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QUASISTATIONARY SOLUTIONS OF THE BOLTZMANN EQUATION

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Abstract

Equations of quasistationary hydrodynamics are derived from the Boltzmann equation by using the modified Hilbert approach. The physical and mathematical meaning of quasistationary solutions are discussed in detail.

1 Introduction

The incompressible Navier-Stokes equations (INSE) were derived in [1,2] from the Boltzmann equation (BE) by using a special time-scaling, precisely $t \to \infty$, the Knudsen number $\mathbf{Kn} \to 0$, $t\mathbf{Kn}$ remains finite. Therefore one can say that INSE describe a quasistationary hydrodynamics for the BE. However an additional restriction was used in the derivation of INSE from BE: the solution of BE was assumed to be absolute Maxwellian distribution in the limiting case $\mathbf{Kn} = 0$. In the present paper, as well as in [3], we refuse of this assumption and derive more general equations of quasistationary hydrodynamics. In contrast to the previous paper [3], where some results of

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the Chapman-Enskog expansion were used, we derive below these equations directly from BE. Besides we discuss in detail the physical and mathematical meaning of quasistationary solutions for both Navier-Stokes and Boltzmann equations.

Let us consider the Boltzmann equation for a distribution function $f(x,v,t)$ ($x \in R^3, v \in R^3, t \in R_+$ denote respectively space coordinate,velocity and time)

$$f_t + v \cdot f_x = \varepsilon^{-1} I(f,f), f_{|t=0} = f_0, \tag{1}$$

where $\cdot$ means a scalar product, $I(f,f)$ denotes the collision integral, ε denotes the Knudsen number, that is a small parameter of this problem. For simplicity we consider in this paper the initial value problem in infinite space R^3 with an equilibrium (absolute Maxwell) distribution in infinity.

The equation (1) is written in dimensionless variables, so that all (except ε) typical parameters of the problem (length, thermal velocity, ets.) are of order of unity. Roughly speaking, one can distinguish three typical time scales: (1) $t_1 \sim \varepsilon$ is the free path time; (2) $t_2 \sim 1$ is the typical macroscopic time(the period of sonic waves); (3) $t_3 \sim \varepsilon^{-1}$ is the typical time of dissipative processes (viscosity,heat transfer). Therefore we can write down formally the general solution of (1) as a function of the three time-variables

$$f(x,v,t) = f_1(x,v;t/\varepsilon,t,\varepsilon t|\varepsilon), \tag{2}$$

that is the standard trick of perturbation theory, the dependence on ε being supposed to be formally-analytical in the neighborhood of the point $\varepsilon = 0$.

When we discuss the so-called normal solutions of the Boltzmann equation [4] we in fact consider the special class of the functions (2) depending on two time-variables only

$$f(x,v,t) = f_2(x,v;t,\varepsilon t|\varepsilon). \tag{3}$$

It is important to consider separately two arguments t and εt inspite of analitic (linear) dependence on ε of the second time variable. Putting $f(x,v,t) = f_2'(x,v;t|\varepsilon)$ we obtain the standard Hilbert expansion [4] that includes some terms increasing with time as εt already in the first order approximation in respect to ε. We can consider the standard Chapman-Enskog method as one of possible ways to take into acccount correct dependence on "slow" time εt.

Finally we can define also the subclass of the normal solutions (3) that includes the dependence on "slow" time εt only, i.e.

$$f(x,v,t) = f_3(x,v;\varepsilon t|\varepsilon). \tag{4}$$

Such solutions will be called quasistationary. Omitting subscript 3 of the function f_3 and changing the time variable $t \to \varepsilon t$, we obtain the quasistationary form of the Boltzmann equation

$$\varepsilon f_t + v \cdot f_x = \varepsilon^{-1} I(f,f). \tag{5}$$

Thus, it is clear that the quasistationary solutions are the special case of the normal solutions of the Hilbert class [4]. For constructing the solutions of (5) we will use the modified Hilbert approach.

