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Algorithmic Methods for Mixed Hodge Modules

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In Erinnerung an meine Omas

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Introduction

The k-th cohomology group of any smooth projective variety X admits a so-called Hodge decomposition

$$H^k(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^q(X,\Omega_X^p).$$

Hodge theory axiomatizes this via Hodge structures of weight k. More precisely, such a Hodge structure is given by a free abelian group $H_{\mathbb{Z}}$, a certainly decreasingly filtered complex vector space $(H, F^\bullet)$ and an isomorphism $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \cong H$. Setting $H^{p,q} := F^pH \cap \overline{F^qH}$, we obtain a Hodge decomposition $H = \bigoplus_{p+q=k} H^{p,q}$ as above. The advantage of endowing H with a filtration instead of a graduation is that the filtration is compatible with families of smooth projective varieties, which led to the introduction of variations of Hodge structure: Replacing $H_{\mathbb{Z}}$ by a local system $\mathcal{H}_{\mathbb{Z}}$ on the complex manifold X and H by a holomorphic vector bundle $\mathcal{H}$ on X with integrable connection satisfying in particular $\mathcal{H} \cong \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_X$, gives the notion of a variation of Hodge structure on X [Gri68] [Gri69]. Deligne extended Hodge structures to mixed Hodge structures to remedy the issue that cohomology groups of singular and non-projective varieties do in general not permit a Hodge decomposition [Del71] [Del74]. A mixed Hodge structure consists essentially of the same data as a Hodge structure and an additional so-called weight filtration $W_\bullet$ on $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that, roughly speaking, the k-th graded part with respect to $W_\bullet$ admits a Hodge structure of weight k.

Considering the filtration $F^\circ_\bullet$ by the order of differential operators on a sheaf of differential operators $\mathcal{D}$, Saito generalized variations of Hodge structure to mixed Hodge modules by combining Hodge theory with well-filtered holonomic $F^\circ_\bullet \mathcal{D}$-modules to deal with families of general varieties [Sai88] [Sai90]: Notice that a $\mathcal{D}$-module already implicitly appears in a variation of Hodge structure as an integrable connection on a holomorphic vector bundle imposes a $\mathcal{D}$-module structure on that bundle. Pure Hodge modules (of weight k) on the complex manifold X play the role of Hodge structures, where a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with a good $F^\circ_\bullet \mathcal{D}_X$-filtration $F_\bullet \mathcal{M}$, called Hodge filtration, replaces H, instead of $H_{\mathbb{Z}}$ we consider a $\mathbb{Q}$-perverse sheaf K and the corresponding isomorphism is replaced by a quasi-isomorphism $\mathrm{DR}(\mathcal{M}) \cong K \otimes_{\mathbb{Q}} \mathbb{C}$. The precise definition of these Hodge modules is very involved and by induction on the dimension of the support of the Hodge module. The most basic example of a pure Hodge module is $\mathcal{O}_X$ with a one-step filtration together with the perverse sheaf $\mathbb{Q}_X[\dim X]$. Considering additionally to $(\mathcal{M}, F_\bullet, K)$ a weight filtration $W_\bullet$ on $\mathcal{M}$ subject to the requirement that the k-th graded part with respect to $W_\bullet$ is, roughly speaking, a pure Hodge module of weight k and using again some recursive definition, gives the notion of a mixed Hodge module. The category $\mathrm{MHM}(X)$ of mixed Hodge modules on an algebraic variety X is abelian and therefore permits a derived category. A key feature of mixed Hodge modules is that they obey the same six-functor formalism as perverse sheaves:

Theorem. *[Sai90, Theorem 0.1] Let X be an algebraic variety. We have natural functors* $f_+, f_!, f^+, f^!, \Psi_g, \Phi_{g,1}, \mathbb{D}, \boxtimes, \otimes$ *and* $\mathcal{H}om$ *between* $D^b\,\mathrm{MHM}(X)$ *the derived categories of mixed Hodge modules, such that these functors are compatible with the corresponding functors on the underlying* $\mathbb{Q}$*-complexes via*

$$\mathrm{rat} : D^b\,\mathrm{MHM}(X) \to D^b\,\mathrm{Perv}(\mathbb{Q}_X) \xrightarrow{\mathrm{real}} D^b_c(\mathbb{Q}_X),$$

where f *is a morphism of algebraic varieties and* $g \in \Gamma(X, \mathcal{O}_X)$*.*

These functors also commute with the forgetful functor assigning a (complex of) mixed Hodge module(s) the underlying (complex of) $\mathcal{D}$-module(s), called Hodge $\mathcal{D}$-module.* Hence we think in this thesis of mixed Hodge modules as a special class of filtered $F^\circ_\bullet\,\mathcal{D}$-modules having good properties. The construction of many of these functors in the filtered setting strongly relies on a third natural filtration on Hodge $\mathcal{D}$-modules, the so-called V-filtration, which behaves by definition of Hodge $\mathcal{D}$-modules "well" with respect to the Hodge filtration.

The main result of this thesis is an algorithm for the V-filtration in the filtered setting.

While some of the above functors, such as the exterior direct product $\boxtimes$, are defined in analogy with the corresponding definition for (filtered) $\mathcal{D}$-modules and their $\mathcal{D}$-module theoretic computation (see [OT01]) is adaptable to filtered Hodge $\mathcal{D}$-modules, the construction of other functors differs completely from the $\mathcal{D}$-module theoretic construction; thus also requiring different algorithmic methods. For example, Saito uses a Beilinson-type resolution to reduce the definition of the direct images f_+ and $f_!$ to quasi-projective morphisms and shows that it suffices to define the cohomological ones $\mathcal{H}^i f_+$ and $\mathcal{H}^i f_!$. Then he factorizes $f = \tilde{f} \circ j$ with $\tilde{f}$ projective and j an open embedding whose complement is a locally principal divisor and sets $\mathcal{H}^i f_+ := (\mathcal{H}^i \tilde{f}_+) j_+$ and $\mathcal{H}^i f_! := (\mathcal{H}^i \tilde{f}_!) j_!$. Considering such an embedding $j : U \hookrightarrow X$ with complement X_0, we have for instance $j_+\,\mathcal{O}_U = \mathcal{O}_X(*X_0)$. If the divisor X_0 is smooth, then the Hodge filtration $F_\bullet\,\mathcal{O}_X(*X_0)$ simply agrees with a pole order filtration [Sai93]. Yet in general, we have only an inclusion and the construction of the Hodge filtration involves taking into account the V-filtration.

We present algorithms for direct images under open embeddings of the above type.

Such algorithms for the computation of j_+ and $j_!$ serve not only as a first step to algorithmically treat the direct image functors, but enable us also to compute inverse images. We describe this for the inverse image f^+, the procedure for $f^!$ is in analogy. Factorize $f = p \circ \iota$ by a projection $p : Y \to Z$ and a closed embedding $\iota : X \to Y$ and set $\mathcal{H}^k f^+\,\mathcal{M} := \mathcal{H}^{k-l} \iota^+ \mathcal{H}^l p^+\,\mathcal{M}$

*We assume in this thesis that all (Hodge) $\mathcal{D}$-modules are defined on smooth varieties. In particular, when talking about direct or inverse images, we assume that the corresponding morphism is a morphism of smooth varieties.

for a Hodge $\mathcal{D}_Z$-module $\mathcal{M}$, where l is the relative dimension of p. The inverse image under the projection is then realized using the exterior tensor product. For the closed embedding, cover the complement by affine opens and use the Čech complex and Kashiwara's equivalence for mixed Hodge modules to calculate ι^+. More precisely, if the image $\iota(X)$ is cut out by the regular functions $g_1, \ldots, g_r$, then $\mathcal{H}^k \iota^+ \mathcal{N}$ for a Hodge $\mathcal{D}_Y$-module $\mathcal{N}$ is the k-th cohomology of the complex $\bigoplus_{|I|=\bullet} (j_I)_+ j_I^{-1} \mathcal{N}$, where $j_I : \bigcap_{i \in I} D(g_i) \hookrightarrow Y$ for $I \subseteq \{1, \ldots, r\}$. Localizations of the form $(j_I)_+ j_I^{-1} \mathcal{N}$ are computable by similar methods as direct images under j_I.

We give an algorithm for localizations along codimension one subvarieties.

Noting that the above complex has cohomology supported on $\iota(X)$, it may be considered as an element of $\mathrm{MHM}(X)$ under Kashiwara's equivalence. Representing a quasi-inverse of this equivalence computationally reduces to computing certain (graded) parts of the V-filtration.

We outline a method to make Kashiwara's equivalence explicit.

We believe that the cohomological inverse image functors $\mathcal{H}^k f^+$ and $\mathcal{H}^k f^!$ are computable by adapting work in [OT01] to represent the exterior tensor product and in [Wal00] to compute the cohomology of the above complex to the filtered setting and combining them with our methods. Being able to compute inverse images under closed embeddings and exterior tensor products allows then the calculation of tensor products. On the other hand, algorithms for graded parts of the V-filtration are used to make the nearby and unipotent vanishing cycles functors Ψ_g and $\Phi_{g,1}$ explicit.

We develop algorithms for the computation of vanishing and nearby cycles.

We describe now the V-filtration and outline the translation of Hodge theoretic constructions, that are based on this filtration, into algorithms by taking the example of direct images under open embeddings of the above type. Given a codimension one inclusion $X_0 \subseteq X$ of smooth equidimensional varieties with defining ideal $\mathcal{I}$, the V-filtration along X_0 on $\mathcal{D}_X$ is defined by

$$V_\bullet \mathcal{D}_X := \{p \in \mathcal{D}_X \mid p(\mathcal{I}^j) \subseteq \mathcal{I}^{j-\bullet} \text{ for all } j \in \mathbb{Z}\},$$

where $\mathcal{I}^j = \mathcal{O}_X$ for $j \leq 0$. The definition of the V-filtration on a $\mathcal{D}_X$-module $\mathcal{M}$ is of local nature. Loosely speaking, the V-filtration $V_\bullet \mathcal{M}$ is a good filtration with respect to $V_\bullet \mathcal{D}_X$ such that locally $(-\partial_t t - \bullet)$ acts nilpotently on $\mathrm{Gr}^V_\bullet \mathcal{M}$, where t is a local generator of $\mathcal{I}$ with corresponding derivation ∂_t. Let us now explain how to use the V-filtration for the computation of direct images under the open embedding $j : U := X \setminus X_0 \to X$, where we also allow singular X_0. Given a Hodge $\mathcal{D}$-module $\mathcal{M}$ on U, we regard $j_+ \mathcal{M} = j \mathcal{M}$ as a $\mathcal{D}_X$-module via the natural isomorphism $\mathcal{D}_X(*X_0) \cong j \mathcal{D}_U$. The Hodge filtration on $j \mathcal{M}$ is for smooth X_0 given by

$$F_\bullet j \mathcal{M} = \sum_{i \in \mathbb{N}} F_i^\circ \mathcal{D}_X \cdot j F_{\bullet - i} V_0 \mathcal{M}.$$

The case of singular X_0 is then reduced to the above situation by considering a certain graph embedding and using Kashiwara's equivalence. So the main task is calculating the Hodge filtration on $V_0\,\mathcal{M}$, where part of the difficulty comes from the fact that while $jF_\bullet\,\mathcal{M}$ is well-filtered as $F^\circ_\bullet\,\mathcal{D}_X(*X_0)$-module, it is not well-filtered as $F^\circ_\bullet\,\mathcal{D}_X$-module. The basic idea of our method for that is to compute the layers $F_kV_0\,\mathcal{M}$ for increasing k stepwise and to use a stopping criterion, which checks based on the computed layers if a set of generators of the filtration has already been determined. The actual computations are performed over certain algebras.

We describe Hodge theoretic constructions in terms of elementary computationally accessible operations over bifiltered algebras.

We give some details of this process in the following: We reduce the above constructions to constructions over non-commutative bifiltered algebras via taking sections of our objects. In classical algorithmic $\mathcal{D}$-module theory this is mainly achieved by considering only affine n-spaces, because the global sections of

$$\mathcal{D}_{\mathbb{C}^n} = \bigoplus_{\alpha\in\mathbb{N}^n} \mathcal{O}_{\mathbb{C}^n}\,\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$$

coincide with the n-th Weyl algebra D_n, which has a well-developed Gröbner basis setup based on the fact that its set of standard monomials forms a $\mathbb{C}$-basis. As $\mathcal{D}_X$ has locally a similar representation and the V-filtration is of local nature, we take certain local sections instead of restricting ourselves to affine spaces. More precisely, there is a computable irreducible affine open cover $\mathcal{U}$ of X with the property that for $U\in\mathcal{U}$ there exist commuting derivations $\theta_1,\dots,\theta_m\in\Theta_X(U)$ such that

$$\mathcal{D}_U = \bigoplus_{\alpha\in\mathbb{N}^m} \mathcal{O}_U\,\theta_1^{\alpha_1}\cdots\theta_m^{\alpha_m}.$$

Identifying U with a closed subvariety of some $\mathbb{C}^n$, these derivations are induced by not necessarily commuting derivations on $\mathbb{C}^n$ generating a $\mathbb{C}[x_1,\dots,x_n]$-subalgebra of D_n. As the corresponding "standard monomials" in these lifted derivations do in general not generate that subalgebra as $\mathbb{C}[x_1,\dots,x_n]$-module, it seems not to be possible to represent the so-called coordinate system ring $\mathcal{D}_X(U)$ as a factor algebra of a PBW-algebra. However, it can be realized as a factor algebra of a free associative $\mathbb{C}$-algebra such that the standard monomials form a set of $\mathbb{C}$-generators subject to some relations.

We introduce the class of so-called PBW-reduction-algebras, which is tailored to capture computations involving coordinate system rings.

These algebras can be thought of as factor algebras of algebras that are "almost" PBW-algebras, but whose set of standard monomials might not be linearly independent.

We develop a comprehensive Gröbner basis framework for this extension of the class of PBW-algebras.

Based on that, we study the interplay of certain filtrations given by weight vectors on these PBW-reduction-algebras. We apply these considerations then to our problems from Hodge theory using that the realization of coordinate system rings as PBW-reduction-algebras can be made such that the V-filtration and the $F^\circ_\bullet$-filtration are induced by weight vectors.

Outline

This thesis is organized as follows:

- Chapter 1 reviews the required background on filtrations and $\mathcal{D}$-modules. This chapter is mainly expository except for Proposition 1.1.15, which is essential for testing whether the already mentioned stopping criterion is satisfied. Moreover, although well-known, a complete account on local coordinates seems to be missing in literature. As these are key players in this thesis, we give a comprehensive and constructive treatment of local coordinate systems.

- Chapter 2 is motivated by the need of a Gröbner basis setup for coordinate system rings. As explained earlier we extend for that the class of PBW-algebras to the new class of PBW-reduction-algebras and develop a Gröbner basis framework for this new class, which mirrors in some aspects that of PBW-algebras, but requires different definitions of the standard terminology. By doing so, we also rectify some errors concerning coordinate system rings and their representation made in [Oak96] (see Remark 2.1.31). Based on that framework, we study weight vector filtrations and their interplay in more generality than has been done for PBW-algebras. This culminates in Algorithm 2.4.15, which is modeled for the computation of the Hodge filtration in the context of localizations.

- We review in Chapter 3 the required theory on V-filtrations, their interaction with $F^\circ_\bullet$-filtrations and localizations following mainly [Sai88] and [SS17]. Building on Kashiwara's, Saito's and Sabbah's work we then translate the material into (mainly local) statements preparing the algorithmic computation of V-filtrations and different types of localizations in both the non-filtered and filtered setting on a sheaf-theoretic level. In this context, we highlight the previously mentioned stopping criterion (see Corollary 3.2.18) and Proposition 3.2.34, which proves that a graph embedding may be used in our setup to deal with direct images under embeddings of complements of non-smooth codimension one subvarieties.

- Finally, Chapter 4 intertwines the sheaf theoretic results from the previous chapter with the computational methods for PBW-reduction-algebras from Chapter 2. For that we first justify passing to global sections in the affine case, consider then a local situation and translate the results from the previous chapter into algorithms strongly relying on our algorithmic framework for PBW-reduction-algebras. A gluing process for filtered free presentations finally patches the local results.

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Notation and Convention

By an (algebraic) variety X we mean a reduced separated scheme of finite type over the complex numbers. We denote its sheaf of regular functions by $\mathcal{O}_X$. *In this context, we draw also attention to Subsection 1.1.1, which explains how we deal conceptually with sheaves on* X.

If X is affine and $I \subseteq \mathcal{O}_X(X)$, we write $V(I)$ for the subvariety of X defined by the vanishing of I and $D(I) := X \setminus V(I)$ for its complement. Similarly, for $f \in \mathcal{O}_X(X)$ we set $V(f) := V(\{f\})$ and $D(f) := D(\{f\})$.

Given a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{F}'$ on X and $U \subseteq X$ open, we write $\mathcal{F}_U$ for the restriction of $\mathcal{F}$ to U and similarly $\phi_U : \mathcal{F}_U \to \mathcal{F}'_U$ for the restriction of ϕ to U. Analogously, we write $\mathcal{F}_p$ for the stalk of $\mathcal{F}$ at $p \in X$ and $\phi_p : \mathcal{F}_p \to \mathcal{F}'_p$ for the induced morphism on the stalks. The kernel and image sheaf of ϕ are denoted by $\ker(\phi)$ and $\operatorname{im}(\phi)$, respectively.

For a regular function $f : X \to \mathbb{C}$, we define the sheaf of rings $\mathcal{O}_X[\overline{f}^{-1}]$ by $U \mapsto (f^{\mathbb{N}})^{-1}\,\mathcal{O}_X(U)$ for $U \subseteq X$ affine open. For an $\mathcal{O}_X$-module $\mathcal{M}$ we write $\mathcal{M}[f^{-1}]$ for the sheaf $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[f^{-1}]$. We denote the global sections of these sheaves by $\mathcal{O}_X(X)[f^{-1}]$ and $\mathcal{M}(X)[f^{-1}]$.

Considering a morphism of algebraic varieties $\phi : X \to Y$, we denote the direct and inverse images in the category of sheaves and of $\mathcal{O}$-modules by ϕ, ϕ^{-1} and ϕ_*, ϕ^*.

Notation 0.0.1. Let X be an algebraic variety, $\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_s$ be sheaf of rings on X, $U \subseteq X$ an open subset, $\mathcal{M}$ a (left, right or two-sided) $\mathcal{A}$-module, $E, E_1, \ldots, E_s$ finite sets and $r \in \mathbb{N}_{>0}$.

(a) The direct sum $\bigoplus_{e \in E} \mathcal{M}(e)$, where (e) is the free generator corresponding to $e \in E$, can be naturally identified with the function space $\mathcal{M}^E$ if X is a one-point space and hence we use the abbreviation

$$\mathcal{M}^E := \bigoplus_{e \in E} \mathcal{M}(e)$$

forgetting the definition of $\mathcal{M}^E$ as function space. Note that we denote for $e \in E$ the corresponding free generator by (e) and not e, because E might contain sections of an $\mathcal{A}$-module itself and we need to distinguish whether we consider e as a free generator or as a section of that $\mathcal{A}$-module.

We write $\{\pi_e \mid e \in E\}$ for the dual basis to E, that is, for $e' \in E$ the $\mathcal{A}$-linear map $\pi_{e'}$ is defined by

$$\pi_{e'} : \mathcal{M}^E \to \mathcal{M},\ (e) \mapsto \delta_{e,e'}.$$

Setting $m_e := \pi_e(m)$ for $m \in \mathcal{M}^E$ and $e \in E$, we write

$$m = \sum_{e \in E} m_e(e).$$

Similarly, for $E' \subseteq E$, we denote by

$$\pi_{E'} : \mathcal{M}^E \to \mathcal{M}^{E'},\ m \mapsto \sum_{e' \in E'} m_{e'}(e')$$

the projection to $\mathcal{M}^{E'}$. We denote $\mathcal{M}^r := \mathcal{M}^{\{e_1,\dots,e_r\}}$. In this case, we also use for $m \in \mathcal{M}^r$ the notation $m = \sum_{1 \le i \le n} m_i(e_i)$ by setting $m_i := m_{e_i}$. Moreover if $G \subseteq \mathcal{M}(U)$, we define $G^E := \{m \in \mathcal{M}(U)^E \mid m_e \in G \text{ for all } e \in E\} \subseteq \mathcal{M}(U)^E = \mathcal{M}^E(U)$.

(b) We identify $\mathcal{A}^1$ with $\mathcal{A}$ as $\mathcal{A}$-module via the canonical map $a \mapsto a_1$. All notations and definitions defined for $\mathcal{A}^r$ are hence implicitly also assumed to be defined for $\mathcal{A}$ via this identification if not said otherwise. Similarly, all notations extend to $\mathcal{A}^{E_1} \oplus \cdots \oplus \mathcal{A}^{E_s}$ by identifying this free $\mathcal{A}$-module with $\mathcal{A}^{\bigsqcup_{1 \le i \le s} E_i}$.

(c) By abuse of notation, for $l \in \mathbb{N}_{>0}$ and $1 \le i_1 < \dots i_l \le s$ the map

$$\pi_{E_{i_1},\dots,E_{i_l}} : \mathcal{A}_1^{E_1} \oplus \cdots \oplus \mathcal{A}_s^{E_s} \to A_{i_1}^{E_{i_1}} \oplus \cdots \oplus \mathcal{A}_{i_l}^{E_{i_l}},\ (a_1, \dots, a_s) \mapsto (a_{i_1}, \dots, a_{i_l})$$

denotes the corresponding projection.

(d) If $\mathcal{M}$ is a left, right or two-sided $\mathcal{A}$-module and $G \subseteq \mathcal{M}(X)$, we denote by ${}_{\mathcal{A}}\langle G\rangle$, $\langle G\rangle_{\mathcal{A}}$ and ${}_{\mathcal{A}}\langle G\rangle_{\mathcal{A}}$ the left, right and two-sided $\mathcal{A}$-submodule of $\mathcal{M}$ generated by G, respectively. If $G = \{g_1, \dots, g_s\}$ we also write ${}_{\mathcal{A}}\langle g_1, \dots, g_s\rangle$ for ${}_{\mathcal{A}}\langle G\rangle$ (and analogously for right and two-sided modules). In the left module case we often write $\sum_{g \in G} \mathcal{A} \cdot g$ for ${}_{\mathcal{A}}\langle G\rangle$. Considering $\mathcal{A}$ as an $\mathcal{A}$-module over itself defines the corresponding notations for $\mathcal{A}$-ideals.

(e) If $\mathcal{N} \subseteq \mathcal{M}$ are $\mathcal{A}$-modules and $m \in \mathcal{M}(X)$, we write $\overline{m}^{\mathcal{N}(X)} \in \mathcal{M}(X)/\mathcal{N}(X) =$ for the residue class of m. If it is clear from the context that $\overline{m}^{\mathcal{N}(X)} \in \mathcal{M}(X)/\mathcal{N}(X)$, we simply write $\overline{m}$. Similarly, for $M' \subseteq \mathcal{M}(X)$ we define $\overline{M'}^{\mathcal{N}(X)} = \{\overline{m}^{\mathcal{N}(X)} \mid m \in M'\}$ and abbreviate $\overline{M'}^{\mathcal{N}(X)}$ by $\overline{M'}$ if this does not cause any ambiguity.

(f) Let $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ be a map between $\mathcal{A}$-modules. Then ϕ^E denotes the map

$$\phi^E : \mathcal{M}_1^E \to \mathcal{M}_2^E,\ \sum_{e \in E} m_e(e) \mapsto \sum_{e \in E} \phi(m_e)(e).$$

(g) For $a_1 \dots, a_k \in \mathcal{A}$ we define $\prod_{i=1,\dots,k} a_i := a_1 \cdots a_k$.

(h) We write $[a, a'] := aa' - a'a$ for the *commutator* of a and $a' \in \mathcal{A}$.

Taking a one-point space for X in the above notation introduces the corresponding notation for rings.

Notation 0.0.2. Let $\alpha, \beta \in \mathbb{Z}^n$ and $\gamma \in \mathbb{Z}^r$ be vectors with integer entries.

(a) We denote all vectors as row vectors and we write $\alpha_i \in \mathbb{Z}$ for the ith component of α for $1 \leq i \leq n$. So in particular, $\alpha = (\alpha_1, \ldots, \alpha_n) = (\alpha_i)_{1 \leq i \leq n}$.

(b) We define $\langle \alpha, \beta \rangle := \sum_{1 \leq i \leq n} \alpha_i \beta_i$ and $|\alpha| := \sum_{1 \leq i \leq n} \alpha_i$.

(c) We set $(\alpha, \gamma) := (\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_r) \in \mathbb{Z}^{n+r}$.

(d) By abuse of notation, we denote the ith unit vector in $\mathbb{Z}^n$ by e_i for $1 \leq i \leq n$.

1 Introduction to filtrations and $\mathcal{D}$-module theory

As already pointed out in the introduction, we treat mixed Hodge modules as a special class of filtered $\mathcal{D}$-modules that has "good properties". Hence we provide in this chapter an introduction to filtered algebras as well as the required $\mathcal{D}$-module theoretic background with focus on filtrations by the order of differential operators, local coordinate systems and direct images. The main references for this chapter are [HTT08], [BGK$^+$87] and [PS08].

The contents of this chapter are as follows: We start in Section 1.1 by investigating $\mathcal{O}$-quasi-coherent locally left Noetherian sheaves of rings and by establishing in certain situations an equivalence of categories between coherent modules over such a sheaf of rings and finitely generated modules over the global sections of that sheaf. This will later in Chapter 4 justify our passage to global sections. Then we consider filtrations on sheaves of algebras and prove in Proposition 1.1.15 a result that will serve as a stopping criterion during the computation of certain induced filtrations in Subsection 2.4.4. Section 1.2 reviews the sheaf of differential operators, its filtration by the order of differential operators and local coordinate systems. As local coordinate systems are a key player in this thesis and as we are not aware of a detailed treatment of them in literature, we give a comprehensive account of local coordinate systems including proofs and algorithmic computations. Section 1.3 is concerned with $\mathcal{D}$-modules, that is, modules over the rings of differential operators. Finally, in Section 1.4 we discuss direct images of (filtered) $\mathcal{D}$-modules with focus on open and closed embeddings.

1.1 Coherent modules and filtrations

We will see that the sheaf of differential operators on the smooth algebraic variety X is a $\mathbb{C}_X$-algebra that is locally free over $\mathcal{O}_X$ and hence in particular $\mathcal{O}_X$-quasi-coherent. Since we need some of the definitions and results in this section not only for the sheaf of differential operators on X, but also for certain tensor products involving it as well as some $\mathcal{O}_X$-submodules of it, we consider in this section a more general setting.

