}, \omega_{\tilde{X}}^{\otimes n}). \quad (5.7)$$

Hence, \tilde{X} is a minimal surface by [BHPVdV15], Theorem VII.5.3. From the free resolution of R as an S -module we deduce that

$$\dim_{\mathbb{k}} R_n = \binom{n}{2} + 1$$

for all $n \geq 2$ and $\dim_{\mathbb{k}} R_1 = 0$. Using the formula of the plurigenera of Proposition 2.3.26, we conclude that \tilde{X} is a minimal surface of general type with $K^2 = 1$ and $p_g = q = 0$. Thus, \tilde{X} is a numerical Godeaux surface with canonical model X . \square

Let R be an S -module fulfilling all assumptions of Theorem 5.0.2. Knowing a minimal free resolution of R as an S -module gives us only the relations of R which are S -linear. However, we are mainly interested in the ring structure of R which exists by the previous results. As before, let us choose $r_0 = 1, r_1 = z_0, \dots, r_4 = z_3, r_5 = w_0, r_6 = w_1, r_7 = w_2$ as module generators for R as an S -module. We want to compute the remaining 28 defining relations of R expressing the products $r_i r_j \in R$, for $1 \leq i \leq j \leq 7$, as S -linear combinations of the module generators (see Lemma 3.3.9). In the following we present a way of determining these relations using the results from Chapter 4. From the proof of Theorem 4.2.1 we know that there exists a commutative diagram

$$\begin{array}{ccccccc}
0 & \longleftarrow & R & \xleftarrow{\gamma_0} & S_2(\mathbf{F})_0 & \xleftarrow{\bar{\delta}_1} & S_2(\mathbf{F})_1 & \xleftarrow{\bar{\delta}_2} & S_2(\mathbf{F})_2 & \longleftarrow & \dots \\
& & \text{id} \downarrow & & \beta_0 \downarrow & & \beta_1 \downarrow & & \beta_2 \downarrow & & \\
0 & \longleftarrow & R & \xleftarrow{d_0} & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 & \longleftarrow & \dots
\end{array}$$

where the second row is exact. Note that we can compute the maps of the first row from the ones in the second row. By definition, the homomorphism γ_0 maps the canonical basis vectors of $S_2(\mathbf{F})_0$ to the elements $r_i r_j$ for $0 \leq i \leq j \leq 7$. Then, the commutativity of the diagram above implies that the images of these basis vectors under the homomorphism β_0 represent the elements $r_i r_j$ as S -linear combinations of the module generators r_0, \dots, r_7 . Consequently, knowing d_1 and $\bar{\delta}_1$, we can compute (a basis for) the homomorphisms of degree 0 in $\text{Hom}_S(\text{coker } \bar{\delta}_1, \text{coker } d_1)$ with the help of a computer algebra system. Furthermore, any S -linear homomorphism $\text{coker } \bar{\delta}_1 \rightarrow \text{coker } d_1$ is also S_Y -linear. Hence, up to scalars, there exists exactly one such homomorphism since the S_Y -algebra structure of R is uniquely determined. Choosing the isomorphism sending $e_0 \otimes e_0 \in S_2(\mathbf{F})_0$ to $e_0 \in F_0$, gives us the desired homomorphism β_0 , and hence the remaining relations.

6 The Minimal Free Resolution Modulo a Regular Sequence

Throughout this chapter X denotes a numerical Godeaux surface with canonical ring $R(X)$ and canonical model $X_{can} = \text{Proj}(R(X))$. Furthermore, $S = \mathbb{k}[x_0, x_1, y_0, \dots, y_3]$ is the graded polynomial ring as defined before.

In Chapter 3 we have seen that the sequence $x_0, x_1 \in S$ is a regular sequence for $R(X)$ and that the minimal free resolution of $R(X)$ as an S -module splits modulo x_0, x_1 into a direct sum of three complexes. Furthermore, the structure theorem of Chapter 4 tells us that there exists a free resolution whose middle map is alternating. The aim of this chapter is to consider such a resolution modulo x_0, x_1 and to describe the direct summands.

6.1 Preliminaries

Let

$$0 \leftarrow R(X) \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

$\underbrace{\hspace{10em}}_{= F_\bullet}$

be a minimal free resolution with $d_2^{tr} = -d_2$. We will now introduce a notation for the graded parts of the maps d_1 and d_2 which will be used throughout the rest of this thesis.

Notation 6.1.1 (The general set-up). We write

$$d_1 = \begin{array}{c|ccc} & 6S(-6) & 12S(-7) & 8S(-8) \\ \hline S & b_0(y) & * & * \\ \hline 4S(-4) & \mathbf{a} & b_1(y) & \mathbf{c} \\ \hline 3S(-5) & 0 & \mathbf{e} & b_2(y) \end{array}$$

$$d_2 = \begin{array}{c|ccc} & 6S(-11) & 12S(-10) & 8S(-9) \\ \hline 6S(-6) & \mathbf{o} & \mathbf{n} & b_3(y) \\ \hline 12S(-7) & -\mathbf{n}^{tr} & b_4(y) & \mathbf{p} \\ \hline 8S(-8) & -b_3(y)^{tr} & -\mathbf{p}^{tr} & 0 \end{array}$$

Note that the matrices \mathbf{o} and b_4 are both skew-symmetric. Since there are no elements of degree 1 in S , the maps $S(-5)^3 \leftarrow S(-6)^6$ and $S(-8)^8 \leftarrow S(-9)^8$ are both zero. The red matrices are the parts of d_1 and d_2 which depend only on the variables y_0, \dots, y_3 . More precisely, for each i , all entries of $b_i(y)$ are linear combinations of y_0, \dots, y_3 with coefficients in \mathbb{k} . By d_1' we denote the matrix obtained from d_1 by erasing the first row. We do not assign names to the matrices indicated by a $*$ since they won't play a role in the following. For the matrices marked in blue we obtain the following characterization:

	degree	entries
a, e, p	2	linear combinations of x_0, x_1
c, n	4	linear combinations of x_0^2, x_0x_1, x_1^2
o	5	linear combinations of $x_iy_j, i = 0, 1, j = 0, \dots, 3$

Let us now briefly recall the notation from Chapter 3. Let

$$\bar{R} = R \otimes S/(x_0, x_1) = \bigoplus_{i=0}^2 \bar{R}^{(i)}$$

and

$$\bar{F}_\bullet = F_\bullet \otimes S/(x_0, x_1) = \bigoplus_{i=0}^2 \bar{F}_\bullet^{(i)}$$

considered as $T = K[y_0, y_1, y_2, y_3]$ -modules. Setting $\bar{d}_i = d_i \otimes T$ for each i yields

$$\bar{d}_1 = \begin{array}{c|ccc} & 6T(-6) & 12T(-7) & 8T(-8) \\ \hline T & b_0(y) & & \\ \hline 4T(-4) & & b_1(y) & \\ \hline 3T(-5) & & & b_2(y) \end{array}$$

$$\bar{d}_2 = \begin{array}{c|ccc} & 6T(-11) & 12T(-10) & 8T(-9) \\ \hline 6T(-6) & & & b_3(y) \\ \hline 12T(-7) & & b_4(y) & \\ \hline 8T(-8) & -b_3(y)^{tr} & & \end{array}$$

So the minimal free resolutions of the $\bar{R}^{(i)}$ are of the form:

$$0 \longleftarrow \bar{R}^{(0)} \longleftarrow T \xleftarrow{b_0(y)} T(-6)^6 \xleftarrow{b_3(y)} T(-9)^8 \xleftarrow{b_2(y)^{tr}} T(-12)^3 \longleftarrow 0$$

$$0 \longleftarrow \bar{R}^{(1)} \longleftarrow T(-4)^4 \xleftarrow{b_1(y)} T(-7)^{12} \xleftarrow{b_4(y)} T(-10)^{12} \xleftarrow{b_1(y)^{tr}} T(-13)^4 \longleftarrow 0$$

$$0 \longleftarrow \bar{R}^{(2)} \longleftarrow T(-5)^3 \xleftarrow{b_2(y)} T(-8)^8 \xleftarrow{-b_3(y)^{tr}} T(-11)^6 \xleftarrow{b_0(y)^{tr}} T(-17) \longleftarrow 0$$

Our aim is to describe the above three minimal resolutions explicitly. Later we will use these results as a starting point for our construction of numerical Godeaux surfaces. More precisely, we will first construct the individual minimal free resolutions modulo x_0, x_1 and then, based on these, the whole resolution.

6.2 The Canonical Ring Modulo a Regular Sequence

In this section we give an explicit description of the canonical ring $R(X)$ and its minimal free resolution modulo the regular sequence x_0, x_1 . Thereby we restrict our study to numerical

Godeaux surfaces fulfilling one additional assumption. As in Section 3.1 we write

$$|2K_X| = |M| + F,$$

where M is a generic member of the moving part of $|2K_X|$ and F is the fixed part. Recall from Proposition 3.1.1 that there are the following possibilities for M and F :

- (i) $F = 0, M^2 = 4,$
- (ii) $M^2 = 2, MF = 2, F^2 = -2,$
- (iii) $M^2 = 0, MF = 4, F^2 = -4.$

In the first case, $|M|$ may have 4 base points (possibly infinitely near) or a single double base point which is then a singular point of M . From now on we assume that our numerical Godeaux surface X satisfies the following conditions:

- (♣) $F = 0$ and $|M|$ has 4 distinct base points P_0, \dots, P_3 .

We call any such numerical Godeaux surface X a *marked numerical Godeaux surface*.

Remark 6.2.1. Note that a base point p of a linear system \mathcal{L} on a smooth surface is called *ordinary* if the general element of \mathcal{L} is smooth at p and two general elements have different tangent directions at p . Since $(2K_X)^2 = M^2 = 4$, Assumption (♣) implies that every base point of $|2K_X|$ is ordinary.

Remark 6.2.2. Let us see how Assumption (♣) fits into the current literature. It is known that the Craighero-Gattazzo surface, a numerical Godeaux surface with $\text{Tors } X = 0$, fulfills this condition (see [CP00], Theorem 5.1). Furthermore, any numerical Godeaux surface X with $\text{Tors } X = \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/5\mathbb{Z}$ satisfies $F = 0$ (see [CCML07], Corollary 4.4). We will see later that the bicanonical system of any numerical Godeaux surface with $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$ has indeed 4 distinct base points. In contrast, a surface X with $\text{Tors } X = \mathbb{Z}/4\mathbb{Z}$ cannot have 4 distinct base points as shown in Lemma 3.1.11.

Now let us consider the birational morphism $\pi: X \rightarrow X_{can}$. Then the 4 base points of $|M|$ are mapped to 4 distinct points under π . Indeed, since we may assume that M is irreducible by Proposition 3.1.1, M does not meet any fundamental cycle of X . Then $\pi|_M$ is an isomorphism onto its image. By abuse of notation, we denote the 4 image points on X_{can} also by P_0, \dots, P_3 . Note that X_{can} is smooth at the points P_0, \dots, P_3 . Furthermore, an arbitrary element $C \in |2K_X|$ is of the form

$$C = C_0 + \sum a_i E_i$$

with $K_X C_0 = 2$ and $K_X E_i = 0$ for all i . Then $M E_i = 2K_X E_i = 0$ for all i and the 4 base points are all contained in C_0 . Thus $|2K_{X_{can}}|$ has also 4 distinct base points.

Lemma 6.2.3. *The points $P_0, \dots, P_3 \in X_{can}$ are mapped to 4 distinct points under $\varphi: X_{can} \rightarrow Y$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X_{can} \\ & \searrow \hat{\varphi} & \downarrow \varphi \\ & & Y \end{array}$$

Note that $\hat{\varphi}$ is a morphism since $F = 0$. So it is enough to show the statement for $\hat{\varphi}$ in place of φ . Since M is smooth at the base points P_0, \dots, P_3 , Bertini's Theorem implies that M is nonsingular. Furthermore we know that M is not hyperelliptic by the results of Pignatelli. Thus M is a smooth non-hyperelliptic curve. Note that here Assumption (\clubsuit) is crucial. By Proposition 3.1.1, M does not contain any base points of $|3K_X|$. Let \hat{M} denote the generic member of $|2K_{X_{can}}|$. Recall that we denote the rational map associated to the systems $|nK_X|$ and $|nK_{X_{can}}|$ by ϕ_n and κ_n , respectively. Let

$$D = \phi_3(M) = \kappa_3(\hat{M}) \subseteq \mathbb{P}^3$$

be the image curve in \mathbb{P}^3 . Note that \hat{M} does also not contain the base points of $|3K_{X_{can}}|$ by Corollary 3.1.5. Now by adjunction we get $K_M = (3K_X)|_M$. Furthermore, since $p_g = q = 0$, the restriction map

$$H^0(X, 3K_X) \rightarrow H^0(M, (3K_X)|_M)$$

is an isomorphism. Hence $\phi_3(M) = \phi_3|_M(M) = \phi_{K_M}(M)$, where ϕ_{K_M} denotes the morphism associated to the canonical system $|K_M|$. Now M being non-hyperelliptic implies that ϕ_{K_M} is an isomorphism. Note that the image curve D is a complete intersection of type $(2, 3)$ of genus 4 in this case (see [Miy76], Proposition 4). Consequently, the points P_0, \dots, P_3 are mapped to 4 different points in \mathbb{P}^3 under ϕ_3 and therefore also to 4 different points in $Y \subseteq \mathbb{P}(2^2, 3^4)$ under $\hat{\varphi}$. \square

Let us denote by p_0, \dots, p_3 the 4 image points of P_0, \dots, P_3 in \mathbb{P}^3 . Moreover, let $J = \bigcap_{i=0}^3 J_i \subseteq T$ be the homogeneous (saturated) ideal of $\{p_0, \dots, p_3\} \subseteq \mathbb{P}^3$, where J_i is the homogeneous ideal of p_i for each i . Our aim is to prove the following result:

Proposition 6.2.4. *For each i , $\bar{R}^{(i)}$ is a Cohen-Macaulay T -module. Furthermore,*

$$\begin{aligned} \bar{R}^{(0)} &\cong T/J, \\ \bar{R}^{(1)} &\cong \bigoplus_{i=0}^3 T/J_i, \\ \bar{R}^{(2)} &\cong \text{Ext}_T^3(\bar{R}^{(0)}, T(-17)). \end{aligned}$$

We will show this statement for every T -module $\bar{R}^{(i)}$ separately. Whenever studying a single module $\bar{R}^{(i)}$ we assume for simplicity that the variables y_i in T all have degree 1.

Let $\bar{f}: T \rightarrow \bar{R}$ be the natural ring homomorphism induced by $f: S \rightarrow R(X)$. Then $\ker(\bar{f}) = \text{ann}_T 1_{\bar{R}} = \text{ann}_T \bar{R}$ and thus

$$\text{Proj}(T/\text{ann}_T \bar{R}) = \{p_0, \dots, p_3\}.$$

On the other hand, since \bar{d}_1 decomposes into a direct sum of the three matrices $b_0(y)$, $b_1(y)$ and $b_2(y)$, we get

$$\text{ann}_T(1_{\bar{R}}) = \text{ann}_T(\text{coker } b_0(y)) = J',$$

where $J' \subseteq T$ is the ideal generated by the entries of the matrix $b_0(y)$. Hence

$$\text{Proj}(T/J') = \{p_0, \dots, p_3\} = \text{Proj}(T/J) \quad (6.1)$$

Proof of Proposition 6.2.4, Part I. First we show that $\bar{R}^{(0)} \cong T/J'$ is Cohen-Macaulay. By the Auslander-Buchsbaum formula we have

$$\text{depth}(\bar{R}^{(0)}) = \dim T - \text{projdim } \bar{R}^{(0)} = 1.$$

On the other hand, $\dim \bar{R}^{(0)} = 1$ by (6.1). Hence, $\bar{R}^{(0)}$ is Cohen-Macaulay. But being Cohen-Macaulay of dimension 1 implies that

$$H_{\mathfrak{m}}^0(\bar{R}^{(0)}) = H_{\mathfrak{m}}^0(T/J') = 0,$$

where \mathfrak{m} is the homogeneous maximal ideal of T . Hence J' is saturated, and $J = J'$ by (6.1). This shows the first part of Proposition 6.2.4. Let us now continue with $\bar{R}^{(2)}$. Applying the functor $\text{Hom}(-, \omega_T)$ to the minimal free resolution

$$0 \leftarrow \bar{R}^{(0)} \leftarrow T \xleftarrow{b_0} T(-6)^6 \xleftarrow{b_3} T(-9)^8 \xleftarrow{b_2^{tr}} T(-12)^3 \leftarrow 0$$

yields a complex

$$0 \leftarrow \text{Ext}_T^3(\bar{R}^{(0)}, \omega_T) \leftarrow \text{Hom}(T(-12)^3, \omega_T) \xleftarrow{b_2} \text{Hom}(T(-9)^8, \omega_T) \\ \xleftarrow{b_3^{tr}} \text{Hom}(T(-6)^6, \omega_T) \xleftarrow{b_0^{tr}} \text{Hom}(T, \omega_T) \leftarrow 0$$

which is exact by Proposition 2.2.11. Now, since $\omega_T \cong T(-12)$, tensoring with $T(-5)$ gives the exact sequence of the form

$$0 \leftarrow \text{Ext}_T^3(\bar{R}^{(0)}, T(-17)) \leftarrow T(-5)^3 \xleftarrow{b_2} T(-8)^8 \xleftarrow{b_3^{tr}} T(-11)^6 \xleftarrow{b_0^{tr}} T(-17) \leftarrow 0.$$

On the other hand, we know that the minimal free resolution of $\bar{R}^{(2)}$ is of type

$$0 \leftarrow \bar{R}^{(2)} \leftarrow T(-5)^3 \xleftarrow{b_2} T(-8)^8 \xleftarrow{-b_3^{tr}} T(-11)^6 \xleftarrow{b_0^{tr}} T(-17) \leftarrow 0.$$

Thus

$$\bar{R}^{(2)} \cong \text{Ext}_T^3(\bar{R}^{(0)}, T(-17)).$$

It remains to show that $\bar{R}^{(2)}$ is a Cohen-Macaulay T -module. Again, by the Auslander-Buchsbaum formula we see that $\text{depth}(\bar{R}^{(2)}) = 1$. Hence $\dim \bar{R}^{(2)} \geq 1$. From

$$J = \text{ann}_T \bar{R} = \bigcap_{i=0}^2 \text{ann}_T(\text{coker } b_i(y)) \subseteq \text{ann}_T(\text{coker } b_2(y))$$

we see that

$$1 = \dim T/J \geq \dim T/\text{ann}_T(\text{coker } b_2(y)) = \dim \bar{R}^{(2)}$$

which shows the claim. \square

So in particular the first summand of the free resolution of \bar{R} resolves the ideal of the points $p_0, \dots, p_3 \in \mathbb{P}^3$. Since the finitely many configurations of 4 distinct points in \mathbb{P}^3 can be distinguished by their minimal free resolution, we can describe this resolution even more precisely:

Lemma 6.2.5. *The 4 points $p_0, \dots, p_3 \in \mathbb{P}^3$ are in general position.*

Proof. We have seen that the minimal free resolution of the homogeneous ideal J of the points p_0, \dots, p_3 is of type

$$0 \leftarrow T/J \leftarrow T \leftarrow T(-2)^6 \leftarrow T(-3)^8 \leftarrow T(-4)^3 \leftarrow 0.$$

But there are only finitely many possible configurations of 4 distinct points in \mathbb{P}^3 whose minimal free resolutions have different Betti numbers:

The 4 points are colinear.

	0	1	2	3
0	1	2	1	.
1
2
3	.	1	2	1
<hr/>				
	1	3	3	1

The 4 points are noncolinear, but three of them are colinear.

	0	1	2	3
0	1	1	.	.
1	.	2	3	1
2	.	1	2	1
<hr/>				
	1	4	5	2

No three of the 4 points are colinear, but all lie on a common hyperplane.

	0	1	2	3
0	1	1	.	.
1	.	2	2	.
2	.	.	1	1
<hr/>				
	1	3	3	1

The 4 points are in general position.

	0	1	2	3
0	1	.	.	.
1	.	6	8	3
<hr/>				
	1	6	8	3

Hence only the last case can occur. \square

General Assumption 1. Recall that we consider X_{can} as a subscheme of $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$. After performing a linear change of coordinates on $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$ (or equivalently, an automorphism of $\mathbb{k}[x_0, x_1, y_0, \dots, y_3, z_0, \dots, z_3, w_0, w_1, w_2]$) if necessary, we assume from now on that the 4 points $P_0, \dots, P_3 \in X_{can}$ are mapped to the 4 coordinate points of \mathbb{P}^3 , more precisely that

$$p_0 = (1 : 0 : 0 : 0), \quad p_1 = (0 : 1 : 0 : 0), \quad p_2 = (0 : 0 : 1 : 0) \text{ and } p_3 = (0 : 0 : 0 : 1).$$

In particular, we assume from now on that

$$J = (y_i y_j \mid 0 \leq i < j \leq 3) = \bigcap_{i=0}^3 J_i,$$

where $J_i = I(p_i)$.

Before proving the remaining part of Proposition 6.2.4, we first compute the support of the T -module $\bar{R}^{(1)}$.

Lemma 6.2.6. $\text{ann}_T \bar{R}^{(1)} = \text{ann}_T(\text{coker } b_1(y)) = J$.

Proof. From the discussion above we know that

$$\text{ann}_T(\text{coker } \bar{d}_1) = \text{ann}_T(\text{coker } b_0(y)) = J$$

and

$$\text{ann}_T(\text{coker } \bar{d}_1) = \bigcap_{i=0}^2 \text{ann}_T(\text{coker } b_i(y)).$$

Thus, $J \subseteq \text{ann}_T(\text{coker } b_1(y))$. To show equality it is enough to prove that

$$V(\text{ann}_T(\text{coker } b_1(y))) = \{p_0, p_1, p_2, p_3\}.$$

Assume that there is a point p_i which is not contained in $V(\text{ann}_T(\text{coker } b_1(y)))$. Then there exists an integer $n_i \geq 1$ such that $y_i^{n_i} \in \text{ann}_T(\text{coker } b_1(y))$. Consequently, for each $j \in \{0, \dots, 3\}$, there is a relation of the form $y_i^{n_i} z_j = 0$ in \bar{R} , and hence a relation of the form

$$d_{j,0}x_0 + d_{j,1}x_1 + y_i^{n_i} z_j = 0$$

in $R(X)$, where $d_{j,0}, d_{j,1} \in R(X)$. Since $y_i^{n_i}(P_i) = y_i^{n_i}(p_i) \neq 0$, all forms z_0, \dots, z_3 must vanish at the point $P_i \in X_{\text{can}}$. But this implies that P_i is a base point of $|4K_{X_{\text{can}}}|$ contradicting Theorem 2.3.28. \square

Using this result, the proof of Proposition 6.2.4 for $\bar{R}^{(1)}$ is a direct consequence of the following statement.

Proposition 6.2.7. *Let N be a finitely generated graded T -module satisfying the following properties:*

(i) $\text{ann}_T N = J = \bigcap_{i=0}^3 J_i$,

(ii) *the minimal free resolution of N is of the following type*

$$0 \leftarrow N \leftarrow T^4 \leftarrow T(-1)^{12} \leftarrow T(-2)^{12} \leftarrow T(-3)^4 \leftarrow 0.$$

Then

$$N \cong \bigoplus_{i=0}^3 T/J_i.$$

Proof. First we will show that N is a Cohen-Macaulay module. Let $\mathfrak{m} = (y_0, \dots, y_3)$ be the homogeneous maximal ideal of T . By the Auslander-Buchsbaum formula we get

$$\text{depth}(\mathfrak{m}, N) = \text{depth}(\mathfrak{m}, T) - \text{projdim}(N) = 4 - 3 = 1.$$

Furthermore, by the second assumption, $\dim N = \dim(T/\text{ann}_T N) = 1$. Consequently,

$$\text{depth}(\mathfrak{m}, N) = \dim N$$

and N is Cohen-Macaulay. Using the long exact sequence relating local and global cohomology we get the exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(N) \rightarrow N \rightarrow \bigoplus_d H^0(\tilde{N}(d)) \rightarrow H_{\mathfrak{m}}^1(N) \rightarrow 0. \quad (6.2)$$

Since $\text{depth}(\mathfrak{m}, N) = 1$ we know that $H_{\mathfrak{m}}^0(N) = 0$ and $H_{\mathfrak{m}}^1(N) \neq 0$. For any coherent sheaf \mathcal{F} on \mathbb{P}^3 , we denote the graded T -module $\bigoplus_{d \geq 0} H^0(\mathbb{P}^3, \mathcal{F}(d))$ by $\Gamma_{\geq 0}(\mathcal{F})$. Now let us first show that

$$N \cong \Gamma_{\geq 0}(\tilde{N}).$$

As $N_d = 0$ for all $d < 0$ the sequence (6.2) yields an injective map $N \rightarrow \Gamma_{\geq 0}(\tilde{N})$. Consequently, it remains to prove that $H_{\mathfrak{m}}^1(N)_d = 0$ for all $d \geq 0$. Let

$$d = \max\{e \mid H_{\mathfrak{m}}^1(N)_e \neq 0\}.$$

Note that the maximum exists by Serre's vanishing theorem. Let us denote by $\text{reg}(N)$ the Castelnuovo-Mumford regularity of N . Then $d + 1 \leq \text{reg}(N)$ by [Eis05], Theorem 4.3. But from the minimal free resolution of N we deduce that $\text{reg}(N) = 0$. Hence $d \leq \text{reg}(N) - 1 < 0$. Let $\eta_i: p_i \hookrightarrow \mathbb{P}^3$ be the inclusion of the closed point p_i in \mathbb{P}^3 . Then $\mathcal{G}_i = \eta_{i*} \mathcal{O}_{p_i} = \widetilde{T/J_i}$ is a skyscraper sheaf on \mathbb{P}^3 with support at the point p_i . We claim that

$$\Gamma_{\geq 0}(\bigoplus_{i=0}^3 \mathcal{G}_i) \cong \Gamma_{\geq 0}(\widetilde{N}).$$

First note that $T/J_i \cong \Gamma_{\geq 0}(\mathcal{G}_i)$ for each i follows from the same arguments as for N above. Now, since all ideals in the support of N are minimal, we can choose for each $i \in \{0, \dots, 3\}$ a homogeneous element $n_i \in N$ such that $\text{ann}_T(n_i) = J_i$. Let $a_i = \deg(n_i)$. Thus, for each i , we obtain an injective map of graded T -modules

$$(T/J_i)(-a_i) \hookrightarrow N,$$

and therefore an injective morphism $\mathcal{G}_i(-a_i) \hookrightarrow \widetilde{N}$ of coherent sheaves on \mathbb{P}^3 . Now since p_i is a point, it is isomorphic to an affine variety. Furthermore, any line bundle on \mathcal{O}_{p_i} is trivial, and hence $\mathcal{G}_i \otimes \mathcal{O}_{\mathbb{P}^3}(d) \cong \mathcal{G}_i$ for all d . Thus, we get an injective morphism $\mathcal{G}_i \hookrightarrow \widetilde{N}$, and hence a morphism

$$\bigoplus_{i=0}^3 \mathcal{G}_i \rightarrow \widetilde{N}.$$

This morphism is again injective since the support of \widetilde{N} is the disjoint union of the supports of the \mathcal{G}_i . By tensoring with $\mathcal{O}_{\mathbb{P}^3}(d)$, taking global sections and the direct sum over all $d \geq 0$, we obtain a homogeneous homomorphism

$$\Gamma_{\geq 0}(\bigoplus_{i=0}^3 \mathcal{G}_i) \hookrightarrow \Gamma_{\geq 0}(\widetilde{N}) \cong N.$$

But for any $d \geq 0$ we have $h^0(\mathbb{P}^3, (\bigoplus_{i=0}^3 \mathcal{G}_i)(d)) = h^0(\mathbb{P}^3, \bigoplus_{i=0}^3 \mathcal{G}_i) = 4 = h^0(\mathbb{P}^3, \widetilde{N}(d)) = \dim_{\mathbb{k}} N_d$. Hence

$$\Gamma_{\geq 0}(\bigoplus_{i=0}^3 \mathcal{G}_i) \cong N.$$

The result follows now from the fact that cohomology commutes with finite direct sums of coherent sheaves on Noetherian schemes:

$$\bigoplus_{i=0}^3 T/J_i \cong \bigoplus_{i=0}^3 \Gamma_{\geq 0}(\mathcal{G}_i) \cong \Gamma_{\geq 0}(\bigoplus_{i=0}^3 \mathcal{G}_i) \cong N. \quad \square$$

6.3 A Standard Resolution of $R(X)$

The aim of this section is to give one possible choice of the maps of the minimal free resolution of each $\overline{R}^{(i)}$ and prove afterwards that there is always a minimal free resolution of $R(X)$ with alternating middle map which is modulo x_0, x_1 the direct sum of the chosen resolutions. We will call such a minimal free resolution then a *standard resolution*.

By duality it is enough to specify the resolutions of $\overline{R}^{(0)}$ and $\overline{R}^{(1)}$. Recall that $\overline{R}^{(0)} \cong T/J$

with $J = (y_i y_j \mid 0 \leq i < j \leq 3)$. So one possible choice for the maps of $\bar{F}_\bullet^{(0)}$ is

$$T \xleftarrow{\begin{pmatrix} y_0 y_1 & y_0 y_2 & y_1 y_2 & y_0 y_3 & y_1 y_3 & y_2 y_3 \end{pmatrix}} T(-6)^6 \quad (6.3)$$

$$T(-6)^6 \xleftarrow{\begin{pmatrix} -y_2 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 \\ y_1 & -y_1 & 0 & 0 & -y_3 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 & 0 & 0 & -y_3 & 0 \\ 0 & 0 & y_1 & -y_1 & y_2 & -y_2 & 0 & 0 \\ 0 & 0 & 0 & y_0 & 0 & 0 & y_2 & -y_2 \\ 0 & 0 & 0 & 0 & 0 & y_0 & 0 & y_1 \end{pmatrix}} T(-9)^8 \quad (6.4)$$

$$T(-9)^8 \xleftarrow{\begin{pmatrix} -y_3 & -y_3 & 0 \\ 0 & -y_3 & 0 \\ y_2 & y_2 & 0 \\ 0 & y_2 & -y_2 \\ -y_1 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & -y_0 & 0 \\ 0 & 0 & -y_0 \end{pmatrix}} T(-12)^3 \quad (6.5)$$

For $\bar{F}_\bullet^{(2)}$ we simply take as maps the dual of the maps above.

For $\bar{R}^{(1)} \cong \bigoplus_{i=0}^3 T/J_i$ we choose first a minimal free resolution of T/J_i for each i and take then the direct sum. We will only specify a free resolution of T/J_0 (the other cases are chosen in the same way).

$$T(-4) \xleftarrow{\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}} T(-7)^3 \xleftarrow{\begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}} T(-10)^3 \xleftarrow{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} T(-13)^3 \leftarrow 0$$

Note that the chosen minimal free resolution of $\bar{R}^{(1)}$ is therefore self-dual (up to a sign).

Now, denoting the three maps (6.3)-(6.5) by \tilde{b}_0, \tilde{b}_3 and \tilde{b}_2^{tr} , and the first two syzygy maps of the minimal free resolution of $\bar{R}^{(1)}$ by \tilde{b}_1 and \tilde{b}_4 , we get a minimal free resolution of \bar{R} of the form

$$0 \leftarrow \bar{R} \leftarrow \bar{F}_0 \xleftarrow{\begin{pmatrix} \tilde{b}_0 & & \\ & \tilde{b}_1 & \\ & & \tilde{b}_2 \end{pmatrix}} \bar{F}_1 \xleftarrow{\begin{pmatrix} & & \tilde{b}_3 \\ & \tilde{b}_4 & \\ -\tilde{b}_3^{tr} & & \end{pmatrix}} \bar{F}_1^\vee \xleftarrow{\begin{pmatrix} \tilde{b}_0^{tr} & & \\ & \tilde{b}_1^{tr} & \\ & & \tilde{b}_2^{tr} \end{pmatrix}} \bar{F}_0^\vee \leftarrow 0 \quad (6.6)$$

Proposition 6.3.1. *There exists a minimal free resolution*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

of $R(X)$ as an S -module such that:

(i) $d_2^{tr} = -d_2$,

(ii) modulo x_0, x_1 the resolution reduces to the minimal free resolution in (6.6).

Proof. Let us first briefly sketch the idea of the proof. We start with a minimal free resolution of $R(X)$ whose middle map is alternating. Reducing this resolution modulo x_0, x_1 gives a minimal free resolution of \bar{R} . Thus, there exists an isomorphism between this complex and the one in (6.6). Any such isomorphism gives an isomorphism between the original resolution of $R(X)$ and another free resolution of $R(X)$ which has the desired form modulo x_0, x_1 . Now the (only) difficult part is to find an isomorphism which preserves the skew-symmetry of the middle matrix.

So let

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{e_1} F_1 \xleftarrow{e_2} F_1^\vee \xleftarrow{e_1^{tr}} F_0^\vee \leftarrow 0 \quad (6.7)$$

be any minimal free resolution of $R(X)$ with $e_2^{tr} = -e_2$. Then there exists a graded isomorphism of complexes

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & \bar{R}^{(0)} & \longleftarrow & \bar{F}_0^{(0)} & \xleftarrow{b_0} & \bar{F}_1^{(0)} & \xleftarrow{b_3} & \bar{F}_2^{(0)} & \xleftarrow{b_2^{tr}} & \bar{F}_3^{(0)} & \longleftarrow & 0 \\ & & \downarrow & & \alpha_0^{(0)} \downarrow & & \alpha_1^{(0)} \downarrow & & \alpha_2^{(0)} \downarrow & & \alpha_3^{(0)} \downarrow & & \\ 0 & \longleftarrow & \bar{R}^{(0)} & \longleftarrow & \bar{F}_0^{(0)} & \xleftarrow{\tilde{b}_0} & \bar{F}_1^{(0)} & \xleftarrow{\tilde{b}_3} & \bar{F}_2^{(0)} & \xleftarrow{\tilde{b}_2^{tr}} & \bar{F}_3^{(0)} & \longleftarrow & 0 \end{array}$$

inducing the identity map on $\bar{R}^{(0)}$, where the first row is the first summand of (6.7) modulo x_0, x_1 . Applying first the functor $\text{Hom}_T(-, T(-17))$ to the diagram above and taking then the inverse of the chain maps yields a graded isomorphism of complexes

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & \bar{R}^{(2)} & \longleftarrow & \bar{F}_0^{(2)} & \xleftarrow{b_2} & \bar{F}_1^{(2)} & \xleftarrow{-b_3^{tr}} & \bar{F}_2^{(2)} & \xleftarrow{b_0^{tr}} & \bar{F}_3^{(2)} & \longleftarrow & 0 \\ & & \downarrow & & \alpha_0^{(2)} \downarrow & & \alpha_1^{(2)} \downarrow & & \alpha_2^{(2)} \downarrow & & \alpha_3^{(2)} \downarrow & & \\ 0 & \longleftarrow & \bar{R}^{(2)} & \longleftarrow & \bar{F}_0^{(2)} & \xleftarrow{\tilde{b}_2} & \bar{F}_1^{(2)} & \xleftarrow{-\tilde{b}_3^{tr}} & \bar{F}_2^{(2)} & \xleftarrow{\tilde{b}_0^{tr}} & \bar{F}_3^{(2)} & \longleftarrow & 0 \end{array}$$

between two minimal free resolution of $\bar{R}^{(2)}$, where the first row is the third summand of resolution (6.7) modulo x_0, x_1 and with $\alpha_i^{(2)} = (\alpha_{3-i}^{(0)})^{-tr}$ for all i . Now let us continue with $\bar{R}^{(1)}$. As above, there is a graded isomorphism of complexes

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & \bar{R}^{(1)} & \longleftarrow & \bar{F}_0^{(1)} & \xleftarrow{b_1} & \bar{F}_1^{(1)} & \xleftarrow{b_4} & \bar{F}_2^{(1)} & \xleftarrow{b_1^{tr}} & \bar{F}_3^{(1)} & \longleftarrow & 0 \\ & & \downarrow & & \alpha_0^{(1)} \downarrow & & \alpha_1^{(1)} \downarrow & & \alpha_2^{(1)} \downarrow & & \alpha_3^{(1)} \downarrow & & \\ 0 & \longleftarrow & \bar{R}^{(1)} & \longleftarrow & \bar{F}_0^{(1)} & \xleftarrow{\tilde{b}_1} & \bar{F}_1^{(1)} & \xleftarrow{\tilde{b}_4} & \bar{F}_2^{(1)} & \xleftarrow{\tilde{b}_1^{tr}} & \bar{F}_3^{(1)} & \longleftarrow & 0 \end{array}$$

inducing the identity map on $\bar{R}^{(1)}$. Our aim is to show that there exists a graded isomorphism between these complexes such that $\alpha_2^{(1)} = (\alpha_1^{(1)})^{-tr}$ and $\alpha_3^{(1)} = (\alpha_0^{(1)})^{-tr}$. Applying the functor $\text{Hom}(-, T(-12))$ and combining the resulting diagram with the original one yields a commutative diagram

$$\begin{array}{ccccccc} & & \bar{F}_0^{(1)} & \xleftarrow{\tilde{b}_1} & \bar{F}_1^{(1)} & \xleftarrow{\tilde{b}_4} & \bar{F}_2^{(1)} & \xleftarrow{\tilde{b}_1^{tr}} & \bar{F}_3^{(1)} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\alpha_3^{(1)})^{tr} & & \bar{F}_0^{(1)} & \xleftarrow{b_1} & \bar{F}_1^{(1)} & \xleftarrow{b_4} & \bar{F}_2^{(1)} & \xleftarrow{b_1^{tr}} & \bar{F}_3^{(1)} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \alpha_0^{(1)} & & \bar{F}_0^{(1)} & \xleftarrow{\tilde{b}_1} & \bar{F}_1^{(1)} & \xleftarrow{\tilde{b}_4} & \bar{F}_2^{(1)} & \xleftarrow{\tilde{b}_1^{tr}} & \bar{F}_3^{(1)} \end{array}$$

where all vertical maps are isomorphisms.

Now $\bar{R}^{(1)} \cong \bigoplus_{i=0}^3 T/J_i$ implies that $\text{Hom}_T(\bar{R}^{(1)}, \bar{R}^{(1)}) \cong \bar{R}^{(1)}$. In particular, since $\bar{R}^{(1)} \cong \text{coker } \tilde{b}_1$, there exist $\mu_0, \dots, \mu_3 \in \mathbb{k}^*$ such that

$$\alpha_0^{(1)}(\alpha_3^{(1)})^{tr} = \begin{pmatrix} \mu_0 & & & \\ & \mu_1 & & \\ & & \mu_2 & \\ & & & \mu_3 \end{pmatrix} \text{ and } \alpha_1^{(1)}(\alpha_2^{(1)})^{tr} = \begin{pmatrix} \mu_0 \text{id}_3 & & & \\ & \mu_1 \text{id}_3 & & \\ & & \mu_2 \text{id}_3 & \\ & & & \mu_3 \text{id}_3 \end{pmatrix}.$$

Now it is just a straightforward but lengthy computation to see that we can modify the original isomorphism of complexes so that $\alpha_0^{(1)}(\alpha_3^{(1)})^{tr} = \text{id}$ and $\alpha_1^{(1)}(\alpha_2^{(1)})^{tr} = \text{id}$. Note that this modification involves the computation of the square roots of μ_i for each i which is possible since we assumed \mathbb{k} to be algebraically closed.

It remains to put the individual isomorphisms together to an isomorphism of the whole complex. Let

$$\alpha_0 = \begin{pmatrix} \alpha_0^{(0)} & & \\ & \alpha_0^{(1)} & \\ & & \alpha_0^{(2)} \end{pmatrix}, \alpha_1 = \begin{pmatrix} \alpha_1^{(0)} & & \\ & \alpha_1^{(1)} & \\ & & \alpha_1^{(2)} \end{pmatrix}, \alpha_2 = \alpha_1^{-tr} \text{ and } \alpha_3 = \alpha_0^{-tr}.$$

Then, setting $d_1 = \alpha_0 e_1 \alpha_1^{-1}$ and $d_2 = \alpha_1 e_2 \alpha_2^{-1} = \alpha_1 e_2 \alpha_1^{tr}$, the above isomorphisms define a graded isomorphism of complexes

$$\begin{array}{ccccccccc} 0 & \longleftarrow & R & \longleftarrow & F_0 & \xleftarrow{e_1} & F_1 & \xleftarrow{e_2} & F_1^\vee & \xleftarrow{e_1^{tr}} & F_0^\vee & \longleftarrow & 0 \\ & & \downarrow & & \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ 0 & \longleftarrow & R & \longleftarrow & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_1^\vee & \xleftarrow{d_1^{tr}} & F_0^\vee & \longleftarrow & 0 \end{array}$$

where, by the definition of the α_i , the second row satisfies all required properties. \square

Definition 6.3.2. Let R be a finitely generated graded S -module with a minimal free resolution of the form

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0, \quad (6.8)$$

where the F_i are as above. If (6.8) satisfies the two properties of Proposition 6.3.1, we call (6.8) a *standard resolution* of R .

Remark 6.3.3. Proposition 6.3.1 shows that the canonical ring of any marked numerical Godeaux surface admits, possibly after a suitable isomorphism as described in General Assumption 1, a standard resolution. Note that such a standard resolution is in general not unique.

7 Constructing Standard Resolutions

Throughout this chapter X denotes a marked numerical Godeaux surface.

In the previous chapter we described explicitly the maps of the minimal resolution of $R(X)$ modulo the regular sequence x_0, x_1 . In this chapter we will first focus on the remaining parts of the maps. Of course we cannot expect to get a complete description of the whole resolution in general. But we can simplify several entries of the maps and give a nice geometric characterization of the relations which are not linear. Afterwards we use this information to set up general matrices for d'_1 and d_2 and solve the relations $d'_1 d_2 = 0$.

7.1 The Relations

In this section we will consider a fixed standard resolution

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0,$$

where R is a finitely generated graded S -module (for example, $R = R(X)$). Recall that d'_1 is the matrix obtained from d_1 by erasing the first row. Then

$$d'_1 = \begin{array}{c|ccc} & 6S(-6) & 12S(-7) & 8S(-8) \\ \hline 4S(-4) & \mathbf{a} & b_1(y) & \mathbf{c} \\ \hline 3S(-5) & - & \mathbf{c} & b_2(y) \end{array} \quad (7.1)$$

$$d_2 = \begin{array}{c|ccc} & 6S(-11) & 12S(-10) & 8S(-9) \\ \hline 6S(-6) & \mathbf{o} & \mathbf{n} & b_3(y) \\ \hline 12S(-7) & -\mathbf{n}^{tr} & b_4(y) & \mathbf{p} \\ \hline 8S(-8) & -b_3(y)^{tr} & -\mathbf{p}^{tr} & - \end{array} \quad (7.2)$$

as introduced in Notation 6.1.1. All matrices marked in red have the form as defined in Section 6.3. We consider the entries of these matrices as the known entries of d'_1 and d_2 . The matrices marked in blue are unknown so far and we treat their entries as variables. In the following we are studying the relations between these variables given by the equation $d'_1 d_2 = 0$.

By fixing the maps for the resolution modulo x_0, x_1 in Section 6.3, we fixed also an ordering on the 4 image points in \mathbb{P}^3 . In general, permuting the points P_0, \dots, P_3 (respectively p_0, \dots, p_3) corresponds to a linear change of coordinates on $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$ (respectively on \mathbb{P}^3):

Lemma 7.1.1. *Let $B_2 = \{p_0, \dots, p_3\} \subseteq \mathbb{P}^3$, and let $G \leq \text{Aut}(\mathbb{P}^3) = \text{PGL}(4, \mathbb{k})$ be the subgroup of automorphisms which leave B_2 invariant. Then $G \cong (\mathbb{k}^*)^3 \rtimes S_4$.*

Proof. Since $\text{PGL}(4, \mathbb{k}) = \text{GL}(4, \mathbb{k})/\mathbb{k}^*$ it is enough to show that

$$\tilde{G} \cong (\mathbb{k}^*)^4 \rtimes S_4,$$

where \tilde{G} is the preimage of G under the natural homomorphism $\mathrm{GL}(4, \mathbb{k}) \rightarrow \mathrm{PGL}(4, \mathbb{k})$. We identify each $\sigma \in S_4$ with the permutation matrix m_σ , where

$$(m_\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we identify $(\mathbb{k}^*)^4$ with the torus T of diagonal matrices in $\mathrm{GL}(4, \mathbb{k})$. Then T, S_4 are clearly subgroups of \tilde{G} with $T \cap S_4 = \{\mathrm{id}\}$ and every element $g \in \tilde{G}$ is of the form $t \circ m_\sigma$ for some $t \in T, m_\sigma \in S_4$. But T being a normal subgroup of \tilde{G} implies then

$$\tilde{G} = T \rtimes S_4. \quad \square$$

From now on we assume that the order of the 4 base points is fixed as in Assumption 1. Then $G \cong (\mathbb{k}^*)^3$ and every element $g = (\lambda_1, \lambda_2, \lambda_3) \in G$ defines an automorphism

$$\begin{aligned} \mathbb{P}^3 &\rightarrow \mathbb{P}^3, \\ (p_0 : p_1 : p_2 : p_3) &\mapsto (p_0 : \lambda_1 p_1 : \lambda_2 p_2 : \lambda_3 p_3). \end{aligned}$$

Later we will lift this morphism to an automorphism ν_g of $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$. Then the canonical model $\nu_g(X_{can})$ satisfies also Assumption 1 and we will study the standard resolution of the corresponding canonical ring which is isomorphic to $R(X)$.

Working with the ordered set of points p_0, \dots, p_3 , we will now introduce a way of labelling the entries of the matrices d'_1 and d_2 which reflects this ordering. For the matrix \mathfrak{a} we have

$$\left(\begin{array}{cccccc|cccc} y_0 y_1 & y_0 y_2 & y_1 y_2 & y_0 y_3 & y_1 y_3 & y_2 y_3 & D_0 & 0 & 0 & 0 \\ & & & & & & 0 & D_1 & 0 & 0 \\ & & \mathfrak{a} & & & & 0 & 0 & D_2 & 0 \\ & & & & & & 0 & 0 & 0 & D_3 \end{array} \right)$$

with $D_0 = (y_1 \ y_2 \ y_3)$, $D_1 = (y_0 \ y_2 \ y_3)$, $D_2 = (y_0 \ y_1 \ y_3)$ and $D_3 = (y_0 \ y_1 \ y_2)$. Each row of the matrix on the right-hand side of \mathfrak{a} stands for a point p_k whose ideal is generated by the entries of the matrix D_k . The element y_k is the only variable of degree 3 which is not contained in this ideal. Hence, we can index the rows of \mathfrak{a} by the integers $k = 0, \dots, 3$. The matrix above of \mathfrak{a} has as entries the elements of the minimal generating set $\{y_i y_j \mid 0 \leq i < j \leq 3\}$ of J . We label the columns of \mathfrak{a} by the tuples (i, j) for $0 \leq i < j \leq 3$. Let

$$N = \{(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3)\}$$

be the set of these column indices.

Notation 7.1.2 (The matrix \mathfrak{a}). Let $k \in \{0, \dots, 3\}$, and let $(i, j) \in N$. Then we denote the entry in row k and column (i, j) of \mathfrak{a} by $\mathfrak{a}_{i,j}^{(k)}$. Hence the matrix \mathfrak{a} is of the form

$$\begin{pmatrix} \mathfrak{a}_{0,1}^{(0)} & \mathfrak{a}_{0,2}^{(0)} & \mathfrak{a}_{1,2}^{(0)} & \mathfrak{a}_{0,3}^{(0)} & \mathfrak{a}_{1,3}^{(0)} & \mathfrak{a}_{2,3}^{(0)} \\ \mathfrak{a}_{0,1}^{(1)} & \mathfrak{a}_{0,2}^{(1)} & \mathfrak{a}_{1,2}^{(1)} & \mathfrak{a}_{0,3}^{(1)} & \mathfrak{a}_{1,3}^{(1)} & \mathfrak{a}_{2,3}^{(1)} \\ \mathfrak{a}_{0,1}^{(2)} & \mathfrak{a}_{0,2}^{(2)} & \mathfrak{a}_{1,2}^{(2)} & \mathfrak{a}_{0,3}^{(2)} & \mathfrak{a}_{1,3}^{(2)} & \mathfrak{a}_{2,3}^{(2)} \\ \mathfrak{a}_{0,1}^{(3)} & \mathfrak{a}_{0,2}^{(3)} & \mathfrak{a}_{1,2}^{(3)} & \mathfrak{a}_{0,3}^{(3)} & \mathfrak{a}_{1,3}^{(3)} & \mathfrak{a}_{2,3}^{(3)} \end{pmatrix}.$$

We prove that if $d'_1 d_2 = 0$, then 12 out of the 24 entries of \mathbf{a} are always zero. More precisely, we show the following:

Proposition 7.1.3. *Let d'_1 and d_2 be as in (7.1) and (7.2) with $d'_1 d_2 = 0$. Moreover, let $k \in \{0, \dots, 3\}$ and $(i, j) \in N$. If $k \notin \{i, j\}$, then $\alpha_{i,j}^{(k)} = 0$.*

Remark 7.1.4. To simplify our presentation, we assume that we have given variables $\alpha_{i,j}^k$ for arbitrary elements $k, i, j \in \{0, \dots, 3\}$ subject to the condition $\alpha_{i,j}^k = \alpha_{j,i}^k$.