2 Modified Hilbert expansion

We consider a formal solution of the equation (5)

$$f = \sum_{n=0}^{\infty} \varepsilon^n f_n(x,v,t) \tag{6}$$

and obtain

$$I(f_0,f_0) = 0, v \cdot \frac{\partial f_0}{\partial x} = I(f_0,f_1) + I(f_1,f_0),$$

$$\frac{\partial f_{n-1}}{\partial t} + v \cdot \frac{\partial f_n}{\partial x} = \sum_{k=0}^{n+1} I(f_k, f_{n+1-k}), n = 1,2,... \tag{7}$$

We introduce the following notations:

$$\rho_n = \int dv f_n(v), J^{(n)} = \int dv f_n(v) v, Q^{(n)} = \frac{1}{2}\int dv f_n(v) v|v|^2,$$

$$P_{ik}^{(n)} = \int dv f_n(v) v_i v_k, p_n = \frac{1}{3}\int dv f_n(v)|v|^2, i,k = 1,2,3.$$

Using the well-known properties of the collision integral

$$\int dv r_i(v)\{I(f,g)+I(g,f)\} = 0, i = 0,1,2,3,4, r_0 = 1, r_4 = |v|^2, r_k = v_k, k = 1,2,3,$$

one can find the solvability conditions for the equations (7)

$$\mathbf{div}J^{(0)} = 0, \mathbf{div}Q^{(0)} = 0, \frac{\partial}{\partial x_i}P_{ik}^{(0)} = 0; k = 1,2,3; \quad (8)$$

$$\frac{\partial \rho_{n-1}}{\partial t} + \mathbf{div}J^{(n)} = 0, \frac{\partial J_k^{n-1}}{\partial t} + \frac{\partial P_{ik}^{(n)}}{\partial x_i} = 0, \frac{3}{2}\frac{\partial p_{n-1}}{\partial t} + \mathbf{div}Q^{(n)} = 0, \quad (9)$$

for $n = 1, ...,$ where the standard summation rule ($i = 1,2,3$) is used. It is clear that f_0 is a locally Maxwellian distribution with parameters satisfying the stationary Euler equations. We choose the Maxwellian distribution with zero mean velocity

$$f_0 = \rho_0(2\pi T_0)^{-3/2}\exp(-|v|^2/2T_0),$$

and with $\rho_0(x,t)$ and $T_0(x,t)$ satisfying the condition $\rho_0(x,t)T_0(x,t) = p_0(t)$. Then the equations (8) are satisfied for any functions $p_0(t)$ and $\rho_0(x,t)$.

Remark. The first difference with the standard Hilbert expansion is the following: the solvability conditions for f_1 are not sufficient for finding f_0, so that we need to consider higher approximations to define f_0.

We choose functions ρ, J and p as basic hydrodynamical parameters and put by definition

$$T(x,t) = T_0(x,t) = p_0(t)/\rho_0(x,t).$$

Then we represent f_n in the form

$$f_n = f_n^{\perp} + (2\pi T)^{-3/2}e^{-|v|^2/2T}\{\rho_n + \frac{v \cdot J^{(n)}}{T} + \frac{p_n - \rho_n T}{2T}(\frac{|v|^2}{T} - 3)\},$$

where

$$\int dv f_n^{\perp}(v) r_i(v) = 0, i = 0, ..., 4.$$

We put

$$P_{ik}^{\perp(n)} = \int dv f_n^{\perp}(v) v_i v_k, Q_i^{\perp(n)} \int dv f_n^{\perp}(v) v_i |v|^2/2$$

and obtain from equations (9)

$$\frac{\partial \rho_{n-1}}{\partial t} + \mathbf{div}J^{(n)} = 0, \frac{\partial J_k^{(n-1)}}{\partial t} + \frac{\partial p_n}{\partial x_k} = -\frac{\partial P_{ik}^{\perp(n)}}{\partial x_i}, \quad (10)$$

$$\frac{\partial p_{n-1}}{\partial t} + \frac{5}{3}\mathbf{div}J^{(n)}T = -\frac{2}{3}\mathbf{div}Q^{\perp(n)}, n = 1, ...$$

