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NONSTANDARD HYDRODYNAMICS FOR THE BOLTZMANN EQUATION

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Abstract

The Boltzmann equation solutions are considered for the small Knudsen number.The main attention is devoted to certain deviations from the classical Navier-Stokes description. The equations for the quasistationary slow flows are derived. These equations do not contain the Knudsen number and provide in this sence a limiting description of hydrodynamical variables. Two well-known special cases are also indicated. In the isothermal case the equations are equivalent to the incompressible Navier-Stokes equations, in stationary case they coincide with the equations of slow non-isothermal flows. It is shown that the derived equations possess all principal properties of the Boltzmann equation in contrast to the Burnett equations. In one dimension the equations reduce to the nonlinear diffusion equation, being exactly solvable for Maxwell molecules. Multidimensional stationary heat-transfer problems are also discussed. It is shown that one can expect an essential difference between the Boltzmann equation solution in the limit of the continuous media and the corresponding solution of the Navier-Stokes equations.

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1 Introduction

It is well-known that for small Knudsen numbers $\mathbf{Kn} \ll 1$ the Boltzmann equation solutions can be approximated by the locally Maxwell disribution with parameters $\rho(x,t)$ (density), $u(x,t)$ (velocity) and $T(x,t)$ (temperature) satisfying hydrodynamical equations. Using the standard Chapman-Enskog method [1,2] we obtain the Euler equations for $\mathbf{Kn} = 0$, then consequently the compressible Navier-Stokes equations (first order in respect to $\mathbf{Kn} \ll 1$), Burnett equations (second order) and so on.Even for the simplest initial value problem in infinite or periodic domain the direct using of the Burnett (and also the next super-Burnett) equations is impossible because of non-physical instability of the global equilibrium state ($\rho = const, u = 0, T = const$) for these equations [3]. Therefore the Euler and the compressible Navier-Stokes equations remain to be basic equations for a description of the Boltzmann equation asymptotics with $\mathbf{Kn} \to 0$. Just for this reason we can term these equations the standard hydrodynamical equations for the Boltzmann equation.The present paper is devoted to the consideration of some deviations (real or imaginary) from these usual equations.

We mention the two well-known such deviations. In the both cases the hydrodynamical values of the main order with $\mathbf{Kn} \to 0$ satisfy not the standard hydrodynamical equations but (1) the incompressible Navier-Stokes equations in the first case [4,5] or (2) the so-called SNIF-equations (SNIF means Slow Non-Isotermal Flows, for a review see [6]). In these cases the typical gas velocity u and the Mach number $\mathbf{M}$ are small ($u \sim \mathbf{M} \sim Kn$), besides the Reynolds number $\mathbf{Re} \to const$ with $\mathbf{Kn} \to 0$. The typical time has in the case (1) the order $\mathbf{Kn}^{-1}$. As to the case (2) the authors of this approach considered mainly the stationary SNIF-equations [6].

In the first case the main attention after the publication of the basic papers [4,5] was devoted to attempts to prove rigorously the corresponding limit transition [7,8]. In the recent paper [9] the compressible Navier-Stokes equations was derived directly from the Hamiltonian dynamics.However, to the best of the author's knowledge, the connection between this approach and SNIF-equations was not discussed.

The SNIF-equations were derived in the beginning of 70th and discussed in detail in the papers of M.N.Kogan, V.S.Galkin and O.G.Freedlender (one can find references in the review [6]). The main attention in these papers was devoted to the so-called thermal stress convection, that seems to be the most important physical effect connected with this theory. Therefore some other interesting aspects of this approach were not discussed in detail, in particular non-stationary problems. The majority of publications on SNIF-theory are devoted to stationary problems. Even in papers [10,11] where the stability of equilibrium solutions is analysed for SNF-equations the authors considered the dispersion relation only and did not define explicitly the complete non-stationary equations.Thus the problem of construction of the correct non-stationary SNIF-equations remains unclear.

One of the goals of the present paper is just an accurate derivation and investigation of such non-stationary (quasistationary as we shall see below) equations. These equations for the stationary case coincide with SNIF-equations. They also admit the particular class of isotermical ($T = const$) solutions that correspond to incompressible Navier-Stokes equations. We unify in such a way the two above mentioned cases (1) and (2) in the more general class of equations that we can term the quasistationary slow flows (QSF) equations. The equations describe the time evolution of the limiting (with $\mathbf{Kn} \to 0$) values and do not contain the Knudsen number.We do not use the term "non-isothermal" for these equations since their isothermal special case (incompressible Navier-Stokes equations) is also very nontrivial. Nevertheless it should be stressed that the QSF-equations are the natural generalization of the stationary SNIF- theory.

In the Sec.2 we formulate the problem connecting with quasistationary solutions of the Boltzmann equation for small Knudsen numbers. Then in the Sec.3 the QSF equations are derived. The derivation is based on the re-expension of the Chapman-Enskog series but it is easy to verify that the result does not change if we use the direct (Hilbert-type) expansion of the Boltzmann equation. The principal properties (conservation laws and H-theorem) of the QSF-equations are proved in the Sec.4, so that these equations are quite correct in contrary to the Burnett equations. We also consider some interesting special cases in the Sec.4 and describe in detail in the Sec.5 a class of solutions depending on the single space variable. Roughly speaking we manage to reduce the equations to a single quasilinear diffusion equation in this simple case. We show also that this diffusion equation can be linearized for Maxwell molecules.In the Sec.6-7 we consider multidimensional problems and discuss in detail the non-Navier-Stokes limit at $\mathbf{Kn} = 0$ of the stationary temperature and density fields for a gas being confined between two non-symmetrical surfaces having different temperatures. It is shown that the SNIF-theory and the Navier-Stokes equations give essentially different results for the limiting case $\mathbf{Kn} = 0$.

