Economics of Downside Risk

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Abstract.
Ever since establishment of portfolio selection theory by Markowitz (1952), the use of standard deviation as a measure of risk has heavily been criticized. The aim of this thesis is to refine classical portfolio selection and asset pricing theory by using a downside deviation risk measure. It is defined as below-target semideviation and referred to as downside risk.

Downside efficient portfolios maximize expected payoff given a prescribed upper bound for downside risk and, thus, are analogs to mean-variance efficient portfolios in the sense of Markowitz. The present thesis provides an alternative proof of existence of downside efficient portfolios and identifies a sufficient criterion for their uniqueness. A specific representation of their form brings structural similarity to mean-variance efficient portfolios to light. Eventually, a separation theorem for the existence and uniqueness of portfolios that maximize the trade-off between downside risk and return is established.

The notion of a downside risk asset market equilibrium (DRAME) in an asset market with finitely many investors is introduced. This thesis addresses the existence and uniqueness problem of such equilibria and specifies a DRAME pricing formula. In contrast to prices obtained from the mean-variance CAPM pricing formula, DRAME prices are arbitrage-free and strictly positive.

The final part of this thesis addresses practical issues. An algorithm that allows for an effective computation of downside efficient portfolios from simulated or historical financial data is outlined. In a simulation study, it is revealed in which scenarios downside efficient portfolios outperform mean-variance efficient portfolios.

> “*But perhaps there is an alternative. Perhaps some other measure of portfolio risk will serve in a two parameter analysis for some of the utility functions which are a problem to variance. For example, in Chapter 9 of Markowitz (1959) I propose the "semi-variance" S as a measure of risk where*
>
> $$\mathbf{S} = \mathbf{E}(\mathbf{Min}(\mathbf{0}, \mathbf{R} - \mathbf{c})^{\mathbf{2}})$$
>
> *where c=E(R) [below-mean semivariance] or c is a constant independent of choice of portfolio [below-target semivariance]. Semi-variance seems more plausible than variance as a measure of risk, since it is concerned only with adverse deviations.*”
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> HARRY M. MARKOWITZ, *FOUNDATIONS OF PORTFOLIO THEORY.*

Nobel Lecture.

December 7, 1990. Baruch College, The City University of New York, New York, USA.

Contents

List of Figures

Introduction

Motivation

The recent financial crisis with its unprecedented economic implications has revived research into the role of risk measures for the stability of financial markets. While alternative measures have long been known, most models of portfolio selection still use the standard deviation for measuring risk. Mean-variance portfolio selection theory, together with the classical capital asset pricing model (CAPM) continues to be the theoretical basis for investment advice, e.g., see Levy (2011). Standard deviation, however, is a rather poor measure of risk, because it does not distinguish between downside and upside deviations. Typically, an investor will only fear the downside risk of receiving payoffs below a certain target value that determines the threshold between gains and losses. In finance, most often the mean payoff serves as such a target but other thresholds such as the payoff of a safe bond are plausible as well. In a similar way, a regulator of a financial system is concerned with the risk of large losses rather than the "risk" of large gains. Her emphasis must be on unfavorable states in which public funds of a government as a lender of last resort are needed to bail out financial institutions.

The significance of the mean-variance analysis until today is remarkable. As is well known, Markowitz (1952, 1959) decided to quantify risk by the variance, but what may less well be known, he favored another measure: semivariance. Already in his seminal book, Markowitz (1959) devoted the entire Chapter 9 to discuss semivariance. There, he argued that "analyses based on S [semivariance] tend to produce better portfolios than those based on V [variance]" and later Markowitz *et al.* (1993) claimed that because "an investor worries about underperformance rather than overperformance, semideviation is a more appropriate measure of investor's risk than variance". What may be the reason why, despite all its well-known limitations, the foundation of finance still rests to a large extent on the mean-variance analysis? Markowitz (1959) suggested because variance has an edge over semivariance "with respect to cost, convenience, and familiarity".

Familiarity has become less of an issue over time and today it is widely accepted in theory and validated empirically that variance as a measure of risk is insufficient when trying to explain the risk-taking behavior of agents. Mao (1970a,b) launched research into decision models beyond mean-variance. He constructed lotteries which all have the same mean and the same variance but differ in skewness. When asked to choose among these lotteries in an experiment, business executives unanimously preferred lotteries with positive skewness, at least when their investment was large. Mao argues that the risk-taking behavior is consistent with below-target semivariance as a concept of risk, cf. Mao (1970b, p. 355). Based on a conceptual comparison of expectation-variance (EV) with expectation-semivariance (ES) decision criteria, Mao examined how portfolio rankings change when the EV criterion is replaced by the ES criterion, see Mao (1970a, p. 673). Building on these findings, Fishburn (1977) formalized a notion of risk associated with below-target returns for univariate random payoffs. Refining Porter (1974), he showed that stochastic dominance of either first, second, or third degree implies mean-semivariance dominance.

The difference in *convenience* stems from the fact that the mean-variance framework allows for closed form solutions, e.g., for efficient portfolios, which can be computed from mean and covariance matrix only. However, as pointed out by Grootveld & Hallerbach (1999), so far nobody succeeded in aggregating semivariances for a portfolio of random payoffs. Therefore, existence results in the mean-semivariance framework, see, e.g., Jin *et al.* (2006); Rockafellar *et al.* (2006b), remain rather abstract. In addition, derivation of mean-semivariance efficient sets requires as input the entire joint distribution of returns, cf. Estrada (2008).

Mean-semivariance efficient portfolios are *costlier* to compute, because an iterative algorithm has to be executed.

This thesis aims at overcoming the weaknesses of standard deviation as a measure of risk and, thus, the limitations of mean-variance analysis. The idea is to improve the methodology of risk measurement while retaining as many features of the mean-variance framework as possible.

According to Rockafellar *et al.* (2006a), there are two categories: general deviation measures as generalizations of standard deviation, and coherent risk measures in the sense of Artzner *et al.* (1999). We will introduce a *downside deviation risk measure*, denoted by $\mathfrak{D}$, as a hybrid measure that combines favorable features of either category. On the one hand side, $\mathfrak{D}$ is a *deviation measure* that measures nonconstancy such as standard deviation. Unlike standard deviation, however, $\mathfrak{D}$ does not average over all possible deviations. Instead, it only takes into account deviations that are below a prespecified threshold. In our context, such *risky deviations* constitute losses. For this reason, it is also a *risk measure* that "evaluates the overall seriousness of possible losses", see Rockafellar *et al.* (2006a). In the sequel, $\mathfrak{D}$ will be referred to as *downside risk* for short.

We argue that our downside risk measure is best suitable for risk management and portfolio selection. We further establish that the capital asset pricing model, whose equilibrium formulation was introduced by Sharpe (1964), Lintner (1965) and Mossin (1966), can be refined when standard deviation is replaced by downside risk. By doing so, we establish a framework within which portfolio selection and asset pricing theory can be performed *conveniently.*

Eventually, we show that, at a little extra computational expense, mean-downside-risk efficient portfolios can be calculated in the same way as mean-variance efficient ones. Furthermore, computational power today is not the bottleneck that is was when modern portfolio theory had been being developed. Hence, financial practitioners can readily apply and benefit from the portfolio selection methodology proposed in this thesis.

We close this introductory section with two motivating examples that point at the limitations of mean-variance analysis. In the course of the present thesis, we elaborate how downside risk can overcome these limitations.

The first example is a numerical illustration of the Borch Paradox. Johnstone & Lindley (2013) state: "the Norwegian insurance theorist and economist Karl Borch (1969) [...] proved, he claimed, that it is impossible to draw indifference curves in the mean-variance (μ, σ^2) or mean-standard-deviation (μ, σ) plane". The Borch Paradox gave rise to an important but generally little-known philosophical literature relating mean-variance analysis and decision theory under uncertainty. Johnstone & Lindley (2013) give a comprehensive overview of this debate, including Baron's (1977) rebuttal of Borch.

Example 0.1 (The Borch Paradox. Johnstone & Lindley (2013)).
Suppose an investor, who considers only mean and standard deviation of an asset, is indifferent between $(\mu_1, \sigma_1) = (10, 15)$ and $(\mu_2, \sigma_2) = (20, 25)$.
Then, we construct two assets. Asset 1 pays 25 € with probability 0.5 and -5 € with probability 0.5. Asset 2 pays 45 € with probability 0.5 and -5 € with probability 0.5.
Obviously, a rational decision maker should prefer asset 2 over asset 1 because both assets yield the same payoff if they lose, they have the same probability of winning (or losing), but asset 2 pays 45 € instead of 25 € if it wins.
Mean and standard deviation of these assets, however, are $(\mu_1, \sigma_1) = (10, 15)$ and $(\mu_2, \sigma_2) = (20, 25)$.
Hence an investor, who considers only mean and standard deviation, would be indifferent between asset 1 and asset 2. □

The numbers are not crucial, since Borch (1969) provided a proof for arbitrary pairs $(\mu_1, \sigma_1), (\mu_2, \sigma_2)$.
Observe that the investor requires a higher mean to compensate for a higher standard deviation. This kind of risk aversion is also not necessary in Borch's demonstration.
Note further that asset 2 strictly stochastically dominates asset 1, but it is not strictly preferred over asset 1 in terms of mean and standard deviation. In Section 2.1, we analyze the interplay between various portfolio selection criteria, one of which is stochastic dominance.
Example 0.1 underlines the weakness of standard deviation as a measure of risk. The standard deviation of asset 2 is larger because its payoff in the winning state is higher. No rational investor

would consider such an upside deviation as risky. The downside deviation measure which we construct in Chapter 1 distinguishes between upside and downside deviations and thus measures what investors actually fear: the risk of losing money.

The second example is an even more distinct contestation against mean-variance analysis. Example 0.1 revealed that an investor may be indifferent between two assets where one of them stochastically dominates the other. In Example 0.2 we show that an even more preposterous situation may occur: an investor may prefer an asset which is almost surely dominated.

EXAMPLE 0.2.
Suppose a risk averse investor who considers mean and standard deviation only. She wants to be compensated for a higher standard deviation with a higher mean payoff.
Let $a \geqslant 1$ and consider two assets,

$$w_a := \begin{cases} a & \text{with probability } \frac{1}{a} \\ 0 & \text{with probability } 1 - \frac{1}{a} \end{cases} \quad \text{and} \quad w_0 := 0.$$

It is immediate that for any $a \geqslant 1$ the relation $w_a \geqslant w_0$ holds, i.e., w_a dominates w_0 almost surely. Figure 1 shows the (μ, σ)-plane. The point $(0, 0)$ represents the (μ, σ)-pattern of w_0 and the solid line depicts all (μ, σ)-profiles between which the investor is indifferent. Since $E[w_a] = 1$ and $\sigma(w_a) = \sqrt{a-1}$, the (μ, σ)-patterns of w_a, for $a \geqslant 1$, are located on the dotted horizontal line starting at $(0, 1)$.

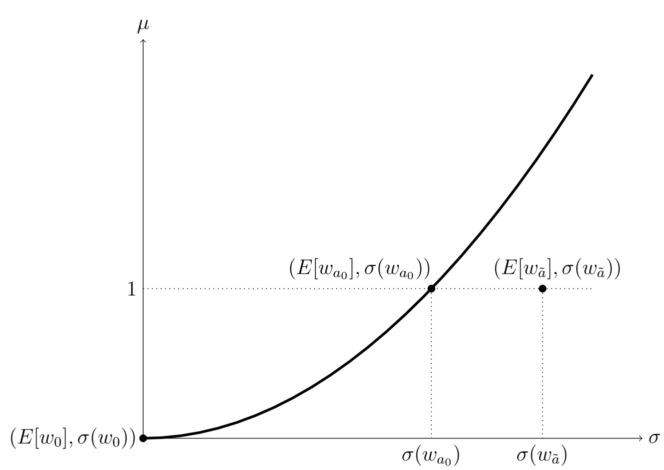

FIGURE 1. Almost sure dominance and mean-variance preferences.

For a risk averse investor, there exists $a_0 > 1$ such that she is indifferent between $(E[w_0], \sigma(w_0))$ and $(E[w_{a_0}], \sigma(w_{a_0}))$. It follows that this investor then prefers $(E[w_0], \sigma(w_0))$ over $(E[w_{\tilde{a}}], \sigma(w_{\tilde{a}}))$, for all $\tilde{a} > a_0$, although $w_{\tilde{a}} \geqslant w_0$.
Hence, the investor prefers an asset that yields a payoff which is less or at best equal in any state of nature. □

Main Results and Structure of the Thesis

In this section, we give a detailed nontechnical overview of the results of this thesis. Further information and all bibliographical references can be found in the respective chapters. In order to improve readability, Chapters 1-4 each have an appendix, which contains lengthy proofs and technical lemmas.

The general setting for this thesis is specified in Chapter 1. A static one-period economy is considered, where a finite number of investors has to transfer their wealth from date 0 to date 1. Each of them can choose between a risk-free asset, for example a bond, which pays a constant return and several risky assets (e.g., stocks) with random payoffs. Investors want to achieve a mean payoff as high as possible, while trying to avoid risk as much as possible.

As a next step, the methodology to measure downside risk is introduced. Downside risk of a random payoff w.r.t. some (nonrandom) target payoff is defined as the square root of the mean squared difference between the payoff and the target *in case the payoff is below target.* The target payoff determines the threshold separating gains from losses. This means that only risky below-target deviations are taken into account when computing downside risk. If the target is chosen to be the expected payoff, downside risk coincides with semideviation, which is the square root of lower semivariance.
On the on hand, downside risk is an asymmetric, surplus invariant deviation measure which fulfills properties such as relevance, positive homogeneity, subadditivity, convexity and continuity. All general deviations measures, in particular standard deviation, have these properties in common, compare Rockafellar *et al.* (2006a). Thus, we can hope that most, if not all, results of portfolio selection and asset pricing theory established in the mean-variance framework remain valid in a similar form when standard deviation is replaced by downside risk.
On the other hand, downside risk averages over risky deviations only, such that it *evaluates the overall seriousness of possible losses.* In that sense it is also a risk measure that fulfills properties like lower range dominance and monotonicity. As a consequence, downside risk overcomes the deficiencies of standard deviation that have been revealed by the examples of the motivating section.
Eventually, we introduce the concept of a downside cosemivariance matrix which is symmetric and positive semidefinite. It is a downside analogon to the covariance matrix.

Chapter 2 develops a portfolio selection theory in the spirit of Markowitz (1952). We suppose that preferences of an investor are formalized by a utility function which is a function of the mean $\mathfrak{M}$ and the downside risk $\mathfrak{D}$. Then, the decision problem of an investor is of the form

$$\max_{\mathbf{x}\in\mathbb{R}^K} U(\mathfrak{M}(\mathbf{x}), \mathfrak{D}(\mathbf{x})).$$

Firstly, we argue that an optimal portfolio $\mathbf{x}^*$ induces a payoff that is almost surely undominated and undominated in the sense of stochastic dominance. Further, if $U(m,d) = m - \alpha d^2$, the payoff of $\mathbf{x}^*$ maximizes expected utility for a properly chosen Bernoulli utility function. For this reason, we confine our analysis to the mean-downside-risk framework.
Secondly, we introduce *downside efficient portfolios*, denoted by $\mathbf{x}^{\mathsf{de}}(d_0)$, that are solutions to

$$\max_{\mathbf{x}\in\mathbb{R}^K} \mathfrak{M}(\mathbf{x}) \text{ s.t. } \mathfrak{D}(\mathbf{x}) \leqslant d_0$$

for some $d_0 \geqslant 0$. These are optimally downside diversified portfolios and analogs to mean-variance efficient portfolios in the sense of Markowitz (1952). We prove existence of downside efficient portfolios and identify invertibility of the downside cosemivariance matrix as sufficient criterion for their uniqueness. Since invertibility of the covariance matrix is the prerequisite for uniqueness of mean-variance efficient portfolios, the requirements do not become stronger when replacing standard deviation by downside risk. Moreover, we characterize the form of downside efficient portfolios which turns out to be structurally equivalent to the form of mean-variance efficient portfolios.
Thirdly, the set of all attainable risk-return profiles is established. In contrast to the mean-variance framework, this set turns out do be an *asymmetric* cone. Its upper boundary is the downside efficient frontier that describes the maximal achievable mean payoff, given a prescribed level of downside risk.
Fourthly, the downside efficient frontier is used to formulate and prove a separation theorem. It states that the investment decision separates into two parts. An investor has to determine her individually optimal amount of downside risk d^* and her individually optimal, i.e., downside efficient, portfolio mix. Then, $\mathbf{x}^* = \mathbf{x}^{\mathsf{de}}(d^*) = d^*\mathbf{x}^{\mathsf{de}}(1)$.
Finally, a portfolio selection theory without a riskless asset is developed. It turns out that the set of all attainable risk-return profiles then has the form of an asymmetric aircraft cone. For d_0 large enough, downside efficient portfolios exist. Existence and uniqueness of a tangential portfolio are not guaranteed and require further prerequisites which are specified in Proposition 2.5. Chapter 2 closes with a reformulation of the separation theorem for the case with only risky assets.

Based on the individually optimal investment given by the separation theorem, **we analyze an asset market with several investors in Chapter 3.** The market portfolio, which constitutes the current stock of shares of risky assets, is distributed among the investors. A *downside risk asset*

market equilibrium (DRAME) is an allocation of assets that is individually optimal and market-clearing. To sidestep the difficulties in the multidimensional asset market, we solve the existence and uniqueness problem for equilibria in the simpler space spanned by the risk-free asset and the payoff of the market portfolio. It turns out that this simple space has the same equilibria as the original asset market.
A DRAME pricing formula is established which is structurally very similar to the mean-variance CAPM pricing formula. The fact that covariance in the CAPM pricing formula is replaced by downside cosemivariance brings some remarkable economic implications. Above all, *equilibrium asset prices are arbitrage-free and strictly positive* in the mean-downside-risk framework. These findings contrast strongly with the traditional CAPM pricing formula which allows for arbitrage opportunities as well as negative stock prices.
A downside security market line is derived with downside beta coefficients in analogy to CAPM betas. We further show that the market portfolio attains the highest possible Sortino ratio at equilibrium prices and, thus, offers the best downside-risk-return trade-off among all potential portfolios.

In the final chapter, practical issues are investigated. As a preliminary step, portfolio selection theory is reformulated in terms of return rather than in terms of prices and payoffs because practitioners usually work with returns. Then, we analyze to what extend downside efficient portfolios improve on mean-variance efficient portfolios. To do so, we carry out a simulation, where three scenarios with increasing default probabilities are simulated.
For each scenario, the downside efficient and the mean-variance efficient portfolio mix are computed. Portfolio compositions differ as the mean-variance efficient portfolio mix spreads its weights more equally whereas the downside efficient mix puts more weight on the safest asset and less weight on all other assets.
To compare the riskiness of the respective portfolios, we re-scale them to have the same mean return. If the asset market is calm in the sense that default rates are zero, downside efficient portfolios are as risky as mean-variance efficient ones. Hence, an investor can just as well stick to mean-variance efficient portfolios. However, when market disruptions and defaults have a strictly positive probability, downside efficient portfolios are much safer. Investors holding a downside efficient portfolio mix are by far better secured against huge losses which may cause bankruptcy.
When compared to mean-variance efficient portfolios, downside efficient portfolios do not perform worse in the calm scenario but strictly better in the dangerous scenarios. Therefore, investors should not spare the small additional computational expense and rely on downside efficient portfolios instead.

Conclusions and prospects for future research close this thesis.

Notation

We briefly introduce the general notation used in the present thesis. Further notation will be introduced in the course of this thesis.

In the present thesis, the word "positive" means $\geqslant 0$ whereas "strictly positive" means > 0. The same convention applies to the terms "negative", "greater", "less", "increasing", "decreasing", "convex", "concave", "is preferred to" $\succcurlyeq$, "is strictly preferred to" $\succ$, "dominates" as well as for the set inclusions "subset" $\subseteq$ and "strict subset" $\subsetneq$. The symbol $:=$ is used for definitions.
We will use some standard abbreviations in this thesis. The abbreviation i.e. stands for *id est* and e.g. stands for *exempli gratia*. By w.r.t. we abbreviate "with respect to" and by w.l.o.g. "without loss of generality". The word "confer" is abbreviated by cf. and s.t. stands for "subject to" and is used when formulating optimization problems.

As usual, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of positive and $\mathbb{R}_{++}$ the set of strictly positive real numbers. $\mathbb{R}_-$ is the set of negative real numbers. The K-fold Cartesian product of $\mathbb{R}$ is termed $\mathbb{R}^K$. We further label multidimensional objects in bold print, e.g., $\mathbf{x} = (x_1, ..., x_K) \in \mathbb{R}^K$, and denote by $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{k=1}^{K} x_k y_k$ the standard Euclidean scalar product on $\mathbb{R}^K$. Moreover, $\nabla_{\mathbf{x}}$ denotes the gradient operator w.r.t. the variable $\mathbf{x}$ and $\frac{\partial}{\partial x_k}$ is the partial derivative operator w.r.t. x_k.
Let V be a real vector space. For $v_1, ..., v_N \in V$, we denote by $\operatorname{span}\{v_1, ..., v_N\}$ the linear span, i.e., the set of all linear combinations of the vectors $v_1, ..., v_N$.

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Ω describes the possible states of nature, $\mathcal{F}$ is the sigma algebra of observable events and $\mathbb{P}$ is their probability measure.
Let $\mathcal{B}(\mathbb{R})$ label the Borel sigma algebra on $\mathbb{R}$. Then, a real random variable is a measurable function $q : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. (In)equalities between real random variables are to be viewed in the sense of holding almost surely.
Let $\mathcal{L}^2(\Omega)$ be the set of square integrable real random variables. For $q, r \in \mathcal{L}^2(\Omega)$, the expected value is given by $E[q] = \int_\Omega q(\omega)\, \mathbb{P}(d\omega)$, the inner product by $E[qr]$, the L^2-norm by $\|q\| = \sqrt{E[q^2]}$ and the positive part by $q_+ = \max\{q, 0\}$. The notation $E_{\mathscr{R}}$ means taking the expectation over the subset $\mathscr{R} \subseteq \Omega$: $E_{\mathscr{R}}[q] := \int_{\mathscr{R}} q(\omega)\, \mathbb{P}(d\omega)$.

A portfolio solving the Markowitz (1952) optimization problem is called efficient, mean-variance efficient or (μ, σ)-efficient portfolio. In continuous text, we often use the term "variance", whereas in formulas the standard deviation σ is employed most of the time. Volatility is a synonym for standard deviation.

Finally, we mention that all numerical simulations in the present thesis are done with the statistical software tool **R**. Moreover, the open source typesetting program LyX is used.

CHAPTER 1

Downside Risk

To start with, the general setting of this thesis is introduced in Section 1.1. We specify the investment possibilities and characterize investors by means of their preferences, beliefs and endowments. In Section 1.2, we define downside risk as a below-target deviation measure, adapted from Fishburn (1977). Afterwards, we discuss crucial properties of downside risk. We argue that it combines beneficial characteristics of general deviation measures as well as risk measures. Therefore, it is eminently suitable for portfolio selection.

1.1. The Model

We consider a one-period economy with dates 0 and 1. Uncertainty occurs at date 1 and is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1.1. Financial Market.

The financial market offers K risky assets with prices $p_1, \dots, p_K \in \mathbb{R}$ at time 0 and random payoffs $q_1, \dots, q_K \in \mathcal{L}^2(\Omega)$ at time 1. Furthermore, there is a riskless investment possibility, e.g., a bond, which pays a constant gross rate of return $r_0 > 0$ per unit. Its price is normalized to 1, so that the bond is taken as numéraire. The vector $\boldsymbol{\pi} := E[\mathbf{q}] - r_0 \mathbf{p} \in \mathbb{R}^K$ represents the *mean excess payoffs* of the risky assets. Let $\mathscr{M} := \operatorname{span}\{r_0, q_1, ..., q_K\}$ denote the *marketed subspace* spanned by the financial assets. $\mathscr{M}$ is a closed finite-dimensional subspace of $\mathcal{L}^2(\Omega)$.
Risky assets are in strictly positive net supply, denoted by $\mathtt{m} \in \mathbb{R}^K_{++}$, and the riskless asset is in zero net supply.[1] We call $\mathtt{m}$ *market portfolio* of the economy and $\epsilon_{\mathtt{m}} := \langle \mathbf{q}, \mathbf{x}_{\mathtt{m}} \rangle$ *market payoff.* There are no short sale constraints.

An investment is a pair $(x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^K$ consisting of the number of shares of the bond x_0 and a *portfolio* $\mathbf{x} = (x_1, ..., x_K)$ of risky assets, where x_k denotes the number of shares of the k^{th} risky asset. Investment $(x_0, \mathbf{x})$ has a date-0 price $x_0 + \langle \mathbf{p}, \mathbf{x} \rangle \in \mathbb{R}$ and promises a random future payoff $r_0 x_0 + \langle \mathbf{q}, \mathbf{x} \rangle \in \mathscr{M}$ at time 1.

[1] As the normal m usually stand for the mean, we use the typewriter type $\mathtt{m}$ to denote the market portfolio. It is the only exception in this thesis where a multidimensional item is not labeled in bold face. Any object related to the market portfolio will receive an $\mathtt{m}$ as index.

1.1.2. Investors.

There is a finite number I of investors operating in the asset market. Each investor $i = 1, ..., I$ is characterized by a utility function U^i, which is a function of mean and risk of future payoff of her investment, and a probability distribution $\mathbb{P}^i$ of the random payoff vector $\mathbf{q}$, which captures her beliefs about the stock market. We assume that the investor takes into consideration only mean and risk when evaluating an investment.[2] This assumption is totally in line with the setting in standard portfolio selection theory à la Markowitz and with CAPM theory built upon it. We only presume that mean is a good and risk is a bad, i.e., investors prefer a higher expected payoff and a less risky payoff. In Chapters 2 and 3, we require the utility functions to fulfill further properties.

Relative to the market, an investor is atomistic. This means, firstly, she has no market power, i.e., she is a price taker. Secondly, her demand for shares of bond and risky assets is small compared to the aggregate supply and can thus be satisfied instantly.

Assume investor i to be endowed with shares of the risk-free and the risky assets $(x_0^i, \mathbf{x}^i) \in \mathbb{R} \times \mathbb{R}^K$. Note that $x_0^i \in \mathbb{R}$ and $\mathbf{x}^i \in \mathbb{R}^K$, which means that an investor can be endowed with short sold assets, i.e., she can be indebted at date 0. Furthermore, an investor's endowment is, in general, risky because shares of risky assets $\mathbf{x}^i \in \mathbb{R}^K$ promise an uncertain payoff $\langle \mathbf{q}, \mathbf{x}^i \rangle$. On the other hand, an investor can very well be endowed risklessly when $\mathbf{x}^i = \mathbf{0}$.
Investor i's endowment $(x_0^i, \mathbf{x}^i)$ yields a date-1 payoff

$$\epsilon^i := r_0 x_0^i + \langle \mathbf{q}, \mathbf{x}^i \rangle \in \mathscr{M}. \tag{1}$$

Thus, we can equivalently suppose an investor to be equipped with a random endowment $\epsilon^i \in \mathscr{M}$, as done by Dana (1999) and others.
We abstract from taxes and transaction costs and assume assets to be liquid. Hence, investor i can monetize her endowment at time 0 to obtain a monetary endowment

$$e^i := x_0^i + \langle \mathbf{p}, \mathbf{x}^i \rangle \in \mathbb{R}. \tag{2}$$

Among others, Wenzelburger (2010) assumed investors to be endowed with $e^i \in \mathbb{R}$ at date 0. He interpreted e^i as units of a nonstorable consumption good and supposed that consumption only takes place at date 1. We consider an investor who wants to transfer wealth from date 0 to date 1 and interpret e^i as money.
Thus, the setting with endowment in shares of the risk-free and the risky assets $(x_0^i, \mathbf{x}^i) \in \mathbb{R} \times \mathbb{R}^K$ can easily be converted to an equivalent endowment with random date-1 payoffs $\epsilon^i \in \mathscr{M}$ or with deterministic monetary endowments $e^i \in \mathbb{R}$. We will switch between these types of endowment whenever appropriate.

[2]This means that she takes into account neither any higher moments, nor the entire probability distribution.

At date 0, an investor can choose any investment whose price does not exceed her monetary endowment. Due to monotonicity of utility, investor i will solely choose an investment $(x_0, \mathbf{x})$ that fulfills the budget constraint

$$e^i = x_0 + \langle \mathbf{p}, \mathbf{x} \rangle \,. \tag{3}$$

Holding a feasible investment $(x_0, \mathbf{x})$ at time 0 promises investor i a random future payoff

$$w^i(\mathbf{x}, \mathbf{p}) := r_0 x_0 + \langle \mathbf{q}, \mathbf{x} \rangle \overset{(3)}{=} r_0 e^i + \langle \mathbf{q} - r_0 \mathbf{p}, \mathbf{x} \rangle \in \mathscr{M} \tag{4}$$

at time 1. Since there are no borrowing constraints on the bond, any portfolio of risky assets $\mathbf{x} \in \mathbb{R}^K$ can be purchased.[3] Future wealth $w^i(\mathbf{x}, \mathbf{p})$ is the crucial figure for investor i. She wants to have achieve a future wealth as high as possible, while avoiding risk as much as possible.

Mean payoff, which actually influences utility, equals

$$\mathfrak{M}^i(\mathbf{x}, \mathbf{p}) := E\left[w^i(\mathbf{x}, \mathbf{p})\right] \overset{(4)}{=} r_0 e^i + \langle \boldsymbol{\pi}, \mathbf{x} \rangle \in \mathbb{R}. \tag{5}$$

The risk of future payoff, which is the second determinant of utility, will be introduced next.

1.2. Definition of Downside Risk

As motivated in the introduction, standard deviation is an insufficient risk measurement method, both from a theoretically point of view and when trying to validate the risk-taking behavior of investors empirically. For this reason, we associate risk with below-target payoffs as introduced by Fishburn (1977) in his landmark paper.

Fishburn (1977) established the following general form of a downside deviation risk measure for one-dimensional square integrable real random variables.

Definition 1.1 (Downside Risk of a Payoff).
Let $w \in \mathscr{M}$ be a random payoff and let F_w be its distribution function. Let further $t \in \mathbb{R}$ be a nonrandom *target payoff* which determines the threshold between gains and losses.
Then, the downside risk of w w.r.t. t is defined by

$$\mathcal{D}(w, t) := \sqrt{\int_{-\infty}^{t} (t - z)^2 \, F_w(dz)}. \tag{6}$$ □

The target payoff may be specified in three different ways.
Firstly, it may be a number, e.g., $t = 0$, with the interpretation that the investor is bankrupt, if she misses a target future payoff of 0.

[3] Since initial endowment e^i and riskless rate of return r_0 are given and investors are price takers, investor i merely chooses a portfolio of risky assets $\mathbf{x}$. Her bond holdings are then given by $x_0 = e^i - \langle \mathbf{p}, \mathbf{x} \rangle$.

Example 1.1 (The Borch Paradox with Downside Risk).
Consider again the two assets from Example 0.1. Asset 1 pays 25 € with probability 0.5 and -5 € with probability 0.5. Asset 2 pays 45 € with probability 0.5 and -5 € with probability 0.5. In terms of mean and standard deviation, these assets are characterized by $(\mu_1, \sigma_1) = (10, 15)$ and $(\mu_2, \sigma_2) = (20, 25)$.
When measuring risk by standard deviation, asset 2 is riskier. When using downside risk with $t = 0$ instead, asset 1 and asset 2 are equally risky, because for both assets a loss of 5 € occurs with probability 0.5. Their downside risk is then

$$\sqrt{5^2 \cdot 0.5} = \frac{5}{\sqrt{2}} \approx 3.5.$$

Hence, asset 2 strictly dominates asset 1 w.r.t. mean and downside risk because its mean is strictly greater and its downside risk is equal when compared to asset 1. A downside-risk averse investor, thus, prefers asset 2 and the Borch Paradox is resolved. □

Secondly, it may be a coefficient derived from the probability distribution of future payoff such as mean future payoff $t = E[w]$. This choice makes downside risk coincident with semideviation defined as the square root of the lower semivariance.
A third possibility is to choose a reference payoff derived from individual endowment, e.g., $t = r_0 e^i$. This choice can be motivated as follows. By investing her whole endowment e^i into the bond, investor i obtains $r_0 e^i$ at time 1 almost surely. By investing a part of her endowment in risky assets, she has the chance to get a higher payoff but faces the risk of getting less than $r_0 e^i$. In view of portfolio theory, $r_0 e^i$ may therefore be seen as the *natural target payoff.*

Formula (6) reveals that downside risk entirely depends on the loss distribution of future wealth. This *surplus invariance* may be formalized as $\mathcal{D}(w, t) = \mathcal{D}(\min\{w, t\}, t)$, meaning that realizations that yield a future wealth above its target are irrelevant for deriving downside risk. Hence, gains cannot compensate the risk stemming from losses. This is a fundamental difference to standard deviation which is obtained by averaging over all possible outcomes. Thus, downside risk is precisely tailored to measure what investors actually fear: losses.

Since investor i's future payoff depends on her portfolio choice $\mathbf{x}$, we introduce the notion of *downside risk of a portfolio.* It is defined as the square root of the mean squared difference between the portfolio payoff $\langle \mathbf{q}, \mathbf{x} \rangle$ and some target payoff $\langle \mathbf{t}, \mathbf{x} \rangle$, *which the portfolio payoff may miss.* Rewriting equation (6), using the well-known change-of-variable formula, e.g., see Ash (1972, Theorem 1.6.12), we obtain the following definition, where we employ $\mathfrak{D}(\mathbf{x}, \mathbf{t})$ as a shorthand notation for $\mathcal{D}(\langle \mathbf{q}, \mathbf{x} \rangle, \langle \mathbf{t}, \mathbf{x} \rangle)$.[4]

[4] We use the calligraphic $\mathcal{D}(w, t)$ to denote downside risk of a one-dimensional random variable w w.r.t. a target payoff $t \in \mathbb{R}$ and the Fraktur $\mathfrak{D}(\mathbf{x}, \mathbf{t})$ for the downside risk of a portfolio $\mathbf{x} \in \mathbb{R}^K$ w.r.t. a target vector $\mathbf{t} \in \mathbb{R}^K$. The same rule also applies to other notations like the mean.

Definition 1.2 (Downside Risk of a Portfolio).
Let $\mathbf{x} \in \mathbb{R}^K$ be a portfolio of risky assets and $\mathbf{t} \in \mathbb{R}^K$ a target vector for the payoff vector $\mathbf{q}$. Then, the downside risk of portfolio $\mathbf{x}$ w.r.t. $\mathbf{t}$ is defined by

$$\mathfrak{D}(\mathbf{x}, \mathbf{t}) := \sqrt{\int_\Omega \left(\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x} \rangle_+\right)^2 \mathbb{P}(d\omega)} = \sqrt{E\left[\left(\langle \mathbf{t} - \mathbf{q}, \mathbf{x} \rangle_+\right)^2\right]}. \tag{7}$$ □

The target vector $\mathbf{t}$ corresponds to the target payoff t in the following way. If $t = E[w]$, then $\mathbf{t} = E[\mathbf{q}]$, so that $\mathfrak{D}(\mathbf{x}, E[\mathbf{q}])$ becomes the semideviation of the portfolio payoff $\langle \mathbf{q}, \mathbf{x} \rangle$.
On the other hand, setting $\mathbf{t} = r_0\mathbf{p}$, the target payoff becomes $r_0 \langle \mathbf{p}, \mathbf{x} \rangle$ which is the amount the investor would have obtained if she had invested $\langle \mathbf{p}, \mathbf{x} \rangle$ prudently in the bond rather than buying the risky portfolio $\mathbf{x}$. With this choice, $\mathfrak{D}(\mathbf{x}, r_0\mathbf{p})$ becomes the below-target semideviation of $\langle \mathbf{q}, \mathbf{x} \rangle$ with target $r_0 \langle \mathbf{p}, \mathbf{x} \rangle$ and corresponds to $t = r_0 e^i$.
Most of the results developed in this thesis will hold true for all choices of $\mathbf{t}$. Some results, however, require a specific form of the target vector, which we will indicate whenever necessary.