To prove Proposition 7.1.3, we introduce also an indexing on the entries of the matrix \mathbf{p} in d_2 and analyze the relations given by $d'_1 d_2 = 0$ which are linear in the unknown entries of \mathbf{a} and \mathbf{p} . So let us consider the matrix \mathbf{p} and its adjacent matrices in d_2 . We have

$$\left(\begin{array}{cccc|cccc} & & & & -y_2 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & y_1 & -y_1 & 0 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 \\ & & & * & 0 & y_0 & 0 & 0 & 0 & 0 & -y_3 & 0 & 0 & 0 \\ & & & & 0 & 0 & y_1 & -y_1 & y_2 & -y_2 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & y_0 & 0 & 0 & y_2 & -y_2 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & y_0 & 0 & 0 & y_1 & 0 \\ \hline E_0 & 0 & 0 & 0 & & & & & & & & & & & \\ 0 & E_1 & 0 & 0 & & & & & & & & & & & \\ 0 & 0 & E_2 & 0 & & & & & & & & & & & \\ 0 & 0 & 0 & E_3 & & & & & \mathbf{p} & & & & & & \\ \hline & & -\mathbf{p}^{tr} & & & & & & & & & & & & \mathbf{0} \end{array} \right)$$

with

$$E_0 = \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_0 \\ y_2 & -y_0 & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & y_3 & -y_1 \\ -y_3 & 0 & y_0 \\ y_1 & -y_0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & y_2 & -y_1 \\ -y_2 & 0 & y_0 \\ y_1 & -y_0 & 0 \end{pmatrix}.$$

For each $k \in \{0, \dots, 3\}$, the matrix E_k is the first syzygy matrix of D_k . Let $h_{k,1} < h_{k,2} < h_{k,3}$ be the indices of the three variables of degree 3 which are contained in D_k . From the definition of the matrix E_k we see that the variable $y_{h_{k,l}}$ does not appear in the l th row of E_k . Then we choose $(k, h_{k,l})$ for $k = 0, \dots, 3$ and $l = 1, 2, 3$ as row indices for the entries of \mathbf{p} and denote the set of these indices by L_1 , that is

$$L_1 = \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 2), (1, 3), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1), (3, 2)\}.$$

Next we continue with the columns of \mathbf{p} . The matrix above of \mathbf{p} is $b_3(y)$ which is the first syzygy matrix of $J = (y_i y_j \mid 0 \leq i < j \leq 3) \subseteq S$. We index the columns of $b_3(y)$ with the elements of the set

$$L_2 = \left\{ \begin{matrix} 1,2 & 0,1 & 1,3 & 0,1 & 2,3 & 0,2 & 2,3 & 1,2 \\ 0, & 2, & 0, & 3, & 0, & 3, & 1, & 3 \end{matrix} \right\}$$

and say that an element $c_e^d \in L_2$ represents the relation

$$y_c(y_d y_e) - y_d(y_c y_e). \quad (7.3)$$

Then, by the choice of the set L_2 , every relation corresponding to a column of $b_3(y)$ is represented by an element of L_2 .

Notation 7.1.5 (The matrix \mathbf{p}). Let $(k, h_{k,l}) \in L_1$ and $c_e^d \in L_2$. Then we denote the element in row $(k, h_{k,l})$ and column c_e^d of the matrix \mathbf{p} by

$$\mathbf{p}_{c_e^d}^{(k, h_{k,l})}.$$

Hence, the first row of a column index of \mathbf{p} consists of the indices (sorted by size) of the variables appearing in the corresponding column of $b_3(y)$. Note that by (7.3), the variable corresponding to the first entry has always a positive sign whereas the second entry corresponds to a variable with a negative sign.

Remark 7.1.6. Note that for any $0 \leq c < d < e \leq 3$ we have $c_e^d \in L_2$ and $d_e^c \in L_2$.

Next we consider those relations given by $d_1' d_2 = 0$ which are linear in the entries of \mathbf{a} and \mathbf{p} and see the advantages from the indexing introduced above. Recall that the matrix of relations is given by

$$d_1' d_2 = \left(\begin{array}{c|c|c} \mathbf{a}\mathbf{o} - \mathbf{b}_1(y)\mathbf{n}^{tr} - \mathbf{c}\mathbf{b}_3(y)^{tr} & \mathbf{a}\mathbf{n} - \mathbf{c}\mathbf{p}^{tr} & \mathbf{a}\mathbf{b}_3(y) + \mathbf{b}_1(y)\mathbf{p} \\ \hline -\mathbf{c}\mathbf{n}^{tr} & \mathbf{c}\mathbf{b}_4(y) - \mathbf{b}_2(y)\mathbf{p}^{tr} & \mathbf{c}\mathbf{p} \end{array} \right).$$

We will first focus on the 4×8 matrix $\mathbf{a}\mathbf{b}_3(y) + \mathbf{b}_1(y)\mathbf{p}$. By the definition of the indexing, the rows of this matrix are indexed by k for $k \in \{0, \dots, 3\}$, whereas the columns are indexed by $c_e^d \in L_2$. Let us first express the entries of the matrices $\mathbf{b}_1(y)$ and $\mathbf{b}_3(y)$ with respect to these indices. If $k \in \{0, \dots, 3\}$ and $l = (l_1, l_2) \in L_1$, then

$$b_1(y)_{k,l} = \begin{cases} 0 & \text{if } k \neq l_1, \\ y_{l_2} & \text{otherwise.} \end{cases}$$

Furthermore, for $n = (n_1, n_2) \in N$ and $\lambda = c_e^d \in L_2$, we have

$$b_3(y)_{n,\lambda} = \begin{cases} 0 & \text{if } \{n_1, n_2\} \neq \{c, e\} \text{ and } \{n_1, n_2\} \neq \{d, e\}, \\ -y_d & \text{if } \{n_1, n_2\} = \{c, e\}, \\ y_c & \text{if } \{n_1, n_2\} = \{d, e\}. \end{cases}$$

Fix an integer $k \in \{0, \dots, 3\}$ and choose 3 different integers $c, d, e \in \{0, \dots, 3\}$ representing an index $c_e^d \in L_2$. As above, let $h_{k,1}, h_{k,2}, h_{k,3}$ be the elements in $\{0, \dots, 3\} \setminus \{k\}$. The entry in row k and column c_e^d of the matrix $\mathbf{b}_1(y)\mathbf{p}$ is then given by

$$\sum_{l \in L_1} b_1(y)_{k,l} \mathbf{p}_{c_e^d}^l = \sum_{j=1}^3 y_{h_{k,j}} \mathbf{p}_{c_e^d}^{(k, h_{k,j})}.$$

The entry in row k and column c_e^d of $\mathbf{a}\mathbf{b}_3(y)$ is then given as

$$\sum_{n \in N} \mathbf{a}_n^{(k)} b_3(y)_{n, c_e^d} = \mathbf{a}_{d,e}^{(k)} y_c - \mathbf{a}_{c,e}^{(k)} y_d.$$

Putting this together we see that the entry in row k and column $\lambda = c_e^d$ of $\mathbf{a}b_3(y) + b_1(y)\mathbf{p}$ is of the form

$$\mathbf{p}_{c,d}^{(k,h_{k,1})} y_{h_{k,1}} + \mathbf{p}_{c,d}^{(k,h_{k,2})} y_{h_{k,2}} + \mathbf{p}_{c,d}^{(k,h_{k,3})} y_{h_{k,3}} + \mathbf{a}_{d,e}^{(k)} y_c - \mathbf{a}_{c,e}^{(k)} y_d.$$

In the following, we will denote this polynomial by $r_{k,\lambda}$ which we consider as a polynomial in the variables y_0, \dots, y_3 whose coefficients are entries of the matrix \mathbf{a} and \mathbf{p} . Then the condition $r_{k,\lambda} = 0$ implies that all these coefficients must be zero. Hence from one polynomial $r_{k,\lambda}$ we get 4 relations. Using these relations, we are now able to prove Proposition 7.1.3.

Proof of Proposition 7.1.3. Let $\mathbf{a}_{c,d}^{(k)}$ be an entry of the matrix \mathbf{a} with $k \notin \{c, d\}$. The idea is to find an element $\lambda \in L_2$ consisting of k, c, d with the additional property that k is contained in the first row of λ . Then the equation $r_{k,\lambda} = 0$ is one of the relations and evaluating this will prove the claim. To begin with, let us sort the three integers k, c, d by size. Since $c < d$ there are three possibilities

- (i) $k < c < d$,
- (ii) $c < k < d$,
- (iii) $c < d < k$.

In the first case we know that $k_d^{k,c} \in L_2$ by Remark 7.1.6. Hence there exists a relation

$$\mathbf{p}_{k,c}^{(k,h_{k,1})} y_{h_{k,1}} + \mathbf{p}_{k,c}^{(k,h_{k,2})} y_{h_{k,2}} + \mathbf{p}_{k,c}^{(k,h_{k,3})} y_{h_{k,3}} + \mathbf{a}_{c,d}^{(k)} y_k - \mathbf{a}_{k,d}^{(k)} y_c = 0. \quad (7.4)$$

By the definition of the elements $h_{k,l}$ we know that $k \notin \{h_{k,1}, h_{k,2}, h_{k,3}\}$. Hence the variable y_k appears exactly once in (7.4). This implies that its coefficient $\mathbf{a}_{c,d}^{(k)}$ must be zero. The remaining cases are proven in the same way. \square

As a consequence of this statement, we modify our original set-up for the matrix d_1' by assuming that

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_{0,1}^{(0)} & \mathbf{a}_{0,2}^{(0)} & 0 & \mathbf{a}_{0,3}^{(0)} & 0 & 0 \\ \mathbf{a}_{0,1}^{(1)} & 0 & \mathbf{a}_{1,2}^{(1)} & 0 & \mathbf{a}_{1,3}^{(1)} & 0 \\ 0 & \mathbf{a}_{0,2}^{(2)} & \mathbf{a}_{1,2}^{(2)} & 0 & 0 & \mathbf{a}_{2,3}^{(2)} \\ 0 & 0 & 0 & \mathbf{a}_{0,3}^{(3)} & \mathbf{a}_{1,3}^{(3)} & \mathbf{a}_{2,3}^{(3)} \end{pmatrix}.$$

Moreover, again by evaluating the relations given by $r_{k,\lambda} = 0$, we see that several entries of the matrix \mathbf{p} are also zero and that any of the remaining entries can be represented by an entry of the matrix \mathbf{a} . More precisely, we obtain the following nice characterization for the entries of \mathbf{p} .

Lemma 7.1.7. *Let $(k, h_{k,l}) \in L_1$ and $\lambda = c_e^d \in L_2$. Evaluating the relations given by $\mathbf{a}b_3(y) + b_1(y)\mathbf{p} = 0$ yields*

$$\mathbf{p}_{c,e}^{(k,h_{k,l})} = \begin{cases} 0 & \text{if } h_{k,l} \notin \{c, d\}, \\ \mathbf{a}_{c,e}^{(k)} & \text{if } h_{k,l} = d, \\ -\mathbf{a}_{d,e}^{(k)} & \text{if } h_{k,l} = c. \end{cases} \quad (7.5)$$

Proof. Clear from the definition of the polynomial $r_{k,\lambda}$ above. \square

Using this lemma, we can assume that

$$p = \begin{pmatrix} -a_{0,2}^{(0)} & a_{0,2}^{(0)} & -a_{0,3}^{(0)} & a_{0,3}^{(0)} & 0 & 0 & 0 & 0 \\ a_{0,1}^{(0)} & 0 & 0 & 0 & -a_{0,3}^{(0)} & a_{0,3}^{(0)} & 0 & 0 \\ 0 & 0 & a_{0,1}^{(0)} & 0 & a_{0,2}^{(0)} & 0 & 0 & 0 \\ 0 & -a_{1,2}^{(1)} & 0 & -a_{1,3}^{(1)} & 0 & 0 & 0 & 0 \\ a_{0,1}^{(1)} & 0 & 0 & 0 & 0 & 0 & -a_{1,3}^{(1)} & a_{1,3}^{(1)} \\ 0 & 0 & a_{0,1}^{(1)} & 0 & 0 & 0 & a_{1,2}^{(1)} & 0 \\ 0 & -a_{1,2}^{(2)} & 0 & 0 & 0 & -a_{2,3}^{(2)} & 0 & 0 \\ -a_{0,2}^{(2)} & a_{0,2}^{(2)} & 0 & 0 & 0 & 0 & 0 & -a_{2,3}^{(2)} \\ 0 & 0 & 0 & 0 & a_{0,2}^{(2)} & 0 & a_{1,2}^{(2)} & 0 \\ 0 & 0 & 0 & -a_{1,3}^{(3)} & 0 & -a_{2,3}^{(3)} & 0 & 0 \\ 0 & 0 & -a_{0,3}^{(3)} & a_{0,3}^{(3)} & 0 & 0 & 0 & -a_{2,3}^{(3)} \\ 0 & 0 & 0 & 0 & -a_{0,3}^{(3)} & a_{0,3}^{(3)} & -a_{1,3}^{(3)} & a_{1,3}^{(3)} \end{pmatrix}.$$

Note that we can write the matrix p as

$$\begin{pmatrix} p^{(0)} \\ p^{(1)} \\ p^{(2)} \\ p^{(3)} \end{pmatrix}, \quad (7.6)$$

where each matrix $p^{(k)}$ has 3 rows whose entries depend only on $a_{k,h_k,1}^{(k)}$, $a_{k,h_k,2}^{(k)}$ and $a_{k,h_k,3}^{(k)}$.

After proceeding similarly with the relations coming from $\epsilon b_4(y) - b_2(y)p^{tr} = 0$, we see that any element of the matrix ϵ is either 0 or an entry of the updated matrix a (up to a sign). Using the updated matrices a , ϵ and p , we obtain a new matrix of relations:

$$d_1' d_2 = \left(\begin{array}{c|c|c} \mathbf{a} \mathbf{o} - b_1(y) \mathbf{n}^{tr} - c b_3(y) \mathbf{t}^{tr} & \mathbf{a} \mathbf{n} - \mathbf{c} \mathbf{p}^{tr} & 0 \\ \hline -\mathbf{c} \mathbf{n}^{tr} & 0 & \mathbf{c} \mathbf{p} \end{array} \right).$$

Furthermore we reduced the problem of describing 156 unknown entries of degree 2 to describing only the 12 unknowns

$$a_{2,3}^{(3)}, a_{1,3}^{(3)}, a_{0,3}^{(3)}, a_{2,3}^{(2)}, a_{1,2}^{(2)}, a_{0,2}^{(2)}, a_{1,3}^{(1)}, a_{1,2}^{(1)}, a_{0,1}^{(1)}, a_{0,3}^{(0)}, a_{0,2}^{(0)}, a_{0,1}^{(0)}. \quad (7.7)$$

In particular, we see from the new matrix of relations that if the entries of the matrix a are known, all the remaining relations are linear in the unknowns and a solution for these relations can be computed using syzygies.

Now let us recall that a possible entry of the matrix a is a linear combination of x_0, x_1 with coefficients in \mathbb{k} . We will think of these coefficients as Stiefel coordinates:

Definition 7.1.8. Let $n \leq m$. We denote by $\text{St}(n, m)$ the *Stiefel manifold of full rank $n \times m$ matrices* (with entries in \mathbb{k} .) The entries of a matrix $l \in \text{St}(n, m)$ are called the *Stiefel coordinates* of l .

We consider $\text{St}(n, m)$ as the open subscheme of \mathbb{A}^{nm} determined by the condition that at least one of the $n \times n$ minors does not vanish. Furthermore, the group $\text{GL}(n, \mathbb{k})$ acts on $\text{St}(n, m)$ by multiplication on the right. The quotient space

$$\text{St}(n, m) / \text{GL}(n, \mathbb{k})$$

is the Grassmannian $\text{Gr}(n, m)$. Throughout this thesis we are mainly interested in the Stiefel variety $\text{St}(2, 12)$ and use the following notation:

Notation 7.1.9. For a matrix $l \in \text{St}(2, 12)$, we denote the linear space (spanned by the rows of l) by $[l] \in \text{Gr}(2, 12)$. Furthermore, considering each row of l as a point in \mathbb{P}^{11} , the matrix l defines a line in \mathbb{P}^{11} , which we denote by \underline{l} .

Remark 7.1.10. Let $l_1, l_2 \in \text{St}(2, 12)$. Then

$$[l_1] = [l_2] \text{ if and only if } \underline{l}_1 = \underline{l}_2.$$

Now a general assignment to the variables in (7.7) gives a matrix $l \in \text{St}(2, 12)$, and hence a line $\underline{l} \subseteq \mathbb{P}^{11}$. On the other hand, let $l \in \text{St}(2, 12)$ be any matrix. Taking the elements of the first row as coefficients of x_0 and the entries of the second row as coefficients of x_1 yields 12 entries for \mathfrak{a} , and hence also for \mathfrak{e} and \mathfrak{p} . In the following, we will denote the corresponding matrices by $\mathfrak{a}(l)$, $\mathfrak{e}(l)$ and $\mathfrak{p}(l)$, respectively. Our aim is to find assignments to the 12 variables in (7.7) which satisfy the given relations, that is the relations coming from $\mathfrak{e}\mathfrak{p} = 0$.

To do so, we consider the product of the updated matrices \mathfrak{e} and \mathfrak{p} which is a 3×8 matrix of the form

$$\begin{pmatrix} 0 & -q_3 & 0 & q_2 & 0 & -q_1 & 0 & 0 \\ q_3 & -q_3 & -q_2 & q_2 & 0 & 0 & 0 & -q_0 \\ 0 & 0 & q_2 & 0 & -q_1 & 0 & q_0 & 0 \end{pmatrix}$$

with

$$\begin{aligned} q_0 &= \mathfrak{a}_{1,2}^{(1)} \mathfrak{a}_{1,3}^{(1)} - \mathfrak{a}_{1,2}^{(2)} \mathfrak{a}_{2,3}^{(2)} + \mathfrak{a}_{1,3}^{(3)} \mathfrak{a}_{2,3}^{(3)}, \\ q_1 &= \mathfrak{a}_{0,2}^{(0)} \mathfrak{a}_{0,3}^{(0)} - \mathfrak{a}_{0,3}^{(3)} \mathfrak{a}_{2,3}^{(3)} + \mathfrak{a}_{0,2}^{(2)} \mathfrak{a}_{2,3}^{(2)}, \\ q_2 &= \mathfrak{a}_{0,1}^{(1)} \mathfrak{a}_{1,3}^{(1)} - \mathfrak{a}_{0,1}^{(0)} \mathfrak{a}_{0,3}^{(0)} + \mathfrak{a}_{0,3}^{(3)} \mathfrak{a}_{1,3}^{(3)}, \\ q_3 &= \mathfrak{a}_{0,1}^{(0)} \mathfrak{a}_{0,2}^{(0)} - \mathfrak{a}_{0,1}^{(1)} \mathfrak{a}_{1,2}^{(1)} + \mathfrak{a}_{0,2}^{(2)} \mathfrak{a}_{1,2}^{(2)} \end{aligned}$$

depending on the 12 variables in (7.7). Let $Q = V(q_0, \dots, q_3) \subseteq \mathbb{P}^{11}$ be the corresponding projective variety. Then with the notation introduced above, the relations coming from $\mathfrak{e}\mathfrak{p} = 0$ have an equivalent geometric description:

Lemma 7.1.11. *Let $l \in \text{St}(2, 12)$ be a matrix, and denote by $\mathfrak{e}(l)$ and $\mathfrak{p}(l)$ the corresponding matrices. Then*

$$\mathfrak{e}(l)\mathfrak{p}(l) = 0 \iff \underline{l} \subseteq Q.$$

Proof. Clear from the definition of the matrices $\mathfrak{e}(l)$, $\mathfrak{p}(l)$ and the variety $Q \subseteq \mathbb{P}^{11}$. \square

Since some of the computations involving the chosen indices were rather technical, let us briefly summarize the achievements of this section.

Summary 7.1.12 (The entries of degree 2). We have seen that in the general set-up of d'_1 and d_2 with $d'_1 d_2 = 0$, the 12 entries of the matrix \mathfrak{a} determine all the other entries of degree 2. Among all relations coming from $d'_1 d_2 = 0$ there are exactly 4 (quadratic) relations involving only the variables in (7.7). Furthermore, in Lemma 7.1.11 we have seen that solving these quadratic relations is equivalent to finding a line $\underline{l} \subseteq Q$.

7.2 Finding Lines in Q

In the previous section we have seen that the variety $Q \subseteq \mathbb{P}^{11}$ plays a central role for describing or constructing standard resolutions. In this section we study the 4 quadrics q_0, \dots, q_3 defining the variety Q . We will see that q_0, \dots, q_3 are Pfaffians of some skew-symmetric matrices. In the end, we will use the special form of the quadrics to describe a procedure for computing lines in Q .

Lemma 7.2.1. *The variety $Q \subseteq \mathbb{P}^{11}$ is an irreducible complete intersection.*

Proof. Using SINGULAR ([DGPS18]) we see that Q is irreducible and that $\dim Q = 7 = 11 - 4$. Hence Q is an irreducible complete intersection. \square

If Q were smooth, it would be a Fano variety of index 4 and would have Kodaira dimension $-\infty$. However, using SINGULAR again, we compute that Q is highly singular with a 3-dimensional reducible singular locus.

7.2.1 Pfaffians

Before studying the single quadratic forms q_i , let us briefly introduce the notion of Pfaffians and some properties. Throughout this section, M denotes a skew-symmetric matrix of order $2n$ defined over a commutative ring. It is well-known that $\det(M)$ is a square of a polynomial in the entries of M . We call this polynomial the *Pfaffian* of M , denoted by $\text{pf}(M)$. There is an explicit expression of $\text{pf}(M)$ in the entries of M .

Lemma 7.2.2. *Let M be as above. Then*

$$\text{pf}(M) = \sum_{\sigma \in \Lambda} \text{sign}(\sigma) m_{i_1 i_2} m_{i_3 i_4} \cdots m_{i_{2n-1} i_{2n}},$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2n \\ i_1 & i_2 & \cdots & i_{2n} \end{pmatrix}$$

and Λ is the set of all permutations in \mathcal{S}_{2n} with

$$i_1 < i_2, i_3 < i_4, \dots, i_{2n-1} < i_{2n}, i_1 < i_3 < \cdots < i_{2n-1}.$$

For example, if $n = 2$ and

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & m_{24} & -m_{34} & 0 \end{pmatrix},$$

then

$$\text{pf}(M) = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}. \quad (7.8)$$

In the following we will write skew-symmetric matrices in a shorter way, where we omit the

entries below the diagonal:

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} \\ & 0 & m_{23} & m_{24} \\ & & 0 & m_{34} \\ & & & 0 \end{pmatrix}.$$

Note that for a given polynomial p which is the Pfaffian of a skew-symmetric matrix, there are in general several skew-symmetric matrices having this polynomial as a Pfaffian:

Proposition 7.2.3. *Let M be a skew-symmetric matrix of order $2n$. If $B \in \text{GL}(2n, \mathbb{k})$ with $\det(B) = 1$, then*

$$\text{pf}(M) = \text{pf}(BMB^T).$$

Now let us return to our 4 quadrics

$$q_0 = \mathbf{a}_{1,2}^{(1)}\mathbf{a}_{1,3}^{(1)} - \mathbf{a}_{1,2}^{(2)}\mathbf{a}_{2,3}^{(2)} + \mathbf{a}_{1,3}^{(3)}\mathbf{a}_{2,3}^{(3)},$$

$$q_1 = \mathbf{a}_{0,2}^{(0)}\mathbf{a}_{0,3}^{(0)} - \mathbf{a}_{0,3}^{(3)}\mathbf{a}_{2,3}^{(3)} + \mathbf{a}_{0,2}^{(2)}\mathbf{a}_{2,3}^{(2)},$$

$$q_2 = \mathbf{a}_{0,1}^{(1)}\mathbf{a}_{1,3}^{(1)} - \mathbf{a}_{0,1}^{(0)}\mathbf{a}_{0,3}^{(0)} + \mathbf{a}_{0,3}^{(3)}\mathbf{a}_{1,3}^{(3)},$$

$$q_3 = \mathbf{a}_{0,1}^{(0)}\mathbf{a}_{0,2}^{(0)} - \mathbf{a}_{0,1}^{(1)}\mathbf{a}_{1,2}^{(1)} + \mathbf{a}_{0,2}^{(2)}\mathbf{a}_{1,2}^{(2)}.$$

Comparing them with Equation (7.8) we see that each q_i is the Pfaffian of a skew-symmetric 4×4 matrix. A possible choice of the matrices is

$$M_0 = \begin{pmatrix} 0 & \mathbf{a}_{1,2}^{(1)} & \mathbf{a}_{1,2}^{(2)} & \mathbf{a}_{1,3}^{(3)} \\ & 0 & \mathbf{a}_{2,3}^{(3)} & \mathbf{a}_{2,3}^{(2)} \\ & & 0 & \mathbf{a}_{1,3}^{(1)} \\ & & & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \mathbf{a}_{0,2}^{(0)} & \mathbf{a}_{0,3}^{(3)} & \mathbf{a}_{0,2}^{(2)} \\ & 0 & \mathbf{a}_{2,3}^{(2)} & \mathbf{a}_{2,3}^{(3)} \\ & & 0 & \mathbf{a}_{0,3}^{(0)} \\ & & & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & \mathbf{a}_{0,1}^{(1)} & \mathbf{a}_{0,1}^{(0)} & \mathbf{a}_{0,3}^{(3)} \\ & 0 & \mathbf{a}_{1,3}^{(3)} & \mathbf{a}_{0,3}^{(0)} \\ & & 0 & \mathbf{a}_{1,3}^{(1)} \\ & & & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & \mathbf{a}_{0,1}^{(0)} & \mathbf{a}_{0,1}^{(1)} & \mathbf{a}_{0,2}^{(2)} \\ & 0 & \mathbf{a}_{1,2}^{(2)} & \mathbf{a}_{1,2}^{(1)} \\ & & 0 & \mathbf{a}_{0,2}^{(0)} \\ & & & 0 \end{pmatrix}.$$

7.2.2 A Las Vegas Algorithm for Computing Lines in Q

Let us now present an algorithm for computing lines which are completely contained in $Q = V(q_0, \dots, q_3) \subseteq \mathbb{P}^{11}$. The idea of the procedure is the following. First choose a nonsingular point p of Q . Then compute the variety $Z = Q \cap T_p Q$ which is a cone with vertex p over a surface in \mathbb{P}^6 . Afterwards choose a second point $q \in Z$ and compute the line \overline{pq} . The correctness of the algorithm follows then from the following statement:

Proposition 7.2.4. *Let $p \in Q$ be a nonsingular point which is also nonsingular for every $Q_i = V(q_i)$. Let $q \in Q \cap T_p Q$ with $q \neq p$. Then $\overline{pq} \subseteq Q$.*

Since $Q = Q_0 \cap \cdots \cap Q_3$ and $T_p Q = T_p Q_0 \cap \cdots \cap T_p Q_3$, the proof is a direct consequence of the following lemma:

Lemma 7.2.5. *Let p be a nonsingular point of Q_i , and let $Z_i = Q_i \cap T_p Q_i$. Let $q \in Z_i$ with $q \neq p$. Then $\overline{pq} \subseteq Q_i$.*

Proof. Let $\ell = \overline{pq}$. Suppose that ℓ is not contained in Q . Then we know from Bézout's theorem that

$$2 = \deg(Q_i) \deg(\ell) = \sum_{x \in Q_i \cap \ell} i(x; Q_i, \ell) \geq i(p; Q_i, \ell) + i(q; Q_i, \ell) \geq 2 + 1 = 3,$$

where $i(-; Q_i, \ell)$ denotes the intersection multiplicity of Q_i and ℓ at a point. But this is a contradiction. Hence $\ell \subseteq Q_i$. \square

Before settling the question how to find a point $p \in Q$, possibly after extending our base field, let us first make some general observations on varieties defined by Pfaffians. Let M denote a general skew-symmetric 4×4 matrix as in the previous section, and let

$$A = \mathbb{k}[m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}].$$

Then $V(\text{pf}(M)) \subseteq \text{Proj}(A) = \mathbb{P}^5$ is a projective variety. Let $0 \neq c = (c_1, \dots, c_4)^{tr} \in \mathbb{k}^4$. Since

$$c^{tr} M c = 0,$$

the vector $M c$ has in general three linearly independent linear entries. For example, if $c_4 \neq 0$, then the first three entries

$$\begin{aligned} b_0 &= c_2 m_{12} + c_3 m_{13} + c_4 m_{14}, \\ b_1 &= -c_1 m_{12} + c_3 m_{23} + c_4 m_{24}, \\ b_2 &= -c_1 m_{13} - c_2 m_{23} + c_4 m_{34} \end{aligned}$$

are linearly independent. Furthermore we have

$$c_1 c_4 \text{pf}(M) = c_1 m_{23} b_0 + (c_2 m_{23} - c_4 m_{34}) b_1 + (c_3 m_{23} + c_4 m_{24}) b_2.$$

Hence, if $c_1 c_4 \neq 0$, then

$$(\text{pf}(M)) \subseteq (b_0, b_1, b_2) \subseteq A. \quad (7.9)$$

We call an element $c \in \mathbb{k}^4$ yielding three linearly independent forms whose ideal contains $\text{pf}(M)$ a *random kernel* for M .

Remark 7.2.6. The word kernel refers to the fact that

$$\begin{aligned} p \in V(\text{pf}(M)) \subseteq \mathbb{P}^5 &\iff \text{rank}(M(p)) < 4 \\ &\iff \text{there exists } 0 \neq c \in \mathbb{k}^4: M(p)c = 0. \end{aligned}$$

Furthermore, since any skew-symmetric matrix has even rank we have

$$p \in V(\text{pf}(M)) \iff \text{rank}(M(p)) = 2.$$

Lemma 7.2.7. *Let $p, q \in V(\text{pf}(M))$ be two different points such that $\text{syz}(M(p)) \cap \text{syz}(M(q))$ is non-trivial. Then $\overline{pq} \subseteq V(\text{pf}(M))$.*

Proof. We have to show that $\text{rank}(M(\lambda p + \mu q)) < 4$ for any $(\lambda : \mu) \in \mathbb{P}^1$. Since $\text{rank}(M(p)) < 4$ and $\text{rank}(M(q)) < 4$ we may assume that $\lambda\mu \neq 0$. Then

$$\begin{aligned} \text{syz}(M(\lambda p + \mu q)) &= \text{syz}(\lambda M(p) + \mu M(q)) \\ &\supseteq \text{syz}(\lambda M(p)) \cap \text{syz}(\mu M(q)) = \text{syz}(M(p)) \cap \text{syz}(M(q)) \\ &\supseteq 0. \end{aligned}$$

Thus $\text{rank}(\lambda M(p) + \mu M(q)) = \text{rank}(M(\lambda p + \mu q)) < 4$. \square

Let us return to our variety $Q \subseteq \mathbb{P}^{11}$. We treat first the case $\mathbb{k} = \mathbb{Q}$. For each $i = 0, 1, 2$ choose a random kernel c_i for M_i such that the nine linear equations define a 2-dimensional space $\Lambda \cong \mathbb{P}^2$. Then $\Lambda \subseteq Q_0 \cap Q_1 \cap Q_2$ by (7.9) and

$$\Lambda \cap Q = \underbrace{\Lambda \cap Q_3}_{=\tilde{Q}_3} \subseteq \Lambda \cong \mathbb{P}^2.$$

The variety \tilde{Q}_3 is defined by a quadratic polynomial in \mathbb{P}^2 and thus is a conic. But finding rational points on a conic is a well-studied problem. We will use the SINGULAR-procedure `rationalPointConic` to choose a point $\tilde{p} \in \tilde{Q}_3$. If there exists a rational point (and hence infinitely many), the algorithm will find one. If there is no such point, the procedure computes a field extension of \mathbb{Q} of degree 2 and chooses a point in this field extension. Let $p \in Q \subseteq \mathbb{P}^{11}$ be the corresponding point. If necessary we repeat the procedure above until we get a nonsingular point $p \in Q$.

As a second step, by Proposition 7.2.4, we have to choose a point $q \in Q \cap T_p Q$. We have $T_p Q \cong \mathbb{P}^7$ and

$$Z = Q \cap T_p Q \subset T_p Q \cong \mathbb{P}^7$$

has expected dimension 3. Let $V(\bar{q}_i) = \bar{Q}_i = Q_i \cap T_p(Q) \subset \mathbb{P}^7$ and let \bar{M}_i be the corresponding matrix such that $\bar{q}_i = \text{pf}(\bar{M}_i)$. Next we do a similar trick as before. Choose an integer $j \in \{0, \dots, 3\}$ and let d_j be a random kernel for \bar{M}_j . By Θ we denote the 4-dimensional linear subspace of \mathbb{P}^7 defined by the three resulting linear equations. Then

$$C_j = \Theta \cap \bar{Q}_0 \cap \dots \cap \bar{Q}_3 \subset \Theta \cong \mathbb{P}^4$$

is a curve of degree 8. Intersecting with some general hyperplane H we get a 0-dimensional variety of degree 8, hence 8 points in a \mathbb{P}^3 which are not necessarily defined over the ground field. We use the SINGULAR-procedure `absPrimdecGTZ` to separate one of these points, possibly over a finite field extension of degree at most 8. Let q be the corresponding point in $Q \subseteq \mathbb{P}^{11}$. Then $\ell = \overline{pq}$ is contained in $Q \subseteq \mathbb{P}_{\mathbb{Q}(\alpha)}^{11}$ as desired, where $\mathbb{Q}(\alpha)$ is a finite field extension of \mathbb{Q} of degree at most 16. Let us summarize the results from this section:

Proposition 7.2.8. *Let $\mathbb{k} = \mathbb{Q}$. There is a Las Vegas algorithm computing lines in Q defined over a finite field extension of \mathbb{Q} of degree at most 16.*

Remark 7.2.9. Varying the random kernels in the first step of the algorithm leads to different conics in \mathbb{P}^2 , and hence we may perform this first step repeatedly to find a conic having a \mathbb{k} -rational point. Thus, in practice, the computed line is usually defined over a field extension of \mathbb{Q} of degree at most 8.

Remark 7.2.10. If $\mathbb{k} = \mathbb{F}_p$ we use similar ideas but proceed simply by trial and error. First we choose random kernels for each skew-symmetric matrix M_i . This yields 12 linear forms which are in general linearly independent. We repeatedly choose kernels until we get only 11

independent linear forms and hence, by the definition of a random kernel, a point p in Q . Then by Remark 7.2.6 there exist $0 \neq c_{i,1}, c_{i,2} \in \mathbb{k}^4$ such that $M_i(p)c_{i,j} = 0$ for $i = 0, \dots, 3$, $j = 1, 2$. Now, by Lemma 7.2.7, as a new random kernel we take random linear combinations of $c_{i,1}$ and $c_{i,2}$. This yields in general 11 independent forms. We repeat this until we get 10 linear independent forms, and hence a line ℓ through p in Q .

7.3 The Fano Scheme of Lines $F_1(Q)$

After having seen how to compute lines in Q , we will now study the scheme whose points correspond to these lines.

Definition 7.3.1. Let X be a projective scheme. We call the Hilbert scheme $\text{Hilb}_{1+t}(X)$ the *Fano scheme of lines* in X , denoted by $F_1(X)$.

Let f_r be the Hilbert polynomial of an r -dimensional linear subspace of \mathbb{P}^n . Then

$$\text{Hilb}_{f_r}(\mathbb{P}^n) \cong \text{Gr}(r, \mathbb{P}^n) = \text{Gr}(r+1, n+1).$$

Hence, if $X \subseteq \mathbb{P}^n$, then we will consider $F_1(X)$ as a subscheme of the Grassmannian $\text{Gr}(r+1, n+1)$. In particular, for $X = Q$, we obtain $F_1(Q) \subseteq \text{Gr}(2, 12)$.

Recall that we have a quotient map $\text{St}(2, 12) \rightarrow \text{Gr}(2, 12)$. We define the set

$$\text{St}(Q) := \{l \in \text{St}(2, 12) \mid [l] \in F_1(Q)\}$$

which is the preimage of $F_1(Q)$ under this quotient map, and hence a closed subset of $\text{St}(2, 12)$. Any line $\ell \subseteq Q$ gives a point in $F_1(Q)$ which we denote by $[\ell]$. Note that, if $\ell = \underline{\ell}$ for some $l \in \text{St}(Q)$, then $[\ell] = [l] \in F_1(Q)$ (see Notation 7.1.9). Moreover, for $l \in \text{St}(2, 12)$ we have

$$l \in \text{St}(Q) \text{ if and only if } \underline{l} \subseteq Q.$$

7.3.1 Generators of $F_1(Q)$

In this subsection we determine defining equations of $F_1(Q)$ locally in open sets of an affine cover of $F_1(Q)$. For the sake of simplicity, we will denote the coordinates of \mathbb{P}^{11} now by t_0, \dots, t_{11} using the substitution

$$(\mathfrak{a}_{2,3}^{(3)}, \mathfrak{a}_{1,3}^{(3)}, \mathfrak{a}_{0,3}^{(3)}, \mathfrak{a}_{2,3}^{(2)}, \mathfrak{a}_{1,2}^{(2)}, \mathfrak{a}_{0,2}^{(2)}, \mathfrak{a}_{1,3}^{(1)}, \mathfrak{a}_{1,2}^{(1)}, \mathfrak{a}_{0,1}^{(1)}, \mathfrak{a}_{0,3}^{(0)}, \mathfrak{a}_{0,2}^{(0)}, \mathfrak{a}_{0,1}^{(0)}) \mapsto (t_0, \dots, t_{11}).$$

First note that

$$F_1(Q) = \bigcap_{i=0}^3 F_1(Q_i)$$

and hence it is enough to describe the Fano scheme of a single hypersurface Q_i . Via the Plücker embedding we can consider $\text{Gr}(2, 12)$ as a subvariety of \mathbb{P}^{65} with coordinates $v_{i,j}$ for $0 \leq i < j \leq 11$. Let $U_{i,j}$ be the open set on which the $v_{i,j}$ -coordinate is non-zero. Then $U_{i,j} \cong \mathbb{A}^{20}$ and we can represent an element of $\text{Gr}(2, 12) \cap U_{i,j}$ by a unique matrix in $\text{St}(2, 12)$ whose 2×2 submatrix with columns i, j is the identity matrix. Using the substitution above, the 4 quadratic

relations are of the form

$$\begin{aligned} \mathfrak{q}_0 &= t_0t_1 - t_3t_4 + t_6t_7, \\ \mathfrak{q}_1 &= t_3t_5 - t_0t_2 + t_9t_{10}, \\ \mathfrak{q}_2 &= t_6t_8 - t_9t_{11} + t_1t_2, \\ \mathfrak{q}_3 &= t_4t_5 - t_7t_8 + t_{10}t_{11}. \end{aligned}$$

Now we replace the variables t_i by $\varrho_{i,0}x_0 + \varrho_{i,1}x_1$, where $\varrho_{i,0}$ and $\varrho_{i,1}$ are the coordinates of \mathbb{A}^{24} . We consider the resulting polynomials as homogeneous polynomials in x_0^2, x_0x_1, x_1^2 with coefficients depending on $\varrho_{i,0}$ and $\varrho_{i,1}$. These coefficients define in total 12 quadratic homogeneous relations and hence a closed subset $W_Q \subseteq \mathbb{A}^{24}$ with $\text{St}(Q) = W_Q \cap \text{St}(2, 12)$. Then $F_1(Q) \cap U_{i,j} \subseteq U_{i,j}$ is defined by these 12 forms in the affine subspace $V(\varrho_{i,0} - 1, \varrho_{j,1} - 1, \varrho_{j,0}, \varrho_{i,1}) \cong \mathbb{A}^{20}$. In particular, we see that on every (affine) open set $U_{i,j} \cong \mathbb{A}^{20}$, $F_1(Q)$ is also defined by quadratic polynomials. Furthermore, with the help of SINGULAR we compute that $\dim(F_1(Q) \cap U_{i,j}) = 8$ for all $0 \leq i < j \leq 11$. Since $(F_1(Q) \cap U_{i,j})_{i,j}$ is an affine cover of $F_1(Q)$ this implies:

Lemma 7.3.2. *The scheme $F_1(Q)$ is 8-dimensional.*

The next question is whether $F_1(Q)$ is irreducible or not. Unfortunately we were not able to test the irreducibility with any of the computer algebra systems SINGULAR, MACAULAY2 or MAGMA. With the help of the software BERTINI ([BHSW]) which uses numerical homotopy continuation methods we computed that $F_1(Q)$ is irreducible. However these computational results are not certified.

7.3.2 Local Properties of $F_1(Q)$

The goal of this subsection is to study $F_1(Q)$ locally. More precisely, we are interested in conditions under which a line $\ell \subseteq Q$ leads to a smooth point $[\ell]$ of $F_1(Q)$. To begin with, let us state some preliminary results on normal sheaves and complete intersections which we use in the following.

Definition 7.3.3. Let X be a scheme over \mathbb{k} , and let $Y \subseteq X$ be a closed subscheme with ideal sheaf \mathcal{I} . The sheaf $\mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is called the *normal sheaf* of Y in X and is denoted by $\mathcal{N}_{Y/X}$. If X is smooth and Y is a locally complete intersection, then $\mathcal{N}_{Y/X}$ is locally free and is called the *normal bundle* of Y in X .

Proposition 7.3.4 ([EH16], Proposition 6.15). *If $Z \subseteq Y \subseteq X$ are schemes such that Y is a locally complete intersection in X , then there is an exact sequence of sheaves*

$$0 \rightarrow \mathcal{N}_{Z/Y} \rightarrow \mathcal{N}_{Z/X} \xrightarrow{\alpha} \mathcal{N}_{Y/X}|_Z.$$

Moreover, if all schemes are smooth, then α is an epimorphism.

Later we will apply this statement to the varieties $\ell \subseteq Q \subseteq \mathbb{P}^{11}$ and show that the morphism α is surjective if and only if Q is smooth along the line ℓ .

Corollary 7.3.5 ([EH16], Corollary 6.16). *Suppose $Z \subseteq Y \subseteq \mathbb{P}^n$ are (not necessarily smooth) complete intersections of hypersurfaces with homogeneous ideals*

$$I_Y = (f_1, \dots, f_t) \subseteq I_Z = (g_1, \dots, g_s), \quad f_i = \sum_{j=1}^s a_{i,j}g_j.$$

If $\deg(f_i) = \gamma_i$ and $\deg(g_j) = \delta_j$, then

$$\mathcal{N}_{Y/\mathbb{P}^n} = \bigoplus_{i=1}^t \mathcal{O}_Y(\gamma_i), \quad \mathcal{N}_{Z/\mathbb{P}^n} = \bigoplus_{j=1}^s \mathcal{O}_Z(\delta_j)$$

and $\mathcal{N}_{Z/Y}$ is the kernel of the morphism

$$\alpha: \mathcal{N}_{Z/\mathbb{P}^n} \rightarrow \mathcal{N}_{Y/\mathbb{P}^n}|_Z$$

given by the matrix $\bar{a} = (\bar{a}_{i,j})$, where $\bar{a}_{i,j}$ denotes the restriction of $a_{i,j}$ to Z .

Let us now apply these results to our setting. Let $\ell \subseteq Q$ be a fixed line. Then there is an exact sequence

$$0 \rightarrow \mathcal{N}_{\ell/Q} \rightarrow \mathcal{N}_{\ell/\mathbb{P}^{11}} \xrightarrow{\alpha} \mathcal{N}_{Q/\mathbb{P}^{11}}|_{\ell}$$

which is the sequence

$$0 \rightarrow \mathcal{N}_{\ell/Q} \rightarrow \bigoplus_{i=1}^{10} \mathcal{O}_{\ell}(1) \xrightarrow{\alpha} \bigoplus_{i=1}^4 \mathcal{O}_{\ell}(2) \quad (7.10)$$

by the previous corollary. Hence, as any subsheaf of a locally free sheaf on a smooth curve, $\mathcal{N}_{\ell/Q}$ is locally free as well. So in our case, where this curve is isomorphic to \mathbb{P}^1 ,

$$\mathcal{N}_{\ell/Q} \cong \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(a_i)$$

for some $m \leq 10$ and integers $a_i \leq 1$. Recall that the singular locus of Q is a 3-dimensional reducible variety. So a general line $\ell \subseteq Q$ will not meet the singular locus. Let $Q_{reg} = Q \setminus \text{Sing}(Q)$ be the open set of regular points of Q .

Proposition 7.3.6. *The map α in sequence (7.10) is surjective if and only if Q is smooth along ℓ . In this case, $\mathcal{N}_{\ell/Q}$ is a locally free sheaf of rank 6 and degree 2.*

Proof. The proof is a generalization of the case where ℓ is contained in a single hypersurface H (see [EH16], Proposition 6.24). We prove that the morphism of stalks α_p is surjective for all $p \in \ell$. For this, after a linear change of coordinates, we may assume that ℓ is defined by the forms t_2, \dots, t_{11} . We denote the 4 quadratic generators of $I(Q)$ still by q_0, \dots, q_3 . Since

$$(q_0, \dots, q_3) = I(Q) \subseteq I(\ell) = (t_2, \dots, t_{11})$$

we can find for each i unique linear forms $a_{i,j}$ depending only on t_0, t_1 and a quadratic form $h_i \in (t_2, \dots, t_{11})^2$ such that

$$q_i = \sum_{j=2}^{11} a_{i,j} t_j + h_i.$$

By Corollary 7.3.5, the map α is given by the matrix $(a_{i,j})_{\substack{0 \leq i \leq 3, \\ 2 \leq j \leq 11}}$ (considered as forms on ℓ).

On the other hand, the Jacobian of Q is the matrix

$$\begin{pmatrix} \sum_{j=2}^{11} \frac{\partial a_{0,j}}{\partial t_0} t_j & \sum_{j=2}^{11} \frac{\partial a_{0,j}}{\partial t_1} t_j & a_{0,2} + \frac{\partial h_0}{\partial t_2} & \cdots & a_{0,11} + \frac{\partial h_0}{\partial t_{11}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=2}^{11} \frac{\partial a_{3,j}}{\partial t_0} t_j & \sum_{j=2}^{11} \frac{\partial a_{3,j}}{\partial t_1} t_j & a_{3,2} + \frac{\partial h_3}{\partial t_2} & \cdots & a_{3,11} + \frac{\partial h_3}{\partial t_{11}} \end{pmatrix}.$$

In particular, the restriction of the Jacobian to ℓ is given by

$$\begin{pmatrix} 0 & 0 & a_{0,2} & \cdots & a_{0,11} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{3,2} & \cdots & a_{3,11} \end{pmatrix}. \quad (7.11)$$

Let $p \in \ell \subseteq Q$. Then Q is smooth at p if and only if there exists a 4×4 minor of (7.11) which does not vanish at p . But this is equivalent to the fact that the matrix corresponding to α_p has an invertible 4×4 minor, and hence that α_p is surjective. \square

Thus if $\ell \subseteq Q_{\text{reg}}$, then

$$\mathcal{N}_{\ell/Q} \cong \bigoplus_{i=1}^6 \mathcal{O}_{\mathbb{P}^1}(a_i) \quad (7.12)$$

is a locally free sheaf of rank 6 and degree $2 = \sum_{i=1}^6 a_i$. The integers a_1, \dots, a_6 determine the splitting type of $\mathcal{N}_{\ell/Q}$:

Definition 7.3.7. Let $\mathcal{F} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(c_i)$ be a locally free sheaf on \mathbb{P}^1 of rank r with $c_1 \geq \dots \geq c_r$. The integers c_1, \dots, c_r are called the *splitting type* of \mathcal{F} . We call \mathcal{F} *balanced* if $|c_i - c_j| \leq 1$ for all i, j .