2 Quasistationary solutions of the Boltzmann equation

We consider the Boltzmann equation for a distribution function $f(x, v, t)(x \in R^3, v \in R^3, t \in R_+$ denote respectively space coordinate,velocity and time)

$$f_t + v \cdot f_x = \varepsilon^{-1} I(f, f), f_{|t=0} = f_0, \tag{1}$$

where $\cdot$ means a scalar product, $I(f, f)$ denotes the collision integral, ε denotes the Knudsen number, that is a small parameter of this problem. For a simplicity we consider in this Section the initial value problem in infinite space R^3 with an equilibrium (absolute Maxwell) distribution in infinity.

The equation (1) is written in dimensionless variables, so that all (except ε) typical parameters of the problem (length, thermal velocity, ets.) are of order of unity. Roughly speaking we can distinguish three typical time scales: (1) $t_1 \sim \varepsilon$ is the free path time; (2)$t_2 \sim 1$ is the typical macroscopic time(the period of sonic waves); (3)$t_3 \sim \varepsilon^{-1}$ is the typical time of dissipative processes (viscosity,heat transfer). Therefore we can write down formally the general solution of (1) as a function of the three time-variables

$$f(x, v, t) = f_1(x, v; t/\varepsilon, t, \varepsilon t|\varepsilon), \tag{2}$$

that is the standard trick of perturbation theory, the dependence on ε being supposed to be formally-analytical in the neighborhood of the point $\varepsilon = 0$.

When we discuss the so-called normal solutions of the Boltzmann equation [2] we in fact consider the partial class of the functions (2) depending on the two time variables only

$$f(x, v, t) = f_2(x, v; t, \varepsilon t|\varepsilon) \tag{3}$$

It is important to consider separately two arguments t and εt inspite of analitic (linear) dependence on ε of the second time variable. Putting

$$f(x, v, t) = f_2'(x, v; t|\varepsilon)$$

we obtain the standard Hilbert expension [2] that includes some terms increasing with time as εt already in the first order approximation in respect to ε. We can consider the standard Chapman-Enskog method as one of possible ways to take into acccount correct dependence on "slow time" εt.

Finally we can define also the sub-class of the normal solutions (3) that includes the dependence on "slow time" εt only, i.e.

$$f(x,v,t) = f_3(x,v;\varepsilon t|\varepsilon). \tag{4}$$

Such solutions will be called quasistationary. Omitting subscript 3 of the function f_3 and changing the time variable $t \to \varepsilon t$, we obtain the quasistationary form of the Boltzmann equation

$$\varepsilon f_t + v \cdot f_x = \varepsilon^{-1} I(f,f). \tag{5}$$

Thus, it is clear that the quasistationary solutions are the special case of the normal solutions of the Hilbert class [2]. Therefore for constructing the solutions of (5) we do not need to do complex direct calculations with the Boltzmann equation since it is possible to use the well-known results of the Chapman-Enskog expansion.

We pass now to the derivation of the basic equations.

3 The derivation of basic equations

We introduce the stansdard notations

$$\rho = \int dv f(v), u = \frac{1}{\rho}\int dv f(v) v, p = \frac{1}{3}\int dv f(v)(v-u)^2 = \rho T \tag{6}$$

for a density ρ, mean velocity $u \in R^3$ and a pressure $p = \rho T, T$ being a gas temperature. It follows from the Boltzmann equation (5) that the hydrodynamical variables $\rho(x,t), u(x,t), p(x,t)$ satisfy the following exact (but unclosed) system of equations:

$$\varepsilon\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i = 0, \varepsilon\frac{\partial}{\partial t}\rho u_k + \frac{\partial}{\partial x_i}(\rho u_i u_k + p\delta_{ik} + \sigma_{ik}) = 0,$$

$$\varepsilon\frac{\partial}{\partial t}(\rho u^2 + 3p) + \frac{\partial}{\partial x_i}[u_i(\rho u^2 + 5p) + 2(u_k\sigma_{ik} + q_i)] = 0 \tag{7}$$

where the stansdard summation rule ($i, k = 1, 2, 3$) and the following notations are used:

$$\sigma_{ik}(x,t) = \int dv f(x,v,t)(c_i c_k - \frac{1}{3}|c|^2 \delta_{ik}), c = v - u(x,t), q = \frac{1}{2}\int dv f(x,v,t) c|c|^2 \tag{8}$$

The system is written in the form of conservation laws, we can also transform it to the more usual form

$$\varepsilon\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\cdot \rho u = 0, \rho(\varepsilon\frac{\partial}{\partial t} + u\cdot\frac{\partial}{\partial x})u_k + \frac{\partial p}{\partial x_k} + \frac{\partial \sigma_{ik}}{\partial x_i} = 0 \tag{9}$$

$$\varepsilon\frac{\partial p}{\partial t} + u\cdot\frac{\partial p}{\partial x} + \frac{5}{3}p\frac{\partial}{\partial x}\cdot u + \frac{2}{3}(\sigma_{ik}\frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x}\cdot q) = 0$$

We shall use the well-known Chapman-Enscog expansion:

$$\sigma_{ik} = \sum_{n=1}^{\infty} \varepsilon^n \sigma_{ik}^{(n)}, q = \sum_{n=1}^{\infty}\varepsilon^n q^{(n)}, \tag{10}$$

where

$$\sigma_{ik}^{(1)} = -\mu(T)(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial}{\partial x}\cdot u), q^{(1)} = -\lambda(T)\frac{\partial T}{\partial x} \tag{11}$$

with the known coefficients of viscosity $\mu(T)$ and heat transfer $\lambda(T)$. The next terms in the sums (10) correspond to higher approximations (Burnett, super-Burnett,ets.). Our goal is to construct the leading asymptotic terms with $\varepsilon \to 0$. Putting in (9) $\varepsilon = 0$ we notice that $\sigma_{ik} = 0$ and $q = 0$ with $\varepsilon = 0$, therefore we obtain for the limiting hydrodynamical values $\rho^{(0)}, u^{(0)}$ and $p^{(0)}$ the stationary Euler equations

$$\frac{\partial}{\partial x}\cdot \rho^{(0)}u^{(0)} = 0, \frac{\partial}{\partial x_i}(\rho^{(0)}u_i^{(0)}u_k^{(0)} + p^{(0)}\delta_{ik}) = 0, u^{(0)}\cdot\frac{\partial}{\partial x}\ln p^{(0)}[\rho^{(0)}]^{-5/3} = 0. \tag{12}$$

