Domain Decomposition for Kinetic Problems with Nonequilibrium States

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October 1995

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Abstract

A nonequilibrium situation governed by kinetic equations with different Knudsen numbers in different subdomains is discussed. We consider a domain decomposition problem for Boltzmann- and Euler equations, establish the correct coupling conditions and prove the validity of the obtained coupled solution. Moreover numerical examples comparing different types of coupling conditions are presented.

1 Introduction

The Boltzmann equation and the more classical gas dynamics equations (such as Euler or Navier-Stokes equations) are used to model hypersonic gas flows. Numerical simulations of such flows are useful for example in the design of space vehicles, in particular in understanding the behavior of the early phases of reentry flights.
Such flows are usually far from any kind of local equilibrium states. This means that variants of the Boltzmann equation have to be used as first principle equations instead of the Euler or Navier-Stokes equations. However, when the mean free path of molecules becomes small, all numerical methods for the Boltzmann equation become exceedingly expensive in computing time. Therefore, gas dynamics equations should be used whenever possible — in other words, near local equilibrium states in situations where the local mean free path is small and outside of shock and boundary layers. These considerations prompt the use of domain decomposition strategies, where the Boltzmann equation is to be solved only in regions others than those mentioned above.
Once the regions described by the gas dynamics equations are determined, the next major problem is the matching of the Boltzmann domain with the Euler or Navier- Stokes domain. This question is far from being an easy one, as the equations to couple and the numerical schemes used to solve them are of very different nature.
The approach usually employed in numerical procedures, see Bourgat et al. [3] or Lukschin et al. [16], is the following: The boundary conditions at the interface for the aerodynamic equation are determined from the Boltzmann distribution function by equalizing the moments or fluxes. The boundary condition for the Boltzmann equation at the interface, i.e. the ingoing function for the Boltzmann region is given by a Maxwellian distribution with

the aerodynamic quantities as parameters. For an equilibrium situation at the interface these coupling conditions are the appropriate ones.
If we consider instead nonequilibrium states at the interface, then the above coupling conditions will not lead to the correct results. Here the matching requires a more exact analysis. It can be done by modelling the interface region by a transition layer. We refer here to Golse [9]. Asymptotic analysis leads to a kinetic linear half space problem. The asymptotic values of the solution of this problem determine the aerodynamic boundary conditions at the interface. The outgoing flux of the half space problem gives the ingoing distribution function for the Boltzmann region at the interface. Kinetic half space problems have been widely considered. A mathematical investigation of these problems is done, e.g., in Arthur and Cercignani [2], Bardos et al. [4], Bensoussan et al. [5], Cercignani [7], Coron et al. [8], Greenberg et al. [12]. Many numerical investigations for various different situations have been performed by Sone and coworkers, see [1, 17].
Obviously the direct solution of the half space problem would be much too expensive. We could as well solve the Boltzmann equation also in the aerodynamic region. Moreover, from the above we see that only the asymptotic states and the outgoing fluxes are really required. In Golse/Klar [11] we developed a fast numerical scheme which computes approximately these two things by a Chapman Enskog type expansion procedure. This makes the approach reasonable from a numerical point of view.
For a different approach to the coupling problem we refer to Illner/Neunzert [13].

The paper is organized as follows:
In Section 2 we describe the physical situation under consideration.
In Section 3 we consider a simple model problem and introduce the coupling conditions given by the analysis of the kinetic layer. We investigate whether the coupled solution obtained with these coupling conditions is the correct one, i.e. whether it is near to the kinetic solution in the whole domain. It is proved that, if the Knudsen number ϵ in the aerodynamic region tends to 0, the analysis of the kinetic layer gives the correct coupling conditions up to order ϵ.
In Section 4 we extend the analysis to the linearized Boltzmann equation coupled with the linearized Euler equations and point out the correct conditions in this case.
In Section 5 we present some numerical examples for the model problem in section 3 comparing the above coupling conditions with the ones obtained by equalizing moments or fluxes. Moreover numerical results for the 3-dimensional BGK model are shown comparing again the different types of coupling conditions.

2 The physical problem

The physical situation in a domain D in $\mathbb{R}^d, d = 1, 2, 3$ is supposed to be described by the Boltzmann equation linearized around a constant Maxwellian state with parameters $\bar{\rho}, \bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3), \bar{T}$. The domain D is divided into two subdomains D_1 and D_2. We assume the mean free paths to be ϵ_1 in D_1 and ϵ_2 in D_2. After shifting velocities $v \to v + \bar{u}$ one obtains the equations

$$\partial_t \varphi^{(i)} + (v+\bar{u})\cdot\nabla_x\varphi^{(i)} + \frac{1}{\epsilon_i}2Q(\bar{M},\varphi^{(i)}) = 0, \tag{2.1}$$

where the index i=1,2 stands for the the regions D_1 and D_2 respectively, $\varphi^{(i)} = \varphi^{(i)}(x,v,t)$, $x \in D_i,\ v = (v_1,v_2,v_3) \in \mathbb{R}^3,\ t \in [0,\infty),\ \bar{u} \in \mathbb{R}^3$. Q is the Boltzmann collision operator with a suitable collision kernel and $\bar{M}$ is the constant Maxwellian state with parameters $(\bar{\rho},\bar{0},\bar{T})$, i.e.,

$$\bar{M}(v) := \frac{\bar{\rho}}{(2\pi\bar{T})^{3/2}}\ exp\left(-\frac{|v|^2}{2\bar{T}}\right).$$