1.1.1 Working with sheaves

Before we start with developing the theory on coherent modules and filtrations, let us explain how we usually deal conceptually with sheaves on the algebraic variety X in this thesis (see [Vak17, Section 13.3]). For this we need the concept of the distinguished affine base of X:

Definition 1.1.1. The *distinguished affine base of* X is the data of the affine open sets of X and the distinguished inclusions (i.e., inclusions of the form $D(f) \subseteq U$ for affine open $U \subseteq X$ and $f \in \mathcal{O}_X(U)$).

We define a "sheaf" (of sets, abelian groups or rings) on the distinguished affine base in analogy to sheaves on topological spaces. Given a sheaf $\mathcal{F}$ on X, we denote the "restriction" of this sheaf to the distinguished affine base by $\mathcal{F}^b$. Then it holds:

Proposition 1.1.2. *[Vak17, Theorem 13.3.2]*

(a) A sheaf $\mathcal{F}$ on the distinguished affine base (of X) determines a unique (up to unique isomorphism) sheaf (on X) which when restricted to the distinguished affine base is $\mathcal{F}$.

(b) A morphism of sheaves on the distinguished affine base uniquely determines a morphism of sheaves.

(c) An $\mathcal{O}_X$-module on the distinguished affine base yields an $\mathcal{O}_X$-module.

In analogy to the proof of the above proposition one shows that other module structures (over sheaves of rings) are defined by the corresponding structures on the distinguished affine base as well.

Using the concept of sheaves on the distinguished affine base, one characterizes $\mathcal{O}_X$-quasi-coherence as follows:

Proposition 1.1.3. *[Vak17, 13.3.3.D] Consider an $\mathcal{O}_X$-module $\mathcal{M}$. Then $\mathcal{M}$ is $\mathcal{O}_X$- quasi-coherent if and only if for each affine open set U and $f \in \mathcal{O}_X(U)$ the natural morphism $\mathcal{M}(U) \otimes \mathcal{O}_X(U)[\overline{f}^{-1}] \to \mathcal{M}(U \cap D(f))$ obtained from the restriction map $\mathcal{M}(U) \to \mathcal{M}(U \cap D(f))$ by the universal property of localization is an isomorphism.*

Remark 1.1.4. Assume that all sheaves under consideration are $\mathcal{O}_X$-quasi-coherent. Then (sheaf theoretic) constructions such as quotient sheaves, images of morphisms, finite sums of subsheaves of a given sheaf, certain product constructions (e.g. given sheaves of rings $\mathcal{S} \subseteq \mathcal{R}$, the $\mathcal{R}$-module $\mathcal{M}$ and the $\mathcal{O}_X$-subsheaf $\mathcal{M}' \subseteq \mathcal{M}$, consider the $\mathcal{S}$-module $\mathcal{S} \cdot \mathcal{M}' \subseteq \mathcal{M}$) and certain tensor products commute on the distinguished affine base with taking sections. So we may e.g. represent sections of the quotient sheaf $(\mathcal{M} \,/\, \mathcal{M}')^b$ as residue classes of sections of $\mathcal{M}^b$.

Hence we usually work when dealing with $\mathcal{O}$-quasi-coherent sheaves implicitly on the restriction of the sheaves to the distinguished affine base and assume that all local sections are local sections on the distinguished affine base. For example, the considerations in Chapter 3 strongly rely on this approach.

Moreover, we often only define sheaves on the distinguished affine base. We demonstrate this in Subsection 1.2.1 and do later so without explicitly saying so.

1.1.2 (Quasi-)coherent modules

Consider an algebraic variety X and morphisms of sheaves of ring $\mathcal{O}_X \to \mathcal{P}_X \to \mathcal{A}_X$ turning $\mathcal{P}_X$ into a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{A}_X$ into a locally $\mathcal{P}_X$-free module. Note in particular that these conditions imply that $\mathcal{A}_X$ is also $\mathcal{O}_X$-quasi-coherent.

Definition 1.1.5. Let $\mathcal{M}$ be a left (right) $\mathcal{A}_X$-module. We say that $\mathcal{M}$ is a left (right) *coherent* $\mathcal{A}_X$-module if it is locally finitely generated and if for any open subset $U \subseteq X$ any locally finitely generated submodule of $\mathcal{M}_U$ is locally finitely presented. We say that $\mathcal{A}_X$ is left (right) coherent if it is left (right) coherent as $\mathcal{A}_X$-module and call $\mathcal{A}_X$ coherent if it is left and right coherent.

Given a left $\mathcal{A}_X$-module that is left coherent, we often say simply that this modules is coherent if that does not cause any ambiguity.

Notation 1.1.6. By $\mathrm{Mod}(\mathcal{A}_X)$ and $\mathrm{Mod}(\mathcal{A}_X^{\mathrm{op}})$ we denote the categories of left and right $\mathcal{A}_X$-modules, respectively. We write $\mathrm{Mod}_{\mathcal{P}_X\text{-}\mathrm{qcoh}}(\mathcal{A}_X)$ and $\mathrm{Mod}_{\mathcal{O}_X\text{-}\mathrm{qcoh}}(\mathcal{A}_X)$ for the categories of left $\mathcal{P}_X$- and $\mathcal{O}_X$-quasi-coherent $\mathcal{A}_X$-modules, respectively, and $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_X)$ stands for the category of coherent $\mathcal{A}_X$-modules (and analogously for right modules). We write $D^{\#}(\mathrm{Mod}_*(\mathcal{A}_X))$ for the corresponding derived categories if they exist, where $\# \in \{\emptyset, b, +, -\}$ and $* \in \{\emptyset, \mathrm{coh}, \mathcal{O}_X\text{-}\mathrm{qcoh}, \mathcal{P}_X\text{-}\mathrm{qcoh}\}$.

We point of that $\mathrm{Mod}_{\mathcal{P}_X\text{-}\mathrm{qcoh}}(\mathcal{A}_X)$ is a subcategory of $\mathrm{Mod}_{\mathcal{O}_X\text{-}\mathrm{qcoh}}(\mathcal{A}_X)$, because every local presentation of an $\mathcal{A}_X$-module by free $\mathcal{P}_X$-modules gives a local presentation by $\mathcal{O}_X$-quasi-coherent modules. As the category of $\mathcal{O}$-quasi-coherent modules on an algebraic variety is abelian and being quasi-coherent is a local property, this shows the claim. Moreover, $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_X)$ is a subcategory of the former categories if $\mathcal{A}_X$ is locally left Noetherian:

We say that $\mathcal{A}_X$ is *locally left (right) Noetherian* if it has an affine open cover $\mathcal{U}$ with the property that $\mathcal{A}_X(U)$ is left (right) Noetherian for all $U \in \mathcal{U}$. By $\mathcal{O}_X$-quasi-coherence and as $\mathcal{O}_X$ acts by restriction of scalars on $\mathcal{A}_X$ this implies that $\mathcal{A}_X(V)$ is also left (right) Noetherian for all affine open $V \subseteq X$ contained distinguishedly in some $U \in \mathcal{U}$. We call $\mathcal{A}_X$ locally Noetherian if it is locally left and right Noetherian.

Proposition 1.1.7. *Let $\mathcal{A}_X$ be a locally left Noetherian sheaf of rings. Then we have:*

(a) The $\mathcal{A}_X$-module $\mathcal{M}$ is $\mathcal{A}_X$-coherent if and only if it is locally finitely generated as $\mathcal{A}_X$-module and $\mathcal{P}_X$-quasi-coherent, or equivalently, if and only if it is locally finitely generated as $\mathcal{A}_X$-module and $\mathcal{O}_X$-quasi-coherent.

(b) The sheaf of rings $\mathcal{A}_X$ is left coherent.

An analogous statement holds for right modules.

Proof. The proof works analogously as the proof of [HTT08, Proposition 1.4.9]:

(a) If $\mathcal{M}$ is $\mathcal{A}_X$-coherent, then it is by definition locally finitely presented as $\mathcal{A}_X$-module. Furthermore, as $\mathcal{A}_X$ is $\mathcal{P}_X$-locally free, $\mathcal{M}$ has a local presentation by free $\mathcal{P}_X$-modules and is thus $\mathcal{P}_X$-quasi-coherent.

Let now $\mathcal{M}$ be locally $\mathcal{A}_X$-finitely generated and quasi-coherent over $\mathcal{O}_X$. For $x \in X$ exists by assumption an affine open neighborhood $U \subseteq X$ of x such that there is a surjective morphism $\mathcal{O}_U^q \to \mathcal{M}_U$ and such that $\mathcal{A}_X(U)$ is left Noetherian. It suffices to prove that the kernel of the $\mathcal{A}_U$-morphism $\phi : \mathcal{A}_U^p \to \mathcal{M}_U$ is finitely generated over $\mathcal{A}_U$ for any $p \in \mathbb{Z}$. As $\mathcal{A}_U(U)$ is a left Noetherian ring, the kernel of $\mathcal{A}_U(U)^p \to \mathcal{M}(U)$ is finitely generated, yielding an exact sequence $\mathcal{A}_U(U)^q \to \mathcal{A}_U(U)^p \to \mathcal{M}(U)$ for some $q \in \mathbb{N}$. Since U is affine and $\mathcal{A}_U$ is $\mathcal{O}_U$-quasi-coherent, and the global section functor on quasi-coherent $\mathcal{O}_U$-modules induces an equivalence of categories with the category of $\mathcal{O}_U(U)$-modules, we obtain the an exact sequence $\mathcal{A}_U^q \to \mathcal{A}_U^p \to \mathcal{M}_U$.

This finishes the proof as every $\mathcal{P}_X$-quasi-coherent module is also $\mathcal{O}_X$-quasi-coherent.

(b) Follows immediately from Part (a).

□

Eventually for computations involving coherent $\mathcal{A}_X$-modules, we wish to pass to the global sections in certain situations. This requires an equivalence of categories

$$\Gamma(X, \bullet) : \mathrm{Mod}_{\mathsf{coh}}(\mathcal{A}_X) \to \mathrm{Mod}_{\mathsf{fg}}(\Gamma(X, \mathcal{A}_X))$$

between the category $\mathrm{Mod}_{\mathsf{coh}}(\mathcal{A}_X)$ and the category $\mathrm{Mod}_{\mathsf{fg}}(\Gamma(X, \mathcal{A}_X))$ of finitely generated $\Gamma(X, \mathcal{A}_X)$-modules.

Definition 1.1.8. We say that an algebraic variety X is $\mathcal{A}_X$-affine if the global section functor

$$\Gamma(X, \bullet) : \mathrm{Mod}_{\mathcal{O}_X\text{-qcoh}}(\mathcal{A}_X) \to \mathrm{Mod}(\Gamma(X, \mathcal{A}_X))$$

is exact, and $\Gamma(X, \mathcal{M}) = 0$ implies $\mathcal{M} = 0$ for $\mathcal{M} \in \mathrm{Mod}_{\mathcal{O}_X\text{-qcoh}}(\mathcal{A}_X)$.

By Serre, X is $\mathcal{O}_X$-affine if and only if it is affine.

Proposition 1.1.9. *Let X be $\mathcal{A}_X$-affine.*

(a) Any $\mathcal{M} \in \mathrm{Mod}_{\mathcal{O}_X\text{-qcoh}}(\mathcal{A}_X)$ is generated over $\mathcal{A}_X$ by its global sections.

(b) The functor

$$\Gamma(X, \bullet) : \mathrm{Mod}_{\mathcal{O}_X\text{-qcoh}}(\mathcal{A}_X) \to \mathrm{Mod}(\Gamma(X, \mathcal{A}_X))$$

is an equivalence of categories.

In particular, the above statements hold for affine X.

Proof. The proof of [HTT08, Proposition 1.4.4] carries over word by word. □

If X is $\mathcal{A}_X$-affine and $\mathcal{A}_X$ is locally left Noetherian, we obtain by Proposition 1.1.7(a) and the above proposition the desired equivalence of categories (for a detailed proof adapt the proof of [HTT08, Proposition 1.4.13] to our situation):

Corollary 1.1.10. *Let $\mathcal{A}_X$ be locally left Noetherian and X be $\mathcal{A}_X$-affine. Then*

$$\Gamma(X, \bullet) : \mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_X) \to \mathrm{Mod}_{\mathrm{fg}}(\Gamma(X, \mathcal{A}_X))$$

is an equivalence of categories. In particular, the above equivalence holds for affine X.

The above equivalence will be crucial in Section 4.1 for the reduction of certain problems involving sheaves of rings to corresponding problems over the global sections of these sheaf of rings.

1.1.3 Filtrations

Filtered $\mathcal{D}$-modules play a key role in this thesis. More generally, we study in this subsection filtrations on the $\mathbb{K}_X$-algebra $\mathcal{A}_X$ for a given algebraic variety X, where $\mathbb{K}_X$ denotes the constant sheaf associated to the field $\mathbb{K}$. Note in particular that our filtrations are by definition exhaustive:

Definition 1.1.11. Let $\mathcal{A}_X$ be a $\mathbb{K}_X$-algebra and $\mathcal{M}$ be an $\mathcal{A}_X$-module.

(a) A family $\mathcal{F}_\bullet \mathcal{A}_X = \{\mathcal{F}_j \mathcal{A}_X\}_{j\in\mathbb{Z}}$ of $\mathbb{K}_X$-vector subspaces of $\mathcal{A}_X$ satisfying for $j, k \in \mathbb{Z}$

  (i) $\mathcal{F}_{j-1} \mathcal{A}_X \subseteq \mathcal{F}_j \mathcal{A}_X$,

  (ii) $\mathcal{F}_j \mathcal{A}_X \cdot \mathcal{F}_k \mathcal{A}_X \subseteq \mathcal{F}_{j+k} \mathcal{A}_X$,

  (iii) $1 \in \mathcal{F}_0 \mathcal{A}_X \setminus \mathcal{F}_{-1} \mathcal{A}_X$ and

  (iv) $\mathcal{A}_X = \bigcup_{j\in\mathbb{Z}} \mathcal{F}_j \mathcal{A}_X$

  is called a *filtration* of $\mathcal{A}_X$. We write $(\mathcal{A}_X, \mathcal{F}_\bullet)$ for the pair $(\mathcal{A}_X, \mathcal{F}_\bullet \mathcal{A}_X)$ and use these notations as well as $\mathcal{F}_\bullet \mathcal{A}_X$ interchangeably. We say that $(\mathcal{A}_X, \mathcal{F}_\bullet)$ is a *sheaf of filtered $\mathbb{K}_X$-algebras* or simply a *filtered $\mathbb{K}_X$-algebra*.

(b) Let $(\mathcal{A}_X, \mathcal{F}_\bullet)$ be a filtered $\mathbb{K}_X$-algebra. A family $\mathcal{G}_\bullet \mathcal{M} = \{\mathcal{G}_\alpha \mathcal{M}\}_{\alpha\in\mathbb{Q}}$ of $\mathbb{K}_X$-vector subspaces of $\mathcal{M}$ is called a *filtration* of $\mathcal{M}$ (with respect to the filtration of $\mathcal{A}_X$) if

  (i) $\mathcal{G}_\alpha \mathcal{M} \subseteq \mathcal{G}_\beta \mathcal{M}$ for all $\alpha, \beta \in \mathbb{Q}$ with $\alpha \leq \beta$,

  (ii) $\mathcal{G}_\bullet \mathcal{M}$ is discretely indexed, i.e., $\mathcal{G}_{<\alpha} \mathcal{M} := \bigcup_{\gamma<\alpha} \mathcal{G}_\gamma \mathcal{M} \subsetneq \mathcal{G}_\alpha \mathcal{M}$ for only finitely many $\alpha \in [k, k+1]$ for every $k \in \mathbb{Z}$,

  (iii) $\mathcal{F}_k \mathcal{A}_X \cdot \mathcal{G}_\alpha \mathcal{M} \subseteq \mathcal{G}_{k+\alpha} \mathcal{M}$ for all $k \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$ and

  (iv) $\mathcal{M} = \bigcup_{\alpha\in\mathbb{Q}} \mathcal{G}_\alpha \mathcal{M}$.

We also write $(\mathcal{M}, \mathcal{G}_\bullet)$ for the pair $(\mathcal{M}, \mathcal{G}_\bullet \mathcal{M})$ and use these notations as well as $\mathcal{G}_\bullet \mathcal{M}$ interchangeably. We say that $(\mathcal{M}, \mathcal{G}_\bullet)$ is a *filtered* $(\mathcal{A}_X, \mathcal{F}_\bullet)$*-module*.

(c) Let $(\mathcal{M}, \mathcal{G}_\bullet)$ be a filtered $(\mathcal{A}_X, \mathcal{F}_\bullet)$-module and $m \in \mathcal{M}$. We define the $\mathcal{G}$*-degree* of m to be

$$\deg_{\mathcal{G}}(m) := \inf\{\alpha \in \mathbb{Q} \mid m \in \mathcal{G}_\alpha \mathcal{M}\} \in \{-\infty\} \cup \mathbb{Q}$$

and say that m has $\mathcal{G}$-degree $\deg_{\mathcal{G}}(m)$.

(d) Let $(\mathcal{M}, \mathcal{G}_\bullet)$ be a filtered $(\mathcal{A}_X, \mathcal{F}_\bullet)$-module. We refer to $(\mathcal{M}, \mathcal{G}_\bullet)$ as a *well-filtered* $(\mathcal{A}_X, \mathcal{F}_\bullet)$*-module* if

 (i) $\mathcal{G}_\alpha \mathcal{M}$ is $\mathcal{F}_0 \mathcal{A}_X$-coherent for all $\alpha \in \mathbb{Q}$ and

 (ii) there exists some $\alpha \gg 0$ such that for all $k \in \mathbb{N}$ and $\beta \in \mathbb{Q}_{\geq \alpha}$

$$\mathcal{F}_k \mathcal{A}_X \cdot \mathcal{G}_\beta \mathcal{M} = \mathcal{G}_{\beta+k} \mathcal{M} \text{ and } \mathcal{F}_{-k} \mathcal{A}_X \cdot \mathcal{G}_{-\beta} \mathcal{M} = \mathcal{G}_{-(\beta+k)} \mathcal{M}\,.$$

In this case, we call $\mathcal{G}_\bullet \mathcal{M}$ also a *good* filtration.

(e) Let $(\mathcal{M}, \mathcal{G}_\bullet)$ and $(\mathcal{M}', \mathcal{G}'_\bullet)$ be filtered $(\mathcal{A}_X, \mathcal{F})$-modules. The $\mathcal{A}_X$-linear morphism $\phi : \mathcal{M} \to \mathcal{M}'$ is called *filtered* if $\phi(\mathcal{G}_\alpha \mathcal{M}) \subseteq \mathcal{G}'_\alpha \mathcal{M}'$ for all $\alpha \in \mathbb{Q}$. We say that ϕ is *strict* if $\phi(\mathcal{M}) \cap \mathcal{G}'_\alpha \mathcal{M}' = \phi(\mathcal{G}_\alpha \mathcal{M})$ for each $\alpha \in \mathbb{Q}$.

(f) We call $\mathcal{A}_X$ *graded* if there are $\mathbb{K}_X$-vector spaces $\mathcal{A}_j$, $j \in \mathbb{Z}$, such that

 (i) $1 \in A_0$,

 (ii) $\mathcal{A}_X = \bigoplus_{j \in \mathbb{Z}} \mathcal{A}_j$ and

 (iii) $\mathcal{A}_j \mathcal{A}_k \subseteq \mathcal{A}_{j+k}$ for all $j, k \in \mathbb{Z}$.

We say that $0 \neq a \in \mathcal{A}$ is *homogeneous (of degree j)* if $a \in \mathcal{A}_j$.

(g) Let $\mathcal{A}_X = \bigoplus_{j \in \mathbb{Z}} \mathcal{A}_j$ be graded. The $\mathcal{A}_X$-module $\mathcal{M}$ is *graded* if there exist $\mathbb{K}_X$-vector spaces $\mathcal{M}_\alpha$, $\alpha \in \mathbb{Q}$, such that

 (i) $\mathcal{M} = \bigoplus_{\alpha \in \mathbb{Q}} \mathcal{M}_\alpha$,

 (ii) $\mathcal{M}_\alpha \neq 0$ for only finitely many $\alpha \in [k, k+1]$ for every $k \in \mathbb{Z}$ and

 (iii) $\mathcal{A}_j \mathcal{M}_\alpha \subseteq \mathcal{M}_{j+\alpha}$ for all $j \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$.

We say that $0 \neq m \in \mathcal{M}$ is *homogeneous (of degree α)* if $m \in \mathcal{M}_\alpha$.

(h) Consider the graded modules $\mathcal{M} = \bigoplus_{\alpha \in \mathbb{Q}} \mathcal{M}_\alpha$ and $\mathcal{M}' = \bigoplus_{\alpha \in \mathbb{Q}} \mathcal{M}'_\alpha$ over the graded $\mathbb{K}_X$-algebra $\mathcal{A}_X = \bigoplus_{j \in \mathbb{Z}} \mathcal{A}_j$. The $\mathcal{A}_X$-linear morphism $\phi : \mathcal{M} \to \mathcal{M}'$ is called *graded* if $\phi(\mathcal{M}_\alpha) \subseteq \mathcal{M}'_\alpha$ for all $\alpha \in \mathbb{Q}$.

The corresponding notations of Definition 1.1.11 for right modules are defined in the canonical way. Moreover, considering the case of X being a one point space, one defines the analogous notations for $\mathbb{K}$-algebras. Given a filtered $\mathbb{K}_X$-algebra $(\mathcal{A}_X, F_\bullet)$, we denote by $\mathrm{Mod}(\mathcal{F}_\bullet\,\mathcal{A}_X)$ and $\mathrm{Mod}(\mathcal{F}_\bullet\,\mathcal{A}_X^{\mathrm{op}})$ the categories of filtered left and right $\mathcal{A}_X$-modules with filtered morphisms, respectively. The corresponding subcategories consisting of well-filtered objects are denoted by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{F}_\bullet\,\mathcal{A}_X)$ and $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{F}_\bullet\,\mathcal{A}_X^{\mathrm{op}})$. For filtered $\mathbb{K}$-algebras (i.e. if X is a one point space), we also use the notation $\mathrm{Mod}_{\mathrm{fg}}(\mathcal{F}_\bullet\,\mathcal{A}_X)$ and $\mathrm{Mod}_{\mathrm{fg}}(\mathcal{F}_\bullet\,\mathcal{A}_X^{\mathrm{op}})$ for the latter two objects.

The remark below explains how to obtain from given filtered algebras or modules new filtered modules:

Remark 1.1.12. Let $(\mathcal{A}_X, \mathcal{F}_\bullet)$ be a filtered $\mathbb{K}_X$-algebra and E a finite set.

(a) Let $\mathbf{s} \in \mathbb{Z}^E$ be a so-called *shift vector*. Then $(\mathcal{A}_X^E, \mathcal{F}[\mathbf{s}]_\bullet)$ with

$$\mathcal{F}[\mathbf{s}]_j\,\mathcal{A}_X^E := \sum_{e\in E} \mathcal{F}_{j-\mathbf{s}_e}\,\mathcal{A}_X\cdot(e)$$

for $j \in \mathbb{Z}$ is a filtered $\mathcal{F}_\bullet\,\mathcal{A}_X$-module indexed by the integers. If $\mathbf{s} = 0$ is the zero vector, we write $\mathcal{F}_\bullet\,\mathcal{A}_X^E = \mathcal{F}[\mathbf{s}]_\bullet\,\mathcal{A}_X^E$.

(b) If $(\mathcal{M}, \mathcal{G}_\bullet)$ is a filtered $(\mathcal{A}_X, \mathcal{F}_\bullet)$-module and $n \in \mathbb{Z}$ an integer, we can shift the filtration by n and define

$$(\mathcal{M}, G_\bullet)(n) := (\mathcal{M}, G_{\bullet-n}).$$

(c) Let $(\mathcal{M}, \mathcal{G}_\bullet)$ be a filtered $(\mathcal{A}_X, \mathcal{F}_\bullet)$-module and $\mathcal{N} \subseteq \mathcal{M}$ an $\mathcal{A}_X$-submodule. Then $\mathcal{G}_\bullet\,\mathcal{N}$ and $\mathcal{G}_\bullet(\mathcal{M}\,/\,\mathcal{N})$ defined by

$$\mathcal{G}_\alpha\,\mathcal{N} := \mathcal{G}_\alpha\,\mathcal{M}\cap\mathcal{N} \text{ and } \mathcal{G}_\alpha(\mathcal{M}\,/\,\mathcal{N}) := (\mathcal{G}_\alpha\,\mathcal{M}+\mathcal{N})/\,\mathcal{N}$$

for $\alpha \in \mathbb{Q}$ are filtered $(\mathcal{A}_X, \mathcal{F}_\bullet)$-modules.

We study now the relationship between filtered and graded modules:

Remark 1.1.13. Gradings and filtrations are related as follows:

(a) Note that gradings induce natural filtrations: Assume that $\mathcal{A}_X = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ is a graded $\mathbb{K}_X$-algebra and $\mathcal{M} = \bigoplus_{\alpha\in\mathbb{Q}} \mathcal{M}_\alpha$ is a graded $\mathcal{A}_X$-module. By setting

$$F_j\,\mathcal{A}_X := \bigoplus_{i\le j} \mathcal{A}_i \text{ and } \mathcal{G}_\beta\,\mathcal{M} := \bigoplus_{\alpha\le\beta} \mathcal{M}_\alpha$$

for $j \in \mathbb{Z}$ and $\beta \in \mathbb{Q}$, we obtain filtrations $\mathcal{F}_\bullet\,\mathcal{A}_X$ (as $\mathbb{K}_X$-algebra) and $\mathcal{G}_\bullet\,\mathcal{M}$ (as filtered $\mathcal{F}_\bullet\,\mathcal{A}_X$-module).

