



The Worst-Case Portfolio Optimization Problem in Discrete-Time

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Abstract

In this thesis, we deal with the worst-case portfolio optimization problem occurring in discrete-time markets.

First, we consider the discrete-time market model in the presence of crash threats. We construct the discrete worst-case optimal portfolio strategy by the indifference principle in the case of the logarithmic utility. After that we extend this problem to general utility functions and derive the discrete worst-case optimal portfolio processes, which are characterized by a dynamic programming equation. Furthermore, the convergence of the discrete worst-case optimal portfolio processes are investigated when we deal with the explicit utility functions.

In order to further study the relation of the worst-case optimal value function in discrete-time models to continuous-time models we establish the finite-difference approach. By deriving the discrete HJB equation we verify the worst-case optimal value function in discrete-time models, which satisfies a system of dynamic programming inequalities. With increasing degree of fineness of the time discretization, the convergence of the worst-case value function in discrete-time models to that in continuous-time models are proved by using a viscosity solution method.

Zusammenfassung

Diese Arbeit befasst sich mit der Worst-Case-Portfoliooptimierung in diskreten Märkten.

Zunächst betrachten wir das zeitdiskrete Marktmodell bei vorhandener Möglichkeit eines Crashes. Im Falle des logarithmischen Nutzens konstruieren wir den diskreten Worst-Case optimalen Portfolioprozess mit Hilfe des Indifferenzprinzips. Danach erweitern wir dieses Problem auf allgemeine Nutzenfunktionen und leiten die diskreten Worst-Case optimalen Portfolioprozesse ab, die durch eine Gleichung ein dynamisches Optimierungsproblems gekennzeichnet sind. Darüber hinaus wird die Konvergenz der diskreten Worst-Case optimalen Portfolioprozesse für verschiedene Nutzenfunktionen untersucht.

Um die Beziehung der Worst-Case-Optimalwertfunktion bei zeitdiskreten Modellen und zeitstetigen Modellen weiter zu untersuchen, wird der Finite-Differenzen-Ansatz verwendet. Durch Herleitung der diskreten HJB-Gleichung verifizieren wir die Worst-Case Optimalwertfunktion in zeitdiskreten Modellen, die ein System von dynamischen Programmierungsgleichungen erfüllt. Mittels einer Viskositätslösungsmethode wird gezeigt, dass die Worst-Case-Wertfunktion in zeitdiskreten Modellen gegen jene aus zeitstetigen Modellen konvergiert, wenn der Feinheitsgrad der Zeitdiskretisierung gegen 0 strebt.

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Chapter 1.

Introduction

In the last few decades, financial mathematics has become an important and rapidly expanding field of modern science, both in mathematics and economics. One of the classical problems in financial mathematics is the portfolio optimization problem, that is, optimizing investments for an investor with a given utility function and a fixed initial endowment. In order to deal with these problems, we need at first to model the financial markets with different mathematical models by considering the possible times of the asset price changes during the time interval. Two kinds of mathematical models, which are discrete-time financial market models and continuous-time financial market models, have been developed and actively investigated to attack the portfolio optimization problem.

The continuous-time models, in which investors are allowed to make investment decisions at any time, were developed from the 70's of the last century. Black and Scholes[5] first used the geometric Brownian motion to model the price processes of stocks in 1973. Subsequently, based on the work of Black and Scholes Merton[33] pioneered the continuous-time approach to the portfolio optimization problem. He applied classical stochastic control methods to the optimal terminal wealth problem in the Black-Scholes market. Since Merton's pioneering work, many complete theories and powerful approaches [for instance, Korn[23], Karatzes and Shreve[21]] have been developed to solve the portfolio optimization problem in the continuous-time setting.

Compared to the continuous-time models, discrete-time models are more preferable from the computational and practical point of view. For studying the portfolio optimization problem in discrete-time models, the single-period market is a nature model and has the advantage of being mathematically simple. Markowitz[31], whose mean-variance portfolio selection is the most important single-period model, is the definitive reference on single-period portfolio management. Multi-period models are much more realistic than single-period ones. The multi-period portfolio optimization problem has also been deeply studied. Samuelson[45] obtained the optimal decision for the discrete-time consumption-investment model with the objective of maximizing the expected utility of consumption by using the stochastic dynamic programming approach. Duffie[12] pro-

vided good treatments of both continuous-time and discrete-time models. The method of dynamic programming, as well as the martingale method for the optimal consumption and investment problems, were developed in Pliska[41]. Sometimes the Markovian property is required to facilitate the study of these models. We can also look at the works by Bertsekas[4], Puterman[44], Bäuerle and Rieder[2] about the Markov decision processes and the applications to finance.

The drawback of the classical stock models is that the model is not able to fully explain extreme stock price movements, which are often observed at the markets and can cause large financial losses for investors. To be prepared for such a situation and avoid large losses for the investor is a desirable goal. Therefore, the modeling of a crash or of large stock price movements incorporated into the optimal portfolio problem has become an active research area in financial mathematics. A natural idea to replace the classical models is the stock price dynamics with a jump diffusion processes(see Merton[35]). And many of the work done relied on modeling stock prices as lévy processes(see Aase[1] and Kallsen[20]). The approaches in these models only lead to optimal strategies which hedge a risk coming from the jump possibility over the investment period. In particular, it is difficult to estimate jump intensities and sizes, moreover, the investor following such strategies may still suffer large losses during a crash. As a contrast to that, we will take the so-called crash model which was firstly introduced by Hua and Wilmott[19]. In this model, it is assumed that only both the maximal number of crashes in a given time interval and the biggest possible size of crashes are known. More precisely, they distinguished between the 'normal times', where the stock prices are assumed to follow general Brownian motion, and the 'crash times', where the stock prices are given by a sudden fall by an unknown factor which they assumed to be bounded by a known constant. Korn and Wilmott[29] took up this crash model and first studied the worst-case portfolio optimization problem in the continuous-time setting. By an indifference argument they showed how to derive the worst-case optimal portfolio processes for logarithmic utility. Korn and Menkens[26] extended this approach to a more general market setting and Korn[24] extended it to the problems in an insurance context. Korn and Steffensen[28] showed that the value function can be found by solving the so-called HJB-system. In Korn and Seifried[27] a new martingale approach is presented to find an indifference strategy. Further studies on the worst-case portfolio problem in continuous-time setting are [50], [10], [11] and [3] .

In contrast to continuous-time models of worst-case portfolio optimization problems, relatively little work has been done in discrete-time models. Nevertheless, some interesting real-life problems are not tractable in the framework of continuous-time setting. The optimal portfolio in continuous-time models is calculated under the assumption that investors can trade continuously. If trading is possible at discrete points in time only, these optimal strategies can no longer be implemented. In practice, however, this can

never be achieved, since trading even on fully electronic systems is only possible at discrete points in time. Furthermore, investors may not want to adjust their portfolio too frequently because of the so-called transaction costs. This motivates us to consider the worst-case portfolio optimization problem in discrete-time setting. Furthermore, if the parameters in the discrete-time financial markets are chosen appropriately, this discrete-time models can be seen as an approximation of the continuous-time models (for instance, Black-Scholes model). This observation serves as another reason for the importance of considering the worst-case portfolio optimization problem in discrete-time setting. The objective of this thesis is to deal with the worst-case portfolio optimization problem for discrete-time market models.

Outline of this thesis

Chapter 2 gives an overview of the portfolio optimization problem in continuous-time models with and without crashes. First of all, we consider the Merton problem in which the investor aims to maximize her expected utility of terminal wealth in a Black-Scholes model. After that, we describe the crash model introduced by Hua and Wilmott and show the optimal worst-case portfolio processes by different approaches.

In Chapter 3 we start by considering the discrete-time financial market model. We define portfolio strategies and characterize the absence of arbitrage in this market. Next, we investigate the dynamic programming method for the discrete-time portfolio optimization problem and present the corresponding numerical examples.

Chapter 4 describes first the set up of the crash model in the discrete-time financial market. In Section 2 we derive the worst-case optimal portfolio processes in discrete-time for the logarithmic utility function by an indifference argument. After this we turn to a more general study of the worst-case portfolio optimization problem in discrete-time. Section 3 is devoted to provide a system of dynamic programming equations and verify the optimal strategies as a system of difference equations. In section 4 these results will be applied to the power-utility, log-utility and exponential-utility functions.

In Chapter 5 we turn our focus to establish a new approach, a finite-difference approach. We first consider the discrete Itô formula introduced by Fujita. In Section 2 the discrete-HJB equation is derived and the relation between the value function of the discrete-time portfolio problem and the discrete-HJB equation is investigated. Moreover, we extend the discrete-time financial market to allow for crashes in the stock price. Our aim is to solve the worst-case portfolio optimization in discrete-time setting by considering the discrete-HJB equation. The result is a verification theorem asserting that a so-called Bellmann system determines the value function. This result and the characterization of the solution are illustrated by some explicit examples. Some connections between

discrete-time and continuous-time crash models are considered in Section 4. We show that, with increasing degree of fineness of the time discretization, the value function in the discrete-time crash model converges to that of the continuous-time crash model.

Finally, our thesis is complemented by a summary and an outlook on future research at the end.

Chapter 2.

The worst-case portfolio optimization in continuous-time

For the portfolio optimization problem in continuous time and in discrete time we have two different financial market settings. Here we will take a look at the portfolio optimization in continuous time. The market setting in discrete time we will discuss in the next chapter. In the following we consider the optimal terminal wealth problem in the Black Scholes market which was first solved in Merton[34]. A more extensive overview of methods and models in continuous time portfolio optimization can be found in R. Korn[23].

2.1. The portfolio optimization in continuous-time

Here we state the Merton model[34], which marks the starting point of stochastic control methods in portfolio optimization.

Let (Ω, \mathcal{F}, P) be a complete probability space with sample space Ω , σ -field \mathcal{F} and probability measure P , and let \mathcal{F} be a filtration which satisfies the usual conditions. (A filtration \mathcal{F} satisfies the usual conditions if it is right-continuous and \mathcal{F}_0 contains all P -null sets of \mathcal{F} .) We furthermore assume that (Ω, \mathcal{F}, P) supports a one-dimension standard Brownian motion $W = (W(t))_{t \geq 0}$ with respect to \mathcal{F} and fix a finite time horizon $T > 0$.

We consider the financial market consisting of a riskless bond and one risky security. The price dynamics of the bond, denoted by B_t , and the price dynamics of the stock, denoted by S_t , are given by

$$dB_t = rB_t dt \tag{2.1}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.2}$$

with constant market coefficients $\mu > r$ and the volatility $\sigma \neq 0$.

Let $x > 0$ be the investor's initial wealth. We assume that π is a self-financing trading

strategy and the strategy π is admissible for the initial value x . We denote the set of all admissible strategies on $[t, T]$ with $X(t) = x$ by $A(t, x)$. Here, we call the portfolio process (π) self-financing if the wealth process $X(t)$ corresponding to an initial capital x satisfies the following stochastic differential equation

$$\begin{cases} dX(t) = rX(t)dt + X(t)\pi(t)(\mu - r)dt + \pi(t)\sigma X(t)dW_t, \\ X(0) = x \end{cases} \quad (2.3)$$

The portfolio process will be called admissible on $[t, T]$ if it is self-financing and has a non-negative wealth process.

So the portfolio problem asks for a self-financing strategy (π) in a suitable class \mathcal{A} of admissible trading strategies which maximizes the expected utility

$$J(x; \pi, c) = E(u(X(T))) \quad (2.4)$$

for some terminal time T and the utility function u . Let

$$V(t, x) = \sup_{\pi \in \mathcal{A}} J(x; \pi) \quad (2.5)$$

be the value function of the portfolio optimization problem. Merton [34] used the so-called Bellman principle:

$$V(t, x) = \sup_{\pi \in \mathcal{A}} E(V(\theta, X(\theta))) \quad (2.6)$$

where θ is a stopping time taking values in $[t, T]$. The Bellman principle which allows us to transform the original problem into two subproblems works well in a discrete time setting. In the continuous-time case, Korn[25] considered the Hamilton-Jacobi-Bellman equation (HJB equation) by using the Itô formula. The corresponding HJB-equation has the form

$$\begin{cases} \sup_{\pi \in \mathcal{A}} \{ \frac{1}{2} \pi' \pi \sigma^2 x^2 V_{xx}(t, x) + ((r + \pi'(\mu - r))x) V_x(t, x) + V_t(t, x) \} = 0, \\ V(T, x) = u(x) \end{cases} \quad (2.7)$$

One can show that under certain conditions this equation has a unique solution and this solution is indeed our value function V . For example, if $U = \frac{1}{\gamma} x^\gamma$, we obtain the optimal strategy (π^*) in a setting with only one stock as

$$\pi^*(t) = \frac{1}{1 - \gamma} \frac{\mu - r}{\sigma^2} \quad (2.8)$$

and for example, if $U = \log x$, the optimal strategy (π^*) is given by

$$\pi^*(t) = \frac{\mu - r}{\sigma^2} \quad (2.9)$$

The form of the optimal trading strategies are illustrated in Figure 2.1. Note that the optimal portfolio strategy $\pi^*(t)$ is constant.

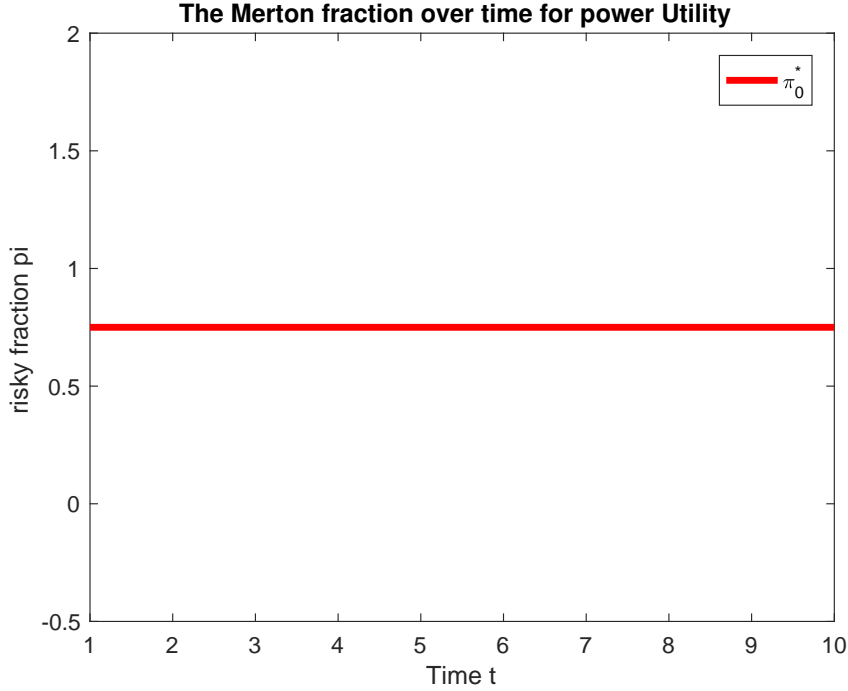


Figure 2.1.: The optimal trading strategies π_0^* for the power utility function in a Merton type market

2.2. The worst-case portfolio optimization in continuous-time

A drawback of the standard geometric Brownian-motion-based models is that the model is not able to fully explain extreme stock price movements, which are often observed at the markets. Sudden price falls of the whole market, so-called crashes, are not incorporated into the standard continuous-path framework.

A natural idea to replace the Brownian motion is the stock price dynamics with a jump diffusion[37] or to consider price processes which are driven by lévy processes(see kallsen [20]). Since in these models the distribution of the jump times and sizes is known to the investor, this leads to optimal strategies which hedge the risk coming from the jump possibility on average over the whole period. In particular it is difficult to estimate jump intensities and sizes. Motivated by the desire to be able to model market crashes, Hua and Wilmott [19] introduced their so-called crash model. The stock prices are assumed to follow geometric Brownian motion in normal times. The crash feature of the stock price at a crash time is given by a sudden fall by an unknown factor, which they assumed to be bounded by an explicitly known constant, but the true distribution of the jumps

remains unknown. Subsequently Korn and Wilmott [29] used the framework of Hua and Wilmott where they focus on the uncertainty of the number, time and height of possible market crashes and optimize over the worst-case bounds of the trading strategies. They assume that the stock and bond dynamics in normal times are given by

$$dB_t = rB_t dt, B_0 = 1 \quad (2.10)$$

$$dS_t = \mu S_t dt + \sigma S_t dw_t, S_0 = p_1 \quad (2.11)$$

with constant market coefficients $\mu > r$ and $\sigma \neq 0$. At a crash time τ , which is modeled as a stopping time, the stock suddenly drops by a relative amount of k with $0 \leq k \leq k^* < 1$. Here, k^* is assumed to be the biggest possible crash height. In a crash scenario (τ, k) , we have

$$S(\tau) = (1 - k)S(\tau-) \quad (2.12)$$

No assumptions are made about the distribution of the crash time or height. Moreover, the investor is assumed to expect the worst-possible crash scenario to occur. Let X^π be the wealth process corresponding to the portfolio process $\pi(t)$ and the initial wealth of x . Instead, the idea is to find the trading strategy which performs the best in the worst-case scenario.

Definition 2.1: Let U be an utility function. The **worst-case portfolio optimization problem** can be expressed as follows:

$$\sup_{\pi \in A} \inf_{\tau, k} E(U(X^\pi(T))) \quad (2.13)$$

where the final wealth $X^\pi(T)$ in the case of a crash of size k at time τ given by

$$X^\pi(T) = (1 - \pi(\tau)k)\tilde{X}^\pi(T) \quad (2.14)$$

with $\tilde{X}^\pi(T)$ the wealth process in the standard crash-free market model and is given as the unique solution to the stochastic differential equation

$$\begin{aligned} d\tilde{X}^\pi(t) &= \tilde{X}^\pi(t)(r + \pi(t)(\mu - r))dt + \pi(t)\sigma\tilde{X}^\pi(t)dw_t, \\ \tilde{X}^\pi(0) &= x \end{aligned}$$

This worst-case portfolio optimization problem could be formulated: For every admissible trading strategy π , determine the crash time τ and size k which yields the minimal expected utility at terminal time. This is the so-called worst-case bound for the strategy π . The optimal trading strategy is then defined to be the trading strategy with the maximal worst-case bound. The optimization problem is hence a version of Wald's maximin model ([51]) which means that the investor chooses a strategy at first and presents this strategy to her opponent market who chooses the crash scenario. In Korn and Menkens[26] and Korn and Seifried [27] we can see an overview of different approaches

to this type of problem.

The worst-case portfolio optimization problem in continuous time has first been studied in Korn and Wilmott[29] for logarithmic utility. Since at most one crash can occur it is straightforward to argue that after the occurrence of the crash the investor should invest according to the optimal strategy in the Merton model. Korn and Wilmott used the indifference strategy to solve this worst-case portfolio problem, where they found a portfolio process that makes the investor indifferent about two extreme cases:

- A crash of maximize size k^* happens immediately.
- No crash happens at all.

In the log-utility case, this can be done by solving the equation

$$E[\log(X_{t,x}^{\pi^*}(T))] = V_0(t, (1 - \pi^*(t)k)x) \quad (2.15)$$

in which the worst-case optimal strategy π satisfies the following differential equation

$$\pi'(t) = -\frac{\sigma^2}{2k^*}(1 - \pi(t)k^*)(\pi(t) - \pi^*)^2 \quad (2.16)$$

with the obvious final condition

$$\pi(T) = 0 \quad (2.17)$$

Then the optimal strategy is given by the solution to this ordinary differential equation up to the crash time. After the crash, the investor invest according to the standard crash-free market model. Korn and Willmott showed that for any other strategy π there exists one crash scenario in which π^* performs better. π^* is indeed optimal.

Theorem 2.2: [29] *In the log-utility case, the portfolio process π^* such that the corresponding expected log-utility after an immediate crash equals the expected log-utility given no crash occurs which is given as the solution π^* of the differential equation*

$$\begin{aligned} \pi'(t) &= \frac{1}{k^*}(1 - \pi(t)k^*)(\pi(t)(\mu - r) - \frac{1}{2}((\pi\sigma)^2 + (\frac{\mu - r}{\sigma})^2)) \\ \pi(T) &= 0 \end{aligned}$$

and satisfies

$$0 \leq \pi^* \leq \frac{1}{k^*}$$

is an optimal portfolio process for the worst-case problem.

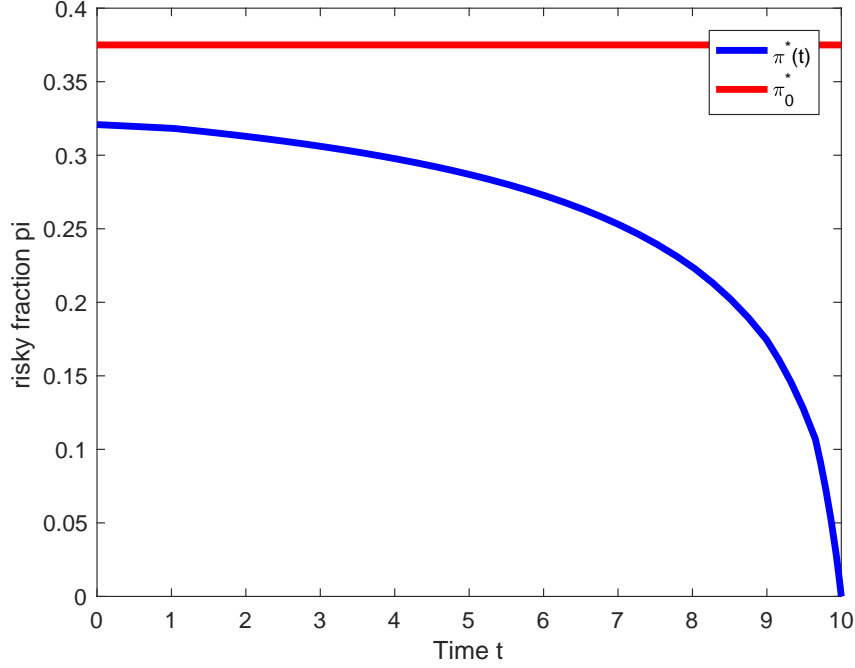


Figure 2.2.: The optimal trading strategies for log utility function with and without crash possibility

Figure 2.2 illustrates the worst-case optimal strategy π^* and the crash-free optimal strategy in a Merton type market π_0 . Note that the worst-case optimal portfolio process π^* is a nonconstant process which is decreasing with time. And even at the initial time $t = 0$ the optimal strategy in the presence of crash is below the crash-free strategy π_0 .

Korn and Wilmott [29] extended these results to $n > 1$ crashes. It can be shown that the strategy π^* in the presence of n crashes is given as the solution of

$$(\pi^{n,*}(t))' = -\frac{\sigma^2}{2k^*}(1 - \pi^{n,*}(t)k^*)(\pi^{n,*}(t) - \pi^{n-1,*}(t))^2 \quad (2.18)$$

with the obvious final condition

$$\pi^{n,*}(T) = 0 \quad (2.19)$$

This approach is extended to a more general market setting by Korn and Menkens [26]. They extend these results to power utility and changing market coefficients after the occurrence of a crash by deriving a dynamic programming equation for the value

function. The price dynamics of the bond and the risky asset after the crash scenario (τ, k) are then assumed to be given by

$$dB_1(t) = r_1 B_1(t)dt, B_1(\tau) = B_0(\tau) \quad (2.20)$$

$$dS_1(t) = \mu_1 S_1(t)dt + \sigma_1 S_1(t)dw_t, S_1(\tau) = (1 - k)S_0(\tau) \quad (2.21)$$

with constant market coefficients r_1, μ_1 and $\sigma_1 \neq 0$ after the crash.

We denote by

$$\Psi_1 = r_1 + \frac{(\mu_1 - r_1)^2}{2\sigma_1^2}$$

and

$$\Psi_0 = r_0 + \frac{(\mu_0 - r_0)^2}{2\sigma_0^2}$$

the utility growth potentials in the respective markets. Then it allows them to show that the crash hedging strategy π^{ch} is given by the solution of the differential equation

$$(\pi^{ch}(t))' = -\frac{1}{k^*}(1 - \pi^{ch}(t)k^*)\left(\frac{(\sigma_0)^2}{2}(\pi^{ch}(t) - \pi_0^*(t))^2 + \Psi_1 - \Psi_0\right) \quad (2.22)$$

with final condition

$$\pi^{ch}(T) = 0$$

This strategy makes the investor indifferent between no crash occurring at all until the investment horizon and the immediate worst possible crash, we have to compare the markets before and after the crash. Note that the equation 2.22 reduces to the equation 2.16 if the market coefficients do not change after a crash. The optimal portfolio strategy before the crash for an investor who wants to solve the worst case portfolio optimization problem is given by

$$\pi^*(t) = \min\{\pi^{ch}(t), \pi_0^*\}$$

A more detailed discussion of the effects of changing market coefficients after a crash can be found in Korn and Menkens [26].

Presently, there are also two other approaches to solve this kind of problem: a control approach as used in Korn and Steffensen [28] and a martingale approach as chosen in Korn and Seifried [27].

Korn and Steffensen interpret the worst-case setting as a game between the market and the investor. While the market is allowed to choose a crash sequence, the investor chooses the portfolio process. The stock price dynamics are modeled by

$$dS_t = \mu S_t dt + \sigma S_t dw_t - k S_t dN_t \quad (2.23)$$

where N_t is a process which counts the number of the crashes. They derive a system of inequalities that they call the HJB-system and thereby obtain optimality of the worst-case portfolio process. For $V \in C^{1,2}$ we define the differential operator $\mathcal{L}^\pi V$ by

$$\mathcal{L}^\pi V(t, x) = V_t(t, x) + V_x(t, x)(r + \pi(\mu - r))x + \frac{1}{2}V_{xx}(t, x)\pi^2\sigma^2x^2$$

and for $n \in N$ we define the value function $V^n(t, x)$ by

$$V^n(t, x) = \sup_{\pi \in A(t, x)} \inf_{N \in B(t, n)} E(U(X^\pi(T)))$$

Theorem 2.3: [28]

- Assume that $v^0(t, x)$ is a classical solution of

$$\begin{aligned} \sup_{\pi \in A(t, x)} \mathcal{L}^\pi v^0(t, x) &= 0 \\ v^0(T, x) &= U(x) \end{aligned}$$

which is polynomially bounded, and that

$$p^0(t, x) = \arg \sup_{\pi \in A(t, x)} \mathcal{L}^\pi v^0(t, x)$$

is an admissible control function. Then we have

$$V^0(t, x) = v^0(t, x)$$

and the optimal control function exists and is given by

$$\pi^{0*}(t, x) = p^0(t, x)$$

- For $n \in N$ and every function $v^n \in C^{1,2}$, define the sets $\mathcal{A}'_n(t, x)$ and $\mathcal{A}''_n(t, x)$ by

$$\mathcal{A}'_n(t, x) = \{\pi : \pi \in \mathcal{A}, 0 \leq \mathcal{L}^\pi v^n(t, x)\} \quad (2.24)$$

$$\mathcal{A}''_n(t, x) = \{\pi : \pi \in \mathcal{A}, 0 \leq v^{n-1}(t, x(1 - k\pi)) - v^n(t, x)\} \quad (2.25)$$

respectively. Assume that there exists a polynomially bounded $C^{1,2}$ -solution of

$$\begin{aligned} 0 &\leq \sup_{\pi \in \mathcal{A}''_n(t, x)} [\mathcal{L}^\pi v^n(t, x)] \\ 0 &\leq \sup_{\pi \in \mathcal{A}'_n(t, x)} [v^{n-1}(t, (1 - \pi k^*)x) - v^n(t, x)] \\ 0 &= \sup_{\pi \in \mathcal{A}''_n(t, x)} [\mathcal{L}^\pi v^n(t, x)] \sup_{\pi \in \mathcal{A}'_n(t, x)} [v^{n-1}(t, (1 - \pi k^*)x) - v^n(t, x)] \\ v^n(T, x) &= U(x) \end{aligned}$$

and that

$$\begin{aligned} p^n(t, x) &:= \arg \sup_{\pi \in \mathcal{A}_n'(t, x)} [\mathcal{L}^\pi v^n(t, x)] \\ \theta^n(t, x) &:= \sup_{s \geq t} [v^{n-1}(s, (1 - \pi k^*)X^\pi(s)) - v^n(s, X^\pi(s)) \leq 0] \end{aligned}$$

where $X^\pi(t) = x$ and s is a stopping time, is a pair of admissible control functions. Then

$$V^n(t, x) = v^n(t, x)$$

and the optimal control functions exist and are given by

$$\begin{aligned} \pi^{n*}(t, x) &:= p^n(t, x) \\ \tau^{n*}(t, x) &:= \theta^n(t, x) \end{aligned}$$

In Korn and Steffensen [28] explicit examples are solved when the utility function is the negative exponential utility function or of the form $U(x) = \frac{1}{\gamma}x^\gamma, \gamma \neq 0$. With the help of the Bellman system, we can drive an ordinary differential equation for the optimal strategy $\pi^{n*}(t)$

$$\begin{aligned} (\pi^{n*}(t))' &= \frac{1}{k}(1 - \pi^{n,*}(t)k)((\mu - r)(\pi^{n,*}(t) - \pi^{n-1,*}(t)) \\ &\quad - \frac{(1 - \gamma)(\sigma_0)^2}{2}((\pi^{n,*}(t))^2 - (\pi^{n-1,*}(t))^2)) \\ \pi^{n*}(T) &= 0 \end{aligned}$$

One can show via induction that its solution satisfies

$$0 \leq \pi^{n*}(t) \leq \pi^{*(n-1)}(t) \leq \dots \leq \pi^{*0}(t)$$

is unique.

Figure 2.3 illustrates the worst-case optimal strategy $\pi^{*1}(t)$ and the crash-free optimal strategy in a Merton type market π_0 for power utility function. They look very similar to the optimal portfolio processes of Figure 2.2. The optimal portfolio process is a nonconstant process which decreases with time.

In contrast to the dynamic programming approach, the martingale approach to the worst-case portfolio problem is based on martingale optimality arguments and the idea that the market acts as an opponent to the investor. Korn and Seifried [27] generalizes the results by considering the worst-case portfolio problem as a controller-vs-stopper game. They set the process $W^\pi(t)$ by

$$W^\pi(t) = V^0(t, 1 - \pi(t)k^*)X^\pi(t) \tag{2.26}$$

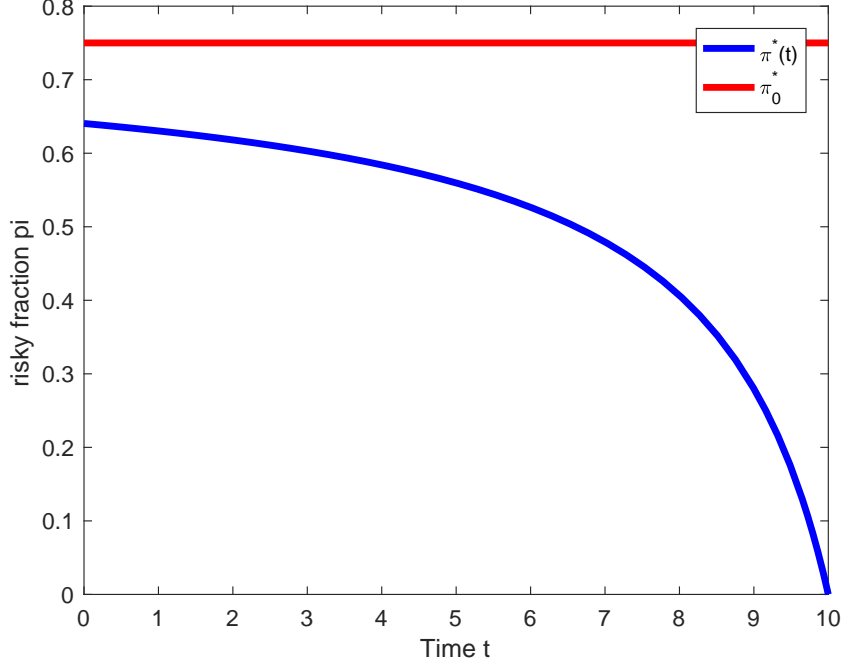


Figure 2.3.: The optimal trading strategies for power utility function with and without crash possibility

for $t \in [0, T]$ and $W^\pi(\infty) := V^0(T, X^\pi(T))$. To construct an indifference strategy $\hat{\pi}$ which turns the process $W^\pi(t)$ into a martingale we obtain the ordinary differential equation

$$\hat{\pi}'(t) = \frac{(1-\gamma)(\sigma)^2}{2k^*} (1 - \hat{\pi}(t)k^*)(\hat{\pi}(t) - \pi^{0*}(t))^2 \quad (2.27)$$

for $\hat{\pi}$ and then show that this is sufficient for $\hat{\pi}$ to be an indifference strategy. The optimal strategy in the pre-crash market for the worst case portfolio problem is given by the indifference strategy $\hat{\pi}$. After the crash, the Merton strategy $\pi^{0*} = \frac{\mu-r}{(1-\gamma)\sigma^2}$.