We assume that $\epsilon_2 = \epsilon$ in D_2 is small such that an approximation of the Boltzmann equation by macroscopic equations is valid. ϵ_1 in D_1 is fixed.
One should keep in mind that from a physical point of view the difference between the mean free paths in D_1 and D_2 should be small for the linearized problem treated here, since we are linearizing around a global Maxwellian state. However we are making here a preliminary attempt to solve the coupling problem, that should work as well for nonlinear problems, where strikingly different mean free paths may appear. Concentrating more on the mathematical aspects of the problem, one may therefore also think of ϵ_1 in D_1 to be large compared to ϵ_2 in D_2.
The aerodynamic equation in D_2 is the compressible Euler equation linearized around the constant state $(\bar{\rho},\bar{u},\bar{T})$.
The aim is now to approximate the global kinetic solution in the whole domain D by the solution of the following coupling problem : In the domain D_1 we solve the kinetic equation (2.1) with mean free path ϵ_1. In the domain D_2 we solve the macroscopic equation . The problem is to determine the correct coupling conditions at the interface between the two equations and to investigate the resulting solution of the coupling problem.

3 Coupling Conditions and Physical Correctness of the Coupled Solution for a Simple Model Problem

We consider here for simplicity a one dimensional geometry $x \in [-L,L]$ with an interface at $x = 0$. The physical situation is described by model kinetic equations with different mean free paths ϵ_1 and ϵ_2 in the domains $D_1 = [-L,0]$ and $D_2 = [0,L]$.
The kinetic equations in $D_i, i = 1,2$ are here

$$\partial_t \varphi^{(i)} + (v_1+u)\partial_x\varphi^{(i)} + \frac{1}{\epsilon_i}(I-K)\varphi^{(i)} = 0, \tag{3.1}$$

where $x \in D_i, v = (v_1,\cdots,v_N) \in S \subseteq \mathbb{R}^N,\ N = 1,2,3,\ t \in [0,\infty)$. S is assumed to be the unit ball around 0 in $\mathbb{R}^N$, $u \in [-1,1]\backslash\{0\}$ is a constant and K an integral operator

$$K\varphi(x,v,t) = \int_S \bar{k}(v,v')\varphi(x,v',t)dv',$$

$\bar{k}$ symmetric in v and $v', 0 < k_1 < \bar{k}(v,v') < k_2$, where k_1, k_2 are some constants, and $\int\limits_S \bar{k}(v,v')dv' = 1$. In particular K is an operator in $\mathcal{L}^\infty(S)$ with $\|K\|_\infty \leq 1$. Moreover, the collision operator $I - K$ has as collision invariants only constants.
We assume again $\epsilon_2 := \epsilon$ in D_2 to be small such that an aerodynamic approximation of the kinetic equation is valid. The aerodynamic equation in D_2 associated to (3.1) is a simple linear advection equation

$$\partial_t \Theta + u \partial_x \Theta = 0, u \in [-1,1] \backslash \{0\}. \tag{3.2}$$

Coupling the solution of (3.1) with ϵ_1 in D_1 and the solution of (3.2) in D_2 one tries to obtain a coupled solution that approximates the global kinetic one. I.e. the reference solution to which the coupled solution is compared will be the solution of the kinetic equation in the whole domain with different mean free paths ϵ_1 in D_1 and $\epsilon_2 = \epsilon$ tending to 0 in D_2.

We describe now more precisely the coupled solution.
Let $\varphi^{(1)}$ be the solution of (3.1) in D_1 and

$$\begin{aligned} \varphi^{(1)}(-L,v,t) &= f_+(v,t), \; v_1 + u > 0 \\ \varphi^{(1)}(x,v,0) &= h(x,v). \end{aligned}$$

Let Θ be the solution of (3.2) in D_2 with

$$\Theta(x,0) = g(x)$$

and

$$\Theta(L,t) = f_-(t),$$

if $u < 0$.
f_+ is assumed to be uniformly bounded in v and t with $v \in S$ and $t \in [0,T], T$ fixed but arbitrary. f_-, h, g are also assumed to be uniformly bounded in x, v, t with $t \in [0,T]$ and $x \in D_1, v \in S$ and $x \in D_2$ respectively. We assume all data to be as smooth as required and the necessary compatibility conditions for initial and boundary conditions to be satisfied in order to avoid problems connected with nonsmoothness.