(b) On the other hand, consider the $\mathbb{K}_X$-algebra $\mathcal{A}_X$ and the $\mathcal{A}_X$-module $\mathcal{M}$ with filtrations $\mathcal{F}_\bullet\,\mathcal{A}_X$ and $\mathcal{G}_\bullet\,\mathcal{M}$, respectively. We define *the associated graded* $\mathbb{K}_X$*-algebra with respect to* $\mathcal{F}_\bullet\,\mathcal{A}_X$ and *the associated graded* $\mathcal{A}_X$*-module with respect to* $\mathcal{G}_\bullet\,\mathcal{M}$ by

$$\operatorname{Gr}^{\mathcal{F}} \mathcal{A}_X := \bigoplus_{i\in\mathbb{Z}} \operatorname{Gr}_i^{\mathcal{F}} \mathcal{A}_X \quad \text{and} \quad \operatorname{Gr}^{\mathcal{G}} \mathcal{M} := \bigoplus_{\alpha\in\mathbb{Q}} \operatorname{Gr}_\alpha^{\mathcal{G}} \mathcal{M}$$

with $\operatorname{Gr}_i^{\mathcal{F}} \mathcal{A}_X := \mathcal{F}_i\,\mathcal{A}_X \,/\, \mathcal{F}_{i-1}\,\mathcal{A}_X$ and $\operatorname{Gr}_\alpha^{\mathcal{G}} \mathcal{M} := \mathcal{G}_\alpha\,\mathcal{M} \,/\, \mathcal{G}_{<\alpha}\,\mathcal{M}$, respectively. Clearly, $\operatorname{Gr}^{\mathcal{F}} \mathcal{A}_X$ and $\operatorname{Gr}^{\mathcal{G}} \mathcal{M}$ are a graded $\mathbb{K}_X$-algebra and a graded $\operatorname{Gr}^{\mathcal{F}} \mathcal{A}$-module, respectively. However, in general we have $\mathcal{A}_X \ncong \operatorname{Gr}^{\mathcal{F}} \mathcal{A}_X$ and $\mathcal{M} \ncong \operatorname{Gr}^{\mathcal{G}} \mathcal{M}$. In particular, not every filtered algebra or module has a natural grading. We remark that if $\mathcal{N} \subseteq \mathcal{M}$ is an $\mathcal{A}_X$-submodule of $\mathcal{M}$ with induced filtration $\mathcal{G}_\bullet\,\mathcal{N}$, then $\operatorname{Gr}^{\mathcal{G}} \mathcal{N}$ can be canonically identified with a $\operatorname{Gr}^{\mathcal{F}} \mathcal{A}_X$-submodule of $\operatorname{Gr}^{\mathcal{G}} \mathcal{M}$ via the isomorphism $\mathcal{G}_\alpha\,\mathcal{N} \,/\, \mathcal{G}_{<\alpha}\,\mathcal{N} \cong (\mathcal{G}_\alpha\,\mathcal{N} + \mathcal{G}_{<\alpha}\,\mathcal{M}) / \,\mathcal{G}_{<\alpha}\,\mathcal{M}$.

The associated graded objects of $(\mathcal{A}_X, \mathcal{F}_\bullet)$ and $(\mathcal{M}, \mathcal{G}_\bullet)$ come with surjective *symbol maps*

$$\sigma^{\mathcal{F}} : \mathcal{A}_X \to \operatorname{Gr}^{\mathcal{F}} \mathcal{A}_X$$

and

$$\sigma^{\mathcal{G}} : \mathcal{M} \to \operatorname{Gr}^{\mathcal{G}} \mathcal{M}\,.$$

Here, the map $\sigma^{\mathcal{F}}$ sends $a \in \mathcal{A}_X$ of finite $\mathcal{F}$-degree to its image under the natural maps $\mathcal{F}_{\deg_{\mathcal{F}}(a)}\,\mathcal{A} \twoheadrightarrow \mathcal{F}_{\deg_{\mathcal{F}}(a)}\,\mathcal{A} \,/\, \mathcal{F}_{\deg_{\mathcal{F}}(a)-1}\,\mathcal{A} \hookrightarrow \operatorname{Gr}^{\mathcal{F}} \mathcal{A}_X$ and to 0 if its $\mathcal{F}$-degree is not finite. The map $\sigma^{\mathcal{G}}$ is defined in complete analogy.

Given a filtered $\mathbb{K}_X$-algebra $(\mathcal{A}_X, \mathcal{F}_\bullet)$, a filtered $(\mathcal{A}_X, \mathcal{F}_\bullet)$-module $(\mathcal{M}, \mathcal{G}_\bullet)$ and two $\mathcal{A}_X$-submodules $\mathcal{N} \subseteq \mathcal{N}' \subseteq M$, then there are two canonical ways to induce a filtration $\mathcal{G}_\bullet$ on $\mathcal{N}' \,/\, \mathcal{N}$, namely by taking either $(\mathcal{N}' \,/\, \mathcal{N}) \cap \mathcal{G}_\bullet(\mathcal{M} \,/\, \mathcal{N})$ or $(\mathcal{G}_\bullet\,\mathcal{N}' + \mathcal{N}) / \,\mathcal{N}$. While these filtrations agree, we investigate now similar constructions in a more general situation that do in general not coincide. So assume moreover that $\mathcal{B}_X$ is a subalgebra of $(\mathcal{A}_X, \mathcal{F}_\bullet)$ with induced filtration $F_\bullet\,\mathcal{B}_X$ and that $\mathcal{L} \subseteq \mathcal{M}$ is a $\mathcal{B}_X$-submodule. The filtration $(\mathcal{M}, \mathcal{G}_\bullet)$ induces via the following diagram naturally two filtrations as $(\mathcal{B}_X, \mathcal{F}_\bullet)$-module on $\mathcal{P} := (\mathcal{L} + \mathcal{N}) / \,\mathcal{N}$:

$$\begin{array}{ccccc}
 & & \mathcal{G}_\bullet\,\mathcal{M} & & \\
 & \swarrow^{\text{subm}}_{\text{filt}} & & \searrow^{\text{quot}}_{\text{filt}} & \\
\mathcal{G}_\bullet\,\mathcal{L} & & & & \mathcal{G}_\bullet(\mathcal{M} \,/\, \mathcal{N}) \\
\downarrow{\scriptstyle \text{quot}\,|\,\text{filt}} & & & & \downarrow{\scriptstyle \text{subm}\,|\,\text{filt}} \\
\mathcal{G}_\bullet^{q(\mathcal{L})}\,\mathcal{P} := (\mathcal{G}_\bullet\,\mathcal{L} + \mathcal{N}) / \,\mathcal{N} & & \hookrightarrow & & \mathcal{G}_\bullet^{s}\,\mathcal{P} := \mathcal{G}_\bullet(\mathcal{M} \,/\, \mathcal{N}) \cap \mathcal{P}\,.
\end{array}$$

One easily sees that indeed $\mathcal{G}_\bullet^{q(\mathcal{L})}\,\mathcal{P} \subseteq \mathcal{G}_\bullet^{s}\,\mathcal{P}$ and that $\mathcal{G}_\bullet^{q(\mathcal{L})}\,\mathcal{P}$ depends on $\mathcal{L}$, while $\mathcal{G}_\bullet^{s}\,\mathcal{P}$ does not. This motivates the following notation similar to the one in Remark 1.1.12:

Notation 1.1.14. Let $(\mathcal{A}_X, \mathcal{F}_\bullet)$ be a filtered $\mathbb{K}_X$-algebra and $\mathcal{B}_X$ a $\mathbb{K}_X$-subalgebra of $\mathcal{A}_X$ with induced filtration $\mathcal{F}_\bullet \mathcal{B}_X$. Given an finite set E, an $\mathcal{A}_X$-submodule $\mathcal{N} \subseteq \mathcal{A}_X^E$ and an $\mathcal{B}_X$-submodule $\mathcal{P}$ of $\mathcal{A}_X^E / \mathcal{N}$, we define for a shift vector $\mathbf{s} \in \mathbb{Z}^E$

$$F[\mathbf{s}]_\bullet \mathcal{P} = \mathcal{P} \cap F[\mathbf{s}]_\bullet (\mathcal{A}_X^E / \mathcal{N})$$

and drop $\mathbf{s}$ if it is the zero vector.

The question whether the other inclusion $\mathcal{G}_\bullet^s \mathcal{P} \subseteq \mathcal{G}_\bullet^{q(\mathcal{L})} \mathcal{P}$ also holds, is related to certain associated graded modules:

Proposition 1.1.15. *We have $\mathcal{G}_\bullet^{q(\mathcal{L})} \mathcal{P} = \mathcal{G}_\bullet^s \mathcal{P}$ if and only if*

$$\operatorname{Gr}^{\mathcal{G}}(\mathcal{L} \cap \mathcal{N}) = \operatorname{Gr}^{\mathcal{G}} \mathcal{L} \cap \operatorname{Gr}^{\mathcal{G}} \mathcal{N}.$$

Proof. Note that the inclusion of the left hand side in the right hand side is always satisfied for each of the two equalities in the statement. As these inclusions are equalities if and only if they are equalities on the stalks and taking quotient and submodule filtrations as well as taking graded objects commutes with passing to stalks, we may assume that X is a one point space and work with modules over $\mathbb{K}$-algebras.

Assume that $\mathcal{G}_\bullet^{q(\mathcal{L})} \mathcal{P} = \mathcal{G}_\bullet^s \mathcal{P}$ and let $0 \neq m \in \operatorname{Gr}_\alpha^{\mathcal{G}} \mathcal{L} \cap \operatorname{Gr}_\alpha^{\mathcal{G}} \mathcal{N}$ for $\alpha \in \mathbb{Q}$. Then there exist $l \in \mathcal{L}$ and $n \in \mathcal{N}$ such that $m = \sigma^{\mathcal{G}}(l) = \sigma^{\mathcal{G}}(n)$. This implies $l - n \in \mathcal{G}_{<\alpha} \mathcal{M}$ and thus $\bar{l} \in \mathcal{P} \cap \mathcal{G}_{<\alpha}(\mathcal{M} / \mathcal{N}) = \mathcal{G}_{<\alpha}^s \mathcal{P} = \mathcal{G}_{<\alpha}^{q(\mathcal{L})} \mathcal{P}$, where the last equality follows by assumption. Hence there is some $l' \in \mathcal{G}_{<\alpha} \mathcal{L}$ and $n' \in \mathcal{N}$ such that $l = l' + n'$. We conclude that $n' \in \mathcal{N} \cap \mathcal{L}$ and $\sigma^{\mathcal{G}}(n') = \sigma^{\mathcal{G}}(l) = m$ showing the first implication.

Conversely, assume $\operatorname{Gr}^{\mathcal{G}}(\mathcal{L} \cap \mathcal{N}) = \operatorname{Gr}^{\mathcal{G}} \mathcal{L} \cap \operatorname{Gr}^{\mathcal{G}} \mathcal{N}$ and consider $p \in \mathcal{M}$ with $0 \neq \overline{p} \in \mathcal{G}_\alpha^s \mathcal{P}$ for $\alpha \in \mathbb{Q}$. By construction of $\mathcal{G}_\bullet^s \mathcal{P}$, there exists $l \in \mathcal{L}, n \in \mathcal{N}$ such that $\overline{p} = \overline{l}$ and $l + n \in \mathcal{G}_\alpha \mathcal{M}$. If $l \in \mathcal{G}_\alpha \mathcal{M}$, we are done. Otherwise $n \notin \mathcal{G}_\alpha \mathcal{M}$ and there is some $\beta > \alpha$ such that $\sigma^{\mathcal{G}}(l) = -\sigma^{\mathcal{G}}(n) \in \operatorname{Gr}_\beta^{\mathcal{G}} \mathcal{L} \cap \operatorname{Gr}_\beta^{\mathcal{G}} \mathcal{N} = \operatorname{Gr}_\beta^{\mathcal{G}}(\mathcal{L} \cap \mathcal{N})$. Hence there exist $m \in \mathcal{L} \cap \mathcal{N}$, $l' \in \mathcal{G}_{<\beta} \mathcal{L}$ and $n' \in \mathcal{G}_{<\beta} \mathcal{N}$ such that $l = m + l'$ and $n = -m + n'$. This gives us a representation $\overline{p} = \overline{l' + n' - n} = \overline{l'}$ with $l' + n' \in \mathcal{G}_\alpha \mathcal{M}$ and $\mathcal{G}$-degree of l' smaller than β. Iteration of the above argument and using that $\mathcal{G}_\bullet \mathcal{M}$ is discretely indexed finish the proof. □

While it is more natural to consider the filtration $\mathcal{G}_\bullet^s \mathcal{P}$, the filtration $\mathcal{G}_\bullet^{q(\mathcal{L})} \mathcal{P}$ can be nevertheless very helpful in certain situations: Namely, in Subsection 2.4.4 we will deal with a setting where $\mathcal{P}$ and $\mathcal{L}$ are finitely generated $\mathcal{B}_X$-modules, but $\mathcal{N}$ is not. As the above proposition implies that $\mathcal{G}^s \mathcal{P} = \mathcal{G}^{q(\mathcal{L}+\mathcal{N})} P$, we approximate $\mathcal{G}_\bullet^s \mathcal{P}$ by computing $\mathcal{G}_\bullet^{q(\mathcal{L}_i)} \mathcal{P}$ for increasing finitely generated $\mathcal{B}_X$-modules $\mathcal{L}_i \subseteq \mathcal{N} + \mathcal{L}$ and use that proposition to check equality.

The statement below follows from the analogous statement for rings:

Proposition 1.1.16. *Let $(\mathcal{A}_X, F_\bullet)$ be an $\mathcal{O}_X$-quasi-coherent filtered $\mathbb{K}_X$-algebra such that $F_k \mathcal{A}_X$ is $\mathcal{O}_X$-quasi-coherent for all k. If $\mathcal{F}_{-1} \mathcal{A}_X = 0$ and $\operatorname{Gr}^{\mathcal{F}} \mathcal{A}_X$ is a locally left (right) Noetherian sheaf of rings then so is $\mathcal{A}_X$.*

We finish this subsection by giving a description for well-filtered modules in a certain situation, which can be proven analogously to [HTT08, Proposition 2.1.1]

Proposition 1.1.17. *Let $\mathcal{A}_X$ be $\mathcal{O}_X$-quasi-coherent and $(\mathcal{A}_X, \mathcal{F}_\bullet)$ be a filtered $\mathbb{K}_X$-algebra such that $\mathcal{F}_{-1}\mathcal{A}_X = 0$, $\mathcal{F}_0\mathcal{A}_X = \mathcal{O}_X$ and $\mathcal{F}_j\mathcal{A}_X$ is $\mathcal{O}_X$-coherent for $j \in \mathbb{Z}$. Assume moreover that $\mathrm{Gr}^{\mathcal{F}}\mathcal{A}_X$ is locally left Noetherian. An $\mathcal{O}_X$-quasi-coherent $(\mathcal{A}_X, \mathcal{F}_\bullet)$-module $(\mathcal{M}, \mathcal{G}_\bullet)$, with the property that $\mathcal{F}_k\mathcal{M}$ is $\mathcal{O}_X$-quasi-coherent for all $k \in \mathbb{Z}$ and $F_k\mathcal{M} = 0$ for $k \ll 0$, is well-filtered if and only if it satisfies one of the following equivalent conditions:*

(i) There exists locally a finite set E, a surjective $\mathcal{A}_X$-linear morphism $\phi : \mathcal{A}_X^E \to \mathcal{M}$ and a vector $\mathbf{s} \in \mathbb{Q}^E$ such that

$$\phi\left(\bigoplus_{e\in E}\mathcal{F}_{\lfloor\beta-\mathbf{s}_e\rfloor}\mathcal{A}_X\cdot(e)\right) = \mathcal{G}_\beta\mathcal{M}$$

for all $\beta \in \mathbb{Q}$.

(ii) $\mathrm{Gr}^{\mathcal{G}}\mathcal{M}$ is a coherent $\mathrm{Gr}^{\mathcal{F}}\mathcal{A}_X$-module.

In particular, $\mathcal{M}$ is a coherent $\mathcal{A}_X$-module if and only if it can be equipped with a good filtration.

1.2 Sheaf of differential operators

We introduce in this section the sheaf of differential operators on smooth affine varieties, study it locally on certain affine open neighborhoods via local coordinates and equip it with the filtration by the order of differential operators.

We assume from now on for the remainder of this chapter that X is a smooth algebraic variety of pure dimension m if not stated otherwise. Similarly, all algebraic varieties are assumed to be smooth and equidimensional unless otherwise specified.

1.2.1 Tangent sheaf and sheaf of differential operators

We construct the sheaf of differential operators $\mathcal{D}_X$ on X by defining it on the distinguished affine base. For $U \subseteq X$ affine open we set $\mathcal{D}_X(U)$ to be the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_X(U))$ generated by $\mathcal{O}_X(U)$ (where we identify $g \in \mathcal{O}_X(U)$ with multiplication by g on $\mathcal{O}_X(U)$) and by the set of derivations $\Theta_X(U) := \mathrm{Der}(\mathcal{O}_X(U))$ on $\mathcal{O}_X(U)$ defined by

$$\mathrm{Der}(\mathcal{O}_X(U)) := \{\theta \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X(U)) \mid \theta(gh) = \theta(g)h + g\theta(h) \text{ for all } g, h \in \mathcal{O}_X(U)\}.$$

The restriction map for the inclusion $D(f) \subseteq U$ of $\mathcal{D}_X$ (with $U \subseteq X$ affine open and $f \in \mathcal{O}_X(U)$) is induced by the ones of $\mathcal{O}_X$ and Θ_X. The restriction map of the latter object is defined by sending derivations on $\mathcal{O}_X(U)$ to their unique extension in $\mathrm{Quot}(\mathcal{O}_X(U))$

(restricted to $\mathcal{O}_X(D(f))$). Such an extension exists since for $\theta \in \Theta_X(U)$ its natural extension defined by

$$\theta\left(\frac{g}{h}\right) := \frac{\theta(g)}{h} - \frac{g\theta(h)}{h^2} \text{ for } \frac{g}{h} \in \operatorname{Quot}(\mathcal{O}_X(U))$$

is indeed a derivation on $\operatorname{Quot}(\mathcal{O}_X(U))$. As $0 = \theta(1) = \theta(\frac{h}{h}) = \theta(h) \cdot \frac{1}{h} + h\theta(\frac{1}{h})$ and hence $\theta(\frac{1}{h}) = -\frac{\theta(h)}{h^2}$, an application of the product rule to $\theta(g\frac{1}{h})$ shows the uniqueness of the extension. Moreover, we point out that these restriction maps are injective. Clearly, the $\Theta_X(U)$ and the $\mathcal{D}_X(U)$ for $U \subseteq X$ affine together with their restriction maps define sheaves of $\mathcal{O}_X$-modules on the distinguished affine base of X. By [Vak17, Theorem 13.3.2] these sheaves extend uniquely to sheaves on X, which we also denote Θ_X and $\mathcal{D}_X$.

Definition 1.2.1. We call $\mathcal{D}_X$ the *sheaf of differential operators* on X and Θ_X the *tangent sheaf* on X.

We will see in Subsection 1.2.3 that $\mathcal{D}_X$ can also be introduced using commutators. While the definition of the sheaves $\mathcal{D}_X$ and Θ_X above is extendable to singular algebraic varieties, the sheaf of differential operators on a singular variety is defined using commutators and does in general not agree with the above construction. In such a case, the sheaf of differential operators might not behave nicely, and hence we restrict ourselves to the smooth case.

If X is the m-affine space, the sheaf of differential operators $\mathcal{D}_X$ is the sheafified version the m-th Weyl algebra:

Example 1.2.2. In the case $X = \mathbb{C}^{\mathrm{m}}$ the global sections of $\mathcal{D}_X$ are isomorphic to the Weyl algebra D_{m}, that is, the free associative $\mathbb{C}$-algebra generated by $x_1, \ldots, x_{\mathrm{m}}, \partial_1, \ldots, \partial_{\mathrm{m}}$ modulo the commutation relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $[\partial_i, x_j] = \delta_{ij}$ for $1 \leq i, j \leq \mathrm{m}$, by identifying ∂_i with the partial derivative $\frac{\partial}{\partial x_i}$. We write from now on also ∂_i for $\frac{\partial}{\partial x_i}$. Abbreviating $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_{\mathrm{m}}]$, we have in particular

$$D_{\mathrm{m}} = \bigoplus_{\alpha \in \mathbb{N}^{\mathrm{m}}} \mathbb{C}[\underline{x}]\partial_1^{\alpha_1} \cdots \partial_{\mathrm{m}}^{\alpha_{\mathrm{m}}}.$$

We will see later that $\mathcal{D}_X$ is $\mathcal{O}_X$-quasi-coherent (see Corollary 1.2.14) and hence we obtain

$$\mathcal{D}_X = \bigoplus_{\alpha \in \mathbb{N}^{\mathrm{m}}} \mathcal{O}_X\, \partial_1^{\alpha_1} \cdots \partial_{\mathrm{m}}^{\alpha_{\mathrm{m}}} \tag{1.2.1}$$

with commuting $\partial_1, \ldots, \partial_{\mathrm{m}}$ and $[\partial_i, f/g] = \frac{\partial}{\partial x_i}(f/g)$ for $1 \leq i \leq n$ and $f, g \in \mathbb{C}[\underline{x}]$ with $g \neq 0$.

Remark 1.2.3. If $\iota : Y \hookrightarrow X$ is a closed embedding of (smooth) varieties with defining ideal $\mathcal{I}$, then we may identify

$$\iota_*\Theta_Y = \operatorname{Der}_{\mathcal{I}}(\mathcal{O}_X)/\mathcal{I}\,\Theta_X.$$

Here $\mathrm{Der}_{\mathcal{I}}(\mathcal{O}_X)$ is defined on $U \subseteq X$ affine open by $\mathrm{Der}_{\mathcal{I}}(\mathcal{O}_X)(U) := \{\theta \in \Theta_X(U) \mid \theta(\mathcal{I}(U)) \subseteq \mathcal{I}(U)\}$. Arguing as above, one shows that this defines indeed a sheaf on the distinguished affine base of X extending uniquely to a sheaf on X.

For the above identification note that by Remark 1.2.5(c) and (e) below we have on U as above $\Theta_Y(Y \cap U) = \mathrm{Der}_{\mathcal{I}}(\mathcal{O}_X)(U)/\mathcal{I}(U)\Theta_X(U)$. As both sheaves are uniquely defined by their values on the distinguished affine base, this shows the claim.

If we drop the assumption of X being an (affine open subset of an) affine space, a similar representation of $\mathcal{D}_X$ as in Equation (1.2.1) exists locally. So in particular $\mathcal{D}_X$ is a locally free $\mathcal{O}_X$-module and hence $\mathcal{O}_X$-quasi-coherent. We will make this explicit in the next subsection. For that purpose we need a dual notation to the tangent sheaf, the so-called cotangent sheaf:

Definition 1.2.4. Let $\pi : X \to Y$ be a morphism of not necessarily smooth algebraic varieties. The *relative cotangent sheaf* $\Omega^1_{X/Y}$ is defined by $\delta^*(\mathcal{I}/\mathcal{I}^2)$, where $\delta : X \to X \times_Y X$ is the diagonal embedding and $\mathcal{I}$ the ideal sheaf of $\delta(X)$ in $X \times_Y X$. We call sections of $\Omega^1_{X/Y}$ *relative differential forms*. If Y is a point, we say that $\Omega^1_{X/Y}$ is the *cotangent sheaf* on X and write also Ω^1_X.

The cotangent sheaf comes with a natural morphism of abelian groups $d : \mathcal{O}_X \to \Omega_{X/Y}$ (see e.g. [Har77, Remark 8.9.2] for a construction of this map via gluing natural derivations of Kähler differentials). We review those properties of the (relative) cotangent sheaf needed in this thesis (see e.g. [Har77, Section II.8] or [Vak17, Chapter 21]):

Remark 1.2.5. Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms of not necessarily smooth algebraic varieties.

(a) If X and Y are affine with coordinate rings A and B, respectively, the global sections $\Omega^1_{X/Y}(X)$ can be identified with the Kähler differentials $\Omega_{A/B}$.

(b) The algebraic variety X is smooth if and only if Ω^1_X is locally free.

(c) The morphism d induces an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to \Theta_X,\ \alpha \mapsto \alpha \circ d.$$

(d) We have for a point p of X that

$$m_{X,p}/m^2_{X,p} \cong \Omega^1_{X,p} \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}/m_{X,p},\ \overline{f} \mapsto df \otimes \overline{1},$$

where $m_{X,p}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

(e) If $X = V(I) \subseteq \mathbb{C}^{\mathrm{n}}$ with $I = \langle f_1, \ldots, f_s \rangle$ radical, we identify by Part (a) the global sections $\Omega^1_X(X)$ with

$$\Big(\bigoplus_{1 \le i \le \mathrm{n}} (\mathbb{C}[x_1, \ldots, x_{\mathrm{n}}]/I) dx_i \Big) / \langle df_1, \ldots, df_s \rangle .$$

(f) There is a natural exact sequence

$$\phi^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

called the relative cotangent sequence.

(g) If ϕ is a closed embedding with ideal sheaf $\mathcal{I}$, then there is the so-called conormal exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \phi^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to 0.$$

If X is smooth over Z, this sequence is also left exact.

1.2.2 Local coordinate systems

Recall that X stands of a smooth variety of pure dimension m. By Remark 1.2.5(b) and (c) the tangent sheaf Θ_X is locally free. This implies an even stronger statement, namely that $\mathcal{D}_X$ is locally free. To proof this we consider so-called local coordinate systems and show that they exist locally:

Definition 1.2.6. Let $p \in X$ be a point and U an affine open neighborhood of p. We call $(f_i, \theta_i)_{1\le i\le \mathrm{m}}$ with $f_i \in \mathcal{O}_X(U)$ and $\theta_i \in \Theta_X(U)$ satisfying

$$\Theta_U = \bigoplus_{1\le i\le \mathrm{m}} \mathcal{O}_U\, \theta_i$$

and

$$[\theta_i, \theta_j] = 0 \text{ and } [\theta_i, f_j] = \delta_{ij} \text{ for } 1 \le i, j \le \mathrm{m}$$

a *local coordinate system* of X at p or a *local coordinate system* on the neighborhood U of X. In this case, we also say that $f_1, \ldots, f_{\mathrm{m}}$ are local coordinates (with differentials $\theta_1, \ldots, \theta_{\mathrm{m}}$) and call $\mathcal{D}_X(U)$ a *coordinate system ring*. If $U = X$, we call $(f_i, \theta_i)_{1\le i\le \mathrm{m}}$ a *global coordinate system* of X.

In the situation of the above definition, we abbreviate $\underline{\theta}^\alpha := \theta_1^{\alpha_1}\cdots\theta_{\mathrm{m}}^{\alpha_{\mathrm{m}}} \in \mathcal{D}_U$ for $\alpha \in \mathbb{N}^{\mathrm{m}}$. Similarly, we write $\underline{f}^\alpha := f_1^{\alpha_1}\cdots f_{\mathrm{m}}^{\alpha_{\mathrm{m}}}$.