The worst-case portfolio optimization problem has also been considered in other situations: Hua and Wilmott[19] considered worst-case option pricing in a discrete-time setting. Korn[24] applied the worst-case modeling approach in the investment for Insurers. Menkens[32] considered the worst-case problem given the probability of the crash. Desmettre, Korn and Seifried[11] analyzed the worst-case consumption portfolio optimization problem over an infinite time horizon and also considered the robust worst case optimal investment with respect to the choice of the maximum crash size[10]. Belak and

Sass[3] considered the worst-case portfolio optimization problem under transaction costs.

Chapter 3.

The discrete-time model and the portfolio optimization in discrete-time

3.1. The discrete-time market model

A discrete time financial model is built on a finite probability space (Ω, \mathcal{F}, P) with sample space Ω , σ -algebra filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0, \dots, N}$ and the probability measure P . The σ -field \mathcal{F}_t usually models the events which can be observed up to time t . From now on we assume that $\mathcal{F}_0 = \emptyset, \Omega, \mathcal{F}_T = P(\Omega)$ and $\forall w \in \Omega, P(w) > 0$.

The market consists of two financial assets, whose prices at time t are given by the non-negative random variables B_t, S_t , measurable with respect to \mathcal{F}_t . The asset $B = \{B_t; t = 0, 1, \dots, T\}$ is the riskless asset and we have $B_0 = 1$. If the return of the riskless asset over time is equal to r , the price of the riskless asset evolves as

$$B_{t+1} = (1 + r_t)B_t = (1 + r)^n \quad (3.1)$$

There is a risky security processes S_t , where S_t is a non-negative stochastic process for each $t = 1, 2, \dots, T$. The price of the risky security S_t evolves according to

$$S_{t+1} = (1 + \tilde{R}_t)S_t \quad (3.2)$$

where \tilde{R}_t is the return process corresponding to the price process S_t and we could equivalently write

$$S_{t+1} = R_t S_t \quad (3.3)$$

with $R_t = 1 + \tilde{R}_t$ and $R_t > 0$ P -almost surely.

We assume that the random vectors R_1, \dots, R_T are independent identically distributed sequences of random variables.

Definition 3.1: A *trading strategy* is defined as a (\mathcal{F}_t) -adapted stochastic process $\phi = (\phi_t^0, \phi_t^1)$ where ϕ_t^i denotes the number of shares of asset i held in the portfolio at time t .

Remark 3.2. Note that the components ϕ_t^i is allowed to be negative. In particular $\phi_t^0 < 0$ implies that a loan is taken such that we receive the amount $|\phi_t^0|$ at time t and pay back the amount $(1+r)|\phi_t^0|$ at time $t+1$. If $\phi_t^1 < 0$ this corresponds to a short sale of the asset.

The value $x := \phi_0^0 B_0 + \phi_0^1 S_0$ is called **the initial value** of ϕ .

The **wealth process** X_t corresponding to ϕ with initial value x is defined by

$$X_t = \phi_t^0 B_t + \phi_t^1 S_t$$

Denote by X_{t-} the wealth process at time t before trading, then we have

$$X_{t-} = \phi_{t-1}^0 B_t + \phi_{t-1}^1 S_t$$

and the wealth process at time t after trading follows

$$X_{t+} = \phi_t^0 B_t + \phi_t^1 S_t$$

Definition 3.3: A strategy is called **self-financing** when the following equation is satisfied for all $t \in \{0, 1, \dots, T-1\}$

$$\phi_{t-1}^0 B_t + \phi_{t-1}^1 S_t = \phi_t^0 B_t + \phi_t^1 S_t$$

Remark 3.4. The self-financing equation can be equivalently expressed as

$$X_{t+1} - X_t = \phi_t^0 \Delta B_t + \phi_t^1 \Delta S_t$$

Definition 3.5: Let ϕ be a self-financing trading strategy with corresponding wealth process $X(t) > 0$ P -a.s. for all t , then the process π_t with

$$\pi_t = \frac{\phi_t^1 S_t}{X_t}$$

is called a **self-financing portfolio process**.

The fraction of wealth invested in the bond is given by

$$1 - \pi_t = \pi_t^0 = \frac{\phi_t^0 B_t}{X_t}.$$

Then the self-financing condition implies

$$\begin{aligned} X_{t+1} &= X_t + \phi_t^0 \Delta B_t + \phi_t^1 \Delta S_t \\ &= X_t + \phi_t^0 B_t r + \phi_t^1 S_t (R_t - 1) \\ &= X_t + X_t (\pi_t^0 r + \pi_t (R_t - 1)) \\ &= X_t (1 - \pi_t^1 + \pi_t^0 r + \pi_t R_t) \\ &= X_t (\pi_t^0 (1 + r) + \pi_t R_t) \\ &= X_t (\pi_t^0 (1 + r) + \pi_t R_t) \\ &= X_t ((1 + r) + \pi_t (R_t - 1 - r)) \end{aligned} \tag{3.4}$$

This recursive formula is important for the wealth evolution which will be used in the following.

From the remark 3.2 borrowing and short-sale is allowed but the wealth process must be positive at all times.

Definition 3.6: A self-financing strategy π or a self-financing portfolio process is **admissible** if the corresponding wealth process satisfies $X_t \geq 0$ for all t .

We denote by $\mathcal{A}(x)$ the set of admissible portfolio processes.

As usual we have to eliminate the arbitrage opportunities in the financial market.

Definition 3.7: A self-financing portfolio strategy π is called an **arbitrage opportunity** with the following property for the initial capital $x_0 \leq 0$

$$X_0^\pi = x_0 \leq 0, P(X_T^\pi \geq 0) = 1, P(X_T^\pi > 0) > 0.$$

A market model is *arbitrage free*, if no arbitrage opportunities exist.

An arbitrage opportunity is an investment strategy which leads to a positive profit with a positive probability. In real markets such arbitrage opportunity exists but it disappears soon after it is found by traders. Therefore, the absence of the arbitrage opportunities is our main assumption in the market.

Remark 3.8. The absence of arbitrage opportunities in market models is also characterized by the existence of the equivalent risk-neutral measure or martingale measure. This equivalence is very important for the pricing and hedging contingent claims in complete markets.

In the following we present two special cases of the discrete-time model.

3.1.1. Binomial model

The binomial model or Cox-Ross-Rubinstein model is an important special case of the discrete-time model. We shall begin by recalling the Bernoulli process. The stochastic process $\{Y_t; t = 1, 2, \dots\}$ is said to be a Bernoulli process with parameter p if the random variables Y_t are i.i.d. and

$$P\{Y_t = 1\} = 1 - P\{Y_t = 0\} = p \in (0, 1)$$

The underlying sample space Ω consists of all the sequences of the form

$$w = (0, 1, 0, 0, 1, 1, \dots)$$

We consider our securities market model which features just a finite number T of periods. Now each state w has T components and the probability measure is given by

$$P(w) = p^n (1 - p)^{T-n}$$

Let the process $\{N_t; t = 1, 2, \dots\}$ be defined as $N_t = Y_1 + Y_2 + \dots + Y_t$, then

$$P(N_t = n) = \binom{t}{n} p^n (1-p)^{t-n}$$

We consider the market with only one risky asset whose price is S_t at time t , and a riskless asset whose return is r over the period of time. S_t is modelled as follow:

$$S_{t+1} = S_t R_t \tag{3.5}$$

where R_t are i.i.d. and such that

$$R_t = \begin{cases} u & \text{with probab. } p \\ d & \text{with probab. } 1-p \end{cases}$$

Then we have the representation

$$R_t = uY_t + d(1 - Y_t)$$

Then the price of the risky asset is given by

$$S_t = S_0 u^{N_t} d^{t-N_t}$$

and

$$P(S_t = S_0 u^{N_t} d^{t-N_t}) = \binom{t}{n} p^n (1-p)^{t-n}$$

Remark 3.9. • In this binomial market model the no-arbitrage condition is satisfied only when the model parameters satisfy:

$$d < 1 + r < u$$

Otherwise there would be an arbitrage opportunity.

If $1 + r \leq d$, we can invest the stock price through a credit at time 0 and get a positive profit in the future time. This is called the arbitrage opportunity.

If $1 + r \geq u$, investing a bond via a stock short selling is also such an arbitrage opportunity.

- It is possible to approximate the price process of the Black-Scholes-Merton model in continuous time by the price process of the binomial model in discrete time when we choose the suitable parameters u , d and p . This approach has been suggested by Cox and Rubinstein[8]. And the detail about the weak convergence in financial market can be found in Prigent[43].

A particular choice of parameters in the binomial model is given by

$$\begin{aligned} u &= \exp(\sigma\sqrt{\Delta t}), & d &= \exp(-\sigma\sqrt{\Delta t}) \\ p &= \frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t}, & 1 + r &= \exp(r_c \Delta t) \end{aligned}$$

which are proposed by Cox[7].

3.1.2. Discrete Black-Scholes model

In the standard Black-Scholes-Merton market model it is assumed that the stock price evolves according to

$$\begin{aligned} dB_t &= rB_t dt \\ dS_t &= \mu S_t dt + \sigma S_t dw_t \end{aligned}$$

where W_t is a Wiener process.

Here we consider a discrete approximation to this continuous time model. If we take a time step Δt , the dynamic processes of the bond price and the stock price satisfy

$$\begin{aligned} B_{t+\Delta t} &= B_t \exp(r\Delta t) \\ S_{t+\Delta t} &= S_t \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma(w_{t+\Delta t} - w_t)\right) \end{aligned}$$

In this case R_t has a lognormal distribution. Let $x_t = \log(R_t) = \log\left(\frac{S_t}{S_{t-1}}\right)$ then we have

$$x_t = \tilde{\mu} + \tilde{\sigma} Z_t \tag{3.6}$$

where $\tilde{\mu} = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t$, $\tilde{\sigma} = \sigma\sqrt{\Delta t}$ and $Z_t = \frac{w_t - w_{t-1}}{\sqrt{\Delta t}}$ is a sequence of standard normal i.i.d. random variables.

3.2. Portfolio optimization in discrete-time

In this section we introduce the classical portfolio problem in discrete time of maximizing the expected utility of terminal wealth.

By the self-financing property the recursive expression of the wealth process X_t with respect to portfolio strategy π_t is given by:

$$X_{t+1} = X_t((1+r) + \pi_t(R_t - 1 - r)) \tag{3.7}$$

Then we obtain the final wealth process X_T as following

$$\begin{aligned} X_T &= X_0((1+r) + \pi_0(R_0 - 1 - r)) \cdots ((1+r) + \pi_{T-1}(R_{T-1} - 1 - r)) \\ &= x \prod_{t=0}^{T-1} ((1+r) + \pi_t(R_t - 1 - r)) \end{aligned}$$

with the initial wealth $X_0 = x$.

Definition 3.10: Let a function $u : D \rightarrow \mathcal{R}$ be strictly concave, strictly increasing and continuous on D , then u is called a **utility function**.

Example 3.11: • the log-utility function. $u(X) = \log(X)$ and $D = (0, \infty)$.

• the power utility function. $u(X) = \frac{1}{\gamma} X^\gamma$ with $0 < \gamma < 1$ and $D = (0, \infty)$.

• the exponential utility function. $u(X) = -e^{-\theta x}$ with $\theta > 0$ and $D = \mathcal{R}$.

◁

Note that for an arbitrage strategy π the expectation of the utility of the terminal wealth is not necessarily defined. Hence the investor maximizes the expected utility of his investment under the constraint that the expectation is finite.

Definition 3.12: Let $u(\cdot)$ be a utility function, and X_t^π be the wealth process. **The portfolio problem in discrete time** is to calculate

$$V_0(t, x) = \sup_{\pi \in \mathcal{A}'(x)} E(u(X_T^\pi)) \quad (3.8)$$

with

$$\begin{cases} X_t = x \\ \mathcal{A}'(x) = \{\pi \in \mathcal{A}(x) \mid E(u(X_T^\pi)) < \infty\} \end{cases}$$

and to find an admissible strategy π^* s.t. $E(u(X_T^{\pi^*})) = V_0$.

We denote by $V_0(t, x)$ the value function of the optimization problem in discrete time.

3.2.1. Dynamic programming

Now we investigate at first the one-period utility maximization problem. Then the formulation of the one-period utility optimization problem is given by :

$$V_0(x) = \sup_{\pi} E(u(x(1 + r + \pi(R - 1 - r)))) \quad (3.9)$$

In order to get the well-defined $\mathcal{A}'(x)$ we assume two cases:

• $D = \mathcal{R}$, u is bounded from above.

• $D = (0, \infty)$, $E(R) < \infty$.

By the Jensen's inequality we have

$$\begin{aligned} E(u(x(1 + r + \pi(R - 1 - r)))) &\leq u(E(x(1 + r + \pi(R - 1 - r)))) \\ &= u((x(1 + r + \pi E(R - 1 - r)))) \\ &\leq C(1 + x(1 + r + \pi E(R - 1 - r))) \end{aligned}$$

with $C \in \mathcal{R}$.

since u is concave, the utility function $u(x)$ can be bounded from above by an affine linear function $c(1 + x)$ which implies the second inequality.

For the one-period utility optimization problem of

$$V_0(x) = \sup_{\pi} E(u(x(1+r+\pi(R-1-r)))) \quad (3.10)$$

we want to show that the existence of the optimal portfolio strategy π^* is equivalent to the absence of arbitrage opportunities. The similar proof of the following theorem is shown in Foeller and Schied[16] and Baeuerle and Rieder[2].

Theorem 3.13: *Let u be a utility function which satisfies the assumption, then it holds:*

- *There are no arbitrage opportunities if and only if there exists a measurable function π^* such that $E(u(x(1+r+\pi^*(R-1-r)))) = V_0(x)$.*
- *There exists at most one maximizer if the market model is non-redundant.*
- *The function $V_0(x)$ is strictly increasing, strictly concave and continuous on D .*

Remark 3.14. A financial market is called non-redundant only when there exists no asset which can be replicated by a linear combination of the other assets.

Now we focus on the multiperiod extension of the portfolio optimization problem in discrete time. Suppose we have the utility function $u : (0, \infty) \rightarrow \mathcal{R}$. From the theorem above we state the following assumption on the financial market which is used throughout this section.

Assumption:

- The market is arbitrary-free.
- $E(\|R_t\|) < \infty$ for all t .

In the discrete-time model, the multi-period portfolio optimization problem can be written as:

$$V_0(t, x) = \sup_{\pi_t, \pi_{t+1}, \dots, \pi_{T-1}} E(u(X_T^\pi)) \quad (3.11)$$

$\pi = (\pi_t, \pi_{t+1}, \dots, \pi_{T-1})$ is the optimal control sequence.

When we are faced with such a sequence of decisions, the method called dynamic programming may reduce the computational difficulties. The dynamic programming idea for the portfolio optimization problem in discrete time is already introduced by Pilska[41]. The main idea of the dynamic programming is that the optimal decision to make now should be consistent with the intention to act optimally in all future periods. If we know the optimal strategy starting at time $t+1$, then the problem of determination of the optimal strategy at time t can be reduced to one-period problem. That means dynamic programming can simplify a multiperiod decision problem by breaking it down into a

sequence of one-priod problems.

In the case of our optimal portfolio problem, let us define $U_t(x)$ as the optimal value process with

$$U_t(x) = \sup_{\pi_t, \dots, \pi_T} \{E(u(X_T^\pi)) \mid \mathcal{F}_t\}. \quad (3.12)$$

$U_t(x)$ is a \mathcal{F}_t measurable random variable.

When $t = T$, we have

$$U_T(x) = u(x), \quad (3.13)$$

and $t < T$

$$\begin{aligned} U_t(x) &= \sup_{\pi_t, \dots, \pi_T} \{E(u(X_T^\pi)) \mid \mathcal{F}_t\} \\ &= \sup_{\pi_t} \{E(U_{t+1}(X_{t+1}(\pi_t))) \mid \mathcal{F}_t\}. \end{aligned}$$

Then from the equation (3.4) we can get the following dynamic programming equation for the sequence π_0, \dots, π_T

$$\begin{cases} U_t(x) = \sup_{\pi_t} \{E(U_{t+1}(x((1+r) + \pi_t(R_t - 1 - r))))\} \\ U_T(x) = u(x). \end{cases} \quad (3.14)$$

Now we state some properties of the optimal value processes $U_t(x)$. We use the formulation in Baeuerle and Rieder[2] and refer to the same book for the proof.

Lemma 3.15: *Let $u(\cdot)$ be a utility function, then for the multiperiod terminal wealth problem it holds: The optimal value function $U_t(x)$ in each stage are strictly increasing, strictly concave and continuous.*

Theorem 3.16: [2] *The value function can be computed recursively by the dynamic programming equations*

$$\begin{cases} U_t(x) = \sup_{\pi_t} \{E(U_{t+1}(x((1+r) + \pi_t(R_t - 1 - r))))\} \\ U_T(x) = u(x) \end{cases} \quad (3.15)$$

and there exist maximizers π_t^ of $U_t(x)$ and the strategy $(\pi_0^*, \dots, \pi_{T-1}^*)$ is optimal for the portfolio optimization problem.*

The dynamic programming equation (3.14) can be used to compute an optimal solution to the problem (3.8) by computing the optimal value functions $U_t(x)$ in a backwards recursive manner. Based on the first-order necessary conditions and the strict concavity of function $U_t(x)$ we obtain maximizers π_t^* of U_{t+1} , and the portfolio strategy $(\pi_0^*, \dots, \pi_{T-1}^*)$ is optimal for the optimization problem. The dynamic programming provides a bonus: you have a solution for all possible values of the initial wealth x .

For some utility function, the portfolio optimization problem (3.8) can be solved rather explicitly.

Power utility:

Let us suppose that the utility function in problem (3.8) is of the form

$$u(x) = \frac{1}{\gamma} x^\gamma$$

with $0 < \gamma < 1$, $x \in [0, \infty)$

Then the one-period optimal value function $U(x)$ satisfies

$$\begin{aligned} U(x) &= \sup_{\pi \in \mathcal{A}(x)} \{E(u(x((1+r) + \pi(R-1-r))))\} \\ &= \sup_{\pi \in \mathcal{A}(x)} \{E(\frac{1}{\gamma} (x((1+r) + \pi(R-1-r)))^\gamma)\} \\ &= \frac{1}{\gamma} x^\gamma \sup_{\pi \in \mathcal{A}(x)} \{E((1+r) + \pi(R-1-r))^\gamma\} \end{aligned}$$

Here we define a function $P_t := \sup_{\pi_t \in \mathcal{A}(x)} \{E((1+r) + \pi_t(R_t-1-r))^\gamma\}$.

From the dynamic programming equation (3.14) we have that

when $t = T$, we have

$$U_T(x) = \frac{1}{\gamma} x^\gamma \tag{3.16}$$

and for $t = T-1$

$$\begin{aligned} U_{T-1}(x) &= \sup_{\pi_{T-1}} \{E(U_T(x((1+r) + \pi(R_{T-1}-1-r))))\} \\ &= \sup_{\pi_{T-1}} \{E(\frac{1}{\gamma} (x((1+r) + \pi(R_{T-1}-1-r)))^\gamma)\} \\ &= \frac{1}{\gamma} x^\gamma \sup_{\pi_{T-1}} \{E((1+r) + \pi(R_{T-1}-1-r))^\gamma\} \\ &= \frac{1}{\gamma} x^\gamma P_{T-1} \end{aligned}$$

Then, the optimal strategy $\pi_{T-1} = \arg P_{T-1}$ is the optimal solution of $P_{T-1}(x)$.

and for $t = T - 2$

$$\begin{aligned}
 U_{T-2}(x) &= \sup_{\pi_{T-2}} \{E(U_{T-2}(x((1+r) + \pi(R_{T-2} - 1 - r))))\} \\
 &= \sup_{\pi_{T-2}} \{E(\frac{1}{\gamma}(x((1+r) + \pi(R_{T-2} - 1 - r)))^\gamma P_{T-1}(x))\} \\
 &= \frac{1}{\gamma} x^\gamma P_{T-1} \sup_{\pi_{T-2}} \{E((1+r) + \pi(R_{T-2} - 1 - r))^\gamma\} \\
 &= \frac{1}{\gamma} x^\gamma P_{T-1} P_{T-2}
 \end{aligned}$$

we can get the optimal strategy $\pi_{T-2} = \arg P_{T-2}$.

and so on, we can conclude that the optimal value function in power utility are given by

$$\begin{cases} U_t(x) = \frac{1}{\gamma} x^\gamma \prod_{s=t}^{T-1} P_s \\ U_T(x) = \frac{1}{\gamma} x^\gamma \end{cases} \quad (3.17)$$

and the optimal portfolio strategy π_t is the optimal solution of P_t .

Therefore if the return of the stock price R_t are identically distributed for all t , then $\pi_t \equiv \pi$ is independent of t .

If we assume that we have one stock and the price process of the stock follows the **binomial model** as described in section 3.1.1. Then we have that

$$R_t = \begin{cases} u & \text{with probab. } p \\ d & \text{with probab. } 1 - p \end{cases}$$

then

$$\begin{aligned}
 P_t &= \sup_{\pi_t \in [0,1]} \{E((1+r) + \pi_t(R_t - 1 - r))^\gamma\} \\
 &= \sup_{\pi \in [0,1]} \{((1+r) + \pi(u - 1 - r))^\gamma p + ((1+r) + \pi(d - 1 - r))^\gamma (1 - p)\}
 \end{aligned}$$

Let $\kappa = \frac{1}{1-\gamma}$, then the optimal portfolio strategy is of the form

$$\pi^* = \frac{(1+r)}{(u-1-r)(1+r-d)} * \frac{p^\kappa(u-1-r)^\kappa - (1-p)^\kappa(1+r-d)^\kappa}{(p^\kappa(u-1-r)^\kappa + (1-p)^\kappa(1+r-d)^\kappa)} \quad (3.18)$$

In the standard continuous-time Black-Scholes-Merton model from the chapter, the optimal portfolio strategy in the case of power utility is of the form

$$\pi_c^*(t) = \frac{1}{1-\gamma} \frac{\mu - r_c}{\sigma^2} \quad (3.19)$$

where μ is the drift of the stock, σ is the volatility and r_c is the risk-free interest rate in continuous time.

Because the Black-Scholes-Merton model can be approximated by the binomial model if we choose the parameters of the binomial model appropriately, we expect that the optimal portfolio strategy (3.18) in binomial model converges to the expression (3.19) in continuous time. If we define:

$$u = \exp(\sigma\sqrt{\Delta t}), \quad d = \exp(-\sigma\sqrt{\Delta t})$$

$$p = \frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t}, \quad 1 + r = \exp(r_c \Delta t)$$

then by using the Taylor series expansion for the exponential function $\exp(\Delta t)$ we can obtain

$$\lim_{\Delta t \downarrow 0} \pi^*(\Delta t) = \frac{1}{1 - \gamma} \frac{\mu - r_c}{\sigma^2} \quad (3.20)$$

Logarithmic utility:

Here we assume that the utility function in problem (3.8) is given by

$$u(x) = \log x$$

Then the one-period optimal value function $U(x)$ satisfies

$$\begin{aligned} U(x) &= \sup_{\pi \in \mathcal{A}(x)} \{E(u(x((1+r) + \pi(R-1-r))))\} \\ &= \sup_{\pi \in \mathcal{A}(x)} \{E \log(x((1+r) + \pi(R-1-r)))\} \\ &= \log x + \sup_{\pi \in [0,1]} \{E \log((1+r) + \pi(R-1-r))\} \end{aligned}$$

Here we define a function $P_t := \sup_{\pi_t \in \mathcal{A}(x)} \{E \log((1+r) + \pi_t(R_t-1-r))\}$.

From the dynamic programming equation (3.14) we have that when $t = T$, we have

$$U_T(x) = \log x \quad (3.21)$$

and for $t = T - 1$

$$\begin{aligned} U_{T-1}(x) &= \sup_{\pi_{T-1}} \{E(U_T(x((1+r) + \pi(R_{T-1}-1-r))))\} \\ &= \sup_{\pi_{T-1}} \{E(\log(x((1+r) + \pi(R_{T-1}-1-r))))\} \\ &= \log x + \sup_{\pi_{T-1}} \{E \log((1+r) + \pi(R_{T-1}-1-r))\} \\ &= \log x + P_{T-1} \end{aligned}$$

we can get the optimal strategy $\pi_{T-1} = \arg P_{T-1}$ is the optimal solution of $P_{T-1}(x)$. and for $t = T - 2$

$$\begin{aligned} U_{T-2}(x) &= \sup_{\pi_{T-2}} \{E(U_{T-2}(x((1+r) + \pi(R_{T-2} - 1 - r))))\} \\ &= \sup_{\pi_{T-2}} \{E(\log(x((1+r) + \pi(R_{T-2} - 1 - r))) + P_{T-1})\} \\ &= \log x + P_{T-1} + \sup_{\pi_{T-2}} \{E \log((1+r) + \pi(R_{T-2} - 1 - r))\} \\ &= \log x + P_{T-1} + P_{T-2} \end{aligned}$$

we can get the optimal strategy $\pi_{T-2} = \arg P_{T-2}$.

and so on. We can conclude that the optimal value function in the case of log utility is of the form

$$\begin{cases} U_t(x) = \log x + \sum_{s=t}^{T-1} P_s \\ U_T(x) = \log x \end{cases} \quad (3.22)$$

and the optimal portfolio strategy π_t is the optimal solution of P_t .

Therefore $\pi_t \equiv \pi$ is also independent of t if the return of the stock price R_t are identically distributed for all t . Specially in the **binomial model** as described in section 3.1.1, we have that

$$\begin{aligned} P_t &= \sup_{\pi_t \in [0,1]} \{E \log((1+r) + \pi_t(R_t - 1 - r))\} \\ &= \sup_{\pi \in [0,1]} \{\log((1+r) + \pi(u - 1 - r))p + \log((1+r) + \pi(d - 1 - r))(1-p)\} \end{aligned}$$

then the optimal portfolio strategy is given by

$$\pi^* = \frac{(1+r)(p(u-d) + d - 1 - r)}{(u - 1 - r)(1 + r - d)} \quad (3.23)$$

The optimal portfolio strategy in the case of log utility in the standard continuous-time Black-Scholes-Merton model is given by

$$\pi_c^*(t) = \frac{\mu - r_c}{\sigma^2} \quad (3.24)$$

where μ is the drift of the stock, σ is the volatility and r_c is the risk-free interest rate in continuous time.

If we define the same parameters as following:

$$\begin{aligned} u &= \exp(\sigma\sqrt{\Delta t}), \quad d = \exp(-\sigma\sqrt{\Delta t}) \\ p &= \frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t}, \quad 1 + r = \exp(r_c \Delta t) \end{aligned}$$

we can get

$$\lim_{\Delta t \downarrow 0} \pi^*(\Delta t) = \frac{\mu - r_c}{\sigma^2} \quad (3.25)$$

Exponential utility:

The utility function in this case is given by

$$u(x) = -e^{-\theta x}$$

for some $\theta > 0$.

Then the one-period optimal value function $U(x)$ satisfies

$$\begin{aligned} U(x) &= \sup_{\pi \in [0,1]} \{E(u(x((1+r) + \pi(R-1-r))))\} \\ &= \sup_{\pi \in [0,1]} \{E(-e^{-\theta x((1+r) + \pi(R-1-r))})\} \\ &= \sup_{\pi \in [0,1]} \{E(-e^{\theta x(1+r) - \theta x \pi(R-1-r)})\} \\ &= -e^{-\theta x(1+r)} \sup_{\pi \in [0,1]} \{E(e^{-\theta x \pi(R-1-r)})\} \end{aligned}$$

Comparing to the examples of the log-utility and power-utility, the situation for the exponential utility is totally different. The separation of the term with respect to x and the term with respect to π in the optimal value function is not possible. Therefore, we consider no longer the portfolio strategy π_t , but the amount of money which is invested in the risky stock at a time with the notation $\pi_t X_t$.

Here we denote that $\hat{\pi}_t = \pi_t X_t$, then define the function $P_t(x)$ as following

$$P_t = \sup_{\hat{\pi}_t \in R} \{E(e^{-\theta(1+r)^{T-t-1} \hat{\pi}_t(R-1-r)})\} \quad (3.26)$$

Because the utility function is bounded from above, we have that the supremum of the equation (3.26) exists.

From the dynamic programming equation (3.14) we have that

when $t = T$, we have

$$U_T(x) = -e^{-\theta x} \quad (3.27)$$

and for $t = T - 1$

$$\begin{aligned} U_{T-1}(x) &= \sup_{\pi_{T-1}} \{E(U_T(x((1+r) + \pi_{T-1}(R-1-r))))\} \\ &= \sup_{\pi_{T-1}} \{E(-e^{-\theta x((1+r) + \pi_{T-1}(R-1-r))})\} \\ &= -e^{-\theta x(1+r)} \sup_{\hat{\pi}_{T-1} \in R} \{E(e^{-\theta \hat{\pi}_{T-1}(R-1-r)})\} \\ &= -e^{-\theta x(1+r)} P_{T-1} \end{aligned}$$

then we can get the optimal strategy $\pi_{T-1} = \arg P_{T-1}$.
and for $t = T - 2$

$$\begin{aligned}
 U_{T-2}(x) &= \sup_{\pi_{T-2}} \{E(U_{T-1}(x((1+r) + \pi_{T-2}(R-1-r))))\} \\
 &= \sup_{\pi_{T-2}} \{E(-e^{-\theta x(1+r)((1+r)+\pi_{T-2}(R-1-r))} P_{T-1})\} \\
 &= -e^{-\theta x(1+r)^2} P_{T-1} \sup_{\hat{\pi}_{T-2} \in R} \{E(e^{-\theta(1+r)\hat{\pi}_{T-2}(R-1-r)})\} \\
 &= -e^{-\theta x(1+r)^2} P_{T-1} P_{T-2}
 \end{aligned}$$

then we can obtain the optimal strategy $\pi_{T-2} = \arg P_{T-2}$.
and so on, we can conclude that the optimal value function in exponential utility are of the form

$$\begin{cases} U_t(x) = -e^{-\theta x(1+r)^{T-t}} \prod_{s=t}^{T-1} P_s \\ U_T(x) = -e^{-\theta x} \end{cases} \quad (3.28)$$

and the optimal portfolio strategy π_t is the optimal solution of P_t .

Chapter 4.

The worst-case portfolio optimization in discrete-time

4.1. The discrete-time crash model

In this section, we specify the discrete-time worst-case market model and formulate the worst-case optimization problem in discrete time. This model is an extension of the discrete-time market model and allows for a crash in stock prices. As in the worst-case market model in continuous time introduced by Hua and Wilmott [19] and taken up by Korn and Wilmott [29], we consider a market consisting of a risk-less bond and one risky security with prices in *normal times* given by

$$\begin{cases} B_{t+1} = (1+r)B_t, & B_0 = 1 \\ S_{t+1} = S_t R_t, & S_0 = s_0 \end{cases} \quad (4.1)$$

with constant market coefficient r , and independent and identically distributed random variables R_t . We assume that the mean of the stock return $E(R_t)$ exceeds the risk-less return factor of $1+r$, i.e.

Assumption (M): *Mean stock return exceeds the risk-less return.*

$$E(R_t) > 1+r > 0 . \quad (4.2)$$

At the *crash time* τ , the stock price can suddenly fall by a relative amount $k \in [0, k^*]$, where $0 < k^* < 1$ (the biggest possible crash height) is given. Then, in a *crash scenario* (τ, k) we have

$$S_{\tau+1} = (1-k)S_\tau . \quad (4.3)$$

Moreover we fix the terminal time $T > 0$. Let further \mathcal{F}_t , $t = 0, 1, \dots, T$ be the filtration generated by the stock price. We then call a real-valued, \mathcal{F}_t -adapted stochastic process a *portfolio process*. As usual, this process describes the fraction of the investor's total wealth $X(t)$ that is allocated to the stock at time t . The corresponding position will then be hold until time $t+1$ where a possible reallocation happens. Obviously, $1 - \pi_t$ equals the fraction of wealth invested in the risk-less asset.

Let $X(t)$ be the wealth process corresponding to the self-financing portfolio strategies π_t , then we have the wealth process at a crash time τ by the following lemma.