Remark 1.1.
In terms of the L^2-norm, downside risk can be expressed as $\mathfrak{D}(\mathbf{x}, \mathbf{t}) = \left\| \langle \mathbf{t} - \mathbf{q}, \mathbf{x} \rangle_+ \right\|$.
Unlike standard deviation, downside risk is *not symmetric*, i.e., in general, $\mathfrak{D}(\mathbf{x}, \mathbf{t}) \neq \mathfrak{D}(-\mathbf{x}, \mathbf{t})$. □

Asymmetry is a favorable feature of downside risk. Since upward and downward movements of risky assets are, in general, not symmetric, purchasing a portfolio should pose a different risk than short selling the same portfolio. Downside risk is able to account for such asymmetric payoff distributions.

We close this section by stating the crucial assumption that no nontrivial portfolio is risk-free. This assumption will be in effect for the whole thesis, except in Chapter 3 where we will relax it. We will not state it explicitly in every assertion.

Assumption 1.
$\mathfrak{D}(\mathbf{x}, \mathbf{t}) > 0$ *whenever* $\mathbf{x} \neq 0$.

Assumption 1 is a simultaneous requirement on the target vector and on the payoff distribution. Given a fixed target, we restrict the probability distribution of the payoffs such that Assumption 1 holds or, vice versa, for a given distribution of payoffs, we allow only target vectors for which Assumption 1 is fulfilled. In Chapter 3, which is about equilibrium asset pricing, the target vector, e.g., for $\mathbf{t} = r_0\mathbf{p}$, may be determined by equilibrium asset prices with a priori no guarantee that Assumption 1 holds. In Chapter 3, we will therefore *prove* that, given a fixed payoff distribution, the equilibrium target vector (whether it is price-dependent or not) does not allow for risk-free nontrivial portfolios.
Assumption 1 implies that *no asset is redundant*, neither the bond nor any risky asset. Indeed,

it is not possible to exactly replicate an asset's payoff from the other assets. Suppose w.l.o.g. that asset 1 were redundant. Then, there exist $\alpha_2, ..., \alpha_K \in \mathbb{R}$ such that $q_1 = \sum_{k=2}^{K} \alpha_k q_k$. Since $\tilde{\mathbf{x}} := (-1, \alpha_2, ..., \alpha_K) \neq \mathbf{0}$ is then a portfolio with $\langle \mathbf{q}, \tilde{\mathbf{x}} \rangle = 0$, we have $\mathfrak{D}(\tilde{\mathbf{x}}, \mathbf{t}) = \langle \mathbf{t}, \tilde{\mathbf{x}} \rangle_+$.[5] As a consequence, $\mathfrak{D}(\tilde{\mathbf{x}}, \mathbf{t}) = 0$ or $\mathfrak{D}(-\tilde{\mathbf{x}}, \mathbf{t}) = 0$. This contradicts Assumption 1. By a similar argument, the risk-free asset is not redundant. Note that there do not have to exist redundant assets when Assumption 1 is not in effect.

1.3. Properties of Downside Risk

In this section, we establish important properties of downside risk and compare them with properties of other popular risk measures such as standard deviation and value-at-risk (VaR). It turns out that, for a suitable choice of the target vector, $\mathfrak{D}$ improves on these other measures of risk in the sense that it fulfills desirable properties such as monotonicity and subadditivity which are violated by standard deviation and VaR, respectively. Moreover, we provide an economic interpretation of the properties of downside risk and outline their significance for risk management.

We start with a central definition.

Definition 1.3 (Risky Set and Downside Cosemivariance Matrix).
The set of all realizations yielding a portfolio payoff below its target,

$$\mathscr{R}(\mathbf{x}, \mathbf{t}) := \{\omega \in \Omega | \, \langle \mathbf{t}, \mathbf{x} \rangle > \langle \mathbf{q}(\omega), \mathbf{x} \rangle\} \in \mathcal{F}, \tag{8}$$

is called *risky set* of portfolio $\mathbf{x}$.
The matrix $\mathbf{C}(\mathbf{x}, \mathbf{t})$, whose $(k, l)^{\text{th}}$ entry is defined as

$$C_{kl}(\mathbf{x}, \mathbf{t}) = \int_{\mathscr{R}(\mathbf{x}, \mathbf{t})} (t_k - q_k(\omega)) \, (t_l - q_l(\omega)) \, \mathbb{P}(d\omega), \tag{9}$$

is called *downside cosemivariance matrix* of $\mathbf{q}$ w.r.t. portfolio $\mathbf{x}$. □

Downside cosemivariance depends on portfolio $\mathbf{x}$, because the choice of the portfolio determines the risky set $\mathscr{R}(\mathbf{x}, \mathbf{t})$. In other words, by choosing a portfolio, the investor defines "upside" and "downside" risk.
The matrix $\mathbf{C}(\mathbf{x}, \mathbf{t})$ turns out to be symmetric and positive semidefinite. It thus becomes the downside analog of the covariance matrix V, whereby $\mathrm{C}_{kl}(\mathbf{x}, \mathbf{t})$ is the *downside cosemivariance* between the payoffs of assets k and l w.r.t. portfolio $\mathbf{x}$.

[5] Note that portfolio $\tilde{\mathbf{x}}$ which yields a constant payoff of 0 is not riskless if $\langle \mathbf{t}, \tilde{\mathbf{x}} \rangle_+ > 0$. This is a fundamental difference to standard deviation where a payoff is riskless if and only if it is constant. It is the reason why we cannot *replace* Assumption 1 by the assumption of non-redundancy.

Lemma 1.1 (Properties of the Downside Cosemivariance Matrix).
The downside cosemivariance matrix fulfills the following properties:

i) Downside risk may be rewritten as

$$\mathfrak{D}(\mathbf{x}, \mathbf{t}) = \sqrt{\langle \mathbf{x}, \mathbf{C}(\mathbf{x}, \mathbf{t})\mathbf{x} \rangle}. \tag{10}$$

ii) The matrix $\mathbf{C}(\mathbf{x}, \mathbf{t})$ is symmetric and positive semidefinite.

iii) $\mathbf{C}(\mathbf{x}, \mathbf{t})$ is invertible if and only if for any $\mathbf{z} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$:

$$\mathbb{P}(\{\omega \in \mathscr{R}(\mathbf{x}, \mathbf{t}) \mid \langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{z} \rangle \neq 0\}) > 0. \tag{11}$$

iv) $\mathbf{C}(\mathbf{x}, \mathbf{t})$ is null-homogeneous in $\mathbf{x}$, i.e., $\mathbf{C}(\lambda\mathbf{x}, \mathbf{t}) = \mathbf{C}(\mathbf{x}, \mathbf{t})$ for all $\lambda > 0$.

Proof.

i)

$$\begin{aligned}\langle \mathbf{x}, \mathbf{C}(\mathbf{x}, \mathbf{t})\mathbf{x} \rangle &= \int_{\mathscr{R}(\mathbf{x}, \mathbf{t})} \left\langle \mathbf{x}, (\mathbf{t} - \mathbf{q}(\omega))(\mathbf{t} - \mathbf{q}(\omega))^\top \mathbf{x} \right\rangle \, \mathbb{P}(d\omega) \\ &= \int_{\mathscr{R}(\mathbf{x}, \mathbf{t})} \langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x} \rangle^2 \, \mathbb{P}(d\omega) \overset{(8)}{=} \int_\Omega \left(\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x} \rangle_+ \right)^2 \, \mathbb{P}(d\omega).\end{aligned}$$

ii) Symmetry follows immediately from equation (9). Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^K$ be arbitrary. Then $\langle \mathbf{z}, \mathbf{C}(\mathbf{x}, \mathbf{t})\mathbf{z} \rangle = \int_{\mathscr{R}(\mathbf{x}, \mathbf{t})} \langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{z} \rangle^2 \, \mathbb{P}(d\omega) \geqslant 0$ implies positive semidefiniteness.

iii) Let $\mathbf{x} \in \mathbb{R}^K$ and $\mathbf{z} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ be arbitrary. $\langle \mathbf{z}, \mathbf{C}(\mathbf{x}, \mathbf{t})\mathbf{z} \rangle = \int_{\mathscr{R}(\mathbf{x}, \mathbf{t})} \langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{z} \rangle^2 \, \mathbb{P}(d\omega)$ is a strictly positive number if and only if $\mathbb{P}(\{\omega \in \mathscr{R}(\mathbf{x}, \mathbf{t}) \mid \langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{z} \rangle \neq 0\}) > 0$.

iv) Let $\lambda > 0$. Since

$$\langle \mathbf{t}, \lambda\mathbf{x} \rangle > \langle \mathbf{q}(\omega), \lambda\mathbf{x} \rangle \iff \langle \mathbf{t}, \mathbf{x} \rangle > \langle \mathbf{q}(\omega), \mathbf{x} \rangle \,,$$

it follows that $\mathscr{R}(\lambda\mathbf{x}, \mathbf{t}) = \mathscr{R}(\mathbf{x}, \mathbf{t})$. This implies $\mathbf{C}(\lambda\mathbf{x}, \mathbf{t}) = \mathbf{C}(\mathbf{x}, \mathbf{t})$. □

Property i) reveals the structural similarity of downside risk to standard deviation which is given by $\sigma(\mathbf{x}) = \sqrt{\langle \mathbf{x}, \mathbf{V}\mathbf{x} \rangle}$, where $\mathbf{V}$ is the covariance matrix of $\mathbf{q}$. Property ii) connotes that the downside cosemivariance matrix has the same algebraic properties as the covariance matrix $\mathbf{V}$. In mean-variance analysis, invertibility of $\mathbf{V}$ is always assumed. Property iii) gives a sufficient criterion for invertibility of $\mathbf{C}(\mathbf{x}, \mathbf{t})$. Null-homogeneity of $\mathbf{C}(\mathbf{x}, \mathbf{t})$ is a technical property which is used in many proofs in this thesis.

One of our main goals is to reformulate portfolio selection and asset pricing theory using downside risk rather than standard deviation. Therefore we show two things.
First, downside risk is constructed as a deviation measure and, thus, has a similar structure to standard deviation. It measures the deviations of the portfolio payoff $\langle \mathbf{q}, \mathbf{x} \rangle$ from the target payoff

$\langle \mathbf{t}, \mathbf{x} \rangle$ in case the target payoff is missed. The structural similarity between downside risk and standard deviation will enable us to solve a portfolio optimization problem in the sense of Markowitz (1952), to formulate a separation theorem and to derive an asset pricing formula.
Second, we establish that downside risk overcomes the shortcomings of standard deviation in terms of risk measurement. Although heavily used in finance to address problems involving risk, deviation measures are not risk measures in the sense of Artzner *et al.* (1999). As pointed out by Rockafellar *et al.* (2006a), deviation measures measure uncertainty, in the sense of nonconstancy, but do not evaluate the *"overall seriousness of possible losses"* as risk measures do. Downside risk is designed to overcome this vulnerability. It is a deviation measure by construction but the crucial point is that only deviations below a target, which constitute losses, contribute to downside risk. Thus, *downside risk* measures risky deviations which justifies its name.

1.3.1. $\mathfrak{D}(\mathbf{x}, \mathbf{t})$ as a Deviation Measure.

General deviation measures, as introduced by Rockafellar *et al.* (2006a), generalize the concept of standard deviation. Downside risk fulfills a set of properties which all general deviation measures have in common.

PROPOSITION 1.1 (Properties of $\mathfrak{D}(\mathbf{x}, \mathbf{t})$ as a Deviation Measure).
Downside risk $\mathfrak{D}(\mathbf{x}, \mathbf{t})$ *satisfies the following properties.*

i) Relevance*:*
$\mathfrak{D}(\mathbf{0}, \mathbf{t}) = 0$ *and* $\mathfrak{D}(\mathbf{x}, \mathbf{t}) > 0$ *for all* $\mathbf{x} \neq 0$.
ii) Positive Homogeneity*:*
for all $\lambda \geqslant 0$, $\mathfrak{D}(\lambda \mathbf{x}, \mathbf{t}) = \lambda \mathfrak{D}(\mathbf{x}, \mathbf{t})$.
iii) Subadditivity*:*
for any portfolios $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^K$, $\mathfrak{D}(\mathbf{x} + \mathbf{x}', \mathbf{t}) \leqslant \mathfrak{D}(\mathbf{x}, \mathbf{t}) + \mathfrak{D}(\mathbf{x}', \mathbf{t})$.
iv) $\mathfrak{D}(\mathbf{x}, \mathbf{t})$ *is finite everywhere and convex, hence also continuous in* $\mathbf{x}$.

PROOF.

i) $\mathfrak{D}(\mathbf{0}, \mathbf{t}) = 0$ is immediate. $\mathfrak{D}(\mathbf{x}, \mathbf{t}) > 0$ for $\mathbf{x} \neq 0$ is equivalent to Assumption 1.
ii) Positive homogeneity follows from null-homogeneity of $\mathbf{C}(\mathbf{x}, \mathbf{t})$, cf. Lemma 1.1.
iii) $\langle \mathbf{t} - \mathbf{q}, \mathbf{x} + \mathbf{x}' \rangle_+ \leqslant \langle \mathbf{t} - \mathbf{q}, \mathbf{x} \rangle_+ + \langle \mathbf{t} - \mathbf{q}, \mathbf{x}' \rangle_+$, together with the triangle inequality of the norm $\|\cdot\|$, implies subadditivity.
iv) $q_k \in \mathcal{L}^2(\Omega)$, for $k = 1, ..., K$ implies finiteness, properties ii) and iii) of Proposition 1.1 imply convexity. For the fact that finite convex functions on $\mathbb{R}^K$ are continuous see Rockafellar (1970, Theorem 10.1). □

The interpretation of Proposition 1.1 is straightforward. Property i) states that any nonzero portfolio poses a downside risk. Property ii) is a scaling property. It implies that, for instance, doubling the amount invested into a portfolio doubles its downside risk. Subadditivity ensures that diversification does not create extra risk. Indeed, the downside risk of the merged portfolio $\mathbf{x}+\mathbf{x}'$ does not exceed the sum of the downside risks of the single portfolios $\mathbf{x}$ and $\mathbf{x}'$. Observe that VaR – a heavily used risk measure based on quantiles – is not subadditive, e.g., see Artzner *et al.* (1999, Section 3.3). In this sense, downside risk has an advantage over VaR. Eventually, positive homogeneity and subadditivity imply convexity of downside risk.

1.3.2. $\mathfrak{D}(\mathbf{x},\mathbf{t})$ as a Measure of Risk.

The following Proposition underpins the ability of $\mathfrak{D}(\mathbf{x},\mathbf{t})$ to measure risk.

Proposition 1.2 (Properties of $\mathfrak{D}(\mathbf{x},\mathbf{t})$ as a Downside Risk Measure).
Downside risk $\mathfrak{D}(\mathbf{x},\mathbf{t})$ *fulfills*

i) Lower Range Dominance:
$\mathfrak{D}(\mathbf{x},\mathbf{t}) \leqslant \langle \mathbf{t},\mathbf{x}\rangle - \inf_{\omega\in\Omega} \langle \mathbf{q}(\omega),\mathbf{x}\rangle$.
ii) Monotonicity in $\mathbf{t}$:
$x_k > 0 \Longrightarrow \frac{\partial}{\partial t_k}\mathfrak{D}(\mathbf{x},\mathbf{t}) > 0$.
iii) If, in addition, $\mathbf{t} = r_0\mathbf{p}$, *then* $\mathfrak{D}$ is monotone in $\mathbf{x}$ *in the sense that*
$w^i(\mathbf{x},\mathbf{p}) \geqslant w^i(\mathbf{x}',\mathbf{p}) \Longrightarrow \mathfrak{D}(\mathbf{x},r_0\mathbf{p}) \leqslant \mathfrak{D}(\mathbf{x}',r_0\mathbf{p})$.

Proof.

i) For any $\tilde\omega \in \Omega : \langle \mathbf{t},\mathbf{x}\rangle - \inf_{\omega\in\Omega} \langle \mathbf{q}(\omega),\mathbf{x}\rangle \geqslant \langle \mathbf{t}-\mathbf{q}(\tilde\omega),\mathbf{x}\rangle_+ \geqslant 0$, due to Assumption 1.
ii) Follows from Proposition 1.3 given in the appendix of this chapter.
iii) Let $\mathbf{x},\mathbf{x}' \in \mathbb{R}^K$ be two portfolios with $w^i(\mathbf{x},\mathbf{p}) \geqslant w^i(\mathbf{x}',\mathbf{p})$. Then, equation (4) yields $\langle r_0\mathbf{p}-\mathbf{q},\mathbf{x}\rangle \leqslant \langle r_0\mathbf{p}-\mathbf{q},\mathbf{x}'\rangle$, implying $\langle r_0\mathbf{p}-\mathbf{q},\mathbf{x}\rangle_+ \leqslant \langle r_0\mathbf{p}-\mathbf{q},\mathbf{x}'\rangle_+$. Consequently, $\left\|\langle r_0\mathbf{p}-\mathbf{q},\mathbf{x}\rangle_+\right\| \leqslant \left\|\langle r_0\mathbf{p}-\mathbf{q},\mathbf{x}'\rangle_+\right\|$. □

Property i) ensures that the downside risk is bounded by the largest possible below-target deviation, which constitutes the worst case. A deviation measure not fulfilling lower range dominance is not reasonable for measuring risk, because it overestimates the risk inherent in a portfolio. Since payoffs cannot be less than $\inf_{\omega\in\Omega} \langle \mathbf{q}(\omega),\mathbf{x}\rangle$, losses cannot be greater than $\langle \mathbf{t},\mathbf{x}\rangle - \inf_{\omega\in\Omega} \langle \mathbf{q}(\omega),\mathbf{x}\rangle$. Standard deviation, which takes into account upside deviations as well as downside deviations, is not dominated by the lower range. Thus, it may assign unreasonably high risk to a financial position.

Monotonicity in the entries of $\mathbf{t}$ has an intuitive interpretation if the target vector is chosen to be $r_0\mathbf{p}$. If an investor wants to buy asset k and its price p_k increases, *ceteris paribus*, the portfolio becomes more expensive. At the same time it becomes more risky because its target payoff $\langle r_0\mathbf{p},\mathbf{x}\rangle$ increases and its realized random payoff $\langle \mathbf{q},\mathbf{x}\rangle$ is more likely to miss this target, given a fixed probability distribution of $\mathbf{q}$. Conversely, for an investor who wants to sell asset k short, an increase in its price

reduces downside risk of the portfolio. By a similar argument, an increasing riskless rate of return r_0 increases downside risk. On the contrary, a decreasing risk-free rate of return (e.g., interest rate) makes an investment in risky assets more attractive, not only because they offer a higher expected excess return but also because their downside risk decreases.
In our view, Property iii) is the most important property when measuring risk, i.e., evaluating prospective losses. If a portfolio $\mathbf{x}$ yields a higher future payoff in any state of nature, its losses are always smaller and its profits are always higher. In particular, its ruin probability is lower: $w^i(\mathbf{x},\mathbf{p}) \geqslant w^i(\mathbf{x}',\mathbf{p})$ implies $\mathbb{P}(w^i(\mathbf{x},\mathbf{p}) \leqslant 0) \leqslant \mathbb{P}(w^i(\mathbf{x}',\mathbf{p}) \leqslant 0)$. Hence, the portfolio $\mathbf{x}$ is less risky compared to portfolio $\mathbf{x}'$. Downside risk accounts for this monotonicity relation whereas standard deviation does not.

We conclude that downside risk with target vector $\mathbf{t} = r_0\mathbf{p}$, which is a specific below-target semideviation, is an excellent choice. It fulfills all important properties of a plausible deviation risk measure. Most importantly, to the best of our knowledge, it is the *only monotone deviation measure*. Thus, it unambiguously outperforms standard deviation.

1.A. Appendix to Chapter 1

LEMMA 1.2.
Let $\mathbf{a},\mathbf{x} \in \mathbb{R}^K$ be arbitrary. Then,

$$\nabla_{\mathbf{x}} \left(\langle \mathbf{a},\mathbf{x}\rangle_+\right)^2 = 2\,\langle \mathbf{a},\mathbf{x}\rangle_+\,\mathbf{a}.$$

PROOF. Let $\mathbf{x}' \in \mathbb{R}^K$ be arbitrary. We consider 3 cases:

i) $\langle \mathbf{a},\mathbf{x}'\rangle > 0$. Then, $\langle \mathbf{a},\mathbf{x}\rangle_+ = \langle \mathbf{a},\mathbf{x}\rangle > 0$ for all $\mathbf{x}$ in the vicinity of $\mathbf{x}'$ and thus

$$\nabla_{\mathbf{x}} \left(\langle \mathbf{a},\mathbf{x}'\rangle_+\right)^2 = \nabla_{\mathbf{x}} \langle \mathbf{a},\mathbf{x}'\rangle^2 = 2\,\langle \mathbf{a},\mathbf{x}'\rangle\,\mathbf{a} = 2\,\langle \mathbf{a},\mathbf{x}'\rangle_+\,\mathbf{a}.$$

ii) $\langle \mathbf{a},\mathbf{x}'\rangle < 0$. Then, $\langle \mathbf{a},\mathbf{x}\rangle_+ = 0$ for all $\mathbf{x}$ in the vicinity of $\mathbf{x}'$ and thus

$$\nabla_{\mathbf{x}} \left(\langle \mathbf{a},\mathbf{x}'\rangle_+\right)^2 = \nabla_{\mathbf{x}} 0 = 0 = 2\,\langle \mathbf{a},\mathbf{x}'\rangle_+\,\mathbf{a}.$$

iii) $\langle \mathbf{a},\mathbf{x}'\rangle = 0$. Then, let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence with $\langle \mathbf{a},\mathbf{x}_n\rangle \neq 0$ and $\lim_{n\to\infty}\mathbf{x}_n = \mathbf{x}'$. Then,

$$\lim_{n\to\infty} \nabla_{\mathbf{x}} \left(\langle \mathbf{a},\mathbf{x}_n\rangle_+\right)^2 \overset{\text{i), ii)}}{=} \lim_{n\to\infty} 2\,\langle \mathbf{a},\mathbf{x}_n\rangle_+\,\mathbf{a} = 2\,\langle \mathbf{a},\mathbf{x}'\rangle_+\,\mathbf{a}.$$

Thus, we have differentiability everywhere and the derivative is continuous. □

PROPOSITION 1.3.
Let $x_k > 0$. Then,

$$\frac{\partial}{\partial t_k}\mathfrak{D}(\mathbf{x},\mathbf{t}) = \frac{x_k}{\mathfrak{D}(\mathbf{x},\mathbf{t})}\int_\Omega \langle \mathbf{t}-\mathbf{q}(\omega),\mathbf{x}\rangle_+\,\mathbb{P}(d\omega).$$

PROOF. Since $x_k > 0 \Rightarrow \mathbf{x} \neq \mathbf{0}$, we can divide by $\mathfrak{D}(\mathbf{x}, \mathbf{t}) > 0$ and the chain rule implies

$$\frac{\partial}{\partial t_k}\mathfrak{D}(\mathbf{x}, \mathbf{t}) = \frac{1}{2\mathfrak{D}(\mathbf{x}, \mathbf{t})}\frac{\partial}{\partial t_k}\int_\Omega \left(\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x}\rangle_+\right)^2 \, \mathbb{P}(d\omega).$$

The integral $\int_\Omega \left(\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x}\rangle_+\right)^2 \, \mathbb{P}(d\omega)$ is continuously differentiable w.r.t. t_k and we can exchange differentiation and integration, compare, e.g., Schilling (2005, Theorem 11.5). Lemma 1.2 implies the assertion. □

CHAPTER 2

Portfolio Selection Theory

In this chapter, we analyze how a downside-risk averse investor selects her individually optimal portfolio. To do so, we formulate a portfolio optimization problem where we assume that preferences of an investor are characterized by a utility function depending on mean payoff and downside risk only, see (12). In the sequel, we compare this ansatz with other portfolio selection criteria. Next, the notion of downside efficient portfolios is introduced. These are optimally downside diversified portfolios and analogs to mean-variance efficient portfolios in the sense of Markowitz (1952). In the subsequent section, all attainable risk-return profiles are characterized. Moreover, a separation theorem is formulated and proven. Eventually, we address the problem of portfolio selection without a risk-free asset.

To improve readability, the following simplified notation will henceforth be in effect. We omit the upper index i since the investment decision of an arbitrary investor is analyzed. Her beliefs about the stock market are captured via the probability distribution $\mathbb{P}^i = \mathbb{P}$.
Asset prices are assumed to be parametrically fixed, thus $\mathbf{p}$ will not be denoted explicitly. So, given prices, an investor, equipped with initial endowment $e \in \mathbb{R}$, chooses a portfolio of risky assets $\mathbf{x}$ which yields a random future payoff $w(\mathbf{x}) \in \mathscr{M}$ with mean $\mathfrak{M}(\mathbf{x})$.
All results of this chapter hold independently of the specific choice of a target vector. Thus, we assume $\mathbf{t}$ to be arbitrary but fixed and analyze an investor's portfolio choice *given* that target. We skip "$\mathbf{t}$" throughout this chapter and employ the notation $\mathfrak{D}(\mathbf{x}), \mathbf{C}(\mathbf{x}), \mathscr{R}(\mathbf{x})$.

As outlined in Section 1.1.2, an investor's preferences are characterized by a utility function U which is a function of mean and downside risk of future payoff. In Chapters 2 and 3, we require the utility function to fulfill the following properties.

ASSUMPTION 2.
$U : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}, (m, d) \longmapsto U(m, d)$ *is continuously differentiable, strictly increasing in mean, strictly decreasing in downside risk and strictly concave.*[6]

Within this setting, the decision problem of an investor takes the form

$$\max_{\mathbf{x} \in \mathbb{R}^K} U(\mathfrak{M}(\mathbf{x}), \mathfrak{D}(\mathbf{x})). \tag{12}$$

[6] The presumption of strict concavity can be mitigated to strict quasiconcavity at the expense of technicalities.

2.1. Portfolio Selection Criteria

Considering the utility induced by mean and downside risk is just one possibility to evaluate portfolios (or their payoffs). In this section, we show how the mean-downside-risk framework is related to concepts like stochastic dominance and expected utility theory.

We analyze the interplay between different portfolio selection criteria in the marketed subspace $\mathscr{M}$. For a payoff $w \in \mathscr{M}$ with distribution function F_w, we denote its mean by $\mathcal{M}(w) = E[w]$, its downside risk by $\mathcal{D}(w) = \sqrt{\int_{-\infty}^{t} (t-z)^2 \, F_w(dz)}$ and its expected utility by $EU(w) := \int_{-\infty}^{\infty} u(z) \, F_w(dz)$, where $u \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a Bernoulli utility function. This representation dates back to von Neumann & Morgenstern (1947). Newer approaches to evaluate random payoffs are prospect theory, developed by Kahneman & Tversky (1979) and refined in 1992, as well as the recent works by Kőszegi & Rabin (2006, 2007) who elaborated on reference-dependent preferences and risk attitudes.

As a first step, several partial order relations on $\mathscr{M}$ are defined.

Definition 2.1 (Partial Order Relations on $\mathscr{M}$).
Let $w, w' \in \mathscr{M}$ and let $F_w, F_{w'}$ be their distribution functions.

i) Almost sure dominance (or statewise dominance) is defined via

$$w \succeq_{AS} w' :\Longleftrightarrow w(\omega) \geqslant w'(\omega) \text{ for all } \omega \in \Omega.$$

ii) M-D dominance is defined via

$$w \succeq_{MD} w' :\Longleftrightarrow \mathcal{M}(w) \geqslant \mathcal{M}(w') \text{ and } \mathcal{D}(w) \leqslant \mathcal{D}(w').$$

iii) Stochastic dominance of first degree is defined via

$$w \succeq_{FS} w' :\Longleftrightarrow F_w(z) \leqslant F_{w'}(z) \text{ for all } z \in \mathbb{R}.$$

iv) Stochastic dominance of second degree is defined via

$$w \succeq_{SS} w' :\Longleftrightarrow \int_{-\infty}^{z} F_w(y) \, dy \leqslant \int_{-\infty}^{z} F_{w'}(y) \, dy \text{ for all } z \in \mathbb{R}.$$

v) Stochastic dominance of third degree is defined via

$$w \succeq_{TS} w' :\Longleftrightarrow \int_{-\infty}^{z} \left(\int_{-\infty}^{y} F_w(x) \, dx \right) dy \leqslant \int_{-\infty}^{z} \left(\int_{-\infty}^{y} F_{w'}(x) \, dx \right) dy \text{ for all } z \in \mathbb{R} \text{ and } \mathcal{M}(w) \geqslant \mathcal{M}(w').$$

□

Strict dominance is defined when at least one equation on the right-hand side in the above definitions is strict, respectively. It is possible to define stochastic dominance of even higher degrees, but we restrict our analysis to the first three degrees.

In another line of research, Jean (1971, 1972, 1973), Ingersoll (1975) and Schweser (1978), among others, extended Markowitz's mean-variance decision model by adding the third moment of a payoff. They formulated a multidimensional optimization problem in which the expected value μ and skewness ν of a payoff is maximized, while its variance σ^2 is minimized. More formally, for two payoffs $w, w' \in \mathscr{M}$ with $(\mu, \sigma, \nu), (\mu', \sigma', \nu')$, their relation is defined via

$$w \succeq_{\mu\sigma\nu} w' :\Longleftrightarrow \mu \geqslant \mu',\, \nu \geqslant \nu' \,\text{and}\, \sigma \leqslant \sigma'.$$

The problem with this setting is that much of the results are confined to a qualitative analysis. It is generally impossible, for example, to derive an explicit representation of the set of undominated portfolios. This is a reason why "$\mu - \sigma - \nu$ dominance" did not become prevalent and why we do not focus on this relation, here.

In the following definition, we specify when an order relation is stronger than another, when two order relations are congruent and we formally define the notion of an efficient set.

DEFINITION 2.2.
Let $\succeq_A, \succeq_B$ be two order relations on $\mathscr{M}$.
We say that $\succeq_A$ *implies* $\succeq_B$ if for any $w, w' \in \mathscr{M}$ with $w \succeq_A w'$ the relation $w \succeq_B w'$ holds. Relation $\succeq_A$ can then be interpreted as a stronger relation than $\succeq_B$.
Two order relations $\succeq_A$ and $\succeq_B$ will be called *congruent* if $\succeq_A$ implies $\succeq_B$ and $\succeq_B$ implies $\succeq_A$.
The *efficient set w.r.t.* $\succeq_A$ (or A efficient set) is defined as the set of all payoffs that are strictly undominated w.r.t $\succeq_A$, i.e., the set $\{w \in \mathscr{M} \mid \nexists w' \in \mathscr{M} : w' \succ_A w\}$. □

Obviously, almost sure dominance implies stochastic dominance of either degree and $\succeq_{FS}$ implies $\succeq_{SS}$ implies $\succeq_{TS}$.

Less obviously, stochastic dominance (of either first, second, or third degree) implies M-D dominance. This result is an immediate corollary of Fishburn (1977, Theorem 3). He refined a similar result by Porter (1974) who showed the assertion for second degree stochastic dominance and semi-variance. As a consequence, the M-D efficient set is a subset of the stochastic dominance efficient set.

In contrast, stochastic dominance does not imply mean-variance dominance, when defined analogously to M-D dominance. Example 0.1 in the introduction provides an illustration for two random variables where one strictly dominates the other in terms of stochastic dominance but not w.r.t. mean and variance. This is one of many points of criticism concerning portfolio selection and asset pricing theory in a mean-variance framework.

Almost sure dominance, which is the strongest of the five orders defined above, also implies M-D dominance. Indeed, $\succeq_{AS}$ implies $\succeq_{FS}$ implies $\succeq_{MD}$. But, almost sure dominance does not imply mean-variance dominance as shown in Example 0.2. Even a random variable that yields a higher payoff in any state of nature is not necessarily preferred by an investor who considers mean and variance only. This fact should be a solemn warning to all theorists and practitioners who apply mean-variance analysis and the CAPM unhesitatingly.

To complete the analysis, we introduce 2 total order relations.

DEFINITION 2.3 (Total Order Relations on $\mathscr{M}$).
Let $w, w' \in \mathscr{M}$, let $F_w, F_{w'}$ be their distribution functions and let U satisfy Assumption 2.

i) M-D utility dominance is defined via

$$w \succeq_{MDU} w' :\Longleftrightarrow U(\mathcal{M}(w), \mathcal{D}(w)) \geqslant U(\mathcal{M}(w'), \mathcal{D}(w')).$$

ii) Expected utility dominance is defined via

$$w \succeq_{EU} w' :\Longleftrightarrow EU(w) \geqslant EU(w').$$

□

Due to monotonicity properties of U, which are given in Assumption 2, M-D dominance implies M-D utility dominance.

Further, it is well-known that stochastic dominance implies expected utility dominance.

LEMMA 2.1 (Stochastic Dominance Implies Expected Utility Dominance).
Let $w, w' \in \mathscr{M}$ and presume sufficient differentiability of u.

i) If $w \succeq_{FS} w'$, then $w \succeq_{EU} w'$ for every Bernoulli utility function with $u' \geqslant 0$.
ii) If $w \succeq_{SS} w'$, then $w \succeq_{EU} w'$ for every Bernoulli utility function with $u' \geqslant 0$ and $u'' \leqslant 0$.
iii) If $w \succeq_{TS} w'$, then $w \succeq_{EU} w'$ for every Bernoulli utility function with $u' \geqslant 0, u'' \leqslant 0$ and $u''' \geqslant 0$.

PROOF. Lemma 1 in Fishburn (1977). □

Now, we analyze the interplay between expected utility theory and mean-downside-risk utility functions satisfying Assumption 2. As pointed out by Fishburn (1977, p. 120): "It is entirely possible that a decision maker's preferences satisfy a mean-risk utility model without also satisfying the von Neumann and Morgenstern axioms for expected utility". Therefore, we specify conditions under which M-D utility dominance is congruent with expected utility dominance.

PROPOSITION 2.1.
Assume $\succeq_{MDU}$ and $\succeq_{EU}$ are congruent. Then, the Bernoulli utility function, which is unique up to positive linear transformations, function takes the form

$$u^{\mathsf{dq}}(z) := \begin{cases} z & \textit{for all}\, z \geqslant t \\ z - \alpha(t-z)^2 & \textit{for all}\, z \leqslant t \end{cases} \tag{13}$$

for some $\alpha > 0$.

PROOF. Theorem 2 in Fishburn (1977). □

The coefficient α describes the investor's downside-risk aversion. Figure 2, which is based on Fishburn (1977, Figure 1), displays the behavior of the *downside quadratic* Bernoulli utility function u^{dq}. For payoff realizations above the target t, the investor is downside-risk neutral – u^{dq} is linear for $z \geqslant t$ – and for below-target realizations she is downside-risk averse, i.e., u^{dq} is strictly concave for $z \leqslant t$. The Bernoulli utility function given in (13) can be considered as downside version of a quadratic Bernoulli utility function

$$u^{\mathsf{q}}(z) := z - \alpha z^2 \tag{14}$$

which makes a mean-variance efficient portfolio coincident with an expected utility maximizing portfolio.

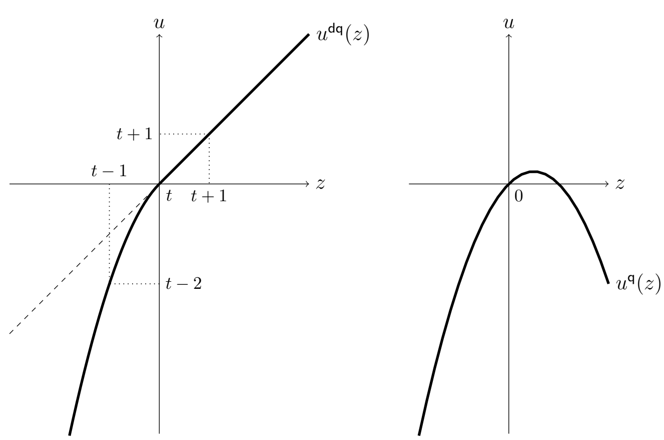

FIGURE 2. Bernoulli utility functions with $\alpha = 1$. Left: u^{dq} given by (13). Right: u^{q} given by (14).

A major drawback of quadratic utility given by (14) is that it exhibits negative marginal utility beyond a point of personal satiation, thereby penalizing too high payoffs, compare Johnstone & Lindley (2013). In contrast, the downside quadratic Bernoulli utility function is strictly monotonically increasing. Thus, u^{dq} models the behavior of an investor who, on the one hand, is risk averse if the payoff constitutes a loss and, on the other hand, prefers high payoffs over low payoffs. It therefore keeps the strength of quadratic Bernoulli utility functions (their ability to model risk aversion) while overcoming their weakness of non-monotonicity.