Returning to our normal bundle $\mathcal{N}_{\ell/Q}$ for $\ell \subseteq Q_{\text{reg}}$, we see that there are only finitely many possible splitting types for $\mathcal{N}_{\ell/Q}$:

a_1	a_2	a_3	a_4	a_5	a_6
1	1	0	0	0	0
1	1	1	0	0	-1
1	1	1	1	-1	-1
1	1	1	1	0	-2
1	1	1	1	1	-3

Note that only the first row leads to a balanced vector bundle on \mathbb{P}^1 . Since being balanced is an open condition in a family of vector bundles on \mathbb{P}^1 with a fixed rank and degree, we can find an open set $U_{\text{bal}} \subseteq F_1(Q)$ such that

$$\mathcal{N}_{\ell/Q} \cong \bigoplus_{i=1}^2 \mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}$$

for any $[\ell] \in U_{\text{bal}}$. Moreover, with the help of the Las Vegas algorithm from the last section, we compute a line ℓ fulfilling this property. Hence U_{bal} is non-empty.

Now let $\ell \subseteq Q_{\text{reg}}$ be a line. The scheme $F_1(Q)$ is smooth at the point $[\ell]$ if and only if

$$\dim_{[\ell]} F_1(Q) = \dim T_{F_1(Q), [\ell]},$$

where $T_{F_1(Q), [\ell]}$ denotes the Zariski tangent space to $F_1(Q)$ at $[\ell]$. We have:

Theorem 7.3.8. Let X be a projective scheme over \mathbb{k} and let $Y \subseteq X$ be a closed subscheme with ideal sheaf \mathcal{I} and Hilbert polynomial f .

(i) There exists an isomorphism of \mathbb{k} -vector spaces

$$T_{H,[Y]} \cong H^0(Y, \mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)) = H^0(Y, \mathcal{N}_{Y/X})$$

where $T_{H,[Y]}$ is the Zariski tangent space to $H = \text{Hilb}_f(X)$ at the point $[Y]$. In particular,

$$\dim_{[Y]} H \leq h^0(Y, \mathcal{N}_{Y/X}).$$

(ii) If $Y \subseteq X$ is a local complete intersection then

$$\dim_{[Y]} H \geq h^0(Y, \mathcal{N}_{Y/X}) - h^1(Y, \mathcal{N}_{Y/X}).$$

In particular, if $h^1(Y, \mathcal{N}_{Y/X}) = 0$, then H is smooth at $[Y]$ of dimension $h^0(Y, \mathcal{N}_{Y/X})$.

Proof. [EH16], Theorem 6.21, [Kol13], Theorem I.2.15. □

Using this result, we get the following statement on smooth points of $F_1(Q)$:

Proposition 7.3.9. *Let $\ell \subseteq Q_{\text{reg}} \subseteq Q$ be a line with normal bundle*

$$\mathcal{N}_{\ell/Q} \cong \bigoplus_{i=1}^6 \mathcal{O}_{\mathbb{P}^1}(a_i),$$

$a_1 \geq a_2 \geq \dots \geq a_6$. If $a_6 \geq -1$, then $F_1(Q)$ is smooth at $[\ell]$ with $\dim_{[\ell]} F_1(Q) = 8$. In particular, for any line $\ell \subseteq Q_{\text{reg}}$ such that $[\ell] \in U_{\text{bal}}$, the point $[\ell]$ is a smooth point of $F_1(Q)$.

Proof. Using Theorem 7.3.8 it is enough to show that $h^1(\ell, \mathcal{N}_{\ell/Q}) = 0$ which is trivially satisfied if $a_6 \geq -1$. Moreover, using again Theorem 7.3.8, we have

$$\dim_{[\ell]} F_1(Q) = h^0(\ell, \mathcal{N}_{\ell/Q}) = 8. \quad \square$$

Note that for any $\ell \subseteq Q_{\text{reg}} \subseteq Q$ we have

$$h^0(\ell, \mathcal{N}_{\ell/Q}) - h^1(\ell, \mathcal{N}_{\ell/Q}) = 8. \quad (7.13)$$

For the sake of completeness, we will also give a formula for the left-hand side of (7.13) for the case where ℓ meets the singular locus of Q . Then we have an exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{N}_{\ell/Q} \rightarrow \bigoplus_{i=1}^{10} \mathcal{O}_{\ell}(1) \xrightarrow{\alpha} \bigoplus_{i=1}^4 \mathcal{O}_{\ell}(2) \rightarrow \mathcal{F} \rightarrow 0,$$

where $\mathcal{F} = \text{coker } \alpha$. The sheaf \mathcal{F} is supported on $\ell \cap \text{Sing}(Q)$. In particular, if $\ell \not\subseteq \text{Sing}(Q)$, then \mathcal{F} is supported on finitely many points. Splitting the above long exact sequence in two short exact sequences

$$0 \rightarrow \mathcal{N}_{\ell/Q} \rightarrow \bigoplus_{i=1}^{10} \mathcal{O}_{\ell}(1) \rightarrow \mathcal{E} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{E} \rightarrow \bigoplus_{i=1}^4 \mathcal{O}_{\ell}(2) \rightarrow \mathcal{F} \rightarrow 0.$$

Taking global sections, we obtain

$$\begin{aligned} 0 \rightarrow H^0(\ell, \mathcal{N}_{\ell/Q}) &\rightarrow H^0(\ell, \bigoplus_{i=1}^{10} \mathcal{O}_{\ell}(1)) \rightarrow H^0(\ell, \mathcal{E}) \\ &\rightarrow H^1(\ell, \mathcal{N}_{\ell/Q}) \rightarrow 0 \rightarrow H^1(\ell, \mathcal{E}) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow H^0(\ell, \mathcal{E}) &\rightarrow H^0(\ell, \bigoplus_{i=1}^4 \mathcal{O}_{\ell}(2)) \rightarrow H^0(\ell, \mathcal{F}) \\ &\rightarrow H^1(\ell, \mathcal{E}) \rightarrow 0. \end{aligned}$$

Thus

$$h^0(\ell, \mathcal{N}_{\ell/Q}) - h^1(\ell, \mathcal{N}_{\ell/Q}) = 8 + h^0(\ell, \mathcal{F}).$$

7.4 Syzygies of the Matrix α

In Section 7.1 we have seen that we can assign to any matrix $l \in \text{St}(2, 12)$ a matrix $\alpha(l)$ whose entries are linear forms in x_0, x_1 . In this section, we study properties of the module $L_l = \text{coker } \alpha(l)$ as a $B = \mathbb{k}[x_0, x_1]$ -module. Our aim is to show that there is an open subset $V_{\text{gensyz}} \subseteq \text{St}(Q)$ such that for any $l \in V_{\text{gensyz}}$, the module L_l has the expected Betti numbers. Then, in the next section, we will further simplify the relations given by $d_1^l d_2 = 0$ for matrices $l \in V_{\text{gensyz}}$.

A priori, the following Betti tables for L_l are possible:

Lemma 7.4.1. *Let $\tilde{\alpha}: B^4 \leftarrow B(-1)^6$ be a homogeneous homomorphism of rank 4. Then the Betti table of $L = \text{coker } \tilde{\alpha}$ is one of the following:*

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 4 & 4 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 5 & 1 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 5 & . \\ 1 & . & . & 1 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 5 & . \\ 2 & . & . & 1 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 5 & . \\ 1 & . & . & . \\ 2 & . & . & . \\ 3 & . & . & 1 \end{array}$$

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 6 & 2 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 6 & 1 \\ 1 & . & . & 1 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 6 & . \\ 1 & . & . & 2 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 6 & 1 \\ 1 & . & . & . \\ 2 & . & . & 1 \end{array}$$

Proof. Since $\text{rank}(\tilde{a}) = 4$, the minimal free resolution of L is of the form

$$\begin{array}{ccccccc}
& & & & B(-2)^{m_2} & & \\
& & & & \oplus & & \\
& & & & B(-3)^{m_2} & & \\
0 \leftarrow L \leftarrow B^4 \leftarrow B(-1)^{4+m} \leftarrow & \oplus & \leftarrow 0, & & & & (7.14) \\
& & & & B(-4)^{m_4} & & \\
& & & & \oplus & & \\
& & & & \vdots & &
\end{array}$$

where $0 \leq m \leq 2$ and $\sum_{i \geq 2} m_i = m$ are integers. Then, for each choice of m , there are only finitely many choices for the integers m_i such that the above sequence is a free resolution of L . Indeed, for $n \gg 0$ we have

$$\begin{aligned}
0 \leq \dim_{\mathbb{k}}(L)_n &= 4(n+1) - (4+m)n + m_2(n-1) + m_3(n-2) + \cdots \\
&= (-m + \sum_{i \geq 2} m_i)n + (4 - \sum_{i \geq 2} (i-1)m_i) \\
&= 4 - \sum_{i \geq 2} (i-1)m_i.
\end{aligned}$$

Now distinguishing between the cases $m = 0, 1$ and 2 , we see that there exist only finitely many choices for the integers m_i such that the right-hand side is non-negative. These possibilities are exactly the Betti numbers as claimed above. Note that the Betti tables in the first row correspond to minimal free resolutions with $m = 0$ or $m = 1$, whereas the Betti tables in the second row correspond to the possible choices having $m = 2$. \square

Remark 7.4.2. Note that for any two matrices $l_1, l_2 \subseteq \text{St}(2, 12)$ with $[l_1] = [l_2] \in \text{Gr}(2, 12)$ we have $\text{rank}(\mathfrak{a}(l_1)) = \text{rank}(\mathfrak{a}(l_2))$. Furthermore, L_{l_1} and L_{l_2} have the same Betti numbers. Indeed, $[l_1] = [l_2]$ implies that there exists an element $u \in \text{GL}(2, \mathbb{k})$ such that $ul_1 = l_2$. The matrix u defines a linear change of coordinates on \mathbb{P}^1 and hence an isomorphism $f_u: B \rightarrow B$. Then $\mathfrak{a}(l_2)$ is just the matrix $\mathfrak{a}(l_1)$, where each entry b is replaced by $f_u(b)$, which shows that these matrices have the same rank and Betti numbers.

Remark 7.4.3. For a matrix $l \in \text{St}(Q)$, we denote the integer m from the proof of Lemma 7.4.1 by $m(l)$. By the previous remark we know that for matrices l_1, l_2 with $[l_1] = [l_2] \in F_1(Q)$ we have $m(l_1) = m(l_2)$. In particular, for any point $[l] \in F_1(Q)$ with representative $l \in \text{St}(Q)$ the following is well-defined: $m([l]) := m(l)$. Recall that we represent an element of $U_{i,j} \cap F_1(Q) \subseteq \mathbb{A}^{20}$, for $0 \leq i < j \leq 11$, by a unique 2×12 matrix l whose 2×2 submatrix with columns i, j is the identity matrix. For example, if $i = 10$ and $j = 11$, then for $[l] \in U_{10,11} \cap F_1(Q) \subseteq \mathbb{A}^{20}$, we obtain a corresponding \mathfrak{a} -matrix

$$\mathfrak{a}(l) = \begin{pmatrix} x_1 & x_0 & 0 & \lambda_9 x_0 + \mu_9 x_1 & 0 & 0 \\ \lambda_8 x_0 + \mu_8 x_1 & 0 & \lambda_7 x_0 + \mu_7 x_1 & 0 & \lambda_6 x_0 + \mu_6 x_1 & 0 \\ 0 & \lambda_5 x_0 + \mu_5 x_1 & \lambda_4 x_0 + \mu_4 x_1 & 0 & 0 & \lambda_3 x_0 + \mu_3 x_1 \\ 0 & 0 & 0 & \lambda_2 x_0 + \mu_2 x_1 & \lambda_1 x_0 + \mu_1 x_1 & \lambda_0 x_0 + \mu_0 x_1 \end{pmatrix}.$$

The conditions $\text{rank}(\mathfrak{a}(l)) = 4$ and $m(l) = 2$ are open conditions on the coordinates of \mathbb{A}^{20} (respectively on the Stiefel coordinates of l). Hence there exists an open subset of $F_1(Q)$ (re-

spectively of $\text{St}(Q)$) whose points lead to \mathfrak{a} -matrices having a full rank and a minimal free resolution with $m(l) = 2$. Using the Las Vegas algorithm from Subsection 7.2.2, we will check by a computation in Section 11.3 that this open set is non-empty.

Lemma 7.4.4. *Let $l \in \text{St}(Q)$ be a matrix and assume that $\text{rank}(\mathfrak{a}(l)) = 4$. Then the Hilbert polynomial of L_l is a constant n_l . Moreover, \tilde{L}_l is supported at n_l points in \mathbb{P}^1 (counted with multiplicity).*

Proof. Since $\text{rank}(\mathfrak{a}(l)) = 4$ there must be a non-zero 4×4 minor of $\mathfrak{a}(l)$. Hence

$$\dim L_l = \dim B / \text{ann}_B(L_l) \leq 1.$$

This implies that the Hilbert polynomial of L_l is a constant $n_l \in \mathbb{N}$. From the structure theorem for coherent sheaves on \mathbb{P}^1 we deduce that

$$\tilde{L}_l \cong \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(s_i) \oplus \bigoplus_{j=1}^m \mathcal{O}_{r_j[p_j]},$$

where $s_i \in \mathbb{Z}$ and $\mathcal{O}_{r_j[p_j]}$ is a torsion sheaf of degree r_j supported on a closed point $p_j \in \mathbb{P}^1$. On the other hand, we know that $h^0(\mathbb{P}^1, \tilde{L}_l(d)) = \dim_{\mathbb{k}}(L_l)_d = n_l$ for $d \gg 0$. Hence $k = 0$ and

$$n_l = \sum_{j=1}^m r_j. \quad \square$$

Remark 7.4.5. Assume L_l has a minimal free resolution corresponding to the first three Betti tables in the second row of Lemma 7.4.1, that is a resolution of type

$$0 \leftarrow L_l \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow \begin{array}{c} B(-2)^h \\ \oplus \\ B(-3)^{2-h} \end{array} \leftarrow 0 \quad (7.15)$$

for $0 \leq h \leq 2$. Then, in each case, the Hilbert polynomial of L_l is h . Hence \tilde{L}_l is supported on h points (counted with multiplicity). We will later observe that this number h is related to the number of hyperelliptic curves in $|2K_X|$.

In the next section, we will see that we can a priori assume that the matrix \mathfrak{c} is zero in d'_1 if and only if the minimal free resolution of L_l is of the form (7.15) with $h = 0$. The following statement shows that "having a minimal free resolution of this form" is again an open condition on $\text{St}(Q)$:

Proposition 7.4.6. *There exists a (non-empty) open subset $V_{\text{gensyz}} \subseteq \text{St}(Q)$ such that for every $l \in V_{\text{gensyz}}$, the module L_l has a minimal free resolution of the form*

$$0 \leftarrow L_l \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow B(-3)^2 \leftarrow 0.$$

Proof. Similarly as before, it is enough to show first that the property stated above is an open condition and give then one example fulfilling this property. Here we will only show the openness and refer for examples to Chapter 11. First recall from Remark 7.4.3 that having a minimal free resolution corresponding to a Betti table in the second row of Lemma 7.4.1 is an open condition. Hence we assume from now on that $m = 2$ in (7.14). In the previous remark, we have seen that the Hilbert polynomial of L_l with a minimal free resolution as in (7.15) is h . Moreover,

we compute that the Hilbert polynomial of a module L_l with minimal free resolution of type

$$0 \leftarrow L_l \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow \begin{array}{c} B(-2) \\ \oplus \\ B(-4) \end{array} \leftarrow 0$$

is the zero polynomial. We show now that, in this setting, having a Hilbert polynomial equal to zero is an open condition. By the previous proposition, this is equivalent to showing that the scheme $\text{Supp}(\tilde{L}_l)$ is empty for $l \in \text{St}(Q)$. But, by the definition of the module L_l , this is equivalent to the fact that the 4×4 minors of $\mathfrak{a}(l)$ do not vanish at any point $p \in \mathbb{P}^1$.

The matrix $\mathfrak{a}(l)$ has 15 minors of size 4 which we write as a product

$$M_4(l) \begin{pmatrix} x_0^4 \\ x_0^3 x_1 \\ x_0^2 x_1^2 \\ x_0 x_1^3 \\ x_1^4 \end{pmatrix},$$

where $M_4(l)$ is a 15×5 matrix depending only on the Stiefel coordinates of l . A point $p \in \mathbb{P}^1$ at which all 4×4 minors of $\mathfrak{a}(l)$ vanish leads to a non-trivial syzygy of the columns of $M_4(l)$. Hence, the 4×4 minors of $\mathfrak{a}(l)$ do not vanish at any point of \mathbb{P}^1 if

$$\text{rank } M_4(l) = 5. \quad (7.16)$$

Since having a full rank is an open condition, (7.16) is an open condition on $\text{St}(Q)$. Thus, on an open subset of $\text{St}(Q)$ the minimal free resolution of L_l is of type

$$0 \leftarrow L_l \leftarrow B^4 \xleftarrow{\mathfrak{a}(l)} B(-1)^6 \leftarrow B(-3)^2 \leftarrow 0$$

or

$$0 \leftarrow L_l \leftarrow B^4 \xleftarrow{\mathfrak{a}(l)} B(-1)^6 \leftarrow \begin{array}{c} B(-2)^1 \\ \oplus \\ B(-4)^1 \end{array} \leftarrow 0.$$

Next we assign to any $l \in \text{St}(Q)$ a coherent sheaf on \mathbb{P}^1 by setting

$$\mathcal{S}_l = \widetilde{\ker(\mathfrak{a}(l))}.$$

Then, on an open subset of $\text{St}(Q)$, we have

$$\mathcal{S}_l \cong \mathcal{O}_{\mathbb{P}^1}(-3)^2 \text{ or } \mathcal{S}_l \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-4).$$

In both cases, \mathcal{S}_l is a vector bundle on \mathbb{P}^1 of rank 2 and degree -6 . But only in the first case, \mathcal{S}_l is a balanced vector bundle. The result follows now from the fact that being balanced is an open condition in a family of vector bundles on \mathbb{P}^1 with fixed rank and degree. Hence, there exists

an open subset of $\text{St}(Q)$ such that for any matrix l in this set, the module L_l has a minimal free resolution as claimed. We denote the largest open subset of $\text{St}(Q)$ on which this condition is satisfied by V_{gensyz} . \square

Remark 7.4.7. In the next section we will also need some information on the syzygies of the matrix $\mathfrak{p}(l)$. With MACAULAY2 we compute that the (updated) matrix \mathfrak{p} has rank 8. Hence, replacing the open set from Proposition 7.4.6 from the last statement by a smaller one, if necessary, we may without loss of generality assume that $\text{rank } \mathfrak{p}(l) = 8$ for any $l \in V_{\text{gensyz}}$. This implies that $\text{coker } \mathfrak{p}(l)$ has a minimal free resolution of the form

$$0 \leftarrow \text{coker } \mathfrak{p}(l) \leftarrow B^{12} \leftarrow B(-1)^8 \leftarrow 0.$$

Furthermore, we assume from now on that $V_{\text{gensyz}} \subseteq \text{St}(Q)$ is the maximal open set on which both open conditions above hold: Hence, if $l \in \text{St}(Q)$ is a matrix with $l \notin V_{\text{gensyz}}$, then L_l or $\text{coker } \mathfrak{p}(l)$ have other Betti numbers than those above.

7.5 Solving the Linear Relations

In this section we study again a standard resolution

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0$$

as in Section 7.1 with the general set-up for d_1' and d_2 as in (7.1) and (7.2). Recall that any matrix $l \in \text{St}(Q)$ completely determines the entries of the matrices \mathfrak{a} , \mathfrak{e} and \mathfrak{p} in d_1' and d_2 . Our updated matrix of relations has then the following form

$$d_1' d_2 = \left(\begin{array}{cc|c} \mathfrak{a}(l)\mathfrak{o} - b_1(y)\mathfrak{n}^{\text{tr}} - \mathfrak{c}b_3(y)^{\text{tr}} & \mathfrak{a}(l)\mathfrak{n} - \mathfrak{c}\mathfrak{p}(l)^{\text{tr}} & 0 \\ -\mathfrak{e}(l)\mathfrak{n}^{\text{tr}} & 0 & 0 \end{array} \right).$$

We see that the relations coming from $d_1' d_2 = 0$ are linear in the entries of the matrices \mathfrak{o} , \mathfrak{n} , \mathfrak{c} . So, after arranging the remaining unknown entries in a vector, they give a syzygy of a matrix, say $\mathcal{M}(l)$, whose entries depend only on l . In this section, we will introduce and study the matrix $\mathcal{M}(l)$ under the assumption that $L_l = \text{coker } \mathfrak{a}(l)$ has a minimal free resolution of the form

$$0 \leftarrow \text{coker } \mathfrak{a}(l) \leftarrow B^4 \xleftarrow{\mathfrak{a}(l)} B(-1)^6 \leftarrow B(-3)^2 \leftarrow 0. \quad (7.17)$$

Recall from Proposition 7.4.6 that this assumption is satisfied for each matrix $l \in V_{\text{gensyz}}$.

Lemma 7.5.1. *Let $\tilde{c}: B^4 \leftarrow B(-2)^{12}$ be an arbitrary homogeneous homomorphism, and let $l \in \text{St}(Q)$ be a matrix such that L_l has a minimal free resolution as in (7.17). Then*

$$\text{im}(\tilde{c}) \subseteq \text{im}(\mathfrak{a}(l)).$$

Proof. The columns of \tilde{c} span a subspace of the 12-dimensional \mathbb{k} -vector space $(B^4)_2$. From the minimal free resolution of L_l we compute that $\dim_{\mathbb{k}}(L_l)_2 = 0$. Hence the image of $\mathfrak{a}(l)$ contains the vector space $(B^4)_2$ which proves the claim. \square

Remark 7.5.2. Note that the previous lemma can be generalized to the following statement. Let

$$\tilde{a}: B^4 \leftarrow B(-1)^6 \text{ and } \tilde{c}: B^4 \leftarrow B(-2)^{12}$$

be two non-zero homogeneous homomorphisms with $\text{rank}(\tilde{a}) = 4$. Then $\text{im}(\tilde{c}) \subseteq \text{im}(\tilde{a})$ if and only if $\text{coker } \tilde{a}$ has Betti numbers as in (7.17) (with \tilde{a} instead of $\mathfrak{a}(l)$).

Now let

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be any standard resolution of R whose \mathfrak{a} -matrix $\mathfrak{a}(l)$ has Betti numbers as in (7.17). Let \tilde{c} denote the \mathfrak{c} -matrix of d_1 . By Lemma 7.5.1, there exists a matrix t_1 such that $\tilde{c} = \mathfrak{a}(l)t_1$. Setting

$$\alpha_1 = \begin{pmatrix} \text{id}_6 & & -t_1 \\ & \text{id}_{12} & \\ & & \text{id}_8 \end{pmatrix},$$

$e_1 = d_1\alpha_1$ and $e_2 = \alpha_1^{-1}d_2\alpha_1^{-tr}$, we obtain another standard resolution

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{e_1} F_1 \xleftarrow{e_2} F_1^\vee \xleftarrow{e_1^{tr}} F_0^\vee \leftarrow 0$$

of R with assigned matrix l and whose \mathfrak{c} -matrix is zero now.

General Assumption 2. *Let*

$$0 \leftarrow R \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be a standard resolution with assigned matrix $l \in \text{St}(Q)$. If $\text{coker } \mathfrak{a}(l)$ has Betti numbers as in (7.17), then we assume from now on that the \mathfrak{c} -matrix of d_1 is zero. Thus, in this case, a standard resolution of R has the additional property $\mathfrak{c} = 0$.

Later we will see that if the canonical ring $R(X)$ admits a standard resolution whose \mathfrak{a} -matrix has Betti numbers as in (7.17), then X is a torsion-free numerical Godeaux surface.

Our next step is to see how the condition $\mathfrak{c} = 0$ effects our original relations. One consequence of this condition is that every entry of \mathfrak{n} can be expressed by the entries of the matrices \mathfrak{a} and \mathfrak{o} . To see this, we will first write the entries of \mathfrak{o} and \mathfrak{n} with respect to the indices introduced in Section 7.1.

Recall that \mathfrak{o} is a square matrix of size 6 whose entries are homogeneous of degree 5. We consider these entries as polynomials in y_0, \dots, y_3 whose coefficients are variables of degree 2. The row indices of \mathfrak{o} should be the same as the column indices of the matrix \mathfrak{a} , namely the elements of the set

$$N = \{(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3)\}.$$

Furthermore, since \mathfrak{o} is skew-symmetric, row and column indices should be the same.

Notation 7.5.3 (The matrix \mathfrak{o}). Let $(r_1, s_1), (r_2, s_2) \in N$. Then the entry in row (r_1, s_1) and column (r_2, s_2) of the matrix \mathfrak{o} is of type:

$$\begin{cases} 0, & \text{if } r_1 = s_1 \text{ and } r_2 = s_2, \\ \sum_{m=0}^3 \mathfrak{o}_{r_2, s_2, m}^{(r_1, s_1)} \cdot y_m, & \text{otherwise.} \end{cases}$$

Furthermore, by the skew-symmetry of \mathfrak{o} , we have

$$\mathfrak{o}_{r_2, s_2, m}^{(r_1, s_1)} = -\mathfrak{o}_{r_1, s_1, m}^{(r_2, s_2)}$$

for all $m = 0, \dots, 3$. So in total, the matrix \mathfrak{o} has 60 unknown entries of degree 2.

Now it remains to consider the matrix \mathbf{n} . Similarly as for the matrix \mathbf{o} , every row of \mathbf{n} should correspond to a column of \mathbf{a} . Furthermore, the column indices of \mathbf{n} should be the same as the column indices of $b_4(y)$. Since the matrix $b_4(y)$ is skew-symmetric, these indices should coincide with the row indices of $b_4(y)$, that is with the elements of the set

$$L_1 = \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 2), (1, 3), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1), (3, 2)\}.$$

Notation 7.5.4 (The matrix \mathbf{n}). Let $(r, s) \in N$ and $(k, l) \in L_1$. We denote the entry in row (r, s) and column (k, l) of \mathbf{n} by $\mathbf{n}_{k,l}^{(r,s)}$.

Next we study the equation $\mathbf{a}\mathbf{o} - b_1(y)\mathbf{n}^{tr} = 0$ which is one of the equations coming from $d'_1 d_2 = 0$ under the additional condition $\mathbf{c} = 0$. Then the entry in row k and column (r, s) of the matrix on the left-hand side is of the form

$$\sum_{n \in N} \mathbf{a}_n^{(k)} \left(\sum_{m=0}^3 \mathbf{o}_{r,s,m}^n y_m \right) - y_{h_{k,1}} \mathbf{n}_{k,h_{k,1}}^{(r,s)} - y_{h_{k,2}} \mathbf{n}_{k,h_{k,2}}^{(r,s)} - y_{h_{k,3}} \mathbf{n}_{k,h_{k,3}}^{(r,s)}. \quad (7.18)$$

Corollary 7.5.5. Let $(r, s) \in N$ and $(k, l) \in L_1$. If $\mathbf{a}\mathbf{o} - b_1(y)\mathbf{n}^{tr} = 0$, then

$$\mathbf{n}_{k,l}^{(r,s)} = \sum_{n \in N} \mathbf{a}_n^{(k)} \mathbf{o}_{r,s,l}^n. \quad (7.19)$$

In particular, we can then express any entry of the matrix \mathbf{n} by elements of \mathbf{a} and \mathbf{o} .

Proof. Clear from Equation (7.18). □

Remark 7.5.6. To obtain the expressions for the entries of \mathbf{n} we used the fact that the coefficients of the variables y_m for $m = h_{k,1}, h_{k,2}, h_{k,3}$ must be zero in (7.18). For $m = k$ we get a further relation

$$\rho_{r,s}^{(k)} := \sum_{n \in N} \mathbf{a}_n^{(k)} \mathbf{o}_{r,s,k}^n = 0. \quad (7.20)$$

Hence, altogether, we obtain 24 relations of degree 4. Furthermore, $\mathbf{a}\mathbf{o} - b_1(y)\mathbf{n}^{tr} = 0$ if and only if (7.19) and (7.20) are satisfied.

Remark 7.5.7. To simplify the following calculations, we assume that we have given variables $\mathbf{o}_{r_2, s_2, m}^{(r_1, s_1)}$ for any $r_1 \neq s_1 \in \{0, \dots, 3\}$ and $r_2 \neq s_2 \in \{0, \dots, 3\}$ subject to the conditions:

$$\mathbf{o}_{r_2, s_2, m}^{(r_1, s_1)} = \mathbf{o}_{r_2, s_2, m}^{(s_1, r_1)} = \mathbf{o}_{s_2, r_2, m}^{(s_1, r_1)} = \mathbf{o}_{s_2, r_2, m}^{(r_1, s_1)}.$$

Substituting the entries of the matrix \mathbf{n} by the expression in Corollary 7.5.5 yields now the following new matrix of relations

$$d'_1 d_2 = \left(\begin{array}{c|cc} 0 & \mathbf{a}\mathbf{n} & 0 \\ \hline -\mathbf{c}\mathbf{n}^{tr} & 0 & 0 \end{array} \right)$$

together with the 24 relations from Remark 7.5.6.

Next we study the relations of degree 6 coming from $\mathbf{an} = 0$. Let $i \in \{0, \dots, 3\}$, and $(k, l) \in L_1$. Then the element in row i and column (k, l) of the updated matrix \mathbf{an} is of the form

$$\begin{aligned} \xi_{k,l}^{(i)} &:= \sum_{n \in N} \mathbf{a}_n^{(i)} \mathbf{n}_{k,l}^n = \sum_{j=1}^3 \mathbf{a}_{i,h_{i,j}}^{(i)} \mathbf{n}_{k,l}^{i,h_{i,j}} \\ &= \sum_{j=1}^3 \mathbf{a}_{i,h_{i,j}}^{(i)} \left(\sum_{m=1}^3 \mathbf{a}_{k,h_{k,m}}^{(k)} \mathbf{o}_{i,h_{i,j},l}^{k,h_{k,m}} \right) \\ &= \sum_{j=1}^3 \sum_{m=1}^3 \mathbf{a}_{i,h_{i,j}}^{(i)} \mathbf{a}_{k,h_{k,m}}^{(k)} \mathbf{o}_{i,h_{i,j},l}^{k,h_{k,m}}. \end{aligned}$$

Corollary 7.5.8. *Let $i \in \{0, \dots, 3\}$ and $(k, l) \in L_1$. Then, modulo the 24 relations from Remark 7.5.6, we have*

$$\xi_{k,l}^{(i)} = \begin{cases} 0 & \text{if } i \in \{k, l\}, \\ -\xi_{i,l}^{(k)} & \text{if } i \notin \{k, l\}. \end{cases}$$

In particular, the equation $\mathbf{an} = 0$ gives 12 additional relations of degree 6.

Proof. If $i = k$ or $i \notin \{k, l\}$, then the claimed expression follows directly from the definition of $\xi_{k,l}^{(i)}$ and the properties of the entries of \mathbf{o} from Notation 7.5.3 without any reduction modulo the relations of degree 4. So it remains to consider the case $l = i$. Then

$$\begin{aligned} \xi_{k,i}^{(i)} &= \sum_{j=1}^3 \sum_{m=1}^3 \mathbf{a}_{i,h_{i,j}}^{(i)} \mathbf{a}_{k,h_{k,m}}^{(k)} \mathbf{o}_{i,h_{i,j},i}^{k,h_{k,m}} \\ &= - \sum_{j=1}^3 \sum_{m=1}^3 \mathbf{a}_{i,h_{i,j}}^{(i)} \mathbf{a}_{k,h_{k,m}}^{(k)} \mathbf{o}_{k,h_{k,m},i}^{i,h_{i,j}} \\ &= - \sum_{m=1}^3 \mathbf{a}_{k,h_{k,m}}^{(k)} \left(\sum_{j=1}^3 \mathbf{a}_{i,h_{i,j}}^{(i)} \mathbf{o}_{k,h_{k,m},i}^{i,h_{i,j}} \right) \\ &= - \sum_{m=1}^3 \mathbf{a}_{k,h_{k,m}}^{(k)} \rho_{k,h_{k,m}}^{(i)} = 0. \quad \square \end{aligned}$$

Finally, from the equation $-\mathbf{en}^{tr} = 0$ we get 18 relations of degree 6 which cannot be reduced any further. Now let us put all our previous results on the unknown entries of d'_1 and d_2 and their relations together.

Summary 7.5.9. We consider d'_1 and d_2 with the updated matrices for \mathbf{a}, \mathbf{e} and \mathbf{p} and with the additional condition $\mathbf{c} = 0$. Then there are in total 72 unknown entries of degree 2: the 12 entries of the matrix \mathbf{a} and the 60 entries in the matrix \mathbf{o} from Notation 7.5.3. Furthermore, there are 58 relations between these 72 variables, namely the 4 quadratic (Pfaffian) relations q_0, \dots, q_3 and the 54 relations deduced above. In particular, after finding a solution for the 4 quadratic relations, we get 54 equations which are linear in the remaining 60 variables.

Writing the 60 unknown entries of the \mathbf{o} -matrix as a column vector $\underline{\mathbf{o}}$, we can represent the 54 linear relations as

$$\mathcal{M}\underline{\mathbf{o}} = 0,$$

where \mathcal{M} is a 54×60 matrix whose entries depends solely on the 12 variables in the matrix \mathbf{a} . Furthermore, we write \mathcal{M} so that the first 24 rows correspond to the relations of degree 4 and

the last 30 rows to the relations of degree 6. With the help of MACAULAY2 we compute that $\text{rank } \mathcal{M} = 38$.

For $l \in V_{\text{gensyz}} \subseteq \text{St}(Q)$, let $\mathcal{M}(l)$ be the matrix obtained from \mathcal{M} by replacing the α -variables with the assignment given by l . Hence, $\mathcal{M}(l)$ depends only on x_0, x_1 and on the Stiefel coordinates of l . In Chapter 11 we consider an example with a matrix $l \in \text{St}(Q)$ such that $\text{coker } \mathcal{M}(l)$ has a minimal free resolution of type

$$0 \leftarrow \text{coker } \mathcal{M}(l) \leftarrow \begin{array}{c} B(-1)^{24} \\ \oplus \\ B^{30} \end{array} \leftarrow B(-2)^{60} \leftarrow \begin{array}{c} B(-3)^4 \\ \oplus \\ B(-4)^{18} \end{array} \leftarrow 0.$$

In general, after replacing V_{gensyz} by a smaller open subset if necessary, we may assume that for any $l \in V_{\text{gensyz}}$ we have $\text{rank}(\mathcal{M}(l)) = 38$ and that $\text{coker } \mathcal{M}(l)$ has a minimal free resolution of the form

$$0 \leftarrow \text{coker } \mathcal{M}(l) \leftarrow \begin{array}{c} B(-1)^{24} \\ \oplus \\ B^{30} \end{array} \xleftarrow{\mathcal{M}(l)} B(-2)^{60} \leftarrow \begin{array}{c} B(-3)^{n_3(l)} \\ \oplus \\ B(-4)^{n_4(l)} \\ \oplus \\ B(-5)^{n_5(l)} \\ \oplus \\ \vdots \end{array} \leftarrow 0, \quad (7.21)$$

where the $n_i(l)$ are non-negative integers with $n_3(l) \leq 4$ and $\sum_{k \geq 3} n_k(l) = 22$. Since the unknown entries of the matrix σ have all degree 2, any possible assignment to these variables gives an element of the vector space $(B(-2)^{60})_3$. Now let

$$\mathcal{V}(l) \subseteq (B(-2)^{60})_3 \cong \mathbb{k}^{120}$$

be the \mathbb{k} -vector space of dimension $n_3(l)$ generated by the columns of the homomorphism $B(-2)^{60} \leftarrow B(-3)^{n_3(l)}$ in (7.21). Hence, $\mathcal{V}(l)$ is the kernel of a linear map of vector spaces

$$\mathbb{k}^{192} \xleftarrow{\mathcal{M}'(l)} \mathbb{k}^{120},$$

where the matrix $\mathcal{M}'(l)$ depends only on the Stiefel coordinates of l .

Then an element $r \in (B(-2)^{60})_3$ is a solution of the system

$$\mathcal{M}(l)\underline{\sigma} = 0 \quad (7.22)$$

if and only if $r \in \mathcal{V}(l)$. Usually we consider $\mathcal{V}(l)$ with a fixed basis and identify it with the vector space $\mathbb{k}^{n_3(l)}$, by sending the standard basis vectors of $\mathbb{k}^{n_3(l)}$ to the chosen basis vectors. Hence, using this identification, we will denote in the following by p both the solution vector in $\mathcal{V}(l)$ and the corresponding point in $\mathbb{k}^{n_3(l)}$. Given such a point p , we write the corresponding solution of the system (7.22) in a matrix as in Notation 7.5.3 and denote this matrix by $\sigma(l, p)$. Moreover, from Corollary 7.5.5 we know that we can express any element of the general matrix n in terms of entries of the matrices α and σ . Now Equation (7.19) with α and σ replaced by $\alpha(l)$

and $\mathfrak{o}(l, p)$ respectively, defines a matrix $\mathfrak{n}(l, p)$. Then

$$\mathfrak{a}(l)\mathfrak{o}(l, p) - b_1(y)\mathfrak{n}(l, p)^{tr} = 0$$

is a direct consequence from the definition of the matrix $\mathfrak{n}(l, p)$ and the fact that the entries of the matrices $\mathfrak{a}(l)$ and $\mathfrak{o}(l, p)$ satisfy (7.20).

Proposition 7.5.10. *Let $l \in V_{\text{gensyz}} \subseteq \text{St}(Q)$ be a matrix, and let $p \in \mathcal{V}(l)$ be any point. Then the matrices*

$$d'_1(l) = \begin{pmatrix} \mathfrak{a}(l) & b_1(y) & 0 \\ 0 & \mathfrak{c}(l) & b_2(y) \end{pmatrix} \text{ and } d_2(l, p) = \begin{pmatrix} \mathfrak{o}(l, p) & \mathfrak{n}(l, p) & b_3(y) \\ \mathfrak{n}(l, p)^{tr} & b_4(y) & \mathfrak{p}(l) \\ -b_3(y)^{tr} & -\mathfrak{p}(l)^{tr} & 0 \end{pmatrix}$$

satisfy

$$d'_1(l)d_2(l, p) = 0.$$

Proof. Clear from the previous results in this section and the discussion on the quadratic relations $\mathfrak{q}_0, \dots, \mathfrak{q}_3$ in Section 7.1. \square

In Remark 7.4.2 we have seen that for matrices $l_1, l_2 \in \text{St}(Q)$ with $[l_1] = [l_2] \in F_1(Q)$, the modules $\text{coker } \mathfrak{a}(l_1)$ and $\text{coker } \mathfrak{a}(l_2)$ have the same Betti numbers. For the corresponding vector spaces $\mathcal{V}(l_1)$ and $\mathcal{V}(l_2)$ we obtain the following:

Lemma 7.5.11. *Let $l_1, l_2 \in V_{\text{gensyz}} \subseteq \text{St}(Q)$ such that $[l_1] = [l_2] \in F_1(Q)$. Then $\mathcal{V}(l_1) \cong \mathcal{V}(l_2)$.*

Proof. The condition $[l_1] = [l_2]$ implies that there exists an invertible matrix

$$u = \begin{pmatrix} u_{0,0} & u_{0,1} \\ u_{1,0} & u_{1,1} \end{pmatrix}$$

such that $l_2 = ul_1$. Then the matrix $\mathcal{M}(l_2)$ is the matrix $\mathcal{M}(l_1)$ with each variable x_i being substituted by $u_{0,i}x_0 + u_{1,i}x_1$. Hence, $\text{coker } \mathcal{M}(l_1)$ and $\text{coker } \mathcal{M}(l_2)$ have the same Betti numbers. In particular, the solution spaces $\mathcal{V}(l_1), \mathcal{V}(l_2) \subseteq (B(-2)^{60})_3$ have the same dimension and there exists an isomorphism

$$s_u : \mathcal{V}(l_1) \rightarrow \mathcal{V}(l_2). \quad \square$$

Furthermore, from any basis of $\mathcal{V}(l_1)$ we get a corresponding basis of $\mathcal{V}(l_2)$ by substituting each variable x_i by $u_{0,i}x_0 + u_{1,i}x_1$ as above. Hence, we assume that s_u maps a chosen basis of $\mathcal{V}(l_1)$ to its corresponding basis in $\mathcal{V}(l_2)$.

Remark and Assumption 7.5.12. Let

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0 \quad (7.23)$$

be a standard resolution of $R(X)$. We can assign a unique matrix $l \in \text{Mat}(\mathbb{k}, 2 \times 12)$ to the \mathfrak{a} -matrix of d_1 (by writing the coefficients of x_0 in the first row, and the ones of x_1 in the second row of l). Suppose that $l \notin \text{St}(2, 12)$. Then there exists an element $(p_0 : p_1) \in \mathbb{P}^1$ inducing a non-trivial relation between the rows of l . Let $I' \subseteq S$ denote the ideal generated by the 7×7 minors of d'_1 as in Notation 5.0.1. Then, from the form of d'_1 modulo x_0, x_1 , we deduce that the

vanishing locus of I' in $\mathbb{P}(2^2, 3^4)$ contains the 4 curves

$$\begin{aligned} C_0 &= V(p_1x_0 + p_0x_1, y_1, y_2, y_3), \quad C_1 = V(p_1x_0 + p_0x_1, y_0, y_2, y_3), \\ C_2 &= V(p_1x_0 + p_0x_1, y_0, y_1, y_3), \quad C_3 = V(p_1x_0 + p_0x_1, y_0, y_1, y_2), \end{aligned}$$

which implies that $\text{depth}(I', S) = \dim S - \dim S/I' \leq 4$. Hence, $R(X)$ does not satisfy the ring condition from Remark 5.2.4. Since, we believe that the canonical ring of any (marked) numerical Godeaux surface X satisfies this condition, we will assume from now on that the assigned matrix to any standard resolution of $R(X)$ is contained in $\text{St}(2, 12)$. In particular, we can assign to (7.23) a unique matrix $l \in \text{St}(Q)$. Furthermore, if $l \in V_{\text{gensyz}}$, then there exists a unique point $p \in \mathcal{V}(l)$ such that

$$d'_1 = d'_1(l) \text{ and } d_2 = d_2(l, p).$$

We call the pair (l, p) the *assigned pair* to the standard resolution (7.23).

We end this section with a brief discussion on the linear relations of $d'_1 d_2 = 0$ without the additional assumption $\mathfrak{c} = 0$:

Remark 7.5.13. If $l \in \text{St}(Q)$ such that $\text{coker } \mathfrak{a}(l)$ has different Betti numbers as in (7.17), we cannot a priori assume that $\mathfrak{c} = 0$. In this case, we use the equation

$$\mathfrak{a}(l)\mathfrak{o} - b_1(y)\mathfrak{n}^{tr} - \mathfrak{c}b_3(y)^{tr} = 0 \quad (7.24)$$

to express every entry of the matrix \mathfrak{c} by entries of the matrices \mathfrak{a} , \mathfrak{o} and \mathfrak{n} . Then the remaining equations are linear in the unknown entries of \mathfrak{o} and \mathfrak{n} and we can represent these relations as

$$\mathcal{M}(l) \begin{pmatrix} \mathfrak{o} \\ \mathfrak{n} \end{pmatrix} = 0. \quad (7.25)$$

As before, we denote the corresponding solution space by $\mathcal{V}(l)$. Then, after choosing a point $p \in \mathcal{V}(l)$, we get matrices $\mathfrak{o}(l, p)$ and $\mathfrak{n}(l, p)$ by writing the solution of (7.25) as entries of the matrices \mathfrak{o} and \mathfrak{n} . Afterwards we use Equation (7.24) to determine a unique matrix \mathfrak{c} which satisfies this equation and denote this matrix by $\mathfrak{c}(l, p)$. Then with

$$d'_1(l, p) = \begin{pmatrix} \mathfrak{a}(l) & b_1(y) & \mathfrak{c}(l, p) \\ 0 & \mathfrak{e}(l) & b_2(y) \end{pmatrix}$$

and $d_2(l, p)$ as in Proposition 7.5.10 we also get

$$d'_1(l, p)d_2(l, p) = 0.$$

In particular, to any standard resolution

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0 \quad (7.26)$$

of $R(X)$ we can assign a unique pair (l, p) with either $d'_1 = d'_1(l)$ or $d'_1 = d'_1(l, p)$, and $d_2 = d_2(l, p)$.

Altogether, the results of this chapter show how to construct matrices d'_1 and d_2 of a particular form satisfying $d'_1 d_2 = 0$. It remains to show how to obtain a finitely generated S -module R and a standard resolution of R from this construction:

Proposition 7.5.14. Let $d'_1: F'_0 \leftarrow F_1$ and $d_2: F_1 \leftarrow F_1^\vee$ be two homogeneous homomorphisms satisfying:

- (i) $d'_1 = d'_1(l, p)$ and $d_2 = d_2(l, p)$, and hence $d'_1 d_2 = 0$,
- (ii) $\text{depth}(I', S) \geq 5$, where I' is the ideal generated by the 7×7 minors of d'_1 ,
- (iii) d_2 is skew-symmetric,
- (iv) d'_1 and d_2 have modulo x_0, x_1 the form as introduced in Section 6.3.

Then there exists an exact sequence

$$F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

with $d_1 = \begin{pmatrix} d_1^{(0)} \\ d'_1 \end{pmatrix}$, where $d_1^{(0)}: S \leftarrow F_1$ is a homogeneous homomorphism. Hence, $R := \text{coker } d_1$ is a finitely generated S -module admitting a standard resolution.

Proof. First note that the skew-symmetry of d_2 implies directly that there exists a further syzygy of d_2 (which cannot be obtained from the columns of $(d'_1)^{tr}$): First we have

$$7 \geq \text{rank}(d'_1) \geq \text{rank}(\bar{d}'_1) = 7$$

and

$$19 \geq \text{rank}(d_2) \geq \text{rank}(\bar{d}_2) = 18.$$

Then, considering d_2 over the quotient field $L = Q(S)$, we deduce that $\text{rank}(d_2) = 18$ since any skew-symmetric matrix with entries in a field is of even rank. Furthermore, the form of d'_1 modulo x_0, x_1 implies that $\text{rank}(d'_1) = 7$ considered as a matrix over L . Hence, there exists exactly one further syzygy defined over L . Multiplying by a common denominator if necessary, we get a further syzygy of d_2 defined over S , and hence a complex

$$S(-k) \xleftarrow{g_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{g_1^{tr}} S(-17+k) \leftarrow 0,$$

where k is chosen minimally. In the following, we will show that $k = 0$.