Lemma 4.1: *The wealth process $X_{\tau+1}$ at crash time τ corresponding to the trading strategy π_τ is given by*

$$X_{\tau+1} = (1 + r - \pi_\tau(r + k))X_\tau \quad (4.4)$$

Proof: Let $\psi = (\psi^0, \psi^1)$ be the trading strategy, then

$$\begin{aligned} X_{\tau+1} &= \psi_{\tau+1}^0 B_{\tau+1} + \psi_{\tau+1}^1 S_{\tau+1} \\ &= \psi_\tau^0 B_{\tau+1} + \psi_\tau^1 S_{\tau+1} \\ &= \psi_\tau^0 B_\tau(1 + r) + \psi_\tau^1 S_\tau(1 - k) \\ &= (1 - \pi_\tau)X_\tau(1 + r) + \pi_\tau X_\tau(1 - k) \\ &= X_\tau((1 + r) - \pi_\tau(r + k)) \end{aligned}$$

□

Therefore for a possible crash scenario (τ, k) with $t \leq \tau \leq T$ the dynamics of the wealth process are given by

$$X_{t+1} = X_t((1 + r) + \pi_t(R_t - 1 - r)) \quad t \in [0, \tau - 1] \cup [\tau + 1, T - 1] \quad (4.5)$$

$$X_{\tau+1} = (1 + r - \pi_\tau(r + k))X_\tau \quad (4.6)$$

where $x > 0$ denotes the initial wealth. We will call a self-financing portfolio process *admissible* if the corresponding wealth process $X(t)$ stays non-negative. We denote this by $\pi \in \mathcal{A}(x)$.

In the following sections, we first restrict ourselves to the case that at most one crash can occur within the investment period $[t, T]$. Details how to extend our results to the general case of at most n crashes by an iterative procedure will be given later.

Let us point out that the optimal portfolio process after the crash has happened coincides with the optimal one in the crash-free setting. Thus, we only have to consider portfolio processes where the final wealth X_T in the case of a crash of size k at time $\tau \leq T - 1$ is given by

$$X_T = x \prod_{t=0}^{\tau-1} ((1+r) + \pi_t(R_t - 1 - r)) * (1+r - \pi_\tau(r+k)) * \prod_{t=\tau+1}^{T-1} ((1+r) + \tilde{\pi}_t^*(R_t - 1 - r)) \quad (4.7)$$

with $\tilde{\pi}_t^*$ being the optimal strategy in the crash-free setting if such a strategy $\tilde{\pi}_t$ exists. To relate the latter one to a corresponding optimization problem in discrete time, let $u(\cdot)$ be a utility function (i.e. a strictly concave and increasing differentiable function)

and X_t^π be the wealth process. Then, the portfolio problem in the crash-free setting is given by its value function

$$V_0(t, x) = \sup_{\pi \in \mathcal{A}(x)} E^{t,x}(u(X^\pi(T))) \quad (4.8)$$

where we simply assume that there is no crash possibility at all. For our considerations in the following, we make the fundamental assumption from now on:

Assumption (O): *Existence of an optimal admissible portfolio.*

We assume that for each pair $(t, x) \in [0, T] \times (0, \infty)$ there exists an optimal admissible deterministic portfolio process $\tilde{\pi}^*$ in the sense of

$$V_0(t, x) = E^{t,x}(u(X^{\tilde{\pi}^*}(T))) . \quad (4.9)$$

This assumption is in particular satisfied for all the examples considered in this article. Further, it is satisfied if the stock price can attain only a finite number of possible prices. However, this is not the definite collection of all examples where this is the case.

To introduce the worst-case problem in the crash setting, the **worst-case bound** for the expected utility from using π before the crash is defined as

$$W(t, x, \pi) = \inf_{t \leq \tau \leq T, 0 \leq k \leq K^*} E^{t,x}(u(X_T^\pi)) \quad (4.10)$$

where we already assume that after the crash an optimal portfolio process in the crash-free setting is followed. The **worst-case portfolio problem in discrete time** then is to calculate

$$V_1(t, x) = \sup_{\pi \in \mathcal{A}(x)} W(t, x, \pi) \quad (4.11)$$

and to find an admissible strategy π^* such that $W(t, x, \pi^*) = V_1(t, x)$. We denote by $V_1(t, x)$ **the value function** of the worst-case portfolio optimization problem.

As motivated by Korn and Wilmott [29] in continuous time, there are two competing effects, a high crash loss if a high portfolio process is chosen and a bad performance if a low one is preferred. To cope with this, they show how to derive the worst-case optimal portfolio strategy by an indifference argument. In the next section, we look for an optimal portfolio strategy by using a similar indifference principle in the worst-case portfolio problem in discrete time in the case of log utility.

4.2. Indifference strategies

In this section, we consider the special case of the logarithmic utility function:

$$u(x) = \ln(x), x > 0 \quad (4.12)$$

of course, still under Assumptions (M) and (O). In this case, we have the following representation of the value function in the discrete-time crash-free model

$$V^0(t, x) = \ln(x) + (T - t)E^{t,x}(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \quad (4.13)$$

with the corresponding optimal portfolio strategy

$$\tilde{\pi}^* = \arg \sup_{\pi \in \mathcal{A}(x)} \{E^{t,x}(\ln((1 + r) + \pi(R - 1 - r)))\}. \quad (4.14)$$

We make the assumption which is e.g. satisfied in the binomial model setting (see e.g. Kröner [30]):

Assumption (L): *Constant log-optimal portfolio.*

The stock price model in the crash-free setting admits a unique positive optimal constant portfolio process $\tilde{\pi}^*$ in Equation (4.14).

Remark 4.2. As π_t is independent of R_t and all the R_t are independent and identically distributed, we can in the following often drop the index t in R_t when only expectations are considered. Note that due to the independence of R_t of the past price history, $R = R_t$ is also independent of every choice of an admissible portfolio process π_t . As the expected value in Equation (4.14) is independent of (t, x) , Assumption (L) mainly can be seen as a reformulation of Assumptions (O) and (M).

Before solving the above worst-case portfolio problem in discrete-time, we consider at first the following two extreme strategies.

- If the investor chooses to use the optimal strategy in the crash-free setting π^{0*} , then the worst-case scenario is given by a crash of maximal height k^* . From the representation of the final wealth X_T we can easily verify that the exact crash time has no impact on the resulting value function. Therefore, we obtain the following worst-case bound from the worst crash scenario happening immediately:

$$\begin{aligned} & V^0(t + 1, x(1 + r - \pi^{0*}(r + k^*))) \\ &= \ln(x) + \ln(1 + r - \pi^{0*}(r + k^*)) + (T - t - 1)E(\ln(1 + r + \pi^{0*}(R - 1 - r))) \end{aligned} \quad (4.15)$$

- If the investor chooses $\hat{\pi}_t = 0$ before the crash, the worst-case scenario is that no crash happens at all. Then the worst-case bound equals

$$\ln(x) + (T - t) \ln(1 + r) \quad (4.16)$$

The Comparison of the worst-case bounds of 4.15 and 4.16 above leads to the following conclusions: Which one of the above strategies yields a better worst-case bound depends on the left investment time $T - t$. If the remaining investment time $T - t$ is big enough,

the worst-case bound of the first strategy is better. Then a strategy which takes more risk leads to a better worst-case bound when $T - t$ is big. When the remaining investment time $T - t$ is small, the pure bond strategy yields the higher worst-case bound. Therefore a strategy which is more risk averse delivers a higher worst-case bound when $T - t$ is small. That means that an optimal strategy should take decreasing risk with the decrease of its remaining investment time. Accordingly, one can easily infer that a constant portfolio process can not be optimal with respect to the worst-case criterion.

4.2.1. Indifference strategy: Optimality

From the conclusion above we devote to look for a portfolio strategy that could balance between good performance of the final wealth process when no crash happens and a corresponding loss when a crash happens. Thus we search for a portfolio strategy which makes the investor indifferent between two extreme cases:

- The crash of maximal size k^* happens immediately.
- No crash happens at all.

This is exactly the **indifference principle** from Korn and Wilmott [29].

Remark 4.3. We consider in this section only the positive portfolio strategies $0 \leq \pi_t \leq \pi^{0*}$. The one reason for this is that the strategy which attain negative values would be dominated by its positive part in the worst-case sense. Additionally, if we take any portfolio process $\pi > \pi^{0*}$, we have the worst-case bound which satisfies

$$\begin{aligned} W(t, x, \pi) &= \inf_{t \leq \tau < T, 0 \leq k \leq K^*} E(u(X_T^\pi)) \\ &= \inf_{t \leq \tau < T, 0 \leq k \leq K^*} E(u(x \prod_{t=0}^{\tau-1} ((1+r) + \pi_t(R-1-r)) * (1+r - \pi_\tau(r+k))) \\ &\quad * \prod_{t=\tau+1}^{T-1} ((1+r) + \pi^{0*}(R-1-r)))) \end{aligned}$$

As the utility function u is strict increasing, the high portfolio strategy $\pi > \pi^{0*}$ at the time of the crash leads to a decrease of the total wealth as well as the optimal portfolio strategy π^{0*} brings a higher utility from the final wealth. Therefore,

$$W(t, x, \pi) < W(t, x, \pi^{0*}).$$

That means, π^{0*} leads to a better worst-case bound than any portfolio process $\pi > \pi^{0*}$.

Before we explore how to derive the optimal strategy of the multi-period portfolio optimization in discrete-time, we consider first the single-period case.

Proposition 4.4: 1. The optimal portfolio process π_0^* for the single-period worst-case portfolio optimization equals 0 .

2. The optimal portfolio strategy above satisfies the indifference principle.

Proof:

1. In the one-period worst-case portfolio problem, the worst-case scenario is a crash with maximal size k^* for every positive $\pi_0 > 0$. Then the worst-case bound of π_0 satisfies:

$$\begin{aligned} E(\ln(X_1)) &= E(\ln(x(1 + r - \pi_0(r + k^*)))) \\ &= \ln(x) + \ln(1 + r - \pi_0(r + k^*)) \end{aligned}$$

Because the utility function $\ln(x)$ increases in x , we have

$$\ln(1 + r - \pi_0(r + k^*)) < \ln(1 + r)$$

Therefore the pure bond strategy leads to a better worst-case bound. We can conclude that the pure bond strategy $\pi_0 = 0$ is the optimal portfolio strategy in the one-period worst-case portfolio problem.

2. The expected utilities of the final wealth for strategy $\pi_0^* = 0$ corresponding to the two extreme cases above satisfy the following representations:

- A crash of maximal size k^* happens immediately

$$E(\ln(X_1)) = \ln(x) + \ln(1 + r - \pi_0^*(r + k^*)) = \ln(x) + \ln(1 + r)$$

- No crash happens at all

$$E(\ln(X_1)) = \ln(x) + E(\ln(1 + r + \pi_0^*(R - 1 - r))) = \ln(x) + \ln(1 + r)$$

These two representations are coincident with each other. Therefore the optimal portfolio strategy in the one-period worst-case portfolio problem satisfies the indifference principle .

□

We now turn to the multi-period setting:

Proposition 4.5: *Under Assumption (L), there exists a portfolio process π^* which satisfies the indifference principle if there exists a solution to the equations*

$$\begin{aligned} \pi_{t+1}^* &= \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_t^* \right) \\ &\quad * e^{E(\ln(1+r+\tilde{\pi}^*(R-1-r))) - E(\ln(1+r+\pi_t^*(R-1-r)))}, \quad 0 < t < T-1 \\ \pi_{T-1}^* &= 0 \end{aligned} \tag{4.17}$$

with

$$0 \leq \pi_t^* \leq \tilde{\pi}^*, \quad t \in \{0, 1, \dots, T-1\} \tag{4.18}$$

and $\tilde{\pi}^*$ being the optimal portfolio process in the crash-free model in discrete time.

Proof: The expected utility of the portfolio process π^* corresponding to the case that a crash of maximal size k^* happens immediately satisfies:

$$\begin{aligned} V^0(t+1, x(1+r-\pi_t^*(r+k^*))) \\ = \ln(x) + \ln(1+r-\pi_t^*(r+k^*)) + (T-t-1)E^{(t,x)}(\ln(1+r+\tilde{\pi}^*(R-1-r))). \end{aligned} \tag{4.19}$$

The expected utility for the portfolio process π^* that corresponds to the scenario that no crash happens at all has the following form:

$$E^{(t,x)}(\ln(\tilde{X}_T^{\pi^*}))(x) = \ln(x) + \sum_{s=t}^{T-1} E^{(t,x)}(\ln(1+r+\pi_s^*(R-1-r))). \tag{4.20}$$

Having these two equations, we now prove the claims of the proposition via backward induction on the time t . For $t = T-1$, the form of π_t^* follows from Proposition 1. We thus consider the

Start of the induction with $t=T-2$:

The equality of the expected utilities of Equations (4.19) and (4.20) is equivalent to

$$\begin{aligned} \ln(1+r-\pi_{T-2}^*(r+k^*)) + E^{(T-2,x)}(\ln(1+r+\tilde{\pi}^*(R-1-r))) \\ = E^{(T-2,x)}(\ln(1+r+\pi_{T-2}^*(R-1-r))) + \ln(1+r). \end{aligned}$$

Collecting all expectations on the right side of the equation and then applying the exponential function leads to

$$\begin{aligned} 1+r-\pi_{T-2}^*(r+k^*) &= (1+r) \exp \left(E^{(T-2,x)}(\ln(1+r+\pi_{T-2}^*(R-1-r))) \right. \\ &\quad \left. - E^{(T-2,x)}(\ln(1+r+\tilde{\pi}^*(R-1-r))) \right). \end{aligned}$$

Dividing both sides of the equation by $r+k^*$ followed by a division by the exponential function term of the right-hand side and shifting all terms to the right side yields the

required form of Equation (4.17). For this, also note that $\pi_{T-1}^* = 0$ then appears implicitly on the left side.

We can now continue with the

Induction step $t + 1 \mapsto t$:

The equality of the expected utilities of Equations (4.19) and (4.20) is equivalent to

$$\begin{aligned} & \ln(1 + r - \pi_t^*(r + k^*)) + (T - t - 1)E^{(t,x)}(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \\ &= \sum_{s=t}^{T-1} E^{(t,x)}(\ln(1 + r + \pi_s^*(R - 1 - r))) . \end{aligned}$$

By induction we now have

$$\begin{aligned} & \ln(1 + r - \pi_t^*(r + k^*)) + (T - t - 1)E^{(t,x)}(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \\ &= E^{(t,x)}(\ln(1 + r + \pi_t^*(R - 1 - r))) + \sum_{s=t+1}^{T-1} E^{(t,x)}(\ln(1 + r + \pi_s^*(R - 1 - r))) \\ &= E^{(t,x)}(\ln(1 + r + \pi_t^*(R - 1 - r))) \\ & \quad + E^{(t,x)}(\ln(1 + r - \pi_{t+1}^*(r + k^*))) + (T - t - 2)E^{(t,x)}(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \end{aligned}$$

which yields

$$\begin{aligned} & \ln(1 + r - \pi_t^*(r + k^*)) + E^{(t,x)}(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \\ &= E^{(t,x)}(\ln(1 + r + \pi_t^*(R - 1 - r))) + \ln(1 + r - \pi_{t+1}^*(r + k^*)) \end{aligned}$$

Collecting the \ln -terms on one side, the expectation terms on the other side of the equation, applying the exponential function, and then solving for π_{t+1}^* yields the desired recursive formula

$$\pi_{t+1}^* = \frac{1 + r}{r + k^*} - \left(\frac{1 + r}{r + k^*} - \pi_t^* \right) * e^{E(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) - E(\ln(1 + r + \pi_t^*(R - 1 - r)))},$$

for all $0 < t < T - 1$. If now there exists a solution π^* to the recursive equations above, the deterministic strategy π^* satisfies the indifference principle by construction

$$E^{(t,x)}(\ln(\tilde{X}_T^{\pi^*})) = V^0(t + 1, x(1 + r - \pi_t^*(r + k^*))) . \quad (4.21)$$

□

Remark 4.6. a) **Existence of an indifference strategy:** It remains to prove the existence of a solution to the recursive equations

$$\begin{aligned} \pi_{t+1}^* &= \frac{1 + r}{r + k^*} - \left(\frac{1 + r}{r + k^*} - \pi_t^* \right) \\ & \quad * e^{E(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) - E(\ln(1 + r + \pi_t^*(R - 1 - r)))}, 0 < t < T - 1 \\ \pi_{T-1}^* &= 0 \end{aligned} \quad (4.22)$$

with

$$0 \leq \pi_t^* \leq \tilde{\pi}^* . \quad (4.23)$$

For this, note that for $\pi_t^* = 0$, the right hand side of Equation 4.22 has the form

$$\frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} \right) e^{E(\ln(1+r+\tilde{\pi}^*(R-1-r))) - E(\ln(1+r+\pi_t^*(R-1-r)))} < 0 \leq \pi_{t+1}^*$$

and for $\pi_t^* = \tilde{\pi}^*$, we obtain the right hand side of Equation (4.22) as

$$\frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \tilde{\pi}^* \right) = \tilde{\pi}^* \geq \pi_{t+1}^* .$$

Moreover, the right hand side of Equation (4.22) is increasing for $\pi_t^* \in [0, \tilde{\pi}^*]$. Therefore, by continuity there exists a solution π_t^* of Equation (4.22).

Even more, by the above considerations there exists a *unique deterministic portfolio process* π_t^* solving Equation (4.22). To see this, note that $\pi_{T-1}^* = 0$ is obviously deterministic. As then by induction the left-hand side of Equation (4.22) is always deterministic, we get the existence of a constant (and thus deterministic) value π_t^* solving Equation (4.22) by using the argument given above to show the existence of a solution as it in particular works for a constant.

b) For the portfolio strategy π^* that satisfies the indifference principle, the representation of the worst-case bound if a crash happens at time τ immediately with $t < \tau < T$ is given by:

$$\begin{aligned} & E^{t,x}(V^0(\tau+1, \tilde{X}_\tau^{\pi^*}(1+r-\pi_\tau^*(r+k^*)))) \\ &= E^{(t,x)}(\ln(\tilde{X}_\tau^{\pi^*})) + \ln(1+r-\pi_\tau^*(r+k^*)) + (T-\tau-1)E(\ln(1+r+\tilde{\pi}^*(R-1-r))) . \end{aligned}$$

As the indifference principle is satisfied for all t , we have

$$\begin{aligned} & E^{t,x}(V^0(\tau+1, \tilde{X}_\tau^{\pi^*}(1+r-\pi_\tau^*(r+k^*)))) = E^{(t,x)}(\ln(\tilde{X}_\tau^{\pi^*})) + \sum_{s=\tau}^{T-1} E(\ln(1+r+\pi_s^*(R-1-r))) \\ &= \ln(x) + \sum_{s=t}^{\tau-1} E^{(t,x)}(\ln(1+r+\pi_s^*(R-1-r))) + \sum_{s=\tau}^{T-1} E^{(t,x)}(\ln(1+r+\pi_s^*(R-1-r))) \\ &= E^{(t,x)}(\ln(\tilde{X}_T^{\pi^*})) = V^0(t+1, x(1+r-\pi_t^*(r+k^*))) . \end{aligned}$$

Therefore, we have exactly the same expected worst-case bound for all possible times of the crash. By the indifference principle, the exact crash time is no longer important for the investor.

As the next step, we prove that the deterministic strategy π^* uniquely determined by the Equations (4.17) indeed solves the worst-case portfolio optimization problem in discrete time (4.11).

Theorem 4.7: (Worst-case optimal portfolio process for logarithmic utility in discrete time)

Under Assumption (L), in the log-utility case, the deterministic portfolio strategy uniquely determined by the Equations (4.17) is optimal for the worst-case portfolio optimization problem in discrete time 4.11.

Proof: Assume that there exists an admissible portfolio process π with a better worst-case bound than the strategy π^* which satisfies the recursive equations (4.17).

From the explicit form of $V_0(t+1, x(1+r-\pi_t(r+k^*)))$ it must satisfy that $\pi_t < \pi_t^*$ almost surely to have a higher worst-case bound if a crash happens immediately.

Furthermore, the expected utility for the portfolio process π corresponding to the scenario if no crash happens at all satisfies:

$$\begin{aligned} E^{(t,x)}(\ln(\tilde{X}_T^\pi)) &= \ln(x) + \sum_{s=t}^{T-1} E(\ln(1+r+\pi_s(R-1-r))) \\ &< \ln(x) + E(\ln(1+r+\pi_t^*(R-1-r))) + \sum_{s=t+1}^{T-1} E(\ln(1+r+\pi_s(R-1-r))) . \end{aligned}$$

The inequality is a consequence of the strictly increasing function $E(\ln(1+r+\pi(t)(R-1-r)))$. If the portfolio strategy π leads to a higher worst-case bound than π^* in the no-crash scenario, then there exists a smallest deterministic time t_m with $t+1 \leq t_m \leq T-1$ so that

$$E(\ln(1+r+\pi_{t_m}(R-1-r))) > E(\ln(1+r+\pi_{t_m}^*(R-1-r))) , \quad (4.24)$$

because π^* has the same worst-case bound in the no-crash scenario according to the indifference property of π^* .

We first want to show that $E(\ln(1+r+\pi_s(R-1-r))) \leq E(\ln(1+r+\pi_s^*(R-1-r)))$, when $E(\pi_s) \leq E(\pi_s^*)$ for $t \leq s \leq T-1$.

If $E(\pi_s) \leq E(\pi_s^*)$, the concavity of the log utility function implies for any such π_s

$$\begin{aligned} &\ln(1+r+\pi_s(R-1-r)) - \ln(1+r+\pi_s^*(R-1-r)) \\ &\leq \ln'(1+r+\pi_s^*(R-1-r))(R-1-r)(\pi_s - \pi_s^*) . \end{aligned}$$

Taking the expectation on the both sides, noting that π_s^* is deterministic and that R is independent of both π_s^* and π_s , we have

$$\begin{aligned} &E(\ln(1+r+\pi_s(R-1-r))) - E(\ln(1+r+\pi_s^*(R-1-r))) \\ &\leq E(\ln'(1+r+\pi_s^*(R-1-r))(R-1-r)) E(\pi_s - \pi_s^*) . \end{aligned}$$

Note that the validity of this relation is implied by the facts that π^* is a deterministic strategy and that π_s^* and π_s are both independent of R .

Using the optimality of $\tilde{\pi}^*$ in the crash-free setting, $\pi_s^* \leq \tilde{\pi}^*$ leads to

$$E(\ln'(1 + r + \pi_s^*(R - 1 - r))(R - 1 - r)) \geq 0 .$$

Therefore, if $E(\pi_s) \leq E(\pi_s^*)$, we obtain

$$E(\ln(1 + r + \pi_s(R - 1 - r))) - E(\ln(1 + r + \pi_s^*(R - 1 - r))) \leq 0 .$$

Hence, the inequality

$$E(\ln(1 + r + \pi_{t_m}(R - 1 - r))) > E(\ln(1 + r + \pi_{t_m}^*(R - 1 - r))) \quad (4.25)$$

implies $E(\pi_{t_m}) > E(\pi_{t_m}^*)$.

The worst-case bound at exactly this time t_m if a crash happens at t_m immediately satisfies:

$$\begin{aligned} & E^{(t,x)}(V_0(t_m + 1, \tilde{X}_{t_m}^\pi(1 + r - \pi_{t_m}(r + k^*)))) \\ &= E^{(t,x)}(\ln(\tilde{X}_{t_m}^\pi)) + E(\ln(1 + r - \pi_{t_m}(r + k^*))) + (T - t_m - 1)E(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \\ &\leq E^{(t,x)}(\ln(\tilde{X}_{t_m}^\pi)) + \ln(1 + r - E(\pi_{t_m})(r + k^*)) + (T - t_m - 1)E(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \\ &< E^{(t,x)}(\ln(\tilde{X}_{t_m}^\pi)) + \ln(1 + r - E(\pi_{t_m}^*)(r + k^*)) + (T - t_m - 1)E(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) . \end{aligned}$$

From the explicit form of the wealth process X_{t_m} , we obtain:

$$E^{(t,x)}(\ln(\tilde{X}_{t_m}^\pi)) = \ln(x) + \sum_{s=t}^{t_m-1} E(\ln(1 + r + \pi_s(R - 1 - r))) .$$

By $E(\ln(1 + r + \pi_s(R - 1 - r))) \leq E(\ln(1 + r + \pi_s^*(R - 1 - r)))$ for all $t < s < t_m$, we get

$$E^{(t,x)}(\ln(\tilde{X}_{t_m}^\pi)) \leq \ln(x) + \sum_{s=t}^{t_m-1} E(\ln(1 + r + \pi_s^*(R - 1 - r))) = E^{(t,x)}(\ln(\tilde{X}_{t_m}^{\pi^*})) ,$$

thus,

$$\begin{aligned} & E^{(t,x)}(V_0(t_m + 1, \tilde{X}_{t_m}^\pi(1 + r - \pi_{t_m}(r + k^*)))) \\ &< E^{(t,x)}(\ln(\tilde{X}_{t_m}^{\pi^*})) + \ln(1 + r - E(\pi_{t_m}^*)(r + k^*)) + (T - t_m - 1)E(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) \\ &= E^{(t,x)}(V_0(t_m + 1, \tilde{X}_{t_m}^{\pi^*}(1 + r - \pi_{t_m}^*(r + k^*)))) . \end{aligned}$$

As we have exactly the same expected worst-case bounds of the optimal strategy π^* for all possible times of the crash, we get a contradiction to our assumption that the admissible strategy π delivers a higher worst-case bound than π^* . \square

Remark 4.8. 1. From the explicit form of the equations 4.17 and the above theorem we can conclude that there only exists one optimal portfolio strategy in our model.

2. For a constant portfolio process which often play an important role in portfolio optimization problem, we can also introduce the optimal constant portfolio strategy π as shown in [Korn, Wilmott][29]. The worst-case bound of the crash that happens immediately before the time horizon T is given by:

$$\begin{aligned} & E^{(t,x)}(\ln(X_T^\pi)) \\ &= \ln(x) + (T-t-1)E(\ln(1+r+\pi(R-1-r))) + \ln(1+r-\pi(r+k^*)) \end{aligned}$$

By taking the first derivative of the right hand side of the above equation with respect to π and setting this derivative equal to zero, we have

$$\begin{aligned} & (T-t-1) \sum_{i=1}^n \frac{R_i - 1 - r}{1+r+\pi(R_i-1-r)} p_i \\ &= \frac{r+k^*}{1+r-\pi(r+k^*)} \end{aligned}$$

Therefore the constant portfolio process π satisfies

$$(T-t-1)E\left(\frac{R-1-r}{1+r+\pi(R-1-r)}\right) = \frac{r+k^*}{1+r-\pi(r+k^*)} \quad (4.26)$$

4.2.2. Numerical example

Example 4.9: (The binomial setting)

To illustrate the performance of the worst-case optimal strategy compared to the crash-free optimal strategy, we assume that the stock price process follows the binomial model with parameters $0 < d < 1+r < u$ (the up- and down-multipliers of the stock price) and $0 < p < 1$ (the probability of a multiplication of the stock price by u at time t). Then, the optimal portfolio $\tilde{\pi}^*$ in the discrete-time crash-free model is given by

$$\tilde{\pi}^* = \frac{(1+r)(p(u-d)+d-1-r)}{(u-1-r)(1+r-d)} . \quad (4.27)$$

The indifference equations (4.17) read as

$$\begin{aligned} \pi_{t+1}^* &= \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_t^* \right) e^{E(\ln(1+r+\tilde{\pi}^*(R-1-r))) - E(\ln(1+r+\pi_t^*(R-1-r)))} \\ &= \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_t^* \right) e^{\ln\left(\frac{1+r+\tilde{\pi}^*(u-1-r)}{1+r+\pi_t^*(u-1-r)}\right)*p + \ln\left(\frac{1+r+\tilde{\pi}^*(d-1-r)}{1+r+\pi_t^*(d-1-r)}\right)*(1-p)} \\ &= \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_t^* \right) \frac{(1+r+\tilde{\pi}^*(u-1-r))^p (1+r+\tilde{\pi}^*(d-1-r))^{1-p}}{(1+r+\pi_t^*(u-1-r))^p (1+r+\pi_t^*(d-1-r))^{1-p}} \end{aligned} \quad (4.28)$$

with

$$\pi_{T-1}^* = 0 .$$

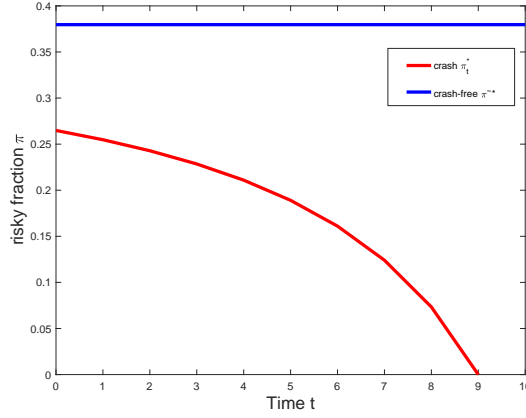


Figure 4.1.: The optimal trading strategies π_t^* with and without crash possibility

Remark 4.6 implies the existence of a unique solution π_t^* of Equation (4.28) which we compute numerically. Figure 4.1 shows that π_t^* is decreasing with time for the choices of $r = 0.05$, $u = 1.4918$, $d = 0.67$, $p = 0.5375$, $k = 0.05$ and $T = 10$. Hence, in the multi-period case the investor always has a positive position in the stock, but decreases it to protect against losses when the time horizon is approached. Only in the last single period, she invests everything in the bond. Furthermore, π_0^* is always smaller than $\tilde{\pi}_0^*$, but the difference is getting smaller as the investment horizon T becomes bigger. \triangleleft

Example 4.10: (The binomial setting for jump model)

In order to show that our techniques can extend to much more general situations. We consider here the case of discontinuous asset prices. We assume that the risk-free bond B_t and risky stock process S_t are modeled as

$$\begin{cases} B_t = e^{rt} B_0, & B_0 = 1 \\ S_t = S_0(1 + \eta)^{N(t)}, & S_0 = s_0 \end{cases}$$

where $N(t)$ is the standard poisson process with parameter λt .

Then the price dynamics of the bond and the stock with respect to the poisson process are given by

$$\begin{cases} dB_t = B_t r dt, \\ dS_t = S_t \eta dN(t). \end{cases}$$

Therefore, the wealth process $X(t)$ with the self-financing portfolio process $\pi(t)$ satisfies the following stochastic differential equation

$$dX^\pi(t) = X^\pi(t)(1 - \pi(t))r dt + X^\pi(t)\pi(t)\eta dN(t) \quad (4.29)$$

with the initial wealth value $X(0) = x$.

Let $T_1, T_2, \dots, T_{N(t)}$ be the successive jump times until time t , the Equation (4.29) is then solved as

$$X^\pi(t) = x e^{\int_0^t (1 - \pi(s)) ds} \prod_{i=1}^{N(t)} (1 + \pi(i)\eta) .$$

In the case of logarithmic utility function we obtain the following expected utility of the final wealth

$$E(\ln X^\pi(T)) = \ln x + E\left(\int_0^T (1 - \pi(s)) ds\right) + \lambda T E(\ln(1 + \pi(T_{N(T)})\eta)) ,$$

then we can get an optimal admissible deterministic portfolio process

$$\pi_{pn}^* = \frac{\lambda\eta - r}{r\eta} ,$$

which is driven by solving the portfolio optimization problem $\max_{\pi \in \mathcal{A}(x)} E(\ln X^\pi(t))$.

By using the theorem of the law of small numbers it can be shown that the stock price movements in binomial model converge to the log-poisson jump model as $n \rightarrow \infty$. Let us choose the appropriate parameters of the binomial model by

$$\begin{aligned} u &= 1 + \eta, & d &= 1 \\ 1 + \tilde{r} &= e^{r\frac{t}{n}}, & p &= \lambda\frac{t}{n}, \end{aligned}$$

then we have the limit of the optimal portfolio $\tilde{\pi}^*$ with the Equation (4.27) as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \tilde{\pi}^* = \frac{\lambda\eta - r}{r\eta} ,$$

which is consistent with the optimal portfolio strategy π_{pn}^* for the jump model above.