Presuming $u(z) = u^{\mathsf{dq}}(z)$, expected utility solely depends on mean and downside risk

$$(15) \quad EU(w) = \int_{-\infty}^{\infty} u(z)\, F_w(dz) \overset{(13)}{=} \int_{-\infty}^{\infty} z\, F_w(dz) - \alpha \int_{-\infty}^{t} (t-z)^2\, F_w(dz) = \mathcal{M}(w) - \alpha \mathcal{D}(w)^2.$$

Hence, the expected utility functional with Bernoulli utility function u^{dq} becomes equivalent to the M-D utility function $U(m,d) = m - \alpha d^2$ which satisfies Assumption 2. Figure 3 illustrates the resulting indifference curves which turn out do be quadratic in the (m,d)-plane.

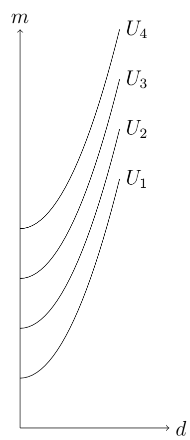

Figure 3. Indifference curves for $\alpha = 1$.

The following corollary is the reverse statement of Proposition 2.1.

Corollary 2.1.
Let $\alpha > 0$ be given and set $u = u^{\mathsf{dq}}$, where u^{dq} is defined by (13), as well as $U(m,d) = m - \alpha d^2$. Then, $\succeq_{MDU}$ and $\succeq_{EU}$ are congruent.

Proof. For $\alpha > 0$, $u = u^{\mathsf{dq}}$ and $U(m,d) = m - \alpha d^2$, we obtain congruence since

$$EU(w) = \mathcal{M}(w) - \alpha \mathcal{D}(w)^2 = U(\mathcal{M}(w), \mathcal{D}(w)).$$

□

The following Theorem summarizes the findings of this section and their implications to portfolio selection theory. It thereby justifies that we analyze an investor who solves $\max_{\mathbf{x}\in\mathbb{R}^K} U(\mathfrak{M}(\mathbf{x}), \mathfrak{D}(\mathbf{x}))$ for a utility function U that fulfills Assumption 2.

Theorem 2.1.
Let U fulfill Assumption 2, let $\mathbf{x}^* \in \mathbb{R}^K$ be a solution to portfolio optimization problem (12) and let $w(\mathbf{x}^*) \in \mathscr{M}$ be its payoff.

i) $w(\mathbf{x}^*)$ is undominated w.r.t. $\succeq_{MDU}$ by construction and thus undominated w.r.t. $\succeq_{MD}$ and w.r.t. $\succeq_{AS}$.
ii) $w(\mathbf{x}^*)$ is undominated in the sense of stochastic dominance of either degree.
iii) If $U(m,d) = m - \alpha d^2$, then $w(\mathbf{x}^*)$ also maximizes expected utility for the downside quadratic Bernoulli utility function given by (13).

As a consequence, $w(\mathbf{x}^*)$ improves on the payoff of a mean-variance efficient portfolio $w(\mathbf{x}^{\mathsf{eff}})$, because $w(\mathbf{x}^{\mathsf{eff}})$ is itself neither undominated w.r.t $\succeq_{AS}$ nor w.r.t. stochastic dominance. Further, $w(\mathbf{x}^{\mathsf{eff}})$ can maximize expected utility only for a quadratic, thus non-monotonic, Bernoulli utility function. In contrast, the downside quadratic Bernoulli utility function (13) is strictly monotonically increasing and concave.

A solution $\mathbf{x}^*$ to (12) yields a payoff that is optimal in terms of mean and downside risk as well as undominated w.r.t $\succeq_{AS}$ and w.r.t. stochastic dominance. Thus we call $\mathbf{x}^*$ not only M-D optimal but the *individually optimal portfolio.*

2.2. Downside Efficient Portfolios

How should an investor diversify a portfolio of risky assets optimally w.r.t. mean and downside risk? Markowitz (1952) was the first to address this problem in the mean-variance framework, when he introduced the notion of mean-variance efficient portfolios. He chose variance as a measure of risk because it is unproblematic to aggregate variances of single assets' payoffs to the variance of a portfolio's payoff. Almost 60 years ago, Markowitz (1959, pp. 188-194 and pp. 287-297) himself already had reservations about variance as a measure of risk. There, he proposed five alternative risk measures, one of which was below-target semivariance, and analyzed a model with semivariances in Chapter 9. But it took nearly five decades until existence of mean-semivariance efficient portfolios could be established by Jin *et al.* (2006). This existence result, however, is rather abstract and an analytically rigorous derivation of a downside analog of a mean-variance efficient portfolio is missing.

The difficulty is to aggregate downside deviation measures for a portfolio of risky assets. As pointed out by Grootveld & Hallerbach (1999), so far nobody succeeded in aggregating downside risks.

However, the representation of downside risk via the downside cosemivariance matrix, see (10), will enable us to aggregate downside risk and to derive downside efficient portfolios.

DEFINITION 2.4 (Downside Efficient Portfolios).
A solution to the optimization problem

$$\max_{\mathbf{x}\in\mathbb{R}^K} \mathfrak{M}(\mathbf{x}) \text{ s.t. } \mathfrak{D}(\mathbf{x}) \leqslant d_0, \tag{16}$$

given some prescribed level of downside risk $d_0 \geqslant 0$, is called *downside efficient portfolio* and denoted by $\mathbf{x}^{\mathsf{de}}(d_0)$.[7] □

We assume $\boldsymbol{\pi} \neq \mathbf{0}$ throughout this section. If $\boldsymbol{\pi} = E[\mathbf{q}] - r_0\mathbf{p} = \mathbf{0}$, then the objective function of (16) is constant, $\mathfrak{M}(\mathbf{x}) = r_0 e$, compare (5). Hence, *every feasible* solution, e.g., $\mathbf{x} = \mathbf{0}$, is optimal. In this case, downside efficient portfolios are not unique.

As a first step, we define and characterize the feasible set and the optimal solution set of (16) which will be referred to as the set of all downside efficient portfolios.

DEFINITION 2.5.
Given $d_0 \geqslant 0$, we denote the *set of all feasible portfolios* of optimization problem (16) by

$$\mathscr{K}(d_0) := \left\{\mathbf{x} \in \mathbb{R}^K \mid \mathfrak{D}(\mathbf{x}) \leqslant d_0\right\} \tag{17}$$

and the *set of all downside efficient portfolios* by

$$\mathscr{S}(d_0) := \left\{\mathbf{x} \in \mathscr{K}(d_0) \mid \mathfrak{M}(\mathbf{x}) = \max_{\mathbf{z}\in\mathscr{K}(d_0)} \mathfrak{M}(\mathbf{z})\right\}. \tag{18}$$ □

Our first result concerns the existence of feasible as well as downside efficient portfolios.

THEOREM 2.2 (Existence of Downside Efficient Portfolios).
The sets (17) and (18) are nonempty, convex and compact. Moreover, both sets are positively homogeneous w.r.t. d:

$$\mathscr{K}(d) = \{d\mathbf{x} \mid \mathbf{x} \in \mathscr{K}(1)\}, \, \mathscr{S}(d) = \{d\mathbf{x} \mid \mathbf{x} \in \mathscr{S}(1)\}.$$

PROOF. Obviously, $\mathbf{0} \in \mathscr{K}(d)$ for all $d \geqslant 0$. Convexity of $\mathscr{K}(d)$ follows from convexity of $\mathfrak{D}$. The set $\mathscr{K}(d)$ is closed, since it is the inverse image of the closed set $[0, d]$ under the continuous function $\mathfrak{D}$. On the principle of Rockafellar (1970, Corollary 8.7.1) and $\mathfrak{D}$ being convex and continuous, if any set of the form $\left\{\mathbf{x} \in \mathbb{R}^K | \mathfrak{D}(\mathbf{x}) \leqslant d\right\}$ is bounded, then all such sets must be bounded. By Proposition 1.1 i), $\mathscr{K}(0) = \{\mathbf{0}\}$. Since $\mathscr{K}(d) \subset \mathbb{R}^K$, closedness and boundedness imply compactness. Positive homogeneity follows from positive homogeneity of $\mathfrak{D}$.

[7]Observe that the case $d_0 = 0$ is trivial as $\mathfrak{D}(\mathbf{x}) = 0 \iff \mathbf{x} = 0$. Alternatively, we also could have considered the optimization problem $\min_{\mathbf{x}\in\mathbb{R}^K} \mathfrak{D}(\mathbf{x})$ s.t. $\mathfrak{M}(\mathbf{x}) \geqslant m_0$ for some $m_0 \geqslant r_0 e$.

Since the feasible set $\mathscr{K}(d)$ is nonempty and compact and $\mathfrak{M}(\mathbf{x})$ is a continuous function, there exists $\mathbf{x}^{\mathsf{de}}(d)$, such that

$$\mathfrak{M}(\mathbf{x}^{\mathsf{de}}(d)) = \max_{\mathbf{z}\in\mathscr{K}(d)} \mathfrak{M}(\mathbf{z}).$$

Any set of type $\{\mathbf{x} \in \mathscr{K}(d) | \langle \boldsymbol{\pi}, \mathbf{x}\rangle \geqslant m\}$ is convex and compact due to convexity and compactness of $\mathscr{K}(d)$ and linearity of $\langle \boldsymbol{\pi}, \mathbf{x}\rangle$. Hence, the set

$$\mathscr{S}(d) = \left\{ \mathbf{x} \in \mathscr{K}(d) \mid \langle \boldsymbol{\pi}, \mathbf{x}\rangle \geqslant \max_{\mathbf{z}\in\mathscr{K}(d)} \langle \boldsymbol{\pi}, \mathbf{z}\rangle \right\}$$

is convex and compact. Let $\mathbf{x}^{\mathsf{de}}(1) \in \mathscr{S}(1)$ and suppose that $\mathbf{x}' \in \mathscr{K}(d)$ with $\langle \boldsymbol{\pi}, \mathbf{x}'\rangle > d \left\langle \boldsymbol{\pi}, \mathbf{x}^{\mathsf{de}}(1)\right\rangle$ existed. Then, $\frac{1}{d}\mathbf{x}' \in \mathscr{K}(1)$ and $\frac{1}{d}\langle \boldsymbol{\pi}, \mathbf{x}'\rangle > \left\langle \boldsymbol{\pi}, \mathbf{x}^{\mathsf{de}}(1)\right\rangle$, which contradicts $\mathbf{x}^{\mathsf{de}}(1) \in \mathscr{S}(1)$. □

Theorem 2.2 establishes existence of downside efficient portfolios. Similar results are derived by Jin *et al.* (2006) for semivariance and by Rockafellar *et al.* (2006b) for general deviation measures. We go one step further by providing a specific representation for downside efficient portfolios, which, to the best of our knowledge, has not been obtained in literature so far.

Proposition 2.2 (Characterization of Downside Efficient Portfolios).
Let $\boldsymbol{\pi} \neq \mathbf{0}$. Then, the following applies.

i) Any $\mathbf{x}^{\mathsf{de}}(d_0)$ is of the form

$$\mathbf{x}^{\mathsf{de}}(d_0) = \frac{d_0}{\mathfrak{D}(\mathbf{x}^{\mathsf{ref}})}\mathbf{x}^{\mathsf{ref}}, \tag{19}$$

where $\mathbf{x}^{\mathsf{ref}} \neq \mathbf{0}$ is a solution to the equation $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$.

ii) Any solution $\mathbf{x}^{\mathsf{ref}}$ to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$ yields a downside efficient portfolio via (19).

iii) If $\mathbf{x}^{\mathsf{ref}}_{\mathbf{1}}$ and $\mathbf{x}^{\mathsf{ref}}_{\mathbf{2}}$ are two solutions to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$, then their downside risks and means are equal, i.e., $\mathfrak{D}(\mathbf{x}^{\mathsf{ref}}_{\mathbf{1}}) = \mathfrak{D}(\mathbf{x}^{\mathsf{ref}}_{\mathbf{2}})$ and $\mathfrak{M}(\mathbf{x}^{\mathsf{ref}}_{\mathbf{1}}) = \mathfrak{M}(\mathbf{x}^{\mathsf{ref}}_{\mathbf{2}})$.

Proof.

i) Observe first that $\mathbf{x}^{\mathsf{de}}(0) = \mathbf{0}$, because it is the only feasible portfolio. Let $d_0 > 0$. Then, each maximizer $\mathbf{x}^{\mathsf{de}}(d_0)$ is different from zero, because it is a boundary solution, i.e., $\mathfrak{D}(\mathbf{x}^{\mathsf{de}}(d_0)) = d_0$.[8] Thus, objective function $\mathfrak{M}(\mathbf{x})$ as well as inequality constraint $\mathfrak{D}(\mathbf{x}) - d_0 \leqslant 0$ of optimization problem (16) are continuously differentiable at $\mathbf{x}^{\mathsf{de}}(d_0)$ with non-vanishing gradient, cf. Proposition 2.6. Hence, the Karush-Kuhn-Tucker (KKT) conditions have to be satisfied.

As a consequence, there exists a constant $\lambda > 0$ such that

$$\lambda\nabla_{\mathbf{x}}(\mathfrak{D}(\mathbf{x}^{\mathsf{de}}(d_0)) - d_0) \overset{(35)}{=} \frac{\lambda}{\mathfrak{D}(\mathbf{x}^{\mathsf{de}}(d_0))}\mathbf{C}(\mathbf{x}^{\mathsf{de}}(d_0))\mathbf{x}^{\mathsf{de}} = \boldsymbol{\pi} = \nabla_{\mathbf{x}}\,\mathfrak{M}(\mathbf{x}^{\mathsf{de}}(d_0)),$$

[8] This follows since the objective function $\mathfrak{M}(\mathbf{x})$ of optimization problem (16) is linear, thus unbounded. So, each maximizer of (16) has to be a boundary solution.

or equivalently, $\mathbf{x}^{\mathsf{de}}(d_0)$ solves

$$\mathbf{C}(\mathbf{x})\mathbf{x} = \frac{d_0}{\lambda}\boldsymbol{\pi}. \tag{20}$$

Null-homogeneity of $\mathbf{C}(\mathbf{x})$ implies that $\mathbf{0} \neq \frac{\lambda}{d_0}\mathbf{x}^{\mathsf{de}}(d_0) =: \mathbf{x}^{\mathsf{ref}}$ solves $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. Further,

$$d_0 = \mathfrak{D}(\mathbf{x}^{\mathsf{de}}(d_0)) = \frac{d_0}{\lambda}\mathfrak{D}(\mathbf{x}^{\mathsf{ref}}) \Longrightarrow \lambda = \mathfrak{D}(\mathbf{x}^{\mathsf{ref}}) > 0.$$

ii) Suppose $\mathbf{x}^{\mathsf{ref}} \neq \mathbf{0}$ solves $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. Then $\frac{d_0}{\mathfrak{D}(\mathbf{x}^{\mathsf{ref}})}\mathbf{x}^{\mathsf{ref}}$ is a solution to equation (20) for $\lambda := \mathfrak{D}(\mathbf{x}^{\mathsf{ref}}) > 0$, i.e., $\frac{d_0}{\mathfrak{D}(\mathbf{x}^{\mathsf{ref}})}\mathbf{x}^{\mathsf{ref}}$ fulfills the KKT conditions of optimization problem (16). Since (16) is a convex optimization problem with linear objective function, the KKT conditions are not only necessary but also sufficient for optimality.

iii) Let $\mathbf{x}_1^{\mathsf{ref}}, \mathbf{x}_2^{\mathsf{ref}}$ be solutions to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. Then,

$$\mathfrak{D}(\mathbf{x}_1^{\mathsf{ref}}) = \frac{\left\langle \mathbf{C}(\mathbf{x}_1^{\mathsf{ref}})\mathbf{x}_1^{\mathsf{ref}}, \mathbf{x}_1^{\mathsf{ref}} \right\rangle}{\sqrt{\left\langle \mathbf{x}_1^{\mathsf{ref}}, \mathbf{C}(\mathbf{x}_1^{\mathsf{ref}})\mathbf{x}_1^{\mathsf{ref}} \right\rangle}} = \frac{\left\langle \boldsymbol{\pi}, \mathbf{x}_1^{\mathsf{ref}} \right\rangle}{\mathfrak{D}(\mathbf{x}_1^{\mathsf{ref}})} = \left\langle \boldsymbol{\pi}, \mathbf{x}_1^{\mathsf{de}}(1) \right\rangle = \left\langle \boldsymbol{\pi}, \mathbf{x}_2^{\mathsf{de}}(1) \right\rangle = \mathfrak{D}(\mathbf{x}_2^{\mathsf{ref}}), \tag{21}$$

where $\langle \boldsymbol{\pi}, \mathbf{x}_1^{\mathsf{de}}(1) \rangle$ is a strictly positive number if and only if $\boldsymbol{\pi} \neq 0$. The last assertion follows from (21), recalling that $\mathfrak{M}(\mathbf{x}) = r_0 e + \langle \boldsymbol{\pi}, \mathbf{x} \rangle$. □

The portfolio $\mathbf{x}^{\mathsf{ref}}$, which will henceforth be referred to as *reference portfolio*, determines the *optimal mix of risky assets*, while the factor $\frac{d_0}{\mathfrak{D}(\mathbf{x}^{\mathsf{ref}})}$ scales the portfolio to risk d_0. In particular, $\mathbf{x}^{\mathsf{ref}}$ is itself downside efficient for $d_0 = \mathfrak{D}(\mathbf{x}^{\mathsf{ref}})$.

From convexity of the optimal solution set $\mathscr{S}(d_0)$ we know that optimization problem (16) admits either a unique solution or a continuum of solutions. To guarantee uniqueness, we need a suitable strict convexity property of the feasible set $\mathscr{K}(d_0)$, at least in a neighborhood of a downside efficient portfolio $\mathbf{x}^{\mathsf{de}}(d_0)$. In general, however, the required version of strict convexity is not available, see, e.g., Rockafellar *et al.* (2006b). We will therefore provide a sufficient criterion for uniqueness of the downside efficient portfolio.

THEOREM 2.3 (Uniqueness of Downside Efficient Portfolios).
Let $\boldsymbol{\pi} \neq \mathbf{0}$ and let $\mathbf{x}^{\mathsf{ref}}$ be a solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. If $\mathbf{C}(\mathbf{x}^{\mathsf{ref}})$ is invertible, then $\mathbf{x}^{\mathsf{ref}}$ is the unique solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$ and $\mathbf{x}^{\mathsf{de}}(d_0) = \frac{d_0}{\mathfrak{D}(\mathbf{x}^{\mathsf{ref}})}\mathbf{x}^{\mathsf{ref}}$ is, thus, the unique downside efficient portfolio.

PROOF. Assume $\mathbf{x}_1^{\mathsf{ref}} \neq \mathbf{x}_2^{\mathsf{ref}}$ are two distinct solutions to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. Let $\tilde{\mathbf{x}} := \mathbf{x}_1^{\mathsf{ref}} - \mathbf{x}_2^{\mathsf{ref}} \neq \mathbf{0}$ be their difference. In Proposition 2.7 in the appendix to this chapter we show

$$\mathbb{P}(\{\omega \in \mathscr{R}(\mathbf{x}_1^{\mathsf{ref}}) \mid \langle \mathbf{t} - \mathbf{q}(\omega), \tilde{\mathbf{x}} \rangle \neq 0\}) = 0.$$

But since $\mathbf{C}(\mathbf{x}_1^{\mathsf{ref}})$ is assumed to be invertible, $\mathbb{P}(\{\omega \in \mathscr{R}(\mathbf{x}_1^{\mathsf{ref}}) \mid \langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{z} \rangle \neq 0\}) > 0$ for any $\mathbf{z} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, due to Lemma 1.1. This is a contradiction. Hence, there can only be one solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. □

Invertibility of $\mathbf{C}(\mathbf{x}^{\mathsf{ref}})$ is a rather mild assumption. In mean-variance analysis, its analog, invertibility of the covariance matrix $\mathbf{V}$, is always required. Moreover, the equation $\mathbf{Vx} = \boldsymbol{\pi}$ has a unique solution if and only if $\mathbf{V}$ is invertible. This solution is then explicit. In contrast, $\mathbf{x}^{\mathsf{ref}} = \mathbf{C}(\mathbf{x}^{\mathsf{ref}})^{-1}\boldsymbol{\pi}$ is still an implicit equation with no a priori guarantee for a closed form solution.

REMARK 2.1.
If $\mathbf{C}(\mathbf{x}^{\mathsf{ref}})$ is invertible, the unique downside efficient portfolio can be represented as

$$\mathbf{x}^{\mathsf{de}}(d_0) = \frac{d_0}{\sqrt{\langle \mathbf{C}(\mathbf{x}^{\mathsf{ref}})^{-1}\boldsymbol{\pi}, \boldsymbol{\pi}\rangle}} \mathbf{C}(\mathbf{x}^{\mathsf{ref}})^{-1}\boldsymbol{\pi}.$$

The structural similarity to mean-variance analysis is now immediate, because a mean-variance efficient portfolio is of the form $\mathbf{x}^{\mathsf{eff}}(\sigma_0) = \frac{\sigma_0}{\sqrt{\langle \mathbf{V}^{-1}\boldsymbol{\pi}, \boldsymbol{\pi}\rangle}} \mathbf{V}^{-1}\boldsymbol{\pi}$, with $\mathbf{V}$ being the covariance matrix of the payoff vector $\mathbf{q}$ and σ_0 denotes the upper bound for standard deviation, see, e.g., Wenzelburger (2010) and references therein. □

In this section, we introduced the notion of downside efficient portfolios and proved their existence. The innovative part is the characterization of $\mathbf{x}^{\mathsf{de}}(d_0)$ in Proposition 2.2 and the sufficient criterion for uniqueness given in Theorem 2.3. Representation of downside efficient portfolios is structurally similar to mean-variance efficient portfolios. This similarity raises hope that most if not all results developed along the lines of mean-variance analysis still hold true when replacing the poor risk measure standard deviation $\sqrt{\langle \mathbf{x}, \mathbf{Vx}\rangle}$ by the superior downside risk $\sqrt{\langle \mathbf{x}, \mathbf{C}(\mathbf{x})\mathbf{x}\rangle}$. Indeed, we are now in a position to establish a separation theorem which will be stated in Theorem 2.4 below.

2.3. Attainable Risk-Return Profiles

The form of the decision problem (12) shows that not the portfolio but its risk-return profile, i.e., its mean payoff and downside risk, determine the investor's utility.[9] This observation gives rise to the following definition whose properties turn out to be crucial for our separation theorem.

DEFINITION 2.6 (Attainable Risk-Return Profiles).
Given some $\boldsymbol{\pi} \in \mathbb{R}^K$, the set of all attainable risk-return profiles is defined by

$$\mathscr{A} := \Big\{ (d, m) \in \mathbb{R}_+ \times \mathbb{R} \mid \exists \mathbf{x} \in \mathbb{R}^K \text{ s.t. } d = \mathfrak{D}(\mathbf{x}) \text{ and } m = \mathfrak{M}(\mathbf{x}) \Big\}. \tag{22}$$ □

Observe that an investor is indifferent between any two portfolios which yield the same mean payoff and the same downside risk. In view of Proposition 2.2 iii), we may thus choose a reference portfolio $\mathbf{x}^{\mathsf{ref}}$ that stipulates the mix of risky assets and take $\mathbf{x}^{\mathsf{de}}(d_0)$ as the corresponding downside efficient portfolio with risk d_0.

[9]Actually, we should speak of a downside-risk-return profile. However, the expression *risk-return profile* gained acceptance in portfolio theory as a fixed term.

As a downside efficient portfolio depends on downside risk d_0, we consider its mean as a function of d_0 and define the *downside efficient frontier* as the curve

$$\mathfrak{e} : \mathbb{R}_+ \longrightarrow \mathbb{R}, \quad d \longmapsto \mathfrak{M}\big(\mathbf{x}^{\mathsf{de}}(d)\big). \tag{23}$$

It describes the maximal mean payoff that is achievable given a prescribed level d of downside risk. By Proposition 2.2 iii), $\mathfrak{e}$ is well-defined even if more than one downside efficient portfolio with risk d exist, as each maximizer of (16) yields the same mean payoff. The downside efficient frontier turns out to be a straight line with slope $\mathfrak{D}(\mathbf{x}^{\mathsf{ref}}) > 0$ and thus becomes the downside analog of the efficient frontier in classical mean-variance analysis, e.g., see Wenzelburger (2010).

Lemma 2.2 (Downside Efficient Frontier).
Let $\boldsymbol{\pi} \neq \mathbf{0}$ and let $\mathbf{x}^{\mathsf{ref}}$ be a solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$.
Then, the downside efficient frontier takes the form

$$\mathfrak{e}(d) = r_0 e + \rho d, \tag{24}$$

where $\rho := \mathfrak{D}(\mathbf{x}^{\mathsf{ref}}) > 0$.

Proof. Using positive homogeneity, we obtain

$$\mathfrak{e}(d) = r_0 e + \langle \boldsymbol{\pi}, \mathbf{x}^{\mathsf{de}}(d)\rangle = r_0 e + d\langle \boldsymbol{\pi}, \mathbf{x}^{\mathsf{de}}(1)\rangle \overset{(21)}{=} r_0 e + \rho d.$$

Since $\mathbf{x}^{\mathsf{ref}} \neq \mathbf{0}$, $\rho = \mathfrak{D}(\mathbf{x}^{\mathsf{ref}})$ is a strictly positive number, due to Assumption 1. □

The downside efficient frontier (24) may also be referred to as downside capital market line, because it is the downside analogon to the mean-variance capital market line.
Observe two characteristics of its slope ρ. First, by Proposition 2.2 iii), ρ does not depend on the particular choice of the reference portfolio $\mathbf{x}^{\mathsf{ref}}$. Second, ρ depends on asset prices $\mathbf{p}$ and on the investor's subjective beliefs about the stock market given by $\mathbb{P}$, because a reference portfolio $\mathbf{x}^{\mathsf{ref}}$ itself depends on $\mathbf{p}$ and $\mathbb{P}$.
We call ρ the *market price of downside risk*, since it denotes the relative price of one unit of d expressed in units of m. Conversely, $\frac{1}{\rho}$ can be interpreted as market price of mean, because to get an additional unit of m an investor has to "pay" the price of $\frac{1}{\rho}$ extra units of d.

We are now in position to characterize the set of all attainable risk-return profiles.

Proposition 2.3 (Attainable Risk-Return Profiles).
Let $\boldsymbol{\pi} \neq \mathbf{0}$ be given. Then, the set of all attainable risk-return profiles takes the form

$$\mathscr{A} = \Big\{(d, m) \in \mathbb{R}_+ \times \mathbb{R} \mid r_0 e - \grave{\rho} d \leq m \leq r_0 e + \rho d\Big\}, \tag{25}$$

where $\grave{\rho} := \mathfrak{D}(\grave{\mathbf{x}}) > 0$ and $\grave{\mathbf{x}} \neq \mathbf{0}$ is a solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = -\boldsymbol{\pi}$.

PROOF. The upper boundary of $\mathscr{A}$ is clear. The lower bound is defined by the solutions to the optimization problem $\min_{\mathbf{x}\in\mathbb{R}^K} \mathfrak{M}(\mathbf{x})$ s.t. $\mathfrak{D}(\mathbf{x}) \leqslant d_0$, where $d_0 > 0$ is some prescribed level of risk. Since this optimization problem is equivalent to $\max_{\mathbf{x}\in\mathbb{R}^K} -\mathfrak{M}(\mathbf{x})$ s.t. $\mathfrak{D}(\mathbf{x}) \leqslant d_0$, existence of the corresponding solutions $\mathbf{x}^{\mathsf{di}}(d_0)$ can be established analogously to the existence of downside efficient portfolios. They take the form

$$\mathbf{x}^{\mathsf{di}}(d_0) = \frac{d_0}{\mathfrak{D}(\grave{\mathbf{x}})}\grave{\mathbf{x}},$$

where $\grave{\mathbf{x}}$ is a solution to the equation $\mathbf{C}(\mathbf{x})\mathbf{x} = -\boldsymbol{\pi}$. These portfolios may be viewed as the most downside inefficient portfolios whose downside risk does not exceed d_0. Using positive homogeneity, we obtain

$$\mathfrak{i}(d) = r_0 e + \langle \boldsymbol{\pi}, \mathbf{x}^{\mathsf{di}}(d)\rangle = r_0 e + d\langle \boldsymbol{\pi}, \mathbf{x}^{\mathsf{di}}(1)\rangle = r_0 e - \grave{\rho} d. \tag{26}$$

Since the objective function $\mathfrak{M}(\mathbf{x})$ is continuous and the set $\mathscr{X}(d) := \{\mathbf{x} \in \mathbb{R}^K \mid \mathfrak{D}(\mathbf{x}) = d\}$ is compact, $\mathfrak{M}(\mathscr{X}(d)) = [\mathfrak{i}(d), \mathfrak{e}(d)]$. ☐

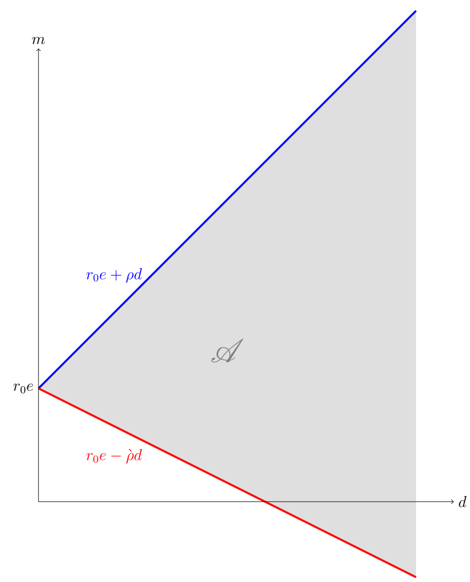

FIGURE 4. Feasible portfolios in the m-d-plane.

Proposition 2.3 states that the set of all attainable risk-return profiles $\mathscr{A}$ is a closed cone with vertex r_0e, depicted in Figure 4. Its upper boundary is the downside efficient frontier $\mathfrak{e}$ that describes the maximal achievable mean payoff, given a prescribed level of downside risk. Its lower boundary is the straight line (26) which characterizes the least downside efficient portfolios possible, given a prescribed level of downside risk. It may therefore be termed "downside inefficient" frontier. Note that in general $\grave{\rho} \neq \rho$. The resulting *asymmetry* of $\mathscr{A}$ is not surprising since downside risk is asymmetric, compare Remark 1.1.

2.4. Separation Theorem

Analogously to the classical mean-variance analysis, it is intuitively clear that, due to the monotonicity of the utility function characterizing risk preferences, an investor will only hold portfolios located on the downside efficient frontier.

To formally establish a well-defined asset demand, recall the concept of a *limiting slope* as introduced in Nielsen (1987). The slope of any indifference curve in the $m-d$ plane is given by the marginal rate of substitution between downside risk and expected payoff

$$S(m,d) := -\frac{\frac{\partial U}{\partial d}(m,d)}{\frac{\partial U}{\partial m}(m,d)}. \tag{27}$$

$S(m,d)$ may be used as a measure of the investor's risk aversion. Due to Assumption 2, $S(m,d)$ is strictly positive, continuous and strictly increasing along indifference curves. By Hiriart-Urruty & Lemarechal (2013, Prop. 3.2.5), all indifference curves have the same limiting slope

$$\rho_U := \sup\left\{S(m,d) \mid (m,d) \in \mathbb{R} \times \mathbb{R}_+\right\} > 0. \tag{28}$$

From convex analysis (see, e.g., Rockafellar (1970)) it is well known that ρ_U is either positive and finite or plus infinity.

The downside efficient frontier (24) is now used to formulate our *separation theorem.* A similar separation theorem in a mean-variance framework was first proved by Tobin (1958) and later by Lintner (1965) and Merton (1972).

Theorem 2.4 (Separation Theorem).
Under the hypothesis of Assumption 2, for any $\mathbf{p} \in \mathbb{R}^K$ *and* $\mathbb{P}$ *such that* $\rho < \rho_U$*, the optimization problem (12) has a maximizer* $\mathbf{x}^* \in \mathbb{R}^K$ *which is given by*

$$\mathbf{x}^* = \begin{cases} d^* \mathbf{x}^{\mathsf{de}}(1) & \text{if } \boldsymbol{\pi} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \boldsymbol{\pi} = \mathbf{0} \end{cases} \tag{29}$$

where

$$d^* := \arg\max_{d \geqslant 0} U(r_0 e + \rho d, d) \tag{30}$$

is the individually optimal amount of downside risk. Moreover, $\mathbf{x}^*$ *is unique if and only if the downside efficient portfolio* $\mathbf{x}^{\mathsf{de}}(1)$ *is unique.*

PROOF. The proof will proceed in several steps

i) At first, we show the assertion for the case $\boldsymbol{\pi} = \mathbf{0}$. Observe that $\boldsymbol{\pi} = \mathbf{0} \Longrightarrow \mathfrak{M}(\mathbf{x}) = r_0 e$ for all $\mathbf{x} \in \mathbb{R}^K$. Since $\mathfrak{D}(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ and $\mathfrak{D}(\mathbf{0}) = 0$, monotonicity properties of U, cf. Assumption 2, now imply $U(r_0 e, 0) > U(\mathfrak{M}(\mathbf{x}), \mathfrak{D}(\mathbf{x}))$ for all $\mathbf{x} \neq \mathbf{0}$. This shows that $\mathbf{x}^* = \mathbf{0}$ is the unique maximizer if $\boldsymbol{\pi} = \mathbf{0}$. If $\boldsymbol{\pi} \neq \mathbf{0}$, then $\mathbf{x}^{\mathsf{de}}(1)$ is well-defined. So, for the following steps, let $\boldsymbol{\pi} \neq \mathbf{0}$.

ii) Now, we show that an optimal solution $\mathbf{x}^*$ of (12) has to be downside efficient, i.e., a maximizer of (16) with $d_0 = \mathfrak{D}(\mathbf{x}^*)$. Assume this is not the case, i.e., there is a feasible portfolio $\mathbf{x}' \in \mathscr{K}(d_0)$ with $\mathfrak{M}(\mathbf{x}') > \mathfrak{M}(\mathbf{x}^*)$. But then, again due to the monotonicity properties of the utility function, $U(\mathfrak{M}(\mathbf{x}'), \mathfrak{D}(\mathbf{x}')) > U(\mathfrak{M}(\mathbf{x}^*), \mathfrak{D}(\mathbf{x}^*))$. This contradicts optimality of $\mathbf{x}^*$. Thus, $\mathbf{x}^* = \mathbf{x}^{\mathsf{de}}(\mathfrak{D}(\mathbf{x}^*)) = \mathfrak{D}(\mathbf{x}^*)\mathbf{x}^{\mathsf{de}}(1)$.

iii) We know $\mathfrak{M}(\mathbf{x}^{\mathsf{de}}(d)) = r_0 e + \rho d$. And as any solution of (12) has to be downside efficient, we can plug in the downside efficient frontier in the utility function (again due to monotonicity properties of U). Thereby, we reduce the multidimensional optimization problem (12) to an equivalent one-dimensional one:

$$\max_{d>0} U(r_0 e + \rho d, d). \tag{31}$$

Hence, $\mathbf{x}^* = d^* \mathbf{x}^{\mathsf{de}}(1)$ is a solution to (12) if and only if d^* solves (31).

iv) The existence and uniqueness of

$$d^* = \arg\max_{d \geqslant 0} U(r_0 e + \rho d, d)$$

can be proven as in the mean-variance case, because optimization problem (31) does not depend on the measure of risk. Compare, e.g., Wenzelburger (2010). This completes the proof. □

The bottom line is that the prerequisites of Theorem 2.4 are the same as in the case of mean-variance analysis, see, e.g., Wenzelburger (2010, Theorem 1): no redundant assets, a continuously differentiable, strictly concave utility function with mean payoff as a good and downside risk as a bad, and the assumption that $\rho < \rho_U$. In particular, the requirements, under which a well-defined utility maximizing portfolio is obtained, do not become more restrictive when replacing standard deviation by downside risk.

Note further that, at a sufficiently low price of risk, the investor's utility maximizing willingness to take on downside risk may be zero.