It remains to fix the coupling conditions. Taking the usual ones, i.e. equality of moments or fluxes as described in Section 5, will in general lead to wrong results. See Section 5 for numerical examples. One has to make a more exact analysis of the situation near the interface. We neglect the boundary layer at $x = L$ and proceed for the transition layer in a way similar to the usual boundary layer expansions, see e.g., Cercignani [6]: We assume the distribution function in the aerodynamic region to be equal up to order ϵ to the solution of the macroscopic equation plus a kinetic layer term concentrated in a region in D_2 around the interface. The size of this region is of order ϵ. This corresponds to a scaling of the space coordinate x in the layer like $\frac{x}{\epsilon}$. This means that we have to find a solution $\Phi(x,v,t)$ of the kinetic equation to order ϵ in the domain D_2 in the form

$$\begin{aligned}\Phi(x,v,t) &:= \Theta(x,t) + \chi(\frac{x}{\epsilon},v,t)\\ &+ \epsilon\hat{W}(\frac{x}{\epsilon},v,t) + \epsilon W(x,v,t).\end{aligned}$$

Φ must fulfill

$$\partial_t\Phi + (v_1+u)\partial_x\Phi + \frac{1}{\epsilon}(I-K)(\Phi) = 0(\epsilon).$$

The distribution function $\varphi^{(1)}$ in D_1 and the distribution function Φ in D_2 are moreover assumed to be continuous at the interface $x=0$, i.e. $\Phi(0,v,t) = \varphi^{(1)}(0,v,t)$ up to order ϵ:

$$\Phi(0,v,t) = \varphi^{(1)}(0,v,t) + 0(\epsilon).$$

Computing $\partial_t\Phi + (v_1+u)\partial_x\Phi$ one obtains

$$\begin{aligned}&\partial_t\left(\Theta + \chi(\frac{x}{\epsilon}) + \epsilon\hat{W}(\frac{x}{\epsilon}) + \epsilon W\right) + (v_1+u)\partial_x\left(\Theta + \epsilon W\right)\\ &+(v_1+u)\frac{1}{\epsilon}\partial_x\chi(\frac{x}{\epsilon}) + (v_1+u)\partial_x\hat{W}(\frac{x}{\epsilon}).\end{aligned}$$

To order ϵ this must be equal to

$$-\frac{1}{\epsilon}(I-K)\left(\Theta + \chi(\frac{x}{\epsilon}) + \epsilon\hat{W}(\frac{x}{\epsilon}) + \epsilon W\right).$$

Comparing the terms of order ϵ^{-1} yields the half space problem

$$(v_1+u)\partial_x\chi + (I-K)(\chi) = 0$$

with $\chi(\infty,v,t) = 0$, because the influence of the layer term must be concentrated near the interface. Assuming an ingoing flux of the form

$$\chi(0,v,t) = \varphi^{(1)}(0,v,t) - a(t), v_1+u > 0$$

with a(t) arbitrary, there is a unique solution χ of this problem for $u > 0$, see e.g., Greenberg et al. [12]. In particular a can not be prescribed, it is determined by the solution. For $u < 0$ the equation has a unique solution, if $a(t)$ is fixed in advance.
Terms of order 0 cancel if

$$\partial_t\Theta + (v_1+u)\partial_x\Theta + (I-K)(W) = 0$$

is satisfied and if a halfspace problem for $\hat{W}$ is fulfilled. The above equation is uniquely solvable, if Θ fulfills the macroscopic equation.
Considering the boundary values at $x=0$ we obtain

$$\Phi(0,v,t) = \Theta(0,t) + \chi(0,v,t) + 0(\epsilon).$$

Now for $u > 0$ it is easily seen that $\Theta(0,t)$, the boundary condition for (3.2), has to be chosen equal to $a(t)$ and $\varphi^{(1)}(0,v,t)$ equal to $\chi(0,v,t) + \Theta(0,t)$ for $v_1 + u < 0$ to achieve to order ϵ

$$\Phi(0,v,t) = \varphi^{(1)}(0,v,t).$$

For $u < 0$ there is no need of a boundary condition for (3.2) at $x = 0$. This corresponds to the solvability of the halfspace equation, if $a(t)$ is prescribed. $a(t)$ must be defined by $\Theta(0,t)$ and $\varphi^{(1)}(0,v,t), v_1 + u < 0$ has to be chosen as before. Then again the same result is obtained.