Hence, we obtain integral equations

$$v \cdot \frac{\partial f_0}{\partial x} = I(f_0, f_1^\perp) + I(f_1^\perp, f_0),$$

$$\frac{\partial f_{n-1}}{\partial t} + v \cdot \frac{\partial f_n}{\partial x} - \sum_{k=1}^{n} I(f_k, f_{n+1-k}) = I(f_0, f_{n+1}^\perp) + I(f_{n+1}^\perp, f_0), n = 1, ...$$

for functions $f_n^\perp (n = 1, ...)$ and differential equations (10) for hydrodynamical parameters $\rho_n, J_n, p_n (n = 0, 1, ...)$. The first equations read

$$J^{(0)} = 0, \frac{\partial p_0}{\partial x} = 0 \Rightarrow p_0 = p_0(t), \frac{\partial \rho_0}{\partial t} + \mathbf{div}J^{(1)} = 0, \frac{\partial p_1}{\partial x_k} = \frac{\partial P_{ik}^{\perp(1)}}{\partial x_i},$$

$$\frac{\partial p_0}{\partial t} + \frac{5}{3}\mathbf{div}J^{(1)}T = -\frac{2}{3}\mathbf{div}Q^{\perp(1)}, \frac{\partial \rho_1}{\partial t} + \mathbf{div}J^{(2)} = 0,$$

$$\frac{\partial p_1}{\partial t} + \frac{5}{3}\mathbf{div}J^{(2)}T = -\frac{2}{3}\mathbf{div}Q^{\perp(2)}, \frac{\partial J_k^{(1)}}{\partial t} + \frac{\partial p_2}{\partial x_k} = -\frac{\partial P_{ik}^{\perp(2)}}{\partial x_i}, ...$$

For simplicity we consider the problem in the whole space with equilibrium conditions in infinity, i.e.

$$\rho_0 \to \rho_\infty = const, J^{(0)} = 0, p_0 \to p_\infty = const, \rho_n \to 0, J^{(n)} \to 0, p_n \to 0 \tag{11}$$

for $|x| \to \infty, n = 1, ...$

Then $p_0(t) = p_0 = const$. The equation for $f_1^\perp$ reads

$$v \cdot \frac{\partial f_0}{\partial x} = \frac{f_0}{T}(v \cdot \frac{\partial T}{\partial x})(\frac{|v|^2}{2T} - \frac{5}{2}) = I(f_0, f_1^\perp) + I(f_1^\perp, f_0).$$

Its solution is well-known (see [4] or any other textbook on kinetic theory):

$$f_1^\perp = -\frac{2}{5}\frac{\lambda(T)}{T^2}(\frac{|v|^2}{2T} - \frac{5}{2})(v \cdot \frac{\partial T}{\partial x})(2\pi T)^{-3/2}\exp(-|v|^2/2T),$$

where $\lambda(T)$ denotes the standard Chapman-Enskog heat transfer coefficient. Hence, we obtain

$$P_{ik}^{\perp(1)} = 0, Q^{\perp(1)} = -\lambda(T)\frac{\partial T}{\partial x}.$$

Taking into account boundary conditions (10) we can consider the first equations (9) and their solutions

$$\rho_0 T = p_0 = const, p_1 = 0, \frac{\partial \rho_0}{\partial t} + \mathbf{div} J^{(1)} = 0, \mathbf{div} J^{(1)} T = \frac{2}{5} \mathbf{div} \lambda(T) \mathbf{grad} T.$$

We introduce the mean velocity $u(x,t)$ by formula $\rho_0(x,t)u(x,t) = J^{(1)}(x,t)$ and write down the above equations in the form

$$\frac{\partial}{\partial t}\frac{1}{T} + \mathbf{div}\frac{u}{T} = 0, \frac{5}{2} p_0 \mathbf{div} u = \mathbf{div}\lambda(T)\mathbf{grad} T. \tag{12}$$