The downside efficient mix of risky assets $\mathbf{x}^{\mathsf{de}}(1)$ is independent of investor's preferences and initial endowment. Given a target vector $\mathbf{t} \in \mathbb{R}^K$ and prices $\mathbf{p} \in \mathbb{R}^K$, it just depends on the investor's beliefs about the asset market, and, in particular, on the choice of the risk measure. Using the same measure of risk and having the same beliefs, rich and poor investors hold the same mix of risky assets, highly risk averse investors hold the same portfolio mix as slightly risk averse ones. Only d^*, which determines the amount invested into the optimal mix of risky assets, depends on preferences and endowment.

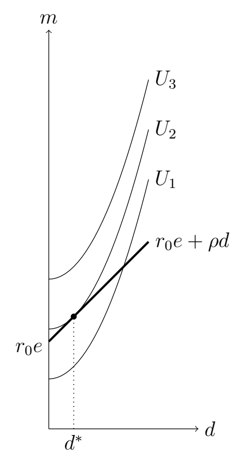

Figure 5. Optimal downside risk.

Figure 5 illustrates how d^* is determined. In optimum, the marginal rate of substitution between downside risk and expected payoff $S(r_0e+\rho d^*, d^*)$ has to be equal to the market price of downside risk ρ.

2.5. Portfolio Selection Without a Risk-Free Asset

A natural question that arises is how portfolio selection theory changes if there is no risk-free asset. This requires certain modifications which will be discussed next. Without a risk-free asset, the whole endowment has to be invested in risky assets, i.e., $e = \langle \mathbf{p}, \mathbf{x} \rangle$. Let

$$\mathscr{P} := \left\{ \mathbf{x} \in \mathbb{R}^K \mid \langle \mathbf{p}, \mathbf{x} \rangle = e \right\} \tag{32}$$

denote the set of all feasible portfolios. The *investor's future wealth* associated with the portfolio $\mathbf{x}$ becomes the following random variable $\hat{w}(\mathbf{x}) := \langle \mathbf{q}, \mathbf{x} \rangle$ with mean $\hat{\mathfrak{M}}(\mathbf{x}) := \langle E[\mathbf{q}], \mathbf{x} \rangle$.[10] Downside risk of portfolio $\mathbf{x}$ remains unchanged $\mathfrak{D}(\mathbf{x}) = \left\| \langle \mathbf{t} - \mathbf{q}, \mathbf{x} \rangle_+ \right\|$.

[10] We use the hat icon ˆ to indicate the setting without bond.

In this section, we assume $\mathbf{p} \neq \mathbf{0}$, because otherwise any portfolio has a price of 0 and the set $\mathscr{P}$ is the empty set unless $e = 0$.

As done in Section 2.2, we define downside efficiency. A solution to the following decision problem is referred to as downside efficient portfolio $\hat{\mathbf{x}}^{\mathsf{de}}(d_0)$

$$\max_{\mathbf{x} \in \mathscr{P}} \hat{\mathfrak{M}}(\mathbf{x}) \text{ s.t. } \mathfrak{D}(\mathbf{x}) \leqslant d_0. \tag{33}$$

As a preliminary step, we characterize the set $\mathscr{P}$ of all feasible portfolios.

PROPOSITION 2.4.
Let $\mathbf{p} \neq \mathbf{0}$. Then the set of feasible portfolios $\mathscr{P}$ has the following properties.

i) $\mathscr{P} \neq \emptyset$.
ii) $\mathscr{P}$ *is a closed set.*
iii) There exists $\mathbf{x}_{\mathsf{min}} \in \mathscr{P}$ *such that* $\mathfrak{D}(\mathbf{x}_{\mathsf{min}}) = \min_{\mathbf{x} \in \mathscr{P}} \mathfrak{D}(\mathbf{x}) =: d_{\mathsf{min}}$.

PROOF. Properties i) and ii) are immediate.
For any $d \geqslant 0$, the feasible set $\hat{\mathscr{K}}(d) := \mathscr{P} \cap \left\{\mathbf{x} \in \mathbb{R}^K \mid \mathfrak{D}(\mathbf{x}) \leqslant d\right\}$ of optimization problem (33) is compact. There exists $\tilde{d}$ large enough, such that $\hat{\mathscr{K}}(\tilde{d}) \neq \emptyset$. Since downside risk $\mathfrak{D}(\mathbf{x})$ is continuous, there exists $\mathbf{x}_{\mathsf{min}}$ such that $\mathfrak{D}(\mathbf{x}_{\mathsf{min}}) = \min_{\mathbf{x} \in \hat{\mathscr{K}}(\tilde{d})} \mathfrak{D}(\mathbf{x})$. For every $\mathbf{x} \in \mathscr{P} \setminus \hat{\mathscr{K}}(\tilde{d})$ it holds that $\mathfrak{D}(\mathbf{x}_{\mathsf{min}}) \leqslant \tilde{d} < \mathfrak{D}(\mathbf{x})$. Since $\hat{\mathscr{K}}(\tilde{d}) \subset \mathscr{P}$, it follows that $\mathfrak{D}(\mathbf{x}_{\mathsf{min}}) = \min_{\mathbf{x} \in \mathscr{P}} \mathfrak{D}(\mathbf{x})$. □

The portfolio $\mathbf{x}_{\mathsf{min}}$ is the downside analog to the minimum variance portfolio in mean-variance analysis and $\mathfrak{D}(\mathbf{x}_{\mathsf{min}}) = d_{\mathsf{min}}$ is the minimal downside risk that can be achieved by purchasing risky assets only. Hence, $\mathbf{x}_{\mathsf{min}}$ is the least downside risky portfolio. In Lemma 2.3 in the appendix, we give a technical characterization of $\mathbf{x}_{\mathsf{min}}$ and show that d_{min} is proportional to e, in particular $d_{\mathsf{min}} = 0 \Longleftrightarrow e = 0$.

We proceed by proving existence of downside efficient portfolios, by characterizing the set of attainable risk-return profiles and by formulating a separation theorem.

THEOREM 2.5 (Existence of Downside Efficient Portfolios Without Bond).
Let $\mathbf{p} \neq \mathbf{0}$.
For $d_0 < d_{\mathsf{min}}$, the optimization problem (33) has no feasible solution. For $d_0 \geqslant d_{\mathsf{min}}$, there exists a downside efficient portfolio $\hat{\mathbf{x}}^{\mathsf{de}}(d_0)$.

PROOF. Since the compact feasible set $\hat{\mathscr{K}}(d_0)$ of (33) is nonempty for $d_0 \geqslant d_{\mathsf{min}}$ and the objective function $\hat{\mathfrak{M}}(\mathbf{x})$ is continuous, there exists $\hat{\mathbf{x}}^{\mathsf{de}}(d_0)$. □

DEFINITION 2.7 (Attainable Risk-Return Profiles Without Bond).
We define the set of all attainable risk-return profiles by

$$\hat{\mathscr{A}} := \Big\{(d,m) \in \mathbb{R}_+ \times \mathbb{R} \mid \exists \mathbf{x} \in \mathscr{P} \text{ s.t. } d = \mathfrak{D}(\mathbf{x}) \text{ and } m = \hat{\mathfrak{M}}(\mathbf{x})\Big\}. \qquad \square$$

The set $\hat{\mathscr{A}}$ has specific properties which can be derived as follows. Analogously to the case with a risk-free asset, it can be shown that there exists a least downside efficient portfolio. This portfolio is a solution to $\min_{\mathbf{x}\in\mathscr{P}} \hat{\mathfrak{M}}(\mathbf{x})$ s.t. $\mathfrak{D}(\mathbf{x}) \leqslant d_0$ and denoted by $\hat{\mathbf{x}}^{\mathsf{di}}(d_0)$. Analogously to the boundaries (23) and (26), we set the functions

$$\begin{aligned} \hat{\mathfrak{e}}: \quad [d_{\mathsf{min}}, \infty) &\to \mathbb{R}, \quad d \mapsto \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{de}}(d)) \\ \hat{\mathfrak{i}}: \quad [d_{\mathsf{min}}, \infty) &\to \mathbb{R}, \quad d \mapsto \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{di}}(d)). \end{aligned}$$

With these definitions we obtain the following result which characterizes the shape of $\hat{\mathscr{A}}$ and specifies conditions under which a tangential portfolio exists.

PROPOSITION 2.5.
Let $e \neq 0$ and $\mathbf{p} \neq \mathbf{0}$. Then, the set of all attainable risk-return profiles is a closed and convex set which takes the form

$$\hat{\mathscr{A}} = \Big\{(d,m) \in \mathbb{R}_+ \times \mathbb{R} \mid d \geq d_{\mathsf{min}} \text{ and } \hat{\mathfrak{i}}(d) \leq m \leq \hat{\mathfrak{e}}(d)\Big\} \tag{34}$$

and has the following properties.

- *i) Its upper boundary $\hat{\mathfrak{e}}$ is a concave, increasing curve starting at $(d_{\mathsf{min}}, \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{de}}(d_{\mathsf{min}})))$ and its lower boundary $\hat{\mathfrak{i}}$ is a convex, decreasing curve starting at $(d_{\mathsf{min}}, \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{di}}(d_{\mathsf{min}})))$.*
- *ii) $\hat{\mathscr{A}} \subsetneq \mathscr{A}$, with $\mathscr{A}$ as given in Definition 2.6.*
- *iii) If $\operatorname{sgn}(\langle \mathbf{p}, \mathbf{x}^{\mathsf{ref}}\rangle) = \operatorname{sgn}(e)$, we set $d_{\mathsf{tan}} := \frac{e\rho}{\langle \mathbf{p}, \mathbf{x}^{\mathsf{ref}}\rangle} \geqslant d_{\mathsf{min}}$. Then, there is a tangential point between $\hat{\mathfrak{e}}$ and $\mathfrak{e}$ at d_{tan}, i.e., $\hat{\mathfrak{e}}(d_{\mathsf{tan}}) = \mathfrak{e}(d_{\mathsf{tan}})$. Moreover, this point of tangency is unique and the corresponding tangential portfolio is given by $\mathbf{x}_{\mathsf{tan}} := \frac{e}{\langle \mathbf{p}, \mathbf{x}^{\mathsf{ref}}\rangle}\mathbf{x}^{\mathsf{ref}}$.*

PROOF. We only prove the properties for $\hat{\mathfrak{e}}$ since the corresponding properties for $\hat{\mathfrak{i}}$ are obtained by an analogous reasoning.
Property ii) is immediate since $(0, r_0 e) \in \mathscr{A}$ but $(0, r_0 e) \notin \hat{\mathscr{A}}$ for $e \neq 0$.
When a riskless asset exists, the downside efficient portfolio $\mathbf{x}^{\mathsf{de}}(d_{\mathsf{tan}}) = \frac{e}{\langle \mathbf{p}, \mathbf{x}^{\mathsf{ref}}\rangle}\mathbf{x}^{\mathsf{ref}}$ has price e. Hence, $\mathbf{x}^{\mathsf{de}}(d_{\mathsf{tan}})$ is also feasible and, thus, downside efficient when there is no bond. This proves $d_{\mathsf{tan}} \geqslant d_{\mathsf{min}}$ and $\hat{\mathfrak{e}}(d_{\mathsf{tan}}) = \mathfrak{e}(d_{\mathsf{tan}})$.
Note that $\Big\langle \mathbf{p}, \mathbf{x}^{\mathsf{de}}(\tilde{d})\Big\rangle \neq e$ for any $\tilde{d} \neq d_{\mathsf{tan}}$. As a consequence $\mathbf{x}^{\mathsf{de}}(\tilde{d})$ is not feasible in the absence of a risk-free asset. Thus, the expected payoff $\mathfrak{e}(\tilde{d})$ cannot be achieved without a bond. This implies $\hat{\mathfrak{e}}(\tilde{d}) < \mathfrak{e}(\tilde{d})$ and proves uniqueness of the tangential point.
The behavior of the curve $\hat{\mathfrak{e}}$ can be constructed as follows. Since $\hat{\mathscr{K}}(d) \subset \hat{\mathscr{K}}(d')$ for $d \leqslant d'$, $\hat{\mathfrak{e}}$ is

increasing. Consider $\hat{\mathbf{x}}^{\mathsf{de}}(d_{\mathsf{min}})$ and $\hat{\mathbf{x}}^{\mathsf{de}}(d_0)$ for some arbitrary $d_0 > d_{\mathsf{min}}$. They are represented via $(d_{\mathsf{min}}, \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{de}}(d_{\mathsf{min}}))), (d_0, \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{de}}(d_0)))$ in the $d-m-$plane and their connecting line is the function

$$\begin{aligned} f : [0,1] &\longrightarrow \mathbb{R}^2, \\ \lambda &\longmapsto \lambda \begin{pmatrix} d_{\mathsf{min}} \\ \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{de}}(d_{\mathsf{min}})) \end{pmatrix} + (1-\lambda) \begin{pmatrix} d_0 \\ \hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{de}}(d_0)) \end{pmatrix}. \end{aligned}$$

Convexity of $\mathfrak{D}$, cf. Proposition 1.1, implies $\mathfrak{D}(\mathbf{x}^{\lambda}) \leqslant \lambda d_{\mathsf{min}} + (1-\lambda)d_0$ for a convex combination $\mathbf{x}^{\lambda} := \lambda\hat{\mathbf{x}}^{\mathsf{de}}(d_{\mathsf{min}}) + (1-\lambda)\hat{\mathbf{x}}^{\mathsf{de}}(d_0), \lambda \in [0,1]$. Due to monotonicity property of downside efficient portfolios, $\hat{\mathfrak{M}}(\hat{\mathbf{x}}^{\mathsf{de}}(\mathfrak{D}(\mathbf{x}^{\lambda}))) \geqslant \hat{\mathfrak{M}}(\mathbf{x}^{\lambda})$. This proves concavity of $\hat{\mathfrak{e}}$ and is illustrated in Figure 6. □

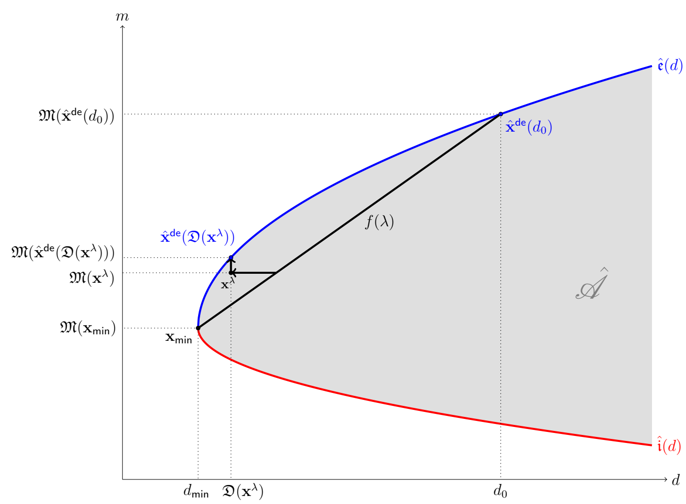

Figure 6. Attainable risk-return profiles without bond.

Figure 6 provides an illustration of the shape of $\hat{\mathscr{A}}$ for $e \neq 0$ in case of a unique downside risk minimizing portfolio. In this case $\hat{\mathbf{x}}^{\mathsf{de}}(d_{\mathsf{min}}) = \mathbf{x}_{\mathsf{min}} = \hat{\mathbf{x}}^{\mathsf{di}}(d_{\mathsf{min}})$. The set of all attainable risk-return profiles $\hat{\mathscr{A}}$ has the form of an asymmetric aircraft cone.

If, however, initial endowment equals zero, then $\mathbf{x}_{\mathsf{min}} = \mathbf{0}$ and $\hat{\mathfrak{e}}(0) = \mathfrak{e}(0)$, i.e., $\mathscr{A}$ and $\hat{\mathscr{A}}$ have a common vertex rather than a tangential point. If, in addition, $\langle \mathbf{p}, \mathbf{x}^{\mathsf{ref}} \rangle = 0$, then any downside efficient portfolio is feasible without a riskless asset since $\langle \mathbf{p}, \mathbf{x}^{\mathsf{de}}(d) \rangle = \frac{d}{\rho} \langle \mathbf{p}, \mathbf{x}^{\mathsf{ref}} \rangle = 0 = e$. As a consequence, $\hat{\mathfrak{e}}(d) = \mathfrak{e}(d)$ for all $d \geqslant 0$ such that uniqueness of the point of contact between $\mathfrak{e}$ and $\hat{\mathfrak{e}}$ does no longer hold.

Figure 7 displays the characteristics of $\hat{\mathfrak{e}}$ (for $e \neq 0$) in case of a unique tangential point with $\mathfrak{e}$ and provides a standard interpretation of the Separation Theorem 2.4. If the individually optimal amount of downside risk $d^* = \arg\max_{d \geqslant 0} U(r_0 e + \rho d, d)$ is greater than d_{tan}, then the utility maximizing portfolio is located on $\mathfrak{e}$ and to the right of the tangential portfolio (depicted as $d_2^* > d_{\mathsf{tan}}$). This means the investor borrows money from the bond and invests it in the downside efficient portfolio mix of risky asset. Conversely, if $d^* = d_1^* < d_{\mathsf{tan}}$, then $\mathbf{x}^*$ lies to the left of $\mathbf{x}^{\mathsf{tan}}$ meaning that money is invested into the riskless asset.

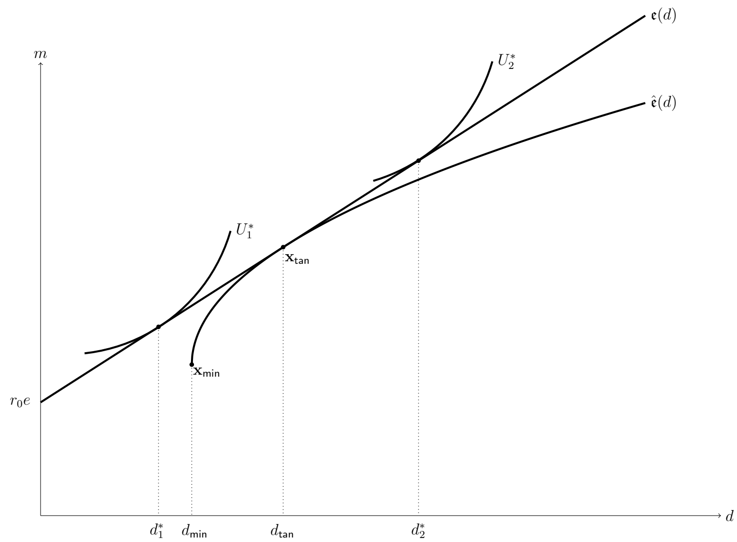

FIGURE 7. Tangential portfolio versus optimal portfolio with bond.

We complete this section with a reformulation of the separation theorem for a utility maximizing investor who is not allowed to invest into a risk-free asset. It shows that such an investor wants to hold a downside efficient portfolio that accounts for the absence of a riskless investment opportunity.

THEOREM 2.6.
Under the hypotheses of Assumption 2, let $\mathbf{p} \neq \mathbf{0}$ *and* $\lim_{d\to\infty} \hat{\mathfrak{e}}'(d) < \rho_U$*. Then, the optimization problem*

$$\max_{\mathbf{x}\in\mathscr{P}} U(\hat{\mathfrak{M}}(\mathbf{x}), \mathfrak{D}(\mathbf{x}))$$

has a maximizer $\hat{\mathbf{x}}^* \in \mathscr{P}$*, given by*

$$\hat{\mathbf{x}}^* = \hat{\mathbf{x}}^{\mathsf{de}}(\hat{d}^*),$$

where

$$\hat{d}^* := \arg\max_{d\geqslant d_{\mathsf{min}}} U\big(\hat{\mathfrak{e}}(d), d\big)$$

is the individually optimal amount of risk.

PROOF. With suitable adjustments, the proof is completely analogous to the proof of Theorem 2.4. $\square$

As done in Sections 2.2 - 2.4, we could show that mean-variance portfolio selection theory without a risk-free asset is structurally preserved when reformulating it in terms of downside risk. Again, prerequisites do not become stronger when replacing standard deviation by downside risk.

2.A. Appendix to Chapter 2

PROPOSITION 2.6 (Gradient of $\mathfrak{D}(\mathbf{x},\mathbf{t})$).
Let $\mathbf{x} \neq 0$. *Then,*

$$\nabla_{\mathbf{x}} \mathfrak{D}(\mathbf{x},\mathbf{t}) = \frac{1}{\mathfrak{D}(\mathbf{x},\mathbf{t})} \mathbf{C}(\mathbf{x},\mathbf{t})\mathbf{x}. \tag{35}$$

PROOF. Since $\mathbf{x} \neq \mathbf{0}$, we can divide by $\mathfrak{D}(\mathbf{x},\mathbf{t}) > 0$ and from the chain rule, we get

$$\nabla_{\mathbf{x}} \mathfrak{D}(\mathbf{x},\mathbf{t}) = \frac{1}{2\mathfrak{D}(\mathbf{x},\mathbf{t})} \nabla_{\mathbf{x}} \int_{\Omega} \left(\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x} \rangle_{+} \right)^2 \mathbb{P}(d\omega).$$

The integral $\int_{\Omega} \left(\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x} \rangle_{+} \right)^2 \mathbb{P}(d\omega)$ is continuously differentiable w.r.t. $x_k, k = 1, ..., K$ and we can exchange differentiation and integration, compare, e.g., Schilling (2005, Theorem 11.5). Lemma 1.2 yields

$$\begin{aligned} \nabla_{\mathbf{x}} \mathfrak{D}(\mathbf{x},\mathbf{t}) &= \frac{1}{\mathfrak{D}(\mathbf{x},\mathbf{t})} \int_{\Omega} \langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x} \rangle_{+} (\mathbf{t} - \mathbf{q}(\omega)) \, \mathbb{P}(d\omega) \\ &= \frac{1}{\mathfrak{D}(\mathbf{x},\mathbf{t})} \mathbf{C}(\mathbf{x},\mathbf{t})\mathbf{x}. \end{aligned}$$ $\square$

PROPOSITION 2.7.
Let $\boldsymbol{\pi} \neq \mathbf{0}$ *and suppose existence of two solutions* $\mathbf{x}_{\mathbf{1}}^{\mathsf{ref}} \neq \mathbf{x}_{\mathbf{2}}^{\mathsf{ref}}$ *to* $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. *Let* $\tilde{\mathbf{x}} := \mathbf{x}_{\mathbf{1}}^{\mathsf{ref}} - \mathbf{x}_{\mathbf{2}}^{\mathsf{ref}}$ *be their difference.*
Then, $\mathbf{0} \neq \tilde{\mathbf{x}} \in \mathbb{R}^K$ *and* $\mathbb{P}(\{\omega \in \mathscr{R}(\mathbf{x}_{\mathbf{1}}^{\mathsf{ref}}) \mid \langle \mathbf{t} - \mathbf{q}(\omega), \tilde{\mathbf{x}} \rangle \neq 0\}) = 0$.

PROOF. Let $\mathbf{x}_{\mathbf{1}}^{\mathsf{ref}}, \mathbf{x}_{\mathbf{2}}^{\mathsf{ref}} \in \mathbb{R}^K$ be two distinct solutions to $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$. The proof will proceed in several steps.

i) As a first step, we show that $\langle \mathbf{t} - \mathbf{q}, \mathbf{x}_{\mathbf{1}}^{\mathsf{ref}} \rangle_{+} = \langle \mathbf{t} - \mathbf{q}, \mathbf{x}_{\mathbf{2}}^{\mathsf{ref}} \rangle_{+} \in \mathcal{L}^2(\Omega)$ although $\mathbf{x}_{\mathbf{1}}^{\mathsf{ref}} \neq \mathbf{x}_{\mathbf{2}}^{\mathsf{ref}}$. Due to Proposition 2.2 ii) $\mathbf{x}_{\mathbf{1}}^{\mathsf{ref}}$ and $\mathbf{x}_{\mathbf{2}}^{\mathsf{ref}}$ have the same mean

$$\mathfrak{M}(\mathbf{x}_{\mathbf{1}}^{\mathsf{ref}}) = \mathfrak{M}(\mathbf{x}_{\mathbf{2}}^{\mathsf{ref}}) =: m^{\mathsf{ref}}$$

and downside risk

$$\left\| \langle \mathbf{t} - \mathbf{q}, \mathbf{x}_{\mathbf{1}}^{\mathsf{ref}} \rangle_{+} \right\| = \left\| \langle \mathbf{t} - \mathbf{q}, \mathbf{x}_{\mathbf{2}}^{\mathsf{ref}} \rangle_{+} \right\| =: d^{\mathsf{ref}}.$$

In particular, $\mathbf{x_1^{\mathsf{ref}}}, \mathbf{x_2^{\mathsf{ref}}} \in \mathscr{S}(d^{\mathsf{ref}})$. For $\lambda \in (0,1)$ define $\mathbf{x}^\lambda := \lambda \mathbf{x_1^{\mathsf{ref}}} + (1-\lambda)\mathbf{x_2^{\mathsf{ref}}}$. Then, $\mathfrak{M}(\mathbf{x}^\lambda) = m^{\mathsf{ref}}$ and, since L^2 is a Hilbert space, and thus strictly convex,

$$\begin{aligned} \mathfrak{D}(\mathbf{x}^\lambda) &= \left\| \left\langle \mathbf{t} - \mathbf{q}, \lambda \mathbf{x_1^{\mathsf{ref}}} + (1-\lambda)\mathbf{x_2^{\mathsf{ref}}} \right\rangle_+ \right\| \\ &\leqslant \left\| \lambda \left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_1^{\mathsf{ref}}} \right\rangle_+ + (1-\lambda) \left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_2^{\mathsf{ref}}} \right\rangle_+ \right\| \\ &< d^{\mathsf{ref}}, \text{ if } \left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_1^{\mathsf{ref}}} \right\rangle_+ \neq \left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_2^{\mathsf{ref}}} \right\rangle_+ . \end{aligned}$$

This is a contradiction to $\mathbf{x_1^{\mathsf{ref}}} \in \mathscr{S}(d^{\mathsf{ref}})$. Hence, $\left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_1^{\mathsf{ref}}} \right\rangle_+ = \left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_2^{\mathsf{ref}}} \right\rangle_+$.

ii) Since the optimal solution set is convex, $\mathbf{x}^\lambda = \mathbf{x_2^{\mathsf{ref}}} + \lambda \tilde{\mathbf{x}} \in \mathscr{S}(d^{\mathsf{ref}})$ for any $\lambda \in [0,1]$. By the same argument as in the first step, we get

$$\left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_2^{\mathsf{ref}}} + \lambda \tilde{\mathbf{x}} \right\rangle_+ = \left\langle \mathbf{t} - \mathbf{q}, \mathbf{x_1^{\mathsf{ref}}} \right\rangle_+ \tag{36}$$

for any $\lambda \in [0,1]$.

iii) For an arbitrary but fixed $\omega \in \Omega$, the function

$$f(\cdot;\omega) : \mathbb{R} \longrightarrow \mathbb{R}_+, \ \lambda \longmapsto f(\lambda;\omega) := \left(\left\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x_2^{\mathsf{ref}}} + \lambda \tilde{\mathbf{x}} \right\rangle_+ \right)^2$$

is continuously differentiable w.r.t λ by Lemma 1.2 and constant on $[0,1]$ by ii). Its derivative equals

$$\frac{\partial f}{\partial \lambda}(\lambda;\omega) = 2 \left\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x_2^{\mathsf{ref}}} + \lambda \tilde{\mathbf{x}} \right\rangle_+ \left\langle \left(\mathbf{t} - \mathbf{q}(\omega)\right), \tilde{\mathbf{x}} \right\rangle . \tag{37}$$

iv) Since $f(\cdot;\omega)$ is constant on $[0,1]$,

$$\begin{aligned} \frac{\partial f}{\partial \lambda}(1;\omega) &= 2 \left\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x_1^{\mathsf{ref}}} \right\rangle_+ \left\langle \mathbf{t} - \mathbf{q}(\omega), \tilde{\mathbf{x}} \right\rangle \\ &= 0. \end{aligned}$$

a) $\omega \notin \mathscr{R}(\mathbf{x_1^{\mathsf{ref}}}) \Longrightarrow \left\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x_1^{\mathsf{ref}}} \right\rangle_+ = 0$.
b) $\omega \in \mathscr{R}(\mathbf{x_1^{\mathsf{ref}}}) \Longrightarrow \left\langle \mathbf{t} - \mathbf{q}(\omega), \mathbf{x_1^{\mathsf{ref}}} \right\rangle_+ > 0 \Longrightarrow \left\langle \mathbf{t} - \mathbf{q}(\omega), \tilde{\mathbf{x}} \right\rangle = 0$.
c) $\Longrightarrow \mathbb{P}(\{\omega \in \mathscr{R}(\mathbf{x_1^{\mathsf{ref}}}) \mid \langle \mathbf{t} - \mathbf{q}(\omega), \tilde{\mathbf{x}} \rangle \neq 0\}) = 0$. □

LEMMA 2.3 (Characterization of $\mathbf{x}_{\mathsf{min}}$).
Let $\mathbf{p} \neq \mathbf{0}$.
Then, the downside risk minimizing portfolio $\mathbf{x}_{\mathsf{min}}$ *takes the following form.*

i) Let $e > 0$ *and let* $\check{\mathbf{x}} \neq \mathbf{0}$ *be a solution to* $\mathbf{C}(\mathbf{x})\mathbf{x} = \mathbf{p}$.
Then, $\mathbf{x}_{\mathsf{min}} = \frac{e}{\langle \mathbf{p}, \check{\mathbf{x}} \rangle} \check{\mathbf{x}}$ *and* $d_{\mathsf{min}} = \frac{e}{\mathfrak{D}(\check{\mathbf{x}})}$. *Moreover,* $\mathbf{x}_{\mathsf{min}}$ *is unique if* $\mathbf{C}(\check{\mathbf{x}})$ *is invertible.*
ii) If $e = 0$, *then* $\mathbf{x}_{\mathsf{min}} = \mathbf{0}$ *and* $d_{\mathsf{min}} = 0$.
iii) Let $e < 0$ *and let* $\check{\check{\mathbf{x}}} \neq \mathbf{0}$ *be a solution to* $\mathbf{C}(\mathbf{x})\mathbf{x} = -\mathbf{p}$.
Then, $\mathbf{x}_{\mathsf{min}} = \frac{e}{\langle \mathbf{p}, \check{\check{\mathbf{x}}} \rangle} \check{\check{\mathbf{x}}}$ *and* $d_{\mathsf{min}} = -\frac{e}{\mathfrak{D}(\check{\check{\mathbf{x}}})}$. *Moreover,* $\mathbf{x}_{\mathsf{min}}$ *is unique if* $\mathbf{C}(\check{\check{\mathbf{x}}})$ *is invertible.*

Proof.

i) We consider the minimization problem min $\frac{1}{2}\mathfrak{D}(\mathbf{x})^2$ s.t. $\langle \mathbf{p}, \mathbf{x}\rangle = e > 0$. A solution exists due to Proposition 2.4 iii), such that we can apply the KKT theorem. Let $\check{\mathbf{x}}$ be a solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = \mathbf{p}$ and $\lambda = \frac{e}{\langle \mathbf{p}, \check{\mathbf{x}}\rangle} = \frac{e}{\mathfrak{D}(\check{\mathbf{x}})^2} > 0$. Then $\mathbf{x}_{\mathsf{min}} := \lambda\check{\mathbf{x}}$ solves

$$\nabla_{\mathbf{x}} \frac{1}{2}\mathfrak{D}(\mathbf{x})^2 \overset{(35)}{=} \mathbf{C}(\mathbf{x})\mathbf{x} = \lambda\mathbf{p} = \lambda\nabla_{\mathbf{x}}(\langle \mathbf{p}, \mathbf{x}\rangle - e).$$

Further,

$$d_{\mathsf{min}} = \mathfrak{D}(\mathbf{x}_{\mathsf{min}}) = \sqrt{\langle \mathbf{x}_{\mathsf{min}}, \mathbf{C}(\mathbf{x}_{\mathsf{min}})\mathbf{x}_{\mathsf{min}}\rangle} = \sqrt{\langle \lambda\check{\mathbf{x}}, \lambda\mathbf{p}\rangle} = \sqrt{\frac{e}{\langle \mathbf{p}, \check{\mathbf{x}}\rangle}\frac{e}{\mathfrak{D}(\check{\mathbf{x}})^2}\langle \check{\mathbf{x}}, \mathbf{p}\rangle} = \frac{e}{\mathfrak{D}(\check{\mathbf{x}})}.$$

If $\mathbf{C}(\check{\mathbf{x}})$ is invertible, then $\check{\mathbf{x}}$ is the unique solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = \mathbf{p}$ (which follows from a analogous argument as used for Theorem 2.3) and $\mathbf{x}_{\mathsf{min}}$ is thus uniquely determined.

ii) If $e = 0$, then $\mathbf{0}$ is a feasible portfolio and it is the only portfolio with downside risk equal to 0.

iii) We consider the minimization problem min $\frac{1}{2}\mathfrak{D}(\mathbf{x})^2$ s.t. $\langle \mathbf{p}, \mathbf{x}\rangle = e < 0$. A solution exists due to Proposition 2.4 iii), such that we can apply the KKT theorem. Let $\check{\check{\mathbf{x}}}$ be a solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = -\mathbf{p}$ and $\lambda = \frac{e}{\langle \mathbf{p}, \check{\check{\mathbf{x}}}\rangle} = \frac{e}{-\mathfrak{D}(\check{\check{\mathbf{x}}})^2} > 0$. Then $\mathbf{x}_{\mathsf{min}} := \lambda\check{\check{\mathbf{x}}}$ solves

$$\nabla_{\mathbf{x}} \frac{1}{2}\mathfrak{D}(\mathbf{x})^2 \overset{(35)}{=} \mathbf{C}(\mathbf{x})\mathbf{x} = \lambda(-\mathbf{p}) = \lambda\nabla_{\mathbf{x}}(e - \langle \mathbf{p}, \mathbf{x}\rangle).$$

Further,

$$d_{\mathsf{min}} = \mathfrak{D}(\mathbf{x}_{\mathsf{min}}) = \sqrt{\langle \mathbf{x}_{\mathsf{min}}, \mathbf{C}(\mathbf{x}_{\mathsf{min}})\mathbf{x}_{\mathsf{min}}\rangle} = \sqrt{\langle \lambda\check{\mathbf{x}}, -\lambda\mathbf{p}\rangle} = \sqrt{\frac{e}{\langle \mathbf{p}, \check{\mathbf{x}}\rangle}\frac{e}{\mathfrak{D}(\check{\mathbf{x}})^2}\langle \check{\mathbf{x}}, \mathbf{p}\rangle} \overset{e\leq 0}{=} \frac{-e}{\mathfrak{D}(\check{\mathbf{x}})}.$$

If $\mathbf{C}(\check{\check{\mathbf{x}}})$ is invertible, then $\check{\check{\mathbf{x}}}$ is the unique solution to $\mathbf{C}(\mathbf{x})\mathbf{x} = -\mathbf{p}$ (which follows from a analogous argument as used for Theorem 2.3) and $\mathbf{x}_{\mathsf{min}}$ is thus uniquely determined. □

It is essential that the Lagrange multiplier λ is strictly positive because $\mathbf{C}(\mathbf{x})$ is null-homogeneous only for $\lambda > 0$, as shown Lemma 1.1. In a mean-variance framework, we have $\check{\mathbf{x}}^{\mathsf{mv}} := \mathbf{V}^{-1}\mathbf{p}$ and $\check{\check{\mathbf{x}}}^{\mathsf{mv}} := \mathbf{V}^{-1}(-\mathbf{p}) = -\check{\mathbf{x}}^{\mathsf{mv}}$ such that $\frac{1}{\langle \mathbf{p}, \check{\mathbf{x}}^{\mathsf{mv}}\rangle}\check{\mathbf{x}}^{\mathsf{mv}} = \frac{1}{\langle \mathbf{p}, \check{\check{\mathbf{x}}}^{\mathsf{mv}}\rangle}\check{\check{\mathbf{x}}}^{\mathsf{mv}}$, i.e., both cases coincide. But when using downside risk, in general $\frac{1}{\langle \mathbf{p}, \check{\mathbf{x}}\rangle}\check{\mathbf{x}} \neq \frac{1}{\langle \mathbf{p}, \check{\check{\mathbf{x}}}\rangle}\check{\check{\mathbf{x}}}$. That is why we need the case analysis, here.