We have for U as above a direct sum representation of $\mathcal{D}_U$ in analogy with Equation (1.2.1):

Lemma 1.2.7. *Let $p \in X$ be a point and U an affine open neighborhood of p such that $(f_i, \theta_i)_{1\le i\le \mathrm{m}}$ with $f_i \in \mathcal{O}_X(U)$ and $\theta_i \in \Theta_X(U)$ is a local coordinate system of X. Then we have*

$$\mathcal{D}_U = \bigoplus_{\alpha\in\mathbb{N}^{\mathrm{m}}} \mathcal{O}_U\, \underline{\theta}^\alpha.$$

Proof. By [Vak17, Theorem 13.3.2] it suffices to show

$$\mathcal{D}_U(U') = \bigoplus_{\alpha\in\mathbb{N}^{\mathrm{m}}} \mathcal{O}_U(U')\underline{\theta}^{\alpha}.$$

for all $U' \subseteq U$ affine open. By definition of $\mathcal{D}_U$ and since $[\theta, g] = \theta(g) \in \mathcal{O}_U(U')$ for $\theta \in \Theta_U(U')$ and $g \in \mathcal{O}_U(U)$, it suffices to proof that the set $\{\underline{\theta}^{\alpha} \mid \alpha \in \mathbb{N}^{\mathrm{m}}\} \subseteq \mathcal{D}_U(U')$ is linearly independent over $\mathcal{O}_U(U')$. So assume that there is a finite set $A \subseteq \mathbb{N}^{\mathrm{m}}$ and $b \in \mathcal{O}_U(U')^A$ with no zero entries such that $\sum_{\alpha\in A} b_\alpha\underline{\theta}^{\alpha} = 0 \in \mathcal{D}_U(U')$. Choosing $\alpha' \in A$ minimal with respect to the natural partial ordering on $\mathbb{N}^{\mathrm{m}}$, we obtain since $\theta_i(f_j) = [\theta_i, f_j] = \delta_{i,j}$ for $1 \leq i, j \leq \mathrm{m}$ by the product rule the contradiction

$$0 = \sum_{\alpha\in A:\alpha\neq\alpha'} b_\alpha \underbrace{\underline{\theta}^{\alpha}(\underline{f}^{\alpha'})}_{=0} + b_{\alpha'} \cdot \theta^{\alpha'}(\underline{f}^{\alpha'}) = b_{\alpha'} \prod_{1\leq i\leq \mathrm{m}} (\alpha'_i!).$$

□

Remark 1.2.8. Let $p \in X$ be a point and U an affine open neighborhood of p such that $(f_i, \theta_i)_{1\leq i\leq \mathrm{m}}$ with $f_i \in \mathcal{O}_X(U)$ and $\theta_i \in \Theta_X(U)$ is a local coordinate system of X. Then it holds:

(a) The f_i define an étale morphism

$$f : U \to \mathbb{C}^{\mathrm{m}},\ u \mapsto (f_1(u), \ldots, f_{\mathrm{m}}(u)) :$$

The exact cotangent sequence (see Remark 1.2.5(f))

$$\begin{aligned} f^*\Omega^1_{\mathbb{C}^m} &\to \Omega^1_U \to \Omega^1_{U/\mathbb{C}^m} \to 0 \\ dx_i &\mapsto df_i \end{aligned}$$

implies that $\Omega^1_{U/\mathbb{C}^m,u} = 0$ (for all $u \in U$) as $df_1, \ldots, df_{\mathrm{m}}$ is a basis of $\Omega^1_{U,u}$ by assumption and Remark 1.2.5(b) and (c). Hence the morphism is G-unramified. The required flatness follows from [Sta18, Tag 07DY] as the regular system of parameters $x_1 - f_1(u), \ldots, x_{\mathrm{m}} - f_{\mathrm{m}}(u) \in \mathcal{O}_{\mathbb{C}^{\mathrm{m}},f(u)}$ is mapped under f to the regular sequence $f_1 - f_1(u), \ldots, f_{\mathrm{m}} - f_{\mathrm{m}}(u) \in \mathcal{O}_{U,u}$ for every $u \in U$ (see Remark 1.2.5(d)).

(b) Note that the $f_i - f_i(u)$ (for $u \in U$) are indeed local coordinates in an analytic neighborhood of u. So we can consider our notion of local coordinates as a counterpart of the notion in the analytic setting and the θ_i are unique lifts of the usual ∂_i in D_{m}. However, the f_i do not separate the points in the Zariski topology.

The following proposition shows the existence of local coordinate systems:

Proposition 1.2.9. *For each point* $\mathrm{p} \in X$ *exists an affine open neighborhood* $U \subseteq X$ *of* p*, regular functions* $f_1, \ldots, f_{\mathrm{m}} \in \mathcal{O}_X(U)$ *and differential operators* $\theta_1, \ldots, \theta_{\mathrm{m}} \in \Theta_X(U)$ *such that* $(f_i, \theta_i)_{1 \leq i \leq \mathrm{m}}$ *is a local coordinate system of* X *at* p*. These regular functions can be chosen to generate the maximal ideal of* $\mathcal{O}_{X,\mathrm{p}}$*.*

Moreover, if $Y \subseteq X$ *is a smooth subvariety of pure dimension* k *containing* p*, we can choose* U *and the local coordinate system such that additionally* $U \cap Y \subseteq U$ *has defining ideal sheaf generated by* $f_{\mathrm{k}+1}, \ldots, f_{\mathrm{m}}$ *and* $(\overline{f_i}, \theta_i)_{1 \leq i \leq \mathrm{k}}$ *is a local coordinate system on* $U \cap Y$*.*

Note that in the situation above the $\overline{f_i}$ and θ_i for $1 \leq i \leq \mathrm{k}$ can indeed by considered as regular functions and derivations on $U \cap Y$: The coordinate ring $\mathcal{O}_Y(U \cap Y)$ of $U \cap Y$ is expressed as $\mathcal{O}_X(U) / \langle f_{\mathrm{k}+1}, \ldots, f_{\mathrm{m}} \rangle$ if $U \cap Y \subseteq U$ has defining ideal sheaf generated by $f_{\mathrm{k}+1}, \ldots, f_{\mathrm{m}}$. Since $\theta_1, \ldots, \theta_{\mathrm{k}} \in \Theta_X(U)$ map the defining ideal of $U \cap Y$ in U to zero, we may interpret them as differentials on $U \cap Y$ by Remark 1.2.3.

Proof. As we want to describe $\mathcal{D}_X$ and $\mathcal{D}_Y$ locally in the neighborhood of a point and as every smooth algebraic variety has an open cover by smooth irreducible affines, we may assume that $Y \subseteq X \subseteq \mathbb{C}^{\mathrm{n}}$ are smooth irreducible affine varieties defined by the vanishing of the prime ideals $I_Y := \langle g_1, \ldots, g_{s_Y} \rangle$ and $I_X := \langle g_1, \ldots, g_{s_X} \rangle$ (with $s_X \leq s_Y$ and $g_1, \ldots, g_{s_Y} \in \mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_{\mathrm{n}}]$).

We construct for a given point $p = (p_1, \ldots, p_{\mathrm{n}}) \in Y \subseteq X$ an affine open neighborhood U in $\mathbb{C}^{\mathrm{n}}$ such that $\Theta_{U \cap X}$ and $\Theta_{U \cap Y}$ are a free $\mathcal{O}_{U \cap X}$- and $\mathcal{O}_{U \cap Y}$-modules of ranks m and k, respectively: Taking $Z \in \{X, Y\}$, and writing $a_p = \langle x_1 - p_1, \ldots, x_{\mathrm{n}} - p_{\mathrm{n}} \rangle \subseteq \mathbb{C}[\underline{x}]$ and, by abuse of notation, $\overline{a_p}$ for its image in $\mathbb{C}[\underline{x}]/I_Z$, we look at the regular local ring

$$(\mathcal{O}_{Z,p}, m_{Z,p}) = ((\mathbb{C}[\underline{x}]/I_Z)_{\overline{a_p}}, \mathcal{O}_{Z,p} \cdot \overline{a_p}).$$

We first determine a basis of the $\mathcal{O}_{Z,p}$-module $\Omega^1_{Z,p}$ using that $\Omega^1_{Z,p} \otimes_{\mathcal{O}_{Z,p}} \mathcal{O}_{Z,p} / m_{Z,p}$ is isomorphic to $m_{Z,p}/m^2_{Z,p}$ (see Remark 1.2.5(b) and (d)) as follows: Considering the canonical $\mathcal{O}_{Z,p} / m_{Z,p}$-vector space isomorphisms

$$\begin{aligned} m_{Z,p}/m^2_{Z,p} &\cong \mathbb{C}[\underline{x}]_{a_p} a_p / \mathbb{C}[\underline{x}]_{a_p} (a_p^2 + I_Z) \\ &\cong \left(\mathbb{C}[\underline{x}]_{a_p} a_p / \mathbb{C}[\underline{x}]_{a_p} a_p^2\right) / \left(\mathbb{C}[\underline{x}]_{a_p} (a_p^2 + I_Z) / \mathbb{C}[\underline{x}]_{a_p} a_p^2\right), \end{aligned}$$

we compute $\mathcal{O}_{Z,p} / m_{Z,p}$-bases of $\mathbb{C}[\underline{x}]_{a_p} a_p / \mathbb{C}[\underline{x}]_{a_p} a_p^2$ and $\mathbb{C}[\underline{x}]_{a_p} (a_p^2 + I_Z) / \mathbb{C}[\underline{x}]_{a_p} a_p^2$ by means of the $\mathbb{C}$-linear homomorphism

$$\lambda : \mathbb{C}[\underline{x}]_{a_p} \to (\mathcal{O}_{Z,p} / m_{Z,p})^{\mathrm{n}}, \ \frac{f}{g} \mapsto \left(\overline{\partial_1 \left(\frac{f}{g}\right)(p)}, \ldots, \overline{\partial_{\mathrm{n}} \left(\frac{f}{g}\right)(p)}\right).$$

We point out that the above morphism is independent of Z as $\mathcal{O}_{X,p} / m_{X,p}$ is canonically isomorphic to $\mathcal{O}_{Y,p} / m_{Y,p}$. This morphism induces an $\mathcal{O}_{Z,p} / m_{Z,p}$-vector space isomorphism

$$\overline{\lambda} : \mathbb{C}[\underline{x}]_{a_p} a_p / \mathbb{C}[\underline{x}]_{a_p} a_p^2 \cong (\mathcal{O}_{Z,p} / m_{Z,p})^{\mathrm{n}},$$

that maps the elements $\overline{x_1 - p_1}, \ldots, \overline{x_\mathrm{n} - p_\mathrm{n}}$ to the canonical basis of $(\mathcal{O}_{Z,p}/m_{Z,p})^\mathrm{n}$. The dimension of the $\mathcal{O}_{Z,p}/m_{Z,p}$-vector subspace $\mathbb{C}[\underline{x}]_{a_p}(a_p^2 + I_Z)/\mathbb{C}[\underline{x}]_{a_p}a_p^2$ equals the dimension of $\lambda(I_Z)$ and is hence the rank of the Jacobian matrix $\left(\overline{\partial_j(g_i)(p)}\right)_{\substack{1\leq i\leq s_Z\\ 1\leq j\leq \mathrm{n}}}$, namely $r_Z := \mathrm{n} - \dim Z$. This implies the existence of sets $T_1^Z := \{i_1, \ldots, i_{r_Z}\} \subseteq \{1, \ldots, s_Z\}$ and $T_2^Z := \{j_1, \ldots, j_{r_Z}\} \subseteq \{1, \ldots, \mathrm{n}\}$ of cardinality r_Z with $T_1^X \subseteq T_1^Y$ and $T_2^X \subseteq T_2^Y$ such that $d_Z := \det\left((\partial_{j_{l'}}(g_{i_l}))_{1\leq l,l'\leq r_Z}\right)$ does not vanish at p. Thus

$$\{\overline{g_i} \mid i \in T_1^Z\} \cup \{\overline{x_i - p_i} \mid i \notin T_2^Z\} \subseteq \mathbb{C}[\underline{x}]_{a_p}a_p/\mathbb{C}[\underline{x}]_{a_p}a_p^2$$

forms a basis of $\mathbb{C}[\underline{x}]_{a_p}a_p/\mathbb{C}[\underline{x}]_{a_p}a_p^2$. Hence a basis of $m_{Z,p}/m_{Z,p}^2$ is given by the residue classes of

$$\{\overline{x_i - p_i} \mid i \notin T_2^Z\} \subseteq m_{Z,p}$$

under the above chain of isomorphisms. Regarding the above basis of $\mathbb{C}[\underline{x}]_{a_p}a_p/\mathbb{C}[\underline{x}]_{a_p}a_p^2$ in the case $Z = Y$, we see that another basis of $m_{X,p}/m_{X,p}^2$ is also given by the residue classes of

$$\{\overline{x_i - p_i} \mid i \notin T_2^Y\} \cup \{\overline{g_i} \mid i \in T_1^Y \setminus T_1^X\} \subseteq \mathbb{C}[\underline{x}]_{a_p}a_p/\mathbb{C}[\underline{x}]_{a_p}a_p^2.$$

Assuming for simplicity $T_1^Z = \{1, \ldots, r_Z\}$ and $T_2^Z = \{\mathrm{n} - r_Z + 1, \ldots, \mathrm{n}\}$ and setting $f_i := x_i - p_i$ for $1 \leq i \leq \mathrm{k}$ and $f_{\mathrm{k}+i} = g_{r_X+i}$ for $1 \leq i \leq \mathrm{m} - \mathrm{k}$, we obtain by Nakayama's lemma

$$m_{Z,p} = \langle \overline{f_1}, \ldots, \overline{f_{\dim Z}} \rangle.$$

Since $m_{Z,p}/m_{Z,p}^2$ is isomorphic to $\Omega^1_{Z,p} \otimes_{\mathcal{O}_{Z,p}} \mathcal{O}_{Z,p}/m_{Z,p}$ as $\mathcal{O}_{X,p}/m_{Z,p}$-vector spaces via the map $\overline{f} \mapsto df \otimes \overline{1}$ by Remark 1.2.5(d), the differential forms $d\overline{f_1}, \ldots, d\overline{f_{\dim Z}}$ are a basis of the free $\mathcal{O}_{Z,p}$-module $\Omega^1_{Z,p}$ (see Remark 1.2.5(b)). As this holds for all $p' \in U_Z := Z \cap U$ for $U = D(d)$ with $d := d_X d_Y$, the $\mathcal{O}_{U_Z}$-module $\Omega^1_{U_Z}$ is free with basis $d\overline{f_1}, \ldots, d\overline{f_{\dim Z}}$. Taking the dual basis $\theta_1, \ldots, \theta_{\dim Z} \in \Theta_{U_Z}$ (see Remark 1.2.5(c)), we get

$$[\theta_i, \overline{f_j}] = \theta_i(\overline{f_j}) = \theta_i(d\overline{f_j}) = \delta_{ij}$$

for $1 \leq i, j \leq \dim Z$ and

$$\Theta_{U_Z} = \bigoplus_{1\leq i\leq \dim Z} \mathcal{O}_{U_Z}\,\theta_i.$$

To see that the θ_i commute note that $[\theta_i, \theta_j]$ is a derivation on $\mathcal{O}_{U_Z}$ for $1 \leq i < j \leq \dim Z$ and that we have hence a representation $[\theta_i, \theta_j] = \sum_{l=1}^{\dim Z} g_{ij}^l \theta_l$ (with $g_{ij}^l \in \mathcal{O}_{U_Z}$). By $[\theta_i, \theta_j](\overline{f_l}) = 0$ for $1 \leq l \leq \dim Z$, we deduce that $[\theta_i, \theta_j] = 0$. This proves that $\overline{f_1}, \ldots, \overline{f_{\dim Z}}$ is indeed a local coordinate system at p.

For the second part of the claim we show that $\mathcal{O}_Z$ is locally a complete intersection defined by the vanishing of $g_1, \ldots, g_{r_Z}$. We have

$$\mathcal{O}_Z(U_Z) = (\mathbb{C}[\underline{x}]/I_Z)[\overline{d}^{-1}] \cong \mathbb{C}[\underline{x}, x_{n+1}]/\tilde{I}_Z,$$

where $\tilde{I}_Z = \langle g_0, \dots, g_{s_Z} \rangle$ with $g_0 := 1 - x_{\mathrm{n}+1} d$. We may drop the $g_{\dim Z+1}, \dots, g_{s_Z}$ after replacing d by a suitable multiple of it as described below: Since the morphism

$$f : U_Z \to \mathbb{C}^{\dim Z},\ u = (u_1, \dots, u_{\mathrm{n}}) \mapsto (f_1(u), \dots, f_{\dim Z}(u))$$

defined by the local coordinates $f_1, \dots, f_{\dim Z}$ is étale by Remark 1.2.8(a), the conormal sequence (see Remark 1.2.5(g)) for the closed embedding $\iota_Z : U_Z \to \mathbb{C}^{\mathrm{n}+1}, u \mapsto (u, \frac{1}{d(u)})$ and the morphism $\mathbb{C}^{\mathrm{n}+1} \to \mathbb{C}^{\dim Z}, (u_1, \dots, u_{\mathrm{n}+1}) \mapsto (f_1(u_1, \dots, u_{\mathrm{n}}), \dots, f_{\dim Z}(u_1, \dots, u_{\mathrm{n}}))$ reads

$$0 \to \tilde{I}_Z / \tilde{I}_Z^2 \to \iota_Z^* \Omega_{\mathbb{C}^{n+1} / \mathbb{C}^{\dim Z}} \to 0.$$

This yields an isomorphism μ fitting into the diagram

$$\begin{array}{ccccc} \bigoplus_{i=0}^{r_Z} \mathcal{O}_Z(U_Z) e_i & \xrightarrow{\psi} & \tilde{I}_Z / \tilde{I}_Z^2 & \xrightarrow[\cong]{\mu} & \bigoplus_{i=\dim Z+1}^{\mathrm{n}+1} \mathcal{O}_Z(U_Z) dx_i. \\ e_i & \longmapsto & \overline{g_i},\ \overline{g} & \longmapsto & \sum_{i=\dim Z+1}^{\mathrm{n}+1} \overline{\partial_i(g)} dx_i \end{array}$$

Here, the map $\pi \circ \psi$ is given by $D_Z := \left(\overline{\partial_j(g_i)} \right)_{\substack{0 \le i \le r_Z \\ \dim Z+1 \le j \le \mathrm{n}+1}}$ with determinant

$$\det D_Z = \pm d \cdot \det \left(\left(\overline{\partial_j g_i} \right)_{\substack{1 \le i \le r_Z \\ \dim Z+1 \le j \le n}} \right) = \pm d \cdot d_Z,$$

that is invertible in $\mathcal{O}_Z(U_Z)$. Thus $\overline{g_0}, \dots, \overline{g_{r_Z}}$ form a basis of the free $\mathcal{O}_Z(U_Z)$-module $\tilde{I}_Z / \tilde{I}_Z^2$ implying that

$$\tilde{I}_Z = \langle g_0, \dots, g_{r_Z} \rangle + \tilde{I}_Z^2. \tag{1.2.2}$$

According to Nakayama's Lemma (see [Sta18, Tag 07RC]) there exists $h_Z \in 1 + \tilde{I}_Z$ such that $\mathbb{C}[\underline{x}, x_{n+1}][h_Z^{-1}] \cdot \tilde{I}_Z = {}_{\mathbb{C}[\underline{x}, x_{\mathrm{n}+1}][h_Z^{-1}]} \langle g_0, \dots, g_{r_Z} \rangle$. Therefore we obtain

$$\begin{aligned} \mathcal{O}_Z(U_Z) &\cong (\mathbb{C}[\underline{x}, x_{n+1}] / \tilde{I}_Z)[\overline{h_Z}^{-1}] \\ &\cong \mathbb{C}[\underline{x}, x_{\mathrm{n}+1}][h_Z^{-1}] / {}_{\mathbb{C}[\underline{x}, x_{\mathrm{n}+1}][h_Z^{-1}]} \langle g_1, \dots, g_{s_Z}, 1 - x_{\mathrm{n}+1} d \rangle \\ &\cong \mathbb{C}[\underline{x}, x_{\mathrm{n}+1}][h_Z^{-1}] / {}_{\mathbb{C}[\underline{x}, x_{\mathrm{n}+1}][h_Z^{-1}]} \langle g_1, \dots, g_{r_Z}, 1 - x_{\mathrm{n}+1} d \rangle \\ &\cong \left(\mathbb{C}[\underline{x}, x_{\mathrm{n}+1}] / {}_{\mathbb{C}[\underline{x}, x_{\mathrm{n}+1}]} \langle g_1, \dots, g_{r_Z}, 1 - x_{\mathrm{n}+1} d \rangle \right) [\overline{h_Z}^{-1}]. \end{aligned}$$

Multiplying h_Z with a suitable power a of d to replace it by a representative of $\overline{d^a h_Z}$ in $\mathbb{C}[\underline{x}]$, we finally get

$$\mathcal{O}_Z(U_Z) \cong \mathbb{C}[\underline{x}][(d h_Z)^{-1}] / {}_{\mathbb{C}[\underline{x}][(d h_Z)^{-1}]} \langle g_1, \dots, g_{r_Z} \rangle.$$

□

Definition 1.2.10. If U has been chosen as in the moreover-part of the above proposition, we call U a *coordinate neighborhood* (of Y in X). If U agrees with X, we say that U is a *global coordinate neighborhood*.

Remark 1.2.11. We keep the notation of the proof of Proposition 1.2.9.

(a) We point out that this proof is constructive if $Y \subseteq X \subseteq \mathbb{C}^{\mathrm{n}}$ are closed subvarieties of $\mathbb{C}^{\mathrm{n}}$. Indeed, all steps except for the determination of h_Z are obviously constructive and h_Z is determined as follows: Starting from Equation (1.2.2) we find an expression $g_i = \sum_{r_Z+1\leq j,l\leq s_Z} c^Z_{jl} g_j g_l + q^Z_i$ for $r_Z + 1 \leq i \leq s_Z$, where $q^Z_i \in \langle g_0, \ldots, g_{r_Z}\rangle$ and $c^Z_{j,l} \in \mathbb{C}[\underline{x}, x_{\mathrm{n}+1}]$ using Gröbner basis theory. From this we obtain a representation $g_i = \sum_{r_Z+1\leq j\leq s_Z} z^Z_{ij} g_j + q_i$ with $z^Z_{ij} \in \tilde{I}_Z$. Setting $h_Z := \det((\delta_{ij} - z^Z_{ij})_{r_Z+1\leq i,j\leq s_Z}) \in 1 + \tilde{I}_Z$, the proof of Nakayama's lemma in [Sta18, Tag 07RC] implies that $h_Z \tilde{I}_Z \subseteq \langle g_0, \ldots, g_{r_Z}\rangle$.

(b) For Y and X as in the proof of Proposition 1.2.9 we extend $f_1, \ldots, f_{\mathrm{m}}$ to a coordinate system on U as follows: One easily checks that setting $f_{\mathrm{m}+1} := g_1, \ldots, f_{\mathrm{n}} := g_{r_X}$ gives the coordinate system $f_1, \ldots, f_{\mathrm{n}}$ on $U \subseteq \mathbb{C}^{\mathrm{n}}$. An explicit representation of the corresponding derivations $\theta_1, \ldots, \theta_{\mathrm{n}}$ in terms of the derivations $\partial_1, \ldots, \partial_n$ is found as follows (see also [Oak96, Section 1]): Setting $\theta_i = \sum_{1\leq l\leq \mathrm{n}} a_{il}\partial_l$ with $a_{il} \in \mathbb{C}[\underline{x}][d^{-1}]$, the a_{il} have to satisfy

$$\Big(a_{il}\Big)_{\substack{1\leq i\leq \mathrm{n}\\ 1\leq l\leq \mathrm{n}}} \cdot \Big(\partial_l(f_j)\Big)_{\substack{1\leq l\leq \mathrm{n}\\ 1\leq j\leq \mathrm{n}}} = \Big(\delta_{ij}\Big)_{\substack{1\leq i\leq \mathrm{n}\\ 1\leq j\leq \mathrm{n}}}. \tag{1.2.3}$$

After performing column switches, the matrix in the middle agrees with

$$\begin{pmatrix} (\delta_{lj})_{\substack{1\leq l\leq \mathrm{k}\\ 1\leq j\leq \mathrm{k}}} & (\partial_l(g_j))_{\substack{1\leq l\leq \mathrm{k}\\ 1\leq j\leq r_Y}} \\ (0)_{\substack{\mathrm{k}+1\leq l\leq \mathrm{n}\\ 1\leq j\leq \mathrm{k}}} & (\partial_l(g_j))_{\substack{\mathrm{k}+1\leq l\leq \mathrm{n}\\ 1\leq j\leq r_Y}} \end{pmatrix}.$$

As $(\partial_l(g_j))_{\substack{\mathrm{k}+1\leq l\leq \mathrm{n}\\ 1\leq j\leq r_Y}}$ is a divisor of d, the above matrix is invertible over $\mathbb{C}[\underline{x}][d^{-1}]$ and hence the a_{il} are uniquely determined by Equation (1.2.3) and can be explicitly computed using Cramer's rule. Also note that the $\theta_1, \ldots, \theta_{\dim Z}$ induce derivations on U_Z which correspond to the coordinates $\overline{f_1}, \ldots, \overline{f_{\dim Z}} \in \mathcal{O}_Z(U_Z)$.

Remark 1.2.12. Consider the (smooth) irreducible affine variety $X \subseteq \mathbb{C}^{\mathrm{n}}$ defined by the vanishing of the prime ideal $I \subseteq \mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_{\mathrm{n}}]$ and its pure codimension one subvariety Y. Moreover, let X be a global coordinate neighborhood of Y with global coordinates $(\overline{f_i}, \theta_i)_{1\leq i\leq \mathrm{m}}$ with $f_i \in \mathbb{C}[\underline{x}]$ such that $Y = V(\overline{f_{\mathrm{m}}})$. By Remark 1.2.3 the θ_i are induced by $\theta^l_i \in \mathrm{Der}(\mathbb{C}[\underline{x}])$. Note that we may assume that f_{m} agrees with some variable x_i: Namely, the map

$$X \hookrightarrow X \times \mathbb{C}_t,\ \underline{x} \mapsto (\underline{x}, f_{\mathrm{m}}(\underline{x}))$$

induces isomorphisms $X \cong V(I \cup \{t - f_{\mathrm{m}}\}) \subseteq \mathbb{C}^{\mathrm{n}} \times \mathbb{C}_t$ and $Y \cong V(I \cup \{t - f_{\mathrm{m}}, t\})$. Furthermore, $(\overline{f_1}, \ldots, \overline{f_{m-1}}, \overline{t}, \theta_1, \ldots, \theta_{\mathrm{m}-1}, \theta_{\mathrm{m}} + \overline{\partial_t})$ is global coordinate system on $V(\{I, t - f_{\mathrm{m}}\})$. We also point out that $\theta^l_{\mathrm{m}} + \partial_t$ is a lift of $\theta_{\mathrm{m}} + \overline{\partial_t}$ and that we have $(\theta^l_{\mathrm{m}} + \partial_t)(t) = 1$ and $\theta^l_i(t) = 0$ for $1 \leq i \leq \mathrm{m} - 1$.