Let $Z = \text{Proj}(S/I')$ and $U = \mathbb{P}(2^2, 3^4) \setminus Z$. Then Z is empty or 0-dimensional by assumption. Consider the exact sequence

$$0 \leftarrow M \leftarrow F'_0 \xleftarrow{d'_1} F_1 \xleftarrow{i} N \leftarrow 0,$$

where $M = \text{coker } d'_1$ and $N = \text{syz}(d'_1)$. Let us split this exact sequence into two short exact sequences:

$$\begin{aligned} 0 \leftarrow M \leftarrow F'_0 \leftarrow E \leftarrow 0, \\ 0 \leftarrow E \xleftarrow{d'_1} F_1 \xleftarrow{i} N \leftarrow 0, \end{aligned}$$

where $E = \text{im } d'_1$. Considering the corresponding exact sequence of sheaves and restricting to U , the fact that $\widetilde{M}|_U = 0$ implies that $\widetilde{N}|_U$ is a locally free \mathcal{O}_U -module of rank

$$19 = \text{rank}(F_1) - \text{rank}(F'_0).$$

Next we apply the functor $\text{Hom}(-, S(-17))$ to the two short exact sequences above. From the first sequence we get

$$\begin{aligned} 0 \rightarrow M^\vee \rightarrow F_0'^\vee \rightarrow E^\vee \\ \rightarrow \text{Ext}^1(M, S(-17)) \rightarrow \text{Ext}^1(F_0', S(-17)) \rightarrow \text{Ext}^1(E, S(-17)) \\ \rightarrow \text{Ext}^2(M, S(-17)). \end{aligned}$$

Now $\text{Ext}^1(F_0', S(-17)) = 0$, and also $\text{Ext}^2(M, S(-17)) = 0$ since $\dim M \leq 1$. Hence $\text{Ext}^1(E, S(-17)) = 0$, and we get an exact sequence

$$0 \rightarrow M^\vee \rightarrow (F_0')^\vee \xrightarrow{i^\vee} E^\vee \rightarrow \text{Ext}^1(M, S(-17)) \rightarrow 0.$$

Using this for the second exact sequence we get an short exact sequence

$$0 \rightarrow E^\vee \xrightarrow{d_1^\vee} F_1^\vee \rightarrow N^\vee \rightarrow 0.$$

Next we consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E^\vee & \longrightarrow & F_1^\vee & \xrightarrow{i^\vee} & N^\vee \longrightarrow 0 \\ & & & & \downarrow d_2 & & \\ & & & & F_1' & \xleftarrow{i} & N \longleftarrow 0 \end{array} \quad (7.27)$$

Now since $d_2 \circ d_1^\vee = 0$ and $F_1^\vee \xrightarrow{i^\vee} N^\vee$ is surjective, there exists a module homomorphism $h: N^\vee \rightarrow N$ which gives commutative square in the diagram above. In particular, we get:

$$\begin{array}{ccccccc} & & S(-17+k) & & & & \\ & & \downarrow g_1^\vee & \searrow n_1^\vee & & & \\ 0 & \longrightarrow & E^\vee & \longrightarrow & F_1^\vee & \xrightarrow{i^\vee} & N^\vee \longrightarrow 0 \\ & & & & \downarrow d_2 & & \downarrow h \\ & & & & F_1' & \xleftarrow{i} & N \longleftarrow 0 \\ & & & & \downarrow g_1 & \swarrow n_1 & \\ & & S(-k) & & & & \end{array} \quad (7.28)$$

where

$$S(-k) \xleftarrow{n_1} N \xleftarrow{h} N^\vee \xleftarrow{n_1^\vee} S(-17+k) \leftarrow 0$$

is a complex of S -modules. Now let us take a point $p \in U$. Then, from $(F_0')_p \cong (\text{im } d_1')_p$ and $(E^\vee)_p \cong (\text{im } d_1^\vee)_p$, we get

$$\begin{aligned} (F_1^\vee)_p &\cong (N^\vee)_p \oplus (F_0'^\vee)_p \\ (F_1)_p &\cong N_p \oplus (F_0')_p \end{aligned}$$

Furthermore, for every $p \in U$, the induced (local) map $(d_2)_p$ is of the form

$$\begin{array}{c|cc}
& (N^\vee)_p & (F_0^{\vee})_p \\
\hline
N_p & h_p & 0 \\
(F_0')_p & 0 & 0
\end{array}$$

where the local homomorphism $h_p: (N^\vee)_p \rightarrow N_p$ is given by a skew-symmetric matrix. Hence, we see that the rank of h drops at exactly at the points of U at which the rank of d_2 drops. Sheafifying the homomorphism $h: N \leftarrow N^\vee$ and restricting to U , we get a morphism

$$(\tilde{N})|_U \xleftarrow{\phi|_U} (\tilde{N}^\vee)|_U$$

of locally free sheaves of rank $19 = 2 \cdot 9 + 1$ which is (locally) alternating. Furthermore, since $\text{rank}(d_2) = 18$ and $\text{depth}(I(d_2), S) \geq 3$, the ideal of the submaximal minors of $\phi|_U$ is of depth 3. Now, after applying the Buchsbaum-Eisenbud structure theorem to $\phi|_U$, a straight forward Chern class computation implies that $k = 0$.

Finally, setting $d_1^{(0)} = g_1$ and $d_1 = \begin{pmatrix} d_1^{(0)} \\ d_1' \end{pmatrix}$, we obtain a complex

$$F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^\vee} F_0^\vee \leftarrow 0$$

which is exact since $\text{rank}(d_1) = \text{rank}(d_1^\vee) = 8$, $\text{rank}(d_2) = 18$ and $\text{depth}(I(d_i), S) \geq 3$ for $i = 1, 2$.

□

8 A Family of Torsion-Free Numerical Godeaux Surfaces

Let X be a marked numerical Godeaux surface. In Proposition 7.5.10 and Remark 7.5.13 we have seen that we can assign to any standard resolution

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

of $R(X)$ a pair (l, p) with $l \in \text{St}(Q)$ and $p \in \mathcal{V}(l)$.

On the other hand, if

$$F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

is an acyclic complex with $d_1' = d_1'(l, p)$ and $d_2 = d_2(l, p)$ for some $l \in \text{St}(Q)$ and $p \in \mathcal{V}(l)$ such that $R := \text{coker } d_1$ satisfies all conditions of Theorem 5.0.2, then $\text{Proj}(R)$ is the canonical model of a marked numerical Godeaux surface. We then call the pair (l, p) *admissible*. More generally, we say that $l \in \text{St}(Q)$ is *admissible* if there exists a point $p \in \mathcal{V}(l)$ such that (l, p) is admissible. Hence, we can assign to the admissible pair (l, p) the isomorphism class $[\text{Proj}(R)]$ in the Gieseker moduli space $\mathfrak{M}_{1,1}$ of numerical Godeaux surfaces. Thus, if we want to calculate the dimension of our constructed family correctly, we have to identify pairs whose assigned surfaces are isomorphic.

The aim of this chapter is to study and characterize (different) admissible pairs which lead to the same point in the moduli space. As in the previous chapter we restrict our study mainly to matrices which are contained in the open set V_{gensyz} . At the end of this chapter, we count the moduli of torsion-free numerical Godeaux surfaces which we obtain by our construction.

Throughout this chapter X denotes a marked numerical Godeaux surface with canonical model X_{can} . Furthermore, we assume that $\mathbb{k} = \mathbb{C}$.

8.1 Standard Resolutions and Pairs (l, p)

In the last chapter we have seen that we can assign a pair (l, p) to any standard resolution of the canonical ring $R(X)$. Of course, we would like to assign such a pair to $R(X)$. But unfortunately, a standard resolution of $R(X)$ is only uniquely determined up to isomorphism. The aim of this section is to characterize pairs which come from isomorphic standard resolutions of $R(X)$. To begin with, we introduce some notation which will be used in the following:

Notation 8.1.1. For $\lambda = (\lambda_0, \dots, \lambda_3) \in (\mathbb{k}^*)^4$ we denote by $E(\lambda)$ the 4×4 diagonal matrix with $\lambda_0, \dots, \lambda_3$ on its diagonal. Furthermore, we denote by $\hat{E}(\lambda)$ the 12×12 block diagonal matrix with $\lambda_0 \text{id}_3, \dots, \lambda_3 \text{id}_3$ on its diagonal.

Now let

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be any standard resolution of $R(X)$ with assigned pair (l, p) . First it is important to emphasize that the pair (l, p) only determines the matrices d_1' and d_2 of a standard resolution but not the first row of the matrix d_1 .

Indeed, if $r: S \xleftarrow{\begin{pmatrix} r_4 & r_5 \end{pmatrix}} S(-4)^4 \oplus S(-5)^3$ is any (non-zero) homogeneous homomorphism, then

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{e_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{e_1^{tr}} F_0^\vee \leftarrow 0,$$

with

$$e_1 = \begin{pmatrix} 1 & r_4 & r_5 \\ & \text{id}_4 & \\ & & \text{id}_3 \end{pmatrix} \circ d_1 = \begin{pmatrix} d_1^{(0)} + r d_1' \\ d_1' \end{pmatrix}$$

is another standard resolution of $R(X)$ with the same assigned pair (l, p) . In general, however, different (isomorphic) standard resolutions of $R(X)$ may lead to different assigned pairs:

Proposition 8.1.2. *Let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.1)$$

be a standard resolution of $R(X)$ with assigned pair (l, p) , where $l \in V_{\text{gensyz}} \subseteq \text{St}(Q)$ and $p \in \mathcal{V}(l)$. If

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{e_1} F_1 \xleftarrow{e_2} F_1^\vee \xleftarrow{e_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.2)$$

is another standard resolution of $R(X)$ with assigned pair (l', p') , then there exist $\mu \in \mathbb{k}^$, $h = (h_0, \dots, h_3) \in H := \{-1, 1\}^4 \cong (\mathbb{Z}/2\mathbb{Z})^4$ and an automorphism*

$$\nu_{\mu, h}: \text{St}(Q) \rightarrow \text{St}(Q)$$

such that

$$l' = \nu_{\mu, h}(l) \text{ and } p' = \frac{1}{\mu^2} p.$$

Moreover, if any two standard resolutions of $R(X)$ have the same assigned matrix l , then they also have the same assigned point $p \in \mathcal{V}(l)$. On the other hand, for any $\mu \in \mathbb{k}^$ and $h \in H$ there exists a standard resolution of $R(X)$ with assigned pair $(\nu_{\mu, h}(l), \frac{1}{\mu^2} p)$.*

Proof. We start by showing the first statement. Since (8.1) and (8.2) are two minimal free resolutions of $R(X)$, there exists an isomorphism of complexes

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & R(X) & \longleftarrow & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_1^\vee & \xleftarrow{d_1^{tr}} & F_0^\vee & \longleftarrow & 0 \\ & & \downarrow & & \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ 0 & \longleftarrow & R(X) & \longleftarrow & F_0 & \xleftarrow{e_1} & F_1 & \xleftarrow{e_2} & F_1^\vee & \xleftarrow{e_1^{tr}} & F_0^\vee & \longleftarrow & 0 \end{array}$$

inducing the identity map on $R(X)$. We use the entries of the maps d_i and e_i modulo x_0, x_1 , to describe the isomorphisms α_i . Note that since the last syzygy map is the transpose of the first one it is enough to consider α_0, α_1 and α_2 . By $\bar{}$ we denote the reduction of a matrix modulo x_0, x_1 as before. Then, since

$$\bar{d}_i = \bar{e}_i \text{ and } \bar{\alpha}_{i-1} \bar{d}_i = \bar{e}_i \bar{\alpha}_i,$$

after dividing by a non-zero scalar if necessary, we get

$$\bar{\alpha}_0 = \begin{pmatrix} 1 & & \\ & E(\lambda) & \\ & & \mu \text{id}_3 \end{pmatrix}, \bar{\alpha}_1 = \begin{pmatrix} \text{id}_6 & & \\ & \hat{E}(\lambda) & \\ & & \mu \text{id}_8 \end{pmatrix} \text{ and } \bar{\alpha}_2 = \begin{pmatrix} \mu \text{id}_6 & & \\ & \hat{E}(\lambda) & \\ & & \text{id}_8 \end{pmatrix}$$

for some $\lambda \in (\mathbb{k}^*)^4$ and $\mu \in \mathbb{k}^*$. Then, simply by degree reasons, this implies that

$$\alpha_0 = \begin{pmatrix} 1 & r_4 & r_5 \\ & E(\lambda) & \\ & & \mu \text{id}_3 \end{pmatrix}, \alpha_1 = \begin{pmatrix} \text{id}_6 & t_1 \\ & \hat{E}(\lambda) \\ & & \mu \text{id}_8 \end{pmatrix} \text{ and } \alpha_2 = \begin{pmatrix} \mu \text{id}_6 & & \\ & \hat{E}(\lambda) & \\ t_2 & & \text{id}_8 \end{pmatrix}$$

where $S(-6)^6 \xleftarrow{t_1} S(-8)^8$, $S(-9)^8 \xleftarrow{t_2} S(-11)^6$ and $r: S \xleftarrow{(r_4 \ r_5)} S(-4)^4 \oplus S(-5)^3$ are homogeneous homomorphisms. Note that this holds for any standard resolution of $R(X)$ regardless whether $l \in V_{\text{gensyz}}$ or not. But now we use the additional property $l \in V_{\text{gensyz}}$ to conclude that $t_1 = 0$ and $t_2 = 0$. First note that $\text{coker } \mathfrak{a}(l')$ has the same Betti numbers as $\text{coker } \mathfrak{a}(l)$. Indeed, from

$$\mathfrak{a}(l') = E(\lambda)\mathfrak{a}(l)$$

we deduce that $\mathfrak{a}(l)$ and $\mathfrak{a}(l')$ have the same rank and their corresponding cokernels the same Betti numbers. Thus, by General Assumption 2, the \mathfrak{c} -matrix of e_1 is also zero. Now let α'_0 be the matrix obtained from α_0 by erasing the first row and the first column. Then

$$\begin{pmatrix} \mathfrak{a}(l') & b_1(y) & 0 \\ 0 & \mathfrak{c}(l') & b_2(y) \end{pmatrix} = e'_1 = \alpha'_0 d'_1 \alpha_1^{-1} = \begin{pmatrix} * & b_1(y) & \mathfrak{a}(l)t_1 \\ 0 & * & b_2(y) \end{pmatrix}$$

implies that $\mathfrak{a}(l)t_1 = 0$. From the definition of the open set V_{gensyz} in Proposition 7.4.6 we know that $\text{coker } \mathfrak{a}(l)$ has a minimal free resolution of the form

$$0 \leftarrow \text{coker } \mathfrak{a}(l) \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow B(-3)^2 \leftarrow 0.$$

Hence, by the definition of the map t_1 , we get $\mathfrak{a}(l)t_1 = 0$ if and only if $t_1 = 0$. Next we consider the maps d_2 and e_2 . From $e_2 = \alpha_1 d_2 \alpha_2^{-1}$ we deduce that

$$\mathfrak{p}(l') = \hat{E}(\lambda)\mathfrak{p}(l)$$

and

$$\mathfrak{p}(l')^{tr} = \mathfrak{p}(l)^{tr} \hat{E}(\mu\lambda^{-1}).$$

Now the particular shape of the matrix \mathfrak{p} as described in (7.6) and the fact that $\text{rank}(\mathfrak{a}(l)) = 4$ imply that $\frac{\mu}{\lambda_i} = \lambda_i$ for all i , or equivalently,

$$\lambda_0^2 = \lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \mu. \quad (8.3)$$

Furthermore $\text{rank}(\mathfrak{p}(l')) = \text{rank}(\mathfrak{p}(l)) = 8$ which implies $l' \in V_{\text{gensyz}}$. Next we show that $t_2 = 0$ by considering the matrix $\mathfrak{n}(l', p')$. We have

$$\mathfrak{n}(l', p') = \mathfrak{n}(l, p)\hat{E}(\lambda^{-1})$$

and

$$\mathfrak{n}(l', p')^{tr} = \hat{E}(\lambda^{-1})\mathfrak{n}(l, p)^{tr} + \hat{E}(\lambda)\mathfrak{p}(l)t_2$$

which implies that $\mathfrak{p}(l)t_2 = 0$. Thus, using Remark 7.4.7, we conclude that $t_2 = 0$. Putting these results together, we see that the first 3 matrices of every isomorphism between two standard resolutions of $R(X)$ are of the form

$$\alpha_0 = \begin{pmatrix} \mu^{-1} & r_4 & r_5 \\ & E(h) & \\ & & \mu \text{id}_3 \end{pmatrix}, \alpha_1 = \begin{pmatrix} \mu^{-1} \text{id}_6 & & \\ & \hat{E}(h) & \\ & & \mu \text{id}_8 \end{pmatrix} \text{ and } \alpha_2 = \begin{pmatrix} \mu \text{id}_6 & & \\ & \hat{E}(h) & \\ & & \mu^{-1} \text{id}_8 \end{pmatrix},$$

where $h = (h_0, \dots, h_3) \in H$, $\mu \in \mathbb{k}^*$ and r_4, r_5 are defined as before. Next let us define the morphism $\nu_{\mu, h}$. The automorphism of \mathbb{A}^{24} defined by

$$\begin{aligned} \mathbb{A}^{24} &\rightarrow \mathbb{A}^{24}, \\ (\dots, \varrho_{i,j,0}^{(i)}, \dots, \varrho_{i,j,1}^{(i)}, \dots) &\mapsto (\dots, \mu h_i \varrho_{i,j,0}^{(i)}, \dots, \mu h_i \varrho_{i,j,1}^{(i)}, \dots) \end{aligned}$$

induces an automorphism of the open subschemes

$$\tilde{\nu}_{\mu, h}: \text{St}(2, 12) \rightarrow \text{St}(2, 12).$$

We claim that $\tilde{\nu}_{\mu, h}(\text{St}(Q)) = \text{St}(Q)$. To prove this, we first define a corresponding automorphism of \mathbb{P}^{11} which restricts to an automorphism of Q . The ring homomorphism

$$\begin{aligned} \mathbb{k}[\mathfrak{a}_{i,j}^{(i)}] &\rightarrow \mathbb{k}[\mathfrak{a}_{i,j}^{(i)}], \\ \mathfrak{a}_{i,j}^{(i)} &\mapsto \mu h_i \mathfrak{a}_{i,j}^{(i)} \end{aligned}$$

induces an automorphism $\tilde{\tau}_{\mu, h}: \mathbb{P}^{11} \rightarrow \mathbb{P}^{11}$ with inverse $\tilde{\tau}_{\mu^{-1}, h^{-1}}$, and satisfying

$$\tilde{\nu}_{\mu, h}(l) = \tilde{\tau}_{\mu, h}(l) \subseteq \mathbb{P}^{11}.$$

Hence, it is enough to show that $\tilde{\tau}_{\mu, h}(Q) = Q$. But this is clear, since for any $p \in Q$, we have $\mathfrak{q}_i(\tilde{\tau}_{\mu, h}(p)) = \mu^2 \mathfrak{q}_i(p) = 0$ for all i . Hence $\nu_{\mu, h} := \tilde{\nu}_{\mu, h}|_{\text{St}(Q)}: \text{St}(Q) \rightarrow \text{St}(Q)$ and $\tau_{\mu, h} := \tilde{\tau}_{\mu, h}|_Q: Q \rightarrow Q$ are automorphisms. Furthermore, by the definition of the isomorphisms α_0 and α_1 , we get

$$l' = \nu_{\mu, h}(l).$$

From the form of the linear relations defining the matrix \mathcal{M} , we see that there is an invertible matrix $\beta_{\mu, h} \in \text{GL}(54, \mathbb{k})$ (depending on μ and h) such that

$$\mathcal{M}(\nu_{\mu, h}(l)) = \beta_{\mu, h} \cdot \mathcal{M}(l).$$

Hence, we may assume that $\mathcal{V}(l) = \mathcal{V}(\nu_{\mu, h}(l))$ (and consider these spaces with the same bases). Then, since $\mathfrak{o}(l', p') = \frac{1}{\mu^2} \mathfrak{o}(l, p)$, we get

$$p' = \frac{1}{\mu^2} p \in \mathcal{V}(l).$$

So, in particular, if we have two standard resolutions of $R(X)$ with assigned pairs (l, p) and (l, p') , then $\mu = \pm 1$, and thus $p' = p$. To show the last statement of the proposition we just

reverse the steps above. More precisely, starting with standard resolution (8.1), we use the isomorphisms α_i , given a fixed $\mu \in \mathbb{k}^*$ and $h \in H$, to construct a new standard resolution of $R(X)$ whose assigned pair is then $(\nu_{\mu,h}(l), \frac{1}{\mu^2}p)$. \square

Since $\tau_{\mu,h}: Q \rightarrow Q$ is a morphism of projective varieties, $\tau_{\mu,h}$ is independent of the choice of $\mu \in \mathbb{k}^*$. Hence we can neglect μ and get the following result:

Corollary 8.1.3. *The finite group H acts on Q (and hence on $F_1(Q)$) by*

$$\begin{aligned} H \times Q &\rightarrow Q, \\ (h, p) &\mapsto \tau_h(p). \end{aligned}$$

Remark 8.1.4. Later we will show that the torus $G = (\mathbb{k}^*)^4$ acts linearly on Q (and on $F_1(Q)$). Furthermore, we will see that the action of H on Q is just the restriction of the action of G , when considering H as a natural subgroup of G .

For any standard resolution of $R(X)$ with assigned pair (l, p) we define the set

$$\begin{aligned} \text{Stab}(l, R(X)) = \{p' \in \mathcal{V}(l) \mid \text{there exists a standard resolution of } R(X) \\ \text{with assigned pair } (l, p')\}. \end{aligned}$$

Then Proposition 7.5.10 shows that if $l \in V_{\text{gensyz}}$, then

$$\text{Stab}(l, R(X)) = \{p\}.$$

If $l \in \text{St}(Q)$ is a matrix such that the module $L_l = \text{coker } \mathfrak{a}(l)$ has Betti numbers which are different from those in Proposition 7.4.6, then $\text{Stab}(l, R(X))$ contains in general more than one point. For example, let us consider the case where $\text{coker } \mathfrak{a}(l)$ has a minimal free resolution of the form

$$\begin{array}{ccccccc} & & & & B(-2) & & \\ & & & & \oplus & & \\ 0 & \leftarrow & \text{coker } \mathfrak{a}(l) & \leftarrow & B^4 \xleftarrow{\mathfrak{a}(l)} & B(-1)^6 & \leftarrow & 0. \end{array} \quad (8.4)$$

$$B(-3)$$

We will show now that in this case there is a family of standard resolutions of $R(X)$ having the same assigned matrix l (or equivalently the same \mathfrak{a} -matrix) but different assigned points $p \in \mathcal{V}(l)$ (or equivalently different σ -matrices).

Remark 8.1.5. Let X be a marked numerical Godeaux surface such that the cokernel of the \mathfrak{a} -matrix of a standard resolution of $R(X)$ has Betti numbers as in (8.4). In Chapter 10 we will see that this implies $\text{Tors } X = 0$ and that there is exactly one hyperelliptic curve in the bicanonical system $|2K_X|$.

Now let $B(-1)^6 \xleftarrow{s_l} B(-2)$ be the homomorphism corresponding to the first column of the syzygy matrix of $\mathfrak{a}(l)$ and let $q = (q_0, \dots, q_7)^{tr} \in \mathbb{k}^8$. Then, similarly as above, we define

$$\alpha_0 = \text{id}_{F_0}, \alpha_1 = \begin{pmatrix} \text{id}_6 & & \gamma_q \\ & \text{id}_{12} & \\ & & \text{id}_8 \end{pmatrix}, \alpha_2 = \begin{pmatrix} \text{id}_6 & & \\ & \text{id}_{12} & \\ -\gamma_q & & \text{id}_8 \end{pmatrix} \text{ and } \alpha_3 = \text{id}_{F_0^\vee},$$

where $\gamma_q = s_l q^{tr}$. Then, setting $e_1 = \alpha_0 d_1 \alpha_1^{-1}$ and $e_2 = \alpha_1 d_2 \alpha_2^{-1}$, we obtain another standard resolution of $R(X)$ and an isomorphism of chain complexes:

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & R(X) & \longleftarrow & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_1^\vee & \xleftarrow{d_1^{tr}} & F_0^\vee & \longleftarrow & 0 \\
& & \downarrow & & \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\
0 & \longleftarrow & R(X) & \longleftarrow & F_0 & \xleftarrow{e_1} & F_1 & \xleftarrow{e_2} & F_1^\vee & \xleftarrow{e_1^{tr}} & F_0^\vee & \longleftarrow & 0
\end{array}$$

From the definition of the isomorphisms α_i we see that

$$e'_1 = d'_1 \text{ and } e_2 = \begin{pmatrix} \mathfrak{o} + b_3\gamma_q^{tr} - \gamma_q b_3^{tr} & \mathfrak{n} - \gamma_q \mathfrak{p}(l)^{tr} & b_3(y) \\ -\mathfrak{n}^{tr} + \mathfrak{p}(l)\gamma_q^{tr} & b_4(y) & \mathfrak{p}(l) \\ -b_3(y)^{tr} & -\mathfrak{p}(l)^{tr} & 0 \end{pmatrix}. \quad (8.5)$$

Hence, the two standard resolutions of $R(X)$ have the same assigned matrix $l \in \text{St}(Q)$. Let $\mathcal{V}(l)$ be the solution space, and let $p \in \mathcal{V}(l)$ such that (l, p) is the pair assigned to the resolution in the first row of the commutative diagram above. Then for any point $q \in \mathbb{k}^8$ there exists a point $p_q \in \mathcal{V}(l)$ such that (l, p_q) is the pair assigned to the second row of the diagram. Hence, we have

$$\text{Stab}(l, R(X)) \supseteq \{p_q \mid q \in \mathbb{k}^8\}.$$

The following result shows that we get indeed more than one point in $\text{Stab}(l, R(X))$ in this case:

Lemma 8.1.6. *If $q_1 \neq q_2 \in \mathbb{k}^8$, then $p_{q_1} \neq p_{q_2}$.*

Proof. It is enough to show that any two different elements in \mathbb{k}^8 lead to different σ -matrices. Let us consider the map

$$\begin{aligned}
g: \mathbb{k}^8 &\rightarrow \text{Mat}(\mathbb{k}, 6 \times 6), \\
q &\mapsto b_3\gamma_q^{tr} - \gamma_q b_3^{tr}.
\end{aligned}$$

Then g is a group homomorphism since $\gamma_{q_1+q_2} = \gamma_{q_1} + \gamma_{q_2}$ for any $q_1, q_2 \in \mathbb{k}^8$. From (8.5) we deduce that the σ -matrices corresponding to two distinct points in \mathbb{k}^8 are different if and only if g is injective. On the other hand $g(q) = b_3\gamma_q^{tr} - (b_3\gamma_q^{tr})^{tr}$. Hence it is enough to show that $b_3\gamma_q^{tr}$ is symmetric if and only if $q = 0$. So let $q \in \mathbb{k}^8$. If we write $s_l = (s_0, \dots, s_5)^{tr}$, then

$$\gamma_q = \begin{pmatrix} s_0 q_0 & \cdots & s_0 q_7 \\ \vdots & \ddots & \vdots \\ s_5 q_0 & \cdots & s_5 q_7 \end{pmatrix}.$$

The condition that the 6×6 matrix $b_3\gamma_q^{tr}$ is symmetric can be expressed by $\binom{6}{2} = 15$ equations which are linear in the variables y_i and whose coefficients are linear combinations of the entries of the matrix γ_q . Now the fact that s_l is a non-zero syzygy implies that there exists at least one $j \in \{0, \dots, 5\}$ such that $s_j \neq 0$. Then by equating the coefficients of the 15 polynomials with the zero polynomial, it is now a straightforward calculation to see that these coefficients are all zero if and only if $q = 0$. \square

Note that $p_0 = p$. For an explicit example of a matrix l , where $\text{coker } \mathfrak{a}(l)$ has Betti numbers as in (8.4), we compute with MACAULAY2 that the solution space $\mathcal{V}(l)$ is 12-dimensional. For

a randomly chosen point $q \in \mathbb{k}^8$ we calculate that

$$p_q = p + r_q,$$

where $r_q \in \mathcal{V}(l)$. Furthermore, we compute that

$$\begin{aligned} r_0 &= 0, \\ r_{q_1+q_2} &= r_{q_1} + r_{q_2}, \\ r_{\lambda q_1} &= \lambda r_{q_1} \end{aligned}$$

for two randomly chosen points $q_1, q_2 \in \mathbb{k}^8$ and $\lambda \in \mathbb{k}$ which suggests that $\{r_q \mid q \in \mathbb{k}^8\}$ is an 8-dimensional vector space. Furthermore, our experimental results suggest that

$$\text{Stab}(l, R(X)) = p + \{r_q \mid q \in \mathbb{k}^8\}.$$

Unfortunately, up to now we were not able to verify this.

Remark 8.1.7. In a future work we want to study the remaining linear relations for the other possible Betti numbers of $\text{coker } \mathfrak{a}(l)$. The case which we have described in the last section is the only one which allows the additional condition $\mathfrak{c} = 0$. So far, we have studied some of the other cases only experimentally. For example in the case (8.4), we can assume that 3 of the 4 rows of \mathfrak{c} are a priori zero and that the entries of the remaining row depend only on one of the quadratic forms x_0^2, x_0x_1, x_1^2 . Similarly as in the previous case, we restrict our study then to standard resolutions whose \mathfrak{c} -matrix fulfills this additional assumption.

8.2 A $\mathbb{P}(\mathcal{V}(l))$ of Choices

In this section we study first how standard resolutions with assigned matrices l_1 and l_2 are related if $[l_1] = [l_2] \in F_1(Q)$. Afterwards, we study standard resolutions with a fixed assigned matrix $l \in \text{St}(Q)$ and points $\lambda p \in \mathcal{V}(l)$, where $p \in \mathcal{V}(l)$ and λ varies over \mathbb{k}^* . We will show that the choice of different representatives of a point in $F_1(Q)$ in the first case (respectively of different representatives of a point in $\mathbb{P}(\mathcal{V}(l))$ in the second case) leads to isomorphic rings.

Proposition 8.2.1. *Let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0 \quad (8.6)$$

be a standard resolution of $R(X)$ with assigned pair (l_1, p_1) , where $l_1 \in V_{\text{gensyz}}$ and $p_1 \in \mathcal{V}(l_1)$. Let $u \in \text{GL}(2, \mathbb{k})$ be arbitrary. There exists a ring R_u isomorphic to $R(X)$ which admits a standard resolution with assigned pair (l_2, p_2) , where $l_2 = ul_1$, $p_2 = s_u(p_1)$ and $s_u: \mathcal{V}(l_1) \rightarrow \mathcal{V}(l_2)$ is the isomorphism from Lemma 7.5.11. Moreover, if $R' = R(X')$ is the canonical ring of another marked numerical Godeaux surface X' which admits a standard resolution with assigned pair $(ul_1, s_u(p_1))$, then

$$R' \cong R(X).$$

Proof. Let us write

$$u = \begin{pmatrix} u_{0,0} & u_{0,1} \\ u_{1,0} & u_{1,1} \end{pmatrix} \text{ and } v = \begin{pmatrix} v_{0,0} & v_{0,1} \\ v_{1,0} & v_{1,1} \end{pmatrix},$$

where v is the inverse of u , and consider the ring homomorphism

$$\begin{aligned} \eta_u: \hat{S} = \mathbb{k}[x_0, x_1, y_0, \dots, y_3, z_0, \dots, z_3, w_0, w_1, w_2] &\rightarrow R(X), \\ x_0 &\mapsto v_{0,0}\bar{x}_0 + v_{1,0}\bar{x}_1, \\ x_1 &\mapsto v_{0,1}\bar{x}_0 + v_{1,1}\bar{x}_1, \\ y_j &\mapsto \bar{y}_j, z_j \mapsto \bar{z}_j, w_k \mapsto \bar{w}_k. \end{aligned}$$

Let I_u be the kernel of η_u , and set $R_u := \hat{S}/I_u$. Then $R(X) \cong R_u$.

Now let e_j be the matrix obtained from d_j by substituting every variable x_i by $u_{0,i}x_0 + u_{1,i}x_1$. Then

$$0 \leftarrow R_u \leftarrow F_0 \xleftarrow{e_1} F_1 \xleftarrow{e_2} F_1^\vee \xleftarrow{e_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.7)$$

is a standard resolution of R_u with assigned matrix $l_2 \in \text{St}(Q)$. Furthermore, from the choice of the isomorphism

$$s_u: \mathcal{V}(l_1) \rightarrow \mathcal{V}(l_2)$$

in Lemma 7.5.11, we see that the assigned point to (8.7) is $p_2 = s_u(p_1)$. This shows the first claim. Finally, let R' be another canonical ring which admits a standard resolution with assigned pair $(ul_1, s_u(p_1))$. Then, since the ring R_u has a standard resolution with the same assigned pair, R_u and R' are isomorphic as S -modules, and hence as rings by Remark 5.2.4. Consequently, $R' \cong R_u \cong R(X)$. \square

Remark 8.2.2. This statement shows that if a pair (l, p) with $l \in V_{\text{gensyz}}$ and $p \in \mathcal{V}(l)$ is admissible, then the pair $(ul, s_u(p))$ is also admissible for every $u \in \text{GL}(2, \mathbb{k})$. In particular, for matrices in V_{gensyz} the property of being admissible is invariant under the group action of $\text{GL}(2, \mathbb{k})$. Note that in the proof of Proposition 8.2.1 we used the condition $l \in V_{\text{gensyz}}$ only to apply the isomorphism in Lemma 7.5.11. Hence, focusing only on the assigned matrices, a matrix $l \in \text{St}(Q)$ is admissible if and only if $ul \in \text{St}(Q)$ is admissible for every $u \in \text{GL}(2, \mathbb{k})$. Thus, we can call a point in $F_1(Q)$ *admissible* if it has an admissible representative in $\text{St}(Q)$.

Next let $l \in V_{\text{gensyz}}$ be some matrix. Assume that there exists a numerical Godeaux surface X whose canonical ring $R(X)$ admits a standard resolution with assigned pair $(l, 0)$. Then the second syzygy matrix is

$$d_2(l, 0) = \begin{pmatrix} 0 & 0 & b_3(y) \\ 0 & b_4(y) & \mathfrak{p}(l) \\ -b_3(y)^{tr} & -\mathfrak{p}(l)^{tr} & 0 \end{pmatrix}.$$

Thus,

$$d_1 = \begin{pmatrix} b_0(y) & 0 & 0 \\ \mathfrak{a}(l) & b_1(y) & 0 \\ 0 & \mathfrak{c}(l) & b_2(y) \end{pmatrix}$$

is a first syzygy matrix for $R(X)$. But this implies that the surface $Y = \text{Proj}(S/\text{ann}_S R(X)) \subseteq \mathbb{P}(2^2, 3^4)$ contains the union of surfaces

$$V(y_0, y_1, y_2) \cup V(y_0, y_1, y_3) \cup V(y_0, y_2, y_3) \cup V(y_1, y_2, y_3)$$

which is a contradiction since Y is irreducible. This shows that if a pair (l, p) is admissible, then $p \neq 0$.

So let $0 \neq p \in \mathcal{V}(l)$. How do the matrices $d'_1(l)$ and $d_2(l, p)$ change if we choose the point λp instead of p for some $0 \neq \lambda \in \mathbb{k}$? Clearly, the matrix d'_1 remains unchanged, whereas

$$d_2(l, \lambda p) = \begin{pmatrix} \sigma(l, \lambda p) & \mathbf{n}(l, \lambda p) & b_3(y) \\ -\mathbf{n}(l, \lambda p)^{tr} & b_4(y) & \mathbf{p}(l) \\ -b_3(y)^{tr} & -\mathbf{p}(l)^{tr} & 0 \end{pmatrix} = \begin{pmatrix} \lambda \sigma(l, p) & \lambda \mathbf{n}(l, p) & b_3(y) \\ -\lambda \mathbf{n}(l, p)^{tr} & b_4(y) & \mathbf{p}(l) \\ -b_3(y)^{tr} & -\mathbf{p}(l)^{tr} & 0 \end{pmatrix}.$$

We notice that the matrix

$$\lambda d_2(l, \lambda p) = \begin{pmatrix} \lambda^2 \sigma(l, p) & \lambda^2 \mathbf{n}(l, p) & \lambda b_3(y) \\ -\lambda^2 \mathbf{n}(l, p)^{tr} & \lambda b_4(y) & \lambda \mathbf{p}(l) \\ -\lambda b_3(y)^{tr} & -\lambda \mathbf{p}(l)^{tr} & 0 \end{pmatrix}$$

is simply the matrix $d_2(l, p)$ with each variable x_i (respectively y_j) being substituted by λx_i (respectively λy_j). This observation motivates the following result:

Proposition 8.2.3. *Let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.8)$$

be a standard resolution of $R(X)$ with assigned pair (l, p) , where $l \in V_{\text{gensyz}}$ and $p \in \mathcal{V}(l)$. For every $\lambda \in \mathbb{k}^*$ there exists a ring R_λ isomorphic to $R(X)$ which admits a standard resolution with assigned pair $(l, \lambda p)$. Moreover, if $R' = R(X')$ is the canonical ring of another marked numerical Godeaux surface X' which admits a standard resolution with assigned pair $(l, \lambda p)$, then $R' \cong R(X)$.

Proof. The proof of this statement is based on the same ideas as the one of Proposition 8.2.1. Let $\lambda \in \mathbb{k}^*$ and consider the ring homomorphism

$$\begin{aligned} \eta_\lambda: \hat{S} &\rightarrow R(X), \\ x_i &\mapsto (1/\lambda)\bar{x}_i, \quad y_j \mapsto (1/\lambda)\bar{y}_j, \\ z_j &\mapsto \bar{z}_j, \quad w_k \mapsto \bar{w}_k. \end{aligned}$$

As above, the ring $R_\lambda := \hat{S}/\ker(\eta_\lambda)$ is isomorphic to $R(X)$ and $\text{Proj}(R_\lambda)$ is a further canonical model satisfying General Assumption 1. Next let δ_1 and δ_2 be the matrices obtained from d_1 and d_2 by substituting every variable x_i by λx_i and every y_j by λy_j . Then $\delta_2 = \lambda d_2(l, \lambda p)$ and $\delta_1 = \lambda d'_1(l)$ as above. Furthermore, by the definition of the homomorphism η_λ , there exists a minimal free resolution of R_λ of the form

$$0 \leftarrow R_\lambda \leftarrow F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_1^\vee \xleftarrow{\delta_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.9)$$

with $\delta_2^{tr} = -\delta_2$ but which is not standard. Dividing δ_1 and δ_2 by λ we get a standard resolution with assigned pair $(l, \lambda p)$. The second statement is proven exactly as in Proposition 8.2.1. \square

So as before we conclude that if a pair (l, p) is admissible, with l and p as usual, then the pair $(l, \lambda p)$ is also admissible for every $\lambda \in \mathbb{k}^*$. In particular, for a fixed (admissible) matrix $l \in V_{\text{gensyz}}$, the condition of being an admissible point in $\mathcal{V}(l)$ is independent under the action of \mathbb{k}^* of $\mathcal{V}(l)$. We call a point in $\text{Proj}(\mathcal{V}(l))$ *admissible* if some (and hence any) representative in $\mathcal{V}(l)$ has this property. All in all, we see that for any matrix $l \in V_{\text{gensyz}}$, we have a $\mathbb{P}^{n_3(l)-1}$ of choices for admissible points p with $d'_1(l)d_2(l, p) = 0$.

8.3 A Group Action on $F_1(Q)$

In the beginning of Chapter 7 we have seen that any element $g \in G = (\mathbb{k}^*)^3$ induces an automorphism of \mathbb{P}^3 which fixes the 4 coordinate points of \mathbb{P}^3 . We will now extend this automorphism to an automorphism of $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$. On the algebraic side, the image of the canonical model under this automorphism corresponds then to a ring R_g which is isomorphic to $R(X)$. In this section we analyze how the standard resolutions of $R(X)$ and R_g are related.

An element $g = (\lambda_0, \dots, \lambda_3) \in (\mathbb{k}^*)^4$ induces the automorphism

$$\begin{aligned} \mathbb{P}(2^2, 3^4, 4^4, 5^3) &\rightarrow \mathbb{P}(2^2, 3^4, 4^4, 5^3), \\ (x_0 : x_1 : y_0 : \dots : y_3 : z_0 : \dots : w_2) &\mapsto (x_0 : x_1 : \lambda_0 y_0 : \dots : \lambda_3 y_3 : z_0 : \dots : w_2). \end{aligned}$$

Note that in contrast to what happens over \mathbb{P}^3 , for any $1 \neq \mu \in \mathbb{k}^*$, the elements g and μg define different automorphisms of $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$. The image of X_{can} under this automorphism is another canonical model of X (fulfilling General Assumption 1). To simplify the following computations, we assume from now on that the automorphism associated to g is

$$\begin{aligned} \mathbb{P}(2^2, 3^4, 4^4, 5^3) &\rightarrow \mathbb{P}(2^2, 3^4, 4^4, 5^3), \\ (x_0 : x_1 : y_0 : \dots : y_3 : z_0 : \dots : w_2) &\mapsto (x_0 : x_1 : \lambda_0^2 y_0 : \dots : \lambda_3^2 y_3 : z_0 : \dots : w_2). \end{aligned}$$

Now restricting ρ_g to X_{can} corresponds to a surjective ring homomorphism

$$\begin{aligned} \eta_g : \hat{S} &\rightarrow R(X), \\ x_i &\mapsto \bar{x}_i, \quad y_j \mapsto \lambda_i^2 \bar{y}_i, \\ z_j &\mapsto \bar{z}_j, \quad w_k \mapsto \bar{w}_k, \end{aligned}$$

where $R_g := \hat{S} / \ker(\eta_g)$ is isomorphic to $R(X)$. Let

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{\delta_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.10)$$

be a standard resolution of $R(X)$ with assigned pair (l, p) , where $l \in V_{\text{gensyz}}$ and $p \in \mathcal{V}(l)$. Our aim is to construct, outgoing from the given standard resolution of $R(X)$, a standard resolution of R_g . To do this, we proceed similarly as in the previous section.

Let δ_1 and δ_2 be the matrices obtained from d_1 and d_2 by substituting every variable y_i by y_i / λ_i^2 . Then

$$0 \leftarrow R_g \leftarrow F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_1^\vee \xleftarrow{\delta_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.11)$$

is a minimal free resolution of R_g as an S -module with $\delta_2^{tr} = -\delta_2$. Modulo x_0, x_1 , the resolution above has now a different form, hence it is not standard. But from Proposition 6.3.1 we know that there is always an isomorphism (of chain complexes) to a standard resolution. Finding such an isomorphism is however a tedious computation as in the proof of Proposition 6.3.1. We omit the derivation of the single matrices giving such an isomorphism and present only the results.

Let $\lambda = \lambda_0 \lambda_1 \lambda_2 \lambda_3$ and $\mu_i = \lambda_i^2$. Then we set

$$\alpha_0 = \begin{pmatrix} \lambda & & \\ & \alpha_0^{(1)} & \\ & & \lambda \text{id}_3 \end{pmatrix} \quad \text{and} \quad \alpha_1 = \begin{pmatrix} \alpha_1^{(0)} & & \\ & \alpha_1^{(1)} & \\ & & \alpha_1^{(2)} \end{pmatrix}$$

and define the individual matrices using the indices introduced in Chapter 7.

The matrices $\alpha_0^{(1)}, \alpha_1^{(0)}, \alpha_1^{(1)}, \alpha_1^{(2)}$ are all diagonal matrices. Hence, we will only specify the elements on the diagonal. The matrix $\alpha_0^{(1)}$ is a 4×4 matrix and for $i \in \{0, \dots, 3\}$ we set

$$(\alpha_0^{(1)})_{i,i} = \frac{\lambda}{\lambda_i}.$$

We index the rows and columns of the 6×6 matrix $\alpha_1^{(0)}$ by the elements of

$$N = \{(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3)\},$$

and set for $r = (r_1, r_2) \in N$

$$(\alpha_1^{(0)})_{r,r} = \frac{\lambda}{\mu_{r_1}\mu_{r_2}}.$$

Next we define the 12×12 matrix $\alpha_1^{(1)}$ whose rows and columns are indexed by

$$L_1 = \{(0, 1), (0, 2), (0, 3), (1, 0), (1, 2), (1, 3), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1), (3, 2)\}.$$

For $s = (s_1, s_2) \in L_1$ we set

$$(\alpha_1^{(1)})_{s,s} = \frac{\lambda}{\lambda_{s_1}\mu_{s_2}}.$$

Finally, the rows and columns of the 8×8 matrix $\alpha_1^{(2)}$ are indexed by the elements of the set

$$L_2 = \left\{ \begin{matrix} 1,2 & 0,1 & 1,3 & 0,1 & 2,3 & 0,2 & 2,3 & 1,2 \\ 0 & 2 & 0 & 3 & 0 & 3 & 1 & 3 \end{matrix} \right\}.$$

Let $\theta = \begin{matrix} c,d \\ e \end{matrix} \in L_2$, and let i_θ be the element in $\{0, \dots, 3\} \setminus \{c, d, e\}$. Then we set

$$(\alpha_1^{(2)})_{\theta,\theta} = \frac{\lambda}{\mu_{i_\theta}}.$$

Furthermore, setting

$$e_1 = \alpha_0 \delta_1 \alpha_1^{-1} \text{ and } e_2 = \alpha_1 \delta_2 \alpha_1^{tr} = \alpha_1 \delta_2 \alpha_1,$$

the matrices α_i induce a graded isomorphism of complexes

$$\begin{array}{ccccccccc} 0 & \longleftarrow & R_g & \longleftarrow & F_0 & \xleftarrow{\delta_1} & F_1 & \xleftarrow{\delta_2} & F_1^\vee & \xleftarrow{\delta_1^{tr}} & F_0^\vee & \longleftarrow & 0 \\ & & \text{id} \downarrow & & \alpha_0 \downarrow & & \alpha_1 \downarrow & & \alpha_1^{-1} \downarrow & & \alpha_0^{-1} \downarrow & & \\ 0 & \longleftarrow & R_g & \longleftarrow & F_0 & \xleftarrow{e_1} & F_1 & \xleftarrow{e_2} & F_1^\vee & \xleftarrow{e_1^{tr}} & F_0^\vee & \longleftarrow & 0 \end{array}$$

where the second row is now a standard resolution of R_g . Hence, we can determine the corresponding matrix $l_g \in \text{St}(Q)$ by considering the \mathfrak{a} -matrix of e_1 . To do so, we proceed similarly as in Proposition 8.1.2. The automorphism of \mathbb{A}^{24} defined by

$$\begin{aligned} \mathbb{A}^{24} &\rightarrow \mathbb{A}^{24}, \\ (\dots, \varrho_{i,j,0}^{(i)}, \dots, \varrho_{i,j,1}^{(i)}, \dots) &\mapsto (\dots, \lambda_i \lambda_j^2 \varrho_{i,j,0}^{(i)}, \dots, \lambda_i \lambda_j^2 \varrho_{i,j,1}^{(i)}, \dots) \end{aligned}$$

induces an automorphism of the open subschemes

$$\tilde{\nu}_g: \text{St}(2, 12) \rightarrow \text{St}(2, 12).$$

We claim that $\tilde{\nu}_g(\text{St}(Q)) = \text{St}(Q)$. As in the proof of Proposition 8.1.2 we obtain a corresponding automorphism

$$\tilde{\tau}_g: \mathbb{P}^{11} \rightarrow \mathbb{P}^{11}$$

satisfying

$$\tilde{\tau}_g(l) = \underline{\tilde{\nu}_g(l)}$$

and whose inverse morphism is $\tilde{\tau}_g^{-1}$. Hence, it is enough to show that $\tilde{\tau}_g(Q) = Q$. For $p \in Q$ and $i \in \{0, \dots, 3\}$ we have

$$\mathfrak{q}_i(\tilde{\tau}_g(p)) = \mu_{h_{i,1}}\mu_{h_{i,2}}\mu_{h_{i,3}}\mathfrak{q}_i(p) = 0,$$

where $h_{i,1}, h_{i,2}, h_{i,3}$ are the 3 distinct integers in $\{0, \dots, 3\} \setminus \{i\}$. In particular, for $p \in Q$ we have $\tilde{\tau}_g(p) \in Q$ for any $g \in G$. Hence, the restrictions

$$\nu_g := \tilde{\nu}_g|_{\text{St}(Q)}: \text{St}(Q) \rightarrow \text{St}(Q) \text{ and } \tau_g := \tilde{\tau}_g|_Q: Q \rightarrow Q$$

are automorphisms.

Now the definition of the isomorphisms α_i implies directly that $l_g = \nu_g(l)$. Furthermore, a lengthy calculation shows that there are invertible matrices $\beta_{g,0} \in \text{GL}(54, \mathbb{k})$ and $\beta_{g,1} \in \text{GL}(60, \mathbb{k})$ (depending on g) such that

$$\mathcal{M}(\nu_g(l)) = \beta_{g,0}\mathcal{M}(l)\beta_{g,1},$$

where \mathcal{M} is the matrix representing the linear relations as defined before. Hence, the matrix $\beta_{g,1}^{-1}$ induces an isomorphism of vector spaces

$$s_g: \mathcal{V}(l) \rightarrow \mathcal{V}(\nu_g(l)). \quad (8.12)$$

Furthermore, comparing the \mathfrak{o} -matrices of d_1 and e_1 , we see that $p_g = s_g(p) \in \mathcal{V}(\nu_g(l))$. Thus, the (canonical) ring R_g admits a standard resolution with assigned pair $(\nu_g(l), s_g(p))$.