CHAPTER 3

Equilibrium Asset Pricing

In this chapter, we develop an equilibrium asset pricing model. Based on Theorem 2.4, we specify conditions under which a unique asset market equilibrium exists. Moreover, we derive a pricing formula which turns out to yield *arbitrage-free and strictly positive* equilibrium asset prices. This finding contrasts strongly with mean-variance pricing as embodied in the capital asset pricing model, compare Levy (2007) who specified conditions for a CAPM equilibrium with positive prices. The equilibrium formulation of the CAPM, which is one of the most central achievements in financial economics, dates back to Sharpe (1964), Lintner (1965) and Mossin (1966). Furthermore, we show that the market portfolio, which constitutes the current stock of shares of assets, attains the highest possible Sortino ratio at equilibrium prices. Hence, it has the best risk-return profile among all possible portfolios. Finally, a downside security market line is derived.

As a first step, we briefly recall the setting and refine notation as well as assumptions for this chapter.[11] There are I investors, characterized by their preferences, beliefs and endowments.
The preferences of any investor $i = 1, ..., I$ are given by a utility function U^i that fulfills Assumption 2. The slope of any indifference curve of investor i is given by the marginal rate of substitution $S^i(m, d)$ and the limiting slope is denoted by ρ_{U^i}.
We suppose that all investors have homogeneous beliefs, i.e., $\mathbb{P}^i = \mathbb{P}$ for all $i = 1, ..., I$.
Further, each investor is endowed with shares of the risk-free and the risky assets $(x_0^i, \mathbf{x}^i) \in \mathbb{R} \times \mathbb{R}^K$, which yields a payoff $\epsilon^i = r_0 x_0^i + \langle \mathbf{q}, \mathbf{x}^i \rangle \in \mathscr{M}$. Endowment ϵ^i has a monetary equivalent $x_0^i + \langle \mathbf{p}, \mathbf{x}^i \rangle$. Risky assets are in strictly positive net supply $\mathtt{m} = \sum_{i=1}^I \mathbf{x}^i \in \mathbb{R}_{++}^K$ and the riskless asset is in zero net supply $\sum_{i=1}^I x_0^i = 0$. We call $\mathtt{m}$ *market portfolio* of the economy and $\epsilon_{\mathtt{m}} = \langle \mathbf{q}, \mathtt{m} \rangle$ *market payoff.*

Assumption 3.
The risky assets are shares of ordinary stocks, i.e., the support of q_k is $\mathbb{R}_+$, for $k = 1, ..., K$. We further assume that neither the risky assets nor the riskless asset is redundant.

Due to Assumption 3, there may now exist risk-free nontrivial portfolios. Let $\mathbf{0} \neq \tilde{\mathbf{x}} \in \mathbb{R}_+^K$, then $\langle \mathbf{q}, \tilde{\mathbf{x}} \rangle \geqslant 0$. If $\tilde{\mathbf{t}} \in \mathbb{R}_-^K$, which implies $\langle \tilde{\mathbf{t}} - \mathbf{q}, \tilde{\mathbf{x}} \rangle \leqslant 0$, then $\mathfrak{D}(\tilde{\mathbf{x}}, \tilde{\mathbf{t}}) = \left\| \langle \tilde{\mathbf{t}} - \mathbf{q}, \tilde{\mathbf{x}} \rangle_+ \right\| = 0$. Thus, we cannot presume Assumption 1 in this chapter, but have to verify it for each assertion.

[11]The setting is introduced in great detail in Section 1.1.

Eventually, we restrict our analysis to the target vector $\mathbf{t} = r_0\mathbf{p}$. It is the target vector that makes downside risk monotone in $\mathbf{x}$ as shown in Proposition 1.2 iii) and can thus be considered as natural target vector. All but one of the results, established in this chapter, carry over to other targets directly.

NOTATION 1.
In this chapter, we express the dependency on prices explicitly, e.g., $\boldsymbol{\pi}(\mathbf{p}) := E[\mathbf{q}] - r_0\mathbf{p}$.
Further, a reference portfolio, i.e., a solution to $\mathbf{C}(\mathbf{x}, r_0\mathbf{p})\mathbf{x} = \boldsymbol{\pi}(\mathbf{p})$ is referred to as $\mathbf{x}^{\mathsf{ref}}(\mathbf{p})$ and

$$\rho(\mathbf{p}) := \mathfrak{D}(\mathbf{x}^{\mathsf{ref}}(\mathbf{p}), r_0\mathbf{p}) \geqslant 0 \tag{38}$$

denotes the market price of downside risk. Moreover, we denote the price of the market portfolio by $p_{\mathtt{m}} := \langle \mathbf{p}, \mathtt{m} \rangle$ and use the alternative notation $\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}) = \mathfrak{D}(\mathtt{m}, r_0\mathbf{p})$ whenever appropriate. □

As a second preliminary step, we reformulate how a downside-risk averse investor chooses her individually optimal amount of downside risk and, thus, her individually optimal portfolio.

PROPOSITION 3.1.
Under the hypothesis of Assumption 2 and for any $\mathbf{p} \in \mathbb{R}^K$ such that $\rho(\mathbf{p}) \in (0, \rho_{U^i})$, investor i's individually optimal investment $\mathbf{x}^{i,*}(\mathbf{p}) \in \mathbb{R}^K$ is given by

$$\mathbf{x}^{i,*}(\mathbf{p}) = \frac{\psi^i(\mathbf{p})}{\rho(\mathbf{p})}\mathbf{x}^{\mathsf{ref}}(\mathbf{p}) \tag{39}$$

where

$$\psi^i(\mathbf{p}) := \arg\max_{d \geqslant 0} U^i(r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i \rangle) + \rho(\mathbf{p})d, d) \tag{40}$$

is the individually optimal amount of downside risk at prices $\mathbf{p}$.

Proposition 3.1 is a reformulation of the Separation Theorem (compare Theorem 2.4) where the dependence on prices is denoted explicitly. The term $\frac{1}{\rho(\mathbf{p})}\mathbf{x}^{\mathsf{ref}}(\mathbf{p})$ in formula (39) is a downside efficient portfolio with downside risk 1 and investor i's individual demand for downside risk is given by $\psi^i(\mathbf{p})$ in formula (40). Note that, due to Assumption 2, $\psi^i(\mathbf{p})$ is well-defined for all $\mathbf{p} \in \mathbb{R}^K$ with $\rho(\mathbf{p}) < \rho_{U^i}$.
In particular, each investor holds the same mix of risky assets. Only the amount of money invested into it differs with different preferences and endowments. This is straightforward if $\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*)$ is unique. However in case of non-uniqueness, we have to *presume* that each investor holds the same downside efficient portfolio mix and call the resulting equilibrium *symmetric*.[12]

[12]Since every individually optimal portfolio has the same downside risk and the same mean payoff, it induces the same utility to an investor. Hence, all individually optimal portfolios are equivalent from the investor's point of view. Thus, the assumption of symmetry is admissible, here.

We are now in a position to establish equilibrium asset pricing theory. In Section 3.1 we introduce two equilibrium concepts. On the one hand, a downside risk asset market equilibrium (DRAME) is given by a market-clearing price vector and an individually optimal allocation of riskless and risky assets among the investors. On the other hand, we define an equilibrium in the space spanned by the risk-free asset and the market payoff. Both equilibrium concepts are equivalent and the latter one represents a mean downside risk equilibrium. In Section 3.2 we prove existence and give a sufficient criterion for uniqueness of these equilibria. Furthermore, we provide an example where a unique DRAME exists. Eventually, in Section 3.3, we discuss economic implications of equilibrium asset pricing theory using downside risk of payoff. Equilibrium prices turn out to be arbitrage-free as well as strictly positive and the market portfolio attains the highest possible Sortino ratio at equilibrium prices. In addition, we establish a downside security market line and a valuation formula to calculate the fair price of financial options.

3.1. Equilibrium Concepts

There is a substantial amount of literature on existence and uniqueness of CAPM equilibria, see, e.g., Nielsen (1987, 1988, 1990a,b), Allingham (1991) and Dana (1993a,b, 1999). We follow the basic line of reasoning in Dana (1999). There, the K-dimensional existence and uniqueness problem for equilibria in the asset market is brought down to a two-dimensional problem in the space generated by the riskless asset and the market portfolio. The latter equilibrium simultaneously represents an equilibrium in the market for mean and downside risk.

3.1.1. Downside Risk Asset Market Equilibria.

A downside risk asset market equilibrium is denoted by its acronym DRAME and defined in the marketed space $\mathscr{M} = \text{span}\{r_0, q_1, ..., q_K\}$.

Observe that there is a bijection between the space of all possible investments $\mathbb{R} \times \mathbb{R}^K$ and the space of all possible payoffs $\mathscr{M}$. We therefore identify an investment $(x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^K$ with its payoff $\epsilon = r_0 x_0 + \langle \mathbf{q}, \mathbf{x} \rangle \in \mathscr{M}$ and call ϵ "investment", too.

Lemma 3.1.
The mapping

$$\mathbb{R} \times \mathbb{R}^K \longrightarrow \mathscr{M},$$
$$(x_0, \mathbf{x}) \longmapsto \epsilon = r_0 x_0 + \langle \mathbf{q}, \mathbf{x} \rangle$$

is bijective.

PROOF. The mapping is surjective, because $\mathscr{M} = \text{span}\{r_0, q_1, ..., q_K\}$.
Suppose a payoff is generated by two different investments, i.e., $r_0x_0 + \langle \mathbf{q}, \mathbf{x}\rangle = \epsilon = r_0x_0' + \langle \mathbf{q}, \mathbf{x}'\rangle$, for $(x_0, \mathbf{x}) \neq (x_0', \mathbf{x}')$. Then, the assumption of non-redundancy, stated in Assumption 3, is violated, since $0 = r_0(x_0 - x_0') + \langle \mathbf{q}, \mathbf{x} - \mathbf{x}'\rangle$. This is a contradiction. Hence, we have injectivity and, thus, bijectivity. □

The investors are endowed with $\epsilon^i = r_0x_0^i + \langle \mathbf{q}, \mathbf{x}^i\rangle \in \mathscr{M}$, for $i = 1, ..., I$, such that $\sum_{1=1}^{I} \epsilon^i = \epsilon_{\mathtt{m}}$. Investor i can choose any investment from her budget set

$$\mathscr{B}(\epsilon^i, \mathbf{p}) := \{\epsilon = r_0x_0 + \langle \mathbf{q}, \mathbf{x}\rangle \in \mathscr{M} \mid x_0 + \langle \mathbf{p}, \mathbf{x}\rangle \leqslant x_0^i + \langle \mathbf{p}, \mathbf{x}^i\rangle\} \subseteq \mathscr{M}. \tag{41}$$

Due to monotonicity properties of investor i's utility function given in Assumption 2, she will always choose $x_0 = x_0^i + \langle \mathbf{p}, \mathbf{x}^i\rangle - \langle \mathbf{p}, \mathbf{x}\rangle$, such that mean and downside risk of her investment are given by

$$\begin{aligned}
\mathcal{M}(\epsilon) &= E[\epsilon] \\
\mathcal{D}(\epsilon, r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i\rangle)) &= \left\|(r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i\rangle) - (r_0x_0 + \langle \mathbf{q}, \mathbf{x}\rangle))_+\right\| \\
&= \left\|(r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i\rangle) - r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i\rangle - \langle \mathbf{p}, \mathbf{x}\rangle) - \langle \mathbf{q}, \mathbf{x}\rangle)_+\right\| \\
&= \left\|\langle r_0\mathbf{p} - \mathbf{q}, \mathbf{x}\rangle_+\right\| = \mathfrak{D}(\mathbf{x}, r_0\mathbf{p}).
\end{aligned}$$

A DRAME is then an allocation of risk-free and risky assets, such that each investor holds an individually optimal investment and the asset market is cleared.

DEFINITION 3.1 (Equilibrium Concept I - DRAME).
Let $\overline{\rho_U} := \min\{\rho_{U^i}, i = 1, \dots, I\} > 0$ denote the smallest limiting slope of all investor's indifference curves.
A DRAME consists of a price vector $\mathbf{p}^* \in \mathbb{R}^K$ with $\rho(\mathbf{p}^*) \in (0, \overline{\rho_U})$ and an allocation of investments $\epsilon^{1,*}, ..., \epsilon^{I,*} \in \mathscr{M}$, such that

i) each $\epsilon^{i,*}$ is individually optimal at prices $\mathbf{p}^*$, i.e., for $i = 1, ..., I$ it solves

$$\max_{\epsilon \in \mathscr{B}(\epsilon^i, \mathbf{p}^*)} U^i\left(\mathcal{M}(\epsilon), \mathcal{D}(\epsilon, r_0(x_0^i + \langle \mathbf{p}^*, \mathbf{x}^i\rangle))\right),$$

ii) and the allocation is feasible, i.e.,

$$\sum_{i=1}^{I} \epsilon^{i,*} = \epsilon_{\mathtt{m}}.$$

□

NOTATION 2.
In the following, we denote by $\mathscr{H} := \text{span}\{r_0, \epsilon_{\mathtt{m}}\} \subseteq \mathscr{M}$ the two-dimensional marketed subspace spanned by the risk-free asset and the market payoff. □

THEOREM 3.1 (Characterization of DRAMEs).
Let the assumptions of this chapter be fulfilled. Let $\mathbf{p}^*, \epsilon^{1,*}, ..., \epsilon^{I,*}$ constitute a DRAME and let $p^*_{\mathtt{m}} = \langle \mathbf{p}^*, \mathtt{m} \rangle$ be the corresponding equilibrium price of the market portfolio.
Then, the following holds:

i) The allocation of assets is given by

$$\epsilon^{i,*} = r_0 \left(x_0^i + \langle \mathbf{p}^*, \mathbf{x}^i \rangle - \frac{\psi^i(\mathbf{p}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} p^*_{\mathtt{m}} \right) + \epsilon_{\mathtt{m}} \frac{\psi^i(\mathbf{p}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} \in \mathscr{H}. \tag{42}$$

ii) The aggregate demand for downside risk equals the aggregate downside risk of the market portfolio, i.e.,

$$\sum_{i=1}^{I} \psi^i(\mathbf{p}^*) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}}) > 0. \tag{43}$$

iii) The market-clearing prices satisfy

$$\mathbf{p}^* = E[\mathbf{q}\kappa(p^*_{\mathtt{m}})], \tag{44}$$

where the price kernel is given by

$$\kappa(p_{\mathtt{m}}) = \frac{1}{r_0} \frac{1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} (r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+}{1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} E[(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+]}. \tag{45}$$

PROOF. See Section 3.A.1 in the appendix to this chapter. □

Due to equation (42), in equilibrium, the individually optimal allocation of risky assets is achieved when each investor holds a positive fraction of the market portfolio $\mathtt{m}$. This fraction is determined by her preferences and her endowment. The bond position is given by the monetization of investor i's initial endowment minus the equilibrium price of her individually optimal fraction of the market portfolio. The equilibrium investment $(x_0^{i,*}, \mathbf{x}^{i,*}) \in \mathbb{R} \times \mathbb{R}^K$, for $i = 1, ..., I$, amounts to

$$\mathbf{x}^{i,*} = \frac{\psi^i(\mathbf{p}^*)}{\mathfrak{D}(\mathtt{m}, r_0 \mathbf{p}^*)} \mathtt{m}, \tag{46}$$

$$x_0^{i,*} = x_0^i + \langle \mathbf{p}^*, \mathbf{x}^i \rangle - \langle \mathbf{p}^*, \mathbf{x}^{i,*} \rangle . \tag{47}$$

This specific portfolio allocation given by (46) and (47) connotes that a DRAME is completely determined by a price vector $\mathbf{p}^*$ which makes the market portfolio the individually optimal portfolio mix and thus collinear to $\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*)$.

Equation (43) has to be understood as market clearing condition for downside risk. Note further that neither the market payoff $\epsilon_{\mathtt{m}}$ nor the individually optimal portfolios are risk-free at equilibrium prices. This observation is nontrivial as pointed out in the consideration after Assumption 3.

Formulas (44) and (45) deserve a separate section.

3.1.2. Pricing Formula and Price Kernel.

Equilibrium asset prices are best understood when formulas (44) and (45) are rearranged in coordinate form[13]:

$$p_k^* = \frac{1}{r_0}\left(E[q_k] - \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)^2} E[(r_0 p_k^* - q_k)(r_0 p_{\mathtt{m}}^* - \epsilon_{\mathtt{m}})_+]\right). \tag{48}$$

The term $E\left[(r_0 p_k^* - q_k)\,(r_0 p_{\mathtt{m}}^* - \epsilon_{\mathtt{m}})_+\right]$ is the downside analogon to the covariance of q_k and $\epsilon_{\mathtt{m}}$. It is referred to as downside cosemivariance between asset k and the market payoff. Note that downside cosemivariance is not symmetric which is not surprising as downside risk is asymmetric. We take the positive part of $(r_0 p_{\mathtt{m}}^* - \epsilon_{\mathtt{m}})$, but not of $(r_0 p_k^* - q_k)$. Since each investor holds a fraction of the market portfolio in equilibrium, only realizations $\omega \in \Omega$ with $r_0 p_{\mathtt{m}}^* > \epsilon_{\mathtt{m}}(\omega)$ pose a risk.

We observe a similarity to classical CAPM pricing. Asset k's equilibrium price equals its discounted expected payoff minus a *risk premium.* This risk premium equals expected excess payoff of the market portfolio divided by its squared downside risk times downside cosemivariance between asset k and the market payoff. The equilibrium expected excess payoff of the market portfolio is strictly positive, otherwise the market portfolio would not be individually optimal. Hence, the sign of the risk premium depends on the *downside correlation* of asset k's payoff with the market payoff in case the market payoff misses its target.

According to Wenzelburger (2009), in equation (48), the term in the parentheses "is called *certainty equivalent* of the k^{th} asset because this value may be treated as the certain amount of the asset's proceeds before discounting it" to obtain p_k^*.

COROLLARY 3.1.

The equilibrium price of the market portfolio is strictly less than its discounted expected payoff and strictly greater than zero,

$$0 < p_{\mathtt{m}}^* < \frac{1}{r_0} E[\epsilon_{\mathtt{m}}].$$

The equilibrium market price of downside risk equals the equilibrium Sortino ratio of the market payoff, i.e.,

$$\rho(\mathbf{p}^*) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} > 0. \tag{49}$$

PROOF. Assume the opposite: $p_{\mathtt{m}}^* \geqslant \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]$. Then, the market portfolio has a negative expected excess payoff and a strictly positive downside risk at equilibrium prices. Thus, it cannot be the individually optimal portfolio mix since the portfolio $\mathbf{x} = \mathbf{0}$ has zero expected excess payoff and zero downside risk. This is a contradiction.

[13] Compare Appendix 3.A.1 for a detailed calculation.

If, on the other hand, $p^*_{\mathtt{m}} \leqslant 0$, then $\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}}) = \|(r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\| = 0$ since $\epsilon_{\mathtt{m}} \geqslant 0 \geqslant r_0 p^*_{\mathtt{m}}$. This however is a contradiction to $\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}}) > 0$, compare Theorem 3.1 and its proof in Section 3.A.1. Equation (49) follows from equation (83) in the appendix. □

The price kernel $\kappa(p_{\mathtt{m}})$ in (45) is a square integrable, real random variable, i.e, an element of $\mathcal{L}^2(\Omega)$ But, in contrast to the analogous theorem in Dana (1999, Proposition 2.2), it is not an element of $\mathscr{H}$. Since $p^*_{\mathtt{m}}$ is just a shorthand notation for $\langle \mathbf{p}^*, \mathtt{m}\rangle$, pricing formula (44) is still implicit. It connotes that knowing $p^*_{\mathtt{m}}$ is sufficient for knowing $\mathbf{p}^*$.

The price kernel $\kappa(p_{\mathtt{m}})$ defines a continuous linear functional $\varphi(p_{\mathtt{m}}) \in \mathcal{L}^2(\Omega)'$ via

$$\begin{aligned} \varphi(p_{\mathtt{m}}) :\ \mathcal{L}^2(\Omega) &\longrightarrow \mathbb{R}, \\ \epsilon &\longmapsto E[\epsilon \kappa(p_{\mathtt{m}})]. \end{aligned}$$

Lemma 3.2.
For any $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$, *one share of the bond is priced by* $\varphi(p_{\mathtt{m}})$ *to 1*

$$E[r_0 \kappa(p_{\mathtt{m}})] = 1 \tag{50}$$

and the market payoff $\epsilon_{\mathtt{m}}$ *to* $p_{\mathtt{m}}$

$$E[\epsilon_{\mathtt{m}} \kappa(p_{\mathtt{m}})] = p_{\mathtt{m}}. \tag{51}$$

Proof. For any $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$ the following holds:

$$\begin{aligned} r_0 p_{\mathtt{m}} &= E[\epsilon_{\mathtt{m}}] - (E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}) = E[\epsilon_{\mathtt{m}}] - \frac{(E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} E[(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+] \\ &= E[\epsilon_{\mathtt{m}}] + \frac{(E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} E[\epsilon_{\mathtt{m}}(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+] - r_0 p_{\mathtt{m}} \frac{(E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} E[(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+] \\ \iff r_0 p_{\mathtt{m}} &\left(1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} E\left[(r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\right]\right) = E\left[\epsilon_{\mathtt{m}} \left(1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\right)\right] \\ \iff p_{\mathtt{m}} &= E\left[\epsilon_{\mathtt{m}} \frac{1}{r_0} \frac{1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} (r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+}{1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} E[(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+]}\right]. \end{aligned}$$

□

When applied at equilibrium $p^*_{\mathtt{m}} := \langle \mathbf{p}^*, \mathtt{m}\rangle$, $\varphi(p^*_{\mathtt{m}})$ prices any payoff $\epsilon \in \mathscr{M}$ consistently to its equilibrium monetization. For an arbitrary $\tilde{\epsilon} = r_0 \tilde{x}_0 + \langle \mathbf{q}, \tilde{\mathbf{x}}\rangle \in \mathscr{M}$:

$$E[\tilde{\epsilon} \kappa(p^*_{\mathtt{m}})] = \tilde{x}_0 \underbrace{E[r_0 \kappa(p^*_{\mathtt{m}})]}_{\overset{(50)}{=} 1} + \left\langle \underbrace{E[\mathbf{q} \kappa(p^*_{\mathtt{m}})]}_{\overset{(44)}{=} \mathbf{p}^*}, \tilde{\mathbf{x}} \right\rangle = \tilde{x}_0 + \langle \mathbf{p}^*, \tilde{\mathbf{x}}\rangle\,. \tag{52}$$

3.1.3. Equilibrium in the Space $\mathscr{H}$.

Theorem 3.1 motivates the introduction of the concept of an equilibrium in the space $\mathscr{H}$. Due to Corollary 3.1 it suffices to consider prices of the market portfolio in the open interval $\left(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right)$. In this section, we follow the lines of Dana (1999) and define an equilibrium in the space spanned by the riskless asset and the market portfolio. Afterwards, we characterize equilibria in the space $\mathscr{H}$. Eventually, we establish that the equilibrium concepts DRAME and equilibrium in $\mathscr{H}$ are equivalent.

To define an equilibrium concept in the space $\mathscr{H}$, we need to project the investors' endowments $\epsilon^1, ..., \epsilon^I \in \mathscr{M}$ on $\mathscr{H}$. Endowment $\epsilon^i = r_0 x_0^i + \langle \mathbf{q}, \mathbf{x}^i \rangle \in \mathscr{M}$ stems from endowment in shares of riskless and risky assets $(x_0^i, \mathbf{x}^i) \in \mathbb{R} \times \mathbb{R}^K$. Initially owned shares of the bond x_0^i are projected with the identity map. The projection of the shares of risky assets, initially owned by investor i, is constructed in such a way that the monetization of her original endowment equals the monetization of her projected endowment. The interpretation is that investor i sells her endowment of risky assets to obtain $\langle \mathbf{p}, \mathbf{x}^i \rangle$ and buys shares of the market portfolio worth $\langle \mathbf{p}, \mathbf{x}^i \rangle$. The projection is thus price-dependent and given by the following mapping which is well-defined for $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right)$,

$$\begin{aligned} \epsilon^i = r_0 x_0^i + \langle \mathbf{q}, \mathbf{x}^i \rangle \in \mathscr{M} \quad &\longmapsto \quad \varepsilon^i = r_0 x_0^i + \epsilon_{\mathtt{m}} x_{\mathtt{m}}^i \in \mathscr{H} \\ x_0^i \quad &\longmapsto \quad x_0^i \\ \mathbf{x}^i \quad &\longmapsto \quad x_{\mathtt{m}}^i := \frac{\langle \mathbf{p}, \mathbf{x}^i \rangle}{p_{\mathtt{m}}}. \end{aligned} \tag{53}$$

We choose this monetization-invariant projection, because investor i's demand for downside risk $\psi^i(\mathbf{p})$ does not depend on her endowment $\epsilon^i = r_0 x_0^i + \langle \mathbf{q}, \mathbf{x}^i \rangle \in \mathscr{M}$ directly. Instead, it depends on the monetization of her endowment $x_0^i + \langle \mathbf{p}, \mathbf{x}^i \rangle \in \mathbb{R}$, compare (40). Hence, it does not matter if an investor is endowed with $\epsilon^i \in \mathscr{M}$ or with $\varepsilon^i \in \mathscr{H}$ as long as the monetization is the same. The monetization-invariant projection guarantees that investor i's demand for downside risk does not change when she is equipped with $\varepsilon^i \in \mathscr{H}$ instead of $\epsilon^i \in \mathscr{M}$.

The investors $i = 1, ..., I$ are endowed with

$$\varepsilon^i = r_0 x_0^i + \epsilon_{\mathtt{m}} x_{\mathtt{m}}^i \in \mathscr{H}, \tag{54}$$

such that $\sum_{1=1}^{I} \varepsilon^i = \epsilon_{\mathtt{m}}$ and thus, in particular $\sum_{1=1}^{I} x_0^i = 0,\ \sum_{1=1}^{I} x_{\mathtt{m}}^i = 1$.
Due to Lemma 3.2, investor i's endowment ε^i has a monetary equivalent

$$x_0^i + p_{\mathtt{m}} x_{\mathtt{m}}^i = E[\varepsilon^i \kappa(p_{\mathtt{m}})], \tag{55}$$

such that she can choose any investment from her budget set

$$\mathscr{B}(\varepsilon^i, p_{\mathtt{m}}) := \{\varepsilon \in \mathscr{H} \mid E[\varepsilon \kappa(p_{\mathtt{m}})] \leqslant E[\varepsilon^i \kappa(p_{\mathtt{m}})]\} \overset{(53)}{=} \mathscr{B}(\epsilon^i, \mathbf{p}) \cap \mathscr{H}. \tag{56}$$

Notation 3.
By $x^i_{\mathtt{m}} \in \mathbb{R}$ we denote the amount of shares of the market portfolio $\mathtt{m} \in \mathbb{R}^K$ initially owned by investor i, i.e., the risky part of her endowment. Furthermore, $x_{\mathtt{m}}$ is the number of shares of the market portfolio in investment $\varepsilon = r_0 x_0 + \epsilon_{\mathtt{m}} x_{\mathtt{m}} \in \mathscr{H}$. □

Hence, we define an equilibrium in the space $\mathscr{H}$ as an allocation of the risk-free asset and the market portfolio, such that each investor holds an individually optimal investment and the asset market is cleared.

Definition 3.2 (Equilibrium in the Space $\mathscr{H}$).
An equilibrium in the space $\mathscr{H}$ consists of a price for the market portfolio $p^*_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$ and an allocation of payoffs $\varepsilon^{1,*}, ..., \varepsilon^{I,*} \in \mathscr{H}$, such that

i) each $\varepsilon^{i,*}$ is individually optimal at price $p^*_{\mathtt{m}}$, i.e., for $i = 1, ...I$ it solves

$$\max_{\varepsilon \in \mathscr{B}(\varepsilon^i, p^*_{\mathtt{m}})} U^i(\mathcal{M}(\varepsilon), \mathcal{D}(\varepsilon, r_0 E[\varepsilon^i \kappa(p^*_{\mathtt{m}})])),$$

ii) and the allocation is feasible, i.e.,

$$\sum_{i=1}^{I} \varepsilon^{i,*} = \epsilon_{\mathtt{m}}.$$ □

The asset market $\mathscr{H}$ offers one riskless and one risky asset: the market portfolio. Thus, it constitutes the special case $K = 1$ and we can apply the separation theorem as developed in Section 2.4. We therefore briefly reformulate portfolio selection theory with one risky asset.

Due to monotonicity properties of investor i's utility function, cf. Assumption 2, she will always choose $x_0 = E[\varepsilon^i \kappa(p_{\mathtt{m}})] - p_{\mathtt{m}} x_{\mathtt{m}}$, such that downside risk of investment $\varepsilon = r_0 x_0 + \epsilon_{\mathtt{m}} x_{\mathtt{m}} \in \mathscr{B}(\varepsilon^i, p_{\mathtt{m}})$ is given by

$$\mathcal{D}(\varepsilon, r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}})]) = \|((r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}}) x_{\mathtt{m}})_+\| = \mathfrak{D}(x_{\mathtt{m}}, r_0 p_{\mathtt{m}}).$$

Let $\pi(p_{\mathtt{m}}) := E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}$ denote the expected excess market payoff. Further, the risky set of investment $\varepsilon = r_0 x_0 + \epsilon_{\mathtt{m}} x_{\mathtt{m}}$ and the downside semivariance of $\epsilon_{\mathtt{m}}$ w.r.t. ε are given by

$$\mathscr{R}(x_{\mathtt{m}}, r_0 p_{\mathtt{m}}) = \{\omega \in \Omega \mid r_0 p_{\mathtt{m}} x_{\mathtt{m}} > \epsilon_{\mathtt{m}}(\omega) x_{\mathtt{m}}\}$$
$$C(x_{\mathtt{m}}, r_0 p_{\mathtt{m}}) = E_{\mathscr{R}(x_{\mathtt{m}}, r_0 p_{\mathtt{m}})}[(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})^2] \geqslant 0.$$

As there is only one risky asset, there is no downside cosemivariance but *downside semivariance.* A solution to $C(x_{\mathtt{m}}, r_0 p_{\mathtt{m}}) x_{\mathtt{m}} = \pi(p_{\mathtt{m}})$, referred to as *reference* ***amount*** *of shares of the market portfolio*, is labeled $x^{\mathsf{ref}}_{\mathtt{m}}(p_{\mathtt{m}})$ and

$$\varrho(p_{\mathtt{m}}) := \mathfrak{D}(x^{\mathsf{ref}}_{\mathtt{m}}(p_{\mathtt{m}}), r_0 p_{\mathtt{m}})$$

is the market price of downside risk.

Lemma 3.3.
For $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$, *we have*

$$x_{\mathtt{m}}^{\mathsf{ref}}(p_{\mathtt{m}}) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} > 0, \tag{57}$$

$$\varrho(p_{\mathtt{m}}) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})} > 0. \tag{58}$$

Moreover, when understood as functions of $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$, $x_{\mathtt{m}}^{\mathsf{ref}}(p_{\mathtt{m}})$ *and* $\varrho(p_{\mathtt{m}})$ *are continuous and strictly decreasing with*

$$\lim_{p_{\mathtt{m}} \to 0} \varrho(p_{\mathtt{m}}) = +\infty, \tag{59}$$

$$\lim_{p_{\mathtt{m}} \to \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]} \varrho(p_{\mathtt{m}}) = 0. \tag{60}$$

Thus, $\varrho(p_{\mathtt{m}})$ *constitutes a bijection between* $\left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$ *and* $\mathbb{R}_{++}$.

Proof. $p_{\mathtt{m}} < \frac{1}{r_0} E[\epsilon_{\mathtt{m}}] \Longrightarrow \pi(p_{\mathtt{m}}) > 0$ and, hence, a solution $x_{\mathtt{m}}^{\mathsf{ref}}(p_{\mathtt{m}})$ to $C(x_{\mathtt{m}}, r_0 p_{\mathtt{m}}) x_{\mathtt{m}} = \pi(p_{\mathtt{m}})$ has to be strictly positive. Thus, $C(x_{\mathtt{m}}^{\mathsf{ref}}(p_{\mathtt{m}}), r_0 p_{\mathtt{m}}) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2 > 0$ and equation (57) follows. Furthermore,

$$\begin{aligned} \varrho(p_{\mathtt{m}}) &= \mathfrak{D}(x_{\mathtt{m}}^{\mathsf{ref}}(p_{\mathtt{m}}), r_0 p_{\mathtt{m}}) \\ &= \left\| ((r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}}) x_{\mathtt{m}}^{\mathsf{ref}})_+ \right\| \\ &= x_{\mathtt{m}}^{\mathsf{ref}}(p_{\mathtt{m}}) \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}). \end{aligned}$$

□

Proposition 3.2 (Separation Theorem for One Risky Asset).
Under the hypothesis of Assumption 2 and for any $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$ *with* $\rho(p_{\mathtt{m}}) \in (0, \rho_{U^i})$, *investor* i*'s individually optimal amount of shares of the market portfolio* $x_{\mathtt{m}}^{i,*}(p_{\mathtt{m}}) \in \mathbb{R}$ *is given by*

$$x_{\mathtt{m}}^{i,*}(p_{\mathtt{m}}) = \frac{\Psi^i(p_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})} \tag{61}$$

where

$$\Psi^i(p_{\mathtt{m}}) := \arg\max_{d \geqslant 0} U^i(r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}})] + \varrho(p_{\mathtt{m}}) d, d) \tag{62}$$

is investor i*'s individually optimal amount of downside risk at price* $p_{\mathtt{m}}$.

Proof. Since $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$, we have $x_{\mathtt{m}}^{\mathsf{ref}}(p_{\mathtt{m}}) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})^2} > 0$ by Lemma 3.3. The assertion follows from the separation theorem stated in Proposition 3.1. □

Similar as in Theorem 3.1, we are now in a position to characterize equilibria in the space $\mathscr{H}$. The proof is the same as for Theorem 3.1 and Corollary 3.1.

COROLLARY 3.2 (Characterization of Equilibria in the Space $\mathscr{H}$).
Let the assumptions of this chapter be fulfilled. Let $p_{\mathtt{m}}^*, \varepsilon^{1,*}, ..., \varepsilon^{I,*}$ constitute an equilibrium in the space $\mathscr{H}$.
Then, the following holds:

i) The allocation of assets is given by

$$\varepsilon^{i,*} = r_0 \left(E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)] - p_{\mathtt{m}}^* \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} \right) + \epsilon_{\mathtt{m}} \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)}. \tag{63}$$

ii) The aggregate demand for downside risk equals the aggregate downside risk of the market payoff, i.e.,

$$\sum_{i=1}^{I} \Psi^i(p_{\mathtt{m}}^*) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*) > 0. \tag{64}$$

iii) The market-clearing prices satisfy

$$p_{\mathtt{m}}^* = \frac{1}{r_0} \left(E\left[\epsilon_{\mathtt{m}}\right] - \varrho(p_{\mathtt{m}}^*) \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*) \right). \tag{65}$$

iv) The market price of downside risk equals the Sortino ratio of the market payoff, i.e.,

$$\varrho(p_{\mathtt{m}}^*) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} > 0. \tag{66}$$

The following theorem confirms that the asset market $\mathscr{M}$ has the same equilibria as the restricted asset market $\mathscr{H}$. Dana (1999) states an analogous assertion, but a formal proof is missing in her paper. The proof, we provide here, is constructive.

THEOREM 3.2 (Equivalence of Equilibria).
Let the assumptions of this chapter be satisfied. Then, the following holds:

i) Let the investors $i = 1, ..., I$ be endowed with $\epsilon^i = r_0 x_0^i + \langle \mathbf{q}, \mathbf{x}^i \rangle \in \mathscr{M}$.
If $\mathbf{p}^* \in \mathbb{R}^K$ and $\epsilon^{1,*}, ..., \epsilon^{I,*} \in \mathscr{M}$ constitute a DRAME, then $p_{\mathtt{m}}^*, \varepsilon^{1,*}, ..., \varepsilon^{I,*}$, where

$$p_{\mathtt{m}}^* = \langle \mathbf{p}^*, \mathbf{x}_{\mathtt{m}} \rangle \in \mathbb{R},$$
$$\varepsilon^i = r_0 x_0^i + \epsilon_{\mathtt{m}} \frac{\langle \mathbf{p}^*, \mathbf{x}^i \rangle}{p_{\mathtt{m}}^*} \in \mathscr{H},$$
$$\varepsilon^{i,*} = \epsilon^{i,*} \in \mathscr{H},$$

for $i = 1, ..., I$ represent an equilibrium in the space $\mathscr{H}$.

ii) Let the investors $i = 1, ..., I$ be endowed with $\varepsilon^i = r_0 x_0^i + \epsilon_{\mathtt{m}} x_{\mathtt{m}}^i \in \mathscr{H}$.
If $p_{\mathtt{m}}^* \in \mathbb{R}$ and $\varepsilon^{1,*}, ..., \varepsilon^{I,*} \in \mathscr{H}$ constitute an equilibrium in the space $\mathscr{H}$, then $\mathbf{p}^*, \epsilon^{1,*}, ..., \epsilon^{I,*}$, given by

$$\epsilon^i = r_0 x_0^i + \langle \mathbf{q}, x_{\mathtt{m}}^i \mathtt{m} \rangle \in \mathscr{M},$$
$$\mathbf{p}^* = E[\mathbf{q}\kappa(p_{\mathtt{m}}^*)] \in \mathbb{R}^K,$$
$$\epsilon^{i,*} = \varepsilon^{i,*} \in \mathscr{M},$$

for $i = 1, ..., I$ represents a DRAME.