Due to this analysis the coupling conditions are found in the following way: Let χ_H be the solution of the kinetic linear half space problem

$$\begin{aligned} (v_1 + u)\partial_x \chi_H + (I - K)\chi_H = 0, \quad & x \in [0, \infty) \\ \chi_H(0,v,t) = \varphi^{(1)}(0,v,t), \quad & v_1 + u > 0. \end{aligned} \tag{3.3}$$

For $u > 0$ the solution χ_H is unique. Solving (3.3) one obtains the asymptotic value $\chi_H(\infty,t)$. We note, that $\chi_H(\infty,t)$ does not depend anymore on v, since the asymptotic value is always a collision invariant and since in this simple case the collision operator has only the constants as collision invariants. Moreover, we get $\chi_H(0,v,t), v_1 + u < 0$. These values will give us the coupling conditions: The condition at the interface for the aerodynamic equation is given by

$$\Theta(0,t) = \chi_H(\infty,t).$$

The condition for the kinetic equation in $[-L, 0]$ at $x = 0$ is

$$\varphi^{(1)}(0,v,t) = \chi_H(0,v,t), \ v_1 + u < 0.$$

For $u < 0$ one needs a constraint to obtain again a unique solution. It will be given by the solution of the aerodynamic equation $\Theta(x,t)$ with boundary condition $f_-(t)$ at $x = L$: $\chi_H(\infty,t) = \Theta(0,t)$. Solving (3.3) for $u < 0$ with this constraint gives $\chi_H(0,v,t), v_1 + u < 0$ and $\varphi^{(1)}(0,v,t), \ v_1 + u < 0$ is obtained as before.

$\varphi^{(1)}$ and Θ fulfilling the coupling and boundary conditions will be called the solution of the coupling problem. The reference solution in $[-L, L]$ to which the coupled solution is compared is the solution of (3.1) in D_1 and D_2 with ϵ_1 fixed and $\epsilon_2 = \epsilon$, the boundary conditions

$$\begin{aligned} \varphi_\epsilon^{(1)}(-L,v,t) &= f_+(v,t), \ v_1 + u > 0 \\ \varphi_\epsilon^{(2)}(L,v,t) &= f_-(t), \quad v_1 + u < 0 \end{aligned}$$

and the coupling conditions

$$\varphi_\epsilon^{(1)}(0,v,t) = \varphi_\epsilon^{(2)}(0,v,t).$$

The initial conditions are

$$\begin{aligned}\varphi_\epsilon^{(1)}(x,v,0) &= h(x,v) \quad \forall x \in D_1\\ \varphi_\epsilon^{(2)}(x,v,0) &= g(x) \qquad \forall x \in D_2.\end{aligned}$$

$\varphi^{(1)}$ is also indexed since it depends on ϵ by the coupling conditions.

The transition and boundary layer terms are modelled by functions, that are solutions of stationary kinetic linear half space problems. The value of these layer functions at infinity is zero, in order to restrict their influence to the interface and boundary regions. The layer functions are, at $x = 0$, the function χ_1

$$\chi_1(x,v,t) := \chi_H(x,v,t) - \chi_H(\infty,t)$$

and, at $x = L$, the function χ_2, where $\chi_2(x,v,t)$ is for $u > 0$ the solution of

$$\begin{aligned}(v_1+u)\partial_x\chi_2 + (I-K)\chi_2 = 0, &\qquad x \in (-\infty,0]\\ \chi_2(0,v,t) = f_-(t) - \Theta(L,t), &\qquad v_1+u<0\end{aligned} \tag{3.4}$$

with the condition

$$\chi_2(-\infty,t) = 0.$$

For $u < 0$ we define

$$\chi_2 = 0.$$

Using a perturbation expansion, like e.g., in Bensoussan et al. [5] or Bardos et al. [4], and the above mentioned results on the linear half space problem one can then prove the following theorem, see Klar [14].

Theorem 3.1

1. *For T fixed but arbitrary there is a unique coupled solution*

   $$(\varphi^{(1)},\Theta) \text{ with } \varphi^{(1)} \in \mathcal{L}^\infty(D_1\times S\times[0,T]) \text{ and } \Theta \in \mathcal{L}^\infty(D_2\times[0,T])$$

   fulfilling the above conditions and a unique reference solution

   $$(\varphi_\epsilon^{(1)},\varphi_\epsilon^{(2)}) \text{ with } \varphi_\epsilon^{(i)} \in \mathcal{L}^\infty(D_i\times S\times[0,T]), i=1,2 \quad \forall\epsilon>0.$$

2. *There exists a constant $C > 0$ such that in D_1 the following is true:*

   $$\|\varphi_\epsilon^{(1)} - \varphi^{(1)}\|_\infty \le \epsilon C.$$

3. In D_2 we get

$$\|\varphi_\epsilon^{(2)} \quad - \quad [\Theta(x,t) + \chi_1(\frac{x}{\epsilon}, v, t) + \chi_2(\frac{x-L}{\epsilon}, v, t)]\|_\infty \leq \epsilon C,$$

where Θ, χ_1 and χ_2 do not depend on ϵ.

Remark: We mention that in general - in contrast to the global kinetic reference solution - there will be a jump in the macroscopic quantities of the coupled solution at the interface. The coupled solution is a correct approximation for small ϵ of the global kinetic solution only outside of boundary and interface layers in D_2.