Now we can drive the optimal worst-case portfolio strategy for the jump model. Defining the same parameters in binomial model, we have

$$\pi_{t+\Delta t}^* = \frac{e^{r\Delta t}}{e^{r\Delta t} - 1 + k^*} - \left(\frac{e^{r\Delta t}}{e^{r\Delta t} - 1 + k^*} - \pi_t^* \right) \frac{(e^{r\Delta t} + \tilde{\pi}^*(1 + \eta - e^{r\Delta t}))^{\lambda\Delta t} (e^{r\Delta t} + \tilde{\pi}^*(1 - e^{r\Delta t}))^{1-\lambda\Delta t}}{(e^{r\Delta t} + \pi_t^*(1 + \eta - e^{r\Delta t}))^{\lambda\Delta t} (e^{r\Delta t} + \pi_t^*(1 - e^{r\Delta t}))^{1-\lambda\Delta t}} ,$$

and for $\Delta t \rightarrow 0$ we obtain the ordinary differential equation of the optimal worst-case portfolio process for the jump model

$$(\pi_t^*)' = \left(1 - \frac{\pi_t^*}{k^*}\right) (r(\pi_{pn}^* - \pi_t^*) + \lambda \log \frac{1 + \pi_t^* \eta}{1 + \pi_{pn}^* \eta}) .$$

◁

4.2.3. Generalizations: An arbitrary number of possible crashes

So far we limited the maximal number of the crashes only to one. We can extend this to an arbitrary upper bound for the number of crashes by a backward induction principle. In such a situation of at most n crashes of size $k \in [0, k^*]$, we have the following theorem:

Theorem 4.11: *If we allow for at most n crashes of size $k \in [0, k^*]$ in the discrete-time market model with the logarithmic utility function, then under Assumption (L) the deterministic worst-case optimal portfolio process $\pi_n^*(t)$ if still at most n crashes can appear is given by the following system of equations:*

$$\begin{aligned} \pi_j^*(t+1) &= \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_j^*(t) \right) \\ &\quad * \exp^{E(\ln(1+r+\pi_{j-1}^*(t+1)(R-1-r))) - E(\ln(1+r+\pi_j^*(t)(R-1-r)))}, \quad 0 < t < T-1 \\ \pi_j^*(T-1) &= 0 \end{aligned} \tag{4.30}$$

with

$$0 \leq \pi_j^*(t) \leq \pi_{j-1}^*(t) .$$

Here, $\pi_j^*(t)$ denotes the worst-case optimal portfolio process if still at most j crashes can occur. Note further that above we used the notation $\pi_0^*(t) = \tilde{\pi}^*$.

Proof: The proof is done via induction on n , the maximum number of crashes. For $n = 1$, all assertions follow from Proposition 4.5. Let us therefore assume that the above claims are satisfied for $n - 1$. Then, the expected utility of the portfolio process corresponding to the case that a crash of maximal size k^* happens immediately satisfies:

$$\begin{aligned} &V^n(t+1, x(1+r-\pi_n^*(t)(r+k^*))) \\ &= \ln(x) + \ln(1+r-\pi_n^*(t)(r+k^*)) + \sum_{s=t+1}^{T-1} E^{(t,x)}(\ln(1+r+\pi_{n-1}^*(s)(R-1-r))) . \end{aligned} \tag{4.31}$$

Using this, we obtain the form of Equations (4.30) similar to those in Proposition 4.5. The reason for the constraints $0 \leq \pi_j^*(t) \leq \pi_{j-1}^*(t)$ follows from our general Assumption (M) and the form of the proof of Theorem 4.7. The rest of the proofs for existence and optimality is totally similar to the case of $n = 1$. \square

4.3. Dynamic programming

In the previous section we showed how to derive the optimal portfolio strategy for the discrete-time worst-case problem by an indifference approach in the case of the logarithmic utility function. For general utility functions $u(x)$, the above methods of proof cannot be imitated directly as they very much benefited from the additive form of both

the value function in the crash-free setting and the expected utility of the final wealth under the assumption of no crash. This, however, is only valid for the logarithmic utility function. We thus present a different approach in this section.

Indeed, we focus on the worst-case portfolio problem in discrete time for general utility functions by applying the dynamic programming approach. The main idea of the dynamic programming approach in portfolio optimization in discrete time is to break a multi-period decision problem up into a sequence of one-period problems. It will help us to reduce the difficulty to verify the optimality.

We only give the basic case when at most one crash can occur within the investment period $[t, T]$. Extending our results to the general case of at most n crashes by an iterative procedure is notationally cumbersome and will be omitted.

Still, **the worst-case portfolio problem in discrete time under the threat of a crash** is defined by its value function:

$$V_1(t, x) = \sup_{\pi_t, \dots, \pi_{T-1}} \inf_{\tau} E^{t,x}(u(X^\pi(T))) . \quad (4.32)$$

To implement the procedure using the dynamic programming principle in the case of our worst-case portfolio problem, we denote by $U_t(x)$ **the worst-case optimal value function at time t** as well as by $\tilde{U}_t(x)$ as **the crash-free optimal value function at time t** . The dynamic programming equation for the discrete-time crash-free model has the form of

$$\begin{aligned} \tilde{U}_T(x) &= u(x) \\ \tilde{U}_t(x) &= \sup_{\pi_t} \{E(\tilde{U}_{t+1}(x((1+r) + \pi_t(R-1-r)))) \mid \mathcal{F}_t\} . \end{aligned} \quad (4.33)$$

To motivate a dynamic programming equation for the worst-case problem in the crash model, let $U_t(x)$ denote the value function when still one crash is possible. Noting that the main principle of dynamic programming for the discrete-time optimization problem is that the optimal decision to make now should be consistent with the intention to act optimally in all future periods, we transfer this to the crash setting. If we know the optimal worst-case portfolio process starting at time $t+1$, then the determination of the optimal worst-case portfolio process starting at time t can be reduced to a one-period problem. In the one-period worst-case portfolio problem at time t there exist only two possible crash scenarios. The first one is that the crash happens immediately at time t . In this case, the value function $U_t^1(x)$ satisfies the following dynamic programming principle

$$U_t^1(x) = \sup_{\pi_t} E(\tilde{U}_{t+1}(x(1+r - \pi_t(r+k^*)))) \mid \mathcal{F}_t . \quad (4.34)$$

If no crash occurs in the next period the representation of the value function $U_t^2(x)$ is given as

$$U_t^2(x) = \sup_{\pi_t} E(U_{t+1}(x(1+r + \pi_t(R-1-r)))) \mid \mathcal{F}_t . \quad (4.35)$$

By combining these two cases we can heuristically derive the worst-case optimal value function $U_t(x)$ based on the worst-case optimal value function $U_{t+1}(x)$:

$$U_t(x) = \sup_{\pi_t} \min \{ E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t), \quad (4.36) \\ E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t) \} .$$

The value of $U_t(x)$ at time $t = T$ satisfies (see also Proposition 4.14)

$$U_T(x) = u(x) .$$

Therefore, the dynamic programming equation for the worst-case portfolio optimization problem under the threat of a crash is given as

$$U_T(x) = u(x) \quad (4.37) \\ U_t(x) = \sup_{\pi_t} \min \{ E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t), E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t) \} .$$

Of course, by this heuristic derivation, we have not shown any kind of optimality. This has to be proved separately. However, if this is shown then by using this dynamic programming equation (4.39), we can compute the optimal worst-case portfolio strategy and the worst-case optimal value function $U_t(x)$ in a recursive way.

Our main aim now is to prove the following theorem that justifies our heuristic approach :

Theorem 4.12: (Verification Theorem)

Let u be a utility function. We further assume that the Assumptions (M) and (O) are satisfied together with Assumption (D):

Let the value function U_t be concave, strictly increasing and continuously differentiable in x , and let the function

$$f(\pi) := E(U_{t+1}(x(1+r+\pi(R-1-r)))) \mid \mathcal{F}_t, \quad t = 0, 1, \dots, T-1 \quad (4.38)$$

be strictly increasing on $[0, \tilde{\pi}_t^]$ with the maximum of $f(\pi)$ attained in $\tilde{\pi}_t^*$, the optimal deterministic portfolio process in the crash-free setting.*

Then there exist unique deterministic maximizers π_t^ of the value function which can be computed recursively by the dynamic programming equation*

$$U_T(x) = u(x) \quad (4.39) \\ U_t(x) = \sup_{\pi_t} \min \{ E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t), E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t) \} ,$$

such that the portfolio strategy $\pi^ = (\pi_0^*, \dots, \pi_{T-1}^*)$ is optimal for the worst-case portfolio problem.*

Remark 4.13. Of course, Assumption (D) is a strong requirement, but will be satisfied in all our examples presented below.

4.3.1. The characterization and the optimality

Before we give the proof of the verification theorem, we will show how to construct the candidates for the optimal strategies appearing in the verification theorem. Afterwards, we show that these candidates are indeed the optimal solutions of the worst-case portfolio problem in discrete time .

The optimal strategy π_{T-1}^* for the single-period worst-case portfolio problem is derived as in the log-utility case.

Proposition 4.14: *The optimal portfolio process π_{T-1} equals 0 for the single-period worst-case portfolio optimization of the terminal time T .*

Proof: In the single-period worst-case portfolio problem at time $T - 1$, the worst-case scenario is a crash with maximal size k^* happening immediately for every positive $\pi_{T-1} > 0$. Then the worst-cse bound of π_{T-1} satisfies:

$$\begin{aligned} & E^{T-1,x}(u(X_T)) \\ &= E(u(x(1+r-\pi_{T-1}(r+k^*))) \mid \mathcal{F}_{T-1}) \\ &= u(x(1+r-\pi_{T-1}(r+k^*))) \end{aligned}$$

Because the utility function $u(x)$ decreases in π_{T-1} , we have

$$u(x(1+r-\pi_{T-1}(r+k^*))) < u(1+r)$$

Therefore the pure bond strategy leads to a better worst-case bound. We can conclude that the pure bond strategy $\pi_{T-1} = 0$ is the optimal portfolio strategy for the single-period worst-case portfolio problem at time $T - 1$. □

Remark 4.15. If we consider the optimal value function $U_{T-1}(x)$ at time $T - 1$ by using the dynamic programming equations 4.39, we have that

$$\begin{aligned} & U_{T-1}(x) \\ &= \sup_{\pi_{T-1}} \min\{E(U_T(x(1+r+\pi_{T-1}(R-1-r))) \mid \mathcal{F}_{T-1}), \\ & \quad E(\tilde{U}_T(x(1+r-\pi_{T-1}(r+k^*))) \mid \mathcal{F}_{T-1})\} \\ &= \sup_{\pi_{T-1}} \min\{E(u(x(1+r+\pi_{T-1}(R-1-r))) \mid \mathcal{F}_{T-1}), \\ & \quad E(u(x(1+r-\pi_{T-1}(r+k^*))) \mid \mathcal{F}_{T-1})\} \end{aligned}$$

For any $\pi_{T-1} \geq 0$, we obtain

$$\begin{aligned} & \sup_{\pi_{T-1}} \min \{ E(u(x(1+r+\pi_{T-1}(R-1-r))) \mid \mathcal{F}_{T-1}), \\ & E(u(x(1+r-\pi_{T-1}(r+k^*))) \mid \mathcal{F}_{T-1}) \} \\ &= \sup_{\pi_{T-1}} u(x(1+r-\pi_{T-1}(r+k^*))) \end{aligned}$$

Therefore,

$$\begin{aligned} U_{T-1}(x) &= \sup_{\pi_{T-1}} u(x(1+r-\pi_{T-1}(r+k^*))) \\ &\leq u(x(1+r)) \end{aligned}$$

Thus $\pi_{T-1} = 0$ is the optimal strategy which delivers the best worst-case bound by dynamic programming equations. This result is coincident with the optimal strategy from the above Proposition.

Next we want to show that the candidate for the optimal strategy obtained as a solution to the dynamic programming equation 4.39 exists.

Lemma 4.16: *Under the assumptions of Theorem 4.12, there exists a portfolio process π_t^* which satisfies*

$$\begin{aligned} & E(U_{t+1}(x(1+r+\pi_t^*(R-1-r))) \mid \mathcal{F}_t) \\ &= E(\tilde{U}_{t+1}(x(1+r-\pi_t^*(r+k^*))) \mid \mathcal{F}_t) \end{aligned}$$

for all $x > 0$ and all $t \in \{0, 1, \dots, T-1\}$.

Proof: Let us start in defining the functions

$$\begin{aligned} f(\pi) &= E(U_{t+1}(x(1+r+\pi(R-1-r))) \mid \mathcal{F}_t), \\ g(\pi) &= E(\tilde{U}_{t+1}(x(1+r-\pi(r+k^*))) \mid \mathcal{F}_t) = \tilde{U}_{t+1}(x(1+r-\pi(r+k^*))). \end{aligned}$$

As it can easily be shown (by induction using the dynamic programming equation) that $\tilde{U}_{t+1}(x)$ is a strictly increasing function, we have that $g(\pi)$ is a strictly decreasing function.

By Assumption (D), the crash-free optimal portfolio strategy $\tilde{\pi}_t^*$ yields the maximum of the function $f(\pi)$. If now the investor chooses the pure bond strategy $\pi = 0$, we have

$$f(0) = U_{t+1}(x(1+r)), \quad g(0) = \tilde{U}_{t+1}(x(1+r)).$$

If the optimal strategy $\tilde{\pi}^*$ in the crash-free model is not worst-case optimal, the value function under the crash-free model is better than the value function of the crash model, if not, the two value functions are at most equal. Therefore, we get

$$g(0) \geq f(0). \tag{4.40}$$

If the investor chooses the optimal deterministic strategy in the crash-free model $\tilde{\pi}_t^*$, we have

$$\begin{aligned} f(\tilde{\pi}_t^*) &= E((U_{t+1}(x(1+r+\tilde{\pi}_t^*(R-1-r)))) \mid \mathcal{F}_t), \\ g(\tilde{\pi}_t^*) &= \tilde{U}_{t+1}(x(1+r-\tilde{\pi}_t^*(r+k^*))) . \end{aligned}$$

The worst-case scenario of this optimal strategy $\tilde{\pi}_t^*$ at time $t+1$ is given by a crash of the maximum size k^* which leads to the worst-case bound of

$$\tilde{U}_{t+1}(x(1+r-\tilde{\pi}_t^*(r+k^*))) .$$

Thus, we obtain the following inequality:

$$E((U_{t+1}(x(1+r+\tilde{\pi}_t^*(R-1-r)))) \mid \mathcal{F}_t) \geq \tilde{U}_{t+1}(x(1+r-\tilde{\pi}_t^*(r+k^*))) .$$

Hence, we arrive at

$$f(\tilde{\pi}_t^*) \geq g(\tilde{\pi}_t^*) . \quad (4.41)$$

The two Inequalities (4.40) and (4.41) imply the existence of a unique deterministic portfolio process $\pi_t^* \in [0, \tilde{\pi}_t^*]$ for all $t \in \{0, 1, \dots, T-2\}$ with

$$E(U_{t+1}(x(1+r+\pi_t^*(R-1-r)))) \mid \mathcal{F}_t = E(\tilde{U}_{t+1}(x(1+r-\pi_t^*(r+k^*)))) \mid \mathcal{F}_t$$

which is what we wanted to show. □

Now let us get back to consider the right side of the value function

$$\sup_{\pi_t} \min\{E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t, E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t\} . \quad (4.42)$$

Lemma 4.16 above yields that the supremum in (4.42) is attained for the smallest π_t which satisfies

$$E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t \geq E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t$$

or the portfolio strategy π_t with the biggest π_t with

$$E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t \leq E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t .$$

The value functions $E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t$ and $E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t$ are both continuous, therefore we obtain the supremum when we have the equality

$$E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) \mid \mathcal{F}_t = E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*)))) \mid \mathcal{F}_t .$$

In Lemma 4.16, we already showed the existence of those portfolio strategies along the dynamic programming equations and derived how to construct the candidates of the optimal portfolio strategies. In the following we show that the derived candidates are indeed the optimal solutions of the worst-case portfolio problem in discrete time.

The proof of Theorem 4.12: Assume that there exists an admissible portfolio process $\pi = (\pi_t, \dots, \pi_{T-1})$ with a better worst-case bound than the portfolio process $\pi^* = (\pi_t^*, \dots, \pi_{T-1}^*)$ obtained by the dynamic programming equation 4.39 as proved by Lemma 4.16. We show the non-existence of such a portfolio process π via backward induction in time.

$t = T - 1$:

Here, we must have $\pi_{T-1} = 0 = \pi_{T-1}^*$ due to Proposition 4.14.

$t = j \in \{0, 1, \dots, T - 2\}$:

Now we assume that the portfolio process $(\pi_j, \dots, \pi_{T-1})$ leads to a better worst-case bound than $(\pi_j^*, \dots, \pi_{T-1}^*)$, and $(\pi_{j+1}, \dots, \pi_{T-1})$ has the same worst-case bound as $(\pi_{j+1}^*, \dots, \pi_{T-1}^*)$. Then, as we have

$$E(\tilde{U}_{j+1}(x((1+r) - \pi_j^*(r + k^*))) \mid \mathcal{F}_j) = E(U_{j+1}(x((1+r) + \pi_j^*(R - 1 - r))) \mid \mathcal{F}_j),$$

we must have both strict inequalities

$$\begin{aligned} E(\tilde{U}_{j+1}(x((1+r) - \pi_j(r + k^*))) \mid \mathcal{F}_j) &> E(\tilde{U}_{j+1}(x((1+r) - \pi_j^*(r + k^*))) \mid \mathcal{F}_j) \quad (4.43) \\ E(U_{j+1}(x((1+r) + \pi_j(R - 1 - r))) \mid \mathcal{F}_j) &> E(U_{j+1}(x((1+r) + \pi_j^*(R - 1 - r))) \mid \mathcal{F}_j) \quad (4.44) \end{aligned}$$

Due to the fact, that both π_j and π_j^* are \mathcal{F} -measurable, the first inequality leads to

$$\tilde{U}_{j+1}(x((1+r) - \pi_j(r + k^*))) > \tilde{U}_{j+1}(x((1+r) - \pi_j^*(r + k^*)))$$

and thus to

$$\pi_j < \pi_j^* \quad (4.45)$$

almost surely (see also the argument for $\tilde{U}_t(x)$ being increasing in x at the beginning of the proof of Lemma 4.16). As the function $f(\pi_t)$ as defined in Lemma 4.16 is increasing, we obtain

$$E(U_{j+1}(x((1+r) + \pi_j(R - 1 - r))) \mid \mathcal{F}_j) \leq E(U_{j+1}(x((1+r) + \pi_j^*(R - 1 - r))) \mid \mathcal{F}_j)$$

which is in contradiction to the strict inequality (4.44). Thus, the assumption of the existence of an admissible portfolio strategy π yielding a bigger worst-case bound than π^* is proved to be wrong. \square

4.3.2. Numerical examples

We present some examples of the solution of the worst-case portfolio problem via the dynamic programming equation to illustrate our theory with the most popular utility functions.

Power utility

Let us start to consider the case of power utility

$$u(x) = \frac{1}{\gamma} x^\gamma, \quad \gamma < 1, \quad \gamma \neq 0.$$

To apply our just obtained results, we have to check if indeed all assumptions of Theorem 4.12 are satisfied. For this, we first look at the crash-free setting. By using the corresponding dynamic programming equations, one can directly show that we have

$$\tilde{U}_t(x) = \frac{1}{\gamma} x^\gamma h(t) \tag{4.46}$$

with

$$h(t) = (E((1 + r + \tilde{\pi}^*(R - r - 1)^\gamma))^{T-t} \tag{4.47}$$

where the constant portfolio process $\tilde{\pi}^*$ is determined as the solution of the maximization problem

$$E((1 + r + \tilde{\pi}^*(R - r - 1)^\gamma) = \sup_{\pi \in (-\infty, \infty)} E((1 + r + \tilde{\pi}(R - r - 1)^\gamma) .$$

By the general assumption (O) on the market model, the supremum is indeed attained. Due to the multiplicative form of the wealth process and the independence of the returns R_t from the past price evolutions combined with the identical distributions of $R_t \sim R$, the optimal portfolio process has to be a constant one. Further, by Assumption (M), we have

$$\tilde{\pi}^* > 0 .$$

We next consider the form of the value function of the worst-case problem and claim that we have

$$U_t(x) = \frac{1}{\gamma} x^\gamma H(t) \tag{4.48}$$

for a suitable positive, deterministic and decreasing function $H(t)$ with $H(t) \leq h(t)$. Starting from $U_T(x) = x^\gamma/\gamma$ and using $\pi_{T-1}^* = 0$, we have

$$U_{T-1}(x) = (1 + r)^\gamma x^\gamma / \gamma$$

which constitutes the start of the induction on $T - t$ with $t = 1$. Let us assume that we have proved the representation (4.48) for $t - 1$. We will now prove it for t . We then have

$$\begin{aligned} U_{T-t}(x) &= \sup_{\pi_{T-t}} \min \{ E(U_{T-t+1}(x(1 + r + \pi_{T-t}(R - 1 - r))) \mid \mathcal{F}_{T-t}), \\ &\quad E(\tilde{U}_{T-t+1}(x(1 + r - \pi_{T-t}(r + k^*))) \mid \mathcal{F}_{T-t}) \} \\ &= \frac{x^\gamma}{\gamma} \sup_{\pi_{T-t}} \min \{ H(T - t + 1) E((1 + r + \pi_{T-t}(R - 1 - r))^\gamma \mid \mathcal{F}_{T-t}), \\ &\quad h(T - t + 1)(1 + r - \pi_{T-t}(r + k^*))^\gamma \} =: \frac{x^\gamma}{\gamma} H(T - t) . \end{aligned}$$

Note that the supremum in the equation above is independent of x and is given by a deterministic function of time as – again – the randomness in the optimization problem is only given by R which is independent of \mathcal{F}_{T-t} . Note that the maximum of $E((1+r+\pi_{T-t}(R-1-r))^\gamma \mid \mathcal{F}_{T-t})$ is attained for the crash-free optimal portfolio process $\tilde{\pi}^*$ and the function increases in π on $[0, \tilde{\pi}^*]$. For the second term $h(T-t+1)(1+r-\pi_{T-t}(r+k^*))^\gamma$ the optimal portfolio value would be zero and decreases in π . As, however, $h(T-t+1) \geq H(T-t+1)$, the optimal value π_{T-t}^* has to be in $[0, \tilde{\pi}^*]$. As the two functions containing π_{T-t} are identical for all times $t < T-1$, but their multipliers $H(T-t+1)$, $h(T-t+1)$ are larger than their counterparts at the next time step, we have also proved that the value of the supremum is bigger at time $T-t$ than at time $T-t+1$. Thus, we have

$$0 < H(T-t) \leq H(T-t+1)$$

where the positivity is implied by the positivity of all ingredients of the optimization problem and the fact that it has a positive lower bound which is attained for choosing $\pi(T-t) = 0$.

Thus, Assumption (D) is satisfied. We can thus make full use of the claims of Theorem 4.12.

Due to Theorem 4.12, we have

$$E(U_{t+1}(x(1+r+\pi_t(R-1-r))) \mid \mathcal{F}_{T-t}) = E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*))) \mid \mathcal{F}_{T-t}) \quad (4.49)$$

for the optimal portfolio process in the crash setting. Using the form of the value function in the crash-free setting, we obtain

$$\begin{aligned} & E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*))) \mid \mathcal{F}_{T-t}) \\ &= \frac{1}{\gamma} (x(1+r-\pi_t(r+k^*)))^\gamma \prod_{s=t+1}^{T-1} E(1+r+\pi_s^*(R-1-r))^\gamma. \end{aligned} \quad (4.50)$$

Applying the dynamic programming equation in $E(U_{t+1}(x(1+r+\pi_t(R-1-r))))$ we

get

$$\begin{aligned}
 & E(U_{t+1}(x(1+r+\pi_t(R-1-r))) \mid \mathcal{F}_t) \\
 &= E(\sup_{\pi_{t+1}} \min\{E(U_{t+2}(x(1+r+\pi_t(R-1-r))(1+r+\pi_{t+1}(R-1-r))) \mid \mathcal{F}_{t+1}), \\
 &\quad E(\tilde{U}_{t+2}(x(1+r+\pi_t(R-1-r))(1+r-\pi_{t+1}(r+k^*))) \mid \mathcal{F}_{t+1})\} \mid \mathcal{F}_t) \\
 &= E(E(U_{t+2}(x(1+r+\pi_t(R-1-r))(1+r+\pi_{t+1}^*(R-1-r))) \mid \mathcal{F}_{t+1}) \mid \mathcal{F}_t) \\
 &= E(E(\tilde{U}_{t+2}(x(1+r+\pi_t(R-1-r))(1+r-\pi_{t+1}^*(r+k^*))) \mid \mathcal{F}_{t+1}) \mid \mathcal{F}_t) \\
 &= E(\tilde{U}_{t+2}(x(1+r+\pi_t(R-1-r))(1+r-\pi_{t+1}^*(r+k^*))) \mid \mathcal{F}_t) \\
 &= \frac{1}{\gamma} x^\gamma E((1+r+\pi_t(R-1-r))^\gamma (1+r-\pi_{t+1}^*(r+k^*))^\gamma \mid \mathcal{F}_t) \\
 &\quad * \prod_{s=t+2}^{T-1} E(1+r+\tilde{\pi}^*(R-1-r))^\gamma . \tag{4.51}
 \end{aligned}$$

By comparing Equations 4.50 and 4.51, the optimal strategy π_t^* has to satisfy

$$\begin{aligned}
 & \frac{1}{\gamma} x^\gamma (1+r-\pi_t^*(r+k^*))^\gamma \prod_{s=t+1}^{T-1} E(1+r+\tilde{\pi}^*(R-1-r))^\gamma \\
 &= \frac{1}{\gamma} x^\gamma E((1+r+\pi_t^*(R-1-r))^\gamma (1+r-\pi_{t+1}^*(r+k^*))^\gamma \mid \mathcal{F}_t) * \prod_{s=t+2}^{T-1} E(1+r+\tilde{\pi}^*(R-1-r))^\gamma
 \end{aligned}$$

which directly leads to

$$\begin{aligned}
 & (1+r-\pi_t^*(r+k^*))^\gamma E(1+r+\tilde{\pi}^*(R-1-r))^\gamma \\
 &= E((1+r+\pi_t^*(R-1-r))^\gamma (1+r-\pi_{t+1}^*(r+k^*))^\gamma \mid \mathcal{F}_t) .
 \end{aligned}$$

As we have that $E(1+r+\pi_t(R-1-r))^\gamma > 0$ in the interval $[0, \tilde{\pi}^*]$, we can transform this into the following recursive relation for the optimal strategy

$$\pi_{t+1}^* = \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_t^* \right) \left(\frac{E(1+r+\tilde{\pi}^*(R-1-r))^\gamma}{E(1+r+\pi_t^*(R-1-r))^\gamma} \right)^{\frac{1}{\gamma}} \tag{4.52}$$

with

$$\pi_{T-1}^* = 0$$

where the latter equation follows from Proposition 4.14.

To show the existence of a solution $\pi_t^* \in [0, \tilde{\pi}^*]$ of Equation 4.52, note first that right hand side of Equation 4.52 is increasing for $\pi_t^* \in [0, \tilde{\pi}^*]$. However, if we choose $\pi_t^* = 0$ on the right hand side, we obtain (by backward induction starting at time $t = T-2$)

$$\frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} \right) * \left(\frac{E(1+r+\tilde{\pi}^*(R-1-r))^\gamma}{(1+r)^\gamma} \right)^{\frac{1}{\gamma}} < 0 \leq \pi_{t+1}^* .$$

For the choice of $\pi_t^* = \tilde{\pi}^*$ on the right hand side of Equation 4.52, we obtain

$$\frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \tilde{\pi}^* \right) = \tilde{\pi}^* \geq \pi_{t+1}^* .$$

Therefore, there indeed exists a unique solution π_t^* of Equation 4.52.

To continue our example, we now have to choose a stock price model so that we can explicitly check the remaining assumptions of Theorem 4.12. If we assume that the price process of the stock follows the binomial model (as in the case of our log-utility example), then the remaining assumptions of Theorem 4.12 follow immediately. Further, by calculating the relevant expectation in Equation 4.47, the crash-free optimal portfolio strategy $\tilde{\pi}^*$ is of the form

$$\tilde{\pi}^* = \frac{(1+r)}{(u-1-r)(1+r-d)} * \frac{p^\kappa(u-1-r)^\kappa - (1-p)^\kappa(1+r-d)^\kappa}{(p^\kappa(u-1-r)^{\kappa\gamma} + (1-p)^\kappa(1+r-d)^{\kappa\gamma})} .$$

Thus, we obtain the recursive formula for the worst-case optimal strategy as follows

$$\begin{aligned} \pi_{t+1}^* &= \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_t^* \right) \\ &\quad * \left(\frac{(1+r+\tilde{\pi}^*(u-1-r))^\gamma p + (1+r+\tilde{\pi}^*(d-1-r))^\gamma (1-p)}{(1+r+\pi_t^*(u-1-r))^\gamma p + (1+r+\pi_t^*(d-1-r))^\gamma (1-p)} \right)^{\frac{1}{\gamma}} \end{aligned}$$

which again has to be solved numerically.

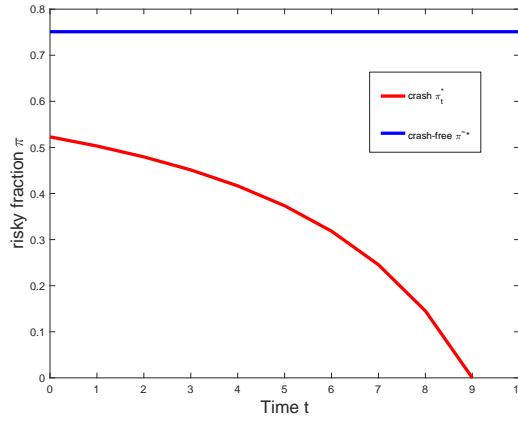


Figure 4.2.: The optimal trading strategies π_t^* with and without crash possibility

Figure 4.2 compares the optimal trading strategies π_t^* with and without crash possibility for power utility for the choices of $r = 0.05$, $u = 1.4918$, $d = 0.67$, $p = 0.5375$, $k = 0.05$ and $T = 10$. The worst-case optimal trading strategies π_t^* is decreasing with time when

we approach the time horizon. Only in the last single period starting in $T - 1$, the fraction of risky investments is reduced to zero.

Approximation of the Black-Scholes-Merton Model. Using the above results in the binomial setting, we now introduce a general time step Δt with the intention to let it tend to zero to approximate the geometric Brownian motion model of the stock price in the Black-Scholes-Merton setting via a sequence of binomial models. For this, we define the parameters of the binomial model by

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}}, & d &= e^{-\sigma\sqrt{\Delta t}} \\ 1 + r &= e^{\tilde{r}\Delta t}, & p &= \frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t}. \end{aligned}$$

For notational simplicity, we will in the following use the abbreviation r for the interest rate again. With the above choice, it is well-known that this sequence of binomial models converges weakly to the geometric Brownian motion with parameters μ and σ^2 .