Remark 1.2.13. We keep the notation of (the proof of) Proposition 1.2.9 and still assume that X is affine.

(a) We compute a finite cover $\{U\}_{U \in \mathcal{U}}$ with $U \subseteq X$ affine open having a global coordinate system by taking the nonvanishing loci of all possible nonzero $r_X \times r_X$-minors of the Jacobian matrix $\left(\overline{\partial_j(g_i)}\right)_{\substack{1 \leq i \leq s_X \\ 1 \leq j \leq \mathrm{n}}}$ as the elements of $\mathcal{U}$ and proceeding as in that proof to determine actual local coordinates. Similarly, this cover can be refined to a cover $\mathcal{U}'$ such that $U' \in \mathcal{U}'$ with the property $U' \cap Y \neq \emptyset$ has a computable coordinate system as in the second part of the statement of Proposition 1.2.9.

(b) We can refine the cover $\mathcal{U}'$ from Part (a) to a cover $\mathcal{U}''$ such that for $U'' \in \mathcal{U}''$, with $U'' \cap Y \neq 0$ and local coordinates $f_1, \ldots, f_{\dim X}$, the $f_1, \ldots, f_k$ for $k \geq \dim Y$ are global coordinates on $U \cap V(f_{k+1}, \ldots, f_{\dim Y})$ with defining ideal sheaf of this subvariety of U generated by $f_{k+1}, \ldots, f_{\dim Y}$.

Corollary 1.2.14. *The sheaves Θ_X and $\mathcal{D}_X$ are $\mathcal{O}_X$-locally free and hence in particular $\mathcal{O}_X$-quasi-coherent.*

Remark 1.2.15. Let $\iota : Y \hookrightarrow X$ be a closed embedding of smooth algebraic varieties with defining ideal $\mathcal{I}$. Then we have for $U \subseteq X$ affine open and $f \in \mathcal{O}_X(U)$ by the quasi-coherence of Θ_X and $\mathcal{I}$ that

$$\mathrm{Der}_{\mathcal{I}}(\mathcal{O}_X)(U \cap D(f)) = \{\theta \in \Theta_X(U)[f^{-1}] \mid \theta(\mathcal{I}(U)[f^{-1}]) \subseteq \mathcal{I}(U)[f^{-1}]\}.$$

As $\mathcal{I}(U)$ is $\mathcal{O}_X(U)$-finitely generated there exists for $\theta \in \Theta_X(U)$ with $\theta(\mathcal{I}(U)[f^{-1}]) \subseteq \mathcal{I}(U)[f^{-1}]$ a natural number $k \in \mathbb{N}$ such that $f^k\theta(\mathcal{I}(U)) \subseteq \mathcal{I}(U)$ showing

$$\mathrm{Der}_{\mathcal{I}}(\mathcal{O}_X)(U \cap D(f)) = \mathrm{Der}_{\mathcal{I}}(\mathcal{O}_X)(U)[f^{-1}].$$

By [Vak17, 13.3.3] this implies that $\mathrm{Der}_{\mathcal{I}}(\mathcal{O}_X)$ is $\mathcal{O}_X$-quasi-coherent.

1.2.3 Order filtration

As already indicated in Subsection 1.2.1, the sheaf of differential operators on X can also be defined using commutators. Namely, set $\mathcal{D}^k_X := 0$ for $k < 0$, inductively define for $k \geq 0$ the sheaves of $\mathcal{O}_X$-modules $\mathcal{D}^k_X$ on the distinguished affine base by

$$\mathcal{D}^k_X(U) := \{P \in \mathrm{End}_{\mathbb{C}}(\mathcal{O}_X(U)) \mid [P, f] \in \mathcal{D}^{k-1}_X(U) \text{ for all } f \in \mathcal{O}_X(U)\}$$

and put

$$\mathcal{D}'_X(U) := \bigcup_{k\in\mathbb{N}} \mathcal{D}^k_X(U)$$

for $U \subseteq X$ affine open. Arguing as in Subsection 1.2.1, $\mathcal{D}'_X$ extends uniquely to a sheaf on X. One can show that $\mathcal{D}'_X$ coincides with $\mathcal{D}_X$ and that $\mathcal{D}^k_X \cdot \mathcal{D}^l_X \in \mathcal{D}^{k+l}_X$ for $k, l \in \mathbb{Z}$. Hence, setting $\mathcal{F}^\circ_k \mathcal{D}_X := \mathcal{D}^k_X$ for $k \in \mathbb{Z}$ turns $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$ into a filtered ring.

Definition 1.2.16. We call $(\mathcal{D}_X, \mathcal{F}^\circ_X)$ the *order filtration* (by the order of differential operators) on $\mathcal{D}_X$.

In local coordinates the order filtration is obviously described as follows:

Lemma 1.2.17. *Let $(f_i, \theta_i)_{1\leq i\leq \mathrm{m}}$ be a local coordinate system on an affine open neighborhood of U of X. Then the order filtration on $\mathcal{D}_X$ is locally represented by*

$$(\mathcal{F}^\circ_\bullet \mathcal{D}_X)_U = \mathcal{F}^\circ_\bullet \mathcal{D}_U = \bigoplus_{\alpha\in\mathbb{N}^{\mathrm{m}}:|\alpha|\leq\bullet} \mathcal{O}_U\, \underline{\theta}^\alpha$$

and

$$\mathrm{Gr}^{\mathcal{F}^\circ}\mathcal{D}_U = \mathcal{O}_U[\zeta_1, \ldots, \zeta_{\mathrm{m}}],$$

where $\zeta_i := \theta_i \mod F^\circ_0 \mathcal{D}_U$ for $1 \leq i \leq \mathrm{m}$.

Note that we used for the representation of the associated graded sheaf $\mathrm{Gr}^{\mathcal{F}^o}\mathcal{D}_U$ the fact that $[p, q] \in \mathcal{F}^\circ_{k+l-1}\mathcal{D}_U$ for $p \in \mathcal{F}^\circ_k \mathcal{D}_U$ and $q \in \mathcal{F}^\circ_l \mathcal{D}_U$.

As $\mathrm{Gr}^{\mathcal{F}^o}\mathcal{D}_X$ is locally isomorphic to a polynomial ring over the commutative ring $\mathcal{O}_X$, it is locally Noetherian and Proposition 1.1.16 implies:

Proposition 1.2.18. *The sheaf of differential operators $\mathcal{D}_X$ is locally Noetherian.*

1.3 $\mathcal{D}$-modules

A $\mathcal{D}$-module is a sheaf of modules over a sheaf of rings of differential operators. It can be considered as an algebraisation of a system of linear partial differential equations.

1.3.1 Introduction to $\mathcal{D}$-modules

Recall our convention that if not stated otherwise, we mean by a $\mathcal{D}_X$-module, also called a $\mathcal{D}$-*module on X*, a left $\mathcal{D}_X$-module. Proposition 1.2.18 and Proposition 1.1.7 give the following characterization of coherent $\mathcal{D}_X$-modules:

Proposition 1.3.1.

(a) $\mathcal{D}_X$ is a coherent ring.

(b) A $\mathcal{D}_X$-module is coherent if and only if it is quasi-coherent over $\mathcal{O}_X$ and locally finitely generated over $\mathcal{D}_X$.

There is in fact an equivalence of categories between the categories of left and right $\mathcal{D}_X$-modules. Before explaining this equivalence, we give examples of some important left and right $\mathcal{D}$-modules, which will be the building blocks of this equivalence as well as of the direct image functor for $\mathcal{D}$-modules.

Example 1.3.2. The sheaf of regular functions $\mathcal{O}_X$ is made a left $\mathcal{D}_X$-module as follows: A differential operator $p \in \mathcal{D}_X$ is by definition a morphism $p : \mathcal{O}_X \to \mathcal{O}_X$ and hence acts on $f \in \mathcal{O}_X$ by applying p to f. We denote this action by $p(f)$. This turns $\mathcal{O}_X$ into a left $\mathcal{D}_X$-module. We point out that it is important to distinguish the action of p on f and the product of p with f inside $\mathcal{D}_X$: For example, in the case $X = \mathbb{C}^2$ with corrdinates x_1 and x_2, we have $\mathcal{D}_X \cong \bigoplus_{\alpha\in\mathbb{N}^2} \mathcal{O}_X \,\underline{\partial}^\alpha$ and $\partial_1(x_2) = 0 \in \mathcal{O}_X$, but $\partial_1 x_2 = x_2\partial_1 \neq 0 \in \mathcal{D}_X$. Using the commutation rules, one easily proves that $\mathcal{O}_X$ is isomorphic to $\mathcal{D}_X \,/\, \mathcal{D}_X \,\Theta_X$ as a left $\mathcal{D}_X$-module.

Example 1.3.3. Our basic example for a right $\mathcal{D}_X$-module is $\omega_X := \bigwedge^{\mathrm{m}} \Omega_X$, which is obviously an $\mathcal{O}_X$-module. The natural right action of $\theta \in \Theta_X$ on $\omega \in \omega_X$ is defined by the Lie-derivative $\mathrm{Lie}\,\theta$, namely

$$\omega\theta := -(\mathrm{Lie}\,\theta)\omega,$$

where, interpreting ω_X as the dual of $\bigwedge^{\dim X} \Theta_X$, the Lie-derivative is given by

$$((\mathrm{Lie}\,\theta)\omega)(\theta_1, \dots, \theta_{\mathrm{m}}) := \theta(\omega(\theta_1, \dots, \theta_{\mathrm{m}})) - \sum_{i=1}^{\mathrm{m}} \omega(\theta_1, \dots, [\theta, \theta_i], \dots, \theta_{\mathrm{m}})$$

for $\theta_1, \dots, \theta_{\mathrm{m}} \in \Theta_X$. By [HTT08], this defines indeed a right $\mathcal{D}_X$-module structure on ω_X. Locally, this operation is given by

$$(g df_1 \wedge \dots \wedge df_{\mathrm{m}})\underline{\theta}^\alpha = ((-1)^{|\alpha|}\underline{\theta}^\alpha(g)) df_1 \wedge \dots \wedge df_{\mathrm{m}},$$

where $(f_i, \theta_i)_{1\le i\le \mathrm{m}}$ is a local coordinate system of $U \subseteq X$ and $g \in \mathcal{O}_U$.

The module ω_X induces so-called side-changing operations on the categories $\mathrm{Mod}(\mathcal{D}_X)$ and $\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$:

Proposition 1.3.4 ([HTT08], 1.2.12)**.** *The correspondence*

$$\omega_X \otimes_{\mathcal{O}_X} (\bullet) : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X^{\mathrm{op}})$$

is an equivalence of categories with quasi-inverse is given by

$$\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \bullet) : \mathrm{Mod}(D_X^{\mathrm{op}}) \to \mathrm{Mod}(D_X).$$

Here, for $\mathcal{M} \in \operatorname{Mod}(\mathcal{D}_X)$, we equip $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ with a right $\mathcal{D}_X$-structure via

$$(\omega \otimes m)\theta = \omega\theta \otimes m - \omega \otimes \theta m,$$

where $m \in \mathcal{M}$, $\omega \in \omega_X$ and $\theta \in \Theta_X$. Similarly, the left action of Θ_X on $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N})$ (with $\mathcal{N} \in \operatorname{Mod}(D_X^{\mathrm{op}})$) is defined by

$$(\theta\varphi)(\omega) = -\varphi(\omega)\theta + \varphi(\omega\theta),$$

where $\theta \in \Theta_X$, $\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N})$ and $\omega \in \omega_X$.

1.3.2 Order filtered $\mathcal{D}$-modules

When talking about (left or right) filtered $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$-modules, we always assume that the filtration on the modules is indexed by the integers. We point out that a $\mathcal{D}_X$-module $\mathcal{M}$ is coherent if and only if a globally defined good $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$-filtration $\mathcal{G}_\bullet \mathcal{M}$ exists (see [HTT08, Theorem 2.1.3]). We equip our two standard examples from Example 1.3.2 and Example 1.3.3 with filtrations as follows:

Example 1.3.5. The one-step filtration

$$\mathcal{F}_j \mathcal{O}_X = \begin{cases} \mathcal{O}_X, & \text{if } j \geq 0 \\ 0, & \text{if } j < 0 \end{cases}$$

turns $(\mathcal{O}_X, \mathcal{F}_\bullet)$ into a well-filtered $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$-module.

Informally speaking, by assigning a differential form degree -1, the right $\mathcal{D}_X$-module ω_X is endowed with a good $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$-module structure via

$$\mathcal{F}_j \omega_X = \begin{cases} \omega_X, & \text{if } j \geq -\dim X \\ 0, & \text{if } j < \dim X. \end{cases}$$

In order to extend the equivalence of categories between left and right $\mathcal{D}_X$-modules in Proposition 1.3.4 to the filtered situation, we first need to define a filtration on the $\mathcal{O}_X$-tensor product of a right and a left $\mathcal{D}_X$-module.

Definition 1.3.6. Let $(\mathcal{M}, \mathcal{F}_\bullet)$ and $(\mathcal{N}, \mathcal{F}'_\bullet)$ be filtered left and right $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$-modules, respectively. We define a filtration $\mathcal{G}_\bullet$ on the $\mathcal{O}_X$-tensor product by

$$\mathcal{G}_\bullet(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}) = \sum_{i \in \mathbb{Z}} \mathcal{F}'_i \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{F}_{\bullet - i} \mathcal{M},$$

where we mean by the right hand side the image of $\sum_{i \in \mathbb{Z}} \mathcal{F}'_i \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{F}_{\bullet - i} \mathcal{M}$ in $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}$. We write $\mathcal{F}'_\bullet \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{F}_\bullet \mathcal{M}$ for $(\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{G}_\bullet)$.

Using the above one-step filtration on ω_X, we induce the following filtration on the associated right object of a left $\mathcal{D}_X$-module:

Definition 1.3.7. Let $(\mathcal{M}, \mathcal{F}_\bullet)$ be filtered $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$-module. We define a filtration $\mathcal{F}_\bullet$ on $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ by setting

$$\mathcal{F}_\bullet(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) := \mathcal{F}_\bullet\, \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}_\bullet\, \mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{F}_{\bullet + \dim X}\, \mathcal{M}\,.$$

In particular, $(\mathcal{M}, \mathcal{F}_\bullet)$ is well $(\mathcal{D}_X, \mathcal{F}^\circ_\bullet)$-filtered, if and only if $(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{F}_\bullet)$ is. Proposition 1.3.4 induces an equivalence of the associated categories of filtered objects

$$\mathcal{F}_\bullet\, \omega_X \otimes_{\mathcal{O}_X} (\bullet) : \mathrm{Mod}(\mathcal{F}^\circ_\bullet\, \mathcal{D}_X) \to \mathrm{Mod}(\mathcal{F}^\circ_\bullet\, \mathcal{D}_X^{\mathrm{op}}) \tag{1.3.1}$$

with quasi-inverse

$$F_\bullet \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \bullet) : \mathrm{Mod}(\mathcal{F}^\circ_\bullet\, \mathcal{D}_X^{\mathrm{op}}) \to \mathrm{Mod}(\mathcal{F}^\circ_\bullet\, \mathcal{D}_X), \tag{1.3.2}$$

where $F_\bullet \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}) := \{\varphi \in \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}) \mid \varphi(\omega_X) \subseteq F_{\bullet - \dim X}\, \mathcal{N}\}$ for the right $F^\circ_\bullet\, \mathcal{D}_X$-module $F_\bullet\, \mathcal{N}$.

1.4 Direct images of $\mathcal{D}$-modules

Consider the morphism $\phi : X \to Y$ of smooth equidimensional algebraic varieties of dimensions m and n. Our aim is to associate to ϕ a direct image functor ϕ_+ from the category of (bounded complexes of) $\mathcal{D}_X$-modules to the category of (bounded complexes of) $\mathcal{D}_Y$-modules. Note that ϕ induces only a morphism of ringed spaces $\phi_* : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and not one of the ringed spaces $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$. In other words, there is in general no map $\mathcal{D}_Y \to \phi\, \mathcal{D}_X$ and hence the sheaf-theoretic direct images under ϕ of $\mathcal{D}_X$-modules do not have the structure of a $\mathcal{D}_Y$-module. However, ϕ^{-1} is left adjoint to ϕ, so there is the natural unit map $\mathcal{D}_Y \to \phi\phi^{-1}\, \mathcal{D}_Y$ allowing us to define a direct image functor as outlined below: To equip a $\mathcal{D}_X$-module with a left $\phi^{-1}\, \mathcal{D}_Y$-structure, we tensor it in the category of $\mathcal{D}_X$-modules with a certain $(\phi^{-1}\, \mathcal{D}_Y, \mathcal{D}_X)$-bimodule called transfer module. The natural unit map then endows the sheaf theoretic direct image of this tensor product with a natural $\mathcal{D}_Y$-structure. This amounts to composing a right exact functor, namely tensoring with the transfer module, with the left exact sheaf theoretic direct image functor. Thus this construction does not commute with composition of morphisms. To remedy this, we work in the corresponding derived categories.

1.4.1 Transfer modules

Given a morphism $\phi : X \to Y$, we introduce the transfer modules $\mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X}$ which turn the right and left $\mathcal{D}_X$-module $\mathcal{N}$ and $\mathcal{M}$ into right and left $\phi^{-1}\, \mathcal{D}_Y$-modules

$\mathcal{N} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ and $\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}$, respectively: We associate to ϕ the so-called *first transfer module* given by

$$\mathcal{D}_{X\to Y} := \phi^* \mathcal{D}_Y = \mathcal{O}_X \otimes_{\phi^{-1}\mathcal{O}_Y} \phi^{-1}\mathcal{D}_Y.$$

This module carries a $(\mathcal{D}_X, \phi^{-1}\mathcal{D}_Y)$-bimodule structure: While its right $\phi^{-1}\mathcal{D}_Y$-structure is simply given by right multiplication on the second factor, the left structure is defined as described below: By the relative cotangent sequence (see Remark 1.2.5(f)) we obtain an $\mathcal{O}_X$-linear map $\phi^*\Omega^1_Y \to \Omega^1_X$ with $\mathcal{O}_X$-dual

$$\alpha : \Theta_X \to \mathcal{H}om_{\mathcal{O}_X}(\phi^*\Omega^1_Y, \mathcal{O}_X) \cong \mathcal{H}om_{\phi^{-1}\mathcal{O}_Y}(\phi^{-1}\Omega^1_Y, \mathcal{O}_X \otimes_{\phi^{-1}\mathcal{O}_Y} \phi^{-1}\mathcal{O}_Y).$$

Since $\phi^{-1}\Omega^1_Y$ is a locally free $\phi^{-1}\mathcal{O}_Y$- module, we have for $U \subseteq Y$ open such that Ω_U is $\mathcal{O}_X$-free, that the $\mathcal{O}_{\phi^{-1}U}$-module $\mathcal{H}om_{\phi^{-1}\mathcal{O}_U}(\phi^{-1}\Omega^1_U, \mathcal{O}_{\phi^{-1}U} \otimes_{\phi^{-1}\mathcal{O}_U} \phi^{-1}\mathcal{O}_U)$ is isomorphic to

$$\begin{aligned}
&\mathcal{H}om_{\phi^{-1}\mathcal{O}_U}\Big(\bigoplus_{1\le i\le \mathrm{n}} \phi^{-1}\mathcal{O}_U, \mathcal{O}_{\phi^{-1}U} \otimes_{\phi^{-1}\mathcal{O}_U} \phi^{-1}\mathcal{O}_U\Big)\\
&\cong \bigoplus_{1\le i\le \mathrm{n}} \mathcal{H}om_{\phi^{-1}\mathcal{O}_U}(\phi^{-1}\mathcal{O}_U, \mathcal{O}_{\phi^{-1}U} \otimes_{\phi^{-1}\mathcal{O}_U} \phi^{-1}\mathcal{O}_U)\\
&\cong \bigoplus_{1\le i\le \mathrm{n}} (\mathcal{O}_{\phi^{-1}U} \otimes_{\phi^{-1}\mathcal{O}_U} \phi^{-1}\mathcal{O}_U)\\
&\cong \mathcal{O}_{\phi^{-1}U} \otimes_{\phi^{-1}\mathcal{O}_U} \mathcal{H}om_{\phi^{-1}\mathcal{O}_U}(\phi^{-1}\Omega^1_U, \phi^{-1}\mathcal{O}_U)\\
&\cong \mathcal{O}_{\phi^{-1}U} \otimes_{\phi^{-1}\mathcal{O}_U} \phi^{-1}\Theta_U,
\end{aligned}$$

where we also write ϕ for the map $\phi^{-1}U \to U, x \mapsto \phi(x)$. Composing $\alpha_{\phi^{-1}U} : \Theta_{\phi^{-1}U} \to \mathcal{H}om_{\mathcal{O}_{\phi^{-1}U}}(\phi^*\Omega^1_U, \mathcal{O}_{\phi^{-1}U})$ with these local isomorphisms, we obtain a map

$$\alpha'_{\phi^{-1}U} : \Theta_{\phi^{-1}U} \to \mathcal{O}_{\phi^{-1}U} \otimes_{\phi^{-1}\mathcal{O}_U} \phi^{-1}\Theta_U,$$

which induces a left $\mathcal{D}_{\phi^{-1}U}$-structure on $(\mathcal{D}_{X\to Y})_{\phi^{-1}U}$ via

$$\theta(a \otimes p) = \theta(a) \otimes p + \sum_j a g_j \otimes \theta_j p,$$

where $\theta \in \Theta_{\phi^{-1}U}, a \in \mathcal{O}_{\phi^{-1}U}, p \in \phi^{-1}\mathcal{D}_U$ and $\alpha'_{\phi^{-1}U}(\theta) = \sum_j g_j \otimes \theta_j$. For a proof that the above formula is well-defined on the tensor product $a \otimes p$ see [CJ93, Subsection 2.1.1]. In local coordinates $\{y_i, \theta_{y_i}\}_{1\le i\le \mathrm{n}}$ on U, we express this action as

$$\theta(a \otimes p) = \theta(a) \otimes p + \sum_{1\le j\le \mathrm{n}} a\theta(y_j \circ \phi) \otimes \theta_{y_j} p,$$

where we may interpret θ_{y_j} as an element of $\phi^{-1}\Theta_U$ since this module is isomorphic to $\bigoplus_{1\le j\le \mathrm{n}} \phi^{-1}\mathcal{O}_U\, \partial_{y_j}$. Note that the above action is indeed independent of the choice of local coordinates (see [BGK+87, VI.4.1]) hence giving a well-defined left $\mathcal{D}_{\phi^{-1}D_Y}$-structure on

$\mathcal{D}_{X\to Y}$. One easily checks that the left $\mathcal{D}_X$ and the right $\phi^{-1}\mathcal{D}_Y$-structure are compatible, thus showing that the first transfer module has the claimed bimodule-structure.

Now we use side-changing operations on both sides to define the *second transfer module*

$$\mathcal{D}_{Y\leftarrow X} := \mathcal{H}om_{\phi^{-1}\mathcal{O}_Y}(\phi^{-1}\omega_Y, \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}),$$

which is a $(\phi^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule. Indeed, the module structure is induced via the left-right transformation by the module structure of $\mathcal{D}_{X\to Y}$: The left $\phi^{-1}\mathcal{D}_Y$-structure is given by

$$(\theta_Y\psi)(w) = -\psi(w)\theta_Y + \psi(w\theta_Y),$$

where $\phi^{-1}\mathcal{D}_Y$ acts on $\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}$ via right multiplication on the second factor, and the right $\mathcal{D}_X$-action is described by

$$(\psi\theta_X)(w) = \sum_i (w_i\theta_X \otimes s_i - w_i \otimes \theta_X s_i)$$

(where $\theta_Y \in \phi^{-1}\Theta_Y$, $\theta_X \in \Theta_X$, $\psi \in \mathcal{D}_{Y\leftarrow X}$, $w \in \phi^{-1}\omega_Y$ and $\psi(w) = \sum_i w_i \otimes s_i$).

Example 1.4.1. If $\iota : U \to X$ is an open embedding, then $\iota^{-1}\mathcal{D}_X = \mathcal{D}_U$ and hence $\mathcal{D}_{U\to X} = \mathcal{D}_U$ and similarly $\mathcal{D}_{X\leftarrow U} = \mathcal{D}_U$ with the canonical bimodule structures given by left and right multiplication.

Example 1.4.2. We describe the transfer modules under the closed embedding of varieties $\iota : X \to Y$ with ideal sheaf $\mathcal{I}$. We have a representation $\mathcal{O}_X = \iota^{-1}\iota\,\mathcal{O}_X \cong \iota^{-1}(\mathcal{O}_Y/\mathcal{I})$ and hence the first transfer module is globally expressed as

$$\mathcal{D}_{X\to Y} \cong \iota^{-1}(\mathcal{D}_Y/\mathcal{I}\mathcal{D}_Y)$$

with canonical right $\iota^{-1}\mathcal{D}_Y$-action and left $\mathcal{D}_X$-action induced by composition of the isomorphism $\mathcal{O}_X \cong \iota^{-1}(\mathcal{O}_Y/\mathcal{I})$ and the natural map $\Theta_X \to \mathcal{O}_X \otimes_{\iota^{-1}\mathcal{O}_Y} \iota^{-1}\Theta_Y \cong \iota^{-1}(\Theta_Y/\mathcal{I}\Theta_Y)$ with left multiplication.