4 Coupling Conditions for Linearized Boltzmann and Euler Equations

We return in this section to the equations considered in section 2 and describe here the extension of the coupling conditions developed in the preceding section to the full Boltzmann and Euler equations linearized around a constant state $\bar{\rho}, \bar{u}, \bar{T}$ with $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$. We restrict ourselves again to a 1-dimensional geometry $x \in [-L, L]$ and refer to the remark at the end of this section for possible extensions to the multidimensional case.
The kinetic equations in $D_i, i = 1, 2$, with $D_1 = [-L, 0]$ and $D_2 = [0, L]$, are the Boltzmann equations from section 2

$$\partial_t \varphi^{(i)} + (v_1 + \bar{u}_1) \cdot \partial_x \varphi^{(i)} + \frac{1}{\epsilon_i} 2Q(\bar{M}, \varphi^{(i)}) = 0. \tag{4.1}$$

The aerodynamic equation in D_2 is the compressible linearized Euler equation

$$\partial_t \Theta + A \partial_x \Theta = 0, \tag{4.2}$$

with $x \in [0, L], \Theta = (\rho, u_1, u_2, u_3, T)$. Let $c > 0$ defined by $c^2 = \frac{5}{3}\bar{T}$ be the speed of sound. A is given by the following matrix

$$A = \begin{bmatrix} \bar{u}_1 & \bar{\rho} & 0 & 0 & 0 \\ \frac{\bar{T}}{\bar{\rho}} & \bar{u}_1 & 0 & 0 & 1 \\ 0 & 0 & \bar{u}_1 & 0 & 0 \\ 0 & 0 & 0 & \bar{u}_1 & 0 \\ 0 & \frac{2}{5}c^2 & 0 & 0 & \bar{u}_1 \end{bmatrix}.$$

Now the coupling of (4.1) in $D_1 = [-L, 0]$ and (4.2) in $D_2 = [0, L]$ is considered and compared to the solution of the Boltzmann equation in the whole domain $[-L, L]$. We concentrate in the following on the coupling conditions at $x = 0$ disregarding the boundary conditions at $x = L$ and $x = -L$.
The system (4.2) is diagonalizable with Eigenvalues $\lambda_i = \bar{u}_1, i = 1, 2, 3, \lambda_4 = \bar{u}_1 + c, \lambda_5 = \bar{u}_1 - c$. One needs at $x = 0$ for (4.2) $0, 1, 4$ or 5 boundary conditions for the characteristic variables according to the value of $\bar{u}_1$, i.e. $\bar{u}_1 < -c, -c < \bar{u}_1 < 0, 0 < \bar{u}_1 < c$, $\bar{u}_1 > c$

respectively. Here and in the following we assume $\bar{u}_1 \neq 0, c, -c$.
Simply using the equality of moments or fluxes at $x = 0$ will - as in Section 3 - not lead to the correct results, as we shall see from the simulations in Section 5. An analysis like in section 3 leads to the consideration of

$$\begin{aligned} (v_1 + \bar{u}_1)\partial_x \chi_H + 2Q(\bar{M}, \chi_H) &= 0, \quad x \in [0, \infty) \\ \chi_H(0, v, t) &= \varphi^{(1)}(0, v, t), \quad v_1 + \bar{u}_1 > 0 \end{aligned} \tag{4.3}$$

where $\varphi^{(1)}(0, v, t)$ is the distribution function in D_1 at $x = 0$. This equation has a unique solution, see Coron et al [8] or Greenberg et al. [12], if - according to the values of $\bar{u}_1$ - a number of constraints is imposed. One needs $5, 4, 1$ or 0 constraints if $\bar{u}_1 < -c$, $-c < \bar{u}_1 < 0, 0 < \bar{u}_1 < c$, $\bar{u}_1 > c$ respectively with $c^2 = \frac{5}{3}\bar{T}$ as before. Solving the half space problem gives the asymptotic value

$$\chi_H(\infty, v, t) = \left(\frac{a_0(t)}{\bar{\rho}} + \sum_{i=1}^{3} \frac{a_i(t)}{\sqrt{\bar{T}}} \frac{v_i}{\sqrt{\bar{T}}} + \frac{a_4(t)}{\bar{T}} \frac{|v|^2 - 3\bar{T}}{2\bar{T}} \right) \bar{M}.$$

According to the number of constraints one already has $5, 4, 1$ or 0 equations for $a_0, \ldots, a_4$. This means that for $\bar{u}_1 < -c, -c < \bar{u}_1 < 0,\ 0 < \bar{u}_1 < c$ or $\bar{u}_1 > c$ we obtain $0, 1, 4$ and 5 new conditions on the asymptotic values $a_0, \ldots, a_4$ respectively. This fits exactly to what is needed for the Euler equations as already mentioned by Golse [9].