Remark 8.3.1. Note that, for any $g \in G$, we have $\nu_g(l) \in V_{\text{gensyz}}$ since the matrix $\nu_g(l)$ satisfies all the open conditions from Proposition 7.4.6 and Remark 7.4.7. Indeed, since

$$\mathfrak{a}(\nu_g(l)) = \alpha_0^{(1)}\mathfrak{a}(l)(\alpha_1^{(0)})^{-1} \text{ and } \mathfrak{p}(\nu_g(l)) = \alpha_1^{(1)}\mathfrak{p}(l)(\alpha_1^{(2)})^{-1},$$

we have

- (i) $\text{rank } \mathfrak{a}(\nu_g(l)) = 4$ and $\text{rank } \mathfrak{p}(\nu_g(l)) = 8$,
- (ii) the module $L_{\nu_g(l)} = \text{coker } \mathfrak{a}(\nu_g(l))$ has a minimal free resolution of the form

$$0 \leftarrow L_{\nu_g(l)} \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow B(-3)^2 \leftarrow 0.$$

Remark 8.3.2. The previous arguments show that the multiplicative group $(\mathbb{k}^*)^4$ acts on Q via the automorphisms τ_g . Furthermore, considering the group H from Proposition 8.1.2 as a subgroup of G , we see that the restriction of the action to H is exactly as in Corollary 8.1.3.

Note that for any $\mu \in \mathbb{k}^*$ and $g \in (\mathbb{k}^*)^4$ we have $\tau_g = \tau_{\mu g}$. Indeed, if $p \in Q$, then

$$\tau_{\mu g}(p) = \mu^3\tau_g(p) = \tau_g(p) \in \mathbb{P}^{11}.$$

Writing every element g of the group $G = (\mathbb{k}^*)^3$ as $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ with $\lambda_0 = 1$ and $\lambda_i \in \mathbb{k}^*$ for $i \geq 1$, we get the following result:

Corollary 8.3.3. *The torus $G = (\mathbb{k}^*)^3$ acts on $Q \subseteq \mathbb{P}^{11}$ by*

$$\begin{aligned} G \times Q &\rightarrow Q, \\ (g, q) &\mapsto g \cdot q := \tau_g(q). \end{aligned} \quad (8.13)$$

Moreover, we get an induced action of G on $F_1(Q)$ by setting

$$g \cdot f = [\nu_g(l)], \quad (8.14)$$

where $f \in F_1(Q)$ and $l \in \text{St}(Q)$ with $f = [l]$.

Proof. Clearly, (8.13) is a morphism of algebraic varieties. It remains to show (8.14) is independent of the choice of the representative in $\text{St}(Q)$. So let $l_1, l_2 \in \text{St}(Q)$ with $[l_1] = [l_2]$. Then there exists an element $u \in \text{GL}(2, \mathbb{k})$ such that $l_2 = ul_1$. Then from the definition of ν_g we see directly

$$\nu_g(l_2) = \nu_g(ul_1) = u\nu_g(l_1).$$

Hence $[\nu_g(l_1)] = [\nu_g(l_2)]$ and (8.14) is well-defined. \square

Proposition 8.3.4. *Let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0 \quad (8.15)$$

be a standard resolution of $R(X)$ with assigned pair (l, p) , where $l \in V_{\text{gensyz}}$ and $p \in \mathcal{V}(l)$. Moreover, let $R' = R(X')$ be the canonical ring of another marked numerical Godeaux surface. Let $g \in G$, and let ν_g and s_g be as above. If R' admits a standard resolution with assigned pair $(\nu_g(l), s_g(p))$, then $R' \cong R(X)$.

Proof. From the previous discussion we know that there is a ring R_g isomorphic to $R(X)$ which has a standard resolution with assigned pair $(\nu_g(l), s_g(p))$. This implies that $R' \cong R_g$ by Remark 5.2.4, and hence $R(X) \cong R'$. \square

In the previous section we identified pairs (l, p) under the action of \mathbb{k}^* on $\mathcal{V}(l) \setminus \{0\}$. More precisely, we concluded that it is enough to choose points in the quotient space $\mathbb{P}(\mathcal{V}(l))$. We want to proceed in a similar way with the group action of G on $F_1(Q)$. Our aim is to define a quotient of $F_1(Q)$ under the action of G which is again a projective scheme and satisfies some nice properties. As a first step we want to linearize the group action of G on $F_1(Q)$:

Definition 8.3.5 (see [New78], §4). Let G be an algebraic group. A *linearization* of an action of G on the projective scheme $V \subseteq \mathbb{P}^n$ is a linear action of G on \mathbb{k}^{n+1} inducing the given action on V . A *linear action* of G on V is an action of G together with a linearization.

Remark 8.3.6. Note that there is in general not a unique linear action of G on \mathbb{k}^{n+1} inducing a given action on $V \subseteq \mathbb{P}^n$. For example, let $G = \mathbb{k}^*$ which acts on $V = \mathbb{P}^2$ by

$$\begin{aligned} G \times \mathbb{P}^2 &\rightarrow \mathbb{P}^2, \\ (\lambda, [t_0 : t_1 : t_2]) &\mapsto [\lambda t_0 : \lambda^{-1} t_1 : \lambda^{-1} t_2]. \end{aligned}$$

Then, for every $i \in \mathbb{Z}$, the linear representation

$$\begin{aligned} f_i: G &\rightarrow \text{GL}(3, \mathbb{k}), \\ \lambda &\mapsto \text{diag}(\lambda^{i+1}, \lambda^{i-1}, \lambda^{i-1}) \end{aligned}$$

defines a linear action of G on \mathbb{k}^3 inducing the given action on V .

Let us now consider $F_1(Q)$ as projective subscheme of $\mathbb{P}^{\binom{12}{2}-1} = \mathbb{P}^{65}$ using the Plücker embedding of $\text{Gr}(2, 12)$ (which contains $F_1(Q)$). Then the action of G on \mathbb{P}^{65} defined by

$$G \times \mathbb{P}^{65} \rightarrow \mathbb{P}^{65},$$

$$(g, [\dots : p_{i,j',k,m}^{(i)(k)} : \dots]) \mapsto [\dots : \lambda_i \lambda_j^2 \lambda_k \lambda_m^2 p_{i,j',k,m}^{(i)(k)} : \dots] \quad (8.16)$$

restricts to the action of G on $F_1(Q) \subseteq \mathbb{P}^{65}$. Next we want to choose a linearization of this group action. The linear action of G on \mathbb{k}^{66} determining the ring homomorphism

$$\mathbb{k}[t_{i,j',k,m}^{(i)(k)}] \rightarrow \mathbb{k}[t_{i,j',k,m}^{(i)(k)}],$$

$$t_{i,j',k,m}^{(i)(k)} \mapsto \frac{\lambda_i \lambda_j^2 \lambda_k \lambda_m^2}{\lambda_0 \lambda_1^2 \lambda_2 \lambda_3^2} t_{i,j',k,m}^{(i)(k)} \quad (8.17)$$

induces the given action on \mathbb{P}^{65} , respectively on $F_1(Q)$, with $\lambda_0 = 1$ as before. For example,

$$t_{0,1',0,3}^{(0)(0)} \mapsto \frac{1}{\lambda_2} t_{0,1',0,3}^{(0)(0)},$$

$$t_{1,2',1,3}^{(1)(1)} \mapsto \lambda_2 t_{1,2',1,3}^{(1)(1)},$$

$$t_{0,1',2,3}^{(0)(2)} \mapsto t_{0,1',2,3}^{(0)(2)},$$

$$t_{2,3',1,3}^{(2)(3)} \mapsto \lambda_3 t_{2,3',1,3}^{(2)(3)},$$

$$t_{0,1',0,2}^{(0)(2)} \mapsto \frac{1}{\lambda_3^2} t_{0,1',0,2}^{(0)(2)},$$

$$t_{1,3',1,2}^{(1)(2)} \mapsto \lambda_1 t_{1,3',1,2}^{(1)(2)},$$

$$t_{0,3',0,2}^{(0)(2)} \mapsto \frac{1}{\lambda_1^2} t_{0,3',0,2}^{(0)(2)},$$

$$\vdots$$
(8.18)

After having defined a linear G -action on $F_1(Q) \subseteq \mathbb{P}^{65}$, we want to construct an appropriate quotient scheme. This involves some notation from Geometric Invariant Theory (GIT):

8.3.1 Good and Geometric Quotients

In the following we will give a brief overview on (projective) GIT. For proofs and further details, we refer to [New78].

Definition 8.3.7. Let G be a reductive group acting linearly on a projective scheme $V \subseteq \mathbb{P}^n$, and let $f \in \mathbb{k}[t_0, \dots, t_n]$ be a G -invariant homogeneous polynomial of degree ≥ 1 . Then the set

$$V_f = \{v \in V \mid f(v) \neq 0\}$$

is an affine open G -invariant subset of V . We say that $v \in V$ is

- (i) *semi-stable* for the action of G , if there exists a homogeneous polynomial $f \in \mathbb{k}[x_0, \dots, x_n]$ as above with $v \in V_f$,
- (ii) *stable* for the action of G , if for the orbit $G.v$ of v we have $\dim G.v = \dim G$, and if there exists a form f as above such that G acts on V_f with closed orbits.

Let $V = \text{Proj}(R)$ with $R = \mathbb{k}[t_0, \dots, t_n]/I(V)$, and let R^G be the graded \mathbb{k} -algebra generated by all G -invariant homogeneous polynomials. Since G is reductive, R^G is a finitely generated \mathbb{k} -algebra by Nagata's theorem. The inclusion $R^G \subseteq R$ gives a rational map

$$V \dashrightarrow \text{Proj}(R^G)$$

whose locus of indeterminacy is the closed subscheme N of V given by the ideal $R_+^G = \bigoplus_{d>0} R_d^G$. From the definition of a semi-stable element, we see that $V \setminus N$ is the set of semi-stable elements in V , and hence open. More generally, we have:

Proposition 8.3.8. *The set V^{ss} (respectively V^s) of semi-stable (respectively stable) points of V is a G -invariant Zariski-open subset of V .*

The open set V^{ss} is the domain of definition of the rational map $V \dashrightarrow \text{Proj}(R^G)$. Note that the sets V^{ss} or V^s may be empty.

Example 8.3.9. Being semi-stable depends in general on the choice of the linearization of the action of G . Let us consider again the action of $G = \mathbb{k}^*$ on $V = \mathbb{P}^2$ from Remark 8.3.6. For the linear action induced by f_2 there exists no non-constant G -invariant polynomial. Hence $R^G = \mathbb{k}$ and $V^{ss} = \emptyset$. For the representation f_0 however, one can easily compute that $R^G = \mathbb{k}[t_0 t_1, t_0 t_2]$. Hence $\text{Proj}(R^G) \cong \mathbb{P}^1$ and

$$V^{ss} = \{v = [v_0 : v_1 : v_2] \in \mathbb{P}^2 \mid v_0 v_1 \neq 0 \text{ or } v_0 v_2 \neq 0\}.$$

Let us now introduce the notion of good and geometric quotients:

Definition 8.3.10. Let $V \subseteq \mathbb{P}^n$ and G be as in Definition 8.3.7, and let W be a scheme. We call a morphism $\pi: V \rightarrow W$ a

(1) *good quotient* of V by G if

- (i) π is G -invariant, that means π is G -equivariant, where G acts trivially on W ,
- (ii) π is affine and surjective,
- (iii) for any affine open subscheme $U \subset W$, the induced homomorphism

$$\mathcal{O}(U) \rightarrow \mathcal{O}(\pi^{-1}(U))^G$$

is an isomorphism,

- (iv) if Y is a closed G -invariant subset of V , then $\pi(Y)$ is closed in W ,
- (v) if $Y_1, Y_2 \subseteq V$ are disjoint closed G -invariant subsets, then $\pi(Y_1) \cap \pi(Y_2) = \emptyset$;

(2) *geometric quotient* of V by G , if it is a good quotient and an *orbit space*, that means for any $w \in W$ the fibre $\pi^{-1}(w)$ is a single G -orbit.

Since any fibre of π as in the previous definition is closed, a geometric quotient has the nice property that every orbit of G is closed and that the orbits are separated in the quotient. Unfortunately, not any projective scheme admits a good or a geometric quotient. Restricting to the open subsets of semi-stable or stable points however, we get one of the central results in geometric invariant theory:

Theorem 8.3.11 (Mumford). *Let $V \subseteq \mathbb{P}^n$ and G be as in Definition 8.3.7. Then*

- (i) *there exists a good quotient $\pi: V^{ss} \rightarrow W$ and W is projective,*
- (ii) *there exists an open subset W^s of W such that $\pi^{-1}(W^s) = V^s$ and W^s is a geometric quotient of V^s ,*
- (iii) *for $v_1, v_2 \in V^{ss}$ we have $\pi(v_1) = \pi(v_2)$ if and only if $\overline{G.v_1} \cap \overline{G.v_2} \neq \emptyset$,*
- (iv) *an element $v \in V^{ss}$ is stable if and only if $\dim G.v = \dim G$ and $G.v$ is closed in V^{ss} .*

For $W = \text{Proj}(R^G)$, the restriction $\pi|_{V^{ss}}: V^{ss} \rightarrow W$ satisfies all the properties of a good quotient. In the following we denote a good quotient of V^{ss} by $V^{ss} // G$, whereas we denote a geometric quotient of V^s by V^s/G . Let us summarize the previous results in the following diagram:

$$\begin{array}{ccccc}
 V^s & \subseteq & V^{ss} & \subseteq & V \\
 \text{geometric} \downarrow & & \text{good} \downarrow & & \downarrow \\
 V^s/G & \subseteq & V^{ss} // G & = & \text{Proj}(R^G)
 \end{array}$$

The theorem shows that the set V^{ss} (respectively V^s) plays a central role in finding a good (respectively geometric) quotient. Determining the set of (semi-)stable elements of V is in general a difficult problem. Fortunately, the Hilbert-Mumford Criterion gives a numerical criterion for (semi-)stability depending on 1-parameter subgroups:

Definition 8.3.12. A 1-parameter subgroup of G is a non-trivial homomorphism $\sigma: \mathbb{k}^* \rightarrow G$ of algebraic groups.

Let $V \subseteq \mathbb{P}^n$ and G be as above. Note that a 1-parameter subgroup $\sigma: \mathbb{k}^* \rightarrow G$ together with the linear representation $G \rightarrow \text{GL}(n+1, \mathbb{k})$ induces a linear action of \mathbb{k}^* on \mathbb{k}^{n+1} which can be diagonalized. That means there exists a basis e_0, \dots, e_n of \mathbb{k}^{n+1} such that for all $\lambda \in \mathbb{k}^*$

$$\lambda.e_i = \lambda^{m_i} e_i$$

for some $m_i \in \mathbb{Z}$. Now let $v \in \mathbb{P}^n$, and let $\tilde{v} = \sum_i v_i e_i$ be a lift to \mathbb{k}^{n+1} . Then for $\lambda \in \mathbb{k}^*$ we have

$$\lambda.\tilde{v} = \sigma(\lambda).\tilde{v} = \sum_i \lambda^{m_i} v_i e_i.$$

Definition 8.3.13. With the notation as above, the *Hilbert-Mumford index* of v and σ is

$$\mu(v, \sigma) = \max\{-m_i \mid v_i \neq 0\}.$$

Note that this is well-defined since the right-hand side is independent of the chosen lift of v and of the basis of \mathbb{k}^{n+1} .

Theorem 8.3.14 (Hilbert-Mumford criterion). *Let $V \subseteq \mathbb{P}^n$ be a projective scheme on which the reductive group G acts linearly, and let $v \in V$. Then*

- (i) *v is semi-stable if and only if $\mu(v, \sigma) \geq 0$ for every 1-parameter subgroup σ of G ,*
- (ii) *v is stable if and only if $\mu(v, \sigma) > 0$ for every 1-parameter subgroup σ of G .*

Now we apply the previous statements to the projective scheme $F_1(Q) \subseteq \mathbb{P}^{65}$ on which the reductive group $G = (\mathbb{k}^*)^3$ acts linearly as defined in (8.17). We want to show that, with the chosen linearization, $F_1(Q)^s$ is not empty. Let $F_1(Q) = \text{Proj}(B)$. The following two statements rely on some computational results which will be verified in Chapter 11.

Lemma 8.3.15. *The open set of semi-stable elements $F_1(Q)^{ss}$ is not empty.*

Proof. Recall that a point $v \in F_1(Q)$ is semi-stable if there exists a non-constant form f in B^G such that $f(v) \neq 0$. From (8.18) we see that $f = t_{0,1^2,2,3}^{(0) (2)}$ is a non-constant G -invariant polynomial in $\mathbb{k}[t_{i,j^k,m}^{(i) (k)}]$. Furthermore, by an explicit computation in Chapter 11, we will see that there are points in $F_1(Q)$ having a non-zero $t_{0,1^2,2,3}^{(0) (2)}$ -coordinate. This implies that f is not contained in the defining ideal of $F_1(Q)$ and thus, $f \in B^G$ is non-zero. Hence, any element in the non-empty open set

$$F_1(Q) \cap D_+(t_{0,1^2,2,3}^{(0) (2)})$$

is semi-stable. \square

To show the existence of stable points in $F_1(Q)$ we use the Hilbert-Mumford criterion.

Lemma 8.3.16. *The open set of stable elements $F_1(Q)^s$ is not empty.*

Proof. Any 1-parameter subgroup $\sigma : \mathbb{k}^* \rightarrow G = (\mathbb{k}^*)^3$ maps an element λ to a diagonal matrix $\text{diag}(\lambda^{a_1}, \lambda^{a_2}, \lambda^{a_3})$, where $a_i \in \mathbb{Z}$ are not all zero. Now let $v \in F_1(Q)$, and let $\tilde{v} \in \mathbb{k}^{66}$ be a lift of v considered with respect to the canonical basis of \mathbb{k}^{66} . Then, using (8.18) again, we get

$$\begin{aligned} \lambda \cdot \tilde{v} &= \sigma(\lambda) \cdot (\dots, v_{0,1^0,3}^{(0) (0)}, \dots, v_{0,1^0,2}^{(0) (2)}, \dots, v_{0,1^2,3}^{(0) (2)}, \dots, v_{0,3^0,2}^{(0) (2)}, \\ &\quad \dots, v_{1,2^1,3}^{(1) (1)}, \dots, v_{1,3^1,2}^{(1) (2)}, \dots, v_{2,3^1,3}^{(2) (3)}, \dots) \\ &= (\dots, \lambda^{-a_2} v_{0,1^0,3}^{(0) (0)}, \dots, \lambda^{-2a_3} v_{0,1^0,2}^{(0) (2)}, \dots, \lambda^0 v_{0,1^2,3}^{(0) (2)}, \dots, \lambda^{-2a_1} v_{0,3^0,2}^{(0) (2)}, \\ &\quad \dots, \lambda^{a_2} v_{1,2^1,3}^{(1) (1)}, \dots, \lambda^{a_1} v_{1,3^1,2}^{(1) (2)}, \dots, \lambda^{a_3} v_{2,3^1,3}^{(2) (3)}, \dots). \end{aligned}$$

Thus, if $v \in F_1(Q) \cap D_+(t_{0,1^0,3}^{(0) (0)} t_{0,1^0,2}^{(0) (2)} t_{0,3^0,2}^{(0) (2)} t_{1,2^1,3}^{(1) (1)} t_{1,3^1,2}^{(1) (2)} t_{2,3^1,3}^{(2) (3)})$, then

$$\mu(v, \sigma) \geq \max\{a_2, 2a_3, 2a_1, -a_2, -a_1, -a_3\} > 0$$

since at least one of the integers a_i is non-zero. Hence,

$$F_1(Q) \cap D_+(t_{0,1^0,3}^{(0) (0)} t_{0,1^0,2}^{(0) (2)} t_{0,3^0,2}^{(0) (2)} t_{1,2^1,3}^{(1) (1)} t_{1,3^1,2}^{(1) (2)} t_{2,3^1,3}^{(2) (3)}) \subseteq F_1(Q)^s$$

and we will verify computationally in Chapter 11 that the open set on the left-hand side is non-empty. \square

Let us now assume that $F_1(Q)$ is irreducible as the computation of BERTINI suggests. Otherwise, we replace $F_1(Q)$ by an irreducible 8-dimensional component.

Theorem 8.3.17. *The quasi-projective scheme $F_1(Q)^s/G$ is a geometric quotient of $F_1(Q)^s$ of expected dimension $\dim F_1(Q)^s - \dim G = 5$.*

Proof. Let $\pi : F_1(Q)^{ss} \rightarrow F_1(Q)^{ss} // G$ be the good quotient of $F_1(Q)^{ss}$, where π is the restriction of the rational map $F_1(Q) \dashrightarrow \text{Proj}(B^G)$, and $F_1(Q) = \text{Proj}(B)$ as before. Furthermore,

let us denote the geometric quotient by $\pi_s: F_1(Q)^s \rightarrow F_1(Q)^s/G$, where $\pi_s = \pi|_{F_1(Q)^s}$. From the previous two lemmas we know that the sets $F_1(Q)^{ss}$ and $F_1(Q)^s$ are not empty. It remains to show that the geometric quotient has the expected dimension. Let $v \in F_1(Q)^s$ be arbitrary. Then the orbit $G.v$ is closed in $F_1(Q)^s$, $\dim G.v = \dim G$ and $\pi_s^{-1}(\pi_s(v)) = G.v$. Hence every fibre of the geometric quotient $\pi_s: F_1(Q)^s \rightarrow F_1(Q)^s/G$ has constant dimension 3. Thus, using the properties of a geometric quotient from Definition 8.3.10, we get

$$\dim F_1(Q)^s/G = \dim F_1(Q)^s - \dim G = \dim F_1(Q) - 3 = 5.$$

Note that this implies also that $\dim F_1(Q)^{ss} // G = 5$. Indeed, by Theorem 8.3.11 (ii), the (irreducible) scheme $F_1(Q)^{ss} // G$ contains the 5-dimensional open subset $F_1(Q)^s/G$. Thus $\dim F_1(Q)^{ss} // G = 5$. \square

8.4 Standard Resolutions and Torsion Groups

After having studied standard resolutions of $R(X)$ and their algebraic properties in detail, we will now focus on the geometric side again. In this section we see how the torsion group $\text{Tors } X$ of a marked numerical Godeaux surface X is determined by a standard resolution of $R(X)$. Recall from Lemma 3.1.11 that there is the following connection between the torsion group $\text{Tors } X$ and the number b of base points of $|3K_X|$

- (i) $\text{Tors } X = 0$ if and only if $b = 0$,
- (ii) $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$ if and only if $b = 1$,
- (iii) $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$ if and only if $b = 2$.

Furthermore, recall that we have a commutative diagram of birational morphisms

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X_{can} \\ & \searrow \hat{\varphi} & \downarrow \varphi \\ & & Y \end{array}$$

where $\hat{\varphi}$ is the morphism defined by the global sections $x_0, x_1, y_0, \dots, y_3$. Thus, any base point P of $|3K_X|$ is mapped to a point $(p_0 : p_1 : 0 : 0 : 0 : 0) \in Y$ under $\hat{\varphi}$. Furthermore, any point in X (respectively X_{can}) lying over such a point $(p_0 : p_1 : 0 : 0 : 0 : 0) \in Y$ must be a base point of $|3K_X|$ (respectively $|3K_{X_{can}}|$). Recall that $Y = \text{Proj}(S/I_Y)$, where $I_Y = \text{ann}_S R(X)$. Then, since

$$\sqrt{\text{ann}_S(R(X))} = \sqrt{\langle 8 \times 8 \text{ minors of } d_1 \rangle},$$

we deduce that a point $p \in \mathbb{P}(2^2, 3^4)$ is contained in Y if and only if all maximal minors of d_1 vanish at p . Hence, using this and the characterization above, we get the following criterion relating minimal free resolutions and torsion groups:

Proposition 8.4.1. *Let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be a minimal free resolution of $R(X)$. Let $\hat{d}_1 = d_1 \otimes S/(y_0, \dots, y_3)$ which we consider as a $B = \mathbb{k}[x_0, x_1]$ -module, and let \hat{I}_8 be the ideal generated by the 8×8 minors of \hat{d}_1 . Then

- (i) $\text{Tors } X = 0$ if and only if $V(\hat{I}_8) = \emptyset$,
- (ii) $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$ if and only if $V(\hat{I}_8) = \{p\} \subseteq \mathbb{P}^1$,
- (iii) $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$ if and only if $V(\hat{I}_8) = \{p, q\} \subseteq \mathbb{P}^1$, with $p \neq q$.

Proof. Part (i) is immediate from the previous arguments. To prove part (ii) and (iii), it remains to show that in the case $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$ the two distinct base points P and Q of $|3K_X|$ are mapped to distinct points $p = (p_0 : p_1 : 0 : 0 : 0 : 0)$ and $q = (q_0 : q_1 : 0 : 0 : 0 : 0)$ in Y , or equivalently that $(p_0 : p_1) \neq (q_0 : q_1) \in \mathbb{P}^1$. But from the description of the base points of $|3K_X|$ in Remark 3.1.6 we know that P and Q are contained in two distinct divisors of $|2K_X|$ which shows the claim. \square

Given a standard resolution of the canonical ring of a marked numerical Godeaux surface X , the last statement gives an easy criterion to determine the torsion group of X . However, from a constructional point of view, we are interested in conditions which we can impose a priori on the system $d'_1 d_2 = 0$ and which result then in numerical Godeaux surfaces with a fixed torsion group. We will see that the matrices \mathfrak{a} and \mathfrak{c} of a standard resolution play a central role for these conditions. To begin with, we establish a relation between the matrix \mathfrak{c} and the torsion group of X .

Lemma 8.4.2. *Let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0$$

be a standard resolution of $R(X)$ with assigned matrix $l \in \text{St}(Q)$. If there is a point $p \in \mathbb{P}^1$ such that all 3×3 minors of $\mathfrak{c}(l)$ vanish at p , then $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$ or $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$. In particular, if there are two different points $p, q \in \mathbb{P}^1$ at which all 3×3 minors of $\mathfrak{c}(l)$ vanish, then $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$.

Proof. Let $p = (p_0 : p_1) \in \mathbb{P}^1$ be a point at which all 3×3 minors of $\mathfrak{c}(l)$ vanish. The first part of the statement will follow from the previous proposition once we show that $p \in V(\hat{I}_8)$. For this, recall that $p \in V(\hat{I}_8)$ if and only if all 8×8 minors of the matrix

$$\hat{d}_1 = \left(\begin{array}{c|c|c} * & 0 & * \\ \hline \mathfrak{a}(l) & 0 & \mathfrak{c}(l, p) \\ \hline 0 & \mathfrak{c}(l) & 0 \end{array} \right)$$

vanish at p . But every 8×8 minor of \hat{d}_1 is a product of a 3×3 minor of the matrix $\mathfrak{c}(l)$ with some other polynomial, and thus every 8×8 minor of \hat{d}_1 vanishes at p . The second part follows now directly from Proposition 8.4.1. \square

The statement above gives a sufficient condition for the torsion group to be non-trivial. To turn this into a usable criterion, we would need to show that this condition is also necessary. Even though we have so far not accomplished this, we believe that this is true for various reasons. First recall that numerical Godeaux surfaces having torsion group $\mathbb{Z}/5\mathbb{Z}$ have been completely described due to the work of Godeaux, Miyaoka and Reid and that there is an 8-dimensional irreducible family of such surfaces. Later we will study the canonical ring of a general element X of this family and we will see that X is a marked numerical Godeaux surface. Furthermore, we are able to give the first syzygy matrix of a minimal free resolution of $R(X)$ as an S -module depending on the parameters describing the 8-dimensional family. We will see that, independent of the parameters, the maximal minors of the matrix \mathfrak{c} vanish at exactly two points. Hence, for the (classical) Godeaux surfaces the converse of Lemma 8.4.2 holds.

For the 8-dimensional family of numerical Godeaux surfaces with torsion group $\mathbb{Z}/3\mathbb{Z}$ we proceed similarly. We first describe the canonical ring with respect to the parameters and compute then explicit examples. We will see that there are marked numerical Godeaux surfaces with a torsion group $\mathbb{Z}/3\mathbb{Z}$ and that the computed examples satisfy the converse of Lemma 8.4.2. Having this in mind, we can formulate the following conjecture:

Conjecture 8.4.3. *Let X be a marked numerical Godeaux surface, and let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be any minimal free resolution of $R(X)$. Then $\text{Tors } X = 0$ if and only if the maximal minors of the ϵ -matrix of d_1 have an empty vanishing locus in \mathbb{P}^1 .

Remark 8.4.4. Note that if the condition on the maximal minors of the matrix ϵ holds for one minimal free resolution of $R(X)$, then it is satisfied for any minimal free resolution and in particular also for a standard resolution of $R(X)$.

If true, this conjecture gives a direct way of proving the following nice relation between the torsion group of X and the open set $V_{\text{gensyz}} \subseteq \text{St}(Q)$:

Proposition 8.4.5. *Let X be a marked numerical Godeaux surface, and let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be a standard resolution of the canonical ring $R(X)$ with assigned matrix $l \in \text{St}(Q)$. If $l \in V_{\text{gensyz}}$, then $\text{Tors } X = 0$.

Before giving the proof, we need a result relating the minors of the matrices \mathfrak{a} and ϵ . With the help of SINGULAR we compute:

Corollary 8.4.6. *Let $m_i(\mathfrak{a})$ be the ideal generated by the $i \times i$ minors of \mathfrak{a} , and let $m_j(\epsilon)$ be the ideal generated by the $j \times j$ minors of ϵ . Then*

$$V(m_3(\mathfrak{a})) \cap Q \subset V(m_3(\epsilon)) \cap Q \subset V(m_4(\mathfrak{a})) \cap Q \subseteq \mathbb{P}^{11}.$$

In particular, for any matrix $l \in \text{St}(Q)$ we have

$$V(m_3(\mathfrak{a}(l))) \subseteq V(m_3(\epsilon(l))) \subseteq V(m_4(\mathfrak{a}(l))) \subseteq \mathbb{P}^1. \quad (8.19)$$

Proof of Proposition 8.4.5. By the choice of the open set V_{gensyz} in Proposition 7.4.6, the condition $l \in V_{\text{gensyz}}$ implies that the 4×4 minors of the matrix $\mathfrak{a}(l)$ define the empty set in \mathbb{P}^1 . Hence, from the inclusions of the vanishing loci in (8.19), we deduce that $V(m_3(\epsilon(l)))$ is empty. The claim follows then from Conjecture 8.4.3. \square

Recall that for a matrix $l \in \text{St}(Q)$ with $\text{rank}(\mathfrak{a}(l)) = 4$, the module $L_l = \text{coker } \mathfrak{a}(l)$ has a minimal free resolution of the form

$$\begin{array}{ccccccc}
& & & & B(-2)^{m_2(l)} & & \\
& & & & \oplus & & \\
& & & & B(-3)^{m_3(l)} & & \\
0 \leftarrow L_l \leftarrow B^4 \leftarrow B(-1)^{4+m(l)} \leftarrow & \oplus & \leftarrow 0 & & & & (8.20) \\
& & & & B(-4)^{m_4(l)} & & \\
& & & & \oplus & & \\
& & & & \vdots & &
\end{array}$$

with $0 \leq m(l) \leq 2$. In Lemma 9.1.6 we will show that $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$ implies that $\text{rank}(\mathfrak{a}(l)) = 4$ and $m(l) = 0$. Furthermore, we will see in Lemma 9.2.4 that for numerical Godeaux surfaces having torsion group $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$, we have $m(l) \leq 1$. Hence, if $m(l) = 2$, then $\text{Tors } X = 0$. Relying on these results, we can give an alternative proof of Proposition 8.4.5 without assuming Conjecture 8.4.3:

Proof of Proposition 8.4.5, II. For any $l \in V_{\text{gensyz}}$ the module L_l has a minimal free resolution of type

$$0 \leftarrow L_l \leftarrow B^4 \xleftarrow{\mathfrak{a}(l)} B(-1)^6 \leftarrow B(-3)^2 \leftarrow 0.$$

Hence, $m(l) = 2$ for any $l \in V_{\text{gensyz}}$ and the statement follows from the discussion above. \square

8.5 Counting the Number of Moduli

In Chapter 7 we have presented a method to construct standard resolutions. The aim of this section is to put this construction and the individual results from the last sections together, to obtain an 8-dimensional family of marked numerical Godeaux surfaces having a trivial torsion group. To do this, we assume that we can compute one marked numerical Godeaux surface having a trivial torsion group via this construction. The existence of such an example will be verified in Chapter 11.

To begin with, let us briefly recall some results on the moduli space of surfaces of general type and the number of moduli.

Theorem 8.5.1 (Gieseker). *There exists a quasi-projective coarse moduli space $\mathfrak{M}_{a,b}$ for surfaces of general type with $K^2 = a$ and $\chi = b$.*

Let X be a minimal surface of general type. In the introduction we have seen that $K_X^2 = \chi(\mathcal{O}_X) = 1$ if and only if $K_X^2 = 1$ and $p_g(X) = q(X) = 0$. Hence the space $\mathfrak{M}_{1,1}$ is a coarse moduli space for numerical Godeaux surfaces.

Definition 8.5.2. Let X be a minimal surface of general type with $K_X^2 = a$ and $\chi(\mathcal{O}_X) = b$. The *number of moduli* of X , denoted by $m(X)$, is the local dimension of the moduli space of $\mathfrak{M}_{a,b}$ at the isomorphism class $[X]$ corresponding to X .

There exist an upper and a lower bound for the number of moduli of X in terms of the tangent sheaf of X :

Proposition 8.5.3. *Let X be a minimal surface of general type, and let \mathcal{T}_X denote the tangent sheaf of X . Then*

$$h^1(X, \mathcal{T}_X) \geq m(X) \geq h^1(X, \mathcal{T}_X) - h^2(X, \mathcal{T}_X) \geq 10\chi - 2K_X^2.$$

Proof. See [Cat84], (1.11). □

Applying this to a numerical Godeaux surface we get:

Corollary 8.5.4. *Let X be a numerical Godeaux surface. Then $m(X) \geq 8$.*

In Chapter 11 we will compute an explicit example of a torsion-free numerical Godeaux surface with properties as follows:

Theorem 8.5.5. *There exists a marked numerical Godeaux surface X satisfying:*

(i) *The canonical ring $R(X)$ has a standard resolution with assigned pair (l, p) satisfying $l \in V_{\text{gensyz}}$, $[l] \in F_1(Q)^s$ and $p \in \mathcal{V}(l) \cong \mathbb{k}^4$.*

(ii) *$\text{Tors } X = 0$.*

(iii) *The birational model $Y \subseteq \mathbb{P}(2^2, 3^4)$ of $X_{\text{can}} = \text{Proj}(R(X))$ is smooth. Hence,*

$$X \cong X_{\text{can}} \cong Y.$$

(iv) *There are no hyperelliptic fibres in the bicanonical system $|2K_X|$.*

In particular, this surface is different from the other existing explicit examples of torsion-free numerical Godeaux surfaces due to Barlow and Craighero-Gattazzo, respectively.

After replacing $F_1(Q)$ by an irreducible component if necessary, we assume from now on that $F_1(Q)$ is 8-dimensional and irreducible.

In the last chapters we have seen that the canonical ring of any marked numerical Godeaux surface admits a standard resolution and that we can assign a matrix $l \in \text{St}(Q)$ and a point $p \in \mathcal{V}(l)$ to any such surface. Now let

$$n_{\min} = \min\{\dim_{\mathbb{k}} \mathcal{V}(l) \mid \text{there exists a torsion-free marked numerical Godeaux surface } X \text{ having an assigned matrix } l \in V_{\text{gensyz}} \text{ with } [l] \in F_1(Q)^s\} - 1.$$

From Theorem 8.5.5 we know that the set on the right-hand side is non-empty and that $n_{\min} \leq 3$. Furthermore, since the Betti numbers are upper semi-continuous, there exists a non-empty open subset $V_{\min} \subseteq V_{\text{gensyz}}$ such that for all $l \in V_{\min}$ we have $\dim_{\mathbb{k}}(\mathcal{V}(l)) = n_{\min} + 1$.

For every $l \in V_{\min}$, the vector space $\mathcal{V}(l)$ is a sub-vector space of $(B(-2)^{60})_3 \cong \mathbb{k}^{120}$ of dimension $n_{\min} + 1$. Hence, we obtain a map

$$V_{\min} \rightarrow \text{Gr}(n_{\min} + 1, 120)$$

sending l to $\mathcal{V}(l)$. Then the pull-back E' of the universal subbundle on $\text{Gr}(n_{\min} + 1, 120)$ is a vector bundle on V_{\min} of rank $n_{\min} + 1$ with $(E')_l \cong \mathcal{V}(l)$ for any $l \in V_{\min}$.

In particular, we can consider a pair (l, p) , with $l \in V_{\min}$, from now on as a point in E' , and call this point *admissible*, if (l, p) is admissible. Furthermore, since all conditions of Theorem 5.0.2 are open conditions, the existence of one admissible pair shows that there exists a non-empty open subset in E' of admissible points.

Recall from Section 8.2 that being admissible is invariant under the action of $\mathrm{GL}(2, \mathbb{k})$ on $\mathrm{St}(Q)$, and invariant under the action of \mathbb{k}^* on $\mathcal{V}(l) \setminus \{0\}$ for a given admissible matrix $l \in V_{\min}$. Having this in mind, we will use E' to construct a $\mathbb{P}^{n_{\min}}$ -bundle over some open subscheme of $F_1(Q)$ whose fibres are isomorphic to the projective spaces $\mathbb{P}(\mathcal{V}(l))$ for $l \in V_{\min}$. Let

$$\pi: \mathrm{St}(2, 12) \rightarrow \mathrm{Gr}(2, 12)$$

be the quotient morphism as before. Let $I \in \Gamma = \{J \subseteq \{0, \dots, 11\} \mid |J| = 2\}$, and let $l \in \mathrm{St}(2, 12)$. In the following we call the 2×2 submatrix of l whose columns are given by the set I , the I th submatrix of l , and its determinant the I th minor of l . For $I \in \Gamma$, the sets

$$V_I = \{l \in \mathrm{St}(2, 12) \mid \text{the } I\text{th minor of } l \text{ is non-zero}\}$$

form an open cover of $\mathrm{St}(2, 12)$. The images $U_I := \pi(V_I)$ form the (standard) open cover of $\mathrm{Gr}(2, 12)$ as considered in the previous section. Furthermore, recall that every point in U_I has a unique representative in V_I whose I th submatrix is the unit matrix. Now let $\tilde{U}_I = U_I \cap F_1(Q)$ and $\tilde{V}_I = V_I \cap \mathrm{St}(Q)$. Then, for any $I \in \Gamma$, there exists a unique smooth section

$$\sigma_I: \tilde{U}_I \rightarrow \tilde{V}_I$$

which maps a point in \tilde{U}_I to its unique representative in \tilde{V}_I introduced above. After replacing V_{\min} with a smaller open subset if necessary, we may assume that $V_{\min} \subseteq \tilde{V}_I$ for some $I \in \Gamma$. Let $U_{\min} \subseteq F_1(Q)$ be the inverse image of V_{\min} under σ_I . Then the pull-back of E' under the map σ_I gives a vector bundle on U_{\min} of rank $n_{\min} + 1$. Then $E = \mathbb{P}(\sigma_I^* E')$ is the desired $\mathbb{P}^{n_{\min}}$ -bundle over U_{\min} .

Proposition 8.5.6. *The minimal dimension is $n_{\min} = 3$. Moreover, there exists an open subset U_{moduli} of the \mathbb{P}^3 -bundle E parametrizing an 8-dimensional family of numerical Godeaux surfaces with a trivial torsion group.*

Proof. By the definition of the open set V_{\min} and the integer n_{\min} there exists a marked numerical Godeaux surface X having an assigned pair (l, p) with $l \in V_{\min}$ (respectively $[l] \in U_{\min}$) and $p \in \mathcal{V}(l)$. Let v denote the corresponding point in E . Now since all the conditions from Theorem 5.0.2 are open, there exists a non-empty open subset $U_{\mathrm{moduli}} \subset E$ whose points define standard resolutions of the canonical ring of (marked) numerical Godeaux surfaces, and hence (marked) numerical Godeaux surfaces. Furthermore, from Proposition 8.4.5 we know that all these surfaces have a trivial torsion group. Hence we obtain a family of torsion-free numerical Godeaux surfaces parametrized by the (irreducible) open set U_{moduli} . Now let us compute the dimension of this family. First we have

$$\dim U_{\mathrm{moduli}} = \dim E = \dim U_{\min} + n_{\min} = 8 + n_{\min}.$$

The dimension of the corresponding family of torsion-free numerical Godeaux surfaces is however lower since we also have to take the action of the 3-dimensional algebraic group $G = (\mathbb{k}^*)^3$ on $F_1(Q)$ into account as explained in Section 8.3. From the definition of the open set U_{\min} and (8.12), we deduce that U_{\min} is G -invariant. Hence, our constructed family of numerical Godeaux surfaces has (at most) dimension $8 + n_{\min} - 3 = 5 + n_{\min}$. Now let us suppose that $n_{\min} < 3$. Then we obtain an open neighborhood of $[X] \in \mathfrak{M}_{1,1}$ of dimension $5 + n_{\min} < 8$ which is a contradiction to $m(X) \geq 8$ by Corollary 8.5.4. Hence $n_{\min} = 3$, and E is a \mathbb{P}^3 -bundle containing the open set U_{moduli} which parametrizes an 8-dimensional family of numerical Godeaux surfaces with $\mathrm{Tors} X = 0$. \square

9 Standard Resolutions and the Families with Non-Trivial Torsion

Throughout this chapter we assume that $\mathbb{k} = \mathbb{C}$.

In this chapter we study numerical Godeaux surfaces having a non-trivial torsion group whose order is odd, that means numerical Godeaux surfaces with torsion groups isomorphic to $\mathbb{Z}/3\mathbb{Z}$ or to $\mathbb{Z}/5\mathbb{Z}$. These surfaces have been explicitly described and, in both cases, they fill up an 8-dimensional, irreducible component of the moduli space of numerical Godeaux surfaces due to the results of Godeaux, Reid and Miyaoka (see [Rei78], [Rei], [Miy76]). For each family, we show how the canonical ring $R(X)$ of a general element X of the family fits into our set-up. In particular, we will prove that the general element X of the family of surfaces with torsion group $\mathbb{Z}/5\mathbb{Z}$ is a marked numerical Godeaux surface.

9.1 The Family of $\mathbb{Z}/5\mathbb{Z}$ -Godeaux Surfaces

In this section we first recall the original construction due to Godeaux and then state the description of numerical Godeaux surfaces with $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$ by Miyaoka (see [Miy76]) and Reid (see [Rei78]).

Construction 9.1.1 (Godeaux). Let ξ be a primitive fifth root of unity. We define an action of $G = \mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}_{\langle u_1, \dots, u_4 \rangle}^3$ by

$$(u_1, u_2, u_3, u_4) \mapsto (\xi u_1, \xi^2 u_2, \xi^3 u_3, \xi^4 u_4).$$

Let Y be the Fermat quintic defined by the form $u_1^5 + u_2^5 + u_3^5 + u_4^5$. Then Y is a smooth surface with $K_Y^2 = 5$, $p_g(Y) = 4$ and $q(Y) = 0$. Furthermore, Y is G -invariant and does not contain the 4 fixed points of the group action. Hence G acts freely on Y and the surface $X = Y/G$ is smooth with $K_X^2 = 1$ and $p_g(X) = q(X) = 0$. Moreover, $\text{Tors } X = \pi_1(X) = \mathbb{Z}/5\mathbb{Z}$, where $\pi_1(X)$ is the topological fundamental group of X .

This construction is the basis for a general definition:

Definition 9.1.2 (See [Miy76], Section 5). Let Y' be a normal quintic surface in \mathbb{P}^3 with only a finite number of rational double points. Assume that the cyclic group $G = \mathbb{Z}/5\mathbb{Z}$ acts freely on Y' . Then the minimal nonsingular model X of Y'/G is called a *Godeaux surface*.

The following statement gives a complete classification of Godeaux surfaces and relates them to numerical Godeaux surfaces:

Theorem 9.1.3. *Let X be a numerical Godeaux surface. Then the following conditions are equivalent:*

- (i) X is a Godeaux surface,
- (ii) $\pi_1(X) \cong \mathbb{Z}/5\mathbb{Z}$,

- (iii) $\text{Tors } X \cong \mathbb{Z}/5\mathbb{Z}$,
- (iv) $|3K_X|$ has two base points,
- (v) $\phi_{|3K_X|}(X)$ is a hypersurface of degree 7 in \mathbb{P}^3 .

Proof. See [Miy76], Theorem 4. □

Next we state a result on the moduli space of Godeaux surfaces:

Theorem 9.1.4. *The moduli space of Godeaux surfaces is a normal unirational variety. It is a quotient space of a Zariski open subset of \mathbb{A}^8 by a finite group.*

Proof. See [Miy76], Theorem 5. □

In his proof, Miyaoka characterized the normal quintic surfaces $Y' \subseteq \mathbb{P}^3$ from Definition 9.1.2 on which G acts freely as in Construction 9.1.1. In the following, we will briefly sketch his proof. Let f be the defining equation of Y' . Since G acts on Y' without fixed points, we can assume that, after a suitable linear change of coordinates on \mathbb{P}^3 , the polynomial f is of the form

$$f = u_1^5 + u_2^5 + u_3^5 + u_4^5 + \dots$$

The 8 monomials

$$u_1^3u_3u_4, u_1u_2^3u_3, u_2u_3^3u_4, u_1u_2u_4^3, u_1^2u_2^2u_4, u_1^2u_2u_3^2, u_2^2u_3u_4^2, u_1u_3^2u_4^2 \quad (9.1)$$

are all the other G -invariant monomials of degree 5. Hence a general surface Y' is given by the G -invariant polynomial

$$f = f_c = u_1^5 + u_2^5 + u_3^5 + u_4^5 + c_0u_1^3u_3u_4 + c_1u_1u_2^3u_3 + c_2u_2u_3^3u_4 + c_3u_1u_2u_4^3 + c_4u_1^2u_2^2u_4 + c_5u_1^2u_2u_3^2 + c_6u_2^2u_3u_4^2 + c_7u_1u_3^2u_4^2 \quad (9.2)$$

with $c = (c_0, \dots, c_7)^{tr} \in \mathbb{A}^8$. Since having at most rational double points is an open condition, there exists an open subset of \mathbb{A}^8 yielding surfaces as described in Definition 9.1.2. Now let f, f' be two quintic polynomials as in (9.2) such that, for $W = V(f)$ and $W' = V(f')$, we have

$$W/G \cong W'/G.$$

Then there exists an automorphism $\gamma \in \text{Aut}(\mathbb{P}^3)$ and an element $g \in G$ (inducing an automorphism of \mathbb{P}^3) such that

$$\gamma(W) = \gamma(W') \text{ and } \gamma \circ g = g^n \circ \gamma \quad (9.3)$$

for some $n \in \{1, 2, 3, 4\}$. The claim follows then from the fact that the automorphisms of \mathbb{P}^3 fulfilling (9.3) form a finite subgroup of $\text{Aut}(\mathbb{P}^3)$.