Thus we obtain two scalar equations for two hydrodynamical variables T and $u = \{u_1, u_2, u_3\}$. In the simplest one-dimensional case $u_1 = u, u_2 = u_3 = 0$ we can reduce the system to a single heat transfer equation

$$u = \frac{2\lambda(T)}{5p_0}\frac{\partial T}{\partial x}, \frac{\partial}{\partial t}\frac{1}{T} = -\frac{\partial}{\partial x}\frac{2\lambda(T)}{5p_0 T}\frac{\partial T}{\partial x}.$$

In multidimensional case the equations (12) are not sufficient to define T and u, therefore we should consider also the equation for $J^{(1)}$

$$\frac{\partial J_k^{(1)}}{\partial t} + \frac{\partial p^{(2)}}{\partial x_k} = -\frac{\partial}{\partial x_i} P_{ik}^{\perp(2)}.$$

Let $p = p^{(2)}/p_0$, then we obtain

$$\frac{\partial}{\partial t}\frac{u_k}{T} + \frac{\partial p}{\partial x_k} = -\frac{\partial}{\partial x_i}\frac{1}{p_0} P_{ik}^{\perp(2)}, \tag{13}$$

with $P_{ik}^{\perp(2)}$ defined by the equation for $f_2^{\perp}$

$$\frac{\partial f_0}{\partial t} + v \cdot \frac{\partial f_1}{\partial x} - I(f_1, f_1) = I(f_0, f_2^{\perp}) + I(f_2^{\perp}, f_0). \tag{14}$$

Using the well-known results of the Chapman-Enskog expansion [4] and some elementary properties of the collision integral we obtain the following formula for $P_{ik}^{\perp(2)}$:

$$P_{ik}^{\perp(2)} = \frac{p_0 u_i u_k}{T} - < 2\mu(T)\frac{\partial u_i}{\partial x_k} - \frac{\mu^2(T)}{p_0}(K_2 \frac{\partial^2 T}{\partial x_i \partial x_k} + \frac{K_3}{T}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_k} >, \tag{15}$$

$$< A_{ik} >= \frac{1}{2}(A_{ik} + A_{ki}) - \frac{1}{3}\delta_{ik}A_{ii}. \qquad (16)$$

with constant coefficients K_2 and $K_3 (K_2 = K_3 = 3$ for Maxwell molecules and $K_2 = 2.418, K_3 = 0.219$ for hard spheres [4,6]).

Hence, we obtain directly from the Boltzmann equation a closed system of hydrodynamical equations (12), (13), (15) for unknown functions T, u, p. These equations were firstly derived in [3] by using the Chapman-Enskog expansion, it is proved above that the same result follows directly from the Boltzmann equation (5), if we assume that its solution admits an asymptotic expansion

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + ...,$$

including terms at least of the third order on ε. In the stationary case the equations are the same as the so-called SNIF (Slow Non-Isothermal Flow) equations [6].

Remark. To solve the equation (14) and other integral equations for $f^{\perp}_{n>1}$ it is very convenient to use the following identities

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2} < |v|^{2k_1}(v\cdot\omega)^{k_2}||v|^{2(n_1-k_1)}(v\cdot\omega)^{n_2-k_2} >= 0, \qquad (17)$$

where $n_{1,2} = 0, 1, ...,$ and ω is any unit vector,

$$< \phi(v)|\psi[v] >= I(f_0\phi, f_0\psi), f_0 = A\exp(-\alpha|v|^2).$$

For the proof it is sufficient to notice that

$$< \exp(\theta|v|^2 + \delta v\cdot\omega)|\exp(\theta|v|^2 + \delta v\cdot\omega) >= 0$$

for any θ and δ. Expanding in power series on θ and δ we obtain identities (17).