We restrict ourselves from now on to supersonic flows; $|\bar{u}_1| > c$.
For $\bar{u}_1 > c$ the half space problem is solved for prescribed incoming fluxes without any constraints. This gives the asymptotic values $a_0(t), a_i(t), a_4(t),\ i = 1, 2, 3$.
The 'macroscopic density function' in D_2 is a linearized Maxwellian with parameters given by the solution of the Euler equations:

$$\varphi_{macro}(x, v, t) := \bar{M}^{lin}_{(\rho(x,t), u(x,t), T(x,t))}(v)$$

with

$$\bar{M}^{lin}_{(\rho,u,T)}(v) := \left(\frac{\rho}{\bar{\rho}} + \sum_{i=1}^{3} \frac{u_i}{\sqrt{\bar{T}}} \frac{v_i}{\sqrt{\bar{T}}} + \frac{T}{\bar{T}} \frac{|v|^2 - 3\bar{T}}{2\bar{T}} \right) \bar{M}.$$

At $x = 0$ this is

$$\bar{M}^{lin}_{(\rho(0,t), u(0,t), T(0,t))}(v).$$

Comparing it to $\chi_H(\infty, v, t)$ one obtains

$$\begin{aligned} \rho(0, t) &= a_0(t) \\ u_i(0, t) &= a_i(t), \quad i = 1, 2, 3 \\ T(0, t) &= a_4(t) \end{aligned} \tag{4.4}$$

Thus the solution of the half space problem gives us the boundary conditions required for the Euler equations with $\bar{u}_1 > c$ at $x = 0$, i.e. $\rho(0, t), u(0, t), T(0, t)$.

Moreover the outgoing flux $\chi_H(0,v,t), v_1 + \bar{u}_1 < 0$ gives $\varphi^{(1)}(0,v,t), v_1 + \bar{u}_1 < 0$, i.e. the boundary condition at $x = 0$ for the domain D_1.

Remark: If $\varphi^{(1)}(0,v,t), v_1 + \bar{u}_1 > 0$ is a linearized Maxwellian

$$\varphi^{(1)}(0,v,t) = \bar{M}^{lin}_{(\rho^{(1)}(t),u^{(1)}(t),T^{(1)}(t))}(v), \quad v_1 + \bar{u}_1 > 0,$$

we get

$$\chi_H(\infty,v,t) = \varphi^{(1)}(0,v,t) = \bar{M}^{lin}_{(\rho^{(1)}(t),u^{(1)}(t),T^{(1)}(t))}(v), \forall v \in \mathbb{R}^3.$$

This yields equality of the macroscopic quantities $\rho(0,t) = \rho^{(1)}(t), u(0,t) = u^{(1)}(t), T(0,t) = T^{(1)}(t)$. For general functions $\varphi^{(1)}(0,v,t), v_1 + \bar{u}_1 > 0$ this is usually not the case. The moments of $\varphi^{(1)}(0,v,t)$ do not coincide with $\rho(0,t), u(0,t), T(0,t)$. One obtains a jump in the macroscopic quantities.

For $\bar{u}_1 < -c$ the situation is the following: To solve the half space problem 5 constraints on the solution are necessary. Comparing the macroscopic density function, with parameters given by the solutions of the Euler equations, with the asymptotic value $\chi_H(\infty,v,t)$, we get the necessary number of constraints. Remark that for $\bar{u}_1 < -c$ we do not need any boundary condition at $x = 0$ for the Euler equation in D_2. We can then solve the half space problem, which yields $\varphi^{(1)}(0,v,t) = \chi_H(0,v,t)$, $v_1 + \bar{u}_1 < 0$.

Remark: In the multidimensional case one can proceed in the same way. Suppose that the interface Σ divides the computational domain Ω into subdomains Ω_1 and Ω_2. At each point $x \in \Sigma$ one has to solve a one dimensional half space problem with coordinate axis along the unit normal $n(x)$ to Σ at the point x. This will lead for each $x \in \Sigma$ to the correct boundary conditions.

5 Numerical Results

In this section we investigate the coupling procedure proposed in Sections 3 and 4 numerically. The coupling conditions described there are compared with the ones obtained by equalizing moments or fluxes. This means that they are determined by the following procedures:

For the model equations in Section 3 and $u > 0$ the following equations are used to determine $\Theta(0,t)$ from $\varphi^{(1)}(0,v,t)$

$$\int_S \varphi^{(1)}(0,v,t)dv = \int_S \Theta(0,t)dv$$

and

$$\int_{v_1+u>0} (v_1 + u)\varphi^{(1)}(0,v,t)dv = \int_{v_1+u>0} (v_1 + u)\Theta(0,t)dv$$

respectively. The ingoing function for the Boltzmann region $\varphi^{(1)}(0,v,t), v_1 + u < 0$ is determined from $\Theta(0,t)$ by

$$\varphi^{(1)}(0,v,t) = \Theta(0,t),\ v_1 + u < 0.$$

$\Theta(0,t)$ must not be prescribed for $u < 0$. $\varphi^{(1)}(0,v,t), v_1 + u < 0$ is given by a function $\tilde{\Theta}(t)$ independent of v s.t. the equality of moments is fulfilled. The equality of fluxes does in this case not give any conditions on the ingoing function. We simply take $\varphi^{(1)}(0,v,t) = \Theta(0,t), v_1 + u < 0$.