Next we consider the canonical ring of a Godeaux surface X . So let Y' be a quintic surface defined by a G -invariant polynomial f as in (9.2) such that $X = Y'/G$. Then

$$R(X) = R(Y')^G,$$

where $R(Y')$ is the canonical ring of the surface Y' . We will now give a set of algebra generators for $R(X)$ which is independent of the choice of the parameter $c \in \mathbb{A}^8$. By the definition of the group action, a monomial $u_1^{a_1}u_2^{a_2}u_3^{a_3}u_4^{a_4} \in \mathbb{k}[u_1, \dots, u_4]$ is G -invariant if and only if

$$a_1 + 2a_2 + 3a_3 + 4a_4 \equiv 0 \pmod{5}.$$

Using this characterization, we compute (either by hand or with the help of a computer algebra system) a basis for all G -invariant homogeneous polynomials up to degree 5:

degree	basis for the invariants	algebra generators for R^G
2	u_1u_4, u_2u_3	u_1u_4, u_2u_3
3	$u_2u_4^2, u_3^2u_4, u_1^2u_3, u_1u_2^2$	$u_2u_4^2, u_3^2u_4, u_1^2u_3, u_1u_2^2$
4	$u_1^2u_4^2, u_1u_2u_3u_4, u_2^2u_3^2, u_2^3u_4, u_3u_4^3, u_1u_3^3, u_1^3u_2$	$u_2^3u_4, u_3u_4^3, u_1u_3^3, u_1^3u_2$
5	$u_4^5, u_3^5, u_2^5, u_1^5$, 8 forms from (9.1)	u_4^5, u_3^5, u_2^5

Let \hat{S} be the positively graded \mathbb{k} -algebra $\mathbb{k}[x_0, x_1, y_0, \dots, y_3, z_0, \dots, z_3, w_0, w_1, w_2]$ as defined before and consider the ring homomorphism

$$\begin{aligned} \eta: \hat{S} &\rightarrow \mathbb{k}[u_1, u_2, u_3, u_4]/(f), \\ x_0 &\mapsto u_1u_4, x_1 \mapsto u_2u_3, y_0 \mapsto u_2u_4^2, \dots, y_3 \mapsto u_1u_2^2, \\ z_0 &\mapsto u_2^3u_4, \dots, z_3 \mapsto u_1^3u_2, w_0 \mapsto u_4^5, w_1 \mapsto u_3^5, w_2 \mapsto u_2^5, \end{aligned} \quad (9.4)$$

which sends the algebra generators of \hat{S} to the G -invariant polynomials in the right column of the table above. Let I be the kernel of this homomorphism. Then $R(X) \cong \hat{S}/I$ and $\text{Proj}(\hat{S}/I)$ is the canonical model of X .

Next we will consider $R(X)$ as an S -module, being generated by the elements

$$1, z_0, \dots, z_3, w_0, w_1, w_2.$$

In the following we present a minimal generating set of the S -linear relations between the chosen module generators of $R(X)$ (depending on $c \in \mathbb{A}^8$). This allows us to completely determine the first syzygy matrix d_1 of a minimal free resolution of $R(X)$ as an S -module. Note that a relation in $R(X)$ is simply an element of the kernel of η . For example, we have:

$$\begin{aligned} \eta(y_0y_3 - x_0z_0) &= u_2u_4^2u_1u_2^2 - u_1u_4u_2^3u_4 = 0, \\ \eta(y_0y_1 - x_1z_1) &= u_2u_4^2u_3^2u_4 - u_2u_3u_3u_4^3 = 0, \\ \eta(x_1^2z_0 - y_1w_2) &= u_2^5u_3^2u_4 - u_2^5u_3^2u_4 = 0, \\ \eta(c_1x_1^2y_3 + c_2x_1^2y_1 + c_3y_3z_1 + (c_4y_2 + c_6y_1)z_0 + (c_5y_3 + c_7y_0)z_2 + (c_0y_1 + y_2)z_3 \\ &\quad + x_1(w_0 + w_1 + w_2)) &= u_2u_3f = 0. \end{aligned} \quad (9.5)$$

Note that the relations in the first three rows do not depend on the choice of the parameters $c_0, \dots, c_7 \in \mathbb{k}$.

From the description of the minimal free resolution of $R(X)$ in Chapter 3, we know that there are 6 S -linear relations of degree 6. Hence, it is enough to find 6 relations of this degree which are linearly independent. The columns of the following matrix represent relations between the chosen module generators:

1	y_0y_1	$y_0y_2 - x_0^2x_1$	y_0y_3	y_1y_2	$y_1y_3 - x_0x_1^2$	y_2y_3
z_0			$-x_0$			
z_1	$-x_1$					
z_2				$-x_0$		
z_3						$-x_1$
w_0						
w_1						
w_2						

Moreover, the columns are clearly linearly independent. Note that the relations do not depend on the parameters.

Now for degree 7 we find 12 linearly independent relations which we represent in the following matrices:

1	$-x_1^2y_0$	$-x_0x_1y_3$		$-x_0^2y_1$	$-x_0x_1y_0$
z_0	y_1	y_2	y_3		
z_1				y_0	y_2
z_2					y_3
z_3					
w_0			$-x_1$		
w_1					
w_2			$-x_0$		

1	$-x_0x_1y_1$	$-x_1^2y_2$	$-x_0^2y_3$	$-x_0x_1y_2$	$c_1x_1^2y_3 + c_2x_1^2y_1$
z_0					$c_4y_2 + c_6y_1$
z_1					c_3y_3
z_2	y_0	y_1	y_3		$c_5y_3 + c_7y_0$
z_3				y_0	y_1
w_0		$-x_0$			x_1
w_1					x_1
w_2					x_1

As a last step we determine also 8 S -linear relations of degree 8 which are linearly independent and which do not result from the relations of degree 6. The following matrix shows one possible choice for such relations:

1		$x_1y_3^2$	$x_1y_1^2$	$x_0y_0^2$	h_1	h_2
z_0	x_1^2					$c_4x_0^2 + c_1x_0x_1$
z_1		x_0^2			$c_3x_0x_1 + c_6x_1^2$	$c_7x_0x_1 + c_2x_1^2$
z_2			x_1^2		$c_0x_0^2 + c_5x_0x_1$	
z_3						x_0^2
w_0		$-y_2$		$-y_3$	y_1	$y_0 + c_3y_3$
w_1			$-y_3$	$-y_0$	$c_2y_0 + y_1$	y_0
w_2	$-y_1$		$-y_2$		y_1	y_0

where

$$h_1 = c_7x_0y_1^2 + x_0y_2^2 + c_4x_0y_1y_3 + c_1x_1y_1y_3,$$

and

$$h_2 = c_6x_1y_0^2 + c_0x_0y_0y_2 + c_5x_0y_1y_3.$$

Putting the single matrices together, we obtain a first syzygy matrix d_1 of $R(X)$.

Next let us draw some conclusions from these technical computations. First we note that the matrix $\bar{d}_1 = d_1 \otimes S/(x_0, x_1)$ is independent of all parameters c_0, \dots, c_7 . Furthermore, we have

$$\text{ann}_T(\text{coker } \bar{d}_1) = \{p_0, \dots, p_3\}, \quad (9.6)$$

where p_0, \dots, p_3 are the 4 coordinate points of \mathbb{P}^3 as in Chapter 6. Hence the canonical model $X_{\text{can}} = \text{Proj}(R(X)) \subseteq \mathbb{P}(2^2, 3^4, 4^4, 5^3)$ contains 4 distinct points whose x_i -coordinates vanish. Now using the fact that the fixed part F of $|2K_X|$ is zero for any Godeaux surface (see Remark 6.2.2), we get the following result:

Proposition 9.1.5. *Any element of the 8-dimensional family of Godeaux surfaces is a marked numerical Godeaux surface.*

In particular, the canonical ring of any element of this 8-dimensional family admits a standard resolution. Now we will consider the \mathfrak{a} -matrix $\tilde{\mathfrak{a}}$ of d_1 . Since we have only determined the first syzygy matrix, we cannot assign a matrix $l \in \text{St}(Q)$ to $\tilde{\mathfrak{a}}$. But since the rank of the \mathfrak{a} -matrix and the Betti numbers of the corresponding module are the same in any minimal free resolution of $R(X)$, we obtain the following result:

Lemma 9.1.6. *Let X be a Godeaux surface, and let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0 \quad (9.7)$$

be a minimal free resolution of $R(X)$ as an S -module with \mathfrak{a} -matrix $\tilde{\mathfrak{a}}$. Then $\text{rank}(\tilde{\mathfrak{a}}) = 4$ and $L = \text{coker } \tilde{\mathfrak{a}}$ has a minimal free resolution of the form

$$0 \leftarrow L \leftarrow B^4 \leftarrow B(-1)^4 \leftarrow 0,$$

where $B = \mathbb{k}[x_0, x_1]$ as before. In particular, if (9.7) is a standard resolution with $\tilde{\mathfrak{a}} = \mathfrak{a}(l)$ for some $l \in \text{St}(Q)$, then $l \notin V_{\text{gensyz}}$, where V_{gensyz} is the open set of $\text{St}(Q)$ defined in Proposition 7.4.6.

Proof. It is enough to show this statement for the \mathfrak{a} -matrix $\tilde{\mathfrak{a}}$ of the syzygy matrix d_1 defined

above. Then

$$\tilde{a} = \begin{pmatrix} 0 & 0 & -x_0 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_1 \end{pmatrix}$$

and all statements are immediate. \square

We end this section by showing that the reverse of the second statement of Lemma 8.4.2 is true:

Lemma 9.1.7. *Let X be a Godeaux surface, and let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0$$

be a minimal free resolution of $R(X)$ as an S -module with ϵ -matrix \tilde{e} . Then the 3×3 minors of \tilde{e} vanish at two distinct points $p, q \in \mathbb{P}^1$.

Proof. As before, we may assume that d_1 is the matrix defined above. But then

$$m_3(\tilde{e}) = (x_0^2 x_1, x_0 x_1^2) \subseteq B,$$

where $m_3(\tilde{e})$ is the ideal generated by the 3×3 minors of \tilde{e} . Thus, the corresponding vanishing locus contains the points $p = (0 : 1)$ and $q = (1 : 0)$. \square

9.2 The Family of $\mathbb{Z}/3\mathbb{Z}$ -Godeaux Surfaces

In this section we consider the family of numerical Godeaux surfaces having a torsion group isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Reid showed that the moduli space of these surfaces is 8-dimensional and irreducible. As before, the idea is to start with a covering surface corresponding to the torsion group: Take a covering surface of general type Y with invariants $K_Y^2 = 3$, $p_g = 2$ and $q = 0$ on which the cyclic group $\mathbb{Z}/3\mathbb{Z}$ acts freely, then Y/G is a numerical Godeaux surface X with $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$ (see [Rei78]). In the following we will briefly present a refined construction by Reid using unprojection (see [Rei13b]). For further details on this method and the theory of unprojections, we refer to [Rei13b], [Rei00] and [CU16].

Let $\mathbb{P} = \mathbb{P}(1^3, 2^3, 3^3)$ be the weighted projective space with coordinates u_i, v_i, t_i for $i = 0, 1, 2$, and let σ be the permutation $(012) \in S_3$. Then σ acts on the coordinates of \mathbb{P} by

$$\sigma : u_i \mapsto u_{\sigma(i)}, v_i \mapsto v_{\sigma(i)}, t_i \mapsto t_{\sigma(i)}$$

which induces an action of $G = \mathbb{Z}/3\mathbb{Z}$ on \mathbb{P} . Given a weighted homogeneous polynomial $f \in \mathbb{k}[u_i, v_i, t_i]$ we will simply write $\sigma(f)$ for the polynomial $f(u_{\sigma(i)}, v_{\sigma(i)}, t_{\sigma(i)})$ and call the set $\{f, \sigma(f), \sigma^2(f)\}$ the *orbit* of f .

Now consider the three weighted homogeneous polynomials

$$\begin{aligned} f_0 &= -u_0 t_0 + v_1 v_2 - r_0 u_1 u_2, \\ g_0 &= -v_0 t_0 + r_1 u_2 v_1 + r_2 u_1 v_2 + s u_1 u_2, \\ h_0 &= -t_1 t_2 + r_0 v_0^2 + s u_0 v_0 + r_1 r_2 u_0^2, \end{aligned}$$

where r_0 is a general weighted homogeneous polynomial of degree 2 with orbit $\{r_0, r_1, r_2\}$ and s is a general G -invariant weighted homogeneous polynomial of degree 3. Furthermore, let $W \subseteq \mathbb{P}$ be the 4-dimensional variety defined by

$$f_0, f_1 = \sigma(f_0), f_2 = \sigma(f_1), g_0, g_1 = \sigma(g_0), g_2 = \sigma(g_1), h_0, h_1 = \sigma(h_0), h_2 = \sigma(h_1).$$

Theorem 9.2.1. *Let $Y \subseteq \mathbb{P}$ be the surface obtained by intersecting W with the G -invariant linear subspace $\{u_0 + u_1 + u_2 = t_0 + t_1 + t_2 = 0\}$. Then $Y \subseteq \mathbb{P}(1^2, 2^3, 3^2)$ is a canonical surface with $p_g = 2$, $K_Y^2 = 3$.*

Proof. See [Rei13b], Theorem 1.1(A). □

Furthermore, Reid showed that for general choices of r_i and s , the surface Y is smooth and irreducible and the action of G on Y is fixed point free. Hence, $X = Y/G$ is a numerical Godeaux surface with $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$. In the following, we refer to such surfaces simply as $\mathbb{Z}/3\mathbb{Z}$ -Godeaux surfaces. Coughlan and Urzúa restated Reid's result as follows:

Proposition 9.2.2. *The coarse moduli space of $\mathbb{Z}/3\mathbb{Z}$ -Godeaux surfaces is irreducible and unirational of dimension 8. It is covered by the 9-dimensional parameter space given by the following forms for r_i and s*

$$\begin{aligned} r_0 &= a_{11}u_1^2 + a_{12}u_1u_2 + a_{22}u_2^2 + b_0v_0 + b_1v_1, \quad r_1 = \sigma(r_0), \quad r_2 = \sigma(r_1), \\ s &= c_2(u_0^2u_1 + u_1^2u_2 + u_2^2u_0) + c_3(u_0^2u_2 + u_1^2u_0 + u_2^2u_1) \\ &\quad + d_2(u_0v_1 + u_1v_2 + u_2v_0) + d_3(u_0v_2 + u_1v_0 + u_2v_1). \end{aligned}$$

Proof. See [CU16], Proposition 2.2. □

Now let

$$R(Y) = \mathbb{k}[u_0, u_1, v_0, v_1, v_2, t_0, t_1]/(f_0, f_1, f_2, g_0, g_1, g_2, h_0, h_1, h_2),$$

where we have substituted u_2 by $-u_0 - u_1$ and t_2 by $-t_0 - t_1$. The results of Reid show that $R(Y)^G$ is the canonical ring of a $\mathbb{Z}/3\mathbb{Z}$ -Godeaux surface X . Now we determine a set of algebra generators of $R(X) \cong R(Y)^G$ depending on $a_{11}, a_{12}, a_{22}, b_0, b_1, c_2, c_3, d_2, d_3$. To do this, we first compute generating sets for all G -invariant homogeneous polynomials of $R(Y)$ up to degree 5 and then use the forms defining Y to decide which of the elements are needed for a minimal set of algebra generators. Unfortunately, these computations are not as straightforward as in the case of the $\mathbb{Z}/5\mathbb{Z}$ -Godeaux surfaces.

Since there are no relations in $R(Y)$ of degree ≤ 3 we compute (either by hand or using SINGULAR) that the polynomials

$$\begin{aligned} \theta_0 &= v_0 + v_1 + v_2, \\ \theta_1 &= u_0^2 + u_0u_1 + u_1^2 \end{aligned}$$

form a basis for the invariants of degree 2, and that

$$\begin{aligned} \gamma_0 &= u_1v_0 - u_0v_1 - u_1v_1 + u_0v_2, \\ \gamma_1 &= u_0v_0 + u_1v_0 - u_0v_1 - u_1v_2, \\ \gamma_2 &= u_0^2u_1 + u_0u_1^2, \\ \gamma_3 &= u_0^3 - 3u_0u_1^2 - u_1^3 \end{aligned}$$

form a basis for the invariants of degree 3. We choose these 6 polynomials as algebra generators of degree ≤ 3 .

Determining the invariants of degree 4 and 5 involves some more work. First, we compute with the help of SINGULAR bases for the invariants of degree 4 and 5 in the polynomial ring $\mathbb{k}[u_0, u_1, v_0, v_1, v_2, t_0, t_1]$. Then we use the defining relations of Y and the invariants of degree 4 (respectively of degree 5) given by the products of θ_0 and θ_1 (respectively of θ_i and γ_j) to compute relations between these invariants which depend on the parameter $\lambda = (a_{11}, a_{12}, a_{22}, b_0, b_1, c_2, c_3, d_2, d_3)^{tr} \in \mathbb{A}^9$. We performed these lengthy calculations with the help of SINGULAR and present only the results here:

Proposition 9.2.3. *Let $\theta_0, \theta_1, \gamma_0, \dots, \gamma_3$ be as above. Furthermore, let*

$$\begin{aligned}\delta_0 &= u_1 t_0 - u_0 t_1, \\ \delta_1 &= u_0 t_0 + u_0 t_1 + u_1 t_1, \\ \delta_2 &= v_0 v_1 + v_0 v_2 + v_1 v_2, \\ \delta_3 &= u_1^2 v_0 + u_0^2 v_1 + 2u_0 u_1 v_1 + u_1^2 v_1 + u_0^2 v_2\end{aligned}$$

which are invariant forms of degree 4 in $R(Y)$, and

$$\begin{aligned}\epsilon_0 &= v_1 t_0 - v_2 t_0 - v_0 t_1 + v_1 t_1, \\ \epsilon_1 &= v_0 t_0 - v_1 t_0 + v_0 t_1 - v_2 t_1, \\ \epsilon_2 &= 2u_0 u_1 t_0 + u_1^2 t_0 + u_0^2 t_1 + 2u_0 u_1 t_1\end{aligned}$$

which are invariant forms of degree 5 in $R(Y)$. Then, if $\lambda \in \{b_1(b_0 - b_1) \neq 0\} \subseteq \mathbb{A}^9$, the invariants $\theta_0, \theta_1, \gamma_0, \dots, \gamma_3, \delta_0, \dots, \delta_3$ and $\epsilon_0, \epsilon_1, \epsilon_2$ generate $R(Y)^G \cong R(X)$ as a \mathbb{k} -algebra.

Now let \hat{S} be the graded polynomial ring $\mathbb{k}[x_0, x_1, y_0, \dots, y_3, z_0, \dots, z_3, w_0, w_1, w_2]$ as before. Furthermore, let $\lambda = (a_{11}, a_{12}, a_{22}, b_0, b_1, c_2, c_3, d_2, d_3)^{tr} \in \mathbb{A}^9$ with $b_1(b_0 - b_1) \neq 0$. We consider the ring homomorphism

$$\begin{aligned}\eta_\lambda: \hat{S} &\rightarrow \mathbb{k}[u_0, u_1, v_0, v_1, v_2, t_0, t_1]/(f_0, f_1, f_2, g_0, g_1, g_2), \\ x_i &\mapsto \theta_i, \quad y_j \mapsto \gamma_j, \quad z_j \mapsto \delta_j, \quad w_k \mapsto \epsilon_k\end{aligned}\tag{9.8}$$

and set $R_\lambda := \hat{S}/\ker(\eta_\lambda)$. Then R_λ is the canonical ring of a $\mathbb{Z}/3\mathbb{Z}$ -Godeaux surface X .

Now let us consider R_λ as an S -module being generated by $1, z_0, \dots, z_3, w_0, w_1, w_2$. In contrast to the Godeaux surfaces considered in the last section, it was computationally not feasible to determine a general first syzygy matrix of R_λ depending on the parameter $\lambda \in \mathbb{A}^9$. We found one general S -linear relation between the module generators of R_λ by studying the algebra generators of degree 2 and 3 (which are completely independent of the choice of λ). We see that

$$\theta_1^3 - 9\gamma_2^2 - 3\gamma_2\gamma_3 - \gamma_3^2 = 0$$

in $\mathbb{k}[u_0, u_1, v_0, v_1, v_2, t_0, t_1]$. Hence we get a relation of the form

$$x_1^3 - 9y_2^2 - 3y_2y_3 - y_3^2 = 0\tag{9.9}$$

in R_λ for every $\lambda \in \mathbb{A}^9$. Using this, we get a weaker version of Lemma 9.1.6 for an element X of this 8-dimensional family of $\mathbb{Z}/3\mathbb{Z}$ -Godeaux surfaces:

Lemma 9.2.4. *Let X be as above, and let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0\tag{9.10}$$

be a minimal free resolution of $R(X)$ as an S -module with \mathfrak{a} -matrix $\tilde{\mathfrak{a}}$. Then for the minimal free resolution of $L = \text{coker } \tilde{\mathfrak{a}}$

$$\begin{array}{ccccccc}
 & & & & B(-2)^{m_2} & & \\
 & & & & \oplus & & \\
 0 & \leftarrow & L & \leftarrow & B^4 & \leftarrow & B(-1)^{4+m} \leftarrow B(-3)^{m_2} \leftarrow 0 \\
 & & & & \oplus & & \\
 & & & & \vdots & &
 \end{array}$$

we have $m \leq 1$. In particular, if X is a marked $\mathbb{Z}/3\mathbb{Z}$ -Godeaux surface and (9.10) is a standard resolution with assigned matrix $l \in \text{St}(Q)$, then $l \notin V_{\text{gensyz}}$.

Proof. Let $\lambda \in \mathbb{A}^9$ such that $R(X) \cong R_\lambda$. Since (9.9) is a non-zero relation in R_λ , there must be a non-zero relation in $R(X)$ of degree 6 only depending on the algebra generators of $R(X)$ of degree 2 and 3. Now let us consider the minimal free resolution of $R(X)$. We may assume that this relation is in the generating set of the 6 S -linear relations of degree 6. But this implies that the 4×6 matrix $\tilde{\mathfrak{a}}$ of the first syzygy matrix d_1 has a zero-column and the claim follows. \square

By computing examples over finite fields, we find several $\lambda \in \mathbb{A}^9$ such that

$$\text{ann}_T(\text{coker } \tilde{d}_1) = \{\tilde{p}_0, \dots, \tilde{p}_3\}, \tag{9.11}$$

where $\tilde{p}_0, \dots, \tilde{p}_3$ are 4 distinct points in \mathbb{P}^3 and $X_{\text{can}} = \text{Proj}(R(X)) \subseteq \mathbb{P}(2^2, 3^4, 4^4, 5^3)$ is smooth at the corresponding points in X_{can} . After performing a suitable linear change of coordinates on $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$, we may assume that these 4 points are mapped to the coordinate points of \mathbb{P}^3 . Hence, we can compute a standard resolution of $R(X)$ and study the properties of the matrices \mathfrak{a} and \mathfrak{e} . In each of the examples, the 3×3 minors of \mathfrak{e} vanish at exactly one point. Hence, these calculations support Conjecture 8.4.3.

10 Torsion-Free Numerical Godeaux Surfaces and Hyperelliptic Fibres

In this chapter we will focus on numerical Godeaux surfaces having a trivial torsion group. We will show that the existence of smooth hyperelliptic bicanonical curves implies that the morphism $\varphi: X_{can} \rightarrow Y$ is not an isomorphism. Furthermore, we will establish relations between the minimal free resolution of $R(X)$ as an S -module and the existence of hyperelliptic curves in the bicanonical system $|2K_X|$.

Recall that we write the bicanonical system as

$$|2K_X| = |M| + F,$$

where F denotes the fixed part and M is a generic member of the moving part of $|2K_X|$. In [CP00], Catanese and Pignatelli described the curves in the bicanonical system and the induced fibration for a numerical Godeaux surface X with $\text{Tors } X = 0$ or $\text{Tors } X = \mathbb{Z}/2\mathbb{Z}$. Their results distinguish between the different possibilities for (F^2, M^2) and some configurations of the base points of $|M|$. Here, we restrict our study to numerical Godeaux surfaces satisfying the following condition

$$(*) \quad F = 0 \text{ and } |M| \text{ has 4 base points (possible infinitely near).}$$

Note that any marked numerical Godeaux surface clearly fulfills this condition.

Throughout this chapter X denotes a numerical Godeaux surface with $\text{Tors } X = 0$ fulfilling condition $()$.*

10.1 Preliminaries

In this section we first study the fibration f induced by the bicanonical system $|2K_X|$ and then state the results by Catanese and Pignatelli on the fibres of f . By \tilde{X} (respectively \widehat{X}_{can}) we denote the blow-up of X (respectively of X_{can}) at the (smooth) base points of $|2K_X|$ (respectively of $|2K_{X_{can}}|$). Then we have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\beta} & X \\ \downarrow \hat{\pi} & & \downarrow \pi \\ \widehat{X}_{can} & \xrightarrow{\hat{\beta}} & X_{can} \end{array}$$

where β (respectively $\hat{\beta}$) denotes the blow-up morphism and π (respectively $\hat{\pi}$) is a contraction of (-2) -curves. Now let us consider the rational map $X \dashrightarrow \mathbb{P}^1$ induced by the bicanonical system. As X satisfies condition $(*)$, the only points at which this rational map is not defined are the 4 base points of $|2K_X|$.

Hence, we get an induced morphism from the blow-up \tilde{X} to \mathbb{P}^1

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\beta} & X \\
 & \searrow f & \downarrow |2K_X| \\
 & & \mathbb{P}^1
 \end{array}$$

such that f is a fibration of genus 4. In his thesis, Pignatelli described the curves in $|2K_X|$:

Lemma 10.1.1. *Let $C \in |2K_X|$. Then one of the following holds:*

- (i) C is embedded by ω_C and $\phi_3(C) = \phi_{\omega_C}(C)$ is the complete intersection of a quadric and a cubic. Moreover, if $\phi_{\omega_C}(C)$ is reducible, it decomposes into a union of two plane cubics intersecting (with multiplicity) in three points.
- (ii) C is honestly hyperelliptic and $\phi_3(C) = \phi_{\omega_C}(C)$ is a (double) twisted cubic curve.

Case (i) is the general one.

Proof. See [Pig00], Lemma 2.3.1. □

Note that we call a Gorenstein curve C *honestly hyperelliptic* if there exists a finite morphism $C \rightarrow \mathbb{P}^1$ of degree 2. Note that this definition does not require that C is smooth or irreducible. A complete characterization of such curves is given in [CFHR99]:

Lemma 10.1.2. *An honestly hyperelliptic curve C of genus $p_a(C) = g \geq 0$ is isomorphic to a divisor C_{2g+2} in the weighted projective space $\mathbb{P}(1, 1, g + 1)$ not passing through the vertex $(0, 0, 1)$, and defined by an equation of type*

$$w^2 + a_{g+1}(x_1, x_2)w + b_{2g+2}(x_1, x_2) = 0.$$

It follows that every point of C is either nonsingular or a plane double point, and that C is either irreducible or of the form $C = D_1 + D_2$ with $D_1D_2 = g + 1$. The projection $\phi: C \rightarrow \mathbb{P}^1$ is a finite double cover, and the inverse image of any point $x \in \mathbb{P}^1$ is a Cartier divisor $Z \subseteq C$ which is a 0-dimensional scheme of length 2. In other words, Z is either two distinct nonsingular points of C , a nonsingular point with multiplicity 2, or a section through a planar double point of C .

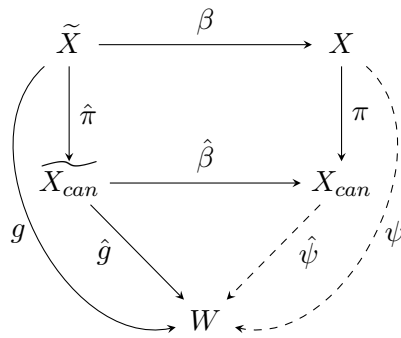
Proof. See [CFHR99], Lemma 3.5. □

Notation 10.1.3. In the following, we will call a smooth honestly hyperelliptic curve simply a hyperelliptic curve.

Next we study the rational map

$$\psi := \phi_2 \times \phi_3: X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^3,$$

where ϕ_n denotes the rational map defined by $|nK_X|$ as before. This map factors through the canonical model. Let $\hat{\psi}$ denote the corresponding map on X_{can} . Since $|3K_X|$ (respectively $|3K_{X_{can}}|$) has no base points, the only points at which ψ (respectively $\hat{\psi}$) is not defined are the base points of $|2K_X|$. Setting $W = \psi(X) = \hat{\psi}(X_{can}) \subseteq \mathbb{P}^1 \times \mathbb{P}^3$, we obtain a commutative diagram



Proposition 10.1.4. *Both maps ψ and $\hat{\psi}$ are birational. Furthermore, if the fibration induced by the bicanonical system on \tilde{X} has a honestly hyperelliptic fibre, then g is not an isomorphism.*

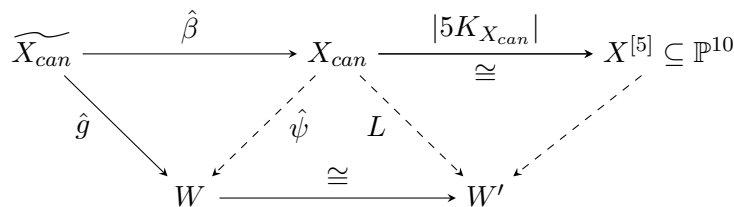
Proof. See [CP00]. □

Pignatelli showed the following result on the number of hyperelliptic fibres of f :

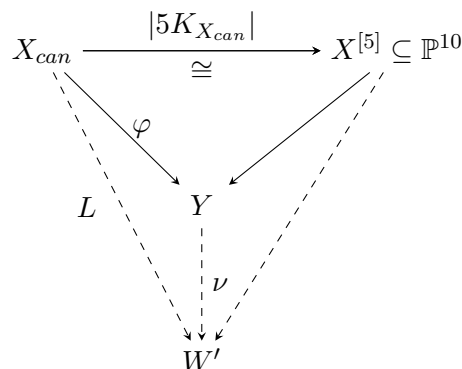
Theorem 10.1.5. *The conductor divisor of the normalization of W is supported on the honestly hyperelliptic fibres of f . If \tilde{h} is the number of the honestly hyperelliptic fibres of f (counted with multiplicity as curves in the conductor divisor), then $0 \leq \tilde{h} \leq 3$. Furthermore, f cannot have 3 distinct hyperelliptic fibres.*

Proof. See [Pig00], Theorem 3.2.1 and Proposition 4.3.1. □

Using the Segre embedding we can identify W with its image W' in \mathbb{P}^7 . Note that the embedding is given by the sections $x_i y_j$, for $i = 0, 1$ and $j = 0, \dots, 3$. Now, let L denote the subsystem of the linear system $|5K_{X_{can}}|$ spanned by these 8 global sections. Then we have a diagram



where $X^{[5]} \dashrightarrow W'$ is the corresponding projection from $X^{[5]}$ to \mathbb{P}^7 . As a next step we want to show how our map $\varphi : X_{can} \rightarrow Y \subseteq \mathbb{P}(2^2, 3^4)$ fits into this diagram. Let $\nu : Y \dashrightarrow \mathbb{P}^7$ be the rational map defined by the forms $x_i y_j$. Then the right-hand side of the previous diagram looks as follows



Suppose that there exists a honestly hyperelliptic fibre C_p of f , where $p \in \mathbb{P}^1$. Then the restriction of $\hat{g}: \widetilde{X_{can}} \rightarrow W$ to C_p is a finite morphism of degree 2. Hence, the non-normal locus of W contains a curve. Now let us consider the finite birational morphism $\varphi: X_{can} \rightarrow Y$, which is the normalization of Y . In the following we will prove that also the morphism φ is not an isomorphism if there exists a smooth hyperelliptic fibre of f . But in contrast to the morphism \hat{g} , the restriction of φ to a hyperelliptic curve is not a degree 2 cover, but the resolution of one double point.

So let $C \in |2K_X|$ be a hyperelliptic curve. Then C is irreducible by Lemma 10.1.2, and hence C does not meet any of the (-2) -curves of X . Thus, we can identify C with its image in X_{can} . Since $g(C) = 4$, the g_2^1 on C is uniquely determined. Let K_0 be the divisor representing the hyperelliptic class. Then $|3K_0| = |K_C|$ and we have the following diagram

$$\begin{array}{ccc}
 & C & \\
 |K_0| \swarrow & \downarrow \phi_{\omega_C} = \phi_3|_C & \\
 \mathbb{P}^1 & \xrightarrow{\nu_3} & \mathbb{P}^3
 \end{array}$$

where ν_3 denotes the 3-uple embedding of \mathbb{P}^1 into \mathbb{P}^3 . The image $D = \phi_{\omega_C}(C) \subseteq \mathbb{P}^3$ is a rational normal curve of degree 3 and $\phi_{\omega_C}: C \rightarrow D$ is a finite degree 2 covering ramified at $2g + 2 = 10$ points. Let $G = \varphi(C) \subseteq Y$, then ϕ_{ω_C} factors through G and we obtain a commutative diagram

$$\begin{array}{ccc}
 C & \searrow \varphi|_C & G \\
 \downarrow \phi_{\omega_C} & & \swarrow \\
 D & &
 \end{array} \tag{10.1}$$

Theorem 10.1.6. *The curve G has exactly one singular point with multiplicity 2. The morphism $\varphi|_C: C \rightarrow G$ is the resolution of this singularity. Moreover, G is an honestly hyperelliptic curve of genus $p_a(G) = 5$.*

To prove this theorem, we will first embed the three curves of Diagram (10.1) in three different projective spaces such that the images of C and G are contained in rational normal scrolls. Then we show that the morphism $\varphi|_C$ corresponds to the restriction of a birational transformation between these scrolls.

10.2 Rational Normal Scrolls in Dimension 2

Let us briefly recall the definition and some properties of rational normal scrolls in dimension 2. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ be a locally free sheaf of rank 2 with $0 \leq a_1 \leq a_2$ and $a_2 > 0$. We consider the corresponding \mathbb{P}^1 -bundle

$$\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^1,$$

which is a rational geometrically ruled surface. The invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is generated by its global sections and defines a morphism to projective space:

Definition 10.2.1. We call the image of $\mathbb{P}(\mathcal{E})$ under the morphism

$$j: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^d = \mathbb{P}(H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)))$$

a *rational normal scroll* of type a_1, a_2 , denoted by $S(a_1, a_2)$.

Note that $d = a_1 + a_2 + 1$. Furthermore, in [Har13] it is shown that $S(a_1, a_2) \subseteq \mathbb{P}^d$ is an irreducible non-degenerate surface of minimal degree, that means

$$\deg(S(a_1, a_2)) = \text{codim}(S(a_1, a_2)) + 1 = a_1 + a_2.$$

If $a_1 > 0$, then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is very ample and $S(a_1, a_2)$ is isomorphic to $\mathbb{P}(\mathcal{E})$. In particular, $S(a_1, a_2)$ is smooth. If $a_1 = 0$, then $S(0, a_2)$ is singular and $j: \mathbb{P}(\mathcal{E}) \rightarrow S(0, a_2)$ is a resolution of singularities.

The Picard group of $\mathbb{P}(\mathcal{E})$ is generated by the class of a fibre $L = [\pi^* \mathcal{O}_{\mathbb{P}^1}(1)]$ and the class of a hyperplane section $H = [j^* \mathcal{O}_{\mathbb{P}^d}(1)]$ with

$$L^2 = 0, LH = 1, H^2 = a_1 + a_2$$

(see [Har77], V.2). In the case where j is an isomorphism, we denote the generators of the Picard group $\text{Pic}(S(a_1, a_2))$ by L and H as well. Furthermore, in the following we will denote both the divisor and its class by L (respectively by H), by abuse of notation.

There is an alternative (equivalent) geometric description of rational normal scrolls which we will briefly explain. Let a_1, a_2 be two integers with $0 \leq a_1 \leq a_2, a_2 > 0$, and set $d = a_1 + a_2 + 1$. In \mathbb{P}^d we choose two complementary linear subspaces W_1 and W_2 of dimension a_1 and a_2 , respectively. Let $\nu_i: \mathbb{P}^1 \rightarrow W_i$ be the a_i -uple embedding of \mathbb{P}^1 into W_i . The image of \mathbb{P}^1 under ν_i is a rational normal curve C_i of degree a_i . Note that if $a_1 = 0$ then C_1 is simply a point. The surface swept out by the lines

$$\overline{\nu_1(t)\nu_2(t)}$$

for t varying over \mathbb{P}^1 is a *rational normal scroll* $S(a_1, a_2)$. If $a_1 = 0$, then $S(0, a_2) \subseteq \mathbb{P}^{a_2+1}$ is the cone over a rational normal curve $C_2 \subseteq \mathbb{P}^{a_2}$ of degree a_2 .

Proposition 10.2.2. *Let $1 \leq a_1 < a_2$. Then:*

- (i) *The rational normal curve $C_1 \subseteq S = S(a_1, a_2)$ occurring in the construction explained above is the unique rational normal curve of degree $< a_2$ on S (other than the lines of the ruling of S). In particular, it is uniquely determined by S . We call this curve the *directrix* of S .*
- (ii) *The image of the scroll $S = S(a_1, a_2)$ under projection from a point $p \in S$ is projectively equivalent to $S(a_1 - 1, a_2)$ if p is contained in the directrix of S , otherwise, the image is projectively equivalent to $S(a_1, a_2 - 1)$.*

Proof. See [Har13], Proposition 8.20. □

Let $S(a_1, a_2) \subseteq \mathbb{P}^{a_1+a_2+1}$ be a rational normal scroll, and let $v_{1,0}, \dots, v_{1,a_1}, v_{2,0}, \dots, v_{2,a_2}$ denote the coordinates of $\mathbb{P}^{a_1+a_2+1}$. Then the ideal of $S(a_1, a_2)$ can be generated by the 2×2 minors of a 1-generic $2 \times (a_1 + a_2)$ matrix of linear forms

$$\left(\begin{array}{ccc|cc} l_{1,0} & \cdots & l_{1,a_1-1} & l_{2,0} & \cdots & l_{2,a_2-1} \\ l_{1,1} & \cdots & l_{1,a_1} & l_{2,1} & \cdots & l_{2,a_2} \end{array} \right), \tag{10.2}$$

such that the 2×2 minors of $\begin{pmatrix} l_{i,0} & \cdots & l_{i,a_i-1} \\ l_{i,1} & \cdots & l_{i,a_i} \end{pmatrix}$ define a rational normal curve of degree a_i in the linear subspace

$$W_i := V(l_{3-i,0}, \dots, l_{3-i,a_{3-i}}) \subseteq \mathbb{P}^{a_1+a_2+1}$$

of dimension a_i , and such that the linear spaces W_1 and W_2 are complimentary (see [Har13], Chapter 9). Hence, any rational normal scroll is a determinantal variety. Furthermore, any such matrix as in (10.2) is conjugate to the matrix

$$\left(\begin{array}{ccc|ccc} v_{1,0} & \cdots & v_{1,a_1-1} & v_{2,0} & \cdots & v_{2,a_2-1} \\ v_{1,1} & \cdots & v_{1,a_1} & v_{2,1} & \cdots & v_{2,a_2} \end{array} \right).$$

10.3 A Criterion for Hyperelliptic Fibres

Now we return to the study of hyperelliptic curves. Eisenbud showed that any hyperelliptic curve C of genus $g \geq 2$ is isomorphic to a curve on a rational normal scroll. To recall the precise statement, let again K_0 denote the divisor on C corresponding to the unique linear system g_2^1 on C . As a consequence of the Riemann-Roch theorem the complete linear system $|(g+k)K_0|$ is very ample if and only if $k \geq 1$. If we write C_k for the image of C under the corresponding embedding, then $C_k \subseteq \mathbb{P}^{g+2k}$ is a non-degenerate, nonsingular curve of degree $2g+2k$.

Theorem 10.3.1. *The curve $C_k \subseteq \mathbb{P}^{g+2k}$ is a divisor of type $2H - (2k-2)L$ on the scroll $S(k-1, g+k)$.*

Proof. See [Eis80], Theorem 3. □

We will apply this result to our given hyperelliptic curve $C \subseteq X_{can}$ of genus 4 and the linear system $|6K_0|$. Let

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be a minimal free resolution of $R(X)$, and let a be the \mathfrak{a} -matrix of d_1 . After applying a linear change of coordinates if necessary, we may assume that

$$C = \text{Proj}(R(X)/(x_0)).$$

Recall that $|6K_0| = |2K_C| = |(6K_X)|_C$. Furthermore, since $h^1(X, \mathcal{O}_X(nK_X)) = 0$ for all n , the sequence

$$0 \rightarrow H^0(X, \mathcal{O}_X((n-2)K_X)) \rightarrow H^0(X, \mathcal{O}_X(nK_X)) \rightarrow H^0(C, \mathcal{O}_C(nK_X|_C)) \rightarrow 0$$

is exact. In particular, this implies that

$$R(X)/(x_0) \cong \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C(nK_X|_C)).$$

From the exact sequence above we compute that $h^0(C, \mathcal{O}_C(6K_0)) = h^0(C, \mathcal{O}_C(6K_X|_C)) = 9$. Hence, there are 6 relations between the 15 global sections

$$x_1^3, \{x_1 z_j\}_{0 \leq j \leq 3}, \{y_i y_j\}_{0 \leq i \leq j \leq 3} \in H^0(C, \mathcal{O}_C(6K_0)).$$

These relations are given by the first 6 columns of $\tilde{d}_1 := d_1 \otimes S/(x_0)$:

$$\begin{array}{c|ccc}
 1 & y_0y_1 + \alpha_1x_1^3 & \dots & y_2y_3 + \alpha_6x_1^3 \\
 \hline
 z_0 & & & \\
 \vdots & & & \\
 z_3 & & a_0 & \\
 \hline
 w_0 & & & \\
 w_1 & & 0 & \\
 w_2 & & &
 \end{array}$$

where $a_0 = a \otimes S/(x_0)$. Note that the entries of the first column represent a set of generators of $R(X)$ as an S -module. Now recall that the image of C under the projection to \mathbb{P}^3 is a rational normal curve whose ideal is generated by 3 quadrics f_0, f_1, f_2 . Hence, these forms are among the relations of degree 6 in $R(X)/(x_0)$ and after multiplying with a suitable invertible matrix from the right, we get the following relations

$$\begin{array}{c|ccc|ccc}
 1 & \tilde{f}_0 & \tilde{f}_1 & \tilde{f}_2 & f_0 & f_1 & f_1 \\
 \hline
 z_0 & & & & & & \\
 \vdots & & & & & & \\
 z_3 & & \tilde{a}_0 & & & 0 &
 \end{array}$$

Lemma 10.3.2. $\text{rank}(\tilde{a}_0) = \text{rank}(a_0) = 3$.

Proof. Since $\text{Tors } X = 0$ by assumption we know from Proposition 8.4.2 that the vanishing locus of the 3×3 minors of e is empty, where e is the ϵ -matrix of d_1 . Then from the inclusions in (8.19) we know that also the vanishing locus of the 3×3 minors of a is empty. Hence, there is no point $p \in \mathbb{P}^1$ with $\text{rank}(a(p)) = 2$. Now assume that $\text{rank}(a_0) < 3$. This implies that every 3×3 minor of $\text{rank}(a_0)$ vanishes, and hence that every 3×3 minor of a vanishes at $p = (0 : 1)$ which is a contradiction. \square

Now this lemma implies that there exist an element $\tilde{z} \in \{z_0, \dots, z_3\}$ and elements $\sigma_6, \sigma_7, \sigma_8 \in \{y_iy_j \mid 0 \leq i < j \leq 3\}$ such that the global sections

$$\sigma_0 = x_1\tilde{z}, \sigma_1 = x_1^3, \sigma_2 = y_0^2, \sigma_3 = y_1^2, \sigma_4 = y_2^2, \sigma_5 = y_3^2, \sigma_6, \sigma_7, \sigma_8$$

form a basis of $H^0(C, \mathcal{O}_C(6K_0))$.

Next let us consider the curve $D \subseteq \mathbb{P}^3$. The invertible sheaf $\mathcal{O}_D(2)$ is very ample. Furthermore, every global section of $\mathcal{O}_D(2)$ is the restriction of a global section of $\mathcal{O}_{\mathbb{P}^3}(2)$, and hence $H^0(\mathbb{P}^3, \mathcal{O}_D(2))$ is 7-dimensional. As a basis of $H^0(\mathbb{P}^3, \mathcal{O}_D(2))$ we can choose the global sections $\sigma_2, \dots, \sigma_8$ (considered as elements of $H^0(\mathbb{P}^3, \mathcal{O}_D(2))$).

As a last step we have to consider the curve $G = \varphi(C) \subseteq \mathbb{P}^7 := \mathbb{P}(2, 3^4)_{\langle x_1, y_0, \dots, y_3 \rangle}$. The line bundle $\mathcal{O}_{\mathbb{P}^7}(6)$ is very ample and $H^0(\mathbb{P}^7, \mathcal{O}_{\mathbb{P}^7}(6))$ is 11-dimensional, where the global sections $x_1^3, y_iy_j, i = 0, 1, j = 0, \dots, 3$, form a basis. Using these global sections, we can embed \mathbb{P}^7 in \mathbb{P}^{10} . On the other hand, the three quadratic forms f_0, f_1, f_2 are among the defining equations of G . Thus, the global sections $\sigma_1, \dots, \sigma_8$ (considered as elements of $H^0(G, \mathcal{O}_G(2))$) define an embedding $G \hookrightarrow \mathbb{P}^7$ with image G' .

All these morphisms fit into a common commutative diagram:

$$\begin{array}{ccccc}
 & & (\sigma_0 : \dots : \sigma_8) & & \\
 & & \cong & & \\
 C & \xrightarrow{\quad} & C_2 & \xrightarrow{\quad} & G' \\
 & \searrow & \searrow \tau_0 & & \\
 & & G & \xrightarrow{(\sigma_1 : \dots : \sigma_8)} & \\
 & & \cong & & \\
 & & D & \xrightarrow{(\sigma_2 : \dots : \sigma_8)} & D_6 \\
 & & \cong & & \\
 & & \nu_3 & & \nu_6 \\
 & & \mathbb{P}^1 & &
 \end{array} \tag{10.3}$$

By Theorem 10.3.1, the curve C_2 is contained in a rational normal scroll $S(1, 6) \subseteq \mathbb{P}^8$. The curve $D_6 \subseteq \mathbb{P}^6$ is a rational normal curve of degree 6. The morphism τ_2 is the restriction of a projection $\hat{\tau}_2$ from the line $V(u_2, \dots, u_8) \subseteq \mathbb{P}^8$ to a complimentary space $V(l_0, l_1) \cong \mathbb{P}^6$, where l_0 and l_1 are linear forms. Hence, $\hat{\tau}_2$ factors through the projection from the point $\Gamma_0 = (1 : 0 : \dots : 0) \in \mathbb{P}^8_{\langle u_0, \dots, u_8 \rangle}$ to $\mathbb{P}^7_{\langle u_1, \dots, u_8 \rangle}$ and the projection from $\Gamma_1 = (1 : 0 : \dots : 0) \in \mathbb{P}^7$ to $V(l_0, l_1) \cong \mathbb{P}^6_{\langle u_2, \dots, u_8 \rangle}$, whose corresponding restrictions are the morphisms τ_0 and τ_1 . This implies that the curve G' is contained in the cone over $D_6 \subseteq \mathbb{P}^6$ with vertex $\Gamma_1 \in \mathbb{P}^7$, hence in a rational normal scroll $S(0, 6) \subseteq \mathbb{P}^7$. Since $\text{Tors } X = 0$, the curve G does not contain any point at which all the y_j vanish. Thus, by the choice of the sections $\sigma_2, \dots, \sigma_8$, the curve G' does not contain the vertex of $S(0, 6)$. The ideal of the rational normal curve $D_6 \subseteq \mathbb{P}^6$ is generated by a 1-generic matrix of linear forms of size 2×6 . Hence, by the previous discussions, there are linear forms m_0, \dots, m_5 in the variables u_2, \dots, u_8 such that the scroll $S(1, 6) \subseteq \mathbb{P}^8$ is defined by the 2×2 minors of the matrix

$$\left(\begin{array}{c|ccc} l_0 & m_0 & \cdots & m_4 \\ l_1 & m_1 & \cdots & m_5 \end{array} \right)$$

with directrix $C_1 = V(m_0, \dots, m_5) = V(u_2, \dots, u_8)$. We can summarize the previous discussions as follows:

Proposition 10.3.3. *The morphism τ_0 is the restriction of the projection from the point Γ_0 on the directrix C_1 of $S(1, 6)$. Hence τ_0 is the restriction of a birational map $S(1, 6) \dashrightarrow S(0, 6)$. Furthermore, the morphism τ_1 is the restriction of the projection from the vertex Γ_1 of the cone $S(0, 6)$ to \mathbb{P}^6 .*

Now from (10.3) we see that the commutative triangle

$$\begin{array}{ccc}
 C_2 & & \\
 \tau_2 \downarrow & \searrow \tau_0 & \\
 & & G' \\
 & \swarrow \tau_1 & \\
 D_6 & &
 \end{array} \tag{10.4}$$

is just the isomorphic “lift” of the triangle (10.1). Hence it is enough to prove Theorem 10.1.6 for the morphism $\tau_0: C_2 \rightarrow G'$.

Remark 10.3.4. Since $\tau_2: C_2 \rightarrow D_6$ is a finite map of degree 2, the inverse image Z_p of any point p in D_6 (respectively G') under τ_1 (respectively τ_0) is a 0-dimensional scheme of length ≤ 2 . Note that Proposition 10.3.3 implies that the length of the inverse image of a point $p \in D_6$ under τ_1 is the number of intersection points (counted with multiplicity) of the line through Γ_1 and p with the curve G' .