Consider now an affine open neighborhood $U \subseteq Y$ with local coordinates $(y_i, \theta_i)_{1\le i\le \mathrm{n}}$ as in Proposition 1.2.9 such that $\iota(X)$ is locally defined by $y_{\mathrm{m}+1} = \cdots = y_{\mathrm{n}} = 0$ and $y_1, \ldots, y_{\mathrm{m}}$ induce local coordinates on $\iota^{-1}U$. Using that ω_X and ω_Y are locally $\mathcal{O}_X$- and $\mathcal{O}_Y$-free, respectively, we obtain

$$\begin{aligned}(\mathcal{D}_{Y\leftarrow X})_{\iota^{-1}U} &= \mathcal{H}om_{\iota^{-1}\mathcal{O}_U}(\iota^{-1}\omega_U, \omega_{\iota^{-1}U} \otimes_{\mathcal{O}_{\iota^{-1}U}} \iota^{-1}(\mathcal{D}_U/\mathcal{I}_U\mathcal{D}_U))\\ &\cong \mathcal{H}om_{\iota^{-1}\mathcal{O}_U}(\iota^{-1}\omega_U, \iota^{-1}(\mathcal{D}_U/\mathcal{I}_U\mathcal{D}_U))\\ &\overset{\psi}{\cong} \iota^{-1}(\mathcal{D}_U/\mathcal{I}_U\mathcal{D}_U)\\ &\cong \iota^{-1}(\mathcal{D}_U/\mathcal{D}_U\mathcal{I}_U)\end{aligned}$$

with map ψ given by $\varphi \mapsto \varphi(dy_1 \wedge \cdots \wedge dy_{\mathrm{n}})$. Under these isomorphisms, the left $\iota^{-1}\mathcal{D}_U$-operation on $\iota^{-1}(\mathcal{D}_U\,/\mathcal{D}_U\mathcal{I}_U)$ is given by left multiplication and the right $\mathcal{D}_{\iota^{-1}U}$-action is induced in analogy to the left $\mathcal{D}_X$-action on the first transfer module. This shows $\mathcal{D}_{Y\leftarrow X} \cong \iota^{-1}(\mathcal{D}_X\,/\,\mathcal{D}_X\,\mathcal{I})$ also globally.

On the other hand, we may represent $\mathcal{D}_{X\to Y}$ as a locally free $\mathcal{D}_X$-module as described below: Setting $x_i = y_i \circ \iota$ for $i = 1, \ldots, \mathrm{m}$ gives local coordinates $x_1, \ldots, x_{\mathrm{m}}$ on $\iota^{-1}U$ with differentials $\theta_{x_1}, \ldots, \theta_{x_{\mathrm{m}}}$ which are sent to $1 \otimes \theta_1, \ldots, 1 \otimes \theta_{\mathrm{m}}$ under the natural map $\Theta_{\iota^{-1}U} \to \mathcal{O}_{\iota^{-1}U} \otimes_{i^{-1}\mathcal{O}_U} i^{-1}\Theta_U$. As we have $\mathcal{D}_U = \bigoplus_{\alpha\in\mathbb{N}^{\mathrm{n}}} \mathcal{O}_U\,\underline{\theta}^{\alpha}$, the first transfer module $(\mathcal{D}_{X\to Y})_{\iota^{-1}U}$ is written as

$$\bigoplus_{\alpha\in\mathbb{N}^{\mathrm{n}}} (\mathcal{O}_{\iota^{-1}U} \otimes_{\iota^{-1}\mathcal{O}_U} \iota^{-1}\,\mathcal{O}_U)\underline{\theta}^{\alpha} \cong \bigoplus_{\alpha\in\mathbb{N}^{\mathrm{n}}} \mathcal{O}_{\iota^{-1}U}\,\underline{\theta}^{\alpha} \cong \mathcal{D}_{\iota^{-1}U} \otimes_{\mathbb{C}} \mathbb{C}[\theta_{\mathrm{m}+1}, \ldots, \theta_{\mathrm{n}}].$$

The left $\mathcal{D}_{\iota^{-1}U}$-action on the right hand side module is given by left multiplication on the first factor, hence showing that the first transfer module is $\mathcal{D}_X$-locally free. Note that the right $\iota^{-1}\mathcal{D}_U$-structure on the above module is described as follows: The differential θ_i $(1 \le i \le \mathrm{m})$ acts via the composition of the map $\theta_i \mapsto \theta_{x_i}$ with right multiplication of $\mathcal{D}_{\iota^{-1}U}$ on the first factor, whereas θ_i for $\mathrm{m}+1 \le i \le \mathrm{n}$ operates by increasing the exponent of θ_i by one. The right action of $f \in i^{-1}\mathcal{O}_U$ on $p\otimes q \in \mathcal{D}_{\iota^{-1}U} \otimes_{\mathbb{C}} \mathbb{C}[\theta_{\mathrm{m}+1}, \ldots, \theta_{\mathrm{n}}]$ is expressed as $\sum_i pf_i \otimes q_i$ if $qf = \sum_i f_i \cdot q_i$ with $q_i \in \mathbb{C}[\theta_{\mathrm{m}+1}, \ldots, \theta_{\mathrm{n}}]$ and $f_i \in \iota^{-1}\mathcal{O}_U$ in the ring $\iota^{-1}\mathcal{D}_U$, where the right action of f_i on p is given by composition of the canonical maps $\iota^{-1}\mathcal{O}_U \to \iota^{-1}(\mathcal{O}_U\,/\,\mathcal{I}_U) \cong \mathcal{O}_{\iota^{-1}U}$ and right multiplication of $\mathcal{O}_{\iota^{-1}U}$ on $\mathcal{D}_{\iota^{-1}U}$.

Using similar arguments as in the global situation, we get an expression

$$(\mathcal{D}_{Y\leftarrow X})_{\iota^{-1}U} = \mathbb{C}[\theta_{\mathrm{m}+1}, \ldots, \theta_{\mathrm{n}}] \otimes_{\mathbb{C}} \mathcal{D}_{\iota^{-1}U}\,.$$

Here, the right $\mathcal{D}_{\iota^{-1}U}$-action given by right $\mathcal{D}_{\iota^{-1}U}$-multiplication on the free $\mathcal{D}_{\iota^{-1}U}$-module and the left $i^{-1}\mathcal{D}_U$-action defined in the same manner as above: The differential θ_i $(1 \le i \le \mathrm{m})$ acts via the composition of the map $\theta_i \mapsto \theta_{x_i}$ with left multiplication of $\mathcal{D}_{\iota^{-1}U}$ on the second factor, whereas θ_i for $\mathrm{m}+1 \le i \le \mathrm{n}$ operates by increasing the exponent of θ_i by one. The left action of $f \in i^{-1}\mathcal{O}_U$ on $q \otimes p \in \mathbb{C}[\theta_{\mathrm{m}+1}, \ldots, \theta_{\mathrm{n}}] \otimes_{\mathbb{C}} \mathcal{D}_{\iota^{-1}U}$ is expressed as $\sum_i q_i \otimes f_i p$ if $fq = \sum_i q_i \cdot f_i$ with $q_i \in \mathbb{C}[\theta_{\mathrm{m}+1}, \ldots, \theta_{\mathrm{n}}]$ and $f_i \in \iota^{-1}\mathcal{O}_U$ in the ring $\iota^{-1}\mathcal{D}_U$, where the left action of f_i on p is given by composition of the canonical maps $\iota^{-1}\mathcal{O}_U \to \iota^{-1}(\mathcal{O}_U\,/\,\mathcal{I}_U) \cong \mathcal{O}_{\iota^{-1}U}$ and left multiplication of $\mathcal{O}_{\iota^{-1}U}$ on $\mathcal{D}_{\iota^{-1}U}$.

Example 1.4.3. A particular case of a closed embedding is a *coordinate change* $\lambda : X \to X$, that is, an automorphism. In this case, $\mathcal{D}_{X\to X} \cong \lambda^{-1}\mathcal{D}_X \cong \mathcal{D}_X$ with left $\mathcal{D}_X$-action on $\mathcal{D}_X$ given by (left) ring multiplication on $\mathcal{D}_X$. The right $\lambda^{-1}\mathcal{D}_X$-action induced on $\mathcal{D}_X$ is described as follows locally: By working on local coordinate neighborhoods, we reduce to the situation that X is an affine irreducible subvariety of $\mathbb{C}^{\mathrm{n}}$ with global coordinate system and $\lambda' = (\lambda'_1, \ldots, \lambda'_{\mathrm{n}}) : \mathbb{C}^{\mathrm{n}} \to \mathbb{C}^{\mathrm{n}}$ a morphism inducing the isomorphism $\lambda : X \cong \lambda'(X)$ with inverse induced by $\psi = (\psi_1, \ldots, \psi_{\mathrm{n}}) : \mathbb{C}^{\mathrm{n}} \to \mathbb{C}^{\mathrm{n}}$. If$(f_i, \theta_i)_{1\le i\le \mathrm{m}}$ is such a global coordinate system on X, then $g_1 := f_1 \circ \psi, \cdots, g_{\mathrm{m}} := f_{\mathrm{m}} \circ \psi$ are global coordinates of $\lambda(X)$ with corresponding derivations $\theta_{g_1}, \dots, \theta_{g_{\mathrm{m}}}$. Now $h \in \lambda^{-1}\mathcal{O}_{\lambda(X)}$ and θ_{g_i} act on $\mathcal{D}_X$ via right multiplication with $h \circ \lambda$ and θ_i, respectively. The actions on the second transfer module $\mathcal{D}_{X \leftarrow X} \cong \mathcal{D}_X$ are described in a similar manner.

The transfer modules are equipped with filtrations as left $(\mathcal{D}_X, F^\circ_\bullet)$-module and as right $f^{-1}(\mathcal{D}_Y, F^\circ_\bullet)$-module as follows: We set

$$\mathcal{F}_\bullet \mathcal{D}_{X \to Y} = F_\bullet \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y, F^\circ_\bullet)$$

(interpreted in analogy to Definition 1.3.6) and $\mathcal{F}_\bullet \mathcal{D}_{Y \leftarrow X}$ is defined via the side-changing operations for filtered modules. We make that filtration explicit for our above examples:

Example 1.4.4. (Continuation of Example 1.4.1) The filtrations are $F_\bullet \mathcal{D}_{U \to X} = F^\circ_\bullet \mathcal{D}_U$ and $F_\bullet \mathcal{D}_{X \leftarrow U} = F^\circ_\bullet \mathcal{D}_U$.

Example 1.4.5. (Continuation of Example 1.4.2) The filtration on the first transfer module is globally given by $F_\bullet \mathcal{D}_{X \to Y} = i^{-1}(F^\circ_\bullet(\mathcal{D}_Y / \mathcal{I}\mathcal{D}_Y))$ and can be locally expressed as

$$F_\bullet \mathcal{D}_{X \to Y} = \bigoplus_{\alpha \in \mathbb{N}^{\mathrm{n}-\mathrm{m}}} \mathcal{F}^\circ_{\bullet - |\alpha|} \mathcal{D}_X \, \theta_{\mathrm{m}+1}^{\alpha_1} \cdots \theta_{\mathrm{n}}^{\alpha_{\mathrm{n}-\mathrm{m}}}.$$

Similarly, $F_\bullet \mathcal{D}_{Y \leftarrow X} = i^{-1}(F^\circ_{\bullet-(\mathrm{n}-\mathrm{m})}(\mathcal{D}_Y / \mathcal{D}_Y \mathcal{I}))$ and locally

$$F_\bullet \mathcal{D}_{Y \leftarrow X} = \bigoplus_{\alpha \in \mathbb{N}^{\mathrm{n}-\mathrm{m}}} \mathcal{F}^\circ_{\bullet - |\alpha| - (\mathrm{n}-\mathrm{m})} \mathcal{D}_X \, \theta_{\mathrm{m}+1}^{\alpha_1} \cdots \theta_{\mathrm{n}}^{\alpha_{\mathrm{n}-\mathrm{m}}}.$$

We point out that the shift $\mathrm{n} - \mathrm{m} = \dim Y - \dim X$ in the above filtration compared to the filtration of the first transfer module comes from the side changing operations.

1.4.2 $\mathcal{D}$-module theoretic direct image functor

Consider the morphism $\phi : X \to Y$ of algebraic varieties of dimensions n and m, respectively. We use the second transfer module and the canonical unit morphism $\phi\phi^{-1}\mathcal{D}_Y \to \mathcal{D}_Y$ to construct the direct image of $\mathcal{D}_X$-modules under ϕ as the composition of the right derived and left derived functors

$$D^b(\mathcal{D}_X) \ni \mathcal{M}^\bullet \mapsto \mathcal{D}_{Y \leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}^\bullet \in D^b(\phi^{-1}\mathcal{D}_Y) \text{ and}$$
$$D^b(\phi^{-1}\mathcal{D}_Y) \ni \mathcal{N}^\bullet \mapsto R\phi(\mathcal{N}^\bullet) \in D^b(\mathcal{D}_Y),$$

where $\otimes^L_{\mathcal{D}_X}$ denotes the left derived functor of the tensor product $\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}$ and $R\phi$ denotes the right derived sheaf theoretic direct image functor. Note that these functors map indeed bounded complexes to bounded complexes by [HTT08, Propositions 1.5.6 and 1.5.4]. More precisely, we define:

Definition 1.4.6. The *($\mathcal{D}$-module theoretic) direct image functor* $\phi_+ : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ is defined by

$$\phi_+ \mathcal{M}^\bullet = R\phi(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}^\bullet).$$

We define $\phi_+ \mathcal{M}$ for the $\mathcal{D}_X$-module $\mathcal{M}$ by identifying $\mathcal{M}$ with the complex whose only non-trivial entry is $\mathcal{M}$ in degree 0.

The direct image functor commutes with composition of morphisms:

Proposition 1.4.7. *[HTT08, Proposition 1.5.21] Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms between algebraic varieties. Then we have*

$$(\psi \circ \phi)_+ = \psi_+\phi_+.$$

We remark that $\phi : X \to Y$ can be written as the composition of the closed embedding $\iota_\phi : X \to X \times Y, x \mapsto (x, \phi(x))$ and the projection $\pi_Y : X \times Y \to Y$. Hence it suffices from a $\mathcal{D}$-module theoretic point of view to study the direct image functors in these situations. We will however focus on closed and open embeddings, because these kind of images show up in the construction of the Hodge theoretic direct image functor for open embeddings of complements of subvarieties of pure codimension one.

Direct images under closed embeddings

Consider the situation of Example 1.4.2, that is, let $\iota : X \to Y$ be a closed embedding defined by the ideal sheaf $\mathcal{I}$ with $\dim X = \mathrm{m}$ and $\dim Y = \mathrm{n}$. Recalling that the second transfer module $\mathcal{D}_{Y\leftarrow X} = \iota^{-1}(\mathcal{D}_Y \,/\, \mathcal{D}_Y \,\mathcal{I})$ is $\mathcal{D}_X$-locally free (see Example 1.4.2) and that the sheaf theoretic direct image functor for closed embeddings is exact, we have for the $\mathcal{D}_X$-module $\mathcal{M}$

$$\iota_+ \mathcal{M} = \iota(\iota^{-1}(\mathcal{D}_Y \,/\, \mathcal{D}_Y \,\mathcal{I}) \otimes_{\mathcal{D}_X} \mathcal{M}) = \iota\iota^{-1}(\mathcal{D}_Y \,/\, \mathcal{D}_Y \,\mathcal{I}) \otimes_{\iota \mathcal{D}_X} \iota \,\mathcal{M}, \tag{1.4.1}$$

where we interpret the right hand side module as the complex with only non-zero entry this module in degree 0. Choosing an affine open neighborhood U of Y with coordinate system $(f_i, \theta_i)_{1\le i\le \mathrm{n}}$ such that $\mathcal{I}$ is locally generated by $f_{\mathrm{m}+1}, \dots, f_{\mathrm{n}}$ and $(f_i, \theta_i)_{1\le i\le \mathrm{m}}$ induce coordinates on $U \cap \iota(X)$, we obtain

$$(\iota_+ \mathcal{M})_U = \mathbb{C}[\theta_{\mathrm{m}+1}, \dots, \theta_{\mathrm{n}}] \otimes_{\mathbb{C}} (\iota \,\mathcal{M})_U. \tag{1.4.2}$$

Note that $x_i := f_i \circ \iota$ for $1 \le i \le \mathrm{m}$ is hence a local coordinate system on X with corresponding differentials denoted by θ_{x_i}. The action of $\mathcal{D}_U$ on the module $\mathbb{C}[\theta_{\mathrm{m}+1}, \dots, \theta_{\mathrm{n}}] \otimes_{\mathbb{C}} (\iota \,\mathcal{M})_U$ is described as follows: The differentials $\theta_{\mathrm{m}+1}, \dots, \theta_{\mathrm{n}}$ act by multiplication on the first factor of the tensor product, whereas $\theta_1, \dots, \theta_{\mathrm{m}}$ operate by left multiplication with $\theta_{x_1} \dots, \theta_{x_{\mathrm{m}}}$ on the second factor, respectively. The element $f \in \mathcal{O}_U$ acts on $q \otimes m \in \mathbb{C}[\theta_{\mathrm{m}+1}, \dots, \theta_{\mathrm{n}}] \otimes_{\mathbb{C}} (\iota \,\mathcal{M})_U$ as $\sum_i q_i \otimes f_i m$, where $fq = \sum_i q_i f_i$ in $\mathcal{D}_U$ with $q_i \in \mathbb{C}[\theta_{\mathrm{m}+1}, \dots, \theta_{\mathrm{n}}]$ and $f_i \in \mathcal{O}_U$ and f_i operates on m via the composition of the canonical maps $\mathcal{O}_U \to (\iota\iota^{-1} \,\mathcal{O}_Y)_U \to (\iota\iota^{-1}(\mathcal{O}_Y \,/\, \mathcal{I}))_U \cong (\iota \,\mathcal{O}_X)_U$ and left multiplication of $(\iota \,\mathcal{O}_X)_U$ on $(\iota \,\mathcal{M})_U$.

The above considerations imply:

Proposition 1.4.8. *Let $\iota : X \to Y$ be a closed embedding of algebraic varieties. Then:*

(a) We have for the $\mathcal{D}_X$-module $\mathcal{M}$ that $H^k(\iota_+ \mathcal{M}) = 0$ for $k \neq 0$. In particular,

$$\iota_+^0 : \operatorname{Mod}(\mathcal{D}_X) \to \operatorname{Mod}(\mathcal{D}_Y),\ \mathcal{M} \mapsto H^0(\iota_+ \mathcal{M})$$

is an exact functor.

(b) The functor ι_+^0 maps $\operatorname{Mod}_{\mathcal{O}_X\text{-}\mathrm{qcoh}}(\mathcal{D}_X)$ to $\operatorname{Mod}_{\mathcal{O}_Y\text{-}\mathrm{qcoh}}(\mathcal{D}_Y)$.

In particular, we may identify for a $\mathcal{D}_X$-module $\mathcal{M}$ the functor ι_+^0 with ι_+. So when writing $\iota_+ \mathcal{M}$, we mean from now on $\iota_+^0 \mathcal{M}$.

Example 1.4.9. (Continuation of Example 1.4.3) Under the reduction in Example 1.4.3 it holds:

(a) The map $\Lambda : \lambda_+ \mathcal{D}_X \to \mathcal{D}_{\lambda(X)}$ given by

$$\overline{x}_i \mapsto \overline{\psi_i} \text{ and } \theta_i \mapsto \theta_{g_i}$$

is an isomorphism of left $\mathcal{D}_{\lambda(X)}$-modules by Example 1.4.3 and hence also of $\mathcal{O}_{\lambda(X)}$-modules. An analogous statement holds for the map Λ^E for any finite set E.

(b) Equipping $\lambda \mathcal{D}_X$ with an $\mathcal{O}_{\lambda(X)}$-structure via the natural isomorphism $\mathcal{O}_{\lambda(X)} \to \lambda \mathcal{O}_X$, we see that $\lambda_+ \mathcal{D}_X$ and $\lambda \mathcal{D}_X$ agree as $\mathcal{O}_{\lambda(X)}$-modules. Thus we may interpret for an $\mathcal{O}_X$-submodule $\mathcal{P}$ of $\mathcal{D}_X^E$ (for a finite set E), $\lambda \mathcal{P}$ as an $\mathcal{O}_{\lambda(X)}$-submodule of $\lambda_+ \mathcal{D}_X^E$ and may consider its image under Λ^E. We identify from now on for an $\mathcal{O}_X$-submodule or $\mathcal{D}_X$-submodule $\mathcal{P}'$ of $\mathcal{D}_X^E$ the direct image $\lambda \mathcal{P}'$ or $\lambda_+ \mathcal{P}'$ with $\Lambda^E(\mathcal{P}')$, respectively.

(c) Given a set $\mathcal{P}' \subseteq \mathcal{D}_X(X)^E$, we have under the above identifications

$$\lambda \left({}_{\mathcal{O}_X}\langle \mathcal{P}' \rangle \right) = {}_{\mathcal{O}_{\lambda(X)}}\left\langle \Lambda^E(\mathcal{P}') \right\rangle \subseteq \mathcal{D}_{\lambda(X)}^E$$

and

$$\lambda_+ \left({}_{\mathcal{D}_X}\langle \mathcal{P}' \rangle \right) = {}_{\mathcal{D}_{\lambda(X)}}\left\langle \Lambda^E(\mathcal{P}') \right\rangle \subseteq \mathcal{D}_{\lambda(X)}^E .$$

These identifications induce for $\mathcal{M} = \mathcal{D}_X^E \,/\, {}_{\mathcal{D}_X}\langle \mathcal{P}' \rangle$ the identification

$$\lambda_+ \mathcal{M} = \mathcal{D}_{\lambda(X)}^E \,/\, {}_{\mathcal{D}_{\lambda(X)}}\left\langle \Lambda^E(\mathcal{P}') \right\rangle$$

and for ${}_{\mathcal{O}_X}\langle \overline{\mathcal{Q}} \rangle \subseteq \mathcal{M}$ with $\mathcal{Q} \subseteq \mathcal{D}_X^E$ we obtain

$$\lambda \left({}_{\mathcal{O}_X}\langle \overline{\mathcal{Q}} \rangle \right) = {}_{\mathcal{O}_{\lambda(X)}}\left\langle \overline{\Lambda^E(\mathcal{Q})} \right\rangle \subseteq \mathcal{D}_{\lambda(X)}^E \,/\, {}_{\mathcal{D}_{\lambda(X)}}\left\langle \Lambda^E(\mathcal{P}') \right\rangle.$$

In view of later applications, we are particularly interested in a certain kind of graph embedding:

Example 1.4.10. Given a regular function $f : X \to \mathbb{C}$ and a $\mathcal{D}_X$-module $\mathcal{M}$, we study the direct image $(i_f)_+ \mathcal{M}$ under the graph embedding $i_f : X \to X \times \mathbb{C}_t : x \mapsto (x, f(x))$. Notice that every system of local coordinates $(f_i, \theta_i)_{1 \leq i \leq \mathrm{m}}$ on the affine open neighborhood U of X can be completed to a system of local coordinates on $U \times \mathbb{C}_t$ by adding the coordinate t of $\mathbb{C}_t$ and its corresponding differential $\partial_t (= \frac{\partial}{\partial t})$. To represent $(i_f)_+ \mathcal{M}$ on this neighborhood, we factorize i_f via the closed embedding i_0 and a coordinate change

$$\begin{array}{ccc} X & \xhookrightarrow{\quad i_f \quad} & X \times \mathbb{C}_t \\ & {\scriptstyle i_0 : x \mapsto (x,0)} \searrow \qquad \nearrow {\scriptstyle \lambda : (x,t) \mapsto (x, t+f)} & \\ & X \times \mathbb{C}_t \,. & \end{array} \tag{1.4.3}$$

By the above considerations, we have $(i_0)_+ \mathcal{M} = \mathbb{C}[\partial_t] \otimes_{\mathbb{C}} i_0 \mathcal{M}$ globally. Locally, $\Theta_{U \times \mathbb{C}}$-acts by

$$\theta_i \cdot (\partial_t^k \otimes m) = \partial_t^k \otimes \theta_i m$$
$$\partial_t \cdot (\partial_t^k \otimes m) = \partial_t^{k+1} \otimes m$$

for $1 \leq i \leq \mathrm{m}$, $m \in (i_0 \mathcal{M})_{U \times \mathbb{C}}$ and $k \in \mathbb{N}$, and $\mathcal{O}_{U \times \mathbb{C}}$ operates as explained after Equation (1.4.2). So in particular

$$t \cdot (\partial_t^k \otimes m) = -k \partial_t^{k-1} \otimes m.$$

If $\mathcal{M}$ is $\mathcal{D}_X$-coherent, then $\mathcal{M}_U$ is of the form $\mathcal{D}_U^E \,/\, _{\mathcal{D}_U}\langle P \rangle$ with $P \in \mathcal{D}_U(U)$ implying $((i_0)_+ \mathcal{M})_{U \times \mathbb{C}} = \mathcal{D}^E_{U \times \mathbb{C}} \,/\, _{\mathcal{D}_{U \times \mathbb{C}}}\langle P, t \rangle$.

Noting that the coordinate change λ maps the local coordinates $f_1, \ldots, f_{\mathrm{m}}, t, \theta_1, \ldots, \theta_{\mathrm{m}}$ and ∂_t on $U \times \mathbb{C}$ to the local coordinates $f_1, \ldots, f_{\mathrm{m}}, t - f, \theta_1 + \theta_1(f)\partial_t, \ldots, \theta_{\mathrm{m}} + \theta_{\mathrm{m}}(f)\partial_t$ and ∂_t on $\lambda(U \times \mathbb{C}) = U \times \mathbb{C}$, we obtain

$$(i_f)_+ \mathcal{M} = \mathbb{C}[\partial_t] \otimes_{\mathbb{C}} i_f \mathcal{M}$$

globally with $\mathcal{D}_{U \times \mathbb{C}}$-module structure given by

$$(\theta_i + \theta_i(f)) \cdot (\partial_t^k \otimes m) = \partial_t^k \otimes \theta_i m$$
$$\partial_t \cdot (\partial_t^k \otimes m) = \partial_t^{k+1} \otimes m$$

for $1 \leq i \leq \mathrm{m}$, $m \in (i_f \mathcal{M})_{U \times \mathbb{C}}$ and $k \in \mathbb{N}$. As $\mathcal{O}_{U \times \mathbb{C}}$ operates as explained after Equation (1.4.2), we have in particular

$$t \cdot (\partial_t^k \otimes m) = \partial_t^k \otimes fm - k \partial_t^{k-1} \otimes m.$$

Applying the coordinate change λ we obtain for coherent $\mathcal{M}$ as above (under the identifications in Example 1.4.9)

$$((i_f)_+ \mathcal{M})_{U\times\mathbb{C}} = \mathcal{D}^E_{U\times\mathbb{C}} \,/\, {}_{\mathcal{D}_{U\times\mathbb{C}}}\langle \Lambda^E(P), t-f\rangle.$$

The direct image functor for the closed embedding $\iota : X \to Y$ even induces an equivalence of categories, called Kashiwara's equivalence, between the categories $\mathrm{Mod}_*(\mathcal{D}_X)$ and category $\mathrm{Mod}_*^{\iota(X)}(\mathcal{D}_Y)$, the subcategory of $\mathrm{Mod}_*(\mathcal{D}_Y)$ consisting of modules supported on $\iota(X)$, for $* \in \{\mathcal{O}_X\text{-}\,\mathrm{qcoh}, \mathrm{coh}\}$. Before we state this equivalence, we introduce the extraordinary inverse image functor which will serve as a quasi-inverse.