Thus we have to choose a certain stationary solution of the Euler equations as the leading asymptotic term. We do not consider discontinuous solutions (shock waves) since this approach is not applied to the description of such solutions. Let us assume that there is no shocks in the domain under consideration and that the limiting solution $(\rho^{(0)}, u^{(0)}, p^{(0)})$ is sufficiently smooth.Then the two essentially different cases are possible: (1)$u^{(0)} \neq 0$ and

(2)$u^{(0)} = 0$. The trivial solution $\rho^{(0)} = const, u^{(0)} = const, p^{(0)} = const$ is included automatically in the case (2) by the transition to the uniformly moving reference system (the Boltzmann equation is invariant under this transformation). We suppose that there exists the Hilbert-type asymptotic expantion

$$\rho = \rho^{(0)} + \rho^{(1)} + ..., u = u^{(0)} + u^{(1)} + ..., p = p^{(0)} + p^{(1)} + ...,$$

where $\rho^{(n)}, u^{(n)}, p^{(n)}, n = 0, 1, ...$ do not depend on ε. In the cases (1) and (2) the limit ($\varepsilon = 0$) Reynolds numbers are respectivaly $\mathbf{Re}_0 = \infty$ and $\mathbf{Re}_0 = const$. We restrict ourselves below to the case (2), that is $u^{(0)} = 0$. Then the system (12) is reduced to the only equation $\mathbf{grad}p^{(0)} = 0$, i.e. its general solution for this case is

$$\rho^{(0)} = \rho^{(0)}(x, t), u^{(0)} = 0, p^{(0)} = p^{(0)}(t) \tag{13}$$

with arbitrary functions $\rho^{(0)}(x, t)$ and $p^{(0)}(t)$.We consider now the equations (7) in the first order on ε

$$\frac{\partial \rho^{(0)}}{\partial t} + \frac{\partial}{\partial x_i}\rho^{(0)} u_i^{(1)} = 0, \frac{\partial p^{(1)}}{\partial x_k} = 0, \frac{\partial p^{(0)}(t)}{\partial t} + \frac{5}{3}p^0(t)\mathbf{div}u^{(1)} = \frac{2}{3}\frac{\partial}{\partial x_i}\lambda(T^{(0)})\frac{\partial T^{(0)}}{\partial x_i}, \tag{14}$$

$T^{(0)} = p^{(0)}/\rho^{(0)}$. To close this system it is necessary to add to it a single (vector) equation of the second order on ε, that defines time evolution of the mean velocity

$$\rho^{(0)}[\frac{\partial}{\partial t} + u^{(1)} \cdot \frac{\partial}{\partial x}]u_k^{(1)} + \frac{\partial p^{(2)}}{\partial x_k} = \frac{\partial}{\partial x_i}[2\mu(T^{(0)}) < \frac{\partial u_i^{(1)}}{\partial x_k} > -\sigma_{ik}^{(2)}(T^{(0)})], \tag{15}$$

where

$$< A_{ik} >= (A_{ik} + A_{ki})/2 - \delta_{ik}(A_{jj}/3), \sigma_{ik}^{(2)}(T^{(0)}) = [\sigma_{ik}^{(2)}]_{|u=0, p=p^{(0)}, T=T^{(0)}}.$$

In other words we can omit in the Burnett expression for $\sigma_{ik}^{(2)}$ [1] certain terms which are proportional to spatial gradients of the mean velocity and the pressure since

$$\frac{\partial u_i}{\partial x_k} = O(\varepsilon), \frac{\partial p}{\partial x_k} = O(\varepsilon^2), \frac{\partial T}{\partial x_k} = O(1).$$

Then we obtain the same formula as in [6]

$$\sigma_{ik}^{(2)} = \frac{\mu^2(T)}{\rho T} < K_2 \frac{\partial^2 T}{\partial x_i \partial x_k} + K_3 \frac{1}{T} \frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_k} > \tag{16}$$

with constant coefficients K_2 and K_3 ($K_2 = K_3 = 3$ for Maxwell molecules and $K_2 = 2.418, K_3 = 0.219$ for hard spheres [6]). The second order (on ε) equations for $\rho^{(1)}(x,t)$ and $p^{(1)}(t)$ do not affect on leading asymptotic terms, therefore we do not consider these equations. The leading asymptotic terms $\rho^{(0)}(x,t), p^{(0)}(t), u^{(1)}(x,t)$ and $p^{(2)}(x,t)$ are defined by the equations (14),(15). Finally we can formulate the following

Proposition. The asymptotic with $\varepsilon \to 0$ expansion of the solution of equations (7), satisfying the additional condition $u = O(\varepsilon)$, has the following form:

$$\rho = \tilde{\rho}(x,t) + ..., u = \varepsilon \tilde{u}(x,t) + ..., p = p_0(t)[1 + \varepsilon \pi(t) + \varepsilon^2 \tilde{p}(x,t) + ...],$$

where points denote higher order terms. The leading asymptotic terms can be obtained from the following equations for the functions $\tilde{\rho}, \tilde{u}, \tilde{p}$ and p_0 (the sign $\sim$ is omitted below):

$$\frac{\partial \rho}{\partial t} + \mathbf{div} \rho u = 0, \frac{\partial p_0(t)}{\partial t} + \frac{5}{3} p_0 \mathbf{div} u = \frac{2}{3} \mathbf{div} \lambda(T) \mathbf{grad} T, T = \frac{p_0}{\rho} \tag{17}$$

$$\frac{1}{T}(\frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x}) u_k + \frac{\partial p}{\partial x_k} = \frac{\partial}{\partial x_i} < 2\kappa \frac{\partial u_i}{\partial x_k} - \kappa^2 (K_2 \frac{\partial^2 T}{\partial x_i \partial x_k} + \frac{K_3}{T} \frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_k}) >, \kappa = \frac{\mu}{p_0(t)}.$$

Remark. Formally speaking the main correction (of order ε) to $p_0(t)$ is defined by the function $\pi(t)$ but it does not change $\mathbf{grad} p$. Therefore the function$\tilde{p}(x,t)$ is more important, as to $\pi(t)$ it can be defined from the equations of the next approximation.