PROOF.

i) The allocation is obviously feasible and $\varepsilon^{i,*} \in \mathscr{B}(\varepsilon^i, p_{\mathtt{m}}^*) = \mathscr{B}(\epsilon^i, \mathbf{p}^*) \cap \mathscr{H}$ because of the monetization-invariant projection of endowments. For $i = 1, ..., I$, $\varepsilon^{i,*} = \epsilon^{i,*} \in \mathscr{H}$ is the individually optimal payoff in the space $\mathscr{M}$ by Theorem 3.1. Thus, it must also be individually optimal in the subspace $\mathscr{H}$.
ii) Feasibility is again clear. In Lemma 3.2, we showed that $\mathbf{p}^* := E[\mathbf{q}\kappa(p_{\mathtt{m}}^*)]$ implies

(67) $$\langle \mathbf{p}^*, \mathtt{m} \rangle = E[\langle \mathbf{q}, \mathtt{m} \rangle \, \kappa(p_{\mathtt{m}}^*)] = E[\epsilon_{\mathtt{m}} \kappa(p_{\mathtt{m}}^*)] \overset{(51)}{=} p_{\mathtt{m}}^*.$$

Moreover, it follows from equations (85) and (84), which are given in the appendix to this chapter, that

$$\boldsymbol{\pi}(\mathbf{p}^*) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)^2} \mathbf{C}(\mathtt{m}, r_0 \mathbf{p}^*)\mathtt{m}.$$

Hence,

(68) $$\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)^2} \mathtt{m}$$

solves $\mathbf{C}(\mathbf{x}, r_0 \mathbf{p}^*)\mathbf{x} = \boldsymbol{\pi}(\mathbf{p}^*)$, i.e., the market portfolio $\mathtt{m}$ is collinear to a reference portfolio at prices $\mathbf{p}^*$. Furthermore,

(69) $$\rho(\mathbf{p}^*) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} = \varrho(p_{\mathtt{m}}^*).$$

Equation (67) implies that the monetary equivalents of endowments are the same, i.e.,

(70) $$E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)] = r_0 x_0^i + p_{\mathtt{m}}^* x_{\mathtt{m}}^i = r_0 x_0^i + \langle \mathbf{p}^*, x_{\mathtt{m}}^i \mathtt{m} \rangle$$

and, together with (69), it follows that

(71) $$\psi^i(\mathbf{p}^*) = \Psi^i(p_{\mathtt{m}}^*).$$

Due to the separation theorem (cf. Proposition 3.1), the individually optimal portfolio of risky assets at prices $\mathbf{p}^*$ equals

(72) $$\mathbf{x}^{i,*}(\mathbf{p}^*) \overset{(39)}{=} \frac{\psi^i(\mathbf{p}^*)}{\rho(\mathbf{p}^*)}\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*) \underset{(69)}{\overset{(68)}{=}} \frac{\psi^i(\mathbf{p}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}\mathtt{m} \overset{(71)}{=} \frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}\mathtt{m}.$$

Consequently, the individually optimal payoff in $\mathscr{M}$ is given by

$$\begin{aligned} \epsilon^{i,*} &= r_0\left(x_0^i + \langle \mathbf{p}^*, x^i_{\mathtt{m}}\mathtt{m}\rangle - p^*_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}\right) + \epsilon_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} \\ &\overset{(70)}{=} r_0\left(E[\varepsilon^i\kappa(p^*_{\mathtt{m}})] - p^*_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}\right) + \epsilon_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} \\ &\overset{(63)}{=} \varepsilon^{i,*} \in \mathscr{H} \end{aligned}$$

which completes the proof. □

Theorem 3.2 connotes that there exists a DRAME if and only if there exists an equilibrium in the space $\mathscr{H}$. In addition, the equilibrium allocation of assets is the same. It follows that a DRAME is unique if and only if the corresponding equilibrium in $\mathscr{H}$ is unique.

In Section 3.2, we will exploit the simpler structure of the space spanned by the risk-free asset and the market portfolio, and address the problem of existence and uniqueness of asset market equilibria in the space $\mathscr{H}$. Before we do so, we argue that an equilibrium in the space $\mathscr{H}$ represents a mean downside risk equilibrium.

3.1.4. Mean Downside Risk Equilibria.

A mean downside risk equilibrium is an individually optimal and market-clearing allocation of mean and downside risk among the I investors. From the Separation Theorem, reformulated for one risky asset in Lemma 3.2, we already know the form of the downside efficient frontier $r_0(x_0^i + p_{\mathtt{m}} x^i_{\mathtt{m}}) + \varrho(p_{\mathtt{m}})d$. It gives the maximal, at price $p_{\mathtt{m}}$ achievable mean s. t. a prescribed level d of downside risk. We further know that $\Psi^i(p_{\mathtt{m}})$ is the individually optimal amount of downside risk for investor i at price $p_{\mathtt{m}}$.

Let $p^*_{\mathtt{m}}, \varepsilon^{1,*}, ..., \varepsilon^{I,*}$ constitute an equilibrium in the space $\mathscr{H}$.
For $i = 1, ..., I$, mean and downside risk of

$$\varepsilon^{i,*} = r_0\left(E[\varepsilon^i\kappa(p^*_{\mathtt{m}})] - p^*_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}\right) + \epsilon_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}$$

are given by

$$\mathcal{M}(\varepsilon^{i,*}) = E[\varepsilon^{i,*}]$$

and

$$\begin{aligned}
\mathcal{D}(\varepsilon^{i,*}, r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)]) &= \left\| \left(r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)] - \varepsilon^{i,*}\right)_+ \right\| \\
&= \left\| \left(r_0 p_{\mathtt{m}}^* \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} - \epsilon_{\mathtt{m}} \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} \right)_+ \right\| \\
&= \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} \left\| (r_0 p_{\mathtt{m}}^* - \epsilon_{\mathtt{m}})_+ \right\| \\
&= \Psi^i(p_{\mathtt{m}}^*).
\end{aligned}$$

It follows that $\sum_{i=1}^I \mathcal{M}(\varepsilon^{i,*}) = E[\epsilon_{\mathtt{m}}]$, since $\sum_{i=1}^I \varepsilon^{i,*} = \epsilon_{\mathtt{m}}$ and

$$\sum_{i=1}^I \mathcal{D}(\varepsilon^{i,*}, r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)]) = \sum_{i=1}^I \Psi^i(p_{\mathtt{m}}^*) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)$$

because of equation (64) in Corollary 3.2. Hence, the market for mean and the market for downside risk are cleared.

Moreover, $\mathcal{D}(\varepsilon^{i,*}, r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)]) = \Psi^i(p_{\mathtt{m}}^*)$ is the individually optimal amount of downside risk by construction. Also,

$$\begin{aligned}
\mathcal{M}(\varepsilon^{i,*}) &= r_0 \left(E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)] - p_{\mathtt{m}}^* \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} \right) + E[\epsilon_{\mathtt{m}}] \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} \\
&\overset{(66)}{=} r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)] + \varrho(p_{\mathtt{m}}^*) \Psi^i(p_{\mathtt{m}}^*)
\end{aligned}$$

is the maximal mean that is achievable given downside risk is smaller than $\Psi^i(p_{\mathtt{m}}^*)$, and thus the individually optimal amount of mean s.t. downside risk is less than $\Psi^i(p_{\mathtt{m}}^*)$.

To sum up, the allocation $\mathcal{M}(\varepsilon^{i,*}), \mathcal{D}(\varepsilon^{i,*}, r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}}^*)])$, for $i = 1, ..., I$, is an allocation of mean and downside risk that is individually optimal at price $p_{\mathtt{m}}^*$ and market-clearing. As a consequence, it represents a mean downside risk equilibrium. The equilibrium market price of downside risk is then given by $\varrho(p_{\mathtt{m}}^*)$.

3.2. Existence and Uniqueness of Downside Risk Asset Market Equilibria

Firstly, we establish that the existence and uniqueness problem for equilibria in the space $\mathscr{H}$ is in fact one-dimensional. Secondly, we prove that such an equilibrium always exists if the assumptions of this chapter are met. As a consequence, a DRAME in the marketed space $\mathscr{M}$ always exists. Thirdly, we give a sufficient criterion which guarantees uniqueness. Eventually, we provide an example where a unique DRAME exists.

From Corollary 3.1 we know that it suffices to consider prices $p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$. The following lemma connotes that the domain of prices of the market portfolio may have to be restricted further.

LEMMA 3.4.
Under the hypothesis of Assumption 2, let $\Psi^i(p_{\mathtt{m}})$ be investor i's individually optimal demand for downside risk and let $\Psi(p_{\mathtt{m}}) := \sum_{i=1}^{I} \Psi^i(p_{\mathtt{m}})$ denote the aggregate demand for downside risk. Then, there exists $\underline{p_{\mathtt{m}}} \in \mathbb{R}_+$ such that $\Psi(p_{\mathtt{m}})$ is well-defined and continuous for all $p_{\mathtt{m}} \in \left(\underline{p_{\mathtt{m}}}, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right)$.

PROOF. Let $i \in \{1, ..., I\}$ be arbitrary. $\Psi^i(p_{\mathtt{m}})$ is well-defined for all $p_{\mathtt{m}} \in \mathbb{R}_{++}$ with $\varrho(p_{\mathtt{m}}) < \rho_{U^i}$. In Lemma 3.3, we showed that

$$\varrho : \left(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right) \longrightarrow \mathbb{R}_{++}$$
$$p_{\mathtt{m}} \longmapsto \varrho(p_{\mathtt{m}})$$

is a continuous bijection. Thus, we can set $\underline{p^i_{\mathtt{m}}}$ to be the unique solution to $\varrho(p_{\mathtt{m}}) = \rho_{U^i}$, if the limiting slope of investor i's indifference curves is finite. If $\rho_{U^i} = \infty$, we set $\underline{p^i_{\mathtt{m}}} = 0$. Then, $\Psi^i(p_{\mathtt{m}})$ is well-defined for all $p_{\mathtt{m}} \in \left(\underline{p^i_{\mathtt{m}}}, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right)$. Continuity of Ψ^i follows from continuity of ϱ and continuity of investor i's utility function U^i, cf. Assumption 2.
As a consequence, $\Psi(p_{\mathtt{m}})$ is well-defined and continuous for all $p_{\mathtt{m}} > \underline{p_{\mathtt{m}}} := \max\left\{\underline{p^i_{\mathtt{m}}} \mid i = 1, ..., I\right\}$. □

The following proposition points out that the existence and uniqueness problem for equilibria in the space $\mathscr{H}$ is one-dimensional.

PROPOSITION 3.3.
If the assumptions of this chapter are fulfilled, then a solution $p^*_{\mathtt{m}} \in \left(\underline{p_{\mathtt{m}}}, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right)$ to

$$\Psi(p_{\mathtt{m}}) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}) \tag{73}$$

completely determines an equilibrium in the space $\mathscr{H}$ by

$$\varepsilon^{i,*} = r_0\left(E[\varepsilon^i \kappa(p^*_{\mathtt{m}})] - p^*_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}\right) + \epsilon_{\mathtt{m}}\frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}$$

for $i = 1, ..., I$.

PROOF. A solution $p^*_{\mathtt{m}} \in \left(\underline{p_{\mathtt{m}}}, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right)$ to $\Psi(p_{\mathtt{m}}) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})$ determines investor i's individually optimal amount of shares of the market portfolio

$$x^{i,*}_{\mathtt{m}}(p^*_{\mathtt{m}}) = \frac{\Psi^i(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})},$$

due to Proposition 3.2 and therewith the individually optimal allocation of payoffs $\varepsilon^{1,*}, ..., \varepsilon^{I,*} \in \mathscr{H}$ among the I investors.

This allocation is market-clearing since

$$\sum_{i=1}^{I} \varepsilon^{i,*} = r_0 \left(\sum_{i=1}^{I} E[\varepsilon^i \kappa(p^*_{\mathtt{m}})] - p^*_{\mathtt{m}} \frac{\Psi(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} \right) + \epsilon_{\mathtt{m}} \frac{\Psi(p^*_{\mathtt{m}})}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}$$

$$= r_0 (E[\epsilon_{\mathtt{m}} \kappa(p^*_{\mathtt{m}})] - p^*_{\mathtt{m}}) + \epsilon_{\mathtt{m}} \overset{(51)}{=} \epsilon_{\mathtt{m}}. \qquad \square$$

Proposition 3.3 is somehow a reversal of Corollary 3.2, where the corresponding assertion reads as follows. If $p^*_{\mathtt{m}}, \varepsilon^{1,*}, ..., \varepsilon^{I,*}$ constitute an equilibrium in the space $\mathscr{H}$, then $p^*_{\mathtt{m}}$ solves (73). Thus, the problem of finding equilibria in the space $\mathscr{H}$ boils down to the problem of finding solutions to the one-dimensional equation (73). Note that the analog equation (43) in the space $\mathscr{M}$ is K-dimensional and, thus, much more difficult to solve.

Theorem 3.3 (Existence of Equilibria).
Let the assumptions of this chapter be fulfilled. Then, there exists an equilibrium in the space $\mathscr{H}$.

Proof. In Section 3.A.2 in the appendix to this chapter, we show that equation (73) has a solution $p^*_{\mathtt{m}}$. This solution determines an equilibrium in the space $\mathscr{H}$ by Proposition 3.3. $\square$

By Theorem 3.2, there exists a DRAME in the marketed space $\mathscr{M}$.

Proposition 3.4 (Uniqueness of Equilibria).
Let, in addition the assumptions of this chapter, Ψ be a monotonically decreasing function. Then, the equilibrium is unique.

Proof. Downside risk of the market portfolio $\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}) = \|(r_0 p_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\|$ is strictly increasing in $p_{\mathtt{m}}$. If Ψ is decreasing in $p_{\mathtt{m}}$, there can only be one solution to equation (73). $\square$

Hence, two central results of CAPM equilibria can be reproduced when replacing standard deviation by downside risk. These equilibria exist if utility functions fulfill some regularity conditions, stated in Assumption 2, and are unique if aggregate demand for downside risk fulfills a monotonicity property. In fact, the monotonicity requirement guaranteeing uniqueness of DRAMEs is weaker than the monotonicity requirement for CAPM equilibria. In Proposition 3.4, we require aggregate demand for downside risk to be a decreasing function whereas for CAPM equilibria *strict* monotonicity of aggregate demand for risk is required, see, e.g., Dana (1999) or Wenzelburger (2010).

In the mean-variance framework with one risky asset, the market price of standard deviation is defined by

$$\varrho^{\mathsf{MV}}(p_{\mathtt{m}}) = \sqrt{\langle \pi(p_{\mathtt{m}}), V_{\mathtt{m}}^{-1} \pi(p_{\mathtt{m}}) \rangle}$$

$$= \frac{\pi(p_{\mathtt{m}})}{\sqrt{V_{\mathtt{m}}}} = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\sigma_{\mathtt{m}}},$$

where $\sigma_{\mathtt{m}}$ is the standard deviation of the market payoff. Similar to $\varrho(p_{\mathtt{m}})$, $\varrho^{\mathsf{MV}}(p_{\mathtt{m}})$ is strictly decreasing in $p_{\mathtt{m}}$.

Let

$$\Psi^{\mathsf{MV}}(p_{\mathtt{m}}) := \sum_{i=1}^{I} \arg\max_{\sigma \geqslant 0} U^i(r_0(x_0^i + p_{\mathtt{m}} x_{\mathtt{m}}^i) + \varrho^{\mathsf{MV}}(p_{\mathtt{m}})\sigma, \sigma)$$

denote the aggregate demand for standard deviation.[14]

The asset market is then cleared for a price $p_{\mathtt{m}}^*$ that solves

$$\Psi^{\mathsf{MV}}(p_{\mathtt{m}}) = \sigma_{\mathtt{m}}, \tag{74}$$

compare Wenzelburger (2010). Aggregate demand for standard deviation has to equal standard deviation of the market payoff.

Condition (74) is structurally equivalent to equation (73). The crucial difference is that $\sigma_{\mathtt{m}}$ is a constant whereas the downside risk of the market payoff $\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})$ is strictly increasing in $p_{\mathtt{m}}$. Therefore, we can mitigate the strict monotonicity requirement that is necessary in the mean-variance framework. The schematic Figure 8 provides an illustration.

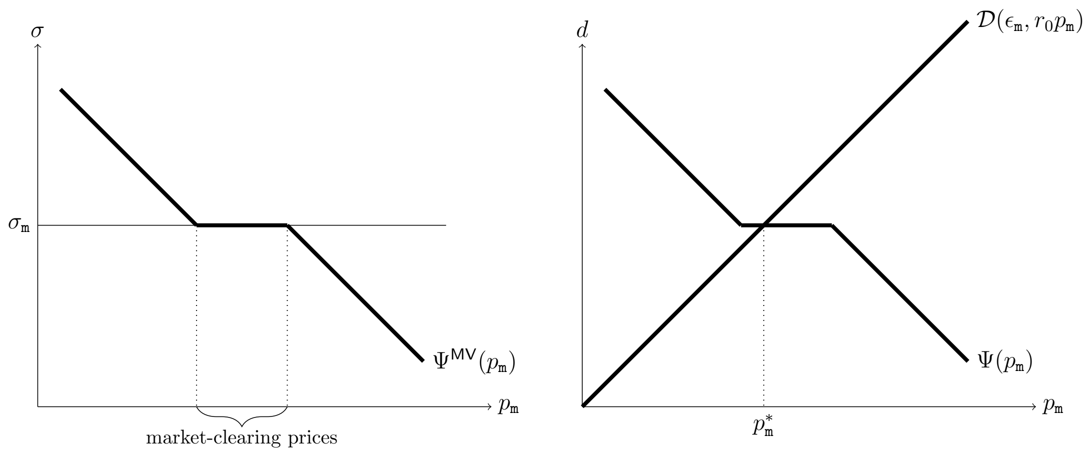

FIGURE 8. Uniqueness of equilibria. Mean-variance versus mean-downside-risk framework.

As a consequence, in the mean-downside-risk framework, asset market equilibria are unique for a broader range of investors' preferences.

[14]Originally, ϱ^{MV} is called market price of risk and Ψ^{MV} aggregate demand for risk, compare Wenzelburger (2010). Since we do not consider standard deviation as risk, we rename these items accordingly.

Example 3.1 (An Additive Separable Utility Function).
Suppose each investor is characterized by a utility function of the form $U^i(m,d) = m - \frac{1}{2\vartheta^i} d^2$, where $\vartheta^i > 0$ denotes investor i's risk tolerance. Let $\vartheta = \sum_{i=1}^I \vartheta^i > 0$ denote aggregate risk tolerance. Consider investor i's willingness to take downside risk

$$\Psi^i(p_{\mathtt{m}}) = \arg\max_{d \geqslant 0} \left(r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}})] + \varrho(p_{\mathtt{m}}) d - \frac{1}{2\vartheta^i} d^2 \right).$$

The first order condition implies $\Psi^i(p_{\mathtt{m}}) = \vartheta^i \varrho(p_{\mathtt{m}})$ which is a strictly decreasing function. Thus, a unique equilibrium exists. The equilibrium allocation of the market portfolio is given by

$$x_{\mathtt{m}}^{i,*} = \frac{\Psi^i(p_{\mathtt{m}}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)} = \frac{\Psi^i(p_{\mathtt{m}}^*)}{\Psi(p_{\mathtt{m}}^*)} = \frac{\vartheta^i \varrho(p_{\mathtt{m}}^*)}{\vartheta \varrho(p_{\mathtt{m}}^*)} = \frac{\vartheta^i}{\vartheta}.$$ □

3.3. Economic Implications

In the previous section, existence and uniqueness of downside risk asset market equilibria have been established. Moreover, a DRAME pricing formula has been specified. In this section, we analyze the ensuing economic implications.
We argue in the complete marketed space $\mathscr{M}$ and consider investments $\epsilon = r_0 x_0 + \langle \mathbf{q}, \mathbf{x} \rangle$.

3.3.1. Arbitrage-Free DRAME Prices.

A major drawback of the CAPM, probably the most severe one, is that equilibrium prices are not arbitrage-free as demonstrated by Dybvig & Ingersoll (1982). Starting from the standard mean-variance CAPM pricing formula, they provide an explicit example which constitutes an arbitrage opportunity. Our DRAME pricing formula (48) differs from the original CAPM pricing formula only through the fact that we replaced asset k's covariance with the market payoff by its downside cosemivariance. Yet, this change makes equilibrium prices arbitrage-free.

Investor i can monetize her endowment and invest this money in the risk-free asset which yields a payoff $r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i \rangle)$. An arbitrage opportunity is a portfolio which yields a future payoff that is almost surely greater than $r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i \rangle)$ and with a strictly positive probability strictly greater than this amount.

Definition 3.3 (Arbitrage Opportunity).
A pair $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$, consisting of a price vector and a portfolio, constitutes an *arbitrage opportunity*, if

$$\mathbb{P}(\langle \mathbf{q}, \tilde{\mathbf{x}} \rangle \geqslant r_0 \langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}} \rangle) = 1, \tag{75}$$

$$\mathbb{P}(\langle \mathbf{q}, \tilde{\mathbf{x}} \rangle > r_0 \langle \tilde{\mathbf{p}}, \tilde{\mathbf{x}} \rangle) > 0. \tag{76}$$

A price vector $\tilde{\mathbf{p}}$ is said to be *arbitrage-free*, if there does not exist a portfolio $\tilde{\mathbf{x}}$ making $(\tilde{\mathbf{p}}, \tilde{\mathbf{x}})$ an arbitrage opportunity. □

Since

$$w^i(\mathbf{x},\mathbf{p}) = r_0(x_0^i + \langle \mathbf{p}, \mathbf{x}^i \rangle) + \langle \mathbf{q} - r_0\mathbf{p}, \mathbf{x} \rangle \,,$$

equations (75) and (76) are equivalent to

$$\begin{aligned} \mathbb{P}(w^i(\tilde{\mathbf{p}},\tilde{\mathbf{x}}) \geqslant r_0(x_0^i + \langle \tilde{\mathbf{p}}, \mathbf{x}^i \rangle)) &= 1 \\ \mathbb{P}(w^i(\tilde{\mathbf{p}},\tilde{\mathbf{x}}) > r_0(x_0^i + \langle \tilde{\mathbf{p}}, \mathbf{x}^i \rangle)) &> 0. \end{aligned}$$

For target vector $\mathbf{t} = r_0\mathbf{p}$, arbitrage opportunities are riskless:

$$\mathfrak{D}(\tilde{\mathbf{x}}, r_0\tilde{\mathbf{p}}) = \left\| \langle r_0\tilde{\mathbf{p}} - \mathbf{q}, \tilde{\mathbf{x}} \rangle_+ \right\| \overset{(75)}{=} 0.$$

Moreover, conditions (75) and (76) imply $\langle \boldsymbol{\pi}(\tilde{\mathbf{p}}), \tilde{\mathbf{x}} \rangle = \langle E[\mathbf{q}] - r_0\tilde{\mathbf{p}}, \tilde{\mathbf{x}} \rangle > 0$.

Theorem 3.4 (Arbitrage-Free DRAME Prices).
DRAME prices $\mathbf{p}^*$, given by formula (44), are arbitrage-free.

Proof. Let $p^*_{\mathtt{m}} = \langle \mathbf{p}^*, \mathtt{m} \rangle$ be the equilibrium price of the market portfolio. Recall that the individually optimal portfolio of risky assets at DRAME prices is $\mathbf{x}^{i,*} = \frac{\psi^i(\mathbf{p}^*)}{\mathfrak{D}(\mathtt{m}, r_0\mathbf{p}^*)}\mathtt{m}$ with mean $\mathfrak{M}^i(\mathbf{x}^{i,*}, \mathbf{p}^*) = r_0 E[\epsilon^i \kappa(p^*_{\mathtt{m}})] + \rho(\mathbf{p}^*)\psi^i(\mathbf{p}^*)$ and downside risk $\mathfrak{D}(\mathbf{x}^{i,*}, r_0\mathbf{p}^*) = \psi^i(\mathbf{p}^*)$, compare formula (46).
Now, assume there is a portfolio $\tilde{\mathbf{x}}$ making $(\mathbf{p}^*, \tilde{\mathbf{x}})$ an arbitrage opportunity. Then, $\langle \boldsymbol{\pi}(\mathbf{p}^*), \tilde{\mathbf{x}} \rangle > 0$ and we can choose $\lambda > 0$ large enough, such that $\lambda \langle \boldsymbol{\pi}(\mathbf{p}^*), \tilde{\mathbf{x}} \rangle > \rho(\mathbf{p}^*)\psi^i(\mathbf{p}^*)$. This implies

$$\mathfrak{M}^i(\lambda\tilde{\mathbf{x}}, \mathbf{p}^*) = r_0 E[\epsilon^i \kappa(p^*_{\mathtt{m}})] + \lambda \langle \boldsymbol{\pi}(\mathbf{p}^*), \tilde{\mathbf{x}} \rangle > \mathfrak{M}^i(\mathbf{x}^{i,*}, \mathbf{p}^*),$$

while $\mathfrak{D}(\lambda\tilde{\mathbf{x}}, r_0\mathbf{p}^*) = 0 \leqslant \mathfrak{D}(\mathbf{x}^{i,*}, r_0\mathbf{p}^*)$. As a consequence, at prices $\mathbf{p}^*$, the portfolio $\lambda\tilde{\mathbf{x}}$ induces a greater utility to investor i than $\mathbf{x}^{i,*}$

$$U^i(\mathfrak{M}^i(\lambda\tilde{\mathbf{x}}, \mathbf{p}^*), \mathfrak{D}(\lambda\tilde{\mathbf{x}}, r_0\mathbf{p}^*)) > U^i(\mathfrak{M}^i(\mathbf{x}^{i,*}, \mathbf{p}^*), \mathfrak{D}(\mathbf{x}^{i,*}, r_0\mathbf{p}^*)).$$

This, however, contradicts individual optimality of $\mathbf{x}^{i,*}$ at DRAME prices $\mathbf{p}^*$. □

The reason why DRAME prices $\mathbf{p}^*$, given by formula (44), are arbitrage-free and CAPM prices are not, stems from the fact that arbitrage is defined in terms of monotonicity. An arbitrage opportunity at equilibrium prices is a portfolio which yields a future payoff that is almost surely *greater* than $r_0 E[\epsilon^i \kappa(p^*_{\mathtt{m}})]$ and with a strictly positive probability *strictly greater* than this amount. For target vector $\mathbf{t} = r_0\mathbf{p}$, downside risk is monotone w.r.t. portfolio choice as shown in Proposition 1.2 iii). Hence, arbitrage opportunities are downside risk-free. Standard deviation, however, is a non-monotonic deviation measure. Thus, arbitrage portfolios are not risk-free when measuring risk by standard deviation, because they create a nonconstant payoff. Moreover, arbitrage portfolios do not even have to be efficient in the sense of Markowitz. As a consequence, CAPM prices allow for arbitrage opportunities.

To asses arbitrage opportunities as "risky" and "inefficient" portfolios emphasizes the dubiousness and danger of using standard deviation as a measure of risk. Portfolio selection and asset pricing theory should not be based on such a doubtful risk measure.

Since the monotonicity property is crucial, the absence of arbitrage opportunities only holds for target vector $\mathbf{t} = r_0\mathbf{p}$. The result stated in Theorem 3.4 is the only result in this chapter that does not carry over to other targets.

3.3.2. Strictly Positive DRAME Prices.

In this chapter, risky assets are assumed to be shares of ordinary stocks. Thus their payoffs are positive, i.e., for $k = 1, ...K$

$$q_k(\omega) \geqslant 0 \text{ for all } \omega \in \Omega.$$

Then, a negative equilibrium price makes asset k downside riskless, since it always yields a payoff above its target $r_0 p_k^*$. In addition, its negative price in combination with its positive payoff makes it an arbitrage opportunity.

The following proposition states that DRAME prices are strictly positive. This result is particularly remarkable because the mean-variance CAPM allows for negative stock prices.

PROPOSITION 3.5 (Strictly Positive DRAME Prices).
Given Assumption 3, DRAME prices are strictly positive, i.e., $p_k^* > 0$ *for all* $k = 1, ...K$.

PROOF. Due to Corollary 3.1, $p_{\mathtt{m}}^* < \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]$ and, by Assumption 3, $q_k \geqslant 0$. Then, in the DRAME pricing formula (44) each term is positive and the assertion follows. □

3.3.3. Downside Security Market Line.

Next, we compute a downside analogon to the beta coefficients, which will yield a downside security market line.

PROPOSITION 3.6 (Downside Security Market Line).
In equilibrium, the downside security market line corresponding to asset k is given by

$$E[q_k] - r_0 p_k^* = \beta_k^* \left(E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^* \right), \tag{77}$$

where

$$\beta_k^* := \frac{E[(r_0 p_k^* - q_k)\,(r_0 p_{\mathtt{m}}^* - \epsilon_{\mathtt{m}})_+]}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)^2}$$

is called downside beta coefficient of asset k *and plays the role of the classical beta coefficients.*

PROOF. The assertion directly follows from equation (48). □

The downside beta coefficient of asset k equals the equilibrium downside cosemivariance of asset k's payoff with the market payoff, scaled by the squared equilibrium downside risk of $\epsilon_{\mathtt{m}}$. It describes the relative riskiness of asset k w.r.t. the market payoff.

Estrada (2007) also introduced *downside betas*. He, however, defined them to be symmetric and used the mean as target:

$$\beta_i^D := \frac{E[\min\{(R_i - \mu_i), 0\} \min\{(R_M - \mu_M), 0\}]}{E[\min\{(R_M - \mu_M), 0\}^2]}.$$

He *defined* his downside beta coefficients as a matter of choice, and symmetry is, no doubt, a nice mathematical property. But, as downside risk is not symmetric, why should downside cosemivariance or downside beta coefficients be? Moreover, his downside beta coefficients are always positive. Thus, they cannot account for negative downside correlations which is their major weakness. We instead *derive* our downside betas from the DRAME pricing formula. If the asset market is in equilibrium, then the corresponding downside security market line (77) dictates the form of the downside beta coefficients, which turn out to be asymmetric.

THEOREM 3.5.
Proposition 3.6 can straightforwardly be generalized to arbitrary portfolios $\mathbf{x} \in \mathbb{R}^K$*:*

$$E[\langle \mathbf{q}, \mathbf{x} \rangle] - r_0 \langle \mathbf{p}^*, \mathbf{x} \rangle = \beta^*(\mathbf{x}) \left(E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}} \right), \tag{78}$$

with

$$\beta^*(\mathbf{x}) := \frac{E[\langle r_0 \mathbf{p}^* - \mathbf{q}, \mathbf{x} \rangle (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+]}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2}$$

being the downside beta coefficient of portfolio $\mathbf{x}$.

Hogan & Warren (1974) derived a similar security market line, using asymmetric cosemivariances. The difference is that their market portfolio is defined as a point of tangency to the capital market line, whereas we use the current stock of shares available on the asset market. Bawa & Lindenberg (1977, Theorem 4) generalize the findings of Hogan & Warren (1974) to a mean-lower-partial-moment framework. In both papers, their security market line is derived under the assumption that the asset market is in equilibrium. A formal proof, under which prerequisites this assumption is met, is missing, there. In this thesis, instead, we establish existence and uniqueness of asset market equilibria in the mean-downside-risk framework before characterizing these equilibria.

3.3.4. Sortino Ratio of the Market Payoff.

The Sortino ratio, introduced by Sortino & Price (1994), is a performance measure in the downside risk framework. It is defined to be the expected excess payoff of a portfolio divided by its downside risk. The Sortino ratio is the downside analogon to the Sharpe ratio, cf. Sharpe (1994).

COROLLARY 3.3.
Let $\mathbf{p}^*$ *denote the vector of DRAME prices,* $p^*_{\mathtt{m}} = \langle \mathbf{p}^*, \mathtt{m} \rangle$ *and let* $\mathbf{x}$ *be an arbitrary portfolio. Then, the equilibrium Sortino ratio of its payoff is bounded from above by the equilibrium Sortino ratio of the market payoff*

$$\frac{E[\langle \mathbf{q}, \mathbf{x} \rangle] - r_0 \langle \mathbf{p}^*, \mathbf{x} \rangle}{\mathfrak{D}(\mathbf{x}, r_0 \mathbf{p}^*)} \leqslant \rho(\mathbf{p}^*). \tag{79}$$

PROOF. Dividing (78) by $\mathfrak{D}(\mathbf{x}, r_0\mathbf{p}^*)$ yields

$$\begin{aligned} \frac{E[\langle \mathbf{q}, \mathbf{x} \rangle] - r_0 \langle \mathbf{p}^*, \mathbf{x} \rangle}{\mathfrak{D}(\mathbf{x}, r_0 \mathbf{p}^*)} &= \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} \frac{E[\langle r_0 \mathbf{p}^* - \mathbf{q}, \mathbf{x} \rangle (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+]}{\mathfrak{D}(\mathbf{x}, r_0 \mathbf{p}^*) \cdot \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} \\ &\overset{(49)}{=} \rho(\mathbf{p}^*) \frac{E[\langle r_0 \mathbf{p}^* - \mathbf{q}, \mathbf{x} \rangle (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+]}{\left\| \langle r_0 \mathbf{p}^* - \mathbf{q}, \mathbf{x} \rangle_+ \right\| \left\| (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+ \right\|} \\ &\leqslant \rho(\mathbf{p}^*) \frac{E[\langle r_0 \mathbf{p}^* - \mathbf{q}, \mathbf{x} \rangle_+ (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+]}{\left\| \langle r_0 \mathbf{p}^* - \mathbf{q}, \mathbf{x} \rangle_+ \right\| \left\| (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+ \right\|}. \end{aligned}$$

The Cauchy-Schwarz inequality implies the assertion. □

This corollary connotes that the market payoff attains the highest possible Sortino ratio at DRAME prices. This means it has the best downside-risk-return profile among all potential portfolios. Therefore, each investor wants to hold a fraction of the market portfolio in equilibrium.

3.3.5. Valuation Formula.

We close this section by stating a valuation formula for arbitrary payoff patterns.

DEFINITION 3.4 (Valuation Formula).
Let $\epsilon \in \mathscr{M}$ be a random payoff and let $p^*_{\mathtt{m}} = \langle \mathbf{p}^*, \mathtt{m} \rangle$ be the equilibrium price of the market portfolio. The *valuation formula* of payoff ϵ at prices $\mathbf{p}^*$ is defined as $v(\epsilon) = E[\epsilon \kappa(p^*_{\mathtt{m}})]$. □

Observe that $v(\epsilon)$ solely depends on $p^*_{\mathtt{m}}$. It can be considered as the monetization of the replicating investment $(x^\epsilon_0, \mathbf{x}^\epsilon)$ of ϵ. The replicating investment of ϵ is defined via $\epsilon = r_0 x^\epsilon_0 + \langle \mathbf{q}, \mathbf{x}^\epsilon \rangle$. Since $\mathscr{M} := \text{span}\{r_0, q_1, ..., q_K\}$, any payoff pattern $\epsilon \in \mathscr{M}$ can be replicated by purchasing or selling marketed assets. In that sense, $v(\epsilon)$ represents the present value of the random future payoff ϵ at DRAME prices $\mathbf{p}^*$. In finance, it is used to calculate the fair date-0 price of financial options.

Dybvig & Ingersoll (1982) construct a random payoff which is always positive but the CAPM valuation formula assigns a strictly negative value to it. Thus, they create an arbitrage opportunity. Here, however, with $\mathbf{t} = r_0 \mathbf{p}^*$, a positive random payoff ϵ will always get a strictly positive equilibrium present value $v(\epsilon)$.