Proof of Theorem 10.1.6. Since G' does not contain the vertex of $S(0, 6)$ we will identify G' with its preimage under the resolution of singularities $j: \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(6)) \rightarrow S(0, 6)$ which is the blow-up of the vertex Γ_1 . Note that the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(6))$ is the 6th Hirzebruch surface X_6 . The exceptional divisor of this blow-up is a curve B with $B \sim H - 6L$. Let $a, b \in \mathbb{Z}$ such that $G' \sim aH + bL$. From Remark 10.3.4 we know that $1 \leq a \leq 2$. Furthermore, we have

$$0 = G'B = (aH + bL)(H - 6L) = b.$$

Hence $G' \sim H$ or $G' \cong 2H$. In the following, we will show that the first case cannot occur. First let $Z_1 \rightarrow S(1, 6)$ be the blow-up of $\Gamma_0 \in S(1, 6)$. Then the induced birational morphism $Z_1 \rightarrow S(0, 6)$ factors through $X_6 \rightarrow S(0, 6)$, where the induced morphism $Z_1 \rightarrow X_6$ is the contraction of a (-1) -curve. Furthermore, since C_2 does not contain Γ_0 , we can identify C_2 with its isomorphic image in Z_1 . Then G' is the image of this curve under the morphism $Z_1 \rightarrow X_6$. In particular, $\tau_0: C_2 \rightarrow G'$ is a birational morphism. Hence, $G' \cong 2H$. Now let ℓ be any line through Γ_0 intersecting the curve C_2 . Suppose that ℓ intersects C_2 in two points (counted with multiplicity). Then, since $S(1, 6)$ is defined by quadrics, Bézout’s Theorem implies that $\ell \subseteq S(1, 6)$. In particular, ℓ is the unique line of the ruling through the point Γ_0 . This implies that the curve G' has exactly one singular point q . Furthermore, q is either a node if ℓ intersects C_2 in two distinct points, or a cusp otherwise. Hence, G' is an honestly hyperelliptic curve with $p_a(G') = 5$. \square

Remark 10.3.5. Note that we can deduce the number of singular points of $G' \subseteq X_6$ and their types also directly from the genus formula. Indeed, we have

$$2p_a(G') - 2 = K_{X_6}G' + (G')^2 = (-2H + 4L)(2H) + (2H)^2 = 8$$

which implies $p_a(G') = 5$.

Proposition 10.3.6. *Let X be a marked numerical Godeaux surface. If there exists a hyperelliptic curve $C \in |2K_X|$, then the finite birational morphism $\varphi: X_{can} \rightarrow Y$ is not an isomorphism. More precisely, for every hyperelliptic curve $C \in |2K_X|$, there exists a unique point $q \in \varphi(C) \subseteq Y$ such that Y is not normal at q .*

Proof. As before we identify C with its isomorphic image in X_{can} . Suppose to the contrary that $\varphi: X_{can} \rightarrow Y$ is an isomorphism. But then also $\varphi|_C$ is an isomorphism of C onto its image $G \subseteq Y$. But this is a contradiction to Theorem 10.1.6. In particular, the point $q \in G \subseteq Y$ is a non-normal point of Y . \square

The last step of this chapter is to translate these results to (algebraic) properties of the minimal free resolution of the canonical ring $R(X)$. In the following, let X be a fixed marked numerical Godeaux surface (with $\text{Tors } X = 0$), and let

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{tr}} F_0^\vee \leftarrow 0$$

be a minimal free resolution of $R(X)$ as an S -module with \mathfrak{a} -matrix a .

Lemma 10.3.7. *If there exists a hyperelliptic curve $C \in |2K_X|$, then there exists a point $q \in Y$ at which the 7×7 minors of d'_1 vanish.*

Proof. In Proposition 10.3.6 we proved that if C is a hyperelliptic curve, then there exists a point $q \in Y$ at which the morphism $(\mathcal{O}_Y)_q \rightarrow (\varphi_* \mathcal{O}_{X_{can}})_q$ is not an isomorphism. Now recall that we have an exact sequence of \mathcal{O}_Y -modules

$$0 \rightarrow \mathcal{O}_Y \rightarrow (\varphi_* \mathcal{O}_{X_{can}}) \rightarrow \widetilde{M} \rightarrow 0,$$

where $M = \text{coker } d'_1$. Consequently,

$$q \in \text{Supp}(\widetilde{M}) \subseteq V(7 \times 7 \text{ minors of } d'_1) \subseteq \mathbb{P}(2^2, 3^4)$$

which shows the claim. \square

Let $0 \neq \tilde{x} \in H^0(X, 2K_X)$, and let p be the corresponding point in \mathbb{P}^1 . We denote the bicanonical curve $\text{Proj}(R(X)/\tilde{x}) \in |2K_{X_{can}}|$ by C_p . If C_p is a hyperelliptic curve, then Lemma 10.3.2 implies that $\text{rank}(a(p)) = 3$. Now we show that the converse of this statement holds as well. This gives a complete characterization of the existence of hyperelliptic curves in terms of the a -matrix of d_1 :

Proposition 10.3.8. *Let $p \in \mathbb{P}^1$ such that the curve C_p is smooth. Then, C_p is hyperelliptic if and only if $\text{rank}(a(p)) = 3$.*

Proof. After a linear change of coordinates we may assume that $p = (0 : 1)$. Let us suppose that $\text{rank}(a(p)) = 3$. Recall from Lemma 10.1.1 that the image D of C_p under $\phi_3: X_{can} \rightarrow \mathbb{P}^3$ is a complete intersection of type $(2, 3)$ if C_p is not hyperelliptic, or a twisted cubic curve otherwise. Hence, it is enough to prove that the ideal of D contains two linearly independent quadrics. To show this, we proceed similarly as in the proof of Lemma 10.3.2. The condition $\text{rank}(a(p)) = 3$ implies that $\text{rank}(a_0) = 3$, where $a_0 = a \otimes S/(x_0)$. Now let us consider the first 6 columns of $\tilde{d}_1 = d_1 \otimes S/(x_0)$:

$$\begin{array}{c|cccc} 1 & y_0 y_1 + \alpha_1 x_1^3 & \dots & y_2 y_3 + \alpha_6 x_1^3 & & \\ \hline z_0 & & & & & \\ \vdots & & & & & \\ z_3 & & & a_0 & & \end{array}$$

After multiplying with a suitable invertible matrix from the right, we can assume that the relations are of the form

$$\begin{array}{c|ccccc} 1 & f_0 + \beta_0 x_1^3 & f_1 & f_2 & f_3 & f_4 & f_5 \\ \hline z_0 & & & & & & \\ \vdots & & & & & & \\ z_3 & & & \tilde{a}_0 & & & \end{array}$$

where f_0, \dots, f_5 do only depend on the variables y_0, \dots, y_3 and $\beta_0 \in \mathbb{k}$. The submatrix of \tilde{a}_0 corresponding to the last 5 columns of the matrix has $\text{rank} \leq 3$. Hence, after a further column operation on the last 5 columns, we obtain a new matrix of relations:

$$\begin{array}{c|ccc|cc}
 1 & f_0 + \beta_0 x_1^3 & \tilde{f}_1 & \tilde{f}_2 & \tilde{f}_3 & \tilde{f}_4 & \tilde{f}_5 \\
 \hline
 z_0 & & & & & & \\
 \vdots & & & & & & \\
 z_3 & & * & & & & 0
 \end{array}$$

Hence, among the defining relations of C_p there are the two linearly independent quadrics \tilde{f}_4, \tilde{f}_5 depending only on y_0, \dots, y_3 which shows the claim. \square

As a final result, we establish a relation between the open set $V_{\text{gensyz}} \subseteq \text{St}(Q)$ and the existence of hyperelliptic bicanonical curves:

Corollary 10.3.9. *Let X be any marked numerical Godeaux surface, and let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0$$

be a standard resolution of $R(X)$ with assigned matrix $l \in \text{St}(Q)$. If $l \in V_{\text{gensyz}}$, then $\text{Tors } X = 0$ and $|2K_X|$ contains no (smooth) hyperelliptic curves.

Proof. From Proposition 8.4.5 we already know that $l \in V_{\text{gensyz}}$ implies that $\text{Tors } X = 0$. But $l \in V_{\text{gensyz}}$ implies also that the vanishing locus of the 4×4 minors of the matrix $\mathfrak{a}(l)$ is empty. Thus, $|2K_X|$ has no hyperelliptic curves by Proposition 10.3.8. \square

Remark 10.3.10. The results of this section can be generalized to the case of an irreducible (not necessarily smooth) honestly hyperelliptic curve C with $p_a(C) = 4$:

The definition of an honestly hyperelliptic curve and Lemma 10.1.2 imply that C is a reduced irreducible Gorenstein curve. Then, for every n , the sheaves $\mathcal{O}_C(K_C)$ and $\mathcal{O}_C(nK_0)$ are invertible, where K_0 is defined as before. Using the Riemann-Roch theorem (for singular curves) we get $h^0(C, \mathcal{O}_C(K_C)) = h^0(C, \mathcal{O}_C(3K_0)) = 4$ and $h^0(C, \mathcal{O}_C(nK_0)) = 2n - 3$ for $n \geq 4$. From the factorization $\mathcal{O}_C(6K_0) = \mathcal{O}_C(K_0) \otimes \mathcal{O}_C(5K_0)$ we obtain a multiplication map

$$\mu: H^0(C, \mathcal{O}_C(K_0)) \otimes H^0(C, \mathcal{O}_C(5K_0)) \rightarrow H^0(C, \mathcal{O}_C(6K_0)).$$

Furthermore, using the base-point-free pencil trick, we obtain an exact sequence

$$0 \rightarrow \mathcal{O}_C(4K_0) \rightarrow H^0(C, \mathcal{O}_C(K_0)) \otimes \mathcal{O}_C(5K_0) \rightarrow \mathcal{O}_C(6K_0) \rightarrow 0.$$

Taking global section we get a sequence

$$0 \rightarrow H^0(C, \mathcal{O}_C(4K_0)) \rightarrow H^0(C, \mathcal{O}_C(K_0)) \otimes H^0(C, \mathcal{O}_C(5K_0)) \rightarrow H^0(C, \mathcal{O}_C(6K_0)) \rightarrow 0$$

which is exact since $h^1(C, \mathcal{O}_C(4K_0)) = 0$. Hence, the multiplication map μ is surjective.

Now let $V = H^0(C, \mathcal{O}_C(6K_0))$, and choose bases $s_0, s_1 \in H^0(C, \mathcal{O}_C(K_0))$ and $t_0, \dots, t_6 \in H^0(C, \mathcal{O}_C(5K_0))$. Then we obtain a 2×7 matrix $M(\mathcal{O}_C(K_0), \mathcal{O}_C(5K_0))$ of linear forms on $\mathbb{P}(V)$ whose entry in row i and column j is $s_i \otimes t_j \in V$. As C is reduced and irreducible, $M(\mathcal{O}_C(K_0), \mathcal{O}_C(5K_0))$ is a 1-generic matrix of linear forms whose 2×2 minors vanish on the image of C under the embedding $C \hookrightarrow \mathbb{P}(V)$ (see [Eis05], Proposition 6.10). Now the variety in $\mathbb{P}(V) \cong \mathbb{P}^8$ defined by the 2×2 minors of $M(\mathcal{O}_C(K_0), \mathcal{O}_C(5K_0))$ is a rational normal scroll of degree 7 which is smooth since μ is surjective, and hence isomorphic to $S(1, 6)$. Then, as in the smooth case, the image of C under this embedding is a divisor on $S(1, 6)$ linearly equivalent to $2H - 2L$. Proceeding along the same lines as for a smooth curve, we obtain the corresponding results for an irreducible honestly hyperelliptic curve C , where the morphism $\varphi|_C: C \rightarrow G$ from Theorem 10.1.6 is now a birational morphism resolving a singularity of multiplicity 2.

11 Explicit Examples with Macaulay2

In this chapter we present some explicit examples of our computations of marked numerical Godeaux surfaces. First we construct an example with torsion group $\mathbb{Z}/5\mathbb{Z}$ and one with torsion group $\mathbb{Z}/3\mathbb{Z}$ using the ideas from Chapter 9. Then we use our methods to compute, in each case, a standard resolution, the associated matrix in $\text{St}(Q)$ and check that Conjecture 8.4.3 holds for these examples.

Afterwards we focus on numerical Godeaux surfaces having a trivial torsion group. We present the main parts of our construction and the corresponding `Macaulay2`-procedures. Furthermore, we want to verify all the assumptions we have made throughout the last chapters, in particular the existence of the non-empty open set $V_{\text{gensyz}} \subseteq \text{St}(Q)$ and the non-emptiness of the set of stable points $F_1(Q)^s$. Afterwards we calculate an explicit example of a torsion-free marked numerical Godeaux surface X having no hyperelliptic bicanonical curves. After that we will briefly sketch the construction of the simply connected Barlow surface and verify computationally that this surface contains two distinct hyperelliptic fibres. In the end, we present an example of a numerical Godeaux surface (over a field with characteristic $p > 0$) having one hyperelliptic fibre.

11.1 An Example with Torsion Group $\mathbb{Z}/5\mathbb{Z}$

In this section we give an explicit example of a numerical Godeaux surface with torsion group $\mathbb{Z}/5\mathbb{Z}$. As a first step, we use the results from Section 9.1 to compute the canonical ring of such a surface as an invariant ring under a free action of $G = \mathbb{Z}/5\mathbb{Z}$. Our procedures are contained in the `Macaulay2`-file `numGodeauxZ5` which we load at the beginning of our session. They work over a finite field $\mathbb{k} = \mathbb{F}_p$ or over $\mathbb{k} = \mathbb{Q}$. However, from the proof of Proposition 6.3.1, we know that the calculation of a standard resolution may involve the computation of some square roots. If these square roots are not contained in \mathbb{k} , we cannot compute a standard resolution over the base field.

To simplify the computations and to avoid coefficient growth we give an example over the finite field \mathbb{F}_{197} . Recall from Section 9.1 that we have to compute a quintic surface $Y' \subseteq \mathbb{P}^3$ depending on 8 parameters on which G acts freely. Afterwards we compute algebra generators of $R(Y')^G \cong R(X)$. Note that the invariants of $R(Y')$ up to degree 5 do not depend on the quintic Y' . So we compute them as a first step:

```
i1 : load "numGodeauxZ5.m2"
i2 : kk = ZZ/197;
i3 : invZ5 = computeZ5Invariants(kk);
i4 : invZ5
o4 = | u_1u_4 u_2u_3 u_2u_4^2 u_3^2u_4 u_1^2u_3 u_1u_2^2
-----
u_2^3u_4 u_3u_4^3 u_1u_3^3 u_1^3u_2 u_4^5 u_3^5 u_2^5 |
```

Next we compute a G -invariant quintic surface $Y' \subseteq \mathbb{P}^3$ depending on a parameter $r \in \mathbb{A}^8$. Note that the definition of a Godeaux surface requires that Y' is normal having at most finitely many rational double points. For the sake of simplicity, we restrict our computations to smooth quintic surfaces.

```
i5 : r = random(kk^1, kk^8)

o5 = | 16 49 -71 97 -78 -43 -69 34 |

i6 : -- compute the corresponding quintic surface Y'
      IY' = computeInvariantQuintic(r);

i7 : gens IY'

o7 = |u_1^5+u_2^5+27u_1u_2^3u_3-20u_1^2u_2u_3^2+u_3^5
-----
+8u_1^2u_2^2u_4+74u_1^3u_3u_4+33u_2u_3^3u_4
-----
-16u_2^2u_3u_4^2-82u_1u_3^2u_4^2+9u_1u_2u_4^3+u_4^5 |
```

Next we determine the canonical ring $R(X)$ by computing the kernel of the homomorphism $\eta: \hat{S} \rightarrow Z/I(Y')$ as in (9.4):

```
i8 : IX = canonicalRing(IY');

o8 : Ideal of Sbig

i9 : betti gens IX

o9 = total: 0 1
          1 54
          0: 1 .
          1: . .
          2: . .
          3: . .
          4: . .
          5: . 6
          6: . 12
          7: . 18
          8: . 12
          9: . 6
```

Note that the number and degrees of the ideal generators of $I(X)$ are exactly as deduced in Proposition 3.3.9.

Next we consider $R(X)$ as an S -module being generated by the elements $1, z_0, \dots, z_3, w_0, w_1, w_2$. We compute a first syzygy matrix d_1 of $R(X)$ as an S -module, or equivalently, a (minimal) generating set for the S -linear relations in $R(X)$. The relations of degree 6 and 7 are simply the generators of the ideal $I(X)$ in the corresponding degrees. Furthermore, we know that there are 8 S -linear (independent) relations of degree 8 which we can compute from the 18 generators of degree 8 of $I(X)$. Note that there are 10 relations in degree 8 which are not S -linear expressing the different products $z_i z_j \in R(X)$ as S -linear combinations of $1, z_0, \dots, z_3, w_0, w_1, w_2$. Knowing the first syzygy matrix we can easily compute a minimal free resolution of $R(X) = \text{coker } d_1$ with Macaulay2 and verify that $R(X)$ has Betti numbers as shown in Theorem 3.3.1:

```
i10 : F = minimalResolution(IX);
```

```

i11 : betti F
      0  1  2  3
o11 = total: 8 26 26 8
      0: 1  .  .  .
      1: .  .  .  .
      2: .  .  .  .
      3: .  .  .  .
      4: 4  .  .  .
      5: 3  6  .  .
      6: . 12  .  .
      7: .  8  8  .
      8: .  . 12  .
      9: .  .  6  3
     10: .  .  .  4
     11: .  .  .  .
     12: .  .  .  .
     13: .  .  .  .
     14: .  .  .  1

```

Next we compute a minimal free resolution having a skew-symmetric middle matrix. From Theorem 4.0.2 we know that such a resolution always exists. Furthermore, we can use the proof of Theorem 4.0.2 to construct this resolution outgoing from the given resolution. So, if F_\bullet denotes the given minimal free resolution, then $G_\bullet = \text{Hom}(F_\bullet, S(-17))$ is also a minimal free resolution of $R(X)$ by Proposition 3.3.3. Hence there exists an isomorphism of chain complexes:

$$\begin{array}{ccccccccccc}
 0 & \longleftarrow & R(X) & \longleftarrow & F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 & \xleftarrow{d_3} & F_3 & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow t_0 & & \downarrow t_1 & & \downarrow t_2 & & \downarrow t_3 & & \\
 0 & \longleftarrow & R(X) & \longleftarrow & F_3^\vee & \xleftarrow{d_3^{tr}} & F_2^\vee & \xleftarrow{d_2^{tr}} & F_1^\vee & \xleftarrow{d_1^{tr}} & F_0^\vee & \longleftarrow & 0
 \end{array}$$

Now we use Macaulay2 to calculate such an isomorphism. More precisely, we compute first the module $\text{Hom}_S(\text{coker } d_1, \text{coker } d_3^{tr})$. This module must have a non-zero homogeneous element of degree 0 which we take as our isomorphism t_0 . Afterwards we can lift t_0 to an isomorphism of complexes $t_\bullet: F_\bullet \rightarrow G_\bullet$. Furthermore, we showed in Chapter 4 that there exists an isomorphism t_\bullet satisfying $t_3^{tr} = -t_0$ and $t_2^{tr} = -t_1$. Choosing such an isomorphism, we compute a skew-symmetric minimal free resolution as in the proof of Theorem 4.0.2 with our procedure `skewsymmetricResolution`:

```
i12 : Fskew = skewsymmetricResolution(F);
```

As a last preliminary step we want to compute a standard resolution of $R(X)$. Our program `standardResolution` follows the (constructive) ideas of the proof of Proposition 6.3.1. The output of this procedure is either a standard resolution of $R(X)$ or an error message if we cannot compute a standard resolution over the chosen base field.

```
i13 : Fstand = standardResolution(Fskew);
```

```
i14 : d1 = Fstand.dd_1;
```

```

      8      26
o14 : Matrix S <--- S

```

```

i15 : d2 = Fstand.dd_2;

          26      26
o15 : Matrix S  <--- S

i16 : d3 = Fstand.dd_3;

          26      8
o16 : Matrix S  <--- S

i17 : d1*d2 == 0

o17 = true

i18 : -- verify that the second matrix is skew-symmetric
      d2 + transpose(d2) == 0

o18 = true

i19 : -- verify that the third matrix is the dual of the first
      d1 - transpose(d3) == 0

o19 = true

```

Having found a standard resolution, we can compute the assigned matrix $l \in \text{St}(Q)$ and the matrix $\alpha(l)$ from the first syzygy matrix d_1 :

```

i20 : l = assignedMatrix(d1)

o20 = {2} | 0  0  0  0  -14  0  0  0  0  -14  0  0 |
      {2} | -1  0  0  0  0   0  0  0  -1  0   0  0 |

i21 : -- compute the a-matrix of d1
      a1 = d1^{1..4}_{0..5}

o21 = {4} | 0   0  0   -14x_0  0  0   |
      {4} | -x_1  0  0   0       0  0   |
      {4} | 0   0 -14x_0  0       0  0   |
      {4} | 0   0  0   0       0 -x_1  |

          4      6
o21 : Matrix S  <--- S

```

Next we see that the minimal free resolution of the module $\text{coker } \alpha(l)$ is of the form as claimed in Lemma 9.1.6. Note that the variables x_0, x_1 have degree 2 in the polynomial ring S which is the reason for the degree “jumps” in the Betti tables below:

```

i22 : betti res coker a1

o22 = total: 4 4
      4: 4 .
      5: . 4

```

Now we check that the 3×3 minors of the matrix $\epsilon(l)$ vanish indeed at two distinct points as shown in Lemma 9.1.7:

```
i23 : -- compute the e-matrix of d1
      e1 = d1^{5..7}_{6..17};
```

```
o23 : Matrix S  <--- S
           3      12
```

```
i24 : me3 = minors(3,e1);
```

```
i25 : decompose me3
```

```
o25 = {ideal x , ideal x }
           0      1
```

Recall that every base point of $|3K_X|$ is contained in a unique divisor of $|2K_X|$. The two points in the vanishing locus of the 3×3 minors of $\epsilon(l)$ (considered as points in \mathbb{P}^1) define exactly these divisors:

```
i26 : use Sbig;
```

```
i27 : base3K = ideal mingens (IX + ideal(y_0..y_3));
```

```
i28 : netList (decompose base3K)
```

```

+-----+
|                5      2
o28 = |ideal(x , x  + w , z , z , z , z , w  + w , w , y , y , ...
|         0  1  2  3  2  1  0  1  2  0  3  2
+-----+
|
|                5      2
|ideal(x , w , z , z , z , z , w , x  + w , y , y , y , y )
|         1  2  3  2  1  0  1  0  0  3  2  1  0
+-----+
```

Let us now consider the birational morphism $\varphi: X_{can} \rightarrow Y \subseteq \mathbb{P}(2^2, 3^4)$. Recall that we have an exact sequence of $\mathcal{O}_{\mathbb{P}(2^2, 3^4)}$ -modules

$$0 \rightarrow \mathcal{O}_Y \rightarrow \varphi_* \mathcal{O}_{X_{can}} \rightarrow \widetilde{M} \rightarrow 0,$$

where $M = \text{coker } d'_1$. We expect that the induced morphism $\mathcal{O}_{Y,p} \rightarrow (\varphi_* \mathcal{O}_{X_{can}})_p$ is not an isomorphism at the image points of the base points of $|3K_X|$ and the base points of $|2K_X|$, hence that these points are contained in $\text{Supp}(\widetilde{M})$. Since

$$\text{Supp}(\widetilde{M}) \subseteq V(7 \times 7 \text{ minors of } d'_1)$$

we first check that the points are contained in the vanishing locus on the right-hand side:

```
i29 : d1' = d1^{1..7};
```

```
i30 : I' = minors(7,d1');
```

```
i31 : netList (decompose I')
```

```

+-----+
o31 = |ideal (x , y , y , y , y ) |
|         1  1  2  3  0 |
+-----+
```

```

+-----+
|ideal (x , y , y , y , y )|
|      0  1  2  3  0 |
+-----+
|ideal (x , x , y , y , y )|
|      0  1  2  3  0 |
+-----+
|ideal (x , x , y , y , y )|
|      0  1  1  3  0 |
+-----+
|ideal (x , x , y , y , y )|
|      0  1  1  2  0 |
+-----+
|ideal (x , x , y , y , y )|
|      0  1  1  2  3 |
+-----+

```

Next let us verify that the base points of $|2K_{X_{can}}|$ and $|3K_{X_{can}}|$ are smooth points of X_{can} , whereas their image points in Y are all singular points. Due to the singularities of the weighted projective space, we cannot determine smoothness directly via the Jacobian criterion. To remedy this situation, we first embed X_{can} (respectively Y) in a standard projective space. From Theorem 2.3.17 we know that the rational map corresponding to $|5K_{X_{can}}|$ gives a closed embedding of X_{can} into the projective space $\mathbb{P}^{P_5-1} = \mathbb{P}^{10}$. The image of X_{can} under this embedding is the surface $\text{Proj}(R^{[5]})$ (see Definition 2.3.16).

To embed Y in a projective space we simply embed $\mathbb{P} = \mathbb{P}(2^2, 3^4)$ in a standard projective space. We know that the invertible sheaf $\mathcal{O}_{\mathbb{P}}(6)$ is very ample. Thus, we can embed \mathbb{P} in \mathbb{P}^{n-1} , where $n = h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6)) = \dim_{\mathbb{k}} S_6 = 14$. We denote the isomorphic image of Y under this embedding by Y' :

```

i32 : R5 = kk[v_(0)..v_(10)];

i33 : -- compute the 5th-canonical morphism and the
      -- 5th-canonical image in P^10
      (IX5,psi5) = fifthCanonicalModel(IX);

i34 : -- compute the images of the base points of |2K_X| (list l2)
      -- and |3K_X| (list l3) in P^10
      (l2,l3) = imageOfBasepoints(psi5);

i35 : jacIX5 = jacobian IX5;

i36 : codim IX5

o36 = 8

i37 : -- check that all base points are smooth
      apply(l2,i->rank(jacIX5 % i) == 8)

o37 = {true, true, true, true}

i38 : apply(l3,i->rank(jacIX5 % i) == 8)

o38 = {true, true}

```

Now we check that the image points in Y are singular points of Y :

```

i39 : -- compute the image Y in P = P(2^2,3^4)
      IY = ann coker d1;

o39: Ideal of S

i40 : S6 = kk[t_(0)..t_(13)];

i41 : -- embed in P^13 via the ample sheaf O_P(6)
      (IY',phi6) = embedInProjectiveSpace(IY);

i42 : (l2',l3') = imageOfBasepoints(phi6);

i43 : jacIY' = jacobian IY';

i44 : codim IY'

o44 = 11

i45 : -- check that the image points are not smooth
      apply(l2',i->rank(jacIY' % i) == 11)

o45 = {false, false, false, false}

i46 : apply(l3',i->rank(jacIY' % i) == 11)

o46 = {false, false}

```

Finally we compute the birational model $W \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ introduced in Section 10.1. To do so, we first compute the image of the rational map $Y \dashrightarrow \mathbb{P}^7$ given by the global sections $x_i y_j$. The image surface W' is contained in $\nu_{1,3}(\mathbb{P}^1 \times \mathbb{P}^3)$, where $\nu_{1,3}: \mathbb{P}^1 \times \mathbb{P}^3 \rightarrow \mathbb{P}^7$ is the Segre embedding. Hence, the surface W is the inverse image of W' under $\nu_{1,3}$.

```

i47 : S13 = kk[x_0,x_1,y_0..y_3,Degrees=>{2:{1,0},4:{0,1}}];

i48 : -- compute the model W in P^1 x P^3
      IW = bihomogeneousModel(IY);

o48 : Ideal of S13

i49 : -- compute the number and degrees of the generators of IW
      tally degrees IW

o49 = Tally{{0, 7} => 1}
      {1, 2} => 1
      {1, 5} => 1
      {2, 3} => 1

```

We see that the set of generators of the ideal of W contains exactly one form depending only on the variables y_0, \dots, y_3 . This form defines the birational tricanonical model of X_{can} in \mathbb{P}^3 . Note that its degree is $7 = (3K_X)^2 - 2$ since $|3K_X|$ has two base points (see also Theorem 9.1.3). Furthermore, we can compute whether W is smooth or not by testing smoothness in every of the 8 affine charts $V_{i,j} = D_+(x_i y_j)$ covering $\mathbb{P}^1 \times \mathbb{P}^3$.

```

i50 : isSmoothBihomModel(IW)

o50 = false

```

11.2 An Example with Torsion Group $\mathbb{Z}/3\mathbb{Z}$

As a next example we consider a numerical Godeaux surface X with torsion group $\mathbb{Z}/3\mathbb{Z}$. All procedures are contained in the Macaulay2-file `numGodeauxZ3` and work over a finite field or the rational numbers. However, to avoid coefficient growth, we work again over a finite field.

To begin with, we use the results from Section 9.2 to compute the canonical ring $R(X)$. Recall that this construction depends on the choice of a parameter $\lambda \in \mathbb{A}^9$. As a first step we want to find an element $\lambda \in \mathbb{A}^9$ such that $\text{Proj}(R_\lambda/(x_0, x_1))$ consists of 4 distinct points, where $R_\lambda \cong R(X)$ is defined as in (9.8).

```
i1 : load "numGodeauxZ3.m2"

i2 : kk = ZZ/97;

i3 : invZ3 = computeZ3Invariants(kk);

i4 : while(
      r = random(kk^1, kk^9);
      Y = computeInvariantSurface(r);
      (IX, lm) = computeBasePoints(Y, mat);
      not (length(lm) == 4)) do();

i5 : r

o5 = | -6 19 -12 -48 1 2 11 -43 9 |
```

From Lemma 6.2.5 we know that the images of the 4 base points of $|2K_X|$ in \mathbb{P}^3 are in general position. Hence, there exists an automorphism ν of $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$ such that the image points of the corresponding points in $\nu(\text{Proj}(R_\lambda))$ are the 4 coordinate points of \mathbb{P}^3 . After applying this automorphism, or algebraically after replacing R_λ by an isomorphic ring, we compute a minimal free resolution of the canonical ring as an S -module as explained in the last section.

```
i6 : IX = computeAutomorphism(IX, lm);

i7 : F = minimalResolution(IX);

i8 : betti F

o8 = total: 8 26 26 8
      0: 1 . . .
      1: . . . .
      2: . . . .
      3: . . . .
      4: 4 . . .
      5: 3 6 . .
      6: . 12 . .
      7: . 8 8 .
      8: . . 12 .
      9: . . 6 3
     10: . . . 4
     11: . . . .
     12: . . . .
     13: . . . .
     14: . . . 1
```


If there exists a standard resolution of $R(X)$ being defined over the field \mathbb{F}_{97} , the procedure `skewsymmetricResolution` computes one as explained in the last section. Otherwise, an error message is printed:

```
i9 : Fskew = skewsymmetricResolution(F);
```

```
i10 : Fstand = standardResolution(Fskew);
```

```
i11 : d1 = Fstand.dd_1;
```

```
o11 : Matrix S  $\begin{matrix} & 8 & & 26 \\ & & & \end{matrix}$  <--- S
```

```
i12 : d2 = Fstand.dd_2;
```

```
o12 : Matrix S  $\begin{matrix} & 26 & & 26 \\ & & & \end{matrix}$  <--- S
```

Having a standard resolution of $R(X)$, we compute the assigned matrix $l \in \text{St}(Q)$ and the matrix $\mathfrak{a}(l)$:

```
i13 : l = assignedMatrix(d1)
```

```
o13 = {2} | 0 -29 29 13 -29 29 -38 38 -33 -38 38 7 |
      {2} | 32 0 0 32 0 0 0 0 -33 0 0 -33 |
```

```
i14 : a1 = d1^{1..4}_{0..5}
```

```
o14 = {4} | 7x_0-33x_1 38x_0 0 -38x_0 0 0 |
      {4} | -33x_0-33x_1 0 38x_0 0 -38x_0 0 |
      {4} | 0 29x_0 -29x_0 0 0 13x_0+32x_1 |
      {4} | 0 0 0 29x_0 -29x_0 32x_1 |
```

Next we compute the Betti numbers of the module $\text{coker } \mathfrak{a}(l)$ and see that they are of the form as claimed in Lemma 9.2.4:

```
i15 : betti res coker a1
```

```
o15 = total: 4 5 1
      4: 4 . .
      5: . 5 .
      6: . . 1
```

Finally we verify that Conjecture 8.4.3 is satisfied in this example, that means that the 3×3 minors of the matrix $\mathfrak{e}(l)$ vanish precisely at one point:

```
i16 : e1 = d1^{5..7}_{6..17};
```

```
i17 : me3 = minors(3, e1);
```

```
i18 : decompose me3
```

```
o18 = {ideal x }
      0
```

This point (considered as a point in \mathbb{P}^1) defines the unique divisor in $|2K_X|$ containing the base point of $|3K_X|$:

```

i19 : use Sbig;

i20 : base3K = ideal mingens (IX + ideal(y_0..y_3));

i21 : netList (decompose base3K)

+-----+
|           5      2
o21 = |ideal(x , - 48x  + w , z , z , z , z , w , w  + w , y , ...
|           0      1  1  2  3  1  0  2  0  1  3
+-----+

```

Next we compute the matrix d'_1 and the vanishing locus of its 7×7 minors:

```

i22 : d1' = d1^{1..7};

i23 : I' = minors(7,d1');

i24 : netList (decompose I')

+-----+
o24 = |ideal (x , y , y , y , y )|
|           0  1  2  3  0 |
+-----+
|ideal (x , x , y , y , y )|
|           0  1  2  3  0 |
+-----+
|ideal (x , x , y , y , y )|
|           0  1  1  3  0 |
+-----+
|ideal (x , x , y , y , y )|
|           0  1  1  2  0 |
+-----+
|ideal (x , x , y , y , y )|
|           0  1  1  2  3 |
+-----+

```

The output shows that the morphism $\mathcal{O}_{Y,p} \rightarrow (\varphi_* \mathcal{O}_{X_{can}})_p$ is an isomorphism except possibly at the points in the vanishing locus of the ideal I' . As in the example of the $\mathbb{Z}/5\mathbb{Z}$ -Godeaux surface from the last section, we compute that the base points of $|2K_{X_{can}}|$ and the single base point of $|3K_{X_{can}}|$ are smooth points of X_{can} . However, in this example, the surface Y is smooth at the corresponding image points of the bicanonical base points but singular at the image of the tricanonical base point. Since these calculations are performed exactly as in the last section, we won't present the Macaulay2-results here. In the end we compute the birational model $W \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ and test this surface on smoothness:

```

i25 : -- compute the image Y in P(2^2,3^4)
      IY = ann coker d1;

o25 : Ideal of S

i26 : S13 = kk[x_0,x_1,y_0..y_3,Degrees=>{2:{1,0},4:{0,1}}];

i27 : IW = bihomogeneousModel(IY);

o27 : Ideal of S13

```

```
i28 : tally degrees IW
o28 = Tally{{0, 8} => 1}
      {1, 3} => 1
      {1, 5} => 1
      {2, 2} => 1
```

As in the previous section, there is exactly one bihomogeneous polynomial among the generators of the ideal of W depending only on the variables y_0, \dots, y_3 . This polynomial defines the birational image of the canonical model in \mathbb{P}^3 . In this case, the degree of this polynomial (or equivalently, of the corresponding surface in \mathbb{P}^3) is $8 = (3K_X)^2 - 1$ since $|3K_X|$ has a single base point. At the end of this section we compute that the surface W is not smooth:

```
i29 : isSmoothBihomModel(IW)
o29 = false
```

11.3 Examples with a Trivial Torsion Group

In this section we present the implementation of our construction method of numerical Godeaux surfaces developed throughout this thesis and give some examples. Let us briefly recall the main steps of this construction: The first step is the computation of two homogeneous homomorphisms

$$d'_1: F'_0 \leftarrow F_1, \quad d_2: F_1 \leftarrow F_1^\vee$$

such that

- (i) $d'_1 d_2 = 0$,
- (ii) d_2 is alternating,
- (iii) modulo x_0, x_1 , the matrices d'_1 and d_2 have the form as chosen in Section 6.3.

As a second step, we use these matrices to compute a standard complex

$$F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0. \quad (11.1)$$

Finally, if $R := \text{coker } d_1$ satisfies all the properties of Theorem 5.0.2, then $\text{Proj}(R)$ is the canonical model of a numerical Godeaux surface.

In Chapter 7 we have seen that the computation of two matrices d'_1 and d_2 as above is equivalent to choosing a matrix $l \in \text{St}(Q)$ and a point p in a vector space $\mathcal{V}(l)$. Furthermore, in Subsection 7.2.2 we described a Las Vegas algorithm for computing lines in Q , and hence elements of $\text{St}(Q)$.

We have implemented our construction using the computer algebra system `Macaulay2`. Up to now, these procedures can be performed over any finite field \mathbb{F}_p . When working over the field $\mathbb{k} = \mathbb{Q}$, the procedures are also applicable except for the procedure which computes lines in Q . According to the Las Vegas algorithm from Subsection 7.2.2 for $\mathbb{k} = \mathbb{Q}$, we first have to choose a point in a conic, and then a point in a 0-dimensional variety Z_8 of degree 8. In general, this variety does not have any \mathbb{k} -rational points. But of course Z_8 has rational points in some finite field extension of degree at most 8. Hence, to choose a point in Z_8 we need to compute an absolute primary decomposition of the corresponding ideal. As far as we know, such a procedure is not implemented in `Macaulay2` up to now. But there exists an efficient implementation in the computer algebra system `SINGULAR`.

So in the case $\mathbb{k} = \mathbb{Q}$, we first compute with the help of SINGULAR a line in Q which is defined over a number field $\mathbb{Q}(\alpha)$ of degree at most 16, and then use our programs in Macaulay2 to finish the construction.

Let us now present the individual procedures by computing an example over a finite field. As a first step we have to introduce several variables representing the unknown entries of the matrices d'_1 and d_2 . We consider first the case where we assume a priori that the matrix \mathfrak{c} is zero. Using the results from Chapter 7, we only have to introduce variables for the entries of the matrices \mathfrak{a} and \mathfrak{o} . For the sake of simplicity, we also introduce variables for the entries of \mathfrak{n} – then our program will compute the substitution of every variable in \mathfrak{n} as in Corollary 7.5.5 and we do not have to do this manually. This initialization is done by our Macaulay2-program `globalVariables`:

```
i1 : load "numGodeaux.m2"

i2 : kk = ZZ/317;

i3 : setRandomSeed(11);

i4 : (A,B) = globalVariables(kk);

          7      26
o4 : Matrix SR <--- SR

          26      26
o5 : Matrix SR <--- SR
```

For example, the initial \mathfrak{a} -matrix of A is of the form:

```
i6 : -- the a-matrix of A
      astart = A^{0..3}_{0..5}

o6 = | a_(0,0,1) a_(0,0,2) 0          a_(0,0,3) 0          0          |
      | a_(1,0,1) 0          a_(1,1,2) 0          a_(1,1,3) 0          |
      | 0          a_(2,0,2) a_(2,1,2) 0          0          a_(2,2,3) |
      | 0          0          0          a_(3,0,3) a_(3,1,3) a_(3,2,3) |
```

As a next step, we construct the complex (11.1) modulo x_0, x_1 as described in Section 6.3. Then, putting the first two syzygy matrices of this complex and the matrices A and B together, we obtain the general set-up for the matrices d'_1 and d_2 as introduced in (7.1) and (7.2). Afterwards we compute a minimal generating set for the relations coming from $d'_1 d_2 = 0$. These steps are performed by the following Macaulay2-procedure:

```
i7 : (rel2, subs1) = setupMarkedGodeaux(R);
```

The matrix `rel2` contains a minimal set of relations coming from the equation $d'_1 d_2 = 0$. The computation shows that there are in total 58 relations as described in Summary 7.5.9:

```
i8 : betti rel2

o8 = total: 0 1
          1 58
          0: 1 .
          1: . .
          2: . .
          3: . 28
          4: . .
          5: . 30
```

Among the 28 relations of degree 4 there are the Pfaffians q_0, \dots, q_3 which we have studied in Section 7.2:

```
i9 : -- the 4 Pfaffians
      pfaffrel = rel2_{24..27}

o9 = |a_(1,1,2)a_(1,1,3)-a_(2,1,2)a_(2,2,3)+a_(3,1,3)a_(3,2,3)
      -----
      a_(0,0,2)a_(0,0,3)+a_(2,0,2)a_(2,2,3)-a_(3,0,3)a_(3,2,3)
      -----
      a_(0,0,1)a_(0,0,3)-a_(1,0,1)a_(1,1,3)-a_(3,0,3)a_(3,1,3)
      -----
      a_(0,0,1)a_(0,0,2)-a_(1,0,1)a_(1,1,2)+a_(2,0,2)a_(2,1,2) |
```

The second output of the program `setupMarkedGodeaux` is a matrix `subs1`. Every entry of this matrix corresponds to a variable of the underlying polynomial ring (possibly being substituted by other values using the relations coming from $d_1' d_2 = 0$). During the construction we update this matrix iteratively. That means, if we have found a possible assignment for the variables in `a` or `o`, we substitute the corresponding entries by these values. As a last initializing step we set up the 4 skew-symmetric matrices from Section 7.2 whose Pfaffians are q_0, \dots, q_3 :

```
i10 : Ms = setupSkewMatrices(rel2);
```

Now we compute a random line in Q (and a representative in $\text{St}(Q)$) with the help of the Las Vegas algorithm from Section 7.2.2. Since we are working over a finite field we do this simply by trial and error as explained at the end of Section 7.2.2. Hence, the runtime of this procedure strongly depends on the characteristic of the base field.

```
i11 : time (subslines,randline) = pickLine(Ms);
      -- used 404.33 seconds
```

```
i12 : randline
```

```
o12 = {2} | -113 64 11 26 37 3 -89 -117 -131 -53 1 0 |
      {2} | -71 -91 62 55 100 41 -61 -57 -33 -142 0 1 |
```

Note that the assumption $c = 0$ is only valid if the a -matrix of d_1' has rank 4 and if the cokernel module has the Betti numbers

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 4 & 6 & . \\ 1 & . & . & 2 \end{array}$$

Our program `pickLine` checks if these conditions are satisfied. Next we verify that the computed matrix l is contained in the open subset $V_{\text{gensyz}} \subseteq \text{St}(Q)$ from Proposition 7.4.6 and Remark 7.4.7:

```
i13 : al = sub(astart,subslines);
```

```
i14 : betti res coker al
```

```
o14 = total: 0 1 2
      4: 4 . .
      5: . 6 .
      6: . . .
      7: . . .
      8: . . 2
```

```

i15 : pstart = B^{6..17}_{18..25};
i16 : pl = sub(pstart,subslines);
i17 : betti res coker pl

```

```

          0 1
o17 = total: 12 8
          7: 12 .
          8: . 8

```

Thus, the matrix l satisfies all the required open conditions, and hence $l \in V_{\text{gensyz}}$.

Next, we want to verify that the point $[l]$ is a stable point of $F_1(Q)$ under the linear action of $G = (\mathbb{k}^*)^3$ defined in (8.17) and (8.18). According to the proof of Lemma 8.3.16 it is enough to show that the respective coordinates of the corresponding point in \mathbb{P}^{65} are non-zero. But, by the definition of the Plücker embedding, these coordinates are the 2×2 minors of l . In our example, we see that all the 2×2 minors are non-zero. Hence, $[l] \in F_1(Q)^s$.

```

i18 : gens minors(2,randline)

o18 = |-72 115 -103 69 -137 -56 -114 -60 74 -69 18 44 -52 -50 -51
-----
      -60 43 92 139 -14 -21 36 -30 -30 -119 81 -129 155 134 -85
-----
      151 7 150 -117 18 -119 -80 37 139 -143 46 -155 -105 -38 52
-----
      71 91 -62 -55 -100 -41 61 57 33 142 -113 64 11 26 37 3 -89
-----
      -117 -131 -53 1 |

          1          66
o18 : Matrix SR <--- SR

```

As a last step we compute a solution of the linear relations given by $d_1^l d_2 = 0$ which is done by our program `pickSection`. This procedure computes first the remaining linear relations, chooses then a basis of the vector space of solutions $\mathcal{V}(l)$, and picks a random point in this space in the end:

```

i19 : time (subpoint,randpoint) = pickSection(subslines);
      -- used 0.459521 seconds

i20 : -- the number of the rows of the chosen point is the
      -- dimension of the solution space
      randpoint

o20 = | -12 |
      | 112 |
      | -120 |
      | 146 |

          4          1
o20 : Matrix SR <--- SR

```

Thus, there exists a matrix $l \in \text{St}(Q)$ such that the vector space $\mathcal{V}(l)$ is 4-dimensional as claimed in Section 7.5. Now, we are able to compute a standard resolution:

```
i21 : Fstand = standardResolution(subspoint);
```

```
i22 : betti Fstand
```

```

      0  1  2  3
o22 = total: 8 26 26 8
      0: 1  .  .  .
      1: .  .  .  .
      2: .  .  .  .
      3: .  .  .  .
      4: 4  .  .  .
      5: 3  6  .  .
      6: . 12  .  .
      7: .  8  8  .
      8: .  . 12  .
      9: .  .  6  3
     10: .  .  .  4
     11: .  .  .  .
     12: .  .  .  .
     13: .  .  .  .
     14: .  .  .  1

```

The procedure `standardResolution` computes first the matrices $d'_1(l)$ and $d_2(l, p)$, where (l, p) is the pair chosen before. Then the program computes one additional syzygy of the matrix $d_2(l, p)$, transforms this to the required form modulo x_0, x_1 and concatenates the resulting row vector with the matrix $d'_1(l)$. All properties of the maps of a standard resolution are checked again by this procedure. We present this for some of the properties:

```
i23 : d1 = Fstand.dd_1;
```

```

      8      26
o23 : Matrix S <--- S

```

```
i24 : d2 = Fstand.dd_2;
```

```

      26      26
o24 : Matrix S <--- S

```

```
i25 : d3 = Fstand.dd_3;
```

```

      26      8
o25 : Matrix S <--- S

```

```
i26 : d1*d2 == 0
```

```
o26 = true
```

```
i27 : (d2 + transpose(d2) == 0, d1 - transpose(d3) == 0)
```

```
o27 = (true, true)
```

```
i28 : (rank(d1) == 8, rank(d2) == 18)
```

```
o28 = (true, true)
```

Now let us verify that the assumptions from Theorem 5.0.2 are satisfied for $R := \text{coker } d_1$. First we verify that R satisfies the ring condition. Let I' denote the ideal of the 7×7 minors

of d'_1 . Recall that R supports the structure of a ring if $\text{depth}(I', S) \geq 5$, or equivalently if $\dim S/I' \leq 1$.

```
i29 : verifyRingCondition(d1)
o29 : true
```

Hence, R is a Gorenstein ring by the proof of the first part of Theorem 5.0.2. Next let us compute the defining ideal I_Y of the surface $Y \subseteq \mathbb{P} := \mathbb{P}(2^2, 3^4)$:

```
i30 : -- compute the image Y in P = P(2^2, 3^4)
      IY = ann coker d1;

o30 : Ideal of S

i31 : -- verify that Y is a surface
      dim(IY) == 3

o31 = true
```

Now set $S_Y = S/I_Y$. By Theorem 5.0.2 we have to verify that x_0, x_1 is a regular sequence for R and that $\text{Proj}(R/(y_0, \dots, y_3)R)$ is empty or 0-dimensional. Since R is Gorenstein (and hence Cohen-Macaulay) it is enough to show that $\dim R/(x_0, x_1)R = \dim R - 2$. Furthermore, since the morphism $\varphi: \text{Proj}(R) \rightarrow Y$ corresponding to the inclusion $S_Y \subseteq R$ is finite, it is enough to show that $\text{Proj}(S_Y/(x_0, x_1))$ is 0-dimensional and that $\text{Proj}(S_Y/(y_0, y_1, y_2, y_3))$ is empty or 0-dimensional. These two checks are performed by our procedure `verifyAssumptions`:

```
i32 : verifyAssumptions(d1)
o32 = true
```

Now it remains to show that I_Y is prime and that $\text{Proj}(R)$ has at most finitely many rational double points. These conditions are (computationally) not as easy to verify as the last ones. Of course, if Y is a smooth surface, then $\text{Proj}(R)$ is smooth as well. Indeed, since $\varphi: \text{Proj}(R) \rightarrow Y$ is finite and birational by the proof of Theorem 5.0.2, then Y being smooth implies that φ is an isomorphism. To determine whether $Y \subseteq \mathbb{P}(2^2, 3^4)$ is smooth or not, we first embed Y in a standard projective space \mathbb{P}^{13} as in Section 11.1. We denote the isomorphic image of Y under this embedding by Y' :

```
i33 : S6 = kk[t_(0)..t_(13)];

i34 : IY' = embedInProjectiveSpace(IY);

o34 : Ideal of S6

i35 : codim IY'

o35 = 11
```

But now the codimension of Y' in this standard projective space is much higher than the one of Y in the weighted projective space $\mathbb{P}(2^2, 3^4)$ and testing smoothness for varieties of a high codimension is again computationally difficult. This problem was one motivation for a new smoothness-test developed by Böhm and Frübis-Krüger (see [BFK18]). The massively parallel implementation of this algorithm (see [BDF⁺18]) verifies that the surface Y is indeed smooth. Hence $\text{Proj}(R) \cong Y$.