Definition 1.4.11. Let $\phi : X \to Y$ be a morphism of algebraic varieties. The *extraordinary inverse image functor* is

$$\phi^! : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X),\ \mathcal{N}^\bullet \mapsto (\mathcal{D}_{X\to Y} \otimes^L_{\phi^{-1}\mathcal{D}_Y} \phi^{-1}\mathcal{N})[\dim X - \dim Y].$$

By applying certain duality functors to the extraordinary inverse image functor and to the direct image functor, one defines the inverse image functor and the extraordinary direct image functor. The reason why $\phi^!$ is called the extraordinary inverse image is that the (extraordinary) inverse image will be left adjoint to the (extraordinary) direct image. Also, this way the functors are compatible with the so-called Riemann-Hilbert correspondence.

Proposition 1.4.12. *[Kas78] Let $\iota : X \to Y$ be a closed embedding with defining ideal sheaf $\mathcal{I}$.*

(a) The functor ι_+ induces equivalences of categories

$$\mathrm{Mod}_{\mathcal{O}_X\text{-}\,\mathrm{qcoh}}(\mathcal{D}_X) \to \mathrm{Mod}^{\iota(X)}_{\mathcal{O}_Y\text{-}\,\mathrm{qcoh}}(\mathcal{D}_Y)$$
$$\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X) \to \mathrm{Mod}^{\iota(X)}_{\mathrm{coh}}(\mathcal{D}_Y)$$

with quasi-inverse $H^0\iota^!$.

(b) We have for $\mathcal{N} \in \mathrm{Mod}^{\iota(X)}_{\mathcal{O}_X\text{-}\,\mathrm{qcoh}}(\mathcal{D}_Y)$ that $H^k\iota^!\mathcal{N} = 0$ for all $k \neq 0$.

(c) We have for $\mathcal{N} \in \mathrm{Mod}_{\mathcal{O}_X\text{-}\,\mathrm{qcoh}}(\mathcal{D}_Y)$ that $H^0\iota_+H^0\iota^!\mathcal{N} = \Gamma_{[X]}(\mathcal{N})$, where $\Gamma_{[X]}(\mathcal{N}) := \{n \in \mathcal{N} \mid$ there exists $i \in \mathbb{N} : \mathcal{I}^i n = 0\}$.

For a proof we refer the reader e.g. to [HTT08, Theorem 1.6.1 and Proposition 1.7.1].

We naively define the filtered direct image under closed embeddings in a way preserving good filtrations:

Definition 1.4.13. Assume that $(\mathcal{M}, F_\bullet)$ is a filtered $(\mathcal{D}_X, F^\circ_\bullet)$-module. Using the filtration $F_\bullet\,\mathcal{D}_{Y\leftarrow X}$ (see Example 1.4.5), we equip $\iota_+\mathcal{M}$ with the $(\mathcal{D}_Y, F^\circ_\bullet)$-filtration

$$F_\bullet \iota_+ \mathcal{M} = \sum_{k\in\mathbb{Z}} \iota F_k\,\mathcal{D}_{Y\to X} \otimes_{\iota\mathcal{O}_X} \iota F_{\bullet-k}\,\mathcal{M} \tag{1.4.4}$$

(where the right hand side is to be understood in analogy to Definition 1.3.6).

Note that we have

$$F_\bullet \iota_+ \mathcal{M} = \sum_{k\in\mathbb{Z}} \iota\iota^{-1}(F_k^\circ(\mathcal{D}_Y \,/\, \mathcal{D}_Y \,\mathcal{I})) \otimes_{\iota\,\mathcal{O}_X} \iota F_{\bullet-k-\dim Y+\dim X}\,\mathcal{M}, \tag{1.4.5}$$

which is in the situation of Equation (1.4.2) expressed as

$$(F_\bullet \iota_+ \mathcal{M})_U = \sum_{\alpha\in\mathbb{N}^{\mathrm{n}\,-\,\mathrm{m}}} \partial_{\mathrm{m}+1}^{\alpha_1}\cdots\partial_{\mathrm{n}}^{\alpha_{\mathrm{n}\,-\,\mathrm{m}}} \otimes (\iota F_{\bullet-|\alpha|-\mathrm{n}+\mathrm{m}}\,\mathcal{M})_U. \tag{1.4.6}$$

Remark 1.4.14. We point out that $F_\bullet \iota_+ \mathcal{M}$ a filtered $(\mathcal{D}_Y, F_\bullet^\circ)$-module that is well-filtered if and only if $F_\bullet \mathcal{M}$ is well-filtered as $(\mathcal{D}_X, F_\bullet^\circ)$-module.

Remark 1.4.15. In the situation of Example 1.4.9(c) it holds for a shift vector $\mathbf{s} \in \mathbb{Z}^E$ that

$$\lambda_+(\mathcal{D}_X^E \,/\, {}_{\mathcal{D}_X}\langle \mathcal{P}' \rangle, F^\circ[\mathbf{s}]_\bullet) = (\mathcal{D}_{\lambda(X)}^E \,/\, {}_{\mathcal{D}_{\lambda(X)}}\langle \Lambda^E(\mathcal{P}') \rangle, F^\circ[\mathbf{s}]_\bullet).$$

Direct images under open embeddings

Let $U \subseteq X$ be an open subset of the variety X with embedding denoted by j and complement $V := X \setminus U$. By Example 1.4.1 the second transfer module $\mathcal{D}_{X\leftarrow U}$ agrees with $\mathcal{D}_U$. Thus the $\mathcal{D}$-module theoretic direct image functor coincides with the sheaf-theoretic direct image functor, i.e.,

$$j_+\,\mathcal{M}^\bullet = Rj\,\mathcal{M}^\bullet$$

for $\mathcal{M}^\bullet \in D^b(\mathcal{D}_U)$ in this situation. The functor j_+ is in general not exact, but it is exact if U is affine as $R^k j\,\mathcal{M} = 0$ for $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_U)$ and $k \neq 0$ in this case. Hence we identify in this case j_+ with $H^0 j_+ = j$ as we did for closed embeddings.

We remark that $j_+\,\mathcal{M}^\bullet$ is not only an complex of $\mathcal{D}_X$-modules, but also of $j\,\mathcal{D}_U$-modules. Working locally we see that

$$j\,\mathcal{D}_U = \mathcal{D}_X \otimes_{\mathcal{O}_X} j\,\mathcal{O}_U = \mathcal{D}_X \otimes_{\mathcal{O}_X} jj^{-1}\,\mathcal{O}_X,$$

where $\mathcal{O}_X$ on the right hand side module acts by left multiplication on $\mathcal{D}_X$ and the ring structure on this module is given by

$$\begin{aligned}(1\otimes f)\cdot(p\otimes g) &= p\otimes fg\\ (\theta\otimes 1)\cdot(p\otimes g) &= \theta p\otimes g + p\otimes\theta(g)\end{aligned}$$

for $f, g \in j\,\mathcal{O}_U$, $p \in \mathcal{D}_X$ and $\theta \in \Theta_X$. In view of the applications to Hodge theory, we are particularly interested in the case of V being a pure codimension one subvariety of X. In this case, $jj^{-1}\,\mathcal{O}_X$ agrees with $\mathcal{O}_X(*V)$, which motivates the following definition:

Definition 1.4.16. Let $V \subseteq X$ be a closed embedding of pure codimension one for not necessarily smooth V and $\mathcal{M}$ a $\mathcal{D}_X$-module. The *localization of* $\mathcal{M}$ *along* V is defined by

$$\mathcal{M}(*V) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*V).$$

It comes with a canonical localization map $i_{(*V)} : \mathcal{M} \to \mathcal{M}(*V)$ sending m to $m \otimes 1$.

In the above situation, $\mathcal{M}(*V)$ is a $\mathcal{D}_X(*V)$-module with $\mathcal{D}_X(*V)$-action defined in analogy to the ring structure of $\mathcal{D}_X(*V)$. In particular, we have $j_{X\setminus V} j_{X\setminus V}^{-1} \mathcal{M} = \mathcal{M}(*V)$, where $j_{X\setminus V} : X \setminus V \to X$ is the corresponding open embedding.

Remark 1.4.17. Let $V \subseteq X$ be a closed embedding of pure codimension one for not necessarily smooth V with defining ideal sheaf $\mathcal{I}$. Then the sheaf of rings $\mathcal{D}_X(*V)$ is locally Noetherian: We define the order filtration $\mathcal{F}^\circ_\bullet \mathcal{D}_X(*V)$ by

$$\mathcal{F}^\circ_\bullet \mathcal{D}_X(*V) = j_{X\setminus V} j_{X\setminus V}^{-1} F^\circ_\bullet \mathcal{D}_X \,.$$

On an affine open subset $U \subseteq X$ with local coordinates $(f_i, \theta_i)_{1\le i\le \mathrm{m}}$ such that $\mathcal{I}_U = {}_{\mathcal{O}_U}\langle g\rangle$, the associated graded sheaf of rings is represented as

$$\mathrm{Gr}^{F^\circ} \mathcal{D}_X(*V)(U) \cong \begin{cases} \mathcal{O}_X(U)[\xi_1, \ldots, \xi_\mathrm{m}], & \text{if } U \cap V = \emptyset, \\ \mathcal{O}_X(U)[g^{-1}][\xi_1, \ldots, \xi_\mathrm{m}], & \text{else.} \end{cases}$$

Hence Proposition 1.1.16 implies the claim and Proposition 1.1.7 shows that $\mathcal{D}_X(*V)$ is a coherent sheaf of rings.

Remark 1.4.18. While it was relatively easy to equip the direct image of a well-filtered module under a closed embedding with a good filtration, it is not so clear how to do this for open embeddings. The first problem is that j_+ is in general not exact, which indicates that we need the notion of a derived category of $\mathrm{Mod}_{\mathrm{coh}}(F^\circ_\bullet \mathcal{D})$ to equip the direct image with a filtration. To circumvent the problem that this category is not abelian, one considers it as an exact category allowing the definition of a corresponding derived category nevertheless (for details see [Lau83]). Yet, the above considerations show that $D^b_{\mathrm{coh}}(\mathcal{D})$ is not preserved under direct images by taking for instance the direct image of the sheaf of differential operators under the natural inclusion $j' : \mathbb{C} \setminus \{0\} \hookrightarrow \mathbb{C}$. This implies that it is not possible to define a filtered $\mathcal{D}$-module theoretic direct image functor preserving $D^b_{\mathrm{coh}}(F^\circ_\bullet \mathcal{D})$ and commuting with the forgetful functor $D^b_{\mathrm{coh}}(F^\circ_\bullet \mathcal{D}) \to D^b_{\mathrm{coh}}(\mathcal{D})$. As the direct image functor preserves complexes with holonomic cohomology one could hope that it is possible to define a direct image functor for the subcategory of $D^b_{\mathrm{coh}}(F^\circ_\bullet \mathcal{D})$ consisting of complexes with holonomic cohomology. But the naive approach by setting for instance for an affine open embedding $j : U \hookrightarrow X$

$$\mathcal{F}_\bullet(j_+ \mathcal{M}) := j(F_\bullet \mathcal{D}_{X\leftarrow U} \otimes_{\mathcal{O}_U} F_\bullet \mathcal{M}) = j\, \mathcal{F}_\bullet \mathcal{M}$$

(with $F_\bullet \mathcal{D}_{X \leftarrow U} \otimes_{\mathcal{O}_U} F_\bullet \mathcal{M}$ being defined in analogy to Definition 1.3.6) does not work, because then the direct image of $(\mathcal{O}_{\mathbb{C}\setminus\{0\}}, F_\bullet)$ would have the filtration

$$\mathcal{F}_i\, j'_+\, \mathcal{O}_{\mathbb{C}\setminus\{0\}} = \begin{cases} \mathcal{O}_{\mathbb{C}}[x^{-1}], & \text{if } i \geq 0 \\ 0, & \text{else,} \end{cases}$$

(where $x \in \mathcal{O}_X$ is the defining equation of $\{0\}$), which is not $\mathcal{O}_{\mathbb{C}}$-coherent for $i \geq 0$. We will see later how to define a good filtration on that module in a way compatible with mixed Hodge module theory.

Considering the case that $V \subseteq X$ has defining ideal generated by the regular function $f : X \to \mathbb{C}$, we investigate the direct image under the corresponding graph embedding of localizations along V:

Lemma 1.4.19. *Let $V \subseteq X$ be a not necessarily smooth subvariety with defining ideal sheaf $\mathcal{I}$ generated by the regular function $f : X \to \mathbb{C}$. Then we have for the direct image of the $\mathcal{D}_X$-module $\mathcal{M}$ under the graph embedding $i_f : X \to X \times \mathbb{C}, x \mapsto (x, f(x))$*

$$(i_f)_+(\mathcal{M}(*V)) \cong ((i_f)_+ \mathcal{M})(*X \times \{0\}).$$

Proof. We set $U := X \setminus V$ and consider its canonical embedding $j_U : U \to X$. As $\mathcal{M}(*V) = (j_U)_+(j_U^{-1} \mathcal{M})$, we obtain by the commutativity of $\mathcal{D}$-module theoretic direct images, by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad i_f \quad} & X \times \mathbb{C} \\ {\scriptstyle j_U}\uparrow & & \uparrow{\scriptstyle j_{X\times\mathbb{C}^*}} \\ U & \xrightarrow{i'_f : x \mapsto (x, f(x))} & X \times \mathbb{C}^* \end{array}$$

and by the isomorphism $(i'_f)_+ j_U^{-1} \mathcal{M} \cong j_{X\times\mathbb{C}^*}^{-1} (i_f)_+ \mathcal{M}$ the claim. Thereby note that the latter isomorphism can be established using local coordinates. □

Remark 1.4.20. For algorithms later on, we need to make the isomorphism in the above lemma for $\mathcal{O}_X$-quasi-coherent $\mathcal{M}$ explicit. We reduce to the embedding i_0 as follows: We keep the setting of the above lemma and decompose $i_f = \lambda \circ i_0$ as in Example 1.4.10. We first construct the isomorphism

$$(i_0)_+(\mathcal{M}(*V)) \cong ((i_0)_+ \mathcal{M})(*V(t+f)) :$$

The left hand side of the above isomorphism is

$$\begin{aligned} (i_0)_+(\mathcal{M}(*V)) &\cong i_0(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[f^{-1}]) \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] \\ &\cong (i_0\, \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{i_0\, \mathcal{O}_X} i_0(\mathcal{O}_X[f^{-1}]) \\ &\cong (i_0\, \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{X\times\mathbb{C}}[f^{-1}], \end{aligned} \tag{1.4.7}$$

while the right hand side can be rewritten as

$$((i_0)_+ \mathcal{M})(*V(t+f)) \cong (i_0 \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{X\times\mathbb{C}}[(t+f)^{-1}]. \quad (1.4.8)$$

An easy calculation shows now that for $a, b \in \mathbb{N}$ and $m \in i_0 \mathcal{M}$

$$(m \otimes \partial_t^a) \otimes f^{-b} = \left(\sum_{k=0}^{c/2} (-1)^k \binom{c}{k} k! f^{c-k-b} m \otimes \partial_t^{a-k} \right) \otimes (t+f)^{-c}$$

(with $c \in 2\,\mathbb{N}$ such that $c/2 \geq a+1, b$) in $(i_0(\mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{X\times\mathbb{C}}[(f(t+f))^{-1}]$ making the above isomorphism explicit. Applying the coordinate change λ to Equations (1.4.7) and (1.4.8), we obtain

$$(i_f)_+(\mathcal{M}(*V)) \cong \lambda_+(i_0)_+(\mathcal{M}(*V)) \cong (i_f(\mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{X\times\mathbb{C}}[f^{-1}]$$

and

$$\lambda_+(((i_0)_+ \mathcal{M})(*V(t+f))) \cong (i_f \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{X\times\mathbb{C}}[t^{-1}] \cong ((i_f)_+ \mathcal{M})(*X \times \{0\}).$$

and the above considerations give the isomorphism

$$\begin{aligned} (i_f \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{X\times\mathbb{C}}[f^{-1}] &\to (i_f \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} \mathcal{O}_{X\times\mathbb{C}}[t^{-1}] \\ m \otimes f^{-b} \otimes \partial_t^a &\mapsto \sum_{k=0}^{c/2} (-1)^k \binom{c}{k} k! f^{c-k-b} m \otimes \partial_t^{a-k} \otimes (t^{-c}) \end{aligned} \quad (1.4.9)$$

(with c as above) representing $(i_f)_+(\mathcal{M}(*V)) \cong ((i_f)_+ \mathcal{M})(*X \times \{0\})$. Its inverse can be presented in a similar manner.

Remark 1.4.21. In the situation of Lemma 1.4.19 let $(\mathcal{M}, F_\bullet)$ be a well-filtered $(\mathcal{D}_X, F_\bullet^\circ)$-module. Then the isomorphism in Lemma 1.4.19 is by Remark 1.4.20 an isomorphism of filtered modules, that is, we have

$$(i_f)_+((\mathcal{M}, F_\bullet) \otimes_{\mathcal{O}_X} F_\bullet \mathcal{O}_X(*V)) \cong ((i_f)_+(\mathcal{M}, F_\bullet)) \otimes_{\mathcal{O}_{X\times\mathbb{C}}} F_\bullet \mathcal{O}_{X\times\mathbb{C}}(*X \times \{0\}),$$

where we equip $\mathcal{O}_X(*V)$ and $\mathcal{O}_{X\times\mathbb{C}}(*X \times \{0\})$ with one-step filtrations in analogy with $\mathcal{O}_X$: Indeed, the map in Equation (1.4.9) is obviously filtered. If on the other hand $m \otimes f^{-b} \otimes \partial_t^a$ with $m \in i_f(F_k \mathcal{M})$ is sent under this map to an element in the F_{k+a-1}-part of the corresponding filtration, then this implies that $f^l m \in i_f(F_{k-1} \mathcal{M})$ for some $l \in \mathbb{N}$ and hence $m \otimes f^{-b} \otimes \partial_t^a = f^l m \otimes f^{-b-l} \otimes \partial_t^a$ is also in the F_{k+a-1}-part of the filtration on the left hand side module of that map.

Remark 1.4.22. We keep the notation of Lemma 1.4.19 and consider the $(\mathcal{D}_X(*V), F^\circ_\bullet)$-module $(\mathcal{N}, F_\bullet)$. As this module is also a $(\mathcal{D}_X, F^\circ_\bullet)$-module, we define $((i_f)_+\mathcal{N}, F_\bullet)$ via Definition 1.4.13. The latter module is in fact a well-filtered $(\mathcal{D}_{X\times\mathbb{C}}(*X \times \{0\}), F^\circ_\bullet)$-module if $(\mathcal{N}, F_\bullet)$ is well-filtered as $(\mathcal{D}_X(*V), F^\circ_\bullet)$-module: We factorize the map i_f via the closed embedding i_0 and the coordinate change λ as in Diagram (1.4.3), and may hence replace $(i_f)_+(\mathcal{N}, F_\bullet)$ and $(\mathcal{D}_{X\times\mathbb{C}}(*X \times \{0\}), F^\circ_\bullet)$ by $(i_0)_+(\mathcal{N}, F_\bullet)$ and $(\mathcal{D}_{X\times\mathbb{C}}(*V(t + f)), F^\circ_\bullet)$, respectively. Then the action of $(t + f)^{-1}$ on $(i_0)_+\mathcal{N} = i_0\mathcal{N} \otimes_\mathbb{C} \mathbb{C}[\partial_t]$ is given by

$$(t + f)^{-1} \cdot (n \otimes \partial_t^a) = \sum_{0\leq i\leq a} \frac{a!}{i!f^{a-i+1}} n \otimes \partial_t^i.$$

If $(\mathcal{N}, F_\bullet)$ is $(\mathcal{D}_X(*V), F^\circ_\bullet)$-good, we may assume that X is affine that there is a finite set $N \subseteq \mathcal{N}(X)$ and $\mathbf{s} \in \mathbb{Z}^N$ with $F_\bullet\mathcal{N} = \sum_{n\in N} F^\circ_{\bullet-\mathbf{s}_n}\mathcal{D}_X(*V) \cdot n$. But then $F_\bullet(\iota_0)_+\mathcal{N} = \sum_{n\in N} F^\circ_{\bullet-\mathbf{s}_n}\mathcal{D}_{X\times\mathbb{C}}(*V(t + f)) \cdot (n \otimes 1)$ because $f^{-k}n \otimes 1 = (t + f)^{-k} \cdot (n \otimes 1)$ for any $k > 0$ showing the claim.

2 PBW-reduction-algebras

Motivated by Saito's theory of mixed Hodge modules, the goal of this chapter is to study the interplay of the filtration by the order of differential operators and a certain V-filtration on modules over the Weyl algebra and more generally on modules over coordinate system rings, and to develop related algorithms. More precisely, on the Weyl algebra D_n over $\mathbb{C}$ in variables $x_1, \dots, x_n$ and corresponding derivations $\partial_1, \dots, \partial_n$, a so-called weight vector $\mathbf{u} \in \mathbb{Z}^{2n}$ with $\mathbf{u}_i + \mathbf{u}_{n+i} \geq 0$ for $1 \leq i \leq n$ induces a filtration $F_\bullet^{\mathbf{u}} D_n$ given by $F_k^{\mathbf{u}} D_n = {}_{\mathbb{C}}\Big\langle \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \mid \alpha, \beta \in \mathbb{N}^n, \langle (\alpha, \beta), \mathbf{u} \rangle \leq k\} \Big\rangle$ for $k \in \mathbb{Z}$. In the case $\mathbf{u} = ((0)_{1 \leq i \leq n}, (1)_{1 \leq i \leq n})$ the corresponding filtration $F_\bullet^{\mathbf{u}} D_n$ is the filtration by the order of differential operators, whereas the weight vector assigning weight 1 to ∂_n, weight -1 to x_n and weight 0 else defines the V-filtration along $\{x_n = 0\}$ on D_n. These filtrations induce not only filtrations on sub- and quotient modules of free modules, but it is also natural to consider $F_0^{\mathbf{u}} D_n$-submodules of such sub- and quotient modules, and investigate the interplay of these structures.

While Weyl algebras can computationally be regarded as a particular case of PBW-algebras with their well-studied Gröbner basis theory, coordinate system rings do not seem to fit into the setting of (quotient algebras of) PBW-algebras or in any other already existing well-developed algorithmic setup that we are aware of. Hence we introduce in this chapter a Gröbner basis theory for a broader class of algebras, called PBW-reduction-algebras. These algebras are certain quotients of free associative $\mathbb{K}$-algebras of type $\mathbb{K}\langle x_1, \dots, x_n \rangle$ by two-sided ideals containing commutation relations with the property that a subset of the set $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha \in \mathbb{N}^n\}$ forms a $\mathbb{K}$-basis of that quotient. We will see that the concept of weight vectors naturally generalizes to PBW-reduction-algebras. We introduce a variant of the Buchberger algorithm for Gröbner bases computations over this new class of algebras and show that many elementary applications thereof, referred to as "Gröbner basics" by Sturmfels, can be adapted from commutative polynomial rings to our setting. With Hodge theoretic constructions in mind, we then study the interplay of structures as above on modules over PBW-reduction-algebras in as much generality as reasonable.

The outline of this chapter is as follows: We introduce PBW-reduction-algebras in Section 2.1 and develop a Gröbner basis theory for well-orderings on such algebras. Section 2.2 addresses the main subject of study in this chapter, namely the already mentioned weight filtrations on PBW-reduction-algebras. Given a weight vector $\mathbf{u}$ on a PBW-reduction-algebra A, we first investigate the subalgebra $F_0^{\mathbf{u}} A$ and prove that this algebra is left and right Noetherian and generated by a finite set of monomials of A. Using homogenized PBW-reduction-algebras with respect to a suitable weight vector, we formulate an algorithm for Gröbner bases computations with respect to non-well-orderings on PBW-reduction-algebras. This allows us to give a computer algebraic proof showing that the filtration $F_\bullet^{\mathbf{u}} A$ induces good filtrations on submodules of free A-modules by considering a $\mathbf{u}$-weighted degree ordering. Given two weight vectors $\mathbf{v}, \mathbf{w}$ on A which satisfy among other conditions $F_0^{\mathbf{w}} A \subseteq F_0^{\mathbf{v}} A$, we explain in Section 2.3 how to determine the intersection of $F_0^{\mathbf{v}} A$- and $F_0^{\mathbf{w}} A$-submodules of a free A-module as well as how to find generators of the filtration induced by $F_\bullet^{\mathbf{w}} A$ on such an $F_0^{\mathbf{v}} A$-submodule. The key to tackle these problems is a translation process to problems over the PBW-reduction-algebra $F_0^{\mathbf{v}} A$. Lastly, in Section 2.4 we consider the same problems as in the previous section, but this time for quotient modules of free A-modules. In many instances these problems can be reduced to the analogous problems for submodules of free A-modules.

In this chapter $\mathbb{K}$ *stands for a field.*

2.1 Gröbner basis framework for PBW-reduction-algebras

PBW-reduction-algebras are certain quotients of free associative $\mathbb{K}$-algebras of type $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ such that a subset of the set of standard monomials $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha \in \mathbb{N}^n\}$ forms a $\mathbb{K}$-basis of this quotient and the multiplication on this basis is defined by certain commutation relations. These algebras can be considered as a generalization of so-called PBW-algebras which are $\mathbb{K}$-algebras of the above type with the set of all standard monomials as $\mathbb{K}$-basis. We adapt in this section the Gröbner basis theory for PBW-algebras to the setting of PBW-reduction-algebras using Bergman's Diamond Lemma [Ber78]. Gröbner bases in the context of PBW-algebras were first studied for the subclass of universal enveloping algebras of finite dimensional Lie algebras in [AL88] and the methods applied there have later been extended to develop a Gröbner basis theory for general PBW-algebras in [KRW90]. The idea behind the corresponding algorithms is that PBW-algebras are still close enough to commutative polynomial rings in order to adopt certain methods from commutative Gröbner basis theory such as the Buchberger algorithm for well-orderings to this setting.

2.1.1 PBW-reduction-algebras

Consider the free associative $\mathbb{K}$-algebra $T_n := \mathbb{K}\langle x_1, \ldots, x_n\rangle$ generated by $x_1, \ldots, x_n$ for $n \in \mathbb{N}$. If $I \subseteq T_n$ we also write $\langle I\rangle$ for the two-sided ideal ${}_{T_n}\langle I\rangle_{T_n}$ generated by I and similarly for two-sided ideals of factor rings of T_n.

Definition 2.1.1. Let E be a finite set.