3 Quasistationary solutions of the Navier-Stokes equations

We consider here for comparison the Navier-Stokes equations for variables

$$\rho = \int dv f(v), u = \frac{1}{\rho}\int dv f(v)v, p = \frac{1}{3}\int dv f(v)(v-u)^2 = \rho T \qquad (18)$$

denoting density ρ, mean velocity $u \in R^3$ and pressure $p = \rho T, T$ being gas temperature. It follows from the Boltzmann equation (1) that the hydrodynamical variables $\rho(x,t), u(x,t), p(x,t)$ satisfy the following exact (but unclosed) system of equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\rho u_i = 0, \frac{\partial}{\partial t}\rho u_k + \frac{\partial}{\partial x_i}(\rho u_i u_k + p\delta_{ik} + \sigma_{ik}) = 0,$$

$$\frac{\partial}{\partial t}(\rho u^2 + 3p) + \frac{\partial}{\partial x_i}[u_i(\rho u^2 + 5p) + 2(u_k \sigma_{ik} + q_i)] = 0 \qquad (19)$$

where the following notations are used: $c = v - u(x,t)$,

$$\sigma_{ik}(x,t) = \int dv f(x,v,t)(c_i c_k - \frac{1}{3}|c|^2 \delta_{ik}), q = \frac{1}{2}\int dv f(x,v,t) c|c|^2 \qquad (20)$$

To close the above written equations we use the Navier-Stokes approximation

$$\sigma_{ik} = -\varepsilon\mu(T)(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\mathbf{div}u), q = -\varepsilon\lambda(T)\mathbf{grad}T, \qquad (21)$$

with the known coefficients of viscosity $\mu(T)$ and heat transfer $\lambda(T)$[4, 5].

We put in (19), (21)

$$\tilde{t} = \varepsilon t, p = p_0[1 + \varepsilon\pi(\tilde{t}) + \varepsilon^2 \tilde{p}(x,\tilde{t}) + ...], u = \varepsilon\tilde{u}(x,\tilde{t}) + ..., \rho = \tilde{\rho}(x,\tilde{t}) + ..$$

and pass to the limit $\varepsilon \to 0$. Then we omit the sign "tilde" and obtain

$$\frac{\partial \rho}{\partial t} + \mathbf{div}\rho u = 0, \frac{5}{2}p_0 \mathbf{div}u = \mathbf{div}\lambda(T)\mathbf{grad}T, p_0 = \rho T = const, \qquad (22)$$

$$\frac{1}{T}(\frac{\partial u_k}{\partial t} + u \cdot \frac{\partial u_k}{\partial x}) + \frac{\partial p}{\partial x_k} = \frac{\partial}{\partial x_i} < 2\kappa \frac{\partial u_i}{\partial x_k} >, \kappa = \frac{\mu(T)}{p_0}.$$

The same equations (22) can be obtained from (12), (13), (15) if we put $K_2 = K_3 = 0$ in (15). Thus, the Burnett terms in (15) indicate the important difference between "exact" equations (12), (13), (15) and their Navier-Stokes approximation (22). The difference disappears if we consider isothermal solutions. Then $T = const, \rho = const$,

$$\mathbf{div}u = 0, (\frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x})u + \frac{\partial \tilde{p}}{\partial x} = \eta\Delta u, \tilde{p} = pT, \eta = T\mu(T) = const \qquad (23)$$

i.e. we obtain in this case the incompressible Navier-Stokes equations.

However if $T \neq const$ then the two different systems of quasistationary hydrodynamics equations can result in essentially different solutions. The difference was discussed in detail in [3], therefore we will not consider this question here.