The recursive formula for the worst-case optimal portfolio process $\pi_{t+\Delta t}^*$ now has the form of

$$\pi_{t+\Delta t}^* = \frac{e^{r\Delta t}}{e^{r\Delta t} - 1 + k^*} - \left(\frac{e^{r\Delta t}}{e^{r\Delta t} - 1 + k^*} - \pi_t^* \right) \left(\frac{A}{B} \right)$$

with

$$\begin{aligned} A &= ((e^{r\Delta t} + \tilde{\pi}^*(e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}))^\gamma \left(\frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} \right) \\ &\quad + (e^{r\Delta t} + \tilde{\pi}^*(e^{-\sigma\sqrt{\Delta t}} - e^{r\Delta t}))^\gamma \left(\frac{1}{2} - \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} \right))^\frac{1}{\gamma}, \\ B &= ((e^{r\Delta t} + \pi_t^*(e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}))^\gamma \left(\frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} \right) \\ &\quad + (e^{r\Delta t} + \pi_t^*(e^{-\sigma\sqrt{\Delta t}} - e^{r\Delta t}))^\gamma \left(\frac{1}{2} - \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} \right))^\frac{1}{\gamma}. \end{aligned}$$

We expect the worst-case optimal discrete-time strategy computed by the dynamic programming equations to be close to the expression in the continuous-time model, at least for small values of Δt . To check this, we compute

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\pi_{t+\Delta t}^* - \pi_t^*}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\left(\frac{e^{r\Delta t}}{e^{r\Delta t} - 1 + k^*} - \pi_t^* \right) - \left(\frac{e^{r\Delta t}}{e^{r\Delta t} - 1 + k^*} - \pi_t^* \right) \frac{A}{B} \right) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left(\left(\frac{e^{r\Delta t}}{e^{r\Delta t} - 1 + k^*} - \pi_t^* \right) \frac{B - A}{B} \right) = \frac{1}{k^*} (1 - k^* \pi_t^*) \lim_{\Delta t \rightarrow 0} \frac{B - A}{B \Delta t}. \end{aligned}$$

To examine, the above limit, let $A = (A_1 + A_2)^{\frac{1}{\gamma}}$ with

$$\begin{aligned} A_1 &= (e^{r\Delta t} + \tilde{\pi}^*(e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}))^\gamma \left(\frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} \right), \\ A_2 &= (e^{r\Delta t} + \tilde{\pi}^*(e^{-\sigma\sqrt{\Delta t}} - e^{r\Delta t}))^\gamma \left(\frac{1}{2} - \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} \right). \end{aligned}$$

Using the Taylor expansion of first order for the exponential function and then binomial series expansion, we have

$$\begin{aligned} A_1 &= \frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} + \frac{1}{2} \gamma (r\Delta t + \tilde{\pi}^*(\sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t - r\Delta t)) \\ &\quad + \frac{1}{2} \gamma \tilde{\pi}^* \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sigma \Delta t + \frac{\gamma(\gamma-1)}{4} (\tilde{\pi}^*)^2 \sigma^2 \Delta t + O(\Delta t^2) \end{aligned}$$

and

$$\begin{aligned} A_2 &= \frac{1}{2} - \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta t} + \frac{1}{2} \gamma (r\Delta t + \tilde{\pi}^*(-\sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t - r\Delta t)) \\ &\quad + \frac{1}{2} \gamma \tilde{\pi}^* \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sigma \Delta t + \frac{\gamma(\gamma-1)}{4} (\tilde{\pi}^*)^2 \sigma^2 \Delta t + O(\Delta t^2). \end{aligned}$$

Using the binomial series expansion again, we obtain

$$\begin{aligned} A &= (A_1 + A_2)^{\frac{1}{\gamma}} = 1 + r\Delta t + \tilde{\pi}^*(\mu - r)\Delta t + \frac{(\gamma-1)}{2} (\tilde{\pi}^*)^2 \sigma^2 \Delta t + O(\Delta t^2), \\ B &= (B_1 + B_2)^{\frac{1}{\gamma}} = 1 + r\Delta t + \pi_t^*(\mu - r)\Delta t + \frac{(\gamma-1)}{2} (\pi_t^*)^2 \sigma^2 \Delta t + O(\Delta t^2). \end{aligned}$$

Therefore, taking the limit of $\Delta t \rightarrow 0$ leads to

$$\lim_{\Delta t \rightarrow 0} (B) = 1 = \lim_{\Delta t \rightarrow 0} (A)$$

and

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{B - A}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{(\pi_t^* - \tilde{\pi}^*)(\mu - r)\Delta t + \frac{(\gamma-1)}{2} ((\pi_t^*)^2 - (\tilde{\pi}^*)^2) \sigma^2 \Delta t + O(\Delta t^2)}{\Delta t} \\ &= (\pi_t^* - \tilde{\pi}^*)(\mu - r) + \frac{(\gamma-1)}{2} ((\pi_t^*)^2 - (\tilde{\pi}^*)^2) \sigma^2 = -\frac{(1-\gamma)}{2} \sigma^2 (\pi_t^* - \tilde{\pi}^*)^2. \end{aligned}$$

This then leads to

$$\lim_{\Delta t \rightarrow 0} \frac{\pi_{t+\Delta t}^* - \pi_t^*}{\Delta t} = -\frac{1}{k^*} (1 - k^* \pi_t^*) \frac{(1-\gamma)}{2} \sigma^2 (\pi_t^* - \tilde{\pi}^*)^2.$$

In particular, the limit on the left hand side of this equation exists and equals $d\pi_t^*$. Thus, the optimal portfolio strategy computed by the dynamic programming equations 4.39 converges to the optimal control of the worst-case portfolio problem in continuous time. Figure 4.3 illustrates this convergence of the worst-case optimal portfolio process in discrete time to the worst-case optimal portfolio process in continuous time for decreasing values of Δt .

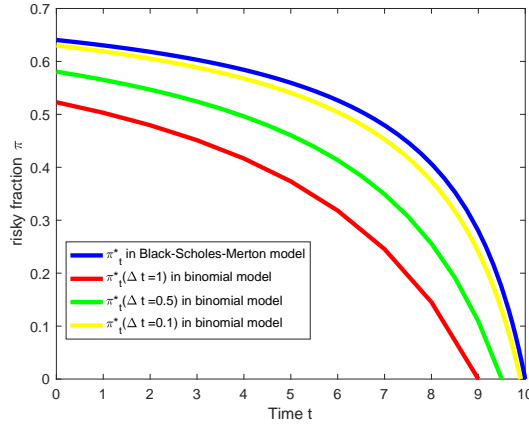


Figure 4.3.: The convergence of the optimal trading strategies π_t^* with crash possibility

Log Utility

In the log utility case,

$$u(x) = \ln(x) ,$$

it can directly be verified that all assumptions of Theorem 4.12 are satisfied that do not depend on the particular choice of the stock price model. The optimal strategy π_t can then be obtained from the dynamic programming equations 4.39 by solving the indifference requirement

$$E(U_{t+1}(x(1+r+\pi_t(R-1-r)))) = E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*))))$$

with

$$\begin{aligned} & E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*))) \mid \mathcal{F}_t) \\ &= \ln(x) + \ln(1+r-\pi_t(r+k^*)) + \sum_{s=t+1}^{T-1} E \ln(1+r+\tilde{\pi}^*(R-1-r)) \end{aligned}$$

and

$$\begin{aligned} & E(U_{t+1}(x(1+r+\pi_t(R-1-r))) \mid \mathcal{F}_t) \\ &= E(\tilde{U}_{t+2}(x(1+r+\pi_t(R-1-r))(1+r-\pi_{t+1}^*(r+k^*))) \mid \mathcal{F}_t) \\ &= \ln x + E \ln(1+r+\pi_t(R-1-r)) + E(\ln(1+r-\pi_{t+1}^*(r+k^*)) \mid \mathcal{F}_t) \\ &\quad + \sum_{s=t+2}^{T-1} E \ln(1+r+\tilde{\pi}^*(R-1-r)) . \end{aligned}$$

Therefore the optimal strategy π_t^* satisfies the following equation:

$$\begin{aligned} & E(\ln(1 + r + \pi_t^*(R - 1 - r))) + E(\ln(1 + r - \pi_{t+1}^*(r + k^*)) \mid \mathcal{F}_t) \\ &= \ln(1 + r - \pi_t^*(r + k^*)) + E(\ln(1 + r + \tilde{\pi}^*(R - 1 - r))) . \end{aligned}$$

Due to $\ln(x)$ being concave and increasing, Assumption (M) yields

$$E \ln(1 + r + \pi_t(R - 1 - r)) > 0$$

in the interval $[0, \tilde{\pi}^*]$. Then, the recursive formula for the optimal strategy is given by

$$\pi_{t+1}^* = \frac{1+r}{r+k^*} - \left(\frac{1+r}{r+k^*} - \pi_t^* \right) * \exp^{E(\ln(1+r+\tilde{\pi}^*(R-1-r))) - E(\ln(1+r+\pi_t^*(R-1-r)))}, \quad \pi_{T-1}^* = 0$$

which is consistent with the result by the indifference approach as shown in previous section.

Exponential Utility

The exponential utility function is given by

$$u(x) = -e^{-\theta x}$$

for some $\theta > 0$.

Compared to the examples of log utility and power utility, the situation for the exponential utility is totally different. First of all, the separation of the term with respect to x and the term with respect to π in the optimal value function is not possible already in the crash-free setting. Therefore, we no longer consider the portfolio strategy π_t to describe the investor's strategy. Instead it will turn out that the amount of money which is invested in the risky stock at time t given by $\pi_t X_t$ is the appropriate term. Further, the exponential utility has a finite slope in $x = 0$. As a consequence, the optimal strategy does no longer automatically ensure the positivity of the corresponding optimal final wealth. On the other hand, this at least does not cause theoretical problems as the maximization problem of the expected terminal wealth is also well-defined in that case, which is not allowed in the previous cases of log utility and power utility.

So let us in the following slightly misuse the notation of π_t (and the corresponding optimal values $\pi_t^*, \tilde{\pi}_t^*$) to now denote the amount of money invested in the risky asset. Then, in the crash-free setting, it can be shown (via induction) that using the corresponding dynamic programming equation, we obtain the value function as

$$\tilde{U}_t(x) = -e^{-\theta x(1+r)^{T-t}} \prod_{s=t}^{T-1} E \left(e^{-\theta \tilde{\pi}_s^*(R-r-1)(1+r)^{T-s-1}} \right) .$$

Here, the values $\tilde{\pi}_s^*$ are determined as the solutions of

$$-E\left(e^{-\theta\tilde{\pi}_s^*(R-r-1)(1+r)^{T-s-1}}\right) = \sup_{\pi \in (-\infty, \infty)} -E\left(e^{-\theta\pi(R-r-1)(1+r)^{T-s-1}}\right).$$

Note that due to the independence of R_t of \mathcal{F}_t and the fact that there is no requirement on the wealth process X_t in the exponential utility case, it is enough to consider the optimization problem for constant values π . Further, due to Assumption (M), the optimal amount of money invested in the stock will be positive at each time s . Even more, in the case of $r = 0$, it is optimal for the crash-free setting to keep the amount of money invested in the risky asset fixed. Gains and losses of stock investment will then always be allocated to the position of the riskless investment.

In principle, the shift from the portfolio process to the process of money invested in the risky asset does not allow a direct application of Theorem 4.12. However, it can be shown that by dropping the requirement of a non-negative wealth process, one can imitate all the steps leading to Theorem 4.12 (compare [24] for the continuous-time case). Thus, the corresponding dynamic programming equations yields the following relation for the optimal amount of money invested in the stock:

$$E(U_{t+1}(x(1+r) + \pi_t^*(R-1-r)) \mid \mathcal{F}_t) = E(\tilde{U}_{t+1}(x(1+r) - \pi_t^*(r+k^*)) \mid \mathcal{F}_t)$$

with

$$\begin{aligned} & E(\tilde{U}_{t+1}(x(1+r) - \pi_t^*(r+k^*)) \mid \mathcal{F}_t) \\ &= -e^{-\theta x(1+r)^{T-t}} e^{\theta(1+r)^{T-t-1}\pi_t^*(r+k^*)} \prod_{s=t+1}^{T-1} E e^{-\theta(1+r)^{T-s-1}\tilde{\pi}_s^*(R-1-r)} \end{aligned}$$

and

$$\begin{aligned} & E(U_{t+1}(x(1+r) + \pi_t^*(R-1-r)) \mid \mathcal{F}_t) \\ &= E(\tilde{U}_{t+2}(x(1+r + \pi_t^*(R-1-r))(1+r - \pi_{t+1}^*(r+k^*))) \mid \mathcal{F}_t) \\ &= -e^{-\theta x(1+r)^{T-t}} E(e^{-\theta(1+r)^{T-t-1}\pi_t^*(R-1-r)} e^{\theta(1+r)^{T-t-2}\pi_{t+1}^*(r+k^*)}) \prod_{s=t+2}^{T-1} E e^{-\theta(1+r)^{T-s-1}\tilde{\pi}_s^*(R-1-r)}. \end{aligned}$$

Therefore we have

$$\begin{aligned} & E(e^{-\theta(1+r)^{T-t-1}\pi_t^*(R-1-r)} e^{\theta(1+r)^{T-t-2}\pi_{t+1}^*(r+k^*)}) \\ &= e^{\theta(1+r)^{T-t-1}\pi_t^*(r+k^*)} E(e^{-\theta(1+r)^{T-t-2}\tilde{\pi}_{t+1}^*(R-1-r)}) \end{aligned}$$

which can be reordered as

$$\frac{e^{\theta(1+r)^{T-t-2}\pi_{t+1}^*(r+k^*)}}{e^{\theta(1+r)^{T-t-1}\pi_t^*(r+k^*)}} = \frac{E(e^{-\theta(1+r)^{T-t-2}\tilde{\pi}_{t+1}^*(R-1-r)})}{E(e^{-\theta(1+r)^{T-t-1}\pi_t^*(R-1-r)})}.$$

This results in a recursive formula for the optimal amount of the money π_t^* :

$$\pi_{t+1}^* = \pi_t^*(1+r) + \frac{1}{\theta(1+r)^{T-t-2}(r+k^*)} \ln\left(\frac{E(e^{-\theta(1+r)^{T-t-2}\tilde{\pi}_{t+1}^*(R-1-r)})}{E(e^{-\theta(1+r)^{T-t-1}\pi_t^*(R-1-r)})}\right)$$

with

$$\pi_{T-1}^* = 0.$$

If we assume that the price process of the stock follows the binomial model, we obtain the crash-free optimal trading strategy (the amount of money invested in the stock) as follows

$$\tilde{\pi}_t^* = \frac{\ln \frac{p(u-1-r)}{(1-p)(1+r-d)}}{\theta(1+r)^{T-t-1}(u-d)}$$

Thus, the recursive Equation for the optimal worst-case strategy is given by

$$\begin{aligned} \pi_{t+1}^* &= \pi_t^*(1+r) + \frac{1}{\theta(1+r)^{T-t-2}(r+k^*)} \\ &\quad * \ln\left(\frac{e^{-\theta(1+r)^{T-t-2}\tilde{\pi}_{t+1}^*(u-1-r)}p + e^{-\theta(1+r)^{T-t-2}\tilde{\pi}_{t+1}^*(d-1-r)}(1-p)}{e^{-\theta(1+r)^{T-t-1}\pi_t^*(u-1-r)}p + e^{-\theta(1+r)^{T-t-1}\pi_t^*(d-1-r)}(1-p)}\right). \end{aligned}$$

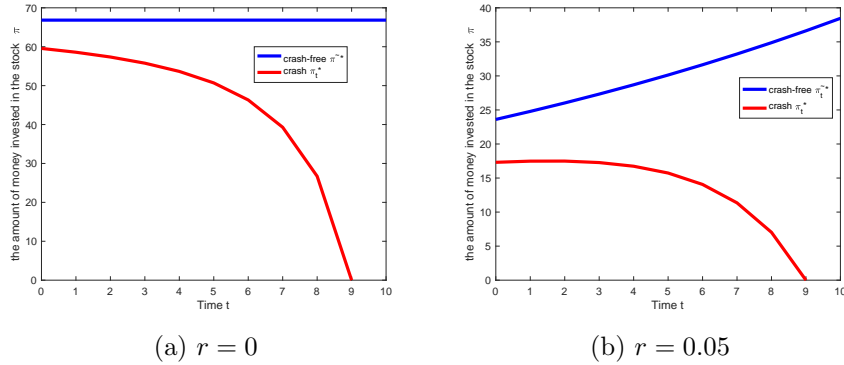


Figure 4.4.: The optimal trading strategies π_t^* with and without crash possibility

The form of the optimal trading strategies are illustrated in Figure 4.4 for the choices of $\theta = 0.01$, $u = 1.4918$, $d = 0.67$, $p = 0.5375$, $k = 0.05$, $T = 10$, $r = 0$ and $r = 0.05$. The curves for $r = 0$ look very similar to the optimal portfolio processes in Figure 4.2. However, note that, we plot here the amount of money invested in the stock. If we would plot the optimal portfolio processes, the curve would be irregular and inversely proportional to the actual wealth processes. The curves for $r = 0.05$ look totally different. The optimal trading strategy π_t^* in the crash setting is no longer decreasing with the

time as the optimal trading strategy $\tilde{\pi}_t^*$ in the crash-free setting is increasing with the time. The optimal worst-case trading strategy π_t^* increases first and then decreases until maturity.

Chapter 5.

Finite-difference approximations

In this chapter we focus on developing the discrete-time HJB equation to solve the portfolio optimization problem and try to exhibit a natural correspondence to the continuous time HJB. The discrete-time HJB can unify the continuous time and discrete time models by regarding the discrete time setting as the finite difference scheme of the continuous time setting.

Before we derive the discrete-time HJB equation, we first review the discrete Itô formula for the simple random walk due to Fujita [17]. Random walks are used as simplified models of Brownian motion, of which the Itô formula is famous for the stochastic calculus and is very useful for the problems in the mathematical finance and stochastic control.

Let W_t be one-dimensional random walk, satisfying $W_0 = 0$ and

$$W_t = Y_1 + Y_2 + \cdots + Y_t,$$

where $\{Y_i\}_{i=1}^\infty$ is an independent Bernoulli sequence such that $P(Y_t = \pm 1) = \frac{1}{2}$.

Theorem 5.1: (see Fujita [17])

For any $f : \mathbb{Z} \rightarrow \mathbb{R}$, we have

$$f(W_{t+1}) - f(W_t) = \frac{f(W_t + 1) - f(W_t - 1)}{2} Y_{t+1} + \frac{f(W_t + 1) - 2f(W_t) + f(W_t - 1)}{2}. \quad (5.1)$$

For any $g : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{R}$, we have

$$\begin{aligned} g(W_{t+1}, t+1) - g(W_t, t) &= \frac{g(W_t + 1, t+1) - g(W_t - 1, t+1)}{2} Y_{t+1} \\ &+ \frac{g(W_t + 1, t+1) - 2g(W_t, t+1) + g(W_t - 1, t+1)}{2} \\ &+ g(W_t, t+1) - g(W_t, t). \end{aligned}$$

With the help of Theorem 5.1, we immediately obtain

Proposition 5.2: For any $f : \mathbb{R} \rightarrow \mathbb{R}$ and any $k \in \mathbb{R}$, let $B_t = kW_t$, then we have

$$f(B_{t+1}) - f(B_t) = \frac{f(B_t + k) - f(B_t - k)}{2} Y_{t+1} + \frac{f(B_t + k) - 2f(B_t) + f(B_t - k)}{2}.$$

For any $f : \mathbb{R} \rightarrow \mathbb{R}$ and any $k, \mu \in \mathbb{R}$, let $B_t = \mu t + kW_t$, then we have

$$\begin{aligned} f(B_{t+1}) - f(B_t) &= \frac{f(B_t + \mu + k) - f(B_t + \mu - k)}{2} Y_{t+1} \\ &+ \frac{f(B_t + \mu + k) - 2f(B_t + \mu) + f(B_t + \mu - k)}{2} + f(B_t + \mu) - f(B_t). \end{aligned}$$

Proof: We shall prove both equations by directly using Equation 5.1. For the first equation, one has

$$\begin{aligned} &f(B_{t+1}) - f(B_t) - \frac{f(B_t + k) - 2f(B_t) + f(B_t - k)}{2} \\ &= f(B_t + k(W_{t+1} - W_t)) - \frac{f(B_t + k) + f(B_t - k)}{2} \\ &= \begin{cases} \frac{f(B_t + k) - f(B_t - k)}{2}, & Y_{t+1} = W_{t+1} - W_t = 1 \\ \frac{f(B_t - k) - f(B_t + k)}{2}, & Y_{t+1} = -1 \end{cases} \\ &= \frac{f(B_t + k) - f(B_t - k)}{2} Y_{t+1}, \end{aligned}$$

so the proof of the first equation is done.

For the second equation, one has

$$\begin{aligned} &f(B_{t+1}) - f(B_t) - \frac{f(B_t + \mu + k) - 2f(B_t + \mu) + f(B_t + \mu - k)}{2} - f(B_t + \mu) + f(B_t) \\ &= f(B_t + \mu + k(W_{t+1} - W_t)) - \frac{f(B_t + \mu + k) + f(B_t + \mu - k)}{2} \\ &= \begin{cases} \frac{f(B_t + \mu + k) - f(B_t + \mu - k)}{2}, & Y_{t+1} = 1 \\ \frac{f(B_t + \mu - k) - f(B_t + \mu + k)}{2}, & Y_{t+1} = -1 \end{cases} \\ &= \frac{f(B_t + \mu + k) - f(B_t + \mu - k)}{2} Y_{t+1} \end{aligned}$$

□

5.1. The finite-difference method for the crash-free model

In this section we will show how to apply the discrete Itô Formula for the simple random walk (Theorem 5.1) to solve the problem of portfolio optimization.

The approximation to Brownian motion by random walks is already proved by Frank

Knight[22]. A sequence of simple, symmetric random walks uniformly converges to Brownian motion on bounded intervals with probability 1.

Here, we assume that the stock price process is governed by the following discrete-time stochastic processes.

$$S_{t+1} - S_t = S_t(\mu + \sigma(W_{t+1} - W_t)) \quad (5.2)$$

with constant market coefficients μ and σ . W_t with $W_0 = 0$ is the one dimension random walk.

By the self-financing property we obtain the wealth equation

$$\begin{aligned} X_{t+1} - X_t &= X_t(r + \pi_t(\mu - r) + \sigma\pi_t(W_{t+1} - W_t)) \\ &= X_t(\mu_t + \sigma_t(W_{t+1} - W_t)) \end{aligned}$$

where π_t are the portfolio processes and $\mu_t = r + \pi_t(\mu - r)$, $\sigma_t = \sigma\pi_t$.

We know that the continuous Itô formula satisfies

$$df(X_t) = (f'(X_t)\mu_t + \frac{1}{2}f''(X_t)(\sigma_t)^2)dt + f'(X_t)\sigma_t dW_t \quad (5.3)$$

where X_t satisfies the Itô process $dX_t = X_t(\mu_t dt + \sigma_t dW_t)$.

We therefore show in Lemma 5.3 that there is a discrete-time analogue.

Lemma 5.3: • For any $f : R \rightarrow R$, we have

$$\begin{aligned} & f(X_{t+1}) - f(X_t) \\ = & \frac{f(X_t + X_t\mu_t + X_t\sigma_t) - f(X_t + X_t\mu_t - X_t\sigma_t)}{2} Y_{t+1} + f(X_t + X_t\mu_t) - f(X_t) \\ + & \frac{f(X_t + X_t\mu_t + X_t\sigma_t) - 2f(X_t + X_t\mu_t) + f(X_t + X_t\mu_t - X_t\sigma_t)}{2} \end{aligned} \quad (5.4)$$

• For any $f : R \times N \rightarrow R$, we have

$$\begin{aligned} & f(X_{t+1}, t+1) - f(X_t, t) \\ = & \frac{f(X_t + X_t\mu_t + X_t\sigma_t, t+1) - f(X_t + X_t\mu_t - X_t\sigma_t, t+1)}{2} Y_{t+1} \\ + & f(X_t + X_t\mu_t, t+1) - f(X_t, t+1) \\ + & \frac{f(X_t + X_t\mu_t + X_t\sigma_t, t+1) - 2f(X_t + X_t\mu_t, t+1) + f(X_t + X_t\mu_t - X_t\sigma_t, t+1)}{2} \\ + & f(X_t, t+1) - f(X_t, t) \end{aligned}$$

Proof: From the lemma above, we can immediately obtain

$$\begin{aligned}
& f(X_{t+1}) - f(X_t) \\
&= f(X_{t+1}) - f(X_t + X_t\mu_t) + f(X_t + X_t\mu_t) - f(X_t) \\
&= f(X_t + X_t(\mu_t + \sigma_t(W_{t+1} - W_t))) - f(X_t + X_t\mu_t) + f(X_t + X_t\mu_t) - f(X_t) \\
&= \frac{f(X_t + X_t\mu_t + X_t\sigma_t) - f(X_t + X_t\mu_t - X_t\sigma_t)}{2} Y_{t+1} \\
&+ \frac{f(X_t + X_t\mu_t + X_t\sigma_t) - 2f(X_t + X_t\mu_t) + f(X_t + X_t\mu_t - X_t\sigma_t)}{2} \\
&+ f(X_t + X_t\mu_t) - f(X_t)
\end{aligned}$$

We then obtain the second equation in the assertion from

$$f(X_{t+1}, t+1) - f(X_t, t) = f(X_{t+1}, t+1) - f(X_t, t+1) + f(X_t, t+1) - f(X_t, t) \quad (5.5)$$

□

From the lemma above we have the discrete Itô formula which satisfies

$$\begin{aligned}
& f(X_{t+1}, t+1) - f(X_t, t) \\
&= \frac{f(X_t + X_t\mu_t + X_t\sigma_t, t+1) - f(X_t + X_t\mu_t - X_t\sigma_t, t+1)}{2} Y_{t+1} \\
&+ f(X_t + X_t\mu_t, t+1) - f(X_t, t+1) \\
&+ \frac{f(X_t + X_t\mu_t + X_t\sigma_t, t+1) - 2f(X_t + X_t\mu_t, t+1) + f(X_t + X_t\sigma_t - X_t\sigma_t, t+1)}{2} \\
&+ f(X_t, t+1) - f(X_t, t)
\end{aligned}$$

The value function of the portfolio optimization 3.8 in the discrete time setting can be computed recursively by the Bellman equation,

$$V(t, x) = \sup_{\pi_t} E(V(t+1, X_{t+1}^\pi)) \quad (5.6)$$

Using the discrete Itô formula above in this Bellman equation we can get

$$\begin{aligned}
V(t, x) &= \sup_{\pi_t} E(V(t+1, X_{t+1}^\pi)) \\
&= \sup_{\pi_t} E\left(V(t, x) + \frac{V(x + x\mu_t + x\sigma_t, t+1) - V(x + x\mu_t - x\sigma_t, t+1)}{2} Y_{t+1} \right. \\
&+ V(x + x\mu_t, t+1) - V(x, t+1) \\
&+ \frac{V(x + x\mu_t + x\sigma_t, t+1) - 2V(x + x\mu_t, t+1) + V(x + x\mu_t - x\sigma_t, t+1)}{2} \\
&+ \left. V(x, t+1) - V(x, t) \right)
\end{aligned}$$

where $X_t = x$.

Because of $E(Y_{t+1}) = 0$ we have that

$$\begin{aligned} 0 &= \sup_{\pi_t} (V(x + x\mu_t, t+1) - V(x, t+1)) \\ &+ \frac{V(x + x\mu_t + x\sigma_t, t+1) - 2V(x + x\mu_t, t+1) + V(x + x\mu_t - x\sigma_t, t+1)}{2} \\ &+ V(x, t+1) - V(x, t) \end{aligned}$$

We define a difference operator ℓ of $V(x, t)$ in discrete time as following:

$$\begin{aligned} &\ell^\pi V(t, x) \\ &= V(x + x\mu_t, t+1) - V(x, t+1) \\ &+ \frac{V(x + x\mu_t + x\sigma_t, t+1) - 2V(x + x\mu_t, t+1) + V(x + x\mu_t - x\sigma_t, t+1)}{2} \\ &+ V(x, t+1) - V(x, t) \end{aligned}$$

Then, the corresponding discrete HJB-equation with respect to π_t has the form

$$\begin{aligned} &\sup_{\pi_t} \ell^\pi V(t, x) \\ &= \sup_{\pi_t} (V(x(1+r+\pi_t(\mu-r)), t+1) - V(x, t+1)) \\ &+ \frac{1}{2} (V(x(1+r+\pi_t(\mu-r)+\sigma\pi_t), t+1) - 2V(x(1+r+\pi_t(\mu-r)), t+1) \\ &+ V(x(1+r+\pi_t(\mu-r)-\sigma\pi_t), t+1)) + V(x, t+1) - V(x, t)) \\ &= 0 \\ &V(T, x) = u(x) \end{aligned}$$

Note how the terms in this equation correspond to the terms in the continuous-time HJB-equation. Thus, we now want to prove the relation between the value function $V(x, t)$ and the discrete HJB-equation, a so-called verification theory.

Theorem 5.4: (*Verification theorem for the solution of the discrete HJB equation*) Let $V(t, x)$ solve the above discrete HJB equation, and

$$J(t, x, \{\pi_s\}_t^{T-1}) = E^{t,x}(u(X_T^\pi)) \quad (5.7)$$

then we have

$$V(t, x) \geq J(t, x, \{\pi_s\}_t^{T-1}) \quad (5.8)$$

for all x and all available strategy π_t . Furthermore, if for all (t, x) there exists a π^* with

$$\pi_s^* \in \arg \sup_{\pi} (\ell^\pi V(s, X_s^*)) \quad (5.9)$$

for all $t \leq s \leq T$, where X_s^* is the controlled process corresponding to π_s^* , then we obtain

$$V(t, x) = J(t, x, \{\pi_s^*\}_t^{T-1}) \quad (5.10)$$

Proof: Since V is a solution of the discrete HJB equation we have

$$\ell^\pi V(t, x) \leq 0 \quad (5.11)$$

Using the discrete Itô formula we have

$$\begin{aligned} & V(T, X_T) - V(t, x) \\ &= \sum_{s=t}^T \ell^\pi V(s, X_s) \\ &+ \sum_{s=t}^T \frac{V(X_s + X_s \mu_s + V_s \sigma_s, s+1) - V(X_s + X_s \mu_s - X_s \sigma_s, s+1)}{2} Y_{s+1} \end{aligned}$$

After taking the expectation we have

$$E^{t,x}(V(T, X_T) - V(t, x)) = \sum_{s=t}^T \ell^\pi V(s, X_s) \leq 0 \quad (5.12)$$

which leads to $V(t, x) \geq E^{t,x}(V(T, X_T)) \geq J(t, x, \{\pi_s\}_t^{T-1})$.

If there exists a π^* with

$$\pi_s^* \in \arg \sup_{\pi} (\ell^\pi V(s, X_s^*)) \quad (5.13)$$

then we obtain equality in

$$E^{t,x}(V(T, X_T) - V(t, x)) = \sum_{s=t}^T \ell^{\pi^*} V(s, X_s^*) = 0 \quad (5.14)$$

and thus the claimed optimality of π^* . \square

In order to illustrate our theory above we present some examples of this discrete HJB-equation.

The power utility

We consider the case of power utility

$$U(x) = \frac{1}{\gamma} x^\gamma, \gamma < 1, \gamma \neq 0$$

In analogy to the continuous-time solution, we try a solution of the form

$$V(t, x) = \frac{1}{\gamma} f(t) x^\gamma$$

Then the discrete-time HJB equation of power utility satisfies

$$\begin{aligned} & \sup_{\pi_t} \ell^\pi V(t, x) \\ &= \sup_{\pi_t} \left(\frac{1}{\gamma} f(t+1) (x(1+r+\pi_t(\mu-r)))^\gamma - \frac{1}{\gamma} f(t+1) x^\gamma \right. \\ &+ \frac{1}{2} \left(\frac{1}{\gamma} f(t+1) (x(1+r+\pi_t(\mu-r)+\sigma\pi_t))^\gamma - 2 \frac{1}{\gamma} f(t+1) (x(1+r+\pi_t(\mu-r)))^\gamma \right. \\ &+ \left. \frac{1}{\gamma} f(t+1) (x(1+r+\pi_t(\mu-r)-\sigma\pi_t))^\gamma \right) + \frac{1}{\gamma} f(t+1) x^\gamma - \frac{1}{\gamma} f(t) x^\gamma \\ &= 0 \\ & V(T, x) = \frac{1}{\gamma} x^\gamma \end{aligned}$$

The first order condition with respect to π_t satisfies

$$\begin{aligned} & f(t+1) (x(1+r+\pi_t(\mu-r)))^{\gamma-1} x(\mu-r) \\ &+ \frac{1}{2} (f(t+1) (x(1+r+\pi_t(\mu-r)+\sigma\pi_t))^{\gamma-1} x(\mu-r+\sigma) \\ &- 2f(t+1) (x(1+r+\pi_t(\mu-r)))^{\gamma-1} x(\mu-r) \\ &+ f(t+1) (x(1+r+\pi_t(\mu-r)-\sigma\pi_t))^{\gamma-1} x(\mu-r-\sigma)) \\ &= 0 \end{aligned}$$

The maximization in the discrete-time HJB equation leads to the candidate

$$\pi_t = \frac{(1+r) \left(\left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{1}{\gamma-1}} - 1 \right)}{\mu-r+\sigma - \left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{1}{\gamma-1}} (\mu-r-\sigma)} \quad (5.15)$$

Inserting π_t of the form 5.15 into the discrete-time HJB equation results in the following equation for $f(t)$

$$\frac{1}{2} f(t+1) \left(\frac{2(1+r)\sigma}{\mu-r+\sigma - \left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{1}{\gamma-1}} (\mu-r-\sigma)} \right)^\gamma \left(\left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{\gamma}{\gamma-1}} + 1 \right) = f(t) \quad (5.16)$$

with final condition $f(T) = 1$.