It is naturally to call the equations (17) the equations of quasistationary slow flows (QSF), they are reduced to the well-known SNIF equations [6] in the stationary case (see also Introduction). We can consider for these equations different initial boundary value problems. These equations include formally the third order derivatives but it is possible to exclude them by the equation

$$2\lambda(T) \Delta T = -2\lambda'(T)(\mathbf{grad} T)^2 + 5 p_0 \mathbf{div} u + 3 \frac{\partial p_0}{\partial t}, \Delta T = \mathbf{divgrad} T.$$

Therefore in the case of purely diffusion reflection we obtain for $\varepsilon \to 0$ the following boundary conditions [6]

$$T_{|\Gamma} = T_w, u_{n|\Gamma} = 0, u_{\tau|\Gamma} = \beta \frac{\partial T_w}{\partial x_\tau}, \tag{18}$$

where $T_{|\Gamma}$ and $u_{|\Gamma}$ denote boundary (on the surface Γ) values of T and u, T_w is the wall (surface Γ) temperature, u_n and u_τ are respectively normal and tangential velocity components.The last relation expresses the known condition of the temperature slip, we obtain the standard condition $u_{|\Gamma} = 0$ for an isotermic wall. The boundary conditions (18) define completely the statement of boundary value problems for the QSF equations (17).

4 Principal properties and special cases of QSF equations

We can exclude the density $\rho(x,t) = p_0/T(x,t)$ from the equations (17) and write them in the form of conservation laws (mass, momentum and energy)

$$\frac{\partial}{\partial t}\frac{p_0(t)}{T} + \frac{\partial}{\partial x_i}\frac{p_0(t)}{T}u_i = 0, \tag{19}$$

$$\frac{\partial}{\partial t}\frac{p_0(t)}{T}u_k + \frac{\partial}{\partial x_i}[\frac{p_0(t)}{T}u_i u_k + p_0(t)p\delta_{ik}] =$$

$$\frac{\partial}{\partial x_i} < 2\mu\frac{\partial u_i}{\partial x_k} - \frac{\mu^2}{p_0(t)}(K_2\frac{\partial^2 T}{\partial x_i \partial x_k} + \frac{K_3}{T}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_k}) >, \tag{20}$$

$$\frac{\partial}{\partial t}\frac{3}{2}p_0(t) + \frac{\partial}{\partial x_i}[\frac{5}{2}p_0(t)u_i - \lambda(T)\frac{\partial T}{\partial x_i}] = 0. \tag{21}$$

We show also that the Boltzmann H-theorem is valid for this system.Putting

$$s = \ln\frac{\rho^{5/3}}{p_0(t)} = \ln p_0^{2/3}T^{-5/3} \tag{22}$$

we obtain

$$\frac{\partial s}{\partial t} + u\cdot\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\cdot\frac{2\lambda(T)}{3p_0(t)}\frac{\partial T}{\partial x} = 0. \tag{23}$$

Hence,

$$\frac{\partial}{\partial t}\rho s + \mathbf{div}\rho s u + \frac{2}{3T}\mathbf{div}\lambda(T)\mathbf{grad}T = 0.$$

Finally, we put

$$h = \rho[\ln \rho T^{-3/2} + C] \tag{24}$$

with non-relevant consant C and obtain the H-theorem in the following form

$$\frac{\partial h}{\partial t} + \mathbf{div}[hu + \frac{\lambda(T)}{T}\mathbf{grad}T] = -\frac{\lambda(T)}{T^2}(\mathbf{grad}T)^2 \leq 0. \tag{25}$$

Thus, the equations (17) possess the analog of the Boltzmann H-theorem on the contrary to the full Burnett equations [3].

First of all we note the two special cases which were considered in detail in above mentioned papers [1, 2]:

(a) stationary solutions, for that our equations coincide with SNIF-equations and

(b) isothermal solutions, i.e.

$$T = const \Rightarrow \rho = const, p_0 = const, \tag{26}$$

for that our equations coincide with incompressible Navier-Stokes equations

$$\mathbf{div}u = 0, (\frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x})u + \mathbf{grad}p = \Delta u \tag{27}$$

after the self-evident change of variables $p \longrightarrow \alpha p, x \longrightarrow \beta x, t \longrightarrow \gamma t$ with $\alpha, \beta, \gamma = const$.

These two cases were studied in detail earlier and therefore we consider now some other cases.

If we consider the problem in infinite space with the equilibrium conditions $p_0 = const, T = const$ in infinity then we have to take into account only

(c) isobaric solutions, i.e. $p_0 = const$. Then the system (17) reads

$$\frac{\partial}{\partial t}\frac{1}{T} + \mathbf{div}\frac{u}{T} = 0, \mathbf{div}(u - \frac{2}{5}\eta\mathbf{grad}T) = 0, \eta = \frac{\lambda(T)}{p_0}, \kappa = \frac{\mu(T)}{p_o}, \tag{28}$$

$$\frac{1}{T}(\frac{\partial}{\partial t}+u\cdot\frac{\partial}{\partial x})u_k+\frac{\partial p}{\partial x_k}=\frac{\partial}{\partial x_i}<2\kappa\frac{\partial u_i}{\partial x_k}-\kappa^2(K_2\frac{\partial_2 T}{\partial x_i\partial x_k}+\frac{K_3}{T}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_k})>. \tag{29}$$

The Prandtl numder $\mathbf{Pr}=\mu c_p/\lambda$ is usually supposed to be equal 2/3 (it is exactly so for the Maxwell molecules only),we also put

$$\lambda=\frac{15}{4}\mu,\ \eta=\frac{15}{4}\kappa.$$

Then we can exclude η and rewrite the second equation in (28) as

$$\mathbf{div}[u-\frac{3}{2}\kappa(T)\mathbf{grad}T]=0. \tag{30}$$

In general case the choice of the function $p_0(t)$ is defined by initial and boundary conditions. Roughly speaking we should at first construct "the general solution" for $T(x,t)$ and $u(x,t)$, that depends on arbitrary function $p_0(t)$ and then define this function from initial and boundary conditions. Let us consider for an illustration some one-dimensional problems.