Let us consider the connection between quasistationary solutions and spectral properties of the linearized Navier-Stokes equations. The linearized near $\rho_0 = 1, p_0 = 1, u_0 = 0$ equations (19), (21) for small fluctuations ρ, u, p read

$$\rho_t + \mathbf{div}u = 0, u_t + \mathbf{grad}p = \frac{\varepsilon\mu_0}{3}\mathbf{graddiv}u + \varepsilon\mu_0\Delta u,$$

$$p_t + \frac{5}{3}\mathbf{div}u = \frac{2}{3}\varepsilon\lambda_0(\Delta p - \Delta\rho), \mu_0 = \mu(1), \lambda_0 = \lambda(1).$$

Using the Fourier transformation

$$\hat{\rho}(k) = \int dx\rho(x)\exp(-ik\cdot x), \hat{u}(k) = \int dxu(x)\exp(-ik\cdot x), \tag{24}$$

$$\hat{p}(k) = \int dxp(x)\exp(-ik\cdot x),$$

we obtain the following system of ODE with constant coefficients:

$$\rho_t + i(k\cdot u) = 0, p_t + \frac{5}{3}(k\cdot u) + \frac{2\varepsilon\lambda_0}{3}|k|^2(p-\rho) = 0,$$

$$u_t + ikp + \frac{\varepsilon\mu_0}{3}|k|^2[\frac{k}{|k|}(\frac{k}{|k|}\cdot u) + 3u] = 0,$$

where the sign "hat" is omitted.

We put

$$u = w\frac{k}{|k|} + u^{\perp}, w = u\cdot\frac{k}{|k|},$$

then denote $k = |k|$ and obtain

$$\rho_t + ikw = 0, p_t + \frac{5}{3}ikw + \frac{2\varepsilon\lambda_0}{3}k^2(p-\rho) = 0 \tag{25}$$

$$w_t + ikp + \frac{4\varepsilon\mu_0}{3}k^2w = 0, u_t^{\perp} + \varepsilon\mu_0k^2u^{\perp} = 0$$

Therefore

$$u^{\perp}(t) = u^{\perp}(0)\exp(-\varepsilon\mu_0 k^2 t), \tag{26}$$

and the general solution for ρ, w, p may be written as a linear combination of three exponentials $\exp(\lambda_{1,2,3}t)$, with $\lambda_{1,2,3}$ (eigenvalues of the system (25)) satisfying the equation

$$3\lambda^3 + 4\varepsilon\lambda^2 k^2(\mu_0 + \lambda_0/2) + 5\lambda k^2 + 2\varepsilon\lambda_0 k^4 = 0.$$

Asymptotics of roots for $\varepsilon \to 0$ reads

$$\lambda_{1,2} \simeq \pm(5/3)^{1/2} ik - (2\varepsilon/3)(\mu_0 + \lambda_0/5)k^2, \lambda_3 \simeq -(2/5)\varepsilon\lambda_0 k^2. \tag{27}$$

It is clear that complex roots $\lambda_{1,2}(\varepsilon)$ correspond to the sound propagation, while the real root $\lambda_3(\varepsilon)$ corresponds to the heat tranfer process. Moreover there exists, in accordance with (26), the double root

$$\lambda_4 = \lambda_5 \simeq -\varepsilon\mu_0 k^2, \tag{28}$$

that corresponds to viscosity of gas.

For brevity we do not consider functions $p(t;\varepsilon)$ and $w(t;\varepsilon)$. In general case the density $\rho(t;\varepsilon)$ satisfying the equations (25) is given by formula

$$\rho(t;\varepsilon) = \sum_{i=1}^{3} \rho_i(\varepsilon)\exp[\lambda_i(\varepsilon)t], \tag{29}$$

where regular at $\varepsilon = 0$ functions $\rho_i(\epsilon)$ depend also on initial values ρ_0, w_0, p_0.

It follows from (27) that quasistationary solutions have the following general form:

$$\rho(t;\varepsilon) = \rho_3(\varepsilon)\exp[\lambda_3(\varepsilon)t], \rho_1 = \rho_2 = 0, \tag{30}$$

since all solutions with $\rho_{1,2} \neq 0$ contain singular at $\varepsilon = 0$ terms in respect to variable $\tilde{t} = \varepsilon t$. In some sense we can consider quasistationary solutions as a result of the averaging over rapid oscillations.