For the Euler equation in Section 4 the boundary values $\rho(0,t), u(0,t)$ and $T(0,t)$ are found for $\bar{u}_1 > c$ by

$$\int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \varphi^{(1)}(0,v,t)dv = \int \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \bar{M}^{lin}_{(\rho(0,t)u(0,t),T(0,t))}(v)dv$$

and

$$\begin{aligned} &\int\limits_{v_1+\bar{u}_1>0} (v_1+\bar{u}_1) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \varphi^{(1)}(0,v,t)dv \\ = &\int\limits_{v_1+\bar{u}_1>0} (v_1+\bar{u}_1) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \bar{M}^{lin}_{(\rho(0,t)u(0,t),T)(0,t))}(v)dv, \end{aligned}$$

respectively.
The ingoing function for the Boltzmann region is for $v_1 + \bar{u}_1 < 0$

$$\varphi^{(1)}(0,v,t) = \bar{M}^{lin}_{(\rho(0,t),u(0,t),T(0,t))}(v).$$

For the other values of $\bar{u}_1$ one can proceed in an analogous way.

Concerning the coupling conditions obtained by the analysis of the kinetic half space problem, it should be remarked that a fast approximate solver, yielding asymptotic states and outgoing distributions, is essential. In particular since for multidimensional problems the halfspace problem has to be solved at each point of the interface. We determine the coupling conditions here by the first step of the numerical scheme mentioned in the introduction and described in detail in Golse/Klar [11], see also Klar [14]. We remark that for $\bar{u}_1 = 0$ the first step of the scheme reduces to the so called variational method, see Golse [10] or Loyalka [15].

In the following figures the mean free paths differ strongly in the two domains. In this case the considered types of coupling conditions give strikingly different results. Concerning the physical aspect of this assumption see the discussion in section 2.
In Figure 1 we consider the model problem (3.1) in section 3 with $v \in S = [-1,1]$ and $K\varphi = \frac{1}{2}\int_{-1}^{1} \varphi(v)dv$. The macroscopic quantity $\int \varphi^{(1)}(x,v,t)dv$ in D_1 and $\Theta(x,t)$ the solution of (3.2) in D_2 is shown at a fixed time $t = T$ so large that a stationary state is obtained. We took $\epsilon_1 = 1$ and $u = 0.3 > 0$. For $t \in [0,T]$ the ingoing function at $x = -L$ was chosen as $f_+(v,t) = v$, at $x = L$ we took $f_-(t) = 1$. The figure shows the 3

kinds of coupling conditions. Moreover the kinetic solution in the whole domain with the parameters $\epsilon_1 = 1$ and $\epsilon_2 = 0.01$ is shown.

8cm

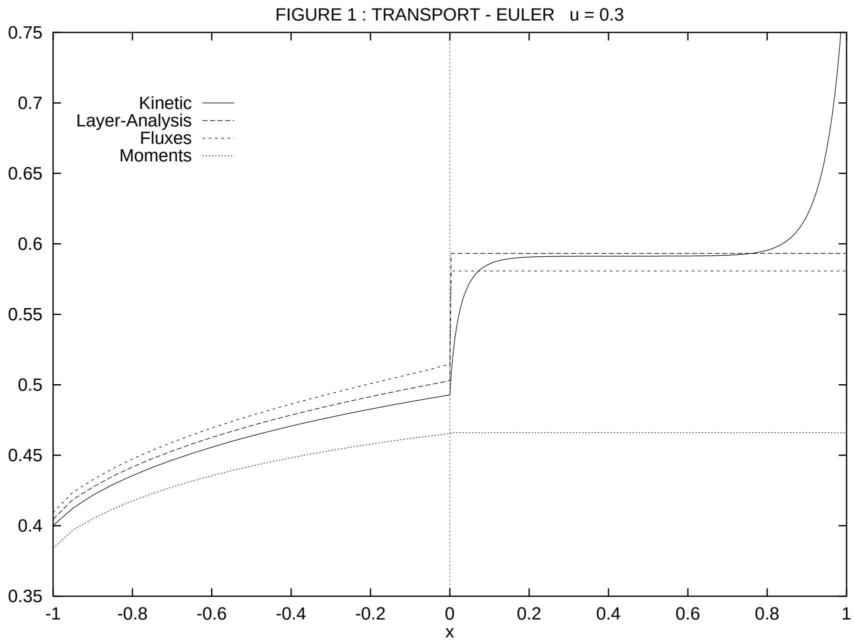
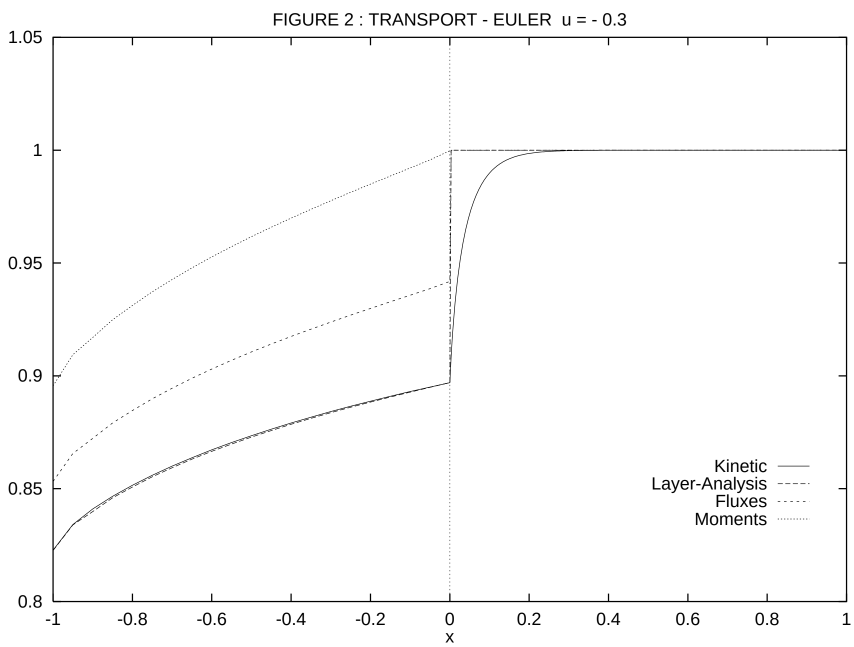