5 One-dimensional problems

Suppose that all functions in (28)-(30) depend on the single space variable x, that is directed along a unit vector $\mathbf{k}$ (in this Section we denote three dimensional vectors by the bold letters and derivatives by corresponding subscripts).Then we can express the velocity by formula

$$\mathbf{u}=u\mathbf{k}+\mathbf{u}_\perp,\mathbf{u}\cdot\mathbf{u}_\perp=0,u=\mathbf{k}\cdot\mathbf{u},|\mathbf{k}|=1, \tag{31}$$

and obtain the following equations for T,u and p:

$$(p_0/T)_t+p_0(u/T)_x=0,\frac{3}{5}(\ln p_0)_t+(u-\frac{3}{2}\kappa T_x)_x=0,\kappa=\mu/p_0, \tag{32}$$

$$\frac{1}{T}(u_t+uu_x)+p_x=\frac{2}{3}[2\kappa u_x-\kappa^2(K_2T_{xx}+\frac{K_3}{T}T_x^2)]_x$$

We obtain from the second equation (32)

$$p_0u(x,t)=(3/2)\mu T_x-\psi(t)-(3/5)x\phi(t)$$

with unknown functions $\psi(t)$ and $\phi(t) = p_0'(t)$. Then the first equation (32) reads

$$p_0(T^{-1})_t + 2\phi/(5T) - (\psi + 3x\phi/5)(T^{-1})_x + (3\mu T_x/2T)_x = 0.$$

Let us consider now the heat transfer between two parallel planes with x-coordinates $x_1 < x_2$, then the boundary conditions are: $u(x_1) = u(x_2) = 0, T(x_1) = T_1, T(x_2) = T_2$. The functions $\psi(t)$ and $\phi(t)$ are defined by the relations

$$\psi + 3x_n\phi/5 = 3\mu(T_n)T'(x_n), n = 1, 2,$$

therefore

$$\psi = \frac{3[x_2\mu(T_1)T'(x_1) - x_1\mu(T_2)T'(x_2)]}{2(x_2 - x_1)}, \phi = \frac{5[\mu(T_2)T'(x_2) - \mu(T_1)T'(x_1)]}{2(x_2 - x_1)}.$$

Finally we introduce new time variable putting $d\tau = dt/p_0(t)$ and reduce the problem to a single equation

$$y_\tau + (2/5)\phi y - [\psi + (3/2)x\phi] = (3/2)[\mu(y^{-1})y^{-1}y_x]_x, x_1 < x < x_2, y = T^{-1}, \tag{33}$$

with boundary conditions $y(x_1) = T_1^{-1}$ and $y(x_2) = T_2^{-1}$ and with given initial data. After the solution $y(x, \tau)$ is found one can define the function $\phi(\tau) = p_0'(t)$ and then put

$$p_0(\tau) = p_0(0)\exp[\int_0^\tau dr\phi(r)], t = p_0(0)\int_0^\tau \exp[\int_0^s dr\phi(r)].$$

In such a way we can obtain the solution of the problem relating to the heat transfer between two parallel walls. It should be note that the momentum equation in (17) is needed in the one-dimensional heat transfer problem only for constructing of non-equilibrium pressure $p(x, t)$.

Let us consider now the more simple initial value problem in infinite domain with the conditions $T \to T_\infty$ with $x \to \mp\infty$. Then $p_0 = const, \phi = \psi = 0$, and we obtain the usual quasilinear diffusion equation for the density $\rho = p_0T^{-1}$

$$\rho_t = [D(\rho)\rho_x]_x, D(\rho) = 3\mu(p_0/\rho)/(2\rho). \tag{34}$$

After the function $\rho(x, t)$, satisfying this equation and given initial conditions, is found we can define the non-equilibrium pressure $p(x, t)$ ($p \to 0$ with $|x| \to \infty$) by formula

$$p(x, t) = 4\kappa u_x/3 - (2\kappa^2/3)(K_2T_{xx} + K_3T_x^2/T) - u^2/T - 3\kappa T_t/(2T). \tag{35}$$

To obtain this formula the identity $[F(T)T_x]_t = [F(T)T_t]_x$ was used. We note that in one-dimensional heat transfer problem the Burnett terms result in the small correction for the equilibrium pressure and do not change the temperature. It will be shown below that the situation is quite different in multi-dimensional case.

Finally we consider the equation for $\mathbf{u}_\perp$(31)

$$\frac{1}{T}(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x})\mathbf{u}_\perp = \frac{\partial}{\partial x}\kappa(T)\frac{\partial \mathbf{u}_\perp}{\partial x}, \mathbf{u}_{\perp|t=0} = \mathbf{u}_\perp^{(0)} \tag{36}$$

If $T(x,t)$ and $u(x,t)$ are known then it is a simple linear equation. Hence, in one-dimensional case the most important step is the solving of the non-linear diffusion equation (34).