As a last step we compute the surface $W \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ from the surface Y as in the last sections.


```

i36 : S13 = kk[x_0,x_1,y_0..y_3,Degrees=>{2:{1,0},4:{0,1}}];
i37 : IW = bihomogeneousModel(IY);
i38 : tally degrees IW
o38 = Tally{{0, 9} => 1 }
      {1, 6} => 2
      {1, 7} => 3
      {2, 5} => 8
      {2, 6} => 3
      {3, 4} => 10
      {4, 3} => 3
      {5, 3} => 2
      {7, 2} => 1

```

We see that the set of generators of the ideal of W contains exactly one form depending only on the variables y_0, \dots, y_3 . The degree of this form is 9 and it defines the image of $\text{Proj}(R)$ in \mathbb{P}^3 . As a last computation we check whether W is smooth:

```

i39 : isSmoothBihomModel(IW)
o39 = true

```

Now, in contrast to the situation in the previous sections, where we presented examples having a non-trivial torsion group, the surface $W \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ is smooth.

11.3.1 A Surface with no Hyperelliptic Fibres

The previous example showed that there exists a ring R and a standard resolution of R defined over the finite field \mathbb{F}_{317} fulfilling all the required properties of Theorem 5.0.2. But of course we want to show that our construction yields a numerical Godeaux surface X over \mathbb{C} . Using the implementation of our Las Vegas algorithm in SINGULAR we compute a line in Q defined over $\mathbb{Q}(\alpha)$, which is a field extension of \mathbb{Q} of degree 8. We use this line to start our construction in Macaulay2:

```

i1 : load "numGodeaux.m2"
i2 : kk1 = QQ;
i3 : V = kk1[u];
i4 : -- the minimal polynomial of the field extension
      minf = 71148218536494892800*u^8-267905284785328397760*u^7
            -191663543691355381443*u^6+660717755665025167872*u^5
            +1148693036616534958317*u^4+340130423478253176036*u^3
            -416450846724240829045*u^2-279258661204766681124*u
            -44344622837306452725;
i5 : -- define the number field kk = Q(alpha)
      kk = toField(V/minf);
i6 : (A,B) = globalVariables(kk);
i7 : astart = A^{0..3}_{0..5};
i8 : (rel2,subs1) = setupMarkedGodeaux(R);

```

```
i9 : load "line1.txt";
i10 : subline = updateRelations(line1);
```

The file `line1.txt` contains a matrix $l \in \text{St}(Q)$ representing the line in Q computed by SINGULAR. The procedure `updateRelations` computes the updated matrix of variables `subline` as explained in the example over \mathbb{F}_{317} .

Next we check that the matrix l is contained in the open set $V_{\text{gensyz}} \subseteq \text{St}(Q)$:

```
i11 : a1 = sub(astart, subline);
i12 : betti res coker a1
```

```
o12 = total: 0 1 2
      4: 4 . .
      5: . 6 .
      6: . . .
      7: . . .
      8: . . 2
```

```
i13 : pstart = B^{6..17}_{18..25};
```

```
i14 : pl = sub(pstart, subline);
```

```
i15 : betti res coker pl
```

```
o15 = total: 0 1
      7: 12 .
      8: . 8
```

Now we determine the solution space $\mathcal{V}(l)$ which is done by a syzygy computation. Naturally, over the rational numbers or over a number field, the intermediate coefficient growth is a problem. As a result, compared to the 0.5 seconds over the field \mathbb{F}_{317} , this procedure takes now more than an hour. To ease the computations further on, we choose a point in \mathbb{Q}^4 and not in $\mathbb{Q}(\alpha)^4$, and compute the corresponding solution in $\mathcal{V}(l)$:

```
i16 : randpoint = sub(matrix{{-99},{81},{117},{63}}, SR)
```

```
o16 = | -99 |
      | 81  |
      | 117 |
      | 63  |
```

```
i17 : time subpoint = computeSection(randpoint, subline);
      -- used 5789.08 seconds
```

The computation of the standard resolution involves the calculation of the syzygies of the 26×26 matrix $d_2(l, p)$ having entries in the polynomial ring $\mathbb{Q}(\alpha)[x_0, x_1, y_0, \dots, y_3]$. In the chosen example the calculation finishes after several hours:

```
i18 : Fstand = standardResolution(subpoint);
      -- used 31639.9 seconds
```

```
i19 : d1 = Fstand.dd_1;
```

```

o19 : Matrix S  <--- S
      8      26
i20 : d2 = Fstand.dd_2;

```

```

o20 : Matrix S  <--- S
      26      26

```

Having a standard resolution, we want to verify that $R := \text{coker } d_1$ supports the structure of a ring:

```

i21 : verifyRingCondition(d1);

o21 : true

```

Hence, R is a Gorenstein ring by Theorem 5.0.2. Next we compute the surface $Y \subseteq \mathbb{P}(2^2, 3^4)$ which is defined by the ideal $I_Y = \text{ann}_S R = \text{ann}_S(\text{coker } d_1)$. Now, since R is a ring, we have

$$\text{ann}_S R = \text{ann}_S 1_R,$$

where the ideal on the right-hand side is usually easier to compute:

```

i22 : d1' = d1^{1..7};

i23 : d10 = d1^{0};

i24 : IY = time ideal mingens ideal (d10*syz(d1'));
      -- used 4947.85 seconds

o24 : Ideal of S

```

Now we verify the assumptions (i) and (ii) of Theorem 5.0.2:

```

i25 : verifyAssumptions(d1)

o25 : true

```

Let us assume for a moment that Y is a smooth surface. Then $\text{Proj}(R)$ is smooth as well and R is the canonical ring of a numerical Godeaux surface X defined over the number field $\mathbb{Q}(\alpha)$. Furthermore, for $X_{\text{can}} = \text{Proj}(R)$ we have

$$Y \cong X_{\text{can}} \cong X.$$

Hence, X is a marked numerical Godeaux surface. Furthermore, since the assigned matrix $l \in \text{St}(Q)$ satisfies $l \in V_{\text{gensyz}}$ we know that $\text{Tors } X = 0$ by Proposition 8.4.1. Furthermore, if the birational model of X in $\mathbb{P}^1 \times \mathbb{P}^3$ is smooth, then the bicanonical system of X has no (honestly) hyperelliptic fibres by Proposition 10.1.4. However, checking smoothness of the surface Y or its isomorphic surface $Y' \subseteq \mathbb{P}^{13}$ over the number field directly is not feasible due to the high codimension and the intermediate coefficient swell. In the following we will argue that Y is smooth using a reduction modulo a prime.

Let $K = \mathbb{Q}(\alpha)$, and let \mathcal{O}_K be the ring of integers of the number field K . The surface $Y = \text{Proj}(S/I_Y)$ is defined over K and we can consider Y as a family of varieties \mathcal{Y} over $\text{Spec}(\mathcal{O}_K)$, where the generic fibre \mathcal{Y}_0 is isomorphic to Y and the special fibre $\mathcal{Y}_{\mathfrak{p}}$ over a closed point $\mathfrak{p} \in \mathcal{O}_K$ corresponds to the reduction modulo \mathfrak{p} . We call the fibre $\mathcal{Y}_{\mathfrak{p}}$ a *specialization* of Y . Now let \mathfrak{p} be a prime in \mathcal{O}_K such that

$$\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_p,$$

where p is a prime number. Furthermore, by replacing Y with its isomorphic surface $Y' \subseteq \mathbb{P}^{13}$, we may assume that Y is a subscheme of a (standard) projective space. Now taking the flat closure of Y in projective space over the local ring of \mathfrak{p} yields a flat family of surfaces over $\text{Spec}(\mathcal{O}_{K,\mathfrak{p}})$ whose generic fibre over the point $\text{Spec}(K)$ is Y , and whose special fibre is a surface Y_p defined over \mathbb{F}_p . Now, since being smooth is an open property, if the special fibre Y_p is smooth, then so is Y .

So we have to find a prime ideal $\mathfrak{p} \in \text{Spec}(\mathcal{O}_K)$ such that $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_p$. Using the characterization of prime ideals in \mathcal{O}_K , it is enough to find a prime number $p \in \mathbb{Z}$ such that the minimal polynomial minf of α has a linear factor (of multiplicity 1) modulo p . Using SINGULAR we compute that for $p = 197$, we have

$$\text{minf} \equiv (u + 36) \cdot f_0 \pmod{p},$$

for some polynomial f_0 of degree 7. Hence, we take as a prime ideal $\mathfrak{p} = (197, u + 36)$. Reducing the computed standard resolution F_\bullet modulo \mathfrak{p} , we obtain a standard resolution over \mathbb{F}_p whose assigned pair is just the reduction of the chosen pair over K . Hence we can apply our construction to this pair and compute the surface in $\mathbb{P}(2^2, 3^4)$ and its isomorphic image in \mathbb{P}^{13} :

```
i1 : load "numGodeaux.m2"
i2 : kk = ZZ/197;
i3 : -- the reduction of the chosen line over K modulo (197,u+36)
      lmod = matrix{{-6, -4, -2, -2, -6, -1, 4, -3, -2, 5, 2, 0},
                   {-53, 16, -88, -12, 76, 71, 67, -48, 71, 0, 74, 94}};
i4 : -- the reduction of the point over K modulo (197,u+36)
      pointmod = matrix{{-99},{81},{117},{63}};
i5 : subspoint = solveRelations(lmod,pointmod);
i6 : Fstand = standardResolution(subspoint);
i7 : d1 = Fstand.dd_1;
i8 : IY = ann coker d1;
i9 : S6 = kk[t_(0)..t_(13)];
i10 : -- compute the isomorphic image of Y in P^13
       IY' = embedInProjectiveSpace(IY);
o10 : Ideal of S6
```

We have verified that Y' is indeed a smooth surface with the help of the algorithm from [BDF⁺18]. Thus, from the discussion above we deduce that Y is a smooth surface over the number field K . Furthermore, modulo p , we compute that the corresponding surface in $\mathbb{P}^1 \times \mathbb{P}^3$ is smooth. Hence, using similar arguments as above, we conclude that the surface $W \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ defined over K is smooth. Let us summarize the results of these computations:

Summary 11.3.1. There exists a marked numerical Godeaux surface X defined over a number field K with

- (i) $\text{Tors } X = 0$,
- (ii) $X \cong X_{\text{can}} \cong Y$,

(iii) the bicanonical system $|2K_X|$ has no hyperelliptic fibres.

In particular, this surface is different from the Barlow surface and the Craighero-Gattazzo surface - the other existing examples of torsion-free numerical Godeaux surfaces.

Note that so far we have only computed the surface $Y \subseteq \mathbb{P}(2^2, 3^4)$ and the S -module $R = \text{coker } d_1$ which supports the structure of a ring. Now we use the ideas introduced at the end of Chapter 5 to compute the remaining defining relations (over \mathbb{F}_p) of R as a ring, and hence the canonical ring $R(X)$:

```
i11: Sbig = kk[x_0, x_1, y_0..y_3, z_0..z_3, w_0..w_2,
           Degrees=>{2:2, 4:3, 4:4, 3:5}];
```

```
i12 : IX = canonicalRing(d1);
```

```
i13 : betti IX
```

```

          0  1
o13 = total: 1 54
          0: 1  .
          1: .  .
          2: .  .
          3: .  .
          4: .  .
          5: .  6
          6: . 12
          7: . 18
          8: . 12
          9: .  6
```

11.3.2 The Barlow Surface

In this subsection we construct the canonical ring $R(X)$ of the Barlow surface and compute a standard resolution of $R(X)$. Recall that the Barlow surface was the first example of a simply connected numerical Godeaux surface. Let us first briefly sketch the construction due to Barlow. Hereby we follow the description given by Lee in [Lee01].

In Section 9.1 we have seen that there is an 8-dimensional family of Godeaux surfaces where each surface is given as the quotient of a quintic in \mathbb{P}^3 under a free action of the group $\mathbb{Z}/5\mathbb{Z}$. In this family there is a 4-dimensional subfamily in which the corresponding quintic is the determinant of a symmetric 5×5 matrix. These symmetric determinantal quintics were studied by Catanese in [Cat81]. The corresponding Godeaux surface is then called a *determinantal Godeaux surface*. Moreover, in this 4-dimensional family there exists a 2-dimensional subfamily in which the group action of $\mathbb{Z}/5\mathbb{Z}$ on the (symmetric determinantal) quintic can be extended to a group action of the dihedral group D_5 . Using a twist of this action, Barlow realized a simply connected numerical Godeaux surface as a quotient of such a quintic. Furthermore, since this twisting works for the whole subfamily, this construction shows the existence of a 2-dimensional family of simply connected numerical Godeaux surfaces.

In the following we briefly recall the description of a symmetric determinantal quintic $\Sigma \subseteq \mathbb{P}^3$ and the definition of the action of D_5 on Σ . Let u_1, \dots, u_4 denote the coordinates of \mathbb{P}^3 , and let ξ be a primitive fifth root of unity as in Section 9.1. Then the group $D_5 = \langle b, a \rangle$ acts on \mathbb{P}^3 via

$$\begin{aligned} b: (u_1 : u_2 : u_3 : u_4) &\mapsto (\xi u_1, \xi^2 u_2, \xi^3 u_3, \xi^4 u_4), \\ a: (u_1 : u_2 : u_3 : u_4) &\mapsto (u_4 : u_3 : u_2 : u_1). \end{aligned} \tag{11.2}$$

A quintic in \mathbb{P}^3 which is invariant under this action of D_5 is the determinant of the symmetric matrix

$$A = \begin{pmatrix} 0 & a_1u_1 & a_2u_2 & a_2u_3 & a_1u_4 \\ a_1u_1 & a_3u_2 & a_4u_3 & a_5u_4 & 0 \\ a_2u_2 & a_4u_3 & a_6u_4 & 0 & a_5u_1 \\ a_2u_3 & a_5u_4 & 0 & a_6u_1 & a_4u_2 \\ a_1u_4 & 0 & a_5u_1 & a_4u_2 & a_3u_3 \end{pmatrix},$$

where $a_1, \dots, a_6 \in \mathbb{k}$ are parameters.

Example 11.3.2 (See [Lee01], Example 1). Setting $a_1 = \dots = a_6 = 1$ in the above matrix A , we get the quintic

$$\Sigma : u_1^5 + u_2^5 + u_3^5 + u_4^5 + 5(u_1u_4 - u_2u_3)(u_1u_2^2 + u_3^2u_4 - u_1^2u_3 - u_2u_4^2) = 0.$$

A generic surface Σ has 20 nodes. Let $\sigma : \Delta \rightarrow \Sigma$ be a double cover which is branched over these nodes. Then there exists the following diagram (see [Lee01], p. 900)

$$\begin{array}{ccc} \Delta/\langle b \rangle & \longrightarrow & B = \Delta/\langle b, a\sigma \rangle \\ \downarrow & & \\ X = \Delta/\langle b, \sigma \rangle & & \end{array} \quad (11.3)$$

where X is a determinantal Godeaux surface and B is a determinantal Barlow surface.

So we can realize the canonical ring of a Barlow surface as the invariant ring under the twisted action of D_5 on Y . The double cover Δ is explicitly described in [Cat81] and [Rei81]: let A be the matrix from above. Furthermore, let

$$R = \mathbb{C}[u_1, \dots, u_4, v_0, \dots, v_5]/I,$$

where $\deg(u_i) = 1$, $\deg(v_j) = 2$ and I is the ideal generated by

$$\begin{aligned} \sum_j A_{i,j}v_j, & \text{ (5 relations of degree 3)} \\ v_jv_k - B_{j,k}, & \text{ (15 relations of degree 4)} \end{aligned} \quad (11.4)$$

where $B_{j,k}$ is the entry in row j and column k of the adjoint matrix of A . Then

$$\Delta = \text{Proj}(R) \subseteq \mathbb{P}(1^4, 2^5)$$

is a smooth surface of general type with $p_g = 4$, $q = 0$ and $K_Y^2 = 10$ (see [Cat81], Proposition 2.11). Moreover, Δ is a double cover of the quintic surface in \mathbb{P}^3 defined by $\det(A)$.

Having settled the theoretical background we can now compute the Barlow surface X . For the calculation of a quintic surface Σ and a double cover Δ we adapt the results and Macaulay2-scripts of [BvBKS12]. In [BvBKS12] a quintic determinantal surface with parameters

$$a_1 = a_2 = a_4 = a_5 = 1, \quad a_3 = a_6 = -4$$

is considered which corresponds to the special surface in [Bar85]. We take the same parameters and perform our computations over the field \mathbb{F}_{521} which contains a fifth root of unity:

```

i1 : load "determinantalBarlow.m2"
i2 : kk = ZZ/521;
i3 : -- a fifth root of unity in kk
     xi = (25)_kk;
i4 : -- compute the double cover in P(1^4,2^5)
     Idelta = doubleCover(kk,xi);
o4 : Ideal of Z

```

Next we compute the canonical ring $R(X)$ as the invariant ring of $Z/I(\Delta)$. We verify that the bicanonical system of the Barlow surface has indeed 4 distinct base points as shown in [CP00]. Afterwards we compute an automorphism of $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$ which maps the 4 base points to the "standard" position as explained in the previous sections:

```

i5 : IX = canonicalRingBarlow(Idelta);
i6 : -- compute the base points of |2K|
     base2K = ideal mingens (IX+ideal(x_0,x_1));
i7 : -- verify that there are 4 distinct base points
     length(decompose base2K) == 4
o7 : true

```

As before, we compute now a minimal free resolution of $R(X) \cong \hat{S}/I(X)$ as an S -module and see whether $R(X)$ admits a standard resolution over \mathbb{F}_{521} :

```

i8 : F = minimalResolution(IX);
i9 : Fskew = skewsymmetricResolution(F);
i10 : Fstand = standardResolution(Fskew);
i11 : d1 = Fstand.dd_1;
o11 : Matrix S  <--- S
           8      26
i12 : d2 = Fstand.dd_2;
o12 : Matrix S  <--- S
           26      26

```

Having a standard resolution, we compute the assigned matrix $l \in \text{St}(Q)$ and the Betti numbers of $\text{coker } \mathfrak{a}(l)$:

```

i13 : l = assignedMatrix(d1)
o13 = |-102 0 235 4 1 0 -51 51 -51 2 -2 -2 |
      |-51 51 -51 2 2 -2 235 0 -102 0 -1 -4 |
i14 : a1 = d1^{1..4}_{0..5};

```

```
i15 : betti res coker al
```

```

      0 1 2
o15 = total: 4 6 2
      4: 4 . .
      5: . 6 .
      6: . . 2

```

In [CP00], Catanese and Pignatelli showed that the bicanonical fibration of the Barlow surface has two hyperelliptic fibres (counted with multiplicities). We can verify that there are indeed two distinct hyperelliptic fibres C_1 and C_2 over the field \mathbb{F}_{521} . Moreover, we compute that the two hyperelliptic curves are smooth and irreducible. Hence, Proposition 10.3.8 implies that there are two points in \mathbb{P}^1 at which the 4×4 minors of the matrix $\alpha(l)$ vanish, but which are not contained in the vanishing locus of the 3×3 minors. The fibres over these two points in \mathbb{P}^1 are exactly the two hyperelliptic curves.

```
i16 : ma4 = minors(4, al);
```

```
i17 : decompose ma4
```

```
o17 = {ideal(x0 + 45x1), ideal(x0 + 220x1)}
```

```
i18 : ma3 = minors(3, al);
```

```
i19 : decompose ma3
```

```
o19 = {ideal(x1, x0)}
```

Next we compute the vanishing locus of the 7×7 minors of d'_1 . Recall that we have the inclusion

$$\text{Supp}(\widetilde{M}) \subseteq V(7 \times 7 \text{ minors of } d'_1),$$

where $M = \text{coker } d'_1$. The algebraic set on the right-hand side contains always the image of the 4 base points of $|2K_X|$. Since the two hyperelliptic fibres in the Barlow surface are smooth, there are two additional points in $V(7 \times 7 \text{ minors of } d'_1)$ by Lemma 10.3.7:

```
i20 : d1' = d1^{1..7};
```

```
i21 : I' = ann coker d1';
```

```
i22 : netList decompose I'
```

```

+-----+
o22 = |ideal(x , y , y , x , y ) |
      |      1  3  2  0  0      |
+-----+
      |ideal(x , y , y , x , y ) |
      |      1  3  2  0  1      |
+-----+
      |ideal(x , y , x , y , y ) |
      |      1  2  0  1  0      |
+-----+

```



```

+-----+
|ideal(x , y , x , y , y )|
|      1  3  0  1  0      |
+-----+
|      3  2|
|ideal(106x  + y , y  - y , x  + 45x , y  - 118y , y  - 118y )|
|      1  3  2  3  0      1  1      3  0      3  |
+-----+
|      3  2|
|ideal(197x  + y , y  - y , x  + 220x , y  + 234y , y  + 234y )|
|      1  3  2  3  0      1  1      3  0      3  |
+-----+

```

The two additional points in the vanishing locus of I' are the singularities of the images of the hyperelliptic curves under φ as shown in Proposition 10.3.6. Furthermore, these points are singular points of the surface Y . We believe that these are the only singularities of Y . We also compute the images of the hyperelliptic curves under the tricanonical morphism $X_{can} \rightarrow \mathbb{P}^3$ and verify that these curves are twisted cubic curves:

```

i23 : -- the ideal of the surface Y in P(2^2,3^4)
      IY = ann coker d1;

i24 : pthyp1 = ideal(x_0+45*x_1);

i25 : pthyp2 = ideal(x_0+220*x_1);

i26 : T = kk[y_0,y_1,y_2,y_3];

i27 : -- compute the images of the hyperell. fibres in P^3
      Dhyp1 = fibreInP3(pthyp1);

i28 : Dhyp2 = fibreInP3(pthyp2);

i29 : (genus(Dhyp1) == 0, genus(Dhyp2) == 0)

o29 = (true, true)

i30 : (betti res Dhyp1, betti res Dhyp2)

      0 1 2      0 1 2
o30 = (total: 1 3 2, total: 1 3 2)
      0: 1 . .      0: 1 . .
      1: . 3 2      1: . 3 2

```

The image of a (general) bicanonical curve is a complete intersection of type $(2, 3)$:

```

i31 : ptgeneral = ideal(x_0+196*x_1);

i32 : Dgeneral = fibreInP3(ptgeneral);

i33 : genus(Dgeneral) == 4

o33 = true

```

```
i34 : betti res Dgeneral
```

```

      0 1 2
o34 = total: 1 2 1
      0: 1 . .
      1: . 1 .
      2: . 1 .
      3: . . 1

```

Finally, we compute the birational model $W \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ and verify that W is singular as expected:

```
i35 : S13 = kk[x_0, x_1, y_0..y_3, Degrees=>{2:{1,0}, 4:{0,1}}];
```

```
i36 : IW = bihomogeneousModel(IY);
```

```
i37 : tally degrees IW
```

```

o37 = Tally{{0, 9} => 1}
      {1, 4} => 1
      {1, 5} => 1
      {2, 3} => 3
      {3, 2} => 1

```

```
i38 : isSmoothBihomModel(IW)
```

```
o38 = false
```

11.3.3 A Surface with one Hyperelliptic Fibre

At the beginning of this section we have constructed a numerical Godeaux surface X with $\text{Tors } X = 0$ whose bicanonical system has no (honestly) hyperelliptic fibres. We have seen that the cokernel of the \mathfrak{a} -matrix $\mathfrak{a}(l)$ of a standard resolution of $R(X)$ has a minimal free resolution of type

$$0 \leftarrow \text{coker } \mathfrak{a}(l) \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow B(-3)^2 \leftarrow 0,$$

where $B = \mathbb{k}[x_0, x_1]$ as before. Furthermore, the vanishing locus of the 4×4 minors of $\mathfrak{a}(l)$ is empty in this case.

On the other hand, in the example of the Barlow surface, we have seen that the corresponding matrix $\mathfrak{a}(l)$ has a minimal free resolution

$$0 \leftarrow \text{coker } \mathfrak{a}(l) \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow B(-2)^2 \leftarrow 0$$

and that the vanishing locus of the 4×4 minors of $\mathfrak{a}(l)$ consists of 2 points in \mathbb{P}^1 .

From the characterization in Proposition 10.3.8 we know that for constructing a torsion-free numerical Godeaux surface with exactly one hyperelliptic fibre, we have to choose a matrix $l \in \text{St}(Q)$ such that the 4×4 minors of $\mathfrak{a}(l)$ vanish at exactly one point (with multiplicity 1). Furthermore, by Remark 7.4.5 and the discussion after Proposition 8.4.5, we are interested in a matrix $l \in \text{St}(Q)$ such that the module $\text{coker } \mathfrak{a}(l)$ has a minimal free resolution of the form:

$$0 \leftarrow \text{coker } \mathfrak{a}(l) \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow \begin{array}{c} B(-2)^1 \\ \oplus \\ B(-3)^1 \end{array} \leftarrow 0. \quad (11.5)$$

There are several possibilities for computing such a matrix $l \in \text{St}(Q)$. The first one is that we simply use the Las Vegas algorithm to compute random lines in Q and stop if we have found a line such that one presentation matrix $l \in \text{St}(Q)$ (and hence any) fulfills the required property. The other possibility is to start with the matrix l associated to the Barlow surface. The corresponding line has two distinguished points in \mathbb{P}^{11} determined by the points in the vanishing locus of the 4×4 minors of $\alpha(l)$. Recall that the Las Vegas algorithm computes first a point $p \in Q$ and then a line in Q through this point. So we can take one of these special points in \mathbb{P}^{11} and start the second part of the Las Vegas algorithm with this point. Then the output is a (presentation matrix of a) line having in general a minimal free resolution as in (11.5).

```
i1 : load "numGodeaux.m2"
i2 : kk = ZZ/521;
i3 : (A,B) = globalVariablesC(kk);
i4 : astart = A^{0..3}_{0..5};
i5 : (rel2,subs1) = setupMarkedGodeaux(R);
```

Note that the procedure `globalVariablesC` works in almost the same manner as the procedure `globalVariables` from the first example of Section 11.3. The only difference is that we do not assume that the \mathfrak{c} -matrix of d_1 is zero. Thus, the starting set-up depends on the unknown entries of the matrices \mathfrak{a} , \mathfrak{o} , \mathfrak{n} and \mathfrak{c} . Next we choose a line in Q which intersects the assigned line of the Barlow surface in the special point corresponding to the point in \mathbb{P}^1 given by $(x_0 + 45x_1)$ (see output `o17` in Section 11.3.2). We verify that the cokernel of the corresponding \mathfrak{a} -matrix has the desired Betti numbers:

```
i6 : l = matrix {{-259, -21, 189, -23, -254, -232, -172, -175,
                 89, -103, 1, 0}, {-130, -201, 71, 8, -65, -242,
                 -24, 25, 4, 221, 0, 1}};
i7 : subsline = updateRelations(l);
i8 : al = sub(astart, subsline);
i9 : betti res coker al

      0 1 2
o9 = total: 4 6 2
      4: 4 . .
      5: . 6 .
      6: . . 1
      7: . . .
      8: . . 1

o10 : ma4 = minors(4, al);
i11 : decompose ma4
o11 = {ideal(x  - 50x )}
      0          1
```

The last step of the construction is the computation of the solution space $\mathcal{V}(l)$. The procedure `pickSectionC` works similar as in the case $\mathfrak{c} = 0$. However, to compute a solution space of the smallest possible dimension, we perform some preliminary calculations to decide which of

the entries of the matrix \mathfrak{c} can be set to zero a priori as mentioned in Remark 8.1.7. Note that these computations depend on the choice of the matrix l . After that we choose a random point in the solution space $\mathcal{V}(l)$ and compute a standard resolution:

```
i12 : (subpoint, randpoint) = pickSectionC(subsline);

i13 : -- a point in a 12-diml solution space
      randpoint

o13 = | 14   |
      | 127  |
      | 229  |
      | 41   |
      | -143 |
      | 116  |
      | 185  |
      | -159 |
      | 195  |
      | 137  |
      | -118 |
      | 168  |

                12      1
o14 : Matrix kk  <--- kk

i15 : Fstand = standardResolution(subpoint);

i16 : d1 = Fstand.dd_1;

                8      26
o16 : Matrix S  <--- S

i17 : d2 = Fstand.dd_2;

                26      26
o17 : Matrix S  <--- S
```

Next we verify the first assumptions of Theorem 5.0.2 as in the example of a numerical Godeaux surface having no hyperelliptic fibres:

```
i18 : verifyRingCondition(d1)

o18 = true

i18 : verifyAssumptions(d1)

o18 = true

i19 : IY = ann coker d1;
```

Hence, $R := \text{coker } d_1$ has a ring structure and $\text{Proj}(R)$ is a surface. Next we compute the vanishing locus of the 7×7 minors of d'_1 :

```
i20 : d1' = d1^{1..7};

i21 : I' = ann coker d1';

i22 : netList decompose I'
```

```

+-----+
|          3      2          |
o22 = |ideal(33x  + y , y  - y , y  + 234y , y  + 234y , x  - 50x )|
|          1      3      2      3      1          3      0          3      0          1      |
+-----+
|ideal(x , y , y , y , x ) |
|          1      3      1      0      0          |
+-----+
|ideal(x , y , y , y , x ) |
|          1      3      2      1      0          |
+-----+
|ideal(x , y , y , y , x ) |
|          1      3      2      0      0          |
+-----+
|ideal(x , y , y , y , x ) |
|          1      2      1      0      0          |
+-----+

```

We see that there is one additional point in the vanishing locus. We compute that the curve $G_1 = \text{Proj}(S_Y/(x_0 - 50x_1)) \subseteq Y$ has arithmetic genus 5 and exactly one singularity $q \in G_1 \subseteq Y$. Furthermore, we verify that q is the only singular point of Y . Let us also compute the image of $G_1 \subseteq Y$ under the projection from Y to \mathbb{P}^3 :

```

i23 : ptspecial = ideal(x_0-50*x_1);
i24 : Dspecial = fibreInP3(ptspecial);
i25 : ptgeneral = ideal(x_0-33*x_1);
i26 : Dgeneral = fibreInP3(ptgeneral);
i27 : (genus Dspecial == 0, genus Dgeneral== 4)
o27 = (true, true)
i28 : betti res Dspecial
          0 1 2
o28 = total: 1 3 2
          0: 1 . .
          1: . 3 2
i29 : betti res Dgeneral
          0 1 2
o29 = total: 1 2 1
          0: 1 . .
          1: . 1 .
          2: . 1 .
          3: . . 1

```

We see that the image of the special curve is a twisted cubic curve, whereas the image of some random curve is a complete intersection of type $(2, 3)$. In the end we compute the surface W in $\mathbb{P}^1 \times \mathbb{P}^3$:

```

i30 : S13 = kk[x_0,x_1,y_0..y_3,Degrees=>{2:{1,0},4:{0,1}}];

```

```
i31 : IW = bihomogeneousModel(IY);
```

```
i32 : tally degrees IW
```

```
o32 = Tally{{0, 9} => 1}
      {1, 6} => 5
      {2, 4} => 4
      {2, 5} => 3
      {3, 3} => 3
      {3, 4} => 3
      {4, 3} => 2
      {5, 2} => 1
```

```
i33 : time isSmoothBihomModel(IW)
      -- used 396.951 seconds
```

```
o33 = false
```

From the proof of Theorem 5.2.1 we know that R admits a unique structure as an S_Y -algebra. Furthermore, from the number and degrees of the module generators of R as an S -module we deduce that R is generated in degree ≤ 5 as a \mathbb{k} -algebra. Hence, the description of the defining relations of the canonical ring of a numerical Godeaux surface from Lemma 3.3.9 holds also for the ring R . Hence we can use our algorithm `canonicalRing` to compute the remaining defining relations of R (as a ring). Recall that this algorithm relies only on Lemma 3.3.9 and the ideas of Chapter 4. Thus, we assume that all statements of Chapter 4 remain true if we replace the canonical ring $R(X)$ by our computed Gorenstein S -algebra R of codimension 3 which can be checked in detail. Moreover, we plan to transfer these results to an even more general setting in a future work.

```
i34: Sbig = kk[x_0,x_1,y_0..y_3,z_0..z_3,w_0..w_2,
             Degrees=>{2:2,4:3,4:4,3:5}];
```

```
i35 : IX = canonicalRing(d1);
```

```
i36 : betti IX
```

```
o36 = total: 1 54
      0: 1 .
      1: . .
      2: . .
      3: . .
      4: . .
      5: . 6
      6: . 12
      7: . 18
      8: . 12
      9: . 6
```

We claim that the surface $\text{Proj}(R)$ is smooth. To verify this, it is enough to check smoothness at the preimages of the one singular point of Y . We compute that there are two distinct points $p_0, p_1 \in \text{Proj}(R)$ lying over the singularity $q \in Y$. Since these two points are not contained in the singular locus of the weighted projective space $\mathbb{P}(2^2, 3^4, 4^4, 5^3)$, we can verify directly that p_0, p_1 are smooth points of $\text{Proj}(R)$ by computing the rank of the Jacobian matrix of $\text{Proj}(R)$ at these points. Thus, $\text{Proj}(R)$ (considered over the algebraic closure of \mathbb{F}_{521}) is the canonical model of a torsion-free numerical Godeaux surface X having exactly one hyperelliptic fibre. Furthermore $X \cong X_{can} = \text{Proj}(R)$.

We end this section with the following conjecture which is suggested by our computational results:

Conjecture 11.3.3. *Let X be a marked numerical Godeaux surface with $\text{Tors } X = 0$. Furthermore, let*

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^\vee \xleftarrow{d_1^{\text{tr}}} F_0^\vee \leftarrow 0$$

be a standard resolution of the canonical ring $R(X)$ with assigned matrix $l \in \text{St}(Q)$. Assume that $\text{coker } \mathfrak{a}(l)$ has a minimal free resolution of type

$$0 \leftarrow \text{coker } \mathfrak{a}(l) \leftarrow B^4 \leftarrow B(-1)^6 \leftarrow \begin{array}{c} B(-2)^h \\ \oplus \\ B(-3)^{2-h} \end{array} \leftarrow 0.$$

Then h is the number of hyperelliptic fibres of the bicanonical fibration (counted with multiplicity).

12 Outlook

We end this thesis by giving a brief outlook on further applications of the presented construction.

In Chapters 7 and 8 we restricted our study mainly to standard resolutions whose assigned matrices are contained in the open subset $V_{\text{gensyz}} \subseteq \text{St}(Q)$. We have shown that such a standard resolution leads to a (marked) numerical Godeaux surface X having a trivial torsion group. Furthermore, we have seen that the existence of (smooth) hyperelliptic curves in the bicanonical system of X is completely determined by the α -matrix a of a standard resolution of $R(X)$, and thus by the choice of the matrix $l \in \text{St}(Q)$. Recall from Proposition 10.3.8 that, for $p \in \mathbb{P}^1$, a smooth bicanonical curve C_p is hyperelliptic if and only if $\text{rank}(a(p)) = 3$. In a future work, we want to establish further relations between the α -matrix and the existence of base points of $|3K_X|$. In Lemma 8.4.2 we have already seen that if there is a point $p \in \mathbb{P}^1$ with $\text{rank}(a(p)) \leq 2$, then $|3K_X|$ has a base point and that the unique bicanonical curve containing this base point is the fibre C_p . We expect that the converse of this statement is also true and that we obtain a similar classification result as for hyperelliptic curves, that means: a bicanonical curve C_p contains a base point of $|3K_X|$ if and only if $\text{rank}(a(p)) = 2$. Furthermore, we expect that our construction gives us also the (known) families of numerical Godeaux surfaces with torsion group $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$, respectively. To see this, we first have to identify subsets $V_3, V_5 \subseteq \text{St}(Q)$ whose elements lead to numerical Godeaux surfaces with torsion group $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$, respectively. Then we have to compute the (minimal) dimension of the vector spaces $\mathcal{V}(l)$ for matrices in these subsets. In explicit examples, we have seen that for $l \notin V_{\text{gensyz}}$, the dimension of the solution space $\mathcal{V}(l)$ is in general much higher than for $l \in V_{\text{gensyz}}$. But in this case, there are the non-trivial sets $\text{Stab}(l, R(X)) \subseteq \mathcal{V}(l)$ (see Lemma 8.1.6) whose points lead to isomorphic standard resolutions of $R(X)$ and which we have to consider in the dimension count at the end.

A further application of our method is the construction of Godeaux curves. A *Godeaux curve* C is a curve of genus 4 marked with an effective divisor Σ such that $3\Sigma = 2K_C$. Reid studied such a curve assuming that C is a complete intersection of type $(2, 3)$ in \mathbb{P}^3 and $\Sigma = p_0 + \dots + p_3$ being the sum of the 4 coordinate points of \mathbb{P}^3 . Hence, a general bicanonical curve of a surface in our constructed 8-dimensional family of torsion-free numerical Godeaux surfaces is a Godeaux curve. More generally, we can modify our method by considering a finitely generated $S' = \mathbb{k}[x_1, y_0, y_1, y_2, y_3]$ -module R' having the same Betti numbers as $R(X)$ (as an S -module). By computing explicit examples with `Macaulay2`, we have seen that the construction of a standard resolution of R' works basically in the same way as for $R(X)$. But, instead of choosing a line in the complete intersection $Q \subseteq \mathbb{P}^1$, we only have to choose a point in Q . Our aim is to get a more detailed description of the solution spaces and to compute the moduli of Godeaux curves which we obtain with our modified construction.

Finally, a further extension is to drop the general assumption that the 4 base points of $|2K_X| = |M|$ are all distinct. To obtain an applicable construction in this case, we first have to describe the possible configurations of the images of the base points in \mathbb{P}^3 and the minimal free resolution of the canonical ring modulo x_0, x_1 as in Chapter 6, and then adapt the original set-up of the matrices d'_1 and d'_2 modulo x_0, x_1 . Then, in particular, our construction should also yield numerical Godeaux surfaces with torsion group $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

Bibliography

- [Bar82] R. Barlow. *Some new surfaces with $p_g = 0$* . PhD thesis, University of Warwick, 1982.
- [Bar85] R. Barlow. A simply connected surface of general type with $p_g = 0$. *Inventiones mathematicae*, 79(2):293–301, 1985.
- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *Journal of the American Mathematical Society*, 23(2):405–468, 2010.
- [BDF⁺18] J. Böhm, W. Decker, A. Frühbis-Krüger, F.-J. Pfreundt, M. Rahn, and L. Ristau. Towards Massively Parallel Computations in Algebraic Geometry. *preprint arXiv:1609.02177*, 2018.
- [BE73] D. A. Buchsbaum and D. Eisenbud. What Makes a Complex Exact? *Journal of Algebra*, 25(2):259–268, 1973.
- [BE77] D. A. Buchsbaum and D. Eisenbud. Algebra Structures for Finite Free Resolutions, and Some Structure Theorems for Ideals of Codimension 3. *American Journal of Mathematics*, 99(3):447–485, 1977.
- [Bea96] A. Beauville. *Complex Algebraic Surfaces*. Cambridge University Press, 1996.
- [BFK18] J. Böhm and A. Frühbis-Krüger. A smoothness test for higher codimensions. *Journal of Symbolic Computation*, 86:153–165, 2018.
- [BH98] W. Bruns and J. Herzog. *Cohen-Macaulay rings*. Cambridge University Press, 1998.
- [BHPVdV15] W. Barth, K. Hulek, C. Peters, and A. Van de Ven. *Compact Complex Surfaces*, volume 4. Springer, 2015.
- [BHSW] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. Bertini: Software for numerical algebraic geometry. Available at bertini.nd.edu.
- [Böh05] C. Böhning. L. Szpiro’s conjecture on Gorenstein algebras in codimension 2. *Journal of Algebra*, 288(2):545–555, 2005.
- [Bom73] E. Bombieri. Canonical Models of Surfaces of General Type. *Publications Mathématiques de l’Institut des Hautes Études Scientifiques*, 42(1):171–219, 1973.
- [BvBKS12] C. Böhning, H.-C. Graf von Bothmer, L. Katzarkov, and P. Sosna. Determinantal Barlow surfaces and phantom categories. *preprint arXiv:1210.0343*, 2012.
- [Cam32] L. Campedelli. Sopra alcuni piani doppi notevoli con curva di diramazione del decimo ordine. *Rend. della R. Acc. Nazionale dei Lincei*, 15:536–542, 1932.

- [Cas96] G. Castelnuovo. Sulle superficie di genere zero. *Memorie della Soc. It. delle Scienze (detta dei XL), ser. III*, 10:7, 1896.
- [Cat81] F. Catanese. Babbage's Conjecture, Contact of Surfaces, Symmetric Determinantal Varieties and Applications. *Inventiones mathematicae*, 63(3):433–465, 1981.
- [Cat84] F. Catanese. On the Moduli Spaces of Surfaces of General Type. *Journal of Differential Geometry*, 19(2):483–515, 1984.
- [CCML07] A. Calabri, C. Ciliberto, and M. Mendes Lopes. Numerical Godeaux surfaces with an involution. *Transactions of the American Mathematical Society*, 359(4):1605–1632, 2007.
- [CFHR99] F. Catanese, M. Franciosi, K. Hulek, and M. Reid. Embeddings of curves and surfaces. *Nagoya Mathematical Journal*, 154:185–220, 1999.
- [CG94] P. Craighero and R. Gattazzo. Quintic surfaces of \mathbb{P}^3 having a non singular model with $q = p_g = 0, P_2 \neq 0$. *Rendiconti del Seminario Matematico della Università di Padova*, 91:185–198, 1994.
- [Cil83] C. Ciliberto. Sul grado dei generatori dell'anello canonico di una superficie di tipo generale. *Rend. Sem. Mat. Univ. Pol. Torino*, 41(3):83–111, 1983.
- [Cou09] S. Coughlan. *Key varieties for surfaces of general type*. PhD thesis, University of Warwick, 2009.
- [CP00] F. Catanese and R. Pignatelli. On simply connected Godeaux surfaces. In *Complex Analysis and Algebraic Geometry*, pages 117–153. de Gruyter, Berlin, 2000.
- [CU16] S. Coughlan and G. Urzúa. On $\mathbb{Z}/3\mathbb{Z}$ -Godeaux Surfaces. *preprint arXiv:1609.02177*, 2016.
- [DGPS18] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 4-1-1 — A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de>, 2018.
- [EH16] D. Eisenbud and J. Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.
- [Eis80] D. Eisenbud. Transcanonical Embeddings of Hyperelliptic Curves. *Journal of Pure and Applied Algebra*, 19:77–83, 1980.
- [Eis05] D. Eisenbud. *The Geometry of Syzygies*. Springer-Verlag, New York, 2005.
- [Eis13] D. Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*, volume 150. Springer Science & Business Media, 2013.
- [EU97] D. Eisenbud and B. Ulrich. Modules that are finite birational algebras. *Illinois Jl. Math*, 141:10–15, 1997.
- [FSWT11] A. J. Frankild, S. Sather-Wagstaff, and A. Taylor. Second symmetric powers of chain complexes. *Bulletin of the Iranian Mathematical Society*, 37:39–75, 2011.
- [Gie77] D. Gieseker. Global moduli for surfaces of general type. *Inventiones mathematicae*, 43(3):233–282, 1977.

- [God34] L. Godeaux. *Les surfaces algébriques non rationnelles de genres arithmétique et géométrie nuls*. Hermann & Cie, 1934.
- [Gra96] M. Grassi. Koszul modules and Gorenstein algebras. *Journal of Algebra*, 180(3):918–953, 1996.
- [Gro67] A. Grothendieck. *Local Cohomology: A Seminar Given by A. Grothendieck, Harvard University, Fall, 1961*, volume 41. Springer, 1967.
- [GS] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [GW78] S. Goto and K. Watanabe. On graded rings, I. *Journal of the Mathematical Society of Japan*, 30(2):179–213, 1978.
- [Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.
- [Har13] J. Harris. *Algebraic Geometry: A First Course*, volume 133 of *Graduate Texts in Mathematics*. Springer Science & Business Media, 2013.
- [HS06] C. Huneke and I. Swanson. *Integral Closure of Ideals, Rings, and Modules*, volume 13. Cambridge University Press, 2006.
- [KLP10] J. Keum, Y. Lee, and H. Park. Construction of surfaces of general type from elliptic surfaces via Q-Gorenstein smoothing. *preprint arXiv:1008.1222*, 2010.
- [Kol13] J. Kollár. *Rational Curves on Algebraic Varieties*, volume 32. Springer Science & Business Media, 2013.
- [Lee01] Y. Lee. Bicanonical Pencil of a Determinantal Barlow Surface. *Transactions of the American Mathematical Society*, 353(3):893–905, 2001.
- [Mat13] K. Matsuki. *Introduction to the Mori Program*. Springer Science & Business Media, 2013.
- [Miy76] Y. Miyaoka. Tricanonical Maps of Numerical Godeaux Surfaces. *Inventiones mathematicae*, 34:99–112, 1976.
- [Mum62] D. B. Mumford. The Canonical Ring of an Algebraic Surface. *Annals of Mathematics*, pages 612–615, 1962.
- [New78] P. E. Newstead. Introduction to Moduli Problems and Orbit Spaces. *Tata Inst. Lecture Notes*, 1978.
- [Pig00] R. Pignatelli. *Low genus fibrations and numerical Godeaux surfaces*. PhD thesis, Università di Pisa, 2000.
- [Rei] M. Reid. Godeaux and Campedelli surfaces. available from <https://homepages.warwick.ac.uk/~masda/surf/more/Godeaux.pdf>.
- [Rei78] M. Reid. Surfaces with $p_g = 0$, $K^2 = 1$. *J. Fac. Sci. Tokyo Univ.*, 25:75–92, 1978. available from <https://homepages.warwick.ac.uk/~masda/Reprint/pg0K21.pdf>.

- [Rei81] M. Reid. A simply connected surface of general type with $p_g = 0$, $K^2 = 1$, due to Rebecca Barlow. 1981. available from <https://homepages.warwick.ac.uk/~masda/surf/more/Barlow.pdf>.
- [Rei85] M. Reid. Young Person's Guide to Canonical Singularities. *Algebraic Geometry, Bowdoin*, 46:345–414, 1985.
- [Rei00] M. Reid. Graded rings and birational geometry. In *Proc. of algebraic geometry symposium (Kinosaki, Oct 2000)*, pages 1–72. K. Ohno (Ed.), 2000.
- [Rei13a] M. Reid. Gorenstein in codimension 4 - The General Structure Theory. In *Algebraic Geometry in East Asia - Taipei 2011*, volume 65 of *Advanced Studies in Pure Mathematics*, pages 201–227, 2013.
- [Rei13b] M. Reid. Parallel unprojection equations for $\mathbb{Z}/3$ Godeaux surfaces. 2013. available from <http://homepages.warwick.ac.uk/~masda/codim4/God3.pdf>.
- [Sch05] F.-O. Schreyer. An Experimental Approach to Numerical Godeaux Surfaces. *Oberwolfach Report "Komplexe Algebraische Geometrie"*, 7/2005:434–436, 2005.
- [Wer94] C. Werner. A surface of general type with $p_g = q = 0$, $K^2 = 1$. *Manuscripta mathematica*, 84(1):327–341, 1994.
- [Wer97] C. Werner. A Four-Dimensional Deformation of a Numerical Godeaux Surface. *Transactions of the American Mathematical Society*, 349(4):1515–1525, 1997.

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