Explicit solution via recursive manner yields

$$f(t) = \left[\frac{2^{\gamma-1} (1+r)^\gamma \sigma^\gamma}{(\mu-r+\sigma - \left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{1}{\gamma-1}} (\mu-r-\sigma))^\gamma} \left(\left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{\gamma}{\gamma-1}} + 1 \right) \right]^{T-t}$$

From the above form we can infer that $f(t)$ is strictly positive which implies that $V(t, x)$ is strictly concave. Thus we have indeed computed the optimal strategy

$$\pi_t^* = \frac{(1+r)((\frac{\sigma-\mu+r}{\mu-r+\sigma})^{\frac{1}{\gamma-1}} - 1)}{\mu-r+\sigma - (\frac{\sigma-\mu+r}{\mu-r+\sigma})^{\frac{1}{\gamma-1}}(\mu-r-\sigma)}$$

The Log utility

In the case of log utility

$$U(x) = \log x$$

we guess the following form of the value function

$$V(t, x) = \log x + f(t)$$

Then the discrete HJB equation of log utility satisfies

$$\begin{aligned} & \sup_{\pi_t} \ell^\pi V(t, x) \\ &= \sup_{\pi_t} (\log(x(1+r+\pi_t(\mu-r))) + f(t+1) - (\log x + f(t+1))) \\ &+ \frac{1}{2} (\log(x(1+r+\pi_t(\mu-r) + \sigma\pi_t)) + f(t+1) - 2(\log(x(1+r+\pi_t(\mu-r))) \\ &+ f(t+1)) + \log(x(1+r+\pi_t(\mu-r) + \sigma\pi_t)) + f(t+1)) \\ &+ \log(x) + f(t+1) - (\log(x) + f(t))) \\ &= 0 \\ &V(T, x) = \log(x) \end{aligned}$$

The first order condition with respect to π_t satisfies

$$\begin{aligned} & \frac{1}{x(1+r+\pi_t(\mu-r))} x(\mu-r) \\ &+ \frac{1}{2} \left(\frac{1}{x(1+r+\pi_t(\mu-r) + \sigma\pi_t)} x(\mu-r+\sigma) - 2 \frac{1}{x(1+r+\pi_t(\mu-r))} x(\mu-r) \right. \\ &+ \left. \frac{1}{x(1+r+\pi_t(\mu-r) - \sigma\pi_t)} x(\mu-r-\sigma) \right) \\ &= 0 \end{aligned}$$

The maximization yields the following candidate for the optimal control π_t^* if $(\mu-r)^2 < \sigma^2$,

$$\pi_t^* = \frac{(\mu-r)(1+r)}{\sigma^2 - (\mu-r)^2} \quad (5.17)$$

Inserting π_t of the form 5.15 into the discrete-time HJB equation we obtain the following equation for $f(t)$

$$f(t+1) + \log(1+r) + \frac{1}{2} \log\left(\frac{\sigma^2}{\sigma^2 - (\mu - r)^2}\right) = f(t)$$

with final condition $f(T) = 0$.

The explicit solution of $f(t)$ satisfies

$$f(t) = [\log(1+r) + \frac{1}{2} \log\left(\frac{\sigma^2}{\sigma^2 - (\mu - r)^2}\right)](T-t)$$

From the above form we can infer that $f(t)$ is strictly positive, which implies that $V(t, x)$ is strictly concave. Thus we have indeed computed the optimal strategy

$$\pi_t^* = \frac{(\mu - r)(1+r)}{\sigma^2 - (\mu - r)^2} \quad (5.18)$$

5.2. The finite-difference method for worst-case portfolio optimization

Now we focus on how to use this discrete HJB equation to solve the worst-case portfolio optimization in discrete-time setting.

We consider the stock price dynamics which are modeled by

$$\begin{cases} S_{t+1} - S_t = S_t(\mu + \sigma(W_{t+1} - W_t)), & t \neq \tau \\ S_{\tau+1} = (1-k)S_\tau \end{cases}$$

where (τ, k) is crash scenario and W_t with $W_0 = 0$ is the one dimension random walk.

Then the wealth process $X(t)$ follows the dynamics

$$\begin{cases} X_{t+1} - X_t = X_t(r + \pi_t(\mu - r) + \pi_t\sigma(W_{t+1} - W_t)), & t \neq \tau \\ X_{\tau+1} = (1+r - \pi_\tau(r+k))X_\tau \end{cases}$$

We assume that the investor chooses a portfolio process to maximize worst-case expectancy of terminal wealth on the sense of the optimization problem

$$\sup_{\pi \in A(x)} \inf_{\tau \in [t, T], k \in [0, k^*]} E(u(X_T^\pi))$$

Then the value function $V^1(t, x)$ satisfies

$$V^1(t, x) = \sup_{\pi \in A(x)} \inf_{\tau \in [t, T], k \in [0, k^*]} E(u(X_T^\pi))$$

Now we derive the dynamic programming principle for the worst-case portfolio optimization in discrete time.

Proposition 5.5: *If $u(x)$ is strictly increasing, then the optimal portfolio process π^* for the single-period worst-case portfolio optimization equals 0.*

The assertion of the Proposition above is trivial by the Proposition 4.14.

Theorem 5.6: (The dynamic programming principle)

If $U(x)$ is strictly increasing in x , then we have

$$V^1(t, x) = \sup_{\pi \in A(x)} \inf_{\tau \in [t, T-1]} E_{t,x}[V^0(\tau + 1, (1 + r - \pi_\tau(r + k^*))X_\tau)] \quad (5.19)$$

Proof: For the case when at most one crash happens, it is optimal to follow the optimal portfolio process of the crash-free setting after a crash. Then the value function $V_1(t, x)$ equals the optimal expected utility of $V^0(\tau + 1, (1 + r - \pi_\tau(r + k^*))X_\tau)$. As $V_0(t, x)$ is increasing in x , then the worst-case scenario is given by a crash of maximal height k^* when the investor follows a non-negative portfolio process at time τ . Moreover, by the proposition above we have

$$\begin{aligned} & E[V^0(T, (1 + r - \pi_{T-1}(r + k^*))X_{T-1}^\pi)] \\ &= E[V^0(T, (1 + r)X_{T-1}^\pi)] \\ &= E[V^0(T, (1 + r + \pi_{T-1}(\mu - r + \sigma(W_t - W_{t-1})))X_{T-1}^\pi)] \\ &= E[V^0(T, X_T^\pi)] \\ &= E(u(X_T^\pi)) \end{aligned}$$

Thus, the case when no crash happens at all is also included in the right hand side of the equation 5.19. Therefore, the equation

$$V^1(t, x) = \sup_{\pi \in A(x)} \inf_{\tau \in [t, T-1]} E[V^0(\tau + 1, (1 + r - \pi_\tau(r + k^*))X_\tau)]$$

is indeed the value function of the worst-case portfolio optimization problem. \square

For each $(t, x) \in [0, T) \times (0, \infty)$ we define

$$\begin{aligned} M'(t, x) &= \{\pi : \pi \in A, \ell^\pi V^1(t, x) \geq 0\} \\ M''(t, x) &= \{\pi : \pi \in A, V^0(t + 1, x(1 + r - \pi_t(r + k))) - V^1(t, x) \geq 0\} \end{aligned}$$

By the dynamic programming principle and discrete HJB equation we can establish a Bellmann system in the spirit of Korn and Steffensen[28] which satisfies

$$\begin{cases} \min\{\sup_{\pi \in M'} [V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x)], \sup_{\pi \in M''} \ell^\pi V^1(t, x)\} = 0 \\ V^1(T, x) = V^0(T, x) = u(x) \end{cases} \quad (5.20)$$

Therefore, for the case when at most one crash happens, our aim is to show that the solution of the so-called discrete Bellmann system above is indeed optimal in the worst-case portfolio optimization problem in discrete time setting.

Theorem 5.7: *(Verification theorem for the solution of the discrete HJB equation in the worst-case setting) Let $V^1(t, x)$ solve the above discrete Bellmann system (5.20), then we have*

$$V^1(t, x) = \sup_{\pi} \inf_{\tau} E(u(X_T^\pi))$$

for all x , all available strategy π_t and all crash time τ . Furthermore, suppose that for each (t, x) there exists a π^* such that

$$\pi^*(t, x) = \arg \sup_{\pi \in M''(t, x)} \ell^\pi V^1(t, x)$$

and a crash time τ^* such that

$$\tau^*(t, x) = \arg \inf_{s: s \geq t} \{V^0(s, x(1+r-\pi_{s-1}(r+k))) - V^1(s-1, x) \leq 0\}$$

where

$$\begin{aligned} M'(t, x) &= \{\pi : \pi \in A, \ell^\pi V^1(t, x) \geq 0\} \\ M''(t, x) &= \{\pi : \pi \in A, V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x) \geq 0\} \end{aligned}$$

then the strategy π^* is worst-case optimal and the corresponding optimal crash time is τ^* .

Proof: Let $V^1(t, x)$ be the solution of the HJB equation (5.20), then we have

$$\begin{cases} \sup_{\pi \in M'} [V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x)] \geq 0 \\ \sup_{\pi \in M''} \ell^\pi V^1(t, x) \geq 0 \end{cases}$$

for all t .

Application of the discrete Itô formula gives the equality

$$\begin{aligned} & V^1(s+1, X^{\pi, \tau}(s+1)) - V^1(s, X^{\pi, \tau}(s)) \\ &= \ell^\pi V^1(s, X^{\pi, \tau}(s)) + \frac{V(X_s + X_s \mu_s + V_s \sigma_s, s+1) - V(X_s + X_s \mu_s - X_s \sigma_s, s+1)}{2} Y_{s+1} \end{aligned}$$

for all $t \leq s < \tau$. Then we obtain

$$\begin{aligned} & V^1(\tau, X_\tau) - V^1(t, x) \\ &= \sum_{s=t}^{\tau} \ell^\pi V^1(s, X_s) \\ &+ \sum_{s=t}^{\tau} \frac{V(X_s + X_s \mu_s + V_s \sigma_s, s+1) - V(X_s + X_s \mu_s - X_s \sigma_s, s+1)}{2} Y_{s+1} \quad (5.21) \end{aligned}$$

Consider the strategy π^* together with an arbitrary τ . $\pi^* \in M''(t, x)$ implies that

$$V^1(\tau, x) - V^0(\tau+1, x(1+r - \pi_\tau^*(r+k))) \leq 0$$

for the arbitrary τ .

Inserting this inequality to the equation (5.21) yields the inequality

$$\begin{aligned} & V^0(\tau+1, X^{\pi^*, \tau}(\tau)(1+r - \pi_\tau^*(r+k))) \\ &\geq V^1(\tau, X_\tau) \\ &= V^1(t, x) + \sum_{s=t}^{\tau} \ell^{\pi^*} V^1(s, X_s) \\ &+ \sum_{s=t}^{\tau} \frac{V(X_s + X_s \mu_s + V_s \sigma_s, s+1) - V(X_s + X_s \mu_s - X_s \sigma_s, s+1)}{2} Y_{s+1} \end{aligned}$$

Since the strategy π^* is a pointwise maximizer of

$$\sup_{\pi^* \in M''(t, x)} \ell^{\pi^*} V^1(t, x) \geq 0$$

we can obtain that

$$\begin{aligned} & V^0(\tau+1, X^{\pi^*, \tau}(\tau)(1+r - \pi_\tau^*(r+k))) \\ &\geq V^1(\tau, X_\tau) \\ &+ \sum_{s=t}^{\tau} \frac{V(X_s + X_s \mu_s + V_s \sigma_s, s+1) - V(X_s + X_s \mu_s - X_s \sigma_s, s+1)}{2} Y_{s+1} \end{aligned}$$

By taking expectation the second term of the right side of the inequality vanishes, leaving us with the inequality

$$V^1(t, x) \leq E_{t, x} \{V^0(\tau+1, X^{\pi^*, \tau}(\tau)(1+r - \pi_\tau^*(r+k)))\}$$

τ is arbitrary, then we have

$$V^1(t, x) \leq \inf_{\tau} E_{t,x} \{V^0(\tau + 1, X^{\pi^*, \tau}(\tau)(1 + r - \pi_{\tau}^*(r + k)))\}$$

and

$$V^1(t, x) \leq \sup_{\pi} \inf_{\tau} E_{t,x} \{V^0(\tau + 1, X^{\pi, \tau}(\tau)(1 + r - \pi_{\tau}(r + k)))\}$$

Now fix the crash time $\theta = \tau^*$, then we have that

$$\begin{aligned} V^0(s + 1, x(1 + r - \pi_s(r + k))) - V^1(s, x) &> 0, t \leq s \leq \tau^* \\ V^0(\tau^* + 1, x(1 + r - \pi_{\tau^*}^*(r + k))) - V^1(\tau^*, x) &\leq 0 \end{aligned} \quad (5.22)$$

Consider again the equation (5.21) we obtain the following inequality

$$\begin{aligned} &V^0(\tau^* + 1, X^{\pi, \tau^*}(\tau^*)(1 + r - \pi_{\tau^*}^*(r + k))) \\ &\leq V^1(t, x) + \sum_{s=t}^{\tau^*} \ell^{\pi} V^1(s, X_s) \\ &+ \sum_{s=t}^{\tau^*} \frac{V(X_s + X_s \mu_s + V_s \sigma_s, s + 1) - V(X_s + X_s \mu_s - X_s \sigma_s, s + 1)}{2} Y_{s+1} \end{aligned} \quad (5.23)$$

That $V^1(t, x)$ is the solution of the HJB equation (5.20) implies that

$$\min \left\{ \sup_{\pi \in M'} [V^0(t + 1, x(1 + r - \pi_t(r + k))) - V^1(t, x)], \sup_{\pi \in M''} \ell^{\pi} V^1(t, x) \right\} = 0$$

If $\ell^{\pi} V^1(s, X_s) \geq 0$, then $\pi \in M'$ and from (5.22) gives us

$$\sup_{\pi \in M'} [V^0(s + 1, x(1 + r - \pi_t(r + k))) - V^1(s, x)] > 0,$$

for all $t \leq s \leq \tau^*$. Then we have that

$$\sup_{\pi \in M''} \ell^{\pi} V^1(s, x) = 0.$$

This implies

$$\ell^{\pi} V^1(s, x) \leq 0.$$

Therefore, in any case, $\ell^{\pi} V^1(s, x) \leq 0$ for all $t \leq s \leq \tau^*$.

Taking expectation on both sides of (5.23) to obtain that

$$V^1(t, x) \geq E_{t,x} \{V^0(\tau^* + 1, X^{\pi, \tau^*}(\tau^*)(1 + r - \pi_{\tau^*}^*(r + k)))\}$$

and

$$V^1(t, x) \geq \inf_{\tau} E_{t,x} \{V^0(\tau + 1, X^{\pi, \tau}(\tau)(1 + r - \pi_{\tau}(r + k)))\}$$

since π was chosen arbitrarily, this implies

$$V^1(t, x) \geq \sup_{\pi} \inf_{\tau} E_{t,x} \{V^0(\tau + 1, X^{\pi, \tau}(\tau)(1 + r - \pi_{\tau}(r + k)))\}$$

Then we can conclude that

$$V^1(t, x) = \sup_{\pi} \inf_{\tau} E_{t,x} \{V^0(\tau + 1, X^{\pi, \tau}(\tau)(1 + r - \pi_{\tau}(r + k)))\}$$

□

Characterization of the solution

Let us now apply the verification theorem (5.7) to construct the value function and the optimal strategies.

By applying the verification theorem we have to solve

$$\min \left\{ \sup_{\pi \in M'} [V^0(t + 1, x(1 + r - \pi_t(r + k))) - V^1(t, x)], \sup_{\pi \in M''} \ell^{\pi} V^1(t, x) \right\} = 0 \quad (5.24)$$

Let us first consider the inequality

$$\sup_{\pi \in M'} [V^0(t + 1, x(1 + r - \pi_t(r + k))) - V^1(t, x)] \geq 0 \quad (5.25)$$

where

$$M'(t, x) = \{\pi : \pi \in M, \ell^{\pi} V^1(t, x) \geq 0\}$$

Since the utility function U is an increasing function and $(r + k) > 0$ we have that $V^0(t + 1, x(1 + r - \pi_t(r + k)))$ is a decreasing function of π_t , then the supremum of (5.25) is attained for the smallest value of π which satisfies the constraint in M' , i.e.

$$\begin{aligned} & \ell^{\pi} V^1(t, x) \\ &= V(x(1 + r + \pi_t(\mu - r)), t + 1) - V(x, t + 1) \\ &+ \frac{1}{2} (V(x + x(r + \pi_t(\mu - r)) + x\pi_t\sigma, t + 1) - 2V(x + x(r + \pi_t(\mu - r)), t + 1) \\ &+ V(x + x(r + \pi_t(\mu - r)) - x\sigma\pi_t, t + 1)) + V(x, t + 1) - V(x, t) \\ &\geq 0 \end{aligned} \quad (5.26)$$

Note that the value function $V^1(t, x)$ is concave and increasing, then the equation $\ell^{\pi} V^1(t, x)$ is also concave and increasing for π . Then the smallest value of π is attained when (5.26) holds as an equality. Thus we have the supremum in (5.25) when the constraint (5.26) keeps equality. If the left hand side of the inequality (5.25) is equal to zero we have that π_t are determined by the set of equations

$$\begin{cases} V^0(t + 1, x(1 + r - \pi_t(r + k))) - V^1(t, x) = 0 \\ \ell^{\pi} V^1(t, x) = 0 \end{cases}$$

When the left hand side of the inequality (5.25) is strictly positive, by the complementarity of the equation (5.24) in Bellmann system we have that

$$\sup_{\pi \in M''} \ell^\pi V^1(t, x) = 0 \quad (5.27)$$

Ignoring the constraint $\pi \in M''$ we can compute the candidate for an optimal portfolio process from this equation as $\tilde{\pi}$.

If $\tilde{\pi}$ satisfies the condition $V^0(t+1, x(1+r-\tilde{\pi}(r+k))) - V^1(t, x) > 0$, then $\tilde{\pi}$ can be indeed considered the maximizer of (5.27). Otherwise we have

$$V^0(t+1, x(1+r-\tilde{\pi}(r+k))) - V^1(t, x) < 0$$

We know that the value function $V^0(t+1, x(1+r-\pi(r+k)))$ is a decreasing function for π , then $\tilde{\pi} > \hat{\pi}$ from $V^0(t+1, x(1+r-\hat{\pi}(r+k))) = V^1(t, x)$. That the function $\ell^\pi V^1(t, x)$ is an increasing function of π implies that if $\tilde{\pi} \notin M''$ then the supremum of the $\sup_{\pi \in M''} \ell^\pi V^1(t, x) = 0$ is obtained for the $\hat{\pi} < \tilde{\pi}$ which satisfies

$$V^0(t+1, x(1+r-\hat{\pi}(r+k))) - V^1(t, x) = 0$$

and consequently π and V are determined by the set of equations

$$\begin{cases} V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x) = 0 \\ \ell^\pi V^1(t, x) = 0 \end{cases}$$

Then the (t, x) space can be decomposed into the set K on which π^* are determined by

$$\begin{cases} V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x) = 0 \\ \ell^\pi V^1(t, x) = 0 \end{cases}$$

and the set N on which π^* are determined by

$$\begin{cases} V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x) > 0 \\ \sup_{\pi \in M} \ell^\pi V^1(t, x) = 0 \end{cases}$$

In our examples below, we will show how to solve the set of equations and whether the set N is empty.

5.3. Numerical examples

We present some numerical examples of the discrete bellmann system in the worst case portfolio optimization to illustrate our verification theory.

Power utility

We consider at first the case of power utility

$$U(x) = \frac{1}{\gamma} x^\gamma, \gamma < 1, \gamma \neq 0$$

We try a solution of the form

$$V(t, x) = \frac{1}{\gamma} f(t) x^\gamma$$

Note that we must have $f(T) = 1$. The discrete Bellmann system satisfies

$$\begin{cases} \min\{ \sup_{\pi \in M'} [V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x)], \sup_{\pi \in M''} \ell^\pi V^1(t, x) \} = 0 \\ V^1(T, x) = V^0(T, x) = \frac{1}{\gamma} x^\gamma \end{cases} \quad (5.28)$$

We consider at first the set N . With the equation $\sup_{\pi \in M} \ell^\pi V^1(t, x) = 0$ we obtain the optimal portfolio strategy

$$\tilde{\pi} = \frac{(1+r)((\frac{\sigma-\mu+r}{\mu-r+\sigma})^{\frac{1}{\gamma-1}} - 1)}{\mu-r+\sigma - (\frac{\sigma-\mu+r}{\mu-r+\sigma})^{\frac{1}{\gamma-1}}(\mu-r-\sigma)} \quad (5.29)$$

then we can get that $\tilde{\pi} = \pi^{0*}$. If we choose the portfolio strategy as $\tilde{\pi}$, then the worst case for the investor is an immediate crash happens under the assumption of $k > 0$. Therefore we have $V^0(t+1, x(1+r-\tilde{\pi}(r+k))) \leq V^1(t, x)$. Thus in the case of $k > 0$ we can conclude that the set N is empty.

Now we can only focus on the set K . From the first equation $V^1(t, x) = V^0(t+1, x(1+r-\pi_t(r+k)))$ we can get that

$$\pi_t^* = \frac{1}{r+k} (1+r - (\frac{f^1(t)}{f^0(t+1)})^{\frac{1}{\gamma}}) \quad (5.30)$$

The discrete Itô formula $\ell^\pi V^1(t, x) = 0$ leads to

$$\frac{f^1(t)}{f^1(t+1)} = \frac{(1+r+\pi_t^*(\mu-r+\sigma))^\gamma + (1+r+\pi_t^*(\mu-r-\sigma))^\gamma}{2} \quad (5.31)$$

with the final condition $f(T) = 1$.

Using this equation (5.31) we can obtain the rekursive difference equation from the

(5.30)

$$\begin{aligned}
 & \pi_{t+1}^* - \pi_t^* \\
 = & \frac{1}{r+k} \left(1+r - \frac{f^1(t+1)^{\frac{1}{\gamma}}}{f^0(t+2)} \right) - \frac{1}{r+k} \left(1+r - \frac{f^1(t)^{\frac{1}{\gamma}}}{f^0(t+1)} \right) \\
 = & \frac{1}{r+k} \frac{f^1(t)^{\frac{1}{\gamma}}}{f^0(t+1)} \left(1 - \left(\frac{f^1(t+1)}{f^1(t)} \frac{f^0(t+1)}{f^0(t+2)} \right)^{\frac{1}{\gamma}} \right) \\
 = & \frac{1}{r+k} (1+r - (r+k)\pi_t^*) \left(1 - \left\{ \frac{(1+r+\pi_t^0(\mu-r+\sigma))^\gamma + (1+r+\pi_t^0(\mu-r-\sigma))^\gamma}{(1+r+\pi_t^*(\mu-r+\sigma))^\gamma + (1+r+\pi_t^*(\mu-r-\sigma))^\gamma} \right\}^{\frac{1}{\gamma}} \right)
 \end{aligned} \tag{5.32}$$

Then we have that

$$\pi_{t+1}^* = \frac{1+r}{r+k} - \left(\frac{1+r}{r+k} - \pi_t^* \right) \left(\frac{(1+r+\pi_t^{0*}(\mu-r+\sigma))^\gamma + (1+r+\pi_t^{0*}(\mu-r-\sigma))^\gamma}{(1+r+\pi_t^*(\mu-r+\sigma))^\gamma + (1+r+\pi_t^*(\mu-r-\sigma))^\gamma} \right)^{\frac{1}{\gamma}}$$

We can show that there exists a solution π_t^* which is bounded by 0 from below and by π^{0*} from above as shown in the example 3.17.

From the equation 5.31 we obtain the following solution for $f^1(t)$

$$f^1(t) = \left[\frac{(1+r+\pi_t^*(\mu-r+\sigma))^\gamma + (1+r+\pi_t^*(\mu-r-\sigma))^\gamma}{2} \right]^{T-t}$$

$f^1(t)$ is always positive for $0 \leq \pi^* \leq \pi^{0*}$ which implies that $V^1(t, x)$ of the above form is concave function in x , as desired. Thus, we have indeed computed the optimal portfolio process which satisfy

$$\pi_{t+1}^* = \frac{1+r}{r+k} - \left(\frac{1+r}{r+k} - \pi_t^* \right) \left(\frac{(1+r+\pi_t^{0*}(\mu-r+\sigma))^\gamma + (1+r+\pi_t^{0*}(\mu-r-\sigma))^\gamma}{(1+r+\pi_t^*(\mu-r+\sigma))^\gamma + (1+r+\pi_t^*(\mu-r-\sigma))^\gamma} \right)^{\frac{1}{\gamma}}$$

We know that $\pi_{T-1}^* = 0$ and the optimal portfolio strategy in crash-free model

$$\pi^{0*} = \frac{(1+r) \left(\left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{1}{\gamma-1}} - 1 \right)}{\mu-r+\sigma - \left(\frac{\sigma-\mu+r}{\mu-r+\sigma} \right)^{\frac{1}{\gamma-1}} (\mu-r-\sigma)} \tag{5.33}$$

Using the difference equation and the final condition $\pi_{T-1}^* = 0$ we can compute the optimal strategies π_t^* for $t = T-1, T-2, \dots, 1$. Now we consider the following parameters throughout this section:

$\mu = 0.11, r = 0.05, \sigma = 0.4, k = 0.05, \gamma = 0.5$ The form of the optimal trading strategies with and without crash possibility are illustrated in Figure 5.1. Note that the worst-case optimal portfolio process π^* is a nonconstant process which decreases with decreasing time to maturity $T-t$. That means the investor reduces the number of shares of stock as he approaches the investment horizon in order to protect against losses due to a crash.

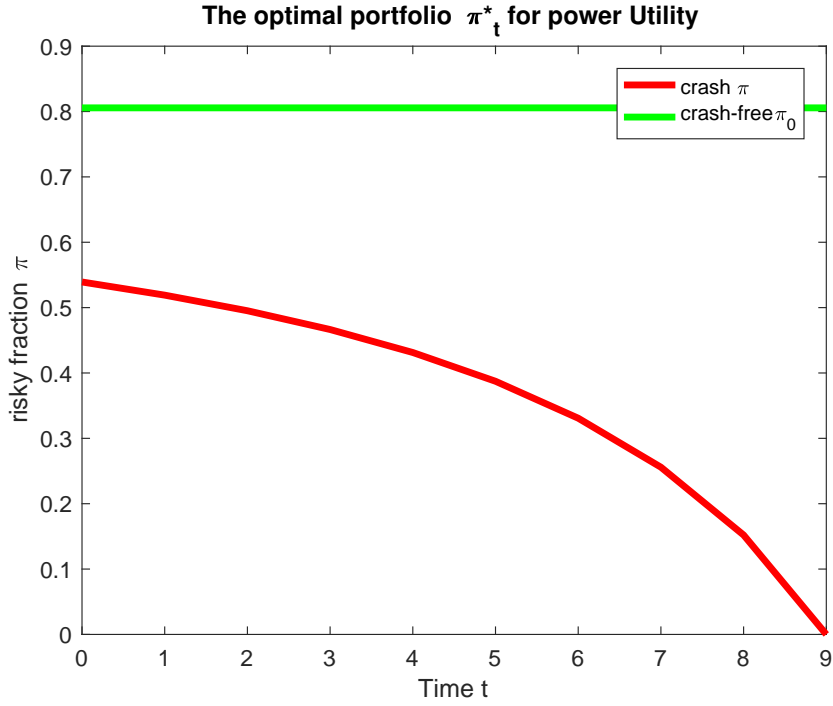


Figure 5.1.: The optimal trading strategies π_t^* for power utility function

Only in last single period $T - 1$ he reduces his fraction of risky investments to zero.

Log utility

Now let us take a look at the the case of logarithmic utility function

$$U(x) = \log x$$

we guesss the following form of the value function

$$V(t, x) = \log x + f(t)$$

Note that we must have $f(T) = 0$.

The discrete Bellmann system satisfies

$$\begin{cases} \min\{ \sup_{\pi \in M'} [V^0(t+1, x(1+r-\pi_t(r+k))) - V^1(t, x)], \sup_{\pi \in M''} \ell^\pi V^1(t, x) \} = 0 \\ V^1(T, x) = V^0(T, x) = \log x \end{cases} \quad (5.34)$$

Under the assumption of $k > 0$ the set of N is empty as in the case of power utility we have already argued. Now we consider the set K . From the first equation $V^1(t, x) =$

$V^0(t+1, x(1+r-\pi_t(r+k)))$ we can get that

$$\pi_t^* = \frac{1}{r+k}(1+r-(\exp(f^1(t)-f^0(t+1)))) \quad (5.35)$$

The discrete Itô formula $\ell^\pi V^1(t, x) = 0$ leads to

$$f^1(t+1) - f^1(t) = -\frac{1}{2}(\log(1+r+\pi_t^*(\mu-r+\sigma)) + \log(1+r+\pi_t^*(\mu-r-\sigma))) \quad (5.36)$$

Inserting the equation (5.36) into the equation (5.35) leads to the equation:

$$\begin{aligned} & \pi_{t+1}^* - \pi_t^* \\ &= \frac{1}{r+k}(\exp(f^1(t)-f^0(t+1)) - \exp(f^1(t+1)-f^0(t+2))) \\ &= \frac{1}{r+k}(1+r-(r+k)\pi_t^*)(1 - (\frac{(1+r+\pi^{0*}(\mu-r+\sigma))(1+r+\pi^{0*}(\mu-r-\sigma))}{(1+r+\pi_t^*(\mu-r+\sigma))(1+r+\pi_t^*(\mu-r-\sigma))})^{\frac{1}{2}}) \end{aligned} \quad (5.37)$$

Then we have that

$$\pi_{t+1}^* = \frac{1+r}{r+k} - (\frac{1+r}{r+k} - \pi_t^*)(\frac{(1+r+\pi^{0*}(\mu-r+\sigma))(1+r+\pi^{0*}(\mu-r-\sigma))}{(1+r+\pi_t^*(\mu-r+\sigma))(1+r+\pi_t^*(\mu-r-\sigma))})^{\frac{1}{2}}$$

As shown in example 4.9 it has a solution π_t^* which satisfies $0 \leq \pi_t^* \leq \pi^{0*}$.

The equation 5.36 leads to a solution for $f^1(t)$

$$f^1(t) = \frac{1}{2}(\log(1+r+\pi_t^*(\mu-r+\sigma)) + \log(1+r+\pi_t^*(\mu-r-\sigma)))(T-t)$$

We infer that $f^1(t)$ is always positive which implies that the value function is a concave function. Thus the optimal portfolio process π_t^* indeed satisfies the following equation

$$\pi_{t+1}^* = \frac{1+r}{r+k} - (\frac{1+r}{r+k} - \pi_t^*)(\frac{(1+r+\pi^{0*}(\mu-r+\sigma))(1+r+\pi^{0*}(\mu-r-\sigma))}{(1+r+\pi_t^*(\mu-r+\sigma))(1+r+\pi_t^*(\mu-r-\sigma))})^{\frac{1}{2}}$$

We already get the optimal portfolio strategy in crash-free model

$$\pi^{0*} = \frac{(\mu-r)(1+r)}{\sigma^2 - (\mu-r)^2} \quad (5.38)$$

Using the difference equation and the final condition $\pi_T^* = 0$ we can compute the optimal strategies π_t^* for $t = T-1, T-2, \dots, 1$. Now we consider the same parameters in our case: $\mu = 0.11$, $r = 0.05$, $\sigma = 0.4$, $k = 0.05$, The form of the optimal trading strategies with and without crash possibility are illustrated in Figure 5.2.

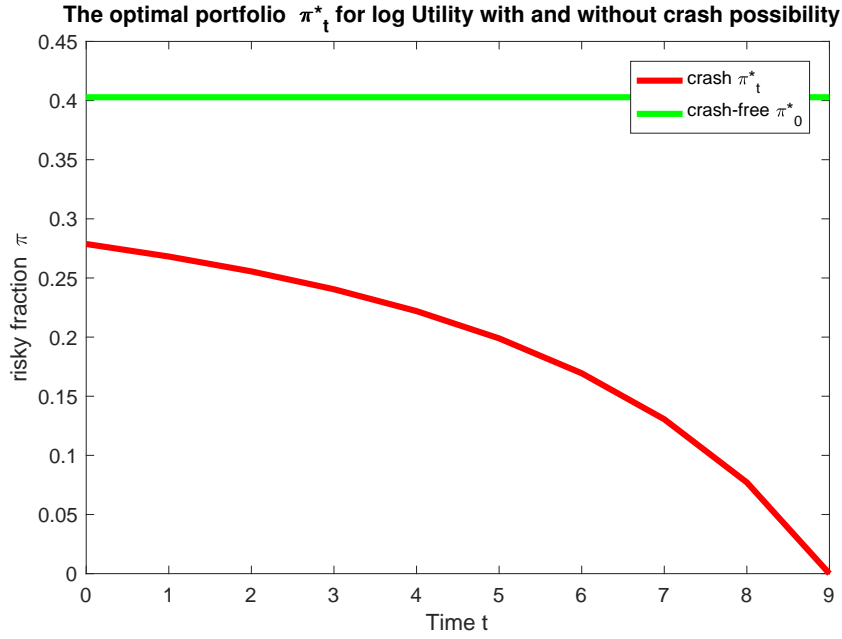


Figure 5.2.: The optimal trading strategies π_t^* for log utility function

5.4. Convergence theory

Here we wish to show that the value function obtained from the worst-case HJB equation in discrete time (the finite-difference scheme) above converges to the value function from the continuous time HJB equation in the worst-case setting when the time step $h \rightarrow 0$. Let $h > 0$ be the time step, for all $s \in [t, T]$, we have $s = t + jh$, $j = 0, 1, \dots, n$. Then the prices of a riskless bond and one risky security satisfy

$$\begin{cases} B_{t+h} - B_t = rhB_t, & B_0 = 1 \\ S_{t+h} - S_t = S_t(\mu h + \sigma\sqrt{h}(W_{t+h} - W_t)), & S_0 = s_0 \end{cases}$$

where W_t with $W_0 = 0$ is the one dimension random walk.