It is remarkable that for the Maxwell molecules we obtain $D(\rho) = D_0\rho^{-2}$ [1,2] and the equation (31) may be linearized by the following way [12]. We put $\rho = z_x, \tau = D_0 t$ in (31) and obtain

$$z_t = z_x^{-2} z_{xx}, \lim_{x\to\mp\infty} z(x,t)/x = \rho_\infty. \tag{37}$$

Let us pass from z(x,t) to inverse function x(z,t) by formulas

$$w(y,t) = x, y = z(x,t).$$

Then

$$w_y = z_x^{-1}, w_{yy} = -z_x^{-3} z_{xx}, w_t = -z_x^{-3} z_t,$$

i.e. the equation (34) is reduced to the usual linear diffusion equation

$$w_t = w_{yy}.$$

Hence, for the Maxwell molecules we are able in principle to construct the exact general solution of the equations (32), (36) with equilibrium boundary conditions $\rho_\infty = const, T_\infty = const$ in the infinity.

6 Multidimensional stationary problems

We consider now the full system of equations (17). The equations are derived from the Boltzmann equation as the leading asymptotic expansion terms with $\varepsilon \to 0$. The following two properties should be emphasized.

(1) The equations (17) include the functions $p_0(t)$ and $T(x,t)$ having the order $O(1)$ with $\varepsilon \to 0$ and also the functions $u(x,t), p(x,t)$ that correspond to small corrections of the order $O(\varepsilon)$ and $O(\varepsilon^2)$ respectively to the limiting values $u = 0$ and $p = p_0$;

(2) The equations (17) do not contain ε, i.e. they define the limiting at $\varepsilon = 0$ values $p_0(t)$ and $T(x,t)$. It is important that these limiting values can not be found without knowledge of the functions $u(x,t)$ and $p(x,t)$, inspite of the fact that these functions are of no importance at $\varepsilon = 0$.

These properties show the important role of the Burnett terms in (17). Let us consider the stationary heat-transfer problem with the boundary conditions

$$T_{|\Gamma_i} = T_i, u_{|\Gamma_i} = 0, i = 1,2,...,N \tag{38}$$

on certain isotermal surfaces Γ_i. In the Navier-Stokes approximation ($K_2 = K_3 = 0$ in (17)) one can put $u = 0, p = 0$ and reduce the problem to the usual stationary boundary value problem for the nonlinear heat transfer equation

$$\mathbf{div}\kappa(T)\mathbf{grad}T = 0, T_{|\Gamma_i} = T_i, \tag{39}$$

that is equivalent to the standard linear Laplace equation. Let us assume now that $K_{2,3} \neq 0$, then the stationary solutions with $u(x,t) \equiv 0$ are admissible only under the following condition: the solution of the boundary value problem (26) guarantees the solvability of the equation for $p(x,t)$

$$p_{x_k} = -[\kappa^2(T)(K_2 T_{x_i x_k} + (K_3/T) T_{x_i} T_{x_k}]_{x_i},$$

i.e. the right hand part has to be a k-th component of gradient vector. The equation can be simplified and written as

$$\nabla\Pi = F(T)(\nabla T)^2 \nabla T, F(T) = K_2(\kappa\kappa'' - \kappa'^2) + (K_3/2)(\kappa/T)^2, \nabla \equiv \mathbf{grad}, \tag{40}$$

with certain function $\Pi(x)$ (see below). Finally we write down the necessary condition of the absence of convection (i.e. $u(x,t) \equiv 0$) as the condition of consistency of the equations

$$\nabla\kappa(T)\nabla T = 0, \mathbf{rot}[F(T)(\nabla T)^2 \nabla T] = 0$$

and boundary conditions (38). This necessary condition was firstly obtained by the authors of [6] and it was proved that this condition is fulfilled only in "very symmetric" cases (concentric spheres or coaxial cylinders). In more general cases the stationary solution of (17) implies $u(x,t) \neq 0$, some partial solutions were described in [6].

Hence, in the general case the thermal stresses induce convection currents, this physical effect is absent in the Navier-Stokes description. The effect is very interesting from physical point of view and was discussed in detail in literature. We note that the corresponding velocity has an order $O(\varepsilon)$ and disappears in the limiting case $\varepsilon = 0$, therefore it is formally a small correction. However there exists another effect that remains nonzero even at the limit $\varepsilon = 0$.

7 Stationary temperature field at the limit $\varepsilon = 0$

Accordingly to the Navier-Stokes equations the temperature $T(x)$ satisfies the stationary heat-transfer equation (39). However it follows from the stationary equations (17) that

$$(3/2)\mathbf{div}\kappa(T)\mathbf{grad}T = \mathbf{div}u = u \cdot \mathbf{grad}\ln T \tag{41}$$

These equations are compatible with (39) only if

$$\mathbf{div}u = 0, u \cdot \mathbf{grad}T = 0, \mathbf{div}\kappa(T)\mathbf{grad}T = 0. \tag{42}$$

These conditions are obviously weaker than the above discussed condition $u = 0$.

We describe below the partial class of solutions of (17) with the temperature satisfying the heat-transfer equation (39). In this case we can find the temperature from the boundary value problem (39) and then consider it as a given function. The Burnett terms in (17) can be expressed as

$$[\kappa^2(K_2 T_{x_i x_k} + (K_3/T)T_{x_i}T_{x_k})]_{x_i} = -K_3(\kappa^2/T)(\nabla T)^2 + F(T)(\nabla T)^2 T_{x_k},$$

with $F(T)$ from (40). Therefore putting in (17)

$$\Pi = p + K_3(\kappa^2/T)(\nabla T)^2$$

we obtain the equations which are similar to the incompressible Navier-Stokes equations

$$\mathbf{div} u = 0, u_{|\Gamma_i} = 0, i = 1, ..., N,$$

$$\frac{1}{T}(u \cdot \frac{\partial}{\partial x})u_k + \frac{\partial \Pi}{\partial x_k} = \frac{\partial}{\partial x_i}(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial xi}) + F(T)(\nabla T)^2 \frac{\partial T}{\partial x_k}. \quad (43)$$

This system of equations with given function $T(x)$ defines the velocity field $u(x)$. However if the resulting function $u(x)$ does not satisfy in general case the orthogonality condition

$$u \cdot \mathbf{grad} T = 0, \quad (44)$$

then the conjecture that the temperature satisfies the heat transfer equation (39) appears to be wrong. It is clear that the additional condition (43) is fulfilled only for very special cases, so that in the general case the Navier-Stokes equation does not result in the correct temperature field $T(x)$ at the limit $\varepsilon = 0$.