Figure 2 shows the same but in contrast to Figure 1 u is here less than $0, u = -0.3$.

8cm



In Figure 3 the 3-dimensional BGK model is considered, i.e. in D_1 we consider equation (4.1) with $\epsilon_1 = 1$ and substitute the following expression for the collision operator $2Q(\bar{M}, \varphi)$:

$$\varphi - \left(\int \varphi dv + \sum_{i=1}^{3} v_i \int v_i \varphi dv + \frac{|v|^2 - 3}{2} \int \frac{|v|^2 - 3}{3} \varphi dv \right) \bar{M}$$

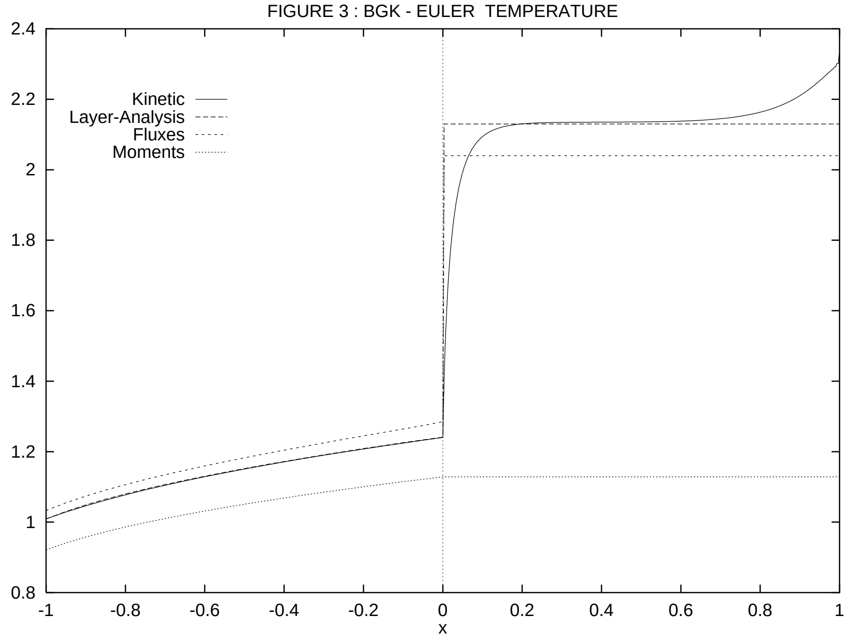
with

$$\bar{M} = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{|v|^2}{2} \right).$$

We linearized here around $\bar{\rho} = \bar{T} = 1$.
In D_2 the Euler equations (4.2) are solved. $\bar{u}_1$ is equal to 1.4, i.e. bigger than $c = \sqrt{\frac{5}{3}}$.
The ingoing function at $x = -L$ is $f_+(v,t) = v_1(|v|^2 - 5)\bar{M}$ for $t \in [0, T]$.
The figure shows the temperature $\int \frac{|v|^2-3}{3} \varphi^{(1)}(x,v,t) dv$ on D_1 and $T(x,t)$ on D_2.
Again the 3 types of coupling conditions are shown together with the kinetic solution in the whole domain with $\epsilon_1 = 1$ and $\epsilon_2 = 0.002$.

8cm

FIGURE 3 : BGK - EULER TEMPERATURE



Remark:
As can be seen in the figures the usual coupling conditions may in certain cases lead to completely wrong results. The kinetic layer analysis combined with only the first step of the above mentioned numerical scheme for half space problems however leads to a considerable improvement. The coupled solution is a good approximation of the true kinetic solution outside of boundary and interface layers.

Acknowledgements
I am grateful to F. Golse for very helpful discussions. I also wish to thank H. Neunzert, who initiated this work and the unknown referee for some useful comments.

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