And in a crash scenario (τ, k) we have

$$S_{\tau+h} = (1 - kh)S_\tau.$$

The dynamics of the wealth process is given by

$$\begin{aligned} X_{t+h} - X_t &= X_t(\mu_t h + \sigma_t \sqrt{h}(W_{t+h} - W_t)) \quad \text{for } t \in [0, T] \text{ and } t \neq \tau \\ X_{\tau+h} - X_\tau &= (rh - \pi_\tau(rh + kh))X_\tau \end{aligned}$$

where $X_0 = x > 0$ denotes the initial wealth and $\mu_t = r + \pi_t(\mu - r)$, $\sigma_t = \sigma\pi_t$.

We denote the value function under the crash model in the discrete time as $V_h^1(t, x)$, and the value function under the crash-free model in the discrete time as $V_h^0(t, x)$ which satisfies

$$V_h^0(t, x) = \sup_{\pi_t} E\{V_h^0(t+h, X^\pi(t+h))\}.$$

Then the discrete differential operator ℓ of the value function $V_h^0(t, x)$ is

$$\begin{aligned} & \ell^\pi V_h^0(t, x) \\ = & V_h^0(x + x\mu_t h, t+h) - V_h^0(x, t+h) \\ & + \frac{V_h^0(x + x\mu_t h + x\sigma_t\sqrt{h}, t+h) - 2V_h^0(x + x\mu_t h, t+h) + V_h^0(x + x\mu_t h - x\sigma_t\sqrt{h}, t+h)}{2} \\ & + V_h^0(x, t+h) - V_h^0(x, t) \end{aligned} \quad (5.39)$$

where X_t satisfies

$$X_{t+h} - X_t = X_t(\mu_t h + \sigma_t\sqrt{h}(W_{t+h} - W_t)) \quad (5.40)$$

Under the time step h we have

$$\begin{aligned} & f(X_{t+h}) - f(X_t) \\ = & \frac{f(X_t + X_t\mu_t h + X_t\sigma_t\sqrt{h}) - f(X_t + X_t\mu_t h - X_t\sigma_t\sqrt{h})}{2} Y_{t+h} \\ & + f(X_t + X_t\mu_t h) - f(X_t) \\ & + \frac{f(X_t + X_t\mu_t h + X_t\sigma_t\sqrt{h}) - 2f(X_t + X_t\mu_t h) + f(X_t + X_t\mu_t h - X_t\sigma_t\sqrt{h})}{2} \end{aligned} \quad (5.41)$$

Now we introduce the following notations. For any function $g(t, x)$, let

$$\Delta_x g = \frac{g(t, x+h) - g(t, x)}{h} \quad (5.42)$$

$$\Delta_x^2 g = \frac{g(t, x+h) - 2g(t, x) + g(t, x-h)}{h^2} \quad (5.43)$$

$$\Delta_t g = \frac{g(t+h, x) - g(t, x)}{h}$$

These are respectively the forward first order difference quotient in x , the second order difference quotient in x and the first order difference quotient in time t .

We have the well known Itô formula in continuous time

$$df(X_t) = (f'(X_t)\mu_t + \frac{1}{2}f''(X_t)(\sigma_t)^2)dt + f'(X_t)\sigma_t dW_t \quad (5.44)$$

where W_t denotes a standard Brownian Motion.

Then the equation (5.41) can be considered as a finite-difference scheme of the Equation (5.44).

Fujita and Kawanishi[18] have already proved the Itô formula for Brownian motion

$$f(w_t) - f(w_0) = \int_0^t f'(w_s)dw_s + \frac{1}{2} \int_0^t f''(w_s)ds$$

using the discrete Itô formula

$$f(Z_{t+1}) - f(Z_t) = \frac{f(Z_t + 1) - f(Z_t - 1)}{2}(Z_{t+1} - Z_t) + \frac{f(Z_t + 1) - 2f(Z_t) + f(Z_t - 1))}{2}$$

by taking an approximation to Brownian motion by random walks. Then we can prove similarly that the discrete Itô formula (5.41) converges to the Itô formula in continuous time (5.44), when we take a suitable approximation to Brownian motion by random walks. Now we want to prove that as $h \rightarrow 0$,

$$\frac{\ell^\pi V(t, x)}{h} \rightarrow L^\pi V(t, x) \quad (5.45)$$

where

$$L^\pi V(t, x) = V_t(t, x) + V_x(t, x)(r + \pi(\mu - r))x + \frac{1}{2}V_{xx}(t, x)\pi^2\sigma^2x^2$$

Proof:

$$\begin{aligned} & \frac{\ell^\pi V(t, x)}{h} - L^\pi V(t, x) \\ = & \frac{V(x, t+h) - V(x, t)}{h} + \frac{V(x + \mu_t h, t+h) - V(x, t+h)}{h} \\ & + \frac{V(x + \mu_t h + \sigma_t \sqrt{h}, t+h) - 2V(x + \mu_t h, t+h) + V(x + \mu_t h - x\sigma_t \sqrt{h}, t+h)}{2h} \\ & - (V_t(t, x) + V_x(t, x)\mu_t + \frac{1}{2}V_{xx}(t, x)(\sigma_t)^2) \\ = & \left(\frac{V(x, t+h) - V(x, t)}{h} - V_t(t, x) \right) + \left(\frac{V(x + \mu_t h, t+h) - V(x, t+h)}{h} - V_x(t, x)\mu_t \right) \\ & + \left(\frac{V(x + \mu_t h + \sigma_t \sqrt{h}, t+h) - 2V(x + \mu_t h, t+h) + V(x + \mu_t h - x\sigma_t \sqrt{h}, t+h)}{2h} \right. \\ & \left. - \frac{1}{2}V_{xx}(t, x)(\sigma_t)^2 \right) \end{aligned}$$

where $\mu_t = (r + \pi(\mu - r))x$ and $\sigma_t = \pi\sigma x$

Because $V(x, t)$ is differentiable at t , we get by the definition of the derivative that

$$\lim_{h \rightarrow 0} \left(\frac{V(x, t+h) - V(x, t)}{h} \right) = V_t(t, x)$$

and we assumed that $|\mu_t| \leq K$ for some constant $K > 0$, then $\mu_t h \rightarrow 0$ as $h \rightarrow 0$, then we obtain

$$\lim_{h \rightarrow 0} \left(\frac{V(x + \mu_t h, t + h) - V(x, t + h)}{\mu_t h} \right) = V_x(t, x)$$

thus

$$\lim_{h \rightarrow 0} \left(\frac{V(x + \mu_t h, t + h) - V(x, t + h)}{h} \right) = V_x(t, x) \mu_t$$

Now we use the Taylor expansions up to second order to achieve that

$$\begin{aligned} & V(x + \mu_t h + \sigma_t \sqrt{h}, t + h) \\ = & V(x + \mu_t h, t + h) + \sigma_t \sqrt{h} V_x(x + \mu_t h, t + h) + (\sigma_t \sqrt{h})^2 \frac{1}{2} V_{xx}(\varepsilon, t + h) \\ & V(x + \mu_t h - \sigma_t \sqrt{h}, t + h) \\ = & V(x + \mu_t h, t + h) - \sigma_t \sqrt{h} V_x(x + \mu_t h, t + h) + (\sigma_t \sqrt{h})^2 \frac{1}{2} V_{xx}(\varepsilon', t + h) \end{aligned}$$

with $x + \mu_t h \leq \varepsilon \leq x + \mu_t h + \sigma_t \sqrt{h}$ and $x + \mu_t h \leq \varepsilon' \leq x + \mu_t h + \sigma_t \sqrt{h}$.

The addition of the two equations above gives us

$$\begin{aligned} & \frac{V(x + \mu_t h + \sigma_t \sqrt{h}, t + h) - 2V(x + \mu_t h, t + h) + V(x + \mu_t h - \sigma_t \sqrt{h}, t + h)}{2h} \\ = & \frac{1}{4} V_{xx}(\varepsilon, t + h) (\sigma_t)^2 + \frac{1}{4} V_{xx}(\varepsilon', t + h) (\sigma_t)^2 \end{aligned}$$

When $h \rightarrow 0$, $\varepsilon \rightarrow x$ and $\varepsilon' \rightarrow x$. As $V(x, t)$ is twice differentiable, we obtain that

$$\begin{aligned} & \lim_{h \rightarrow 0} \left(\frac{V(x + \mu_t h + \sigma_t \sqrt{h}, t + h) - 2V(x + \mu_t h, t + h) + V(x + \mu_t h - \sigma_t \sqrt{h}, t + h)}{2h} \right) \\ = & \frac{1}{2} V_{xx}(x, t) (\sigma_t)^2 \end{aligned}$$

thus we get the result

$$\lim_{h \rightarrow 0} \left(\frac{\ell^\pi V(t, x)}{h} \right) = L^\pi V(t, x)$$

□

Here we will show at first that the value function $V_h^0(t, x)$ obtained from the discrete HJB equation

$$\begin{cases} \sup \ell^\pi V_h^0(t, x) = 0 \\ V_h^0(\pi_t, x) = u(x) \end{cases} \quad (5.46)$$

converges to the value function $V^0(t, x)$ from the continuous time HJB equation

$$\begin{cases} \sup_{\pi_t} L^\pi V^0(t, x) = 0 \\ V^0(T, x) = u(x) \end{cases} \quad (5.47)$$

when the time step $h \rightarrow 0$.

Before we prove the convergence theory, we need the following definition of viscosity sub- and super solutions.[14]

Definition 5.8: Let $O \in R$ be an open set, $Q = [t_0, t_1) \times O$, $w \in C(\bar{Q})$, and let $F : [0, T) \times O \times R \times R \times R \times R \rightarrow R$ be a continuous function which satisfying

$$X \leq X' \Rightarrow F(t, x, r, q, p, X) \geq F(t, x, r, q, p, X')$$

and

$$q \leq q' \Rightarrow F(t, x, r, q, p, X) \geq F(t, x, r, q', p, X)$$

Let $w : [0, T) \times O \rightarrow R$ be continuous. We can consider the equation

$$F(t, x, w, D_t w, D_x w, D_x^2 w) = 0 \quad (5.48)$$

1. w is called a viscosity subsolution of the equation (5.48), if for each $(t, x) \in [0, T) \times O$ and all $\varphi \in C^{1,2}([0, T) \times \bar{O})$ with $\varphi \geq w$ satisfying $\varphi(t, x) = w(t, x)$, we have

$$F(t, x, \varphi, D_t \varphi, D_x \varphi, D_x^2 \varphi) \leq 0$$

2. w is called a viscosity supersolution of the Equation (5.48), if for each $(t, x) \in [0, T) \times O$ and all $\varphi \in C^{1,2}([0, T) \times \bar{O})$ with $\varphi \leq w$ satisfying $\varphi(t, x) = w(t, x)$, we have

$$F(t, x, \varphi, D_t \varphi, D_x \varphi, D_x^2 \varphi) \geq 0$$

3. w is called a viscosity solution of the Equation (5.48), if it is both a viscosity subsolution and a viscosity supersolution.

Here in our case we can set that

$$F(t, x, \varphi, D_t \varphi, D_x \varphi, D_x^2 \varphi) = - \sup_{\pi_t} L^\pi \varphi(t, x)$$

Then we have that when $D_x^2\varphi \geq D_x^2\varphi'$,

$$\begin{aligned}
 & F(t, x, \varphi, D_t\varphi, D_x\varphi, D_x^2\varphi) \\
 = & -\sup_{\pi_t} L^\pi \varphi(t, x) \\
 = & -\sup_{\pi_t} (D_t\varphi(t, x) + D_x\varphi(t, x)(r + \pi(\mu - r))x + \frac{1}{2}D_x^2\varphi(t, x)\pi^2\sigma^2x^2) \\
 \leq & -\sup_{\pi_t} (D_t\varphi(t, x) + D_x\varphi(t, x)(r + \pi(\mu - r))x + \frac{1}{2}D_x^2\varphi'(t, x)\pi^2\sigma^2x^2) \\
 = & F(t, x, \varphi, D_t\varphi, D_x\varphi, D_x^2\varphi')
 \end{aligned}$$

and when $D_t\varphi \geq D_t\varphi'$,

$$\begin{aligned}
 & F(t, x, \varphi, D_t\varphi, D_x\varphi, D_x^2\varphi) \\
 = & -\sup_{\pi_t} L^\pi \varphi(t, x) \\
 = & -\sup_{\pi_t} (D_t\varphi(t, x) + D_x\varphi(t, x)(r + \pi(\mu - r))x + \frac{1}{2}D_x^2\varphi(t, x)\pi^2\sigma^2x^2) \\
 \leq & -\sup_{\pi_t} (D_t\varphi'(t, x) + D_x\varphi(t, x)(r + \pi(\mu - r))x + \frac{1}{2}D_x^2\varphi(t, x)\pi^2\sigma^2x^2) \\
 = & F(t, x, \varphi, D_t\varphi', D_x\varphi, D_x^2\varphi')
 \end{aligned}$$

Now we define

$$(V^0)^*(t, x) = \limsup_{(s, y) \rightarrow (t, x), h \downarrow 0} V_h^0(s, y) \quad (5.49)$$

$$(V^0)_*(t, x) = \liminf_{(s, y) \rightarrow (t, x), h \downarrow 0} V_h^0(s, y) \quad (5.50)$$

The basic proof of the viscosity subsolution and viscosity supersolution is shown in Fleming and Soner[14]. In our financial market we can show it for our discrete-time HJB equation as follows.

Lemma 5.9: $(V^0)^*$ is a viscosity subsolution of the continuous-time HJB equation, and $(V^0)_*$ is a viscosity supersolution.

Proof: Let $(t_0, x_0) \in [0, T] \times O$, O is a open set with $O \in R$ and $w \in C^{1,2}([0, T] \times \bar{O})$ (note that $\bar{O} = O \cup \partial O$) with $w \geq (V^0)^*$ be a function satisfying $w(t_0, x_0) = (V^0)^*(t_0, x_0)$. To show that $(V^0)^*$ is a viscosity subsolution of the continuous-time HJB system (5.47), we want to show that

$$-\sup_{\pi_t} L^\pi w(t_0, x_0) \leq 0$$

From $w(t_0, x_0) = (V^0)^*(t_0, x_0)$ and $w \geq (V^0)^*$, we can get that (t_0, x_0) is a maximizer of $(V^0)^* - w$.

Now we choose a sequence h with $h \rightarrow 0$, then for every h , $(V^0)_h - w$ has a maximum at (t_h, x_h) . We want to show that $(t_h, x_h) \rightarrow (t_0, x_0)$ when $h \rightarrow 0$.

Since $(t_h, x_h) \in [0, T] \times O$ which is compact, the sequence (t_h, x_h) has limit points (\tilde{t}, \tilde{x}) . Then, we want to show that $(\tilde{t}, \tilde{x}) = (t_0, x_0)$.

Indeed pick $h \rightarrow 0$ and $(s_h, y_h) \rightarrow (t_0, x_0)$ satisfying

$$(V^0)^*(t_0, x_0) - w(t_0, x_0) = \lim_{h \rightarrow 0} [V_h^0(s_h, y_h) - w(s_h, y_h)]$$

By the definition and the maximum properties, we have that

$$\begin{aligned} (V^0)^*(t_0, x_0) - w(t_0, x_0) &= \lim_{h \rightarrow 0} [V_h^0(s_h, y_h) - w(s_h, y_h)] \\ &\leq \lim_{h \rightarrow 0} [V_h^0(t_h, x_h) - w(t_h, x_h)] \\ &= (V^0)^*(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x}) \end{aligned}$$

If we assume that the maximum is strict, then we have that $(\tilde{t}, \tilde{x}) = (t_0, x_0)$. Therefore $V_h^0 - w$ has a maximum at (t_h, x_h) which tends to (t_0, x_0) when $h \rightarrow 0$. This directly implies

$$\begin{aligned} V_h^0(t_h, x_h) - w(t_h, x_h) &\geq V_h^0(t_h + h, X_{t_h+h}) - w(t_h + h, X_{t_h+h}) \\ w(t_h + h, X_{t_h+h}) - w(t_h, x_h) &\geq V_h^0(t_h + h, X_{t_h+h}) - V_h^0(t_h, x_h) \end{aligned} \quad (5.51)$$

then we can take the supremum and the expectation on the both sides of the Equation (5.51) and obtain that

$$\sup_{\pi} E_{t_h, x_h} [w(t_h + h, X_{t_h+h}) - w(t_h, x_h)] \geq \sup_{\pi} E_{t_h, x_h} [V_h^0(t_h + h, X_{t_h+h})] - V_h^0(t_h, x_h)$$

By using the dynamic programming principle in discrete time model 3.14, we have that

$$\sup_{\pi} E_{t_h, x_h} [V_h^0(t_h + h, X_{t_h+h})] = V_h^0(t_h, x_h)$$

then

$$\sup_{\pi} E_{t_h, x_h} [w(t_h + h, X_{t_h+h}) - w(t_h, x_h)] \geq 0$$

and the application of the discrete Itô formula gives the equality

$$\begin{aligned} &w(t_h + h, X_{t_h+h}) - w(t_h, x_h) \\ &= \ell^\pi w(t_h, x_h) \\ &+ \frac{w(X_h + X_h \mu_h + w_h \sigma_h, t_h + h) - w(X_h + X_h \mu_h - X_h \sigma_h, t_h + h)}{2} (W(t_h + h) - W(t_h)) \end{aligned}$$

Since $E(W(t_h + h) - W(t_h)) = 0$, we have that

$$\sup_{\pi} E_{t_h, x_h} [w(t_h + h, X_{t_h+h}) - w(t_h, x_h)] = \sup_{\pi} \ell^\pi w(t_h, x_h) \geq 0$$

We divide by h and let $h \downarrow 0$, by (5.45)

$$-\sup_{\pi} L^{\pi} w(t_0, x_0) \leq 0$$

Thus, $(V^0)^*$ is a viscosity subsolution of the continuous-time HJB equation.

Similarly, $(V^0)_*$ is a viscosity supersolution of the continuous-time HJB equation. \square

Using the lemma above we can prove the following theorem:

Theorem 5.10: *The value function $V_h^0(t, x)$ obtained from this discrete HJB system (5.46) converges to the value function $V^0(t, x)$ from the continuous time HJB equation (5.47) when the time step $h \rightarrow 0$, that is*

$$\lim_{(s,y) \rightarrow (t,x), h \downarrow 0} V_h^0(s, y) = V^0(t, x)$$

Proof: The lemma above implies that $(V^0)^*$ is a viscosity subsolution and $(V^0)_*$ is a viscosity supersolution respectively, hence we have that

$$(V^0)^* \leq V^0$$

similarly, $(V^0)_* \geq V^0$. By a comparison we have

$$(V^0)^* \leq V^0 \leq (V^0)_*$$

Since by construction we have

$$\begin{aligned} (V^0)^*(t, x) &= \limsup_{(s,y) \rightarrow (t,x), h \downarrow 0} V_h^0(s, y) \\ &\geq \liminf_{(s,y) \rightarrow (t,x), h \downarrow 0} V_h^0(s, y) \\ &= (V^0)_*(t, x) \end{aligned}$$

therefore we get

$$(V^0)^* = V^0 = (V^0)_*$$

hence we can conclude that

$$\lim_{(s,y) \rightarrow (t,x), h \downarrow 0} V_h^0(s, y) = V^0(t, x)$$

\square

Remark 5.11. If the value function $V^0(t, x) \in C^{1,2}$ is the solution of the continuous-time HJB equation (5.47), the function $V^0(t, x) \in C^{1,2}$ is then the unique viscosity solution of the HJB equation (5.47). Therefore, if $V_h^0(t, x)$ converges to the viscosity solution of the HJB equation (5.47), we can obtain that $V_h^0(t, x)$ converges to the value function $V^0(t, x)$.

It is well-known that in two following examples of log-utility and power-utility the continuous-time value functions are sufficiently smooth to apply Theorem 5.10. Thus, the discrete-time value functions converge. We will show that in these two examples even the sequence of optimal strategies will converge, too.

The Log utility. From the above section of numerical examples the optimal strategy in the case of log utility satisfies

$$\pi_h^* = \frac{(\mu - r)(1 + rh)}{\sigma^2 - (\mu - r)^2 h^2} \quad (5.52)$$

If $h \rightarrow 0$, we can easy get that

$$\lim_{h \downarrow 0} \pi_h^* = \frac{(\mu - r)}{\sigma^2} = \pi^* \quad (5.53)$$

the optimal strategy in discrete time converges to the optimal strategy in continuous time.

The Power utility. We have the optimal strategy in the case of power utility in the following

$$\pi_h^* = \frac{(1 + rh) \left(\left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} - 1 \right)}{\mu h - rh + \sigma\sqrt{h} - \left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} (\mu h - rh - \sigma\sqrt{h})} \quad (5.54)$$

For $h \rightarrow 0$, we have that

$$\begin{aligned} & \lim_{h \downarrow 0} \pi_h^* \\ &= \lim_{h \downarrow 0} \frac{(1 + rh) \left(\left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} - 1 \right)}{\mu h - rh + \sigma\sqrt{h} - \left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} (\mu h - rh + \sigma\sqrt{h})} \end{aligned}$$

From

$$\lim_{h \downarrow 0} \frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} = \lim_{h \downarrow 0} \frac{\sigma - \mu\sqrt{h} + r\sqrt{h}}{\mu\sqrt{h} - r\sqrt{h} + \sigma} = 1$$

we have that

$$\begin{aligned} & \lim_{h \downarrow 0} \pi_h^* \\ &= \lim_{h \downarrow 0} \frac{[(1 + rh) \left(\left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} - 1 \right)]'}{[\mu h - rh + \sigma\sqrt{h} - \left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} (\mu h - rh + \sigma\sqrt{h})]'} \end{aligned}$$

Let $A = \frac{\sigma\sqrt{h}-\mu h+rh}{\mu h-rh+\sigma\sqrt{h}}$, then

$$(A^{\frac{1}{\gamma-1}})' = \frac{1}{\gamma-1}(A)^{\frac{2-\gamma}{\gamma-1}} \frac{-(\mu-r)\sigma\frac{1}{\sqrt{h}}}{((\mu-r)\sqrt{h}+\sigma)^2}$$

and

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{[(1+rh)((\frac{\sigma\sqrt{h}-\mu h+rh}{\mu h-rh+\sigma\sqrt{h}})^{\frac{1}{\gamma-1}} - 1)]'}{[\mu h - rh + \sigma\sqrt{h} - (\frac{\sigma\sqrt{h}-\mu h+rh}{\mu h-rh+\sigma\sqrt{h}})^{\frac{1}{\gamma-1}}(\mu h - rh + \sigma\sqrt{h})]'} \\ &= \lim_{h \downarrow 0} \frac{(A^{\frac{1}{\gamma-1}} - 1) + (1+rh)(\frac{1}{\gamma-1}(A)^{\frac{2-\gamma}{\gamma-1}} \frac{-(\mu-r)\sigma\frac{1}{\sqrt{h}}}{((\mu-r)\sqrt{h}+\sigma)^2})}{\mu - r + \frac{\sigma}{2\sqrt{h}} - (\frac{1}{\gamma-1}(A)^{\frac{2-\gamma}{\gamma-1}} \frac{-(\mu-r)\sigma\frac{1}{\sqrt{h}}}{((\mu-r)\sqrt{h}+\sigma)^2})(\mu h - rh + \sigma\sqrt{h}) - A^{\frac{1}{\gamma-1}}(\mu - r + \frac{\sigma}{2\sqrt{h}})} \\ &= \lim_{h \downarrow 0} \frac{\sqrt{h}(A^{\frac{1}{\gamma-1}} - 1) + (1+rh)(\frac{1}{\gamma-1}(A)^{\frac{2-\gamma}{\gamma-1}} \frac{-(\mu-r)\sigma}{((\mu-r)\sqrt{h}+\sigma)^2})}{(\mu - r)\sqrt{h} + \frac{\sigma}{2} - (\frac{1}{\gamma-1}(A)^{\frac{2-\gamma}{\gamma-1}} \frac{-(\mu-r)\sigma}{((\mu-r)\sqrt{h}+\sigma)^2})(\mu h - rh + \sigma\sqrt{h}) - A^{\frac{1}{\gamma-1}}((\mu - r)\sqrt{h} + \frac{\sigma}{2})} \\ &= \frac{\frac{1}{\gamma-1} \frac{-(\mu-r)}{\sigma}}{\sigma} \\ &= \frac{(\mu-r)}{(1-\gamma)\sigma^2} \end{aligned}$$

Therefore we obtain

$$\lim_{h \downarrow 0} \pi_h^* = \frac{(\mu-r)}{(1-\gamma)\sigma^2} = \pi^*$$

the optimal strategy of power utility in discrete time converges also to the optimal strategy in continuous time.

Now we focus on showing that the value function obtained from the worst-case HJB equation in discrete time above converges to the value function from the continuous time HJB equation when the time step $h \rightarrow 0$.

We consider the discrete worst-case HJB system under the time step h as following

$$\begin{cases} \min\{\sup_{\pi \in M'} [V_h^0(t+1, x(1+r-\pi_t(r+k))) - V_h^1(t, x)], \sup_{\pi \in M''} \ell^\pi V_h^1(t, x)\} = 0 \\ V_h^1(T, x) = V_h^0(T, x) = u(x) \end{cases} \quad (5.55)$$

and the continuous worst-case HJB system satisfying

$$\begin{cases} \min\{\sup_{\pi \in M'} [V^0(t, x(1-k\pi(t))) - V^1(t, x)], \sup_{\pi \in M''} L^\pi V^1(t, x)\} = 0 \\ V^1(T, x) = V^0(T, x) = u(x) \end{cases} \quad (5.56)$$

Before we show that the value function $V_h^1(t, x)$ obtained from this worst-case discrete HJB system converges to the value function $V^1(t, x)$ from the continuous time HJB equation when the time step $h \rightarrow 0$, we define

$$(V^1)^*(t, x) = \limsup_{(s, y) \rightarrow (t, x), h \downarrow 0} V_h^1(s, y) \quad (5.57)$$

$$(V^1)_*(t, x) = \liminf_{(s, y) \rightarrow (t, x), h \downarrow 0} V_h^1(s, y) \quad (5.58)$$

Here we can also set that

$$F(t, x, \varphi, D_t \varphi, D_x \varphi, D_x^2 \varphi) = -\min\left\{\sup_{\pi \in M'} [V^0(t, x(1 - k\pi(t))) - V^1(t, x)], \sup_{\pi \in M''} L^\pi V^1(t, x)\right\}$$

then we have that

$$F(t, x, \varphi, D_t \varphi, D_x \varphi, D_x^2 \varphi) = \max\left\{-\sup_{\pi \in M'} [V^0(t, x(1 - k\pi(t))) - V^1(t, x)], -\sup_{\pi \in M''} L^\pi V^1(t, x)\right\}$$

Lemma 5.12: $(V^1)^*$ is a viscosity subsolution of the continuous-time HJB system (5.56).

Proof: Let $(t_0, x_0) \in [0, T] \times O$ and let $w \in C^{1,2}([0, T] \times \bar{O})$ with $w \geq (V^1)^*$ and $w(t_0, x_0) = (V^1)^*(t_0, x_0)$. To show that $(V^1)^*$ is a viscosity subsolution of the continuous-time HJB system (5.56), we want to show that

$$\max\left\{-\sup_{\pi \in M'} [V^0(t_0, x_0(1 - k\pi(t))) - w(t_0, x_0)], -\sup_{\pi \in M''} L^\pi w(t_0, x_0)\right\} \leq 0$$

We have

$$\begin{aligned} w(t_0, x_0) &= (V^1)^*(t_0, x_0) \\ &= \limsup_{(s, y) \rightarrow (t_0, x_0), h \downarrow 0} V_h^1(s, y) \\ &\leq \limsup_{(s, y) \rightarrow (t_0, x_0), h \downarrow 0} \sup_{\pi} E(V_h^0(s + h, y(1 + rh - \pi(rh + k)))) \\ &= \limsup_{(s, y) \rightarrow (t_0, x_0), h \downarrow 0} \sup_{\pi} V_h^0(s + h, y(1 + rh - \pi(rh + k))) \\ &= \sup_{\pi} V^0(t_0, x_0(1 - k\pi)) \end{aligned} \quad (5.59)$$

the inequality is obtained from the Dynamic Programming Principle (5.19).

Then we obtain $-\sup_{\pi} [V^0(t_0, x_0(1 - k\pi)) - w(t_0, x_0)] \leq 0$. We only have to show that $-\sup_{\pi} L^\pi w(t_0, x_0) \leq 0$.

We can get that (t_0, x_0) is a maximizer of $(V^1)^* - w$ with $w(t_0, x_0) = (V^1)^*(t_0, x_0)$ and $w \geq (V^1)^*$. And from the proof of the Lemma (5.9) we can have that there exist (t_h, x_h) so that $V_h^0 - w$ has a maximum at (t_h, x_h) which tends to (t_0, x_0) when $h \rightarrow 0$, then we have

$$\begin{aligned} V_h^1(t_h, x_h) - w(t_h, x_h) &\geq V_h^1(t_h + h, X_{t_h+h}) - w(t_h + h, X_{t_h+h}) \\ w(t_h + h, X_{t_h+h}) - w(t_h, x_h) &\geq V_h^1(t_h + h, X_{t_h+h}) - V_h^1(t_h, x_h) \end{aligned} \quad (5.60)$$

Now we assume that $\sup \ell^\pi w(t_h, x_h) < 0$.

By taking the supremum and expectation on the both sides of the Equation (5.60), we get

$$\sup_{\pi_{t_h}} E_{t_h, x_h} [w(t_h + h, X_{t_h+h}) - w(t_h, x_h)] \geq \sup_{\pi_{t_h}} E_{t_h, x_h} [V_h^1(t_h + h, X_{t_h+h})] - V_h^1(t_h, x_h)$$

Using the assumption and the discrete Itô formula shows that

$$\sup_{\pi_{t_h}} E_{t_h, x_h} [w(t_h + h, X_{t_h+h}) - w(t_h, x_h)] = \sup \ell^\pi w(t_h, x_h) < 0$$

Then we have

$$V_h^1(t_h, x_h) > \sup_{\pi_{t_h}} E_{t_h, x_h} [V_h^1(t_h + h, X_{t_h+h})] \quad (5.61)$$

From the dynamic programming principle for the worst-case portfolio optimization 4.39 we have that

$$\begin{aligned} V_h^1(t_h, x_h) &= \sup_{\pi_{t_h}} \min \{ E(V_h^1(t_h + h, X_{t_h+h}) \mid \mathcal{F}_{t_h}), E(V_h^0(t_h + h, x_h(1 + rh - \pi_{t_h}(rh + k^*))) \mid \mathcal{F}_{t_h}) \} \\ &\leq \sup_{\pi_{t_h}} \{ E(V_h^1(t_h + h, X_{t_h+h})) \} \end{aligned}$$

which contradicts the inequation (5.61).

Therefore we have that

$$\sup \ell^\pi w(t_h, x_h) \geq 0$$

We divide by h and let $h \downarrow 0$,

$$-\sup_{\pi} L^\pi w(t_0, x_0) \leq 0$$

Thus, $(V^1)^*$ is a viscosity subsolution. □

Lemma 5.13: $(V^1)_*$ is a viscosity supersolution of the continuous-time HJB system (5.56).