The valuation formula can be rearranged such that $v(\epsilon)$ is a solution to

$$v = \frac{1}{r_0}\left(E[\epsilon] - \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} E[(r_0 v - \epsilon)\,(r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})]_+\right).$$

The equilibrium present value of ϵ is thus its discounted expected value minus a risk premium.

3.A. Appendix to Chapter 3

3.A.1. Proof of Theorem 3.1.

From Proposition 3.1, we get the form of the individually optimal payoff at DRAME prices

$$\epsilon^{i,*} = r_0\left(x^i_0 + \langle \mathbf{p}^*, \mathbf{x}^i\rangle - \frac{\psi^i(\mathbf{p}^*)}{\rho(\mathbf{p}^*)}\langle \mathbf{p}^*, \mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*)\rangle\right) + \frac{\psi^i(\mathbf{p}^*)}{\rho(\mathbf{p}^*)}\langle \mathbf{q}, \mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*)\rangle \in \mathscr{M}. \tag{80}$$

The market clearing condition

$$\langle \mathbf{q}, \mathtt{m}\rangle = \epsilon_{\mathtt{m}} = \sum_{i=1}^{I} \epsilon^{i,*} \overset{(80)}{=} r_0\left(\langle \mathbf{p}^*, \mathtt{m}\rangle - \frac{\sum_{i=1}^I \psi^i(\mathbf{p}^*)}{\rho(\mathbf{p}^*)}\langle \mathbf{p}^*, \mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*)\rangle\right) + \frac{\sum_{i=1}^I \psi^i(\mathbf{p}^*)}{\rho(\mathbf{p}^*)}\langle \mathbf{q}, \mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*)\rangle$$

together with the assumption of non-redundancy stated in Assumption 3, implies

$$\frac{\sum_{i=1}^I \psi^i(\mathbf{p}^*)}{\rho(\mathbf{p}^*)}\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*) = \mathtt{m}. \tag{81}$$

Hence, it follows that in equilibrium the market portfolio has to be collinear with the reference portfolio. Note that equation (81) implies $\sum_{i=1}^I \psi^i(\mathbf{p}^*) > 0$, since $\mathtt{m} \in \mathbb{R}^K_{++}$.
Combining equations (39) and (81) yields

$$\mathbf{x}^{i,*}(\mathbf{p}^*) = \frac{\psi^i(\mathbf{p}^*)}{\sum_{i=1}^I \psi^i(\mathbf{p}^*)}\mathtt{m}. \tag{82}$$

By computing downside risk, we get

$$\mathfrak{D}(\mathbf{x}^{i,*}(\mathbf{p}^*), r_0\mathbf{p}^*) = \psi^i(\mathbf{p}^*)$$

from formula (39) and

$$\mathfrak{D}(\mathbf{x}^{i,*}(\mathbf{p}^*), r_0\mathbf{p}^*) = \frac{\psi^i(\mathbf{p}^*)}{\sum_{i=1}^I \psi^i(\mathbf{p}^*)}\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})$$

from equation (82). This implies $\sum_{i=1}^I \psi^i(\mathbf{p}^*) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})$ and equations (42) and (43) are proven.

Multiplying equation (81) with $\mathbf{C}(\mathtt{m}, r_0\mathbf{p}^*) = \mathbf{C}(\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*), r_0\mathbf{p}^*)$ yields

$$\mathbf{C}(\mathtt{m}, r_0\mathbf{p}^*)\mathtt{m} = \frac{\sum_{i=1}^I \psi^i(\mathbf{p}^*)}{\rho(\mathbf{p}^*)}\mathbf{C}(\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*), r_0\mathbf{p}^*)\mathbf{x}^{\mathsf{ref}}(\mathbf{p}^*) \overset{(43)}{=} \frac{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}{\rho(\mathbf{p}^*)}\boldsymbol{\pi}(\mathbf{p}^*),$$

which implies

$$\mathbf{p}^* = \frac{1}{r_0}\left(E[\mathbf{q}] - \frac{\rho(\mathbf{p}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}\mathbf{C}(\mathtt{m}, r_0\mathbf{p}^*)\mathtt{m}\right). \tag{83}$$

Equation (49) follows immediately from

$$r_0 p^*_{\mathtt{m}} = r_0 \langle \mathbf{p}^*, \mathtt{m} \rangle \overset{(83)}{=} E[\langle \mathbf{q}, \mathtt{m} \rangle] - \frac{\rho(\mathbf{p}^*)}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})} \langle \mathbf{C}(\mathtt{m}, r_0 \mathbf{p}^*)\mathtt{m}, \mathtt{m} \rangle = E[\epsilon_{\mathtt{m}}] - \rho(\mathbf{p}^*)\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}}).$$

When we plug in

$$\begin{aligned} \mathbf{C}(\mathtt{m}, r_0 \mathbf{p}^*)\mathtt{m} &= E[(r_0 \mathbf{p}^* - \mathbf{q}) \langle r_0 \mathbf{p}^* - \mathbf{q}, \mathtt{m} \rangle_+] \\ &= E[(r_0 \mathbf{p}^* - \mathbf{q}) (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+] \end{aligned} \tag{84}$$

and $\rho(\mathbf{p}^*) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})}$ in equation (83), we get the coordinate form of equilibrium asset prices given in (48).

The following computation gives the form of the price kernel (44)

$$\begin{aligned} r_0 \mathbf{p}^* &= E[\mathbf{q}] - \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} E\left[(r_0 \mathbf{p}^* - \mathbf{q}) (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\right] \\ &= E[\mathbf{q}] - r_0 \mathbf{p}^* \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} E\left[(r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\right] + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} E\left[\mathbf{q} (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\right] \\ &\Longleftrightarrow \end{aligned}$$

$$r_0 \mathbf{p}^* \left(1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} E\left[(r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\right]\right) = E\left[\mathbf{q}\left(1 + \frac{E[\epsilon_{\mathtt{m}}] - r_0 p^*_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})^2} (r_0 p^*_{\mathtt{m}} - \epsilon_{\mathtt{m}})_+\right)\right]. \tag{85}$$

This completes the proof. □

3.A.2. Proof of Theorem 3.3.

As a preliminary step, we analyze the individually optimal demand for downside risk.

For investor i, equipped with $\varepsilon^i = r_0 x^i_0 + \epsilon_{\mathtt{m}} x^i_{\mathtt{m}} \in \mathscr{H}$, let

$$\delta^i(p_{\mathtt{m}}) := x^i_{\mathtt{m}} \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}) \tag{86}$$

denote the amount of shares of the market portfolio initially owned by investor i multiplied with the downside risk of the market payoff at price $p_{\mathtt{m}}$.

The market portfolio is the only risky asset available on the asset market $\mathscr{H}$. Consequently, the downside risk of investor i's endowment solely stems from the number of shares $x^i_{\mathtt{m}}$ of the market portfolio initially owned by her. If $x^i_{\mathtt{m}} \geqslant 0$, then $\delta^i(p_{\mathtt{m}}) = \mathcal{D}(\varepsilon^i, r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}})])$, but for $x^i_{\mathtt{m}} < 0$, it holds $0 > \delta^i(p_{\mathtt{m}}) \neq \mathcal{D}(\varepsilon^i, r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}})]) > 0$. Hence, $\delta^i(p_{\mathtt{m}})$ is a combined measure of the downside risk of investor i's endowment and of the downside correlation between investor i's endowment ε^i and the market payoff $\epsilon_{\mathtt{m}}$ (which is either $+1$ or -1). It thus evaluates the *relative riskiness* of ε^i w.r.t $\epsilon_{\mathtt{m}}$ at price $p_{\mathtt{m}}$.

The following corollary reveals that $\delta^i(p_{\mathtt{m}})$ can be interpreted as an allocation of the downside risk of the market portfolio at price $p_{\mathtt{m}}$ among the I investors.

Corollary 3.4.
Since $\sum_{i=1}^{I} \varepsilon^i = \epsilon_{\mathtt{m}}$*, we have* $\sum_{i=1}^{I} x^i_{\mathtt{m}} = 1$*, and thus*

$$\sum_{i=1}^{I} \delta^i(p_{\mathtt{m}}) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}).$$

Lemma 3.5.
Individual demand for downside risk can be rewritten as

$$\Psi^i(p_{\mathtt{m}}) = \arg\max_{d \geqslant 0} U^i(E[\epsilon^i] + \varrho(p_{\mathtt{m}})(d - \delta^i(p_{\mathtt{m}})), d). \tag{87}$$

Proof. Due to equation (58) in Lemma 3.3, $\varrho(p_{\mathtt{m}}) = \frac{E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})} > 0$. Note further that

$$r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}})] = r_0(x^i_0 + p_{\mathtt{m}} x^i_{\mathtt{m}}) = r_0 x^i_0 + E[\epsilon_{\mathtt{m}}] x^i_{\mathtt{m}} - (E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}) x^i_{\mathtt{m}} = E[\varepsilon^i] - \pi(p_{\mathtt{m}}) x^i_{\mathtt{m}}.$$

Therewith, we have

$$\begin{aligned} \Psi^i(p_{\mathtt{m}}) &= \arg\max_{d \geqslant 0} U^i(r_0 E[\varepsilon^i \kappa(p_{\mathtt{m}})] + \varrho(p_{\mathtt{m}}) d, d) \\ &= \arg\max_{d \geqslant 0} U^i(E[\varepsilon^i] - \pi(p_{\mathtt{m}}) x^i_{\mathtt{m}} + \varrho(p_{\mathtt{m}}) d, d) \\ &= \arg\max_{d \geqslant 0} U^i(E[\varepsilon^i] + \varrho(p_{\mathtt{m}})(d - \delta^i(p_{\mathtt{m}})), d). \end{aligned}$$

□

Note that $\varrho(p_{\mathtt{m}})\delta^i(p_{\mathtt{m}})$ is a risk premium which has to be subtracted in the utility function, since investor i is endowed with shares of risky assets rather than with money. Compare pricing formula (48) and the discussion thereafter on the role of the risk premium. $E[\varepsilon^i] + \varrho(p_{\mathtt{m}})(d - \delta^i(p_{\mathtt{m}}))$ in equation (87) is a reformulation of the downside efficient frontier (24) which describes the maximal achievable mean given a prescribed level d of downside risk.

Lemma 3.6.
For $i = 1, ..., I$, $\delta^i(p_{\mathtt{m}})$ *is bounded on* $\left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$*, i.e.,*

$$\overline{\delta^i} := \sup \left\{ \left|\delta^i(p_{\mathtt{m}})\right| \mid p_{\mathtt{m}} \in \left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right) \right\} < \infty.$$

Proof. $\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})$ is continuous in $p_{\mathtt{m}}$, with $\mathcal{D}(\epsilon_{\mathtt{m}}, 0) = 0 < \infty$, because $\epsilon_{\mathtt{m}} \geqslant 0$ by Assumption 3, and $\mathcal{D}(\epsilon_{\mathtt{m}}, E[\epsilon_{\mathtt{m}}]) < \infty$ since $\epsilon_{\mathtt{m}} \in \mathcal{L}^2(\Omega)$. It follows that $\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}})$ is bounded on $\left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$ and, thus, $\delta^i(p_{\mathtt{m}})$ is bounded on $\left(0, \frac{1}{r_0} E[\epsilon_{\mathtt{m}}]\right)$. □

We are now in a position to establish that equation (73) always has a solution.

Proposition 3.7.
Provided that the Assumptions 2 and 3 are fulfilled, there exists a solution $p^*_{\mathtt{m}}$ to equation (73).

Proof. The basic line of reasoning in this proof is adopted from Koch-Medina & Wenzelburger (2018).
Observe that $\Psi(\frac{1}{r_0}E[\epsilon_{\mathtt{m}}]) = 0 < \mathcal{D}(\epsilon_{\mathtt{m}}, E[\epsilon_{\mathtt{m}}])$. If we show existence of some $p^0_{\mathtt{m}} \in \Big(\underline{p_{\mathtt{m}}}, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\Big)$ with $\Psi(p^0_{\mathtt{m}}) \geqslant \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^0_{\mathtt{m}})$, the intermediate value theorem for continuous functions implies existence of $p^*_{\mathtt{m}}$ with $\Psi(p^*_{\mathtt{m}}) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p^*_{\mathtt{m}})$.
For $i = 1, ..., I$, $d \in \mathbb{R}_+$ and $p_{\mathtt{m}} \in \Big(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\Big)$, we define

$$f^i(d, p_{\mathtt{m}}) := U^i(E[\epsilon^i] + \varrho(p_{\mathtt{m}})(d - \delta^i(p_{\mathtt{m}})), d).$$

The function $d \mapsto f^i(d, p_{\mathtt{m}})$ is continuously differentiable as well as strictly concave, due to Assumption 2, and therefore either strictly increasing, strictly decreasing or attains a unique inner maximum at $\Psi^i(p_{\mathtt{m}}) = \arg\max_{d\geqslant 0} f^i(d, p_{\mathtt{m}})$. We distinguish two cases: finite and infinite limiting slope of indifference curves.

Case 1. Let investor j be the investor whose limiting slope of indifference curves is smallest and finite: $\rho_{U^j} = \varrho(\underline{p_{\mathtt{m}}}) < \infty$. Note that $\varrho(\underline{p_{\mathtt{m}}}) < \infty \iff \underline{p_{\mathtt{m}}} > 0$. We show that $\lim_{p_{\mathtt{m}} \searrow \underline{p_{\mathtt{m}}}} \Psi^j(p_{\mathtt{m}}) = \infty > \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 \underline{p_{\mathtt{m}}})$, which implies the assertion.
Assume the opposite, i.e., $\lim_{p_{\mathtt{m}} \searrow \underline{p_{\mathtt{m}}}} \Psi^j(p_{\mathtt{m}}) < \infty$. Then there exists $\bar{d} \in \mathbb{R}_+$ and a sequence $(p_n)_{n\in\mathbb{N}} \subset \Big(\underline{p_{\mathtt{m}}}, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\Big)$ with $\lim_{n\to\infty} p_n = \underline{p_{\mathtt{m}}}$, such that $\Psi^j(p_n) =: d_n \underset{n\to\infty}{\longrightarrow} \bar{d}$. By construction, $\frac{\partial f^j}{\partial d}(d_n, p_n) \leqslant 0$ for $n \in \mathbb{N}$ and, by continuity, $\frac{\partial f^j}{\partial d}(\bar{d}, \underline{p_{\mathtt{m}}}) \leqslant 0$. Hence, $d \mapsto f^j(d, \underline{p_{\mathtt{m}}})$ cannot be a strictly increasing function on $\Big(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\Big)$.
On the other hand, $\frac{\partial f^j}{\partial d}(d, \underline{p_{\mathtt{m}}}) \geqslant 0$ is equivalent to $\varrho(\underline{p_{\mathtt{m}}}) \geqslant S^j(E[\epsilon^j] + \varrho(\underline{p_{\mathtt{m}}})(d - \delta^j(\underline{p_{\mathtt{m}}})), d)$. This, however, is the case for all $d \in \mathbb{R}_+$ since $\varrho(\underline{p_{\mathtt{m}}})$ equals the *limiting* slope of investor j's indifference curves ρ_{U^j}. Thus $f^j(d, \underline{p_{\mathtt{m}}})$ is an increasing and therewith, due to strict concavity, a strictly increasing function, which is a contradiction.

Case 2. When all investors are characterized by indifference curves with infinite limiting slopes, then $\underline{p_{\mathtt{m}}} = 0$.
Since limiting slopes of indifference curves are continuous, and, due to Lemma 3.6, $\delta^i(p_{\mathtt{m}})$ is bounded on $\Big(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\Big)$, we have

$$\overline{S^i} := \sup\left\{S^i(E[\epsilon^i], \delta^i(p_{\mathtt{m}})) \mid p_{\mathtt{m}} \in \left(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\right) : \delta^i(p_{\mathtt{m}}) \geqslant 0\right\} < \infty.$$

Set $\overline{S} := \max\{\overline{S^i} \mid i = 1, ..., I\} < \infty$ and choose $p^0_{\mathtt{m}} \in \Big(0, \frac{1}{r_0}E[\epsilon_{\mathtt{m}}]\Big)$ such that $\varrho(p^0_{\mathtt{m}}) > \overline{S}$. This choice is possible since $\lim_{p_{\mathtt{m}} \searrow 0} \varrho(p_{\mathtt{m}}) = \infty$, cf. equation (59).

If $\delta^i(p_{\mathtt{m}}^0) < 0$, then $\Psi^i(p_{\mathtt{m}}^0) \geqslant 0 > \delta^i(p_{\mathtt{m}}^0)$.
If $\delta^i(p_{\mathtt{m}}^0) \geqslant 0$, then

$$\Psi^i(p_{\mathtt{m}}^0) > \delta^i(p_{\mathtt{m}}^0) \Longleftrightarrow \frac{\partial f^i}{\partial d}(\delta^i(p_{\mathtt{m}}^0), p_{\mathtt{m}}^0) > 0 \Longleftrightarrow S^i(E[\epsilon^i], \delta^i(p_{\mathtt{m}}^0)) < \varrho(p_{\mathtt{m}}^0).$$

This condition, however, is always fulfilled because, we set $p_{\mathtt{m}}^0$, such that

$$\varrho(p_{\mathtt{m}}^0) > \overline{S} \geqslant \overline{S^i} \geqslant S^i(E[\epsilon^i], \delta^i(p_{\mathtt{m}}^0)).$$

Hence, $\Psi(p_{\mathtt{m}}^0) = \sum_{i=1}^I \Psi^i(p_{\mathtt{m}}^0) > \sum_{i=1}^I \delta^i(p_{\mathtt{m}}^0) = \mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^0)$. □

CHAPTER 4

Practical Issues

A natural question that arises is to what extent the theoretical results obtained in the previous chapters can be applied by financial practitioners. Moreover, it is of interest how much the performance of a downside efficient portfolio improves on the performance of a mean-variance efficient portfolio. Is it worth to determine downside efficient portfolios which is computationally more costly? Or could large investors, e.g., banks or pension funds, just as well rely on the simpler and well-understood concept of mean-variance efficiency?

Because practitioners usually work with returns, we briefly reformulate the portfolio selection methodology developed in Chapter 2 in terms of returns rather than in terms of prices and payoffs. Concepts like downside cosemivariance matrix, reference portfolio, downside efficiency and market price of downside risk can be reproduced. In contrast to Section 3.3, where DRAME prices were assumed, we establish a downside security market line without the hypothesis of equilibrium. Furthermore, downside efficient portfolios attain the highest possible Sortino ratio, regardless of whether asset prices are in equilibrium or not. These findings are particularly remarkable, because equilibrium asset prices are the crucial presumption in mean-variance capital asset pricing theory.

In Section 4.2, we perform a simulation study to investigate how far downside efficient portfolios differ from mean-variance efficient portfolios. To do so, three scenarios with increasing probabilities of default for risky assets are considered.
Although portfolio compositions differ in each scenario, downside efficient portfolios are not better than efficient ones, when default probabilities are zero. They yield the same mean and are equally risky. Thus, when asset markets are calm, the concept of (μ, σ)-diversification is sufficient to secure investors and there is no need to do the costlier computation of downside efficient portfolios.
If, however, market disruptions and defaults have a strictly positive probability, downside efficient portfolios unambiguously outperform efficient portfolios in the sense of Markowitz. Yielding the same mean return, downside efficient portfolios protect investors better against large losses which may cause bankruptcy.
Eventually, we provide an iterative algorithm to compute the downside efficient portfolio mix and give some remarks concerning implementation and computational expense.

4.1. Portfolio Selection Using Returns

We briefly reformulate the setting and essential concepts of portfolio selection using returns rather than prices and payoffs. Proofs and interpretations can be reviewed in Chapter 2 and Section 3.3.

In this chapter, the K risky assets are not described by date-0 prices and random date-1 payoffs, but they are characterized by a random return vector $\mathbf{r} = (r_1, \ldots, r_K)$. For investing 1 € in asset k at date 0, an investor will get a positive random gross return of r_k € at date 1. The return of each risky asset is modeled as a real square integrable random variable, i.e., $r_k \in \mathcal{L}^2(\Omega)$ for $k = 1, ..., K$.

An investor splits her initial wealth among the risk-free and the risky assets. In this chapter we denote by $\mathbf{y} = (y_1, ..., y_K) \in \mathbb{R}^K$ a vector of portfolio weights, i.e., y_k is the *relative* amount of money invested in risky asset k.[15] The relative amount invested in the risk-free asset is then given by $y_0 = 1 - \sum_{k=1}^K y_k$.

The realized rate of return on portfolio $\mathbf{y}$ amounts to

$$R(\mathbf{y}) := \sum_{k=0}^{K} r_k y_k = r_0 + \langle \mathbf{r} - \mathbf{r_0}, \mathbf{y} \rangle \,, \tag{88}$$

where $\mathbf{r_0} = (r_0, \ldots, r_0) \in \mathbb{R}^K$. It is decomposed in the risk-free rate of return and the excess return of portfolio $\mathbf{y}$.
Its mean is given by

$$\mathfrak{M}(\mathbf{y}) := E[R(\mathbf{y})] \overset{(88)}{=} r_0 + \langle \boldsymbol{\pi}, \mathbf{y} \rangle \,, \tag{89}$$

where $\boldsymbol{\pi} := E[\mathbf{r}] - \mathbf{r_0}$ denotes the *mean excess return vector.*

Downside risk of return $R \in \text{span}\{r_0, ..., r_K\}$ w.r.t. some target return $R_T \in \mathbb{R}$ is defined as

$$\mathcal{D}(R, R_T) := \left\| (R_T - R)_+ \right\| . \tag{90}$$

R_T determines the threshold between gains and losses, e.g., $R_T = 1$, $R_T = E[R]$ which yields lower semideviation or $R_T = r_0$. We choose $R_T = r_0$ in this chapter, because it may be seen as an economically natural target. By investing everything into the bond, an investor obtains $R(\mathbf{0}) = r_0$ for sure. By investing a part of her wealth in risky assets, she has the chance to realize a higher return but faces the risk of getting less than r_0.

[15] To avoid ambiguity, we use y_k to denote the relative portion of the investor's wealth that is invested in asset k, whereas x_k labels the number of shares of asset k that are held in portfolio $\mathbf{x}$. Thus, y_k is a relative number and x_k is an absolute number.

Observe that downside risk is increasing in the target return:

$$r_0 > r_0' \Longrightarrow \left\|(r_0 - R)_+\right\| > \left\|(r_0' - R)_+\right\|.$$

Thus, a lower riskless rate of return (e.g., interest rate) makes portfolios less risky when investors use downside risk as a measure of risk. Besides higher expected excess return, this further explains why investors increase their demand for risky assets when interest rates are low. The following example provides a numerical illustration.

EXAMPLE 4.1.
Suppose the financial market offers a riskless investment opportunity with $r_0 = 1.02$ and let a portfolio $\tilde{\mathbf{y}}$ be given, whose realized rate of return $\tilde{R} := R(\tilde{\mathbf{y}})$ is uniformly distributed on the interval $[1.01, 1.03]$. Then, $\mathcal{D}(\tilde{R}, r_0) > 0$ because the realized return can drop below the safe return of the risk-free asset.
If, however, the riskless rate of return declines to $r_0' = 1.002$, then the portfolio $\tilde{\mathbf{y}}$ becomes downside-risk-free, i.e., $\mathcal{D}(\tilde{R}, r_0') = 0$, because its realized rate of return cannot be worse than r_0'.
Note that the distribution of $\tilde{R}$ and, thus, its standard deviation does not change while its downside risk decreases when the target return declines. □

DEFINITION 4.1 (Portfolio's Downside Risk).
For a given portfolio $\mathbf{y} \in \mathbb{R}^K$, we define its downside risk by

$$\mathfrak{D}(\mathbf{y}) := \mathcal{D}(R(\mathbf{y}), r_0) = \left\|\langle \mathbf{r_0} - \mathbf{r}, \mathbf{y}\rangle_+\right\|. \tag{91}$$
□

The setting of this chapter is as general as possible. There are no assumptions on the investor's preferences such as a specific form of utility function. The only suppose is that investors prefer higher expected return and lower downside risk. This is a very natural assumption. Clearly, there are people who like gambling. Thus, the hypothesis that investors prefer lower volatility can be questioned. But no rational investor prefers a higher risk of losing money.
Furthermore, we do not presume the asset market to be in equilibrium. Equilibrium asset prices are the crucial requirement making the standard CAPM market portfolio efficient. However, the theory developed in the course of this section will apply to an equilibrium as well as a non-equilibrium setting.

ASSUMPTION 4.
In this chapter, we, firstly, assume that an investor considers only $\mathfrak{M}(\mathbf{y})$ as a good and $\mathfrak{D}(\mathbf{y})$ as a bad when evaluating portfolio $\mathbf{y}$, and, secondly, that no nontrivial portfolio is risk-free, i.e., $\mathbf{y} \neq \mathbf{0} \Rightarrow \mathfrak{D}(\mathbf{y}) > 0$.

4.1.1. Scaled Reference Portfolio and Downside Efficiency.
As a preliminary step, we introduce the downside cosemivariance matrix of returns in the same way as we did in Definition 1.3. It will be used to introduce the notion of a scaled reference portfolio.

Definition 4.2 (Risky Set and Downside Cosemivariance Matrix of Returns).
The set of all realizations yielding a future return below its target

$$\begin{aligned}\mathscr{R}(\mathbf{y}) &:= \{\omega \in \Omega \mid R(\mathbf{y}) < r_0\}, \\ &= \{\omega \in \Omega \mid \langle \mathbf{r}(\omega) - \mathbf{r_0}, \mathbf{y}\rangle < 0\} \in \mathcal{F}\end{aligned}$$

is referred to as risky set of portfolio $\mathbf{y}$.

The symmetric and null-homogeneous matrix $\mathbf{C}(\mathbf{y})$, whose $(k,l)^{\text{th}}$ entry is defined as

$$C_{kl}(\mathbf{y}) := E_{\mathscr{R}(\mathbf{y})}\left[(r_0 - r_k)(r_0 - r_l)\right], \tag{92}$$

is called downside cosemivariance matrix of $\mathbf{r}$ w.r.t. portfolio $\mathbf{y}$. □

The scaled reference portfolio is a scaled version of the reference portfolio which has been defined in Section 2.2.

Definition 4.3 (Scaled Reference Portfolio).
As shown in Proposition 2.2, there is a solution $\mathbf{y}^{\mathsf{ref}}$ to the equation

$$\mathbf{C}(\mathbf{y})\mathbf{y} = \boldsymbol{\pi}.$$

With the strictly positive scaling factor

$$\upsilon := \begin{cases} 1 & \text{if } \sum_{k=1}^K y_k^{\mathsf{ref}} = 0 \\ \frac{1}{\left|\sum_{k=1}^K y_k^{\mathsf{ref}}\right|} & \text{otherwise} \end{cases},$$

we define the *scaled reference portfolio*

$$\mathbf{y}^{\mathsf{sref}} := \upsilon \mathbf{y}^{\mathsf{ref}}. \tag{93}$$ □

Since there are no assumptions on the probability distribution of the asset returns, $\mathbf{y}^{\mathsf{ref}}$ as well as the scaled reference portfolio may have negative entries. Consequently, $\sum_{k=1}^K y_k^{\mathsf{ref}}$ can be strictly positive, zero or strictly negative.
Thus, the relative amount invested into or borrowed from the bond $y_0^{\mathsf{sref}} = 1 - \sum_{k=1}^K y_k^{\mathsf{sref}}$ is either 0, 1 or 2.
If $\sum_{k=1}^K y_k^{\mathsf{ref}} > 0$, then $\mathbf{y}^{\mathsf{sref}}$ describes an investment of one monetary unit into risky assets and nothing in the bond. For $\sum_{k=1}^K y_k^{\mathsf{ref}} < 0$, one monetary unit of risky assets is sold short and $y_0^{\mathsf{sref}} = 2$. When $\sum_{k=1}^K y_k^{\mathsf{ref}} = 0$, then $\mathbf{y}^{\mathsf{sref}} = \mathbf{y}^{\mathsf{ref}}$ is a costless investment into risky asset and, thus, one monetary unit is invested in the bond.

Notation 4.
In this chapter, the scaled reference portfolio turns out to be the crucial building block. Therefore, we denote its return by $R^{\mathsf{sref}} := R(\mathbf{y}^{\mathsf{sref}})$ and its downside risk by $\mathcal{D}(R^{\mathsf{sref}}) = \mathfrak{D}(\mathbf{y}^{\mathsf{sref}})$. □

Next, we establish that downside efficient portfolios can be characterized by the scaled reference portfolio $\mathbf{y}^{\mathsf{sref}}$. A solution to the optimization problem

$$\max_{\mathbf{y}\in\mathbb{R}^K} \mathfrak{M}(\mathbf{y}) \text{ s.t. } \mathfrak{D}(\mathbf{y}) \leqslant d_0, \tag{94}$$

for some upper bound $d_0 \geqslant 0$, is called a downside efficient portfolio and denoted by $\mathbf{y}^{\mathsf{de}}(d_0)$. The corresponding amount of money allocated to or borrowed from the risk-free asset is then given by $y_0^{\mathsf{de}}(d_0) = 1 - \sum_{k=1}^{K} y_k^{\mathsf{de}}(d_0)$.

The following two assertions are immediate corollaries of portfolio selection theory developed in Chapter 2. Their interpretation remains the same and can be reviewed in Sections 2.2 and 2.3.

Corollary 4.1 (Downside Efficient Portfolios).
Let $\boldsymbol{\pi} \neq \mathbf{0}$. Then, there exists a downside efficient portfolio $\mathbf{y}^{\mathsf{de}}(d_0)$. It can be represented as

$$\mathbf{y}^{\mathsf{de}}(d_0) = \frac{d_0}{\mathfrak{D}(\mathbf{y}^{\mathsf{sref}})}\mathbf{y}^{\mathsf{sref}}, \tag{95}$$

i.e., downside efficient portfolios are positive multiples of $\mathbf{y}^{\mathsf{sref}}$. If $\mathbf{C}(\mathbf{y}^{\mathsf{sref}})$ is invertible, the downside efficient portfolio is unique.

Proof. The assertion follows from Theorem 2.2, Proposition 2.2 and Theorem 2.3 when we apply the following identities

$$\mathbf{C}(\mathbf{y}^{\mathsf{sref}}) = \mathbf{C}(\upsilon\mathbf{y}^{\mathsf{ref}}) = \mathbf{C}(\mathbf{y}^{\mathsf{ref}}),$$
$$\mathfrak{D}(\mathbf{y}^{\mathsf{sref}}) = \mathfrak{D}(\upsilon\mathbf{y}^{\mathsf{ref}}) = \upsilon\mathfrak{D}(\mathbf{y}^{\mathsf{ref}}).$$

□

As already established in Chapter 2, downside efficient portfolios are boundary solutions to optimization problem (94), i.e.,

$$\mathfrak{D}(\mathbf{y}^{\mathsf{de}}(d)) = d. \tag{96}$$

Notation 5.
We use $\mathbf{y}^{\mathsf{de}} := \mathbf{y}^{\mathsf{de}}(1)$ as a shorthand notation. □

Corollary 4.2 (Price of Risk and Downside Efficient Frontier).
The market price of downside risk is defined as the following positive constant

$$\rho := \mathfrak{D}(\mathbf{y}^{\mathsf{ref}}) = \left\langle \boldsymbol{\pi}, \mathbf{y}^{\mathsf{de}} \right\rangle. \tag{97}$$

Therewith, the downside efficient frontier, which is defined as the function

$$\mathfrak{e} : \mathbb{R}_+ \longrightarrow \mathbb{R},\ d \longmapsto \mathfrak{M}\big(\mathbf{y}^{\mathsf{de}}(d)\big),$$

takes the form

$$\mathfrak{e}(d) = r_0 + d\langle \boldsymbol{\pi}, \mathbf{y}^{\mathsf{de}} \rangle = r_0 + \rho d. \tag{98}$$

4.1.2. Sortino Ratio and Downside Security Market Line.

Sortino ratio and downside security market line have already been introduced in Section 3.3 under the hypothesis of equilibrium asset prices. In this section, we show that these notions can still be established without the equilibrium assumption.

Proposition 4.1 (The Sortino Ratio of a Downside Efficient Portfolio).
For any $d > 0$,

$$\mathfrak{S}(\mathbf{y}^{\mathsf{de}}(d)) := \frac{E[R(\mathbf{y}^{\mathsf{de}}(d))] - r_0}{\mathfrak{D}(\mathbf{y}^{\mathsf{de}}(d))} \underset{(96)}{\overset{(89)}{=}} \frac{\big\langle \boldsymbol{\pi}, \mathbf{y}^{\mathsf{de}}(d) \big\rangle}{d} \overset{(95)}{=} \big\langle \boldsymbol{\pi}, \mathbf{y}^{\mathsf{de}} \big\rangle \overset{(97)}{=} \rho. \tag{99}$$

In particular, $\mathfrak{S}(\mathbf{y}^{\mathsf{sref}}) = \rho$.

Thus, the Sortino ratio of every downside efficient portfolio equals the market price of downside risk. Since DRAME prices make the market portfolio downside efficient, the equilibrium Sortino ratio of the market portfolio equals the equilibrium market price of downside risk as stated in equation (49) in Theorem 3.1.

The CAPM security market line is by construction an equilibrium concept. Utilizing the scaled reference portfolio $\mathbf{y}^{\mathsf{sref}}$ we are able to relax the assumption of equilibrium asset prices and still establish the notion of a downside security market line.

Proposition 4.2 (Downside Security Market Line for Asset k).
The downside security market line for asset k takes the form

$$E[r_k] - r_0 = \beta_k \left(E[R^{\mathsf{sref}}] - r_0\right), \tag{100}$$

where

$$\beta_k = \frac{E[(r_0 - r_k)\left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathcal{D}(R^{\mathsf{sref}})^2} \tag{101}$$

is called downside beta coefficient of asset k w.r.t. the scaled reference portfolio $\mathbf{y}^{\mathsf{sref}}$ *and plays the role of the classical β coefficient.*

PROOF.

$$
\begin{aligned}
E[r_k] - r_0 = \pi_k = \left(\mathbf{C}(\mathbf{y}^{\mathsf{ref}})\mathbf{y}^{\mathsf{ref}}\right)_k &= \sum_{l=1}^{K} E_{\mathscr{R}(\mathbf{y}^{\mathsf{ref}})}\left[(r_0 - r_k)(r_0 - r_l)\right] y_l^{\mathsf{ref}} \\
&= E[(r_0 - r_k)\left\langle \mathbf{r}_0 - \mathbf{r}, \mathbf{y}^{\mathsf{ref}} \right\rangle_+] \underbrace{\frac{\left\langle \boldsymbol{\pi}, \mathbf{y}^{\mathsf{ref}} \right\rangle}{\mathfrak{D}(\mathbf{y}^{\mathsf{ref}})^2}}_{=1} \\
&= \frac{E[(r_0 - r_k)\left\langle \mathbf{r_0} - \mathbf{r}, \mathbf{y}^{\mathsf{sref}} \right\rangle_+]}{\mathfrak{D}(\mathbf{y}^{\mathsf{sref}})^2} \left\langle \boldsymbol{\pi}, \mathbf{y}^{\mathsf{sref}} \right\rangle \\
&= \frac{E[(r_0 - r_k)\left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathcal{D}(R^{\mathsf{sref}})^2} \left(E[R^{\mathsf{sref}}] - r_0\right). \qquad \square
\end{aligned}
$$

The form of the downside security market line for an arbitrary portfolio $\mathbf{y}$ follows immediately.

COROLLARY 4.3 (Downside Security Market Line for a Portfolio).
The downside security market line for an arbitrary portfolio $\mathbf{y} \in \mathbb{R}^K$ *is given by*

(102)
$$
E[R(\mathbf{y})] - r_0 = \beta(\mathbf{y})\left(E[R^{\mathsf{sref}}] - r_0\right),
$$

where

(103)
$$
\beta(\mathbf{y}) = \frac{E[(r_0 - R(\mathbf{y}))\left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathcal{D}(R^{\mathsf{sref}})^2}
$$

is called downside beta coefficient of portfolio $\mathbf{y}$ w.r.t. the scaled reference portfolio.