We consider in more detail the two-dimensional flows with the stream function B(x,y) such that

$$u = (u_x, u_y), u_x = B_y, u_y = -B_x, T = T(x, y), p = p(x, y).$$

Then it follows from the orthogonality condition (43) that

$$B_y T_x - B_x T_y = 0,$$

i.e. the stream function depends on x, y only through the temperature $T(x, y), B \equiv B(T)$. The streamlines coinside with isotherms. Introducting a new function $\Phi(T)$ such that $\Phi'(T) = \kappa(T)$ we notice that the function $\phi(x, y) = \Phi[T(x, y)]$ satisfies the Laplace equation $\Delta\phi = 0$. Therefore we can consider an analytic function

$$f(z) = \phi(x, y) + \psi(x, y), \Delta\phi = \Delta\psi = 0,$$

where ϕ and ψ are conjugate harmonic functions. The streamlines and isoterms coincide with the lines $Ref = const$. Let us introduce new unknown function $A(\phi)$ by formulas

$$u_x = A(\phi)\phi_y, u_y = -A(\phi)\phi_x.$$

We consider the functions $T, \kappa(T), F(T)$ as given functions of $\phi : T = T(\phi), \kappa(T) = \alpha(\phi), F(T) = C(\phi)$. Finally we exclude the function $\Pi(x,y)$ from the equation (42) and obtain the equation

$$\frac{\partial}{\partial y}[-\frac{1}{T}(u\cdot\frac{\partial}{\partial x})u_y + C(\phi)(\nabla T)^2\frac{\partial T}{\partial x} + \frac{\partial}{\partial x_i}\alpha(\phi)(\frac{\partial u_i}{\partial x} + \frac{\partial u_x}{\partial x_i})] =$$

$$\frac{\partial}{\partial x}[-\frac{1}{T}(u\cdot\frac{\partial}{\partial x})u_y + C(\phi)(\nabla T)^2\frac{\partial T}{\partial y} + \frac{\partial}{\partial x_i}\alpha(\phi)(\frac{\partial u_i}{\partial y} + \frac{\partial u_y}{\partial x_i})], x_1 = x, x_2 = y.$$

It is possible to pass to new independent variables $r = \phi(x,y), s = \psi(x,y)$ (conformal transformation by the analitic function $f(z)$). In these variables the last equation is reduced to the ordinary differential equation of the third order for the unknown function $A(r)$

$$\sum_{n=0}^{3} k_n(r,s)A^{(n)}(r) + k_4(r,s)A(r)A'(r) + k_5(r,s)A^2(r) + k_6(r,s) = 0,$$

with given coeffitients $k_n(r,s), n = 0,...,6$. The solution of this equation is in general case a function of the both variables r and s. Therefore the condition $A = A(r)$ defines certain additional conditions imposed on the coeffitients $k_n(x,t)$. In particular the example described in [6] (gas flow between two sides of the angle) corresponds to such degenerate case.

Finally we would like to stress once more that in general case the limiting (at $\varepsilon = 0$) temperature does not satisfy the standard heat transfer equation. Of course, the above described equations and also the known SNIF-equations are derived formally on the basis of some conjectures and therefore all consequences of these equations should be considered with certain care. We discuss this question in detail in the next Section.

8 Conclusions

We have considered above some special cases of the rarefied gas flows with small Knudsen numbers $\varepsilon \to 0$, when the time evolution of hydrodinamical parameters ρ, u, T is described by QSF equations (17), which are more complex than the usual Navier-Stokes equations. The QSF equations can be considered as the non-stationary version of the SNIF equations [6]. In the isotermal case the equations (17) are equivalent to the incompressible Navier-Stokes equations.

The QSF equations are derived from the Boltzmann equation on the basis of three conjectures:

(1) quasistationarity, i.e. $f(x,v,t|\varepsilon)=\tilde{f}(x,v,\varepsilon t|\varepsilon), t\to\infty, \varepsilon\to 0, \tilde{t}=\varepsilon t$ is finite;

(2) the distribution function $\tilde{f}(x,v,\tilde{t}|\varepsilon)$ admits an asymptotic expansion

$$\tilde{f}=\tilde{f}_0+\varepsilon\tilde{f}_1+\varepsilon^2\tilde{f}_2+\varepsilon^3\tilde{f}_3+...,$$

including the terms not less than of the third order on ε;

(3) slowness, i.e.

$$\int dv\tilde{f}_0(x,v,\tilde{t})v=0.$$

The incompressible Navier-Stokes equations can be considered as a partial case of QSF equations that corresponds to the additional conjecture

(4') isothermality, i.e.

$$T_0=\frac{1}{3\rho_0}\int dv\tilde{f}_0(x,v,\tilde{t})=const.$$

The last two conjectures result in the fact that the limiting distribution function $\tilde{f}$ coincides with the absolute Maxwell distribution, that corresponds to the approach used earlier in [4],[5]. We note that the incompressible Navier-Stokes equations can be easily derived from the compressible Navier-Stokes equations by a silmilar way without any connection to the Boltzmann equation (M.N. Kogan first called my attention to this fact in 1991).

Finally the well-known SNIF equations [6] can be also consedered as a partial case of the equations (17) that corresponds to the substitution of the conjecture (1) by the stronger conjecture

(1') stationarity, i.e. $f(x,v,t|\varepsilon)=\tilde{f}(x,v|\varepsilon)$.