Proof: Let $(t_0, x_0) \in [0, T] \times O$ and let $w \in C^{1,2}([0, T] \times \bar{O})$ with $w \leq (V^1)_*$ and $w(t_0, x_0) = (V^1)_*(t_0, x_0)$. To show that $(V^1)_*$ is a viscosity subsolution of the continuous-time HJB system (5.56), we want to show that

$$\max\left\{-\sup_{\pi \in M'} [V^0(t_0, x_0(1 - k\pi(t))) - w(t_0, x_0)], -\sup_{\pi \in M''} L^\pi w(t_0, x_0)\right\} \geq 0$$

By (5.59) we have

$$w(t_0, x_0) = (V^1)_*(t_0, x_0) \leq \sup_{\pi} V^0(t_0, x_0(1 - k\pi_t)) \quad (5.62)$$

If the equality holds we have

$$\max\{0, -\sup_{\pi} L^\pi w(t_0, x_0)\} \geq 0$$

then we are done. Therefore, it is left to show that

$$-\sup_{\pi} L^\pi w(t_0, x_0) \geq 0$$

under the assumption of

$$w(t_0, x_0) = (V^1)_*(t_0, x_0) < \sup_{\pi} V^0(t_0, x_0(1 - k\pi)) \quad (5.63)$$

From the definition of w we have that (t_0, x_0) is a minimizer of $(V^1)^* - w$ and from the proof of the Lemma (5.9) we can obtain that there exist (t_h, x_h) so that $V_h^0 - w$ has a minimum at (t_h, x_h) which tends to (t_0, x_0) when $h \rightarrow 0$. Then we get

$$\begin{aligned} V_h^1(t_h, x_h) - w(t_h, x_h) &\leq V_h^1(t_h + h, X_{t_h+h}) - w(t_h + h, X_{t_h+h}) \\ w(t_h + h, X_{t_h+h}) - w(t_h, x_h) &\leq V_h^1(t_h + h, X_{t_h+h}) - V_h^1(t_h, x_h) \end{aligned} \quad (5.64)$$

By taking expectation on the both sides of the Equation (5.64), we get

$$E_{t_h, x_h}[w(t_h + h, X_{t_h+h}) - w(t_h, x_h)] \leq E_{t_h, x_h}[V_h^1(t_h + h, X_{t_h+h})] - V_h^1(t_h, x_h)$$

Let us now assume that we have

$$\sup_{\pi_{t_h}} \ell^\pi w(t_h, x_h) > 0$$

under the assumption of

$$(V_h^1)(t_h, x_h) < \sup_{\pi_{t_h}} V_h^0(t_h + h, x_h(1 + rh - \pi_{t_h}(rh + k^*))) \quad (5.65)$$

Using the assumption and the discrete Ito formula shows that

$$\sup_{\pi_{t_h}} E_{t_h, x_h} [w(t_h + h, X_{t_h+h}) - w(t_h, x_h)] = \sup_{\pi_{t_h}} \ell^\pi w(t_h, x_h) > 0$$

Then we have

$$V_h^1(t_h, x_h) < \sup_{\pi_{t_h}} E_{t_h, x_h} [V_h^1(t_h + h, X_{t_h+h})] \quad (5.66)$$

The dynamic programming principle for the worst-case portfolio optimization 4.39 implies that

$$V_h^1(t_h, x_h) = \sup_{\pi_{t_h}} \min \{ E(V_h^1(t_h+h, X_{t_h+h}) \mid \mathcal{F}_{t_h}), E(V_h^0(t_h+h, x_h(1+rh-\pi_{t_h}(rh+k^*))) \mid \mathcal{F}_{t_h}) \}$$

The assumption

$$(V_h^1)(t_h, x_h) < \sup_{\pi_{t_h}} V_h^0(t_h + h, x_h(1 + rh - \pi_{t_h}(rh + k^*)))$$

leads to

$$V_h^1(t_h, x_h) = \sup_{\pi_{t_h}} E_{t_h, x_h} [V_h^1(t_h + h, X_{t_h+h})]$$

which contradicts (5.66).

Therefore, we must have

$$\ell^\pi w(t_h, x_h) \leq 0$$

under the assumption of

$$(V_h^1)(t_h, x_h) < V_h^0(t_h + h, x_h(1 + rh - \pi_{t_h}(rh + k^*))) \quad (5.67)$$

Deviding the Equation 5.67 by h and let $h \downarrow 0$, we get

$$-\sup_{\pi} L^\pi w(t_0, x_0) \geq 0.$$

Thus, $(V^1)^*$ is a viscosity supersolution.

□

Using the lemma above we can prove the following theorem:

Theorem 5.14: *The value function $V_h^1(t, x)$ obtained from this worst-case discrete HJB system (5.55) converges to the value function $V^1(t, x)$ from the continuous time HJB equation (5.56) when the time step $h \rightarrow 0$, that is*

$$\lim_{(s,y) \rightarrow (t,x), h \downarrow 0} V_h^1(s, y) = V^1(t, x)$$

Proof: The lemma above implies that $(V^1)^*$ is a viscosity subsolution and $(V^1)_*$ is a viscosity supersolution respectively, hence we have that

$$(V^1)^* \leq V^1$$

similarly, $(V^1)_* \geq V^1$. By a comparison we have

$$(V^1)^* \leq V^1 \leq (V^1)_*$$

Since by construction we have

$$\begin{aligned} (V^1)^*(t, x) &= \limsup_{(s, y) \rightarrow (t, x), h \downarrow 0} V_h^1(s, y) \\ &\geq \liminf_{(s, y) \rightarrow (t, x), h \downarrow 0} V_h^1(s, y) \\ &= (V^1)_*(t, x) \end{aligned}$$

therefore we get

$$(V^1)^* = V^1 = (V^1)_*$$

hence we can conclude that

$$\lim_{(s, y) \rightarrow (t, x), h \downarrow 0} V_h^1(s, y) = V^1(t, x)$$

□

Remark 5.15. From the Remark 5.11 we can also obtain that if $V_h^1(t, x)$ converges to the viscosity solution of the HJB equations (5.56), then $V_h^1(t, x)$ converges also to the value function $V^1(t, x)$.

Now we want to study this convergence theory numerically to determine the optimal strategy in discrete time when the time step $h \rightarrow 0$ and compare them to the optimal strategies in continuous time.

The Power utility

The discrete Bellmann system of the power utility satisfies

$$\begin{cases} \min_{\pi \in M'} \{ \sup [V_h^0(t+h, x(1+rh - \pi_t(rh+k))) - V^h(t, x)], \sup_{\pi \in M''} \ell^\pi V^h(t, x) \} = 0 \\ V^1(T, x) = V^0(T, x) = \frac{1}{\gamma} x^\gamma \end{cases} \quad (5.68)$$

Then form of the optimal strategy follows

$$\begin{cases} \pi_{t+h}^* = \frac{1+rh}{rh+k} - \left(\frac{1+rh}{rh+k} - \pi_t^* \right) \left(\frac{(1+rh+\pi_t^{0*}((\mu-r)h+\sigma\sqrt{h}))^\gamma + (1+rh+\pi_t^{0*}((\mu-r)h-\sigma\sqrt{h}))^\gamma}{(1+rh+\pi_t^*((\mu-r)h+\sigma\sqrt{h}))^\gamma + (1+rh+\pi_t^*((\mu-r)h-\sigma\sqrt{h}))^\gamma} \right)^{\frac{1}{\gamma}} \\ \pi_{T-h}^* = 0 \end{cases} \quad (5.69)$$

with the optimal portfolio strategy in crash-free model

$$\pi^{0*} = \frac{(1 + rh) \left(\left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} - 1 \right)}{\mu h - rh + \sigma\sqrt{h} - \left(\frac{\sigma\sqrt{h} - \mu h + rh}{\mu h - rh + \sigma\sqrt{h}} \right)^{\frac{1}{\gamma-1}} (\mu h - rh - \sigma\sqrt{h})} \quad (5.70)$$

When $h \rightarrow 0$, applying the Taylor series we have that

$$\lim_{h \downarrow 0} \frac{\pi_{t+h}^* - \pi_t^*}{h} = -\frac{\sigma^2}{2k^*} (1 - \gamma) (1 - \pi_t^* k^*) \left(\pi_t^* - \frac{\mu - r}{\sigma^2(1 - \gamma)} \right)^2 \quad (5.71)$$

the optimal strategy of power utility in discrete time converges also to the optimal strategy in continuous time.

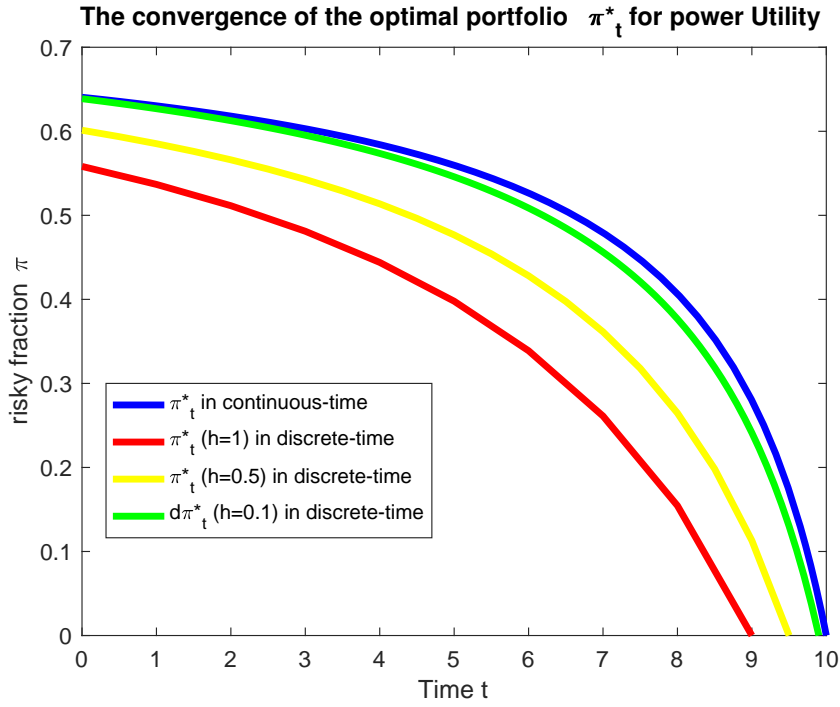


Figure 5.3.: The convergence of the optimal trading strategies π_t^* for power utility function

The convergence of the optimal trading strategies for power utility is illustrated in Figure 5.3. From the figure it is showed that the lines of optimal worst-case portfolio processes in discrete-time are below the blue line of optimal worst-case portfolio processes in continuous-time. In the presence of crashes the investor who trades in discrete-time plays

safer and reallocates more wealth from risky assets to riskless bonds. The worst-case optimal portfolio process in discrete-time goes nearly to the optimal one in continuous-time as the number of trading times goes to infinity.

The Log utility

From the above section the discrete Bellmann system in the case of log utility satisfies

$$\begin{cases} \min\{ \sup_{\pi \in M'} [V_h^0(t+h, x(1+r-\pi_t(r+k))) - V^1(t, x)], \sup_{\pi \in M''} \ell^\pi V_h^1(t, x) \} = 0 \\ V^1(T, x) = V^0(T, x) = \log x \end{cases} \quad (5.72)$$

Then the optimal strategy follows

$$\begin{cases} \pi_{t+h}^* = \frac{1+rh}{rh+k} - (\frac{1+rh}{rh+k} - \pi_t^*) \left(\frac{(1+rh+\pi_t^{0*}((\mu-r)h+\sigma\sqrt{h}))(1+rh+\pi_t^{0*}((\mu-r)h-\sigma\sqrt{h}))}{(1+rh+\pi_t^*((\mu-r)h+\sigma\sqrt{h}))(1+rh+\pi_t^*((\mu-r)h-\sigma\sqrt{h}))} \right)^{\frac{1}{2}} \\ \pi_{T-h}^* = 0 \end{cases} \quad (5.73)$$

with the optimal portfolio strategy in crash-free model

$$\pi^{0*} = \frac{(\mu-r)(1+rh)}{\sigma^2 - (\mu-r)^2 h^2} \quad (5.74)$$

When $h \rightarrow 0$, using the Taylor series we have that

$$\lim_{h \downarrow 0} \frac{\pi_{t+h}^* - \pi_t^*}{h} = -\frac{\sigma^2}{2k^*} (1 - \pi_t^* k^*) \left(\pi_t^* - \frac{\mu-r}{\sigma^2} \right)^2 \quad (5.75)$$

the optimal strategy in discrete time converges to the optimal strategy in continuous time.

The convergence of the optimal trading strategies for the log utility function are showed in Figure 5.4.

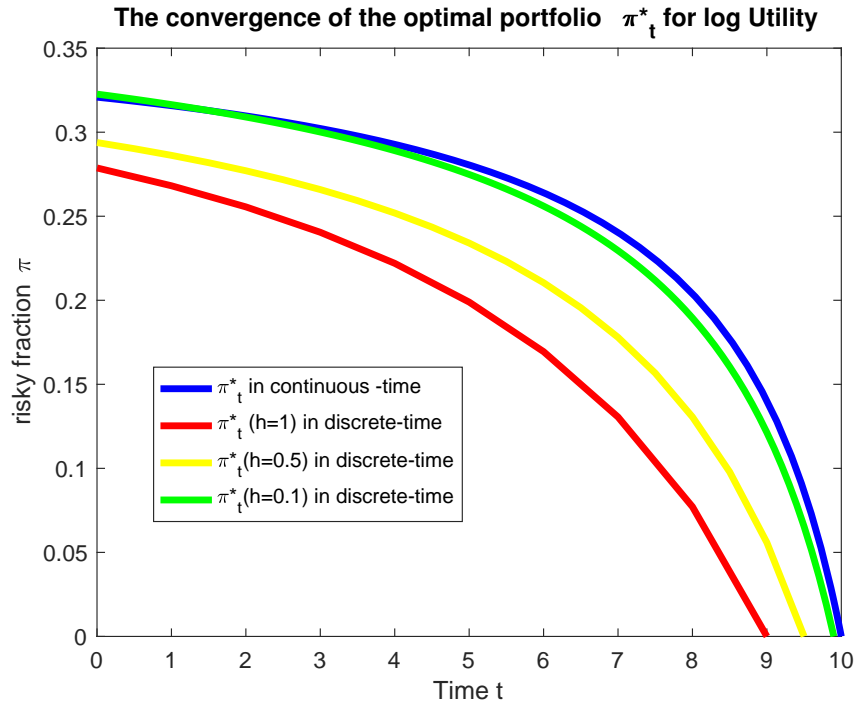


Figure 5.4.: The convergence of the optimal trading strategies π_t^* for log utility function

Chapter 6.

Conclusion

In this chapter we summarize the main contributions of this thesis as well as discuss the possible problems which might be subject of future research.

First, we turn to the introduction of the discrete-time model for stochastic control taking into account system crashes in Chapter 4. Our task is to make the decision in respect of the worst-case scenario which can be an immediate crash of maximum size or no crash at all. In Section 2 we derive an optimal portfolio strategy π^* which makes the investor indifferent between an immediate crash and no crash at all in the case of the logarithmic utility function.

$$V^0(t+1, x(1+r-\hat{\pi}(t)(r+k^*))) = E^{(t,x)}(\ln(\tilde{X}_T^{\hat{\pi}}))(x)$$

In order to extend these results to general utility functions, we use the classical discrete-time method, dynamic programming method, which simplifies a multiperiod decision problem by breaking it down into a sequence of single-period problems. The optimal portfolio process is derived by computing the optimal value function in a backward recursive way:

$$E(U_{t+1}(x(1+r+\pi_t(R-1-r))) \mid \mathcal{F}_t) = E(\tilde{U}_{t+1}(x(1+r-\pi_t(r+k^*))) \mid \mathcal{F}_t).$$

We demonstrate the usefulness of dynamic programming by solving the examples of power-utility, log-utility and exponential-utility functions explicitly. The optimal worst-case portfolio process π^* in the case of logarithmic utility by this recursive equation is identical with the one by indifference approach. Furthermore, the optimal worst-case portfolio process π^* for the explicit power-utility function is showed to convergent to the expression in the continuous-time model, if we approximate the stock price process of the Black-Scholes model by the stock price process in a binomial model by choosing appropriate parameters in the discrete-time model.

In order to further study the limit behavior of the optimal value function in the discrete-time crash models, we establish a new approach, the finite difference approach, in Chapter 5. Applying the discrete-Itô formula for the simple random walk we derive the

discrete HJB-equation for the discrete-time portfolio optimization problem by considering the stock price as a stochastic process which follows the random walk. The discrete HJB equation can be seen as a finite-difference approximation scheme of the continuous HJB equation. By the discrete HJB equation, we verify the optimal value function V for worst-case portfolio optimization problem in discrete time, which satisfies a system of dynamic programming inequalities. In order to investigate the connection between the discrete-time and continuous-time crash models, a viscosity solution method is used to prove the convergence of the worst-case value function in discrete-time to that in continuous-time. We prove that the limit of the upper semi-continuous envelope V^* and the limit of the lower semi-continuous envelope V_* are a viscosity subsolution and a viscosity supersolution of the continuous-time HJB system respectively. Furthermore, the convergence of the optimal portfolio processes is also proved and illustrated in the explicit examples of log utility and power utility.

There are still many possible problems with the worst-case portfolio optimization in the discrete-time setting which are worth to research. Our problem can be extended to the problem including the possibility for consumption. With the existence of the transaction costs some most successful continuous-time portfolio choices are no longer implemented, which makes it promising to derive worst-case portfolio theory in discrete-time setting under the transaction costs.

Appendices

Appendix A.

Basic definitions of probability theory for financial mathematics

The aim of this appendix is to recall some main concepts of Probability Theory as they are needed for financial mathematics. We follow [46] and [25] in our exposition.

We consider a probability space (Ω, \mathcal{A}, P) .

Definition A.1: An increasing family $\mathcal{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ of σ -algebras \mathcal{F}_t , i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t$, $s, t \in \mathcal{T}$ for all $s < t$, is called a **filtration**. $(\Omega, \mathcal{A}, \mathcal{F}, P)$ is a filtered probability space.

The σ -field \mathcal{F}_t usual models the information available at time t . Thus, we can determine the value of a given random variable X_t at time t , if and only if X_t is \mathcal{F}_t -measurable.

Definition A.2: A family of R^n -value random variables $\{X_t\}_{t \in \mathcal{T}}$ is called **stochastic process**. If $X(t)$ is \mathcal{F}_t -measurable for all $t \in \mathcal{T}$, then X is adapted to a filtration \mathcal{F} .

The usual choices for \mathcal{T} are $\mathcal{T} = [0, \infty)$, or $\mathcal{T} = [0, T]$ with $0 < T < \infty$.

Definition A.3: Support that X is \bar{R} -valued random variable with $X \geq 0$ or $E | X | < \infty$ and that $\mathcal{F} \subset \mathcal{A}$ is a σ -algebra. A \mathcal{F} -measurable random variable $Y : \Omega \rightarrow \bar{R}$ is an **\mathcal{F} -conditional expectation of X** , if

$$E(1_F Y) = E(1_F X)$$

for every $F \in \mathcal{F}$.

Noptation: $Y = E(X | \mathcal{F})$, $Y = E(X | Y_1, \dots, Y_t)$ if $\mathcal{F} = \sigma(Y_1, \dots, Y_t) = \mathcal{F}_c^Y$.

Proposition A.4: Let $J \subseteq R$ be an open interval, $X : \Omega \rightarrow J$ in \mathcal{L}^1 and $\varphi : J \rightarrow R$ convex, then $E(X | \mathcal{F})$ is J -valued and if $\varphi(X) \geq 0$ or in \mathcal{L}^1 , then

$$\varphi(E[X | \mathcal{F}]) \leq E[\varphi(X) | \mathcal{F}]$$

Let a set of times $\mathcal{T} \in [0, \infty]$ and a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be given.

Definition A.5: A random variable $\tau : \Omega \rightarrow \mathcal{T} \cup \infty$ is called **stopping time** w.r.t. \mathcal{F} if

$$\tau \leq t \in \mathcal{F}_t$$

for all $t \in \mathcal{T}$. τ is bounded if $P(\tau \leq c) = 1$ for a constant c and finite if $P(\tau < \infty) = 1$.

The above definition means that by using only the available information we can decide whether we stop or not, i.e., the event that we have stopped before or at t is an event of the σ -algebra \mathcal{F}_t .

For our purposes the most important example of a stochastic process is the Brownian motion.

Definition A.6: A (standard) Brownian motion $W = (W_t)_{t \in [0, T]}$ is a stochastic process satisfying

- $W_0 = 0$ P-a.s,
- (independence of Increments)
 $W_t - W_s$ is independent of $W_{s_n} - W_{s_{n-1}}, \dots, W_{s_1} - W_{s_0}$ for $0 \leq s_0 \leq s_n \leq s \leq t$,
- (stationarity of Increments)
 $W_t - W_s \sim N(0, t - s)$, i.e., is normally distributed with mean 0 and variance $t - s$,
 $t > s \geq 0$,
- (Continuity of Paths) W is continuous, i.e.e, $t \rightarrow W_t(\omega)$ is continuous for all $\omega \in \Omega$.

Appendix B.

Derivation of CRR model parameters

Goal: Establish weak convergence of discrete-time CRR model against the continuous-time Black-Scholes model, when we take the limit as n tends toward infinity.

Black-Scholes Model:

For interest rate $r \in \mathcal{R}$, trend parameter $\mu \in \mathcal{R}$, volatility $\sigma > 0$, the stock price of the Black-Scholes model is given by

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right)$$

where $W = (W_t)_{t \in [0, T]}$ is a standard Brownian motion.

CRR model:

The stock price of a CRR model with N time periods satisfies

$$S_n^N = S_{n-1}^N Y_n^N = S_0 \prod_{k=1}^n Y_k^N,$$

where Y_1^N, \dots, Y_N^N are iid with $P(Y_n^N = u) = 1 - P(Y_n^N = d) = p \in (0, 1)$.

Consider

$$\log\left(\frac{S_n^N}{S_{n-1}^N}\right) = \log(Y_n^N),$$

then we have

$$a_N := E\left(\log\left(\frac{S_n^N}{S_{n-1}^N}\right)\right) = E(\log(Y_n^N)) = p \log u + (1 - p) \log d,$$

and

$$b_N^2 := \text{Var}\left(\log\left(\frac{S_n^N}{S_{n-1}^N}\right)\right) = \text{Var}(\log(Y_n^N)) = p(\log u)^2 + (1 - p)(\log d)^2 - E^2(\log(Y_n^N)),$$

Note that with

$$\eta_n^N := \frac{\sum_{k=1}^n \log(Y_k^N) - na_N}{b_N \sqrt{\frac{1}{\Delta}}}$$

where $\Delta = T/N$. we can rewrite

$$S_n^N = S_0 \exp\left(\frac{b_N}{\sqrt{\Delta}} \eta_n^N + n\Delta \frac{a_N}{\Delta}\right)$$

Then a Donsker type theorem can be used to show that in distribution

$$\eta_n^N \approx W_{n\Delta}$$

for N large enough.

Therefore, if we choose p, u, d such that

$$\sigma = \frac{b_N}{\sqrt{\Delta}}, \mu - 1/2\sigma^2 = \frac{a_N}{\Delta} \quad (\text{B.1})$$

we can get in distribution

$$S_n^N \approx S_{n\Delta}$$

where $S_{n\Delta}$ is the Black-Scholes stock price at $t = n\Delta$.

In Equation B.1 there are three unknown variables u, d and p , but only two equations.

Cox, Ross and Rubinstein(CRR,1979) suggested $u * d = 1$. It means "the jump sizes compensate each other". The parameters are

$$\begin{aligned} u &= \exp(\sigma\sqrt{\Delta}), d = \exp(-\sigma\sqrt{\Delta}) \\ p &= \frac{1}{2} + \frac{1}{2} \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} \sqrt{\Delta}. \end{aligned}$$

Jarrow and Rudd(JR, 1983) suggested $q = 1/2$. It means "equal probability for up and down jumps". The parameters are

$$\begin{aligned} u &= \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\sqrt{\Delta}\right), \\ d &= \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta - \sigma\sqrt{\Delta}\right) \\ p &= \frac{1}{2}. \end{aligned}$$

Appendix C.

Viscosity solutions

This appendix is to introduce the notion of viscosity solutions and to state some fundamental results. We refer to Pham[39] and Fleming and Soner[14] in our exposition.

We consider the following function. Let $\mathcal{O} \subset \mathcal{R}^n$ be open and let

$$F : \mathcal{O} \times \mathcal{R} \times \mathcal{R}^n \times S^n \rightarrow \mathcal{R}$$

be continuous. The function F is assumed to satisfy the following ellipticity condition: For all $x \in \mathcal{O}$, $r \in \mathcal{R}$, $q \in \mathcal{R}^n$ and $M, M' \in S^n$,

$$M \leq M' \Rightarrow F(x, r, q, M) \geq F(x, r, q, M')$$

For time-dependent problems, a point in \mathcal{R}^n must be understood as a time variable t and a space variable x . Furthermore, the function $F(t, x, r, p, q, M)$ must satisfy the following parabolicity condition: for all $t \in [0, T)$, $x \in \mathcal{O}$, $r \in \mathcal{R}$, $p, p' \in \mathcal{R}$, $q \in \mathcal{R}^n$ and $M \in S^n$,

$$p \leq p' \Rightarrow F(t, x, r, p, q, M) \geq F(t, x, r, p', q, M)$$

Let us now consider a function $w \in C^{1,2}([0, T)) \times \mathcal{O}$ and assume that w satisfies the following second-order differential equation

$$F(t, x, w(t, x), D_t w(t, x), D_x w(t, x), D_t^2 w(t, x)) = 0 \quad (\text{C.1})$$

for each $(t, x) \in [0, T) \times \mathcal{O}$.

And let $\varphi \in C^{1,2}([0, T)) \times \mathcal{O}$ be another smooth function and $(\bar{t}, \bar{x}) \in C^{1,2}([0, T)) \times \mathcal{O}$ be a maximum point of $w - \varphi$. In this case, the first- and second-order optimality conditions imply:

$$\begin{aligned} D_t w(\bar{t}, \bar{x}) &\leq D_t \varphi(\bar{t}, \bar{x}), \\ D_x w(\bar{t}, \bar{x}) &= D_x \varphi(\bar{t}, \bar{x}), \\ D_x^2 w(\bar{t}, \bar{x}) &\leq D_x^2 \varphi(\bar{t}, \bar{x}). \end{aligned}$$

where the first equality holds if $\bar{t} > 0$.

By the ellipticity condition and parabolicity condition we have

$$\begin{aligned} & F(\bar{t}, \bar{x}, w(\bar{t}, \bar{x}), D_t \varphi(\bar{t}, \bar{x}), D_x \varphi(\bar{t}, \bar{x}), D_t^2 \varphi(\bar{t}, \bar{x})) \\ & \leq F(\bar{t}, \bar{x}, w(\bar{t}, \bar{x}), D_t w(\bar{t}, \bar{x}), D_x w(\bar{t}, \bar{x}), D_t^2 w(\bar{t}, \bar{x})) \\ & = 0 \end{aligned}$$

In order to make sense of the inequality

$$F(\bar{t}, \bar{x}, w(\bar{t}, \bar{x}), D_t \varphi(\bar{t}, \bar{x}), D_x \varphi(\bar{t}, \bar{x}), D_t^2 \varphi(\bar{t}, \bar{x})) \leq 0$$

we only need w to be upper semi-continuous. Similarly, we only need w to be lower semi-continuous if $w - \varphi$ attains a local minimum at (\bar{t}, \bar{x}) .

The above arguments lead to the notion of viscosity solutions. We assume that w is locally bounded and define its upper semi-continuous envelope w^* and its lower semi-continuous envelope w_* by

$$\begin{aligned} w^* &:= \limsup_{(t,x) \rightarrow (\bar{t}, \bar{x})} w(t, x) \\ w_* &:= \liminf_{(t,x) \rightarrow (\bar{t}, \bar{x})} w(t, x) \end{aligned}$$

respectively.

Definition C.1: Let $w : [0, T) \times \mathcal{O}$ be locally bounded.

- w is a viscosity subsolution of equation C.1 if

$$F(\bar{t}, \bar{x}, w^*(\bar{t}, \bar{x}), D_t \varphi(\bar{t}, \bar{x}), D_x \varphi(\bar{t}, \bar{x}), D_t^2 \varphi(\bar{t}, \bar{x})) \leq 0$$

for all $(\bar{t}, \bar{x}) \in [0, T) \times \mathcal{O}$ and for all $\varphi \in C^{1,2}([0, T)) \times \mathcal{O}$ such that (\bar{t}, \bar{x}) is a local maximum point of $w^* - \varphi$.

- w is a viscosity supersolution of equation C.1 if

$$F(\bar{t}, \bar{x}, w_*(\bar{t}, \bar{x}), D_t \varphi(\bar{t}, \bar{x}), D_x \varphi(\bar{t}, \bar{x}), D_t^2 \varphi(\bar{t}, \bar{x})) \geq 0$$

for all $(\bar{t}, \bar{x}) \in [0, T) \times \mathcal{O}$ and for all $\varphi \in C^{1,2}([0, T)) \times \mathcal{O}$ such that (\bar{t}, \bar{x}) is a local minimum point of $w_* - \varphi$.

- we say that w is a viscosity solution of equation C.1 if it is a viscosity subsolution as well as a viscosity supersolution.

Remark C.2. Without loss of generality we can assume that $w^*(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$ and $w_*(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$ respectively. Then (\bar{t}, \bar{x}) is a local maximum (resp. minimum) of $w^* - \varphi$ (resp. $w_* - \varphi$).

Now we want to state the stability property of viscosity solutions. This property shows that if the viscosity solution w^h of approximate equations depending on h are uniformly convergent as $h \rightarrow 0$, then the limiting function w is a viscosity solutions of the limit equation. We use the formulation in Fleming and Soner[14].

Lemma C.3: (*Stability*)

Let w^h be a viscosity subsolution (or a supersolution) of

$$F^h(t, x, w^h(t, x), D_t w^h(t, x), D_x w^h(t, x), D_t^2 w^h(t, x)) = 0$$

in $([0, T]) \times \mathcal{O}$ with some continuous function F^h satisfying the ellipticity condition and parabolicity condition. Suppose that F^h uniformly converges to F , and w^h uniformly converges to w . Then w is a viscosity subsolution (or a supersolution) of the limiting equation.

In order to prove uniqueness of viscosity solutions we define the second-order superjet $J^{2,+}w^*(\bar{t}, \bar{x})$ of the upper semicontinuous envelope w^* of w at (\bar{t}, \bar{x}) to be the set of all $(p, q, M) \in \mathcal{R} \times \mathcal{R}^n \times S^n$ such that

$$\limsup_{(t, x) \rightarrow (\bar{t}, \bar{x})} \frac{1}{|\bar{t} - t| + |\bar{x} - x|} [w^*(\bar{t}, \bar{x}) - w(t, x) - p(\bar{t} - t) - \langle q, \bar{x} - x \rangle - \frac{1}{2} \langle M(\bar{x} - x), \bar{x} - x \rangle] \leq 0$$

and the second-order superjet $J^{2,-}w_*(\bar{t}, \bar{x})$ of the lower semicontinuous envelope w_* of w by

$$J^{2,-}w_*(\bar{t}, \bar{x}) := -J^{2,+}(-w_*)(\bar{t}, \bar{x})$$

We define the closure $\bar{J}^{2,+}w^*(\bar{t}, \bar{x})$ of the superjet $J^{2,+}w^*(\bar{t}, \bar{x})$ as the set of all $(p, q, M) \in \mathcal{R} \times \mathcal{R}^n \times S^n$ for which we can find a sequence $(t_j, x_j, p_j, q_j, M_j)_{j \in \mathcal{N}}$ such that $t_j \in [0, T]$, $x_j \in \mathcal{O}$ and $(p_j, q_j, M_j) \in J^{2,+}w^*(\bar{t}, \bar{x})$ for all $j \in \mathcal{N}$ and

$$\lim_{j \rightarrow \infty} (t_j, x_j, w^*(t_j, x_j), p_j, q_j, M_j) = (\bar{t}, \bar{x}, w^*(\bar{t}, \bar{x}), p, q, M).$$

The closure $\bar{J}^{2,-}w_*(\bar{t}, \bar{x})$ of $J^{2,-}w_*(\bar{t}, \bar{x})$ is defined analogously.

Then we state the following theorem which is the main tool in proving uniqueness of viscosity solutions.

Theorem C.4: (*Ishii's Lemma*) Let u be an upper semi-continuous function on $[0, T] \times \mathcal{O}$, let v be a lower semi-continuous function on $[0, T] \times \mathcal{O}$ and let $\phi \in C^{1,1,2,2}([0, T]^2 \times \mathcal{R}^n \times \mathcal{R}^n)$. Suppose that (t_0, s_0, x_0, y_0) is a local maximum of $u(t, x) - v(s, y) - \phi(t, s, x, y)$. Then for each $\varepsilon > 0$ there exist $M, N \in S^n$ such that

$$\begin{aligned} (D_t \phi(t_0, s_0, x_0, y_0), D_x \phi(t_0, s_0, x_0, y_0), M) &\in \bar{J}^{2,+}u(t_0, x_0), \\ (-D_s \phi(t_0, s_0, x_0, y_0), -D_y \phi(t_0, s_0, x_0, y_0), N) &\in \bar{J}^{2,-}v(t_0, y_0), \end{aligned}$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq D_{x,y}^2 \phi(t_0, s_0, x_0, y_0) + \varepsilon (D_{x,y}^2 \phi(t_0, s_0, x_0, y_0))^2.$$

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Declaration

I, Lihua Chen, hereby declare that this thesis "The Worst-Case Portfolio Optimization Problem in discrete-time" and the work reported herein was composed by myself and originated entirely from me. Information derived from the published and unpublished work of others has been acknowledged in the text and references are given in the list of sources. Moreover, I declare that this thesis was not used in the same or in a similar version to achieve an academic grading or is being published elsewhere.

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