Hence, the physical meaning of quasistationary solutions is connected with a special choice of initial conditions, such that their time evolution are defined by viscosity and heat transfer processes, but the sound propagation. These solutions may be considered as constructed on dissipative modes belonging to the orthogonal subspace in respect to sonic modes, i.e. eigenvectors coresponding to eigenvalues $\lambda_{1,2}$ in (28).

The situation is more complicated in the non-linear case. However we can understand it at least qualitatively on the basis of the Poincare normal form theorem [7]. Roughly speaking we can assume that for "small" initial data there exists a formally-analitical transformation $F(\tilde{\rho}, \tilde{u}, \tilde{p})$ of the general solution $(\tilde{\rho}, \tilde{u}, \tilde{p})$ of linearized equations to the general solution (ρ, u, p) of nonlinear equations. Then it becomes clear that a quasistationary solution of nonlinear equations appears to be a result of the transformation $F(\tilde{\rho}_q, \tilde{u}_q, \tilde{p}_q)$ of certain quasistationary solution $F(\tilde{\rho}_q, \tilde{u}_q, \tilde{p}_q)$ of linearized equations. Thus, it is a class of special solutions of the nonlinear Navier-Stokes equations constructed on dissipative modes (viscosity and heat transfer) only.

All considerations were made for the Fourier transformed functions $\hat{\rho}, \hat{u}, \hat{p}$ since the solutions seem to be more simple in the Fourier representation. However it is not difficult to generalize the considerations to functions ρ, u, p in the physical space.

4 Quasistationary solutions and spectrum of the linearized Boltzmann operator

We consider briefly the same problem for the Boltzmann equation.Putting

$$f = f_M(1 + F), f_M = (2\pi)^{-3/2} \exp(-|v|^2/2), F \to 0 for |x| \to \infty \qquad (31)$$

we obtain from (1)

$$F_t + v \cdot F_x = \frac{1}{\varepsilon}[L(F) + Q(F, F)], \qquad (32)$$

with

$$f_M L(F) = I(f_M, f_M F) + I(f_M F, f_M), f_M Q(F, F) = I(f_M F, f_M F).$$

For comparison we will also consider the linearized equation, i.e. the lincar part of (32)

$$\Phi_t + v \cdot \Phi_x = \frac{1}{\varepsilon} L(\Phi). \qquad (33)$$

We put

$$\hat{\Phi}(k) = \int dx \Phi(x) \exp(-ik \cdot x), \qquad (34)$$

and then consider the Fourier transformed linerized equation and the corresponding eigenvalue problem

$$\Phi_t + ik \cdot v\Phi = \frac{1}{\varepsilon} L(\Phi), [\lambda(k) + ik \cdot v]\Phi = \frac{1}{\varepsilon} L(\Phi). \tag{35}$$

It is well known that $Re\lambda(k) \leq 0$, moreover the following fact is valid for intermolecular potentials with finite radius of action [5].

If ε is small enough and $|k|$ is bounded, then there exist exactly 5 bounded (for $\varepsilon \to 0$) eigenvalues, the rest of the spectrum satisfies inequality $Re\lambda(k) < -\beta/\varepsilon, \beta > 0$. Therefore the general solution of the equation (35) for $\varepsilon \to 0$ reads

$$\Phi(t;\varepsilon) = \sum_{n=1}^{5} \Phi_i(\varepsilon) \exp[\lambda(\varepsilon)t] + O[\exp(-\beta t/\varepsilon)] \tag{36}$$

An asymptotic behaviour for $\varepsilon \to 0$ of these bounded eigenvalues $\lambda_i(\varepsilon)(i = 1, ..., 5)$ is given by the same formulas (27),(28), as for the Navier-Stokes equations.