In connection with stationary problems the interesting open problem should be mentioned. It is clear that according to the Navier-Stokes equations the stationary temperature distribution in the limiting case $\varepsilon=0$ satisfies the heat transfer equation

$$\mathbf{div}\lambda(T)\mathbf{grad}T=0. \qquad (45)$$

However it is not so according to the SNIF equations (and our conjectures (1'), (2), (3)), it is necessary to solve much more complex system of equations. The solution of this system does not coinside with the solution of the heat transer equation except some degenerate cases.

Thus the SNIF-theory [6] predicts an absence of gas convection for $\varepsilon = 0$, that is in complete agreement with the Navier-Stokes equations, however it also predicts the non-Navier - Stokes temperature field at the same limit. At the same time the simple heat transfer equation is very customary in physics and it is difficult to refuse of it. Is it possible that this equation remains valid ? In principle the answer can be positive if we weaken the conjecture (2) and substitute it by

(2') $\tilde{f}(x, v, \tilde{t}|\varepsilon)$ admits an asymptotic expansion

$$\tilde{f} = \tilde{f}_0 + \varepsilon\tilde{f}_1 + \varepsilon^2\tilde{f}_2 + ...,$$

including the terms not less than of the second order on ε.

Then the stationary hydrodinamics equations will have the form

$$p = const, \mathbf{div}\frac{u}{T} = 0, p\mathbf{div}u = \frac{2}{5}\mathbf{div}\lambda(T)\mathbf{grad}T. \qquad (46)$$

If we suppose that the temperature satisfies the equation (45) then the velocity satisfies the equations

$$\mathbf{div}u = 0, u \cdot \mathbf{grad}T = 0. \qquad (47)$$

As we mentioned above the general solution of these equation for the plane case $T = T(x, y), u = u(x, y), u_z = 0$ reads

$$u_x = F(T)\frac{\partial T}{\partial y}, u_y = -F(T)\frac{\partial T}{\partial x} \qquad (48)$$

with an arbitrary function F(T). Let us suppose that the conjecture (2'), not (2),is valid. For example,

$$\tilde{f} = \tilde{f}_0 + \varepsilon\tilde{f}_1 + \varepsilon^2\tilde{f}_2 + \varepsilon^3 \ln\varepsilon\tilde{f}_3 + ...$$

Then the SNIF equations are not valid but the equations (46) remain correct. In this case the velocity should be found not from the SNIF equations but in some different way. It is obviously possible that the temperature satisfies the equation (45) and then the velocity satisfies (47),(48) or even $u = 0$.

We note that the standard Chapman-Enskog expansion are essentially non-stationary one, therefore the validity of the stationary Navier-Stokes equations is not self-evident *apriori*. It was proved in [14] that they are valid for certain class of one dimensional problems but the same question becomes much more difficult in two-dimensional case. At the same time it should be stressed that there is no contradiction between the above considered equations and the Navier-Stokes equations in one-dimensional case, the contradiction appears for multidimensional problems only.

Thus it is desirable to clarify this question and to obtain the definite answer as regards to the limiting with $\varepsilon \to 0$ temperature field in the heat transfer problems for the Boltzmann equation. Besides a rigorous mathematical analysis of this limiting case it is possible to use the numerical experiment. For instance, in the problem of the heat-transfer between two non-coaxial cylinders, that was solved numerically in [13], one can compare the temperature field with the solution of the heat-transfer equation (45). This equation is equivalent to the usual Laplace equation and therefore can be solved without any serious difficulties. If some stable with $\varepsilon \to 0$ deviations from the Laplace equation solution would be observed then it could be considered as a confirmation of the validity of the SNIF and QSF equations.

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References

[1] S.Chapman and T.G.Cowling. *The Mathematical Theory of Nonuniform Gases.*(Oxford University Press,1951).

[2] H.Grad. Principles of the Kinetic theory of gases. *Handbuch der Physik,***12**:205-294 (Springer, Berlin, 1958).

[3] A.V.Bobylev. The Chapman-Enskog and Grad methods for solving the Boltzmann equation. *Sov.Phys.Dokl.***27**(1):29-31(1982).

[4] A.De Masi, R.Esposito and J.L.Lebowitz. Incompressible Navier-Stokes and Euler limits of the Boltzmann equation. *Commun. Pure. Appl. Math.* **42**:1189-1214 (1989).

[5] C.Bardos, F.Golse and D.Levermore. Fluid dynamic limits of kinetic equartions I. Formal derivation. *J.Stat.Phys.* **63**(1/2): 323-344 (1991).

[6] M.N.Kogan, V.S.Galkin and O.G.Fridlender. Stresses produced in gasses by temperature and concentration inhomogeneities. New types of free convection. *Sov. Phys. Usp.* **19** (5): 420 - 428 (1976).

[7] C.Bardos, F.Golse and D.Levermore. *Fluid dynamical limits of kinetic equations II. Convergence proof for the Boltzmann equation.* Preprint(1991)

[8] C.Bardos and S.Ukai. *The classical incompressible Navier-Stokes limit of the Boltzmann equation.* Preprint(1991).

[9] R.Esposito and R.Marra. On the derivation of the incompressible Navier-Stokes equation for Hamiltonian particle system. *J. Stat.Phys.* **74**(5/6): 981-1004 (1994).

[10] A.Yu.Boris, V.S.Galkin,V.A.Kravchenko and O.G.Fridlender. On stability of solutions of the slow non-isotermal gas flow equations. *J. Tech. Phys.* **57** (7): 1255-1259 (1987).

[11] O.G.Fridlender. On thermal-stress gas flows **Arch. Mech. 42** (4/5):475-482 (1990).

[12] N.H.Ibragimov.
Groups of Transformations in Mathematical Physics.(Nauka, Moscow, 1985).

[13] K.Aoki, Y.Sone and T.Yano. Numerical analysis of a flow induced in rarefied gas between noncoaxial cylinders with different temperatures for the entire range of the Knudsen number. *Phys.Fluids A* **1** (2): 409-419 (1989).

[14] R.Esposito, J.L.Lebowitz, R.Marra. *Hydrodynamic limit of the stationary Boltzmann equation in a slab.* Preprint (1992).