PROOF.

$$
\begin{aligned}
E[R(\mathbf{y})] - r_0 = \langle \mathbf{y}, \boldsymbol{\pi} \rangle &= \sum_{k=1}^{K} \pi_k y_k \\
&\overset{(100)}{=} \sum_{k=1}^{K} \beta_k \left(E[R^{\mathsf{sref}}] - r_0\right) y_k \\
&\overset{(101)}{=} \frac{E[\langle \mathbf{r_0} - \mathbf{r}, \mathbf{y} \rangle \left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathcal{D}(R^{\mathsf{sref}})^2} \left(E[R^{\mathsf{sref}}] - r_0\right) \\
&= \frac{E[(r_0 - R(\mathbf{y}))\left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathcal{D}(R^{\mathsf{sref}})^2} \left(E[R^{\mathsf{sref}}] - r_0\right). \qquad \square
\end{aligned}
$$

In Corollary 3.3, we showed that no portfolio can have a higher Sortino ratio than the market portfolio at DRAME prices. Now, we are in a position to establish that no portfolio can have a larger Sortino ratio than the $\mathbf{y}^{\mathsf{sref}}$, no matter if the asset market is in equilibrium or not.

PROPOSITION 4.3 (Boundedness of Sortino Ratio).
Let $\mathbf{y}$ *be an arbitrary portfolio. Then its Sortino Ratio is bounded from above*

$$\mathfrak{S}(\mathbf{y}) \leqslant \rho. \tag{104}$$

PROOF. Dividing (102) by $\mathfrak{D}(\mathbf{y})$ yields

$$\begin{aligned}
\mathfrak{S}(\mathbf{y}) = \frac{E[R(\mathbf{y})] - r_0}{\mathfrak{D}(\mathbf{y})} &\overset{(103)}{=} \frac{E[(r_0 - R(\mathbf{y}))\left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathfrak{D}(\mathbf{y})\mathfrak{D}(\mathbf{y}^{\mathsf{sref}})} \frac{E[R^{\mathsf{sref}}] - r_0}{\mathfrak{D}(\mathbf{y}^{\mathsf{sref}})} \\
&\overset{(104)}{=} \frac{E[(r_0 - R(\mathbf{y}))\left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathfrak{D}(\mathbf{y})\mathfrak{D}(\mathbf{y}^{\mathsf{sref}})} \mathfrak{S}\left(\mathbf{y}^{\mathsf{sref}}\right) \\
&\leqslant \frac{E[(r_0 - R(\mathbf{y}))_+\left(r_0 - R^{\mathsf{sref}}\right)_+]}{\mathfrak{D}(\mathbf{y})\mathfrak{D}(\mathbf{y}^{\mathsf{sref}})} \mathfrak{S}\left(\mathbf{y}^{\mathsf{sref}}\right).
\end{aligned}$$

The Cauchy-Schwarz inequality implies the assertion. □

Proposition 4.3 connotes that the scaled reference portfolio has the highest possible Sortino ratio, i.e., it has the best downside-risk-return profile among all tradeable portfolios. Consequently, an investor wants to hold a positive multiple of $\mathbf{y}^{\mathsf{sref}}$.

4.2. A Simulation Study

We employ simulated asset returns to compare efficient portfolios in the sense of Markowitz with downside efficient portfolios. The simulation settings are adapted from Jarrow & Zhao (2006). Monte-Carlo simulations with a sample size of $100,000$ are executed. This extraordinarily large sample size is chosen to circumvent difficulties in the course of estimating covariance and downside cosemivariance matrices. We emphasize that sample covariance matrix as well as sample downside cosemivariance matrix are estimated from the simulated data. In fact, the downside cosemivariance matrix has to be estimated from a smaller effective sample size, because only realizations that yield a return below its target are taken into account for estimation. Hence, its mean squared estimation error is at least as large as the covariance matrix's one. Thus, improvements caused by replacing efficient by downside efficient portfolios do not stem from a more accurate estimation but from a better diversification structure.

When estimating the downside efficient scaled reference portfolio, the challenge is that the matrix equation $\mathbf{C}(\mathbf{y})\mathbf{y} = \boldsymbol{\pi}$ may have multiple solutions. If, however, $\mathbf{C}(\mathbf{y}^{\mathsf{ref}})$ is invertible, then $\mathbf{y}^{\mathsf{ref}}$ is the unique solution to $\mathbf{C}(\mathbf{y})\mathbf{y} = \boldsymbol{\pi}$, cf. Corollary 4.1. The advanced statistical method of *shrinkage*, see, e.g., Ledoit & Wolf (2004), guarantees invertibility and numerical stability of the sample downside cosemivariance matrix (and for the sample covariance matrix as well). But even if the downside cosemivariance matrix is invertible, the implicit equation $\mathbf{y} = \mathbf{C}(\mathbf{y})^{-1}\boldsymbol{\pi}$ has to be solved numerically. The iterative algorithm to calculate $\mathbf{y}^{\mathsf{sref}}$ is outlined in the appendix to this chapter.

Hence, when applied to simulated or historical financial data, the shrinkage technique ensures that the downside efficient portfolio can be computed in the same way as the efficient portfolio in the sense of Markowitz. The only difference is that $\mathbf{y}^{\mathsf{sref}}$ has to be calculated iteratively, cf. Algorithm 1, which is computationally more costly, whereas $\mathbf{y}^{\mathsf{eff}}$ can be calculated directly by formula (105).
In our idealized setting, we obtained a numerically stable, invertible estimate of $\mathbf{C}(\cdot)$ and thus a unique scaled reference portfolio in any Monte-Carlo simulation such that shrinkage was not necessary.

We investigate portfolios consisting of $K = 5$ risky assets, called A, B, C, D and E, and a riskless asset with $r_0 = 1$. We assume them to be independent. Each risky asset has a default probability ϑ_k and its non-default return distribution is assumed to be log-normal with mean μ_k and variance σ_k^2 for $k = 1, ..., 5$.
A realization of the return vector $\mathbf{r}$ is generated as follows. We draw a random vector $\boldsymbol{\theta}$, consisting of 5 independent Bernoulli random variables with parameters $\vartheta_1, ..., \vartheta_5$ and a random vector $\mathbf{n}$ of 5 independent log-normally distributed random variables with parameters $(\mu_1, \sigma_1), ..., (\mu_5, \sigma_5)$. Then,

$$\mathbf{r} = (\mathbf{1} - \boldsymbol{\theta}) \odot \mathbf{n}.$$

where $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^5$ and $\odot$ denotes the term-by-term multiplication of two vectors.

The parameters are calibrated to mimic a time frame of one year. Volatility of the non-default component $\mathbf{n}$ is set to be 10 % for all 5 assets and their expected excess returns range from 3 % to 23 %.
Three scenarios are considered. First, a scenario without defaults, i.e., $\vartheta_1 = ... = \vartheta_5 = 0$, is implemented. In this scenario, returns are log-normally distributed. Secondly, we mirror settings from Jarrow & Zhao (2006). There, default rates are 0 %, 0.3 %, 1 %, 3 %, 7 %. Thirdly, we adapt Jarrow's settings by doubling default probabilities and call this scenario "dangerous". Parameters of the simulated returns are reported in Table 1 on the following page.

Already in the log-normal scenario, we detect downside risk's superiority over standard deviation. From asset A to E nothing changes except the increasing mean. A higher mean together with a constant standard deviation makes an asset safer, because this upwards shift makes returns less likely to be smaller than 1 (which are losses). Downside risk accounts for this feature but standard deviation ignores it. In other words, 52 % of its deviation are risky for asset A, while only 3 % are risky for asset E. The remaining 97 % of deviation of asset E, which do not involve losses, constitute chance rather than risk! In scenarios with defaults, we observe that standard deviation as well as downside risk increase with increasing default probability. This is not surprising since the additional possibility of a default increases dispersion. More remarkably, downside risk increases much stronger than standard deviation. For instance in Jarrow's scenario, downside risk of E is

Scenario		A	B	C	D	E
Log-Normal	Default Probability	0.000	0.000	0.000	0.000	0.000
	Mean	1.030	1.041	1.060	1.115	1.230
	Standard Deviation	0.100	0.100	0.100	0.100	0.100
	Downside Risk	0.052	0.047	0.038	0.019	0.003
Jarrow	Default Probability	0.000	0.003	0.010	0.030	0.070
	Mean	1.030	1.037	1.050	1.081	1.144
	Standard Deviation	0.100	0.116	0.145	0.215	0.329
	Downside Risk	0.052	0.073	0.106	0.175	0.265
Dangerous	Default Probability	0.000	0.006	0.020	0.060	0.140
	Mean	1.030	1.034	1.039	1.048	1.057
	Standard Deviation	0.100	0.128	0.179	0.283	0.438
	Downside Risk	0.052	0.090	0.147	0.247	0.375

Table 1. Asset statistics.

more than 5 times as high as A's downside risk while standard deviation is only slightly more than tripled.

Next, we compare an investor holding a (μ, σ)-efficient portfolio with an investor holding a downside efficient portfolio. Therefore, the optimal portfolio weights for an efficient scaled reference portfolio in the sense of Markowitz[16]

$$\mathbf{y}^{\mathsf{eff}} := \frac{1}{|\langle \mathbf{1}, \mathbf{V}^{-1}\boldsymbol{\pi}\rangle|}\mathbf{V}^{-1}\boldsymbol{\pi}, \tag{105}$$

and for the downside efficient scaled reference portfolio $\mathbf{y}^{\mathsf{sref}}$ given by (93), respectively, are computed and comparatively displayed in Figure 9. Dark bars represent portfolio weights of the (μ, σ)-efficient portfolio mix. Light bars display portfolio weights of the downside efficient portfolio mix.

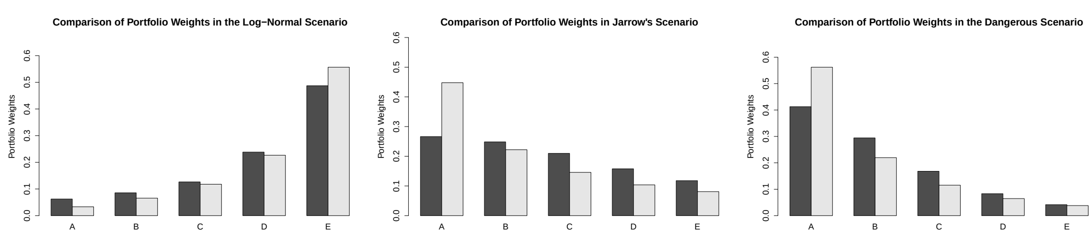

Figure 9. Portfolio weights in three scenarios.

[16] $\mathbf{V}$ denotes the covariance matrix of the return vector $\mathbf{r}$. In any simulation, we obtained $|\langle \mathbf{1}, \mathbf{V}^{-1}\boldsymbol{\pi}\rangle| > 0$, such that $\mathbf{y}^{\mathsf{eff}}$ is well-defined.

In any scenario, the efficient portfolio mix spreads the weights more equally, while the downside efficient portfolio mix puts more weight on the safest asset and less weight on the other assets. We also observe that the absolute difference in portfolio composition is larger in scenarios with defaults. The largest absolute difference occurs for the portfolio weight of the safest asset and amounts to 0.069 in the log-normal scenario, 0.181 in Jarrow's scenario and 0.149 in the dangerous scenario. Note that relative differences are remarkably high, e.g., almost 70 % for asset A in Jarrow's scenario. Portfolio weights' differences are summarized in Table 2.

Scenario		A	B	C	D	E
Log-Normal	$\mathbf{y}^{\mathsf{eff}}$	0.062	0.086	0.126	0.238	0.487
	$\mathbf{y}^{\mathsf{sref}}$	0.033	0.066	0.118	0.227	0.557
	absolute difference	−0.029	−0.020	−0.009	−0.012	**0.069**
	relative difference	**−0.469**	−0.232	−0.068	−0.049	0.143
Jarrow	$\mathbf{y}^{\mathsf{eff}}$	0.266	0.248	0.210	0.158	0.118
	$\mathbf{y}^{\mathsf{sref}}$	0.448	0.222	0.146	0.104	0.081
	absolute difference	**0.181**	−0.026	−0.064	−0.054	−0.037
	relative difference	**0.681**	−0.106	−0.305	−0.343	−0.314
Dangerous	$\mathbf{y}^{\mathsf{eff}}$	0.413	0.294	0.168	0.083	0.042
	$\mathbf{y}^{\mathsf{sref}}$	0.562	0.220	0.115	0.065	0.038
	absolute difference	**0.149**	−0.075	−0.053	−0.019	−0.003
	relative difference	**0.362**	−0.254	−0.313	−0.223	−0.083

TABLE 2. Differences in portfolio weights of efficient versus downside efficient portfolio mix. The largest difference is highlighted in bold print each.

To facilitate comparing efficient and downside efficient portfolio compositions concerning their riskiness, we rescale the downside efficient portfolios to the same mean return as the efficient ones. Thus, we obtain the following *scaled* downside efficient portfolios

$$\mathbf{y}^{\mathsf{sde}} := \frac{\mathfrak{M}(\mathbf{y}^{\mathsf{eff}})}{\mathfrak{M}(\mathbf{y}^{\mathsf{sref}})}\mathbf{y}^{\mathsf{sref}}.$$

In the log-normal scenario, the remaining money is invested into the risk-free bond, in the scenarios with default, money is borrowed from the bond (see, Table 3).

$\mathbf{y}^{\mathsf{sde}}$	A	B	C	D	E	Bond
Log-Normal	0.031	0.061	0.109	0.210	0.515	0.075
Jarrow	0.526	0.261	0.171	0.122	0.095	−0.174
Dangerous	0.583	0.228	0.120	0.067	0.040	−0.037

TABLE 3. Portfolio weights of scaled downside efficient portfolios.

Eventually, we compare $\mathbf{y}^{\mathsf{eff}}$ and $\mathbf{y}^{\mathsf{sde}}$ concerning their riskiness. Yielding the same mean, the efficient portfolio has (by construction) lower variance but higher downside risk. For each scenario, a boxplot illustrates the distribution of the returns. The log-normal scenario is displayed in green, Jarrow's scenario in blue and the dangerous scenario in black. The length of the whiskers is chosen to be the twofold interquartile range. Realized returns outside the whiskers are considered to be outliers. These extraordinarily large, respectively small, returns are of particular interest. Figure 10 shows that there is almost no difference between efficient and scaled downside efficient return distribution in the log-normal scenario. In scenarios with positive default probabilities, however, there are discernible differences, especially in the tails. We observe more exceedingly large and less very small returns realized by the scaled downside efficient portfolio. In particular, the largest gains are achieved by $\mathbf{y}^{\mathsf{sde}}$ while the largest losses arise from $\mathbf{y}^{\mathsf{eff}}$.

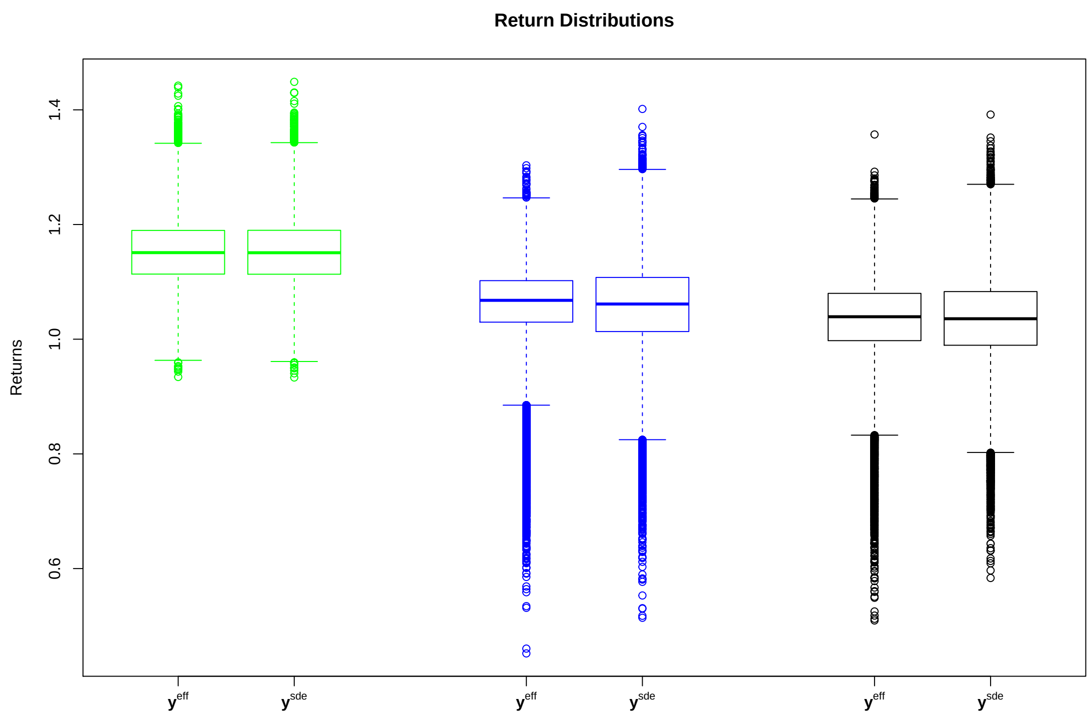

Figure 10. Boxplots in three scenarios.

In scenarios with default, a downside-risk averse investor purchasing $\mathbf{y}^{\mathsf{sde}}$ borrows money from the bond. Despite more money being invested into risky assets, $\mathbf{y}^{\mathsf{sde}}$ is safer than $\mathbf{y}^{\mathsf{eff}}$, because the scaled downside efficient portfolio exhibits a better diversification structure.

A volatility-averse investor tries to minimize standard deviation by purchasing assets with a low (or even negative) correlation, such that a low return of a certain asset is compensated by a large return of another asset. The other side of that coin is that not only small returns (which are potential losses) but also large returns (gains) are mitigated by the diversification effect.
However, when minimizing downside risk, an investor does not necessarily want to hold a portfolio mix of risky assets with a weak *overall* correlation. Instead, she prefers assets with a small *downside correlation*, which is the correlation in case of a loss. By focusing on realizations with $R(\mathbf{y}^{\mathsf{sde}}) < 1$, the downside correlation is smaller than the overall correlation, which explains why the smallest returns of $\mathbf{y}^{\mathsf{sde}}$ are greater than those of $\mathbf{y}^{\mathsf{eff}}$. On the other hand, the portfolio structure of $\mathbf{y}^{\mathsf{sde}}$ ignores correlations in case of returns larger than 1, such that gains are not mitigated by "upside diversification". Therefore, we observe more extraordinarily large returns for $\mathbf{y}^{\mathsf{sde}}$ than for $\mathbf{y}^{\mathsf{eff}}$.
This approach of downside-risk averse investors is called *downside diversification*.

A numerical analysis confirms the graphical insights. Although both portfolios differ, their riskiness is the same in the log-normal scenario – only for the maximal loss (that occurred in 100,000 simulations) we observe a negligible difference of 0.1 %. The reason is that the log-normal scenario is not risky (enough). The largest loss is less than 7 % and the expected shortfall is even negative. As downside risk takes into account the shortfall of an investment, i.e., realized returns below $r_0 = 1$, there is no opportunity to improve on the efficient portfolio w.r.t. riskiness.

	log-normal		Jarrow		dangerous	
	$\mathbf{y}^{\mathsf{eff}}$	$\mathbf{y}^{\mathsf{sde}}$	$\mathbf{y}^{\mathsf{eff}}$	$\mathbf{y}^{\mathsf{sde}}$	$\mathbf{y}^{\mathsf{eff}}$	$\mathbf{y}^{\mathsf{sde}}$
Mean	1.152	1.152	1.057	1.057	1.035	1.035
Standard Deviation	**0.057**	0.057	**0.072**	0.077	**0.070**	0.073
Downside Risk	0.001	**0.001**	0.038	**0.034**	0.039	**0.036**
Maximum Loss	**0.066**	0.067	0.548	**0.486**	0.491	**0.416**
$q_{1\,\%}(R(\cdot))$	**1.028**	1.028	0.830	**0.840**	0.817	**0.847**
$ES_{1\,\%}(R(\cdot))$	**−0.011**	−0.011	0.227	**0.210**	0.249	**0.203**
$\mathbb{P}(R(\cdot) < 0.9)$	0.000	0.000	0.045	**0.033**	0.035	**0.033**
$\mathbb{P}(R(\cdot) < 0.75)$	0.000	0.000	0.0027	**0.0016**	0.0042	**0.0014**

TABLE 4. Riskiness of $\mathbf{y}^{\mathsf{eff}}$ and $\mathbf{y}^{\mathsf{sde}}$.

However, in scenarios with default, other measures of riskiness, which are displayed in detail in Table 4, point out the superiority of $\mathbf{y}^{\mathsf{sde}}$. Maximal loss as well as expected shortfall at level 1 % are lower, the 1 %-quantile is higher. More precisely, maximal loss is more than 6 % smaller in Jarrow's scenario and about 7.5 % smaller in the dangerous scenario. Expected shortfall is 1.7 % down in Jarrow's scenario, respectively 4.6 % in the dangerous one. These numbers do not seem to be very impressive at first glance. But be aware of the scale of the return data. Mean net returns are 5.7 % and 3.5 %. In such a setting, single-figure percentages are of great importance. Even an absolute difference of just one percent is highly significant. Additionally, the probability of a loss of more than 25 %, which may constitute a state of insolvency, is much higher for efficient portfolios. In the dangerous scenario, it is approximately three times as high as for the scaled downside efficient portfolio!

Eventually, we compare Jarrow's scenario with the dangerous scenario where default probabilities are twice as high. For the (μ, σ)-efficient portfolio, expected shortfall increases from 22.7 % to 24.9 % and the probability of a very large loss of more than 25 % rises from 0.27 % to 0.42 % (which is a relative increase of over 50 %!). These observations may not be surprising, since the higher probability of default should cause larger losses. Therefore it is particularly remarkable that the expect shortfall and the probability of a loss of more than 25 % *decrease* for the scaled downside efficient portfolio! Although the scenario becomes more dangerous, the investment in $\mathbf{y}^{\mathsf{sde}}$ does not.

We summarize findings of this section.

Portfolio compositions remarkably differ between the (μ, σ)-efficient and the downside efficient portfolio mix. Downside efficient portfolios put more weight on the safest asset and less weight on all other assets.

When scaled to the same mean, the riskiness of efficient compared to downside efficient portfolios differ, depending on the nature of the scenario. In the calm scenario, i.e., for log-normally distributed returns, downside efficient portfolios are as risky as the efficient ones. In more dangerous scenarios with defaults, downside efficient portfolios are much safer than efficient portfolios. Investors holding a downside efficient portfolio mix are better secured against exceedingly large losses, in particular against insolvency.

Improvement in security becomes the larger, the more the return distribution deviates from log-normality. The higher default probabilities are, the more distinct the decrease of the probability of very large losses when holding $\mathbf{y}^{\mathsf{sde}}$ instead of $\mathbf{y}^{\mathsf{eff}}$, compare Table 4.

All in all, if the financial market is calm and the probability distribution of asset returns is (at least close to) log-normal, it does not matter if an investor holds an efficient or a downside efficient portfolio: mean return and riskiness are the same. Since the asset market is in such a calm mode most of the time, we do not see many banks going bankrupt. But in a turbulent market environment where disruptions and defaults have a strictly positive probability, e.g., during the recent financial crisis, some financial institutes became insolvent. In such a dangerous scenario the downside efficient portfolio mix unambiguously outperforms the efficient portfolio mix. Yielding the same mean return, it offers a better protection against downside shocks. A better downside diversified portfolio structure might have saved several investors from bankruptcy.

Even Markowitz (1959) himself stated that "semi-variance is the more plausible measure of risk [than variance]" but he decided to formulate portfolio optimization in terms of mean and variance, because variance is mathematically easier to handle. Since nowadays computational power is not

the bottleneck that is was when modern portfolio theory was developed, there is no need to still rely on variance as a measure of risk. The high rate of convergence of Algorithm 1 together with the arithmetic speed of modern computers enable banks, insurance companies or pension funds to apply the concept of optimal downside diversification when making investment decisions.
Due to the considerations of this section, the downside efficient portfolio mix may be considered as a general improvement over the efficient portfolio mix. It is not worse if the financial market is calm but it is strictly better in dangerous scenarios. Consequently, we advise investors to choose downside efficient portfolios rather than (μ, σ)-efficient ones.

4.A. Appendix to Chapter 4

Given $r_0 = 1$ and $100,000$ simulated return vectors $\mathbf{r}^{(1)}, ..., \mathbf{r}^{(100,000)} \in \mathbb{R}^5$, the downside efficient scaled reference portfolio $\mathbf{y}^{\mathsf{sref}}$ is calculated as follows.

Algorithm 1 Computation of $\mathbf{y}^{\mathsf{sref}}$.

i) Estimate the expected excess return vector

$$\boldsymbol{\pi} := \frac{1}{100,000} \sum_{j=1}^{100,000} \mathbf{r}^{(j)} - \mathbf{r}_0.$$

ii) Choose an appropriate initial portfolio $\mathbf{y}^{(1)}$.

iii) In the n^{th} iteration, i.e. for $n = 1, 2, 3, ...$

a) Determine the risky realizations, i.e., introduce the following indicator variable for all $j \in \{1, ..., 100,000\}$

$$\xi_j^{(n)} := \begin{cases} 1 & \text{if } \left\langle \mathbf{r}^{(j)} - \mathbf{r_0}, \mathbf{y}^{(n)} \right\rangle < 0 \\ 0 & \text{otherwise} \end{cases}.$$

b) Estimate the downside cosemivariance matrix $\mathbf{C}^{(n)}$ via

$$C_{kl}^{(n)} := \frac{1}{100,000} \sum_{j=1}^{100,000} \xi_j^{(n)} (r_0 - r_k^{(j)})(r_0 - r_l^{(j)})$$

for $k, l = 1, ..., 5$.

c) Update the portfolio vector

$$\mathbf{y}^{(n+1)} := \frac{1}{\left|\left\langle \mathbf{1}, (\mathbf{C}^{(n)})^{-1} \boldsymbol{\pi} \right\rangle\right|} (\mathbf{C}^{(n)})^{-1} \boldsymbol{\pi}.$$

iv) Repeat step iii) until a stopping criterion is fulfilled.

There are several possible choices of the initial portfolio $\mathbf{y}^{(1)}$. For instance, $\mathbf{y}^{(1)} = (0.2, 0.2, 0.2, 0.2, 0.2)$ or $\mathbf{y}^{(1)} = \boldsymbol{\pi}$ are reasonable approaches. We choose

$$\mathbf{y}^{(1)} = \mathbf{y}^{\mathsf{eff}},$$

because we supposed that, though not identical, the structure of $\mathbf{y}^{\mathsf{sref}}$ and $\mathbf{y}^{\mathsf{eff}}$ might be similar if covariance and downside cosemivariance patterns do not differ too much. Figure 9 justifies our approach. We also tried $\mathbf{y}^{(1)} = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $\mathbf{y}^{(1)} = \boldsymbol{\pi}$ which yielded the same results for $\mathbf{y}^{\mathsf{sref}}$ but more iteration had to be executed until the algorithm stopped.

We specify

$$\left\| \mathbf{y}^{(n+1)} - \mathbf{y}^{(n)} \right\|_1 < 0.000001$$

as stopping criterion, which is a very conservative choice. The stopping rule triggered termination of the algorithm after 4-7 iterations. This high rate of convergence underlines the direct applicability of portfolio selection using downside risk to practical issues.

Conclusion

By his seminal contributions, Markowitz (1952, 1959) pioneered modern portfolio theory. Building on his ideas, Tobin (1958), Sharpe (1964), Lintner (1965), Mossin (1966) and Merton (1972), to name but a few, further developed portfolio selection and equilibrium asset pricing theory. Markowitz (1959) decided to quantify risk by variance but, from the very beginning, he favored another measure of risk: semivariance, cf. Estrada (2008). In this thesis, we quantified risk by the square root of the below-target semivariance, referred to as downside risk. Thereby we showed how to improve the risk measurement methodology while retaining all theoretical insights from portfolio selection and equilibrium asset pricing theory. Furthermore, the mean-downside-risk framework can be applied to financial data with a little additional computational expense.

All concepts developed in the mean-variance framework can be refined by downside risk. Two tabular comparisons summarize how portfolio selection and equilibrium asset pricing theory structurally preserve.

When replacing the covariance matrix of payoffs $\mathbf{V}$ by the portfolio-dependent downside cosemivariance matrix $\mathbf{C}(\mathbf{x})$, compare Definition 1.3, we obtained the following.

Mean-Variance Framework	Mean-Downside-Risk Framework
There exists a solution to $\max_{\mathbf{x}\in\mathbb{R}^K} \mathfrak{M}(\mathbf{x})$ s.t. $\sqrt{\langle \mathbf{x}, \mathbf{V}\mathbf{x}\rangle} \leqslant \sigma_0$.	There exists a solution to $\max_{\mathbf{x}\in\mathbb{R}^K} \mathfrak{M}(\mathbf{x})$ s.t. $\sqrt{\langle \mathbf{x}, \mathbf{C}(\mathbf{x})\mathbf{x}\rangle} \leqslant d_0$.
$\mathbf{x}^{\mathsf{eff}}(\sigma_0) = \frac{\sigma_0}{\sqrt{\langle \tilde{\mathbf{x}}^{\mathsf{ref}}, \mathbf{V}\tilde{\mathbf{x}}^{\mathsf{ref}}\rangle}}\tilde{\mathbf{x}}^{\mathsf{ref}}$, where $\tilde{\mathbf{x}}^{\mathsf{ref}} \neq \mathbf{0}$ is a solution to the equation $\mathbf{V}\mathbf{x} = \boldsymbol{\pi}$.	$\mathbf{x}^{\mathsf{de}}(d_0) = \frac{d_0}{\sqrt{\langle \mathbf{x}^{\mathsf{ref}}, \mathbf{C}(\mathbf{x}^{\mathsf{ref}})\mathbf{x}^{\mathsf{ref}}\rangle}}\mathbf{x}^{\mathsf{ref}}$, where $\mathbf{x}^{\mathsf{ref}} \neq \mathbf{0}$ is a solution to the equation $\mathbf{C}(\mathbf{x})\mathbf{x} = \boldsymbol{\pi}$.
$\mathbf{x}^{\mathsf{eff}}(\sigma_0)$ is unique, if $\mathbf{V}$ is invertible.	$\mathbf{x}^{\mathsf{de}}(d_0)$ is unique, if $\mathbf{C}(\mathbf{x}^{\mathsf{ref}})$ is invertible.
Efficient frontier: $\mathfrak{e}(\sigma) = r_0 e + \sigma\sqrt{\langle \tilde{\mathbf{x}}^{\mathsf{ref}}, \mathbf{V}\tilde{\mathbf{x}}^{\mathsf{ref}}\rangle}$.	Downside efficient frontier: $\mathfrak{e}(d) = r_0 e + d\sqrt{\langle \mathbf{x}^{\mathsf{ref}}, \mathbf{C}(\mathbf{x}^{\mathsf{ref}})\mathbf{x}^{\mathsf{ref}}\rangle}$.

TABLE 5. Efficiency.

Similar to the mean-variance framework, we established a separation theorem for downside-risk averse investors. Downside efficient portfolios have an advantage over mean-variance efficient portfolios, because their payoffs $w(\mathbf{x}^{\mathsf{de}}(d_0))$ are almost surely undominated and undominated w.r.t. stochastic dominance. This need not be the case for $w(\mathbf{x}^{\mathsf{eff}}(\sigma_0))$. Furthermore, $w(\mathbf{x}^{\mathsf{de}}(d_0))$ maximizes expected utility for a strictly increasing Bernoulli utility function whereas $w(\mathbf{x}^{\mathsf{eff}}(\sigma_0))$ is consistent with expected utility theory for quadratic, i.e., non-monotonic, Bernoulli utility functions only.

A downside risk asset market equilibrium (DRAME) is defined analogously to a mean-variance CAPM equilibrium. It is an individually optimal and market-clearing allocation of assets.

Mean-Variance Framework	Mean-Downside-Risk Framework
A mean-variance CAPM equilibria exists.	A DRAME exists.
It is unique if the aggregate demand for standard deviation $\Psi^{\mathsf{MV}}(p_{\mathtt{m}})$ is *strictly* decreasing.	It is unique if the aggregate demand for downside risk $\Psi(p_{\mathtt{m}})$ is decreasing.
CAPM pricing formula: $p_k^* = \frac{1}{r_0}\Big(E\left[q_k\right] - \frac{E[\epsilon_{\mathtt{m}}]-r_0 p_{\mathtt{m}}^*}{\sigma_{\mathtt{m}}^2}\times$ $E[(E[q_k]-q_k)(E[\epsilon_{\mathtt{m}}]-\epsilon_{\mathtt{m}})]\Big)$.	DRAME pricing formula: $p_k^* = \frac{1}{r_0}\Big(E[q_k] - \frac{E[\epsilon_{\mathtt{m}}]-r_0 p_{\mathtt{m}}^*}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)^2}\times$ $E[(r_0 p_k^* - q_k)(r_0 p_{\mathtt{m}}^* - \epsilon_{\mathtt{m}})_+]\Big)$.
The market payoff $\epsilon_{\mathtt{m}}$ attains the highest possible *Sharpe* ratio at equilibrium prices.	The market payoff $\epsilon_{\mathtt{m}}$ attains the highest possible *Sortino* ratio at equilibrium prices.
Security market line: $E[q_k] - r_0 p_k^* = \beta_k\left(E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*\right)$, $\beta_k := \frac{E[(E[q_k]-q_k)(E[\epsilon_{\mathtt{m}}]-\epsilon_{\mathtt{m}})]}{\sigma_{\mathtt{m}}^2}$	Downside security market line: $E[q_k] - r_0 p_k^* = \beta_k^*\left(E[\epsilon_{\mathtt{m}}] - r_0 p_{\mathtt{m}}^*\right)$, $\beta_k^* := \frac{E\left[(r_0 p_k^* - q_k)(r_0 p_{\mathtt{m}}^* - \epsilon_{\mathtt{m}})_+\right]}{\mathcal{D}(\epsilon_{\mathtt{m}}, r_0 p_{\mathtt{m}}^*)^2}$

TABLE 6. Equilibria.

We did not only construct a pricing formula and a security market line under the hypothesis of equilibrium in the mean-downside-risk framework.[17] Instead, we *proved* existence of DRAMEs and *derived* the corresponding pricing formula and downside security market line. Note further that uniqueness is achieved under a milder requirement. The major improvement is that DRAME prices are always arbitrage-free and strictly positive whereas the CAPM pricing formula, in general, allows for arbitrage opportunities and negative stock prices.

Financial practitioners can readily apply and benefit from the portfolio selection methodology using downside risk. The usage of the shrinkage technique ensures a numerically stable and invertible estimate of the downside cosemivariance matrix. Hence, the downside

[17] This was already done by Hogan & Warren (1974) and generalized by Bawa & Lindenberg (1977).

efficient portfolio can be derived in the same way as the mean-variance efficient one. Since downside efficient portfolios are characterized implicitly, numerical calculations are costlier. The effective implementation via Algorithm 1 (see appendix to Chapter 4), its high rate of convergence and the increased arithmetic speed of modern computers minimize the additional computational expense.
In a simulation study, we recalibrated (μ, σ)-efficient and downside efficient portfolios, such that they yielded the same mean return. We verified that downside efficient portfolios protect investors better from large losses in turbulent market scenarios. When market disruptions and defaults of assets have a strictly positive probability, downside efficient portfolios unambiguously outperform mean-variance efficient portfolios. Value-at-risk, expected shortfall, maximal loss and the probability of a loss of more than 25 %, every single one of them is greater for the mean-variance efficient portfolio. Above all, we found no scenario where downside efficient portfolios perform worse.
Yielding the same mean payoff, downside efficient portfolios turned out to be as safe as mean-variance efficient portfolios in calm scenarios and substantially safer in dangerous scenarios. Thus, investors should rely on downside efficient portfolios.

A number of further research avenues are possible.
An immediate extension is to analyze a dynamic multiperiod model instead of the static model with two dates which is proposed in this thesis. Wenzelburger (2017) suggests a dynamic approach where returns are generated by an exogenous stochastic process and develops mean-variance portfolio theory, there. This setting seems to allow for a refinement of portfolio theory using downside risk.
Further, it would be interesting to examine existence and uniqueness of downside risk asset market equilibria under the paradigm of heterogeneous beliefs. This assumption is more realistic but analytically much more demanding than the setup with homogeneous beliefs.
Another extension would be to incorporate short sale constraints or other types of regulatory framework such as capital adequacy requirements for investors. This question to what extend downside risk can be incorporated in financial regulation and banking theory calls for future research.
All these possible advancements are outside the scope of the present thesis. They are open questions to succeeding generations of financial economists.

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Publications

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