We obtain from (36) an asymptotic formula

$$\Phi_{as}(t;\varepsilon) = \sum_{n=1}^{5} \Phi_i(\varepsilon) \exp[\lambda_i(\varepsilon)t], \tag{37}$$

that is similar to formulas (26), (29) for the Navier-Stokes equations. We notice that $\Phi_i(\varepsilon)(i = 1, ..., 5)$ are projections of the initial condition $\Phi(t = 0) = \Phi_0$ onto the proper subspace of the Boltzmann linearized operator, that corresponds to eigenvalues $\lambda_i(i = 1, ..., 5)$. Therefore to obtain the asymptotic solution (37) we do not need to know exactly the initial condition $\Phi_0(k, v)$, it is sufficient to know its projection onto 5-dimensional proper subspace of the Boltzmann linearized operator. It is the exact mathematical contents of the so-called "reduction of description" in kinetic theory of gases [4,5]. At the limit point $\varepsilon = 0$ the projections $\Phi_i(\varepsilon)$ are equivalent to the standard hydrodynamical quantities, i.e. density, mean velocity and temperature.

Passing now to the "slow" time-variable εt we discard in (37) rapidly oscillating terms and obtain quasistationary solution of the linearized Boltzmann equation in the form

$$\Phi_q(t;\varepsilon) = \sum_{n=3}^{5} \Phi_i(\varepsilon) \exp[\lambda_i(\varepsilon)t], \tag{38}$$

that is quite similar to formulas (26),(30) for the Navier-Stokes equations. We notice that in the quasistationary case we obtain the additional "reduction of description" since the solution (38) is completely defined by the projection of the initial condition onto 3-dimensional proper subspace.

Hence, we can define the normal solutions of the Hilbert class as solutions constructed on hydrodynamical modes (two sonic modes and three dissipative ones) and the quasistationary solutions as solutions constructed on dissipative modes only. We can also consider the special solutions constructed on two viscous modes and separately the solutions constructed on single heat transfer mode. Then in the first case the incompressible Navier-Stokes equations will arise while in the second case we shall obtain the heat transfer equation.

Finally we remark that a qualitative generalization of these considerations to the nonlinear Boltzmann equation (32) can be done by the same approach as described above for the Navier-Stokes equations. We put

$$\hat{F}(k) = \int dx F(x) \exp(-ik \cdot x)$$

and obtain the equation

$$F_t + ik \cdot vF = \frac{1}{\varepsilon}[L(F) + \tilde{Q}(F, F)], \tag{39}$$

where $\tilde{Q}$ denotes the Fourier representation of the operator Q, the sign "hat" is omitted. Following the same way, as in the previous section, we assume that there exists a formally-analytical transformation A, such that the function

$$F = A[\Phi] = \Phi + \sum_{n=1}^{\infty} A_n(\Phi, ..., \Phi)$$

satisfies the nonlinear equation (39) for any solution $\Phi(k, v, t)$ of the linear equation (35). The notation $A_n(...)$ means independent on time t n-linear operator. Then an asymptotic behaviour of F for $\varepsilon \to 0$ can be formally described as

$$F_{as}(t; \varepsilon) = A\{\sum_{n=1}^{5} \Phi_i(\varepsilon) \exp[\lambda_i(\varepsilon)t]\} \tag{40}$$

in accordance with formula (37). It is a formal representation of the normal Hilbert solution to the nonlinear Boltzmann equation (39). Therefore we

can conclude that such solutions in the nonlinear case are also constructed on hydrodynamical modes only. As to quasistationary solutions F_q they are given by formula

$$F_q(t;\varepsilon) = A\{\sum_{n=3}^{5} \Phi_i(\varepsilon)\exp[\lambda_i(\varepsilon)t]\},$$

so that they are constructed on dissipative modes similarly to the linear case. If in this formula$\Phi_3 = 0$, then such solution is constructed on viscous modes only, its leading asymptotic terms for $\varepsilon \to 0$ being described by the incompressible Navier-Stokes equations.

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