(a) We denote by

$$\mathrm{Mon}(T_n^E) := \{x_{i_1} \cdots x_{i_k}(e) \mid k \in \mathbb{N}, 1 \leq i_1, \ldots, i_k \leq n, e \in E\} \subseteq T_n^E$$

the set of *monomials* of T_n^E and

$$\mathrm{SMon}(T_n^E) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n}(e) \mid \alpha \in \mathbb{N}^n, e \in E\} \subseteq T_n^E$$

is called the set of *standard monomials* of T_n^E. We write for the element $t \in T_n^E$ also $t = \sum_{m \in \mathrm{Mon}(T_n^E)} t_m m$ with $t_m \in \mathbb{K}$. Abbreviating $\underline{x}^\alpha(e) := x_1^{\alpha_1} \cdots x_n^{\alpha_n}(e)$ for $e \in E$ and $\alpha \in \mathbb{N}^n$, we often use for $p \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E) \rangle$ the multi-index notation $p = \sum_{e,\alpha} p_{e,\alpha} \underline{x}^\alpha(e)$ with $p_{e,\alpha} \in \mathbb{K}$ (by implicitly assuming that e runs through E and α through $\mathbb{N}^n$).

We point out that we have $\mathrm{SMon}(T_n) = \{\underline{x}^\alpha \mid \alpha \in \mathbb{N}^n\}$ and $\mathrm{Mon}(T_n) = \{x_{i_1} \cdots x_{i_k} \mid k \in \mathbb{N}, 1 \leq i_1, \ldots, i_k \leq n\}$ under the convention in Notation 0.0.1(b).

(b) A total order $\prec$ on $\mathrm{Mon}(T_n^E)$ is called a *monomial well-ordering* if it holds for all $m, m', p, q \in \mathrm{Mon}(T_n)$ and $e, e' \in E$

(i) $(e) \preceq m(e)$ and

(ii) $m(e) \prec m'(e')$ implies $pmq(e) \prec pm'q(e')$.

A total order $\prec$ on $\mathrm{Mon}(T_n^E)$ is called a *monomial ordering* if it satisfies Condition (bii) and a monomial ordering that violates Condition (bi) is called a *monomial non-well-ordering*. We also say that the corresponding monomial ((non)-well) ordering is a ((non)-well) ordering on T_n^E.

(c) We say that the total order $\prec$ on $\mathrm{SMon}(T_n^E)$ is a *monomial well-ordering* if it holds for all $\alpha, \alpha', \gamma \in \mathbb{N}^n$ and $e, e' \in E$ that

(i) $(e) \preceq \underline{x}^\alpha(e)$ and

(ii) $\underline{x}^\alpha(e) \prec \underline{x}^{\alpha'}(e')$ implies $\underline{x}^{\alpha+\gamma}(e) \prec \underline{x}^{\alpha'+\gamma}(e')$.

A total order $\prec$ on $\mathrm{SMon}(T_n^E)$ is called a *monomial ordering* if it satisfies Condition (cii) and a monomial ordering that violates Condition (ci) is called a *monomial non-well-ordering*. We also say that the corresponding monomial ((non)-well) ordering is a ((non)-well) ordering on ${}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E) \rangle$.

(d) Let $\prec$ be a monomial ordering on $\mathrm{Mon}(T_n^E)$. If $0 \neq t = \sum_{e \in E, m \in \mathrm{Mon}(T_n)} t_{e,m} m(e) \in T_n^E$ with $t_{e,m} \in \mathbb{K}$ and $m'(e') := \max_\prec\{m(e) \mid t_{e,m} \neq 0\}$, then we define

- $\mathrm{lm}_\prec(t) := m'(e')$ (*leading monomial* of t),
- $\mathrm{lt}_\prec(t) := t_{e',m'} m'(e')$ (*leading term* of t),
- $\mathrm{lc}_\prec(t) := t_{e',m'}$ (*leading coefficient* of t),
- $\mathrm{lcomp}_\prec(t) := e'$ (*leading component* of t),
- $\mathrm{tail}_\prec(t) := t - \mathrm{lt}_\prec(t)$ (*tail* of t),
- $\mathrm{le}_\prec^{\mathrm{com}}(t) := \sum_{1 \leq j \leq k} e_{i_j} \in \mathbb{N}^n$ if $m' = x_{i_1} \cdots x_{i_k}$,

- $\mathrm{ele}^{\mathrm{com}}_{\prec}(t) := (\sum_{1\leq j\leq k} e_{i_j}, e')$ if $m' = x_{i_1}\cdots x_{i_k}$.

If $\mathrm{lc}_{\prec}(t) = 1$, we say that t is *$\prec$-monic*. By abuse of notation, we assume that the expressions $\mathrm{lm}_{\prec}(0) \prec \mathrm{lm}_{\prec}(t)$ and $\mathrm{lm}_{\prec}(0) \preceq \mathrm{lm}_{\prec}(t')$ for all $0 \neq t \in T_n^E$ and $t' \in T_n^E$ are true.

If $m' = \underline{x}^{\alpha} \in \mathrm{SMon}(T_n)$, we denote moreover

- $\mathrm{le}_{\prec}(t) := \alpha$ (*leading exponent* of t).
- $\mathrm{ele}_{\prec}(t) := (\alpha, e')$ (*extended leading exponent* of t).

We sometimes omit the index $\prec$ if it is clear from the context.

(e) The corresponding notations from Part (d) are defined analogously for a monomial ordering $\prec'$ on $\mathrm{SMon}(T_n^E)$ and $0 \neq p \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$. We denote the ordering induced by $\prec'$ on $\mathbb{N}^n \times E$ via the bijection $\underline{x}^{\alpha}(e) \mapsto (\alpha, e)$ also $\prec'$ and adapt an analogous convention for $\mathrm{le}_{\prec'}(0)$, $\mathrm{le}^{\mathrm{com}}_{\prec'}(0)$, $\mathrm{ele}_{\prec'}(0)$ and $\mathrm{ele}^{\mathrm{com}}_{\prec'}(0)$ as we did for $\mathrm{lm}_{\prec'}(0)$. Moreover, we introduce for $G \subseteq {}_{\mathbb{K}}\big\langle \mathrm{SMon}(T_n^E)\big\rangle$ the set

$$L_{\prec'}(G) := \{\mathrm{ele}_{\prec'}(g) + \mathbb{N}^n \mid g \in G \setminus \{0\}\} \subseteq \mathbb{N}^n \times E,$$

where we define $(\alpha, e) + \beta := (\alpha + \beta, e)$ for $\alpha, \beta \in \mathbb{N}^n$ and $e \in E$ and write sometimes also $L(G)$ for $L_{\prec'}(G)$ is the corresponding ordering is understood from the context.

Convention 2.1.2. In the situation of Definition 2.1.1(d) and (e), we define for simplicity (when dealing with Gröbner bases) by abuse of notation $\alpha + \mathrm{ele}_{\prec'}(0) := \mathrm{ele}_{\prec'}(0)$, $\alpha + \mathrm{le}_{\prec'}(0) := \mathrm{le}_{\prec'}(0)$, $\alpha + \mathrm{le}^{\mathrm{com}}_{\prec}(0) := \mathrm{le}_{\prec}(0)$ and $\alpha + \mathrm{ele}^{\mathrm{com}}_{\prec}(0) := \mathrm{ele}_{\prec}(0)$ for any $\alpha \in \mathbb{N}^n$.

Remark 2.1.3.

(a) By the natural identification of T_n and T_n^1 as T_n-modules and the convention of Notation 0.0.1(b) everything defined in Definition 2.1.1 carries over to T_n, but the notations of leading components, extended leading exponents and the definition of $L_{\prec'}(G)$. In this case, we define $L_{\prec'}(G)$ by replacing $\mathrm{ele}()$ by $\mathrm{le}()$.

(b) By canonically identifying the commutative polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ with the $\mathbb{K}$-module ${}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ as $\mathbb{K}$-modules, we may consider $\mathbb{K}[x_1, \ldots, x_n]$ as a $\mathbb{K}$-submodule of T_m for any $m \geq n$. Note that the definition of monomial orderings on the set of monomials of $\mathbb{K}[x_1, \ldots, x_n]$ is compatible with the definition of such orderings on $\mathrm{SMon}(T_m)$ under this identification.

Remark 2.1.4. Let E be a finite set.

(a) Clearly the ordering defined by

$$x_{i_1}\cdots x_{i_k} \prec' x_{j_1}\cdots x_{j_l} \text{ if and only if } k < l \text{ or } k = l \text{ and } (i_1, \ldots, i_k) <_{\text{lex}} (j_1, \ldots, j_k)$$

is a monomial well-ordering on $\mathrm{Mon}(T_n)$. So in particular, monomial well-orderings on $\mathrm{Mon}(T_n)$ exist.

(b) We can refine monomial orderings on $\mathrm{SMon}(T_n^E)$ to monomial orderings on $\mathrm{Mon}(T_n^E)$. More precisely, if $\prec$ and $\prec'$ are monomial orderings on $\mathrm{SMon}(T_n^E)$ and $\mathrm{Mon}(T_n)$, respectively, then $(\prec, \prec')$ defined by

$$x_{i_1} \cdots x_{i_k}(e)(\prec, \prec')x_{j_1} \cdots x_{j_l}(e') \text{ if and only if } \underline{x}^{\sum_{1\le p\le k} e_{i_p}}(e) \prec \underline{x}^{\sum_{1\le p\le l} e_{j_p}}(e')$$
$$\text{or } \underline{x}^{\sum_{1\le p\le k} e_{i_p}}(e) = \underline{x}^{\sum_{1\le p\le l} e_{j_p}}(e')$$
$$\text{and } x_{i_1} \cdots x_{i_k} \prec' x_{j_1} \cdots x_{j_l}$$

is a monomial ordering on $\mathrm{Mon}(T_n^E)$. If $\prec$ and $\prec'$ are well-orderings, $(\prec, \prec')$ is also a monomial well-ordering. If $\prec'$ is the ordering introduced in Part (a), we sometimes denote the ordering $(\prec, \prec')$ also by $\prec$ if it is understood from the context the we consider it as an ordering on $\mathrm{Mon}(T_n^E)$.

(c) Let $\prec$ be a monomial ordering on $\mathrm{(S)Mon}(T_n^E)$. Then $\prec_e$ defined by

$$x_{i_1} \cdots x_{i_k} \prec_e x_{j_1} \cdots x_{j_l} \text{ if and only if } x_{i_1} \cdots x_{i_k}(e) \prec x_{j_1} \cdots x_{j_l}(e)$$

for $e \in E$ is a monomial ordering on $\mathrm{(S)Mon}(T_n)$. This ordering is a well-ordering if $\prec$ is one.

Eventually, we will restrict ourselves to monomial orderings on $\mathrm{SMon}(T_n^E)$ and refine them to $\mathrm{Mon}(T_n^E)$ as outlined in Remark 2.1.4(b) above if necessary. The following remark lists some of the orderings on $\mathrm{SMon}(T_n^E)$ which we will use frequently throughout this thesis:

Remark 2.1.5. Let $E_1, \ldots, E_s$ and E be finite sets.

(a) Given an ordering $\prec$ on $\mathrm{SMon}(T_n)$ and a total order $<$ on E, the pair $(\prec, <)$ induces the following orderings on $\mathrm{SMon}(T_n^E)$:

(i) *Term over position ordering* (TOP-ordering):

$$\underline{x}^\alpha(e) \prec^E_{top,<} \underline{x}^\beta(e') \text{ if and only if } \underline{x}^\alpha \prec \underline{x}^\beta$$
$$\text{or } \underline{x}^\alpha = \underline{x}^\beta \text{ and } e < e',$$

where $\alpha, \beta \in \mathbb{N}^n$ and $e, e' \in E$.

(ii) *Position over term ordering* (POT-ordering):

$$\underline{x}^\alpha(e) \prec^E_{pot,<} \underline{x}^\beta(e') \text{ if and only if } e < e'$$
$$\text{or } e = e' \text{ and } \underline{x}^\alpha \prec \underline{x}^\beta,$$

where $\alpha, \beta \in \mathbb{N}^n$ and $e, e' \in E$.

These orderings are well-orderings if and only if $\prec$ is a well-ordering.

(b) Many of our computations rely on so-called *(module) block-orderings*: Let $\prec_1^{E_1}, \ldots, \prec_s^{E_s}$ be orderings on $\mathrm{SMon}(T_n^{E_1}), \ldots, \mathrm{SMon}(T_n^{E_s})$, respectively. By abuse of notation, we define the ordering $\prec_{1,\ldots,s}^{E_1,\ldots,E_s} = (\prec_1^{E_1}, \ldots, \prec_s^{E_s})$ on $\mathrm{SMon}(T_n^{E_1 \sqcup \cdots \sqcup E_S})$ by

$$\underline{x}^\alpha(e) \prec_{1,\ldots,s}^{E_1,\ldots,E_s} \underline{x}^\beta(e') \text{ if and only if } i > j \text{ or } i = j \text{ and } \underline{x}^\alpha(e) \prec_i^{E_i} \underline{x}^\beta(e'),$$

where $e \in E_i, e' \in E_j$ and $\alpha, \beta \in \mathbb{N}^n$. Notice that $\prec_{1,\ldots,s}^{E_1,\ldots,E_s}$ is a well-ordering if and only if all $\prec_i^{E_i}$ are well-orderings.

Convention 2.1.6. Let $E_1, \ldots, E_s$ and E be finite sets. If we write from now on $\prec^E$, we implicitly assume that $\prec^E$ is some ordering on $\mathrm{SMon}(T_n^E)$. Similarly, $(\prec_1^{E_1}, \ldots, \prec_s^{E_s})$ always denotes a block ordering on $\mathrm{SMon}(T_n^{E_1 \sqcup \cdots \sqcup E_s})$.

Under the identification $T_n^{E_1} \oplus \cdots \oplus T_n^{E_s} \cong T_n^{E_1 \sqcup \cdots \sqcup E_s}$, we define the set of (standard) monomials of the former module as well as monomial orderings on them.

Definition 2.1.7. Let E be a finite set and $\prec$ a monomial ordering on T_n^E.

(a) We call $S \subseteq T_n^E \setminus \{0\}$ with $\mathrm{lc}_\prec(s) = 1$ for all $s \in S$ a *reduction system* (with respect to $\prec$). For $s \in S$ and $m, m' \in \mathrm{Mon}(T_n)$ we define the $\mathbb{K}$-linear map

$$\rho_{m,s,m'} : T_n^E \to T_n^E,$$
$$x_{i_1} \cdots x_{i_l}(e) \mapsto \begin{cases} m(-\mathrm{tail}_\prec(s))m', & \text{if } x_{i_1} \cdots x_{i_l}(e) = m\,\mathrm{lm}(s)m' \\ x_{i_1} \cdots x_{i_l}(e), & \text{else} \end{cases}$$

and say that $\rho_{m,s,m'}$ is a *reduction (map) (with respect to S)*.

(b) Let $S \subseteq T_n^E$ be a reduction system, $t \in T_n^E$ and ρ a finite composition of reductions. Then we call $\rho(t)$ a *reduction* of t (under S) and say that t *reduces* to $\rho(t)$ (under S).

(c) If we have for $\prec$ and a reduction system S with respect to $\prec$

(i) $x_i x_j(e) \prec x_j x_i(e)$ for all $1 \le i < j \le n$ and $e \in E$,

(ii) there exist elements $x_j x_i(e) - c_{ij} x_i x_j(e) - d_{ij} \in S$ with $c_{ij} \in \mathbb{K}^*$ and $d_{ij} \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E) \rangle$ such that $\mathrm{lm}_\prec(d_{ij}) \prec x_i x_j(e)$ for all $1 \le i < j \le n$ and $e \in E$, and

(iii) every element in T_n^E can be reduced to an element in ${}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E) \rangle$,

then we call S a *standard reduction system* (with respect to $\prec$). In this case, the reductions $\rho_{m, x_j x_i(e) - c_{ij} x_i x_j(e) - d_{ij}, m'}$ with $x_j x_i(e) - c_{ij} x_i x_j(e) - d_{ij}$ as above and $m, m' \in \mathrm{Mon}(T_n)$ are called *commutation reductions*.

(d) Let $I \subseteq T_n$ and $A := T_n/\langle I\rangle$. We say that the reduction system $S \subseteq T_n^E$ is a reduction system for A^E if ${}_{T_n}\langle I^E\rangle_{T_n} = {}_{T_n}\langle S\rangle_{T_n}$.

(e) Let $S \subseteq T_n^E$ be a reduction system. We say that $t \in T_n^E$ is *irreducible* (with respect to S) if all reductions ρ act *trivially* on t, that is, $\rho(t) = t$. We denote the $\mathbb{K}$-submodule of all irreducible elements of T_n^E by $(T_n^E)^{\mathrm{irr}}_{S,\prec}$ and write sometimes also $(T_n^E)^{\mathrm{irr}}_S$ for the latter module if the ordering is understood. A sequence of reductions $\rho_1, \dots, \rho_k$ is called *final* on t if $\rho_k \circ \cdots \circ \rho_1(t) \in (T_n^E)^{\mathrm{irr}}_S$.

(f) Let $S \subseteq T_n^E$ be a reduction system. We call $t \in T_n^E$ *reduction-finite* if for any infinite sequence of reductions $\rho_1, \rho_2, \dots$, the reduction ρ_i acts trivially on $\rho_{i-1} \circ \cdots \circ \rho_1(t)$ for i big enough. We say that t is *reduction-unique* if it is reduction-finite and its images under all final sequences on t are the same. This common value is denoted by $\rho_{S,\prec}(t)$ or $\rho_S(t)$ if the ordering is clear from the context.

Remark 2.1.8. Let $S \subseteq T_n^E$ be a reduction system with respect to the monomial ordering $\prec$.

(a) If $\prec$ is a well-ordering, then all elements of T_n^E are reduction-finite. Moreover, if S is additionally finite, a final sequence of reductions for a given element is effectively computable.

(b) If S is a standard reduction system with respect to $\prec$, then $(T_n^E)^{\mathrm{irr}}_S \subseteq {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle$. Also note that Definition 2.1.7(ciii) follows immediately from Definition 2.1.7(ci) and (cii) if $\prec$ is a well-ordering.

Reduction uniqueness can be tested with the help of so-called ambiguities:

Definition 2.1.9. Let $S \subseteq T_n^E$ be a reduction system with respect to the monomial ordering $\prec$.

(a) A tuple $(s_1, s_2, m_1, m_2, m_3)$ with $s_1, s_2 \in S$ such that $e := \mathrm{lcomp}_\prec(s_1) = \mathrm{lcomp}_\prec(s_2)$ and $m_1, m_2, m_3 \in \mathrm{Mon}(T_n) \setminus \{1\}$ satisfying $\mathrm{lm}_\prec(s_1) = m_1 m_2(e)$ and $\mathrm{lm}_\prec(s_2) = m_2 m_3(e)$ is called an *overlap ambiguity* of S. We say that this ambiguity is *resolvable* if there exist compositions of reductions ρ, ρ' such that $\rho \circ \rho_{1,s_1,m_3}(m_1 m_2 m_3(e)) = \rho' \circ \rho_{m_1,s_2,1}(m_1 m_2 m_3(e))$.

(b) A tuple $(s_1, s_2, m_1, m_2, m_3)$ with $s_1, s_2 \in S$ such that $s_1 \neq s_2$, $e := \mathrm{lcomp}_\prec(s_1) = \mathrm{lcomp}_\prec(s_2)$ and $m_1, m_2, m_3 \in \mathrm{Mon}(T_n)$ satisfying $\mathrm{lm}_\prec(s_1) = m_2(e)$ and $\mathrm{lm}_\prec(s_2) = m_1 m_2 m_3(e)$ is called an *inclusion ambiguity* of S. We say that this ambiguity is *resolvable* if there are compositions of reductions ρ, ρ' such that $\rho \circ \rho_{m_1,s_1,m_3}(m_1 m_2 m_3(e)) = \rho' \circ \rho_{1,s_2,1}(m_1 m_2 m_3(e))$.

Remark 2.1.10. Let $S \subseteq T_n^E$ be a reduction system with respect to the monomial ordering $\prec$. If all elements of the T_n-module ${}_{T_n}\langle S\rangle_{T_n} \subseteq T_n^E$ are reducible to zero, then all ambiguities of

S are resolvable: Indeed, consider for instance an overlap ambiguity as in Definition 2.1.9(a). Then $d_{(s_1,s_2,m_1,m_2,m_3)} := \rho_{1,s_1,m_3}(m_1m_2m_3(e)) - \rho_{m_1,s_2,1}(m_1m_2m_3(e)) \in {}_{T_n}\langle S\rangle_{T_n}$ reduces to zero, say by the composition of reductions σ. Choosing ρ and ρ' in Definition 2.1.9(a) as σ, we see that the overlap ambiguity is resolvable since the reduction maps are additive. In particular, if S is the set of all $\prec$-monic elements of a two-sided ideal of T_n^E, then S is ambiguity resolvable.

The so-called Diamond Lemma relates reduction-uniqueness and resolvability of ambiguities:

Proposition 2.1.11. *[Ber78, Theorem 1.2] Let $S \subseteq T_n$ be a reduction system with respect to the monomial well-ordering $\prec$. The following are equivalent:*

(a) All ambiguities of S are resolvable.

(b) All elements of T_n are reduction-unique under S.

(c) A set of representatives in T_n of the algebra $A = T_n/\langle S\rangle$ is given by the $\mathbb{K}$-submodule $(T_n)_S^{\mathrm{irr}}$ spanned by the irreducible (with respect to S) elements of $\mathrm{Mon}(T_n)$.

When these conditions hold, A may be identified with the $\mathbb{K}$-module $(T_n)_S^{\mathrm{irr}}$, made a $\mathbb{K}$-algebra by the multiplication $t \cdot t' := \rho_S(tt')$ for $t, t' \in (T_n)_S^{\mathrm{irr}}$.

The Diamond Lemma and Remark 2.1.10 imply:

Corollary 2.1.12. *Let $S \subseteq T_n$ be a reduction system with respect to the monomial well-ordering $\prec$. Then the following are equivalent:*

(a) All ambiguities of S are resolvable.

(b) Every $t \in {}_{T_n}\langle S\rangle_{T_n}$ can be reduced to zero under S.

(c) For every $t \in {}_{T_n}\langle S\rangle_{T_n}$ exists a finite set $P \subseteq T_n \times S \times T_n$ such that

$$t = \sum_{(p,s,q)\in P} psq \text{ with } \mathrm{lm}_\prec(psq) \preceq \mathrm{lm}_\prec(t).$$

Proof. If all ambiguities of S are resolvable, then then the equivalence of (a) and (c) in the Diamond Lemma implies Condition (b). The converse direction follows from Remark 2.1.10. Obviously, if Condition (b) holds, then Condition (c) is also satisfied. Conversely assume that the latter condition holds and consider $0 \neq t \in {}_{T_n}\langle S\rangle_{T_n}$. Then there exists a finite set $P \subseteq T_n \times S \times T_n$ such that $t = \sum_{(p,s,q)\in P} psq$ and $\mathrm{lm}_\prec(psq) \preceq \mathrm{lm}_\prec(t)$. Choose $(p,s,q) \in P$ such that $\mathrm{lm}_\prec(t) = \mathrm{lm}_\prec(psq)$. Then $\rho_{p,s,q}(t) \in {}_{T_n}\langle S\rangle_{T_n}$ has leading monomial strictly smaller than $\mathrm{lm}_\prec(t)$ and Condition (b) follows by induction on $\prec$. $\square$

We are particularly interested in the following class of $\mathbb{K}$-algebras:

Definition 2.1.13. Let $S := \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \leq i < j \leq n\} \subseteq T_n$ be a standard reduction system with respect to the monomial well-ordering $\prec = (\prec, \prec')$ (see Remark 2.1.4(a) and (b)).

(a) Then the $\mathbb{K}$-algebra

$$A := T_n / \langle R \rangle ,$$

where $S \subseteq \langle R \rangle \subseteq T_n$, is called a *PBW-reduction-algebra* and we say that $\prec$ is a *well-ordering* on A. If $I \subseteq {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n) \rangle$ is a finite set satisfying

(i) ${}_{T_n}\langle I \cup S \rangle_{T_n} = {}_{T_n}\langle R \rangle_{T_n}$ and

(ii) $\underline{x}^\alpha \in T_n$ for $\alpha \in \mathbb{N}^n$ is irreducible with respect to the $\prec$-monic elements of ${}_{T_n}\langle R \rangle_{T_n}$ and $\prec$ if and only if

$$\alpha \notin L_\prec(I),$$

then we call the tuple $(T_n, S, I, \prec)$ *PBW-reduction datum* of A and write $A = (A, \prec) = (T_n, S, I, \prec)$. We refer to (the elements of) S as *commutation relations*.

(b) Given that A is a PBW-reduction-algebra, we moreover define: If S is a standard reduction system with respect to the monomial ordering $\prec'' = (\prec'', \prec')$, we say that $\prec''$ is an *ordering* on the PBW-reduction-algebra A. Given $I' \subseteq {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n) \rangle$ satisfying Conditions (ai) and (aii) after replacing I and $\prec$ by I' and $\prec''$, respectively, we call $(T_n, S, I', \prec'')$ also *PBW-reduction datum* of A.

Remark 2.1.14. Note that given a PBW-reduction-algebra A with PBW-reduction datum $(T_n, S, I', \prec')$ the notation $A = (T_n, S, I', \prec'')$ is reserved for the case that $\prec''$ is a well-ordering.

Remark 2.1.15.

(a) One easily checks that Definition 2.1.13(aii) is equivalent to

$$L_\prec(I) = \{\mathrm{le}(r) \mid 0 \neq r \in {}_{T_n}\langle R \rangle_{T_n}, \mathrm{lm}(r) \in \mathrm{SMon}(T_n)\}. \qquad (2.1.1)$$

Also note that by construction $L_\prec(I)$ is always included in the right hand side of Equation (2.1.1) because if $r \in I \subseteq {}_{T_n}\langle R \rangle_{T_n}$ with $\mathrm{le}(r) = \alpha$, then we can apply commutation reductions to $\underline{x}^\gamma r$ to find an element $r' \in {}_{T_n}\langle R \rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n) \rangle$ with $\mathrm{le}(r') = \alpha + \gamma$ for any $\gamma \in \mathbb{N}^n$. For convenience we also observe that the right hand set in Equation (2.1.1) agrees with

$$\{\mathrm{le}(r) \mid 0 \neq r \in {}_{T_n}\langle R \rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n) \rangle\},$$

since given an element $0 \neq r \in {}_{T_n}\langle R \rangle_{T_n}$ with $\mathrm{lm}(r) \in \mathrm{SMon}(T_n)$, we can apply commutation reductions to $\mathrm{tail}(r)$ to reduce r to an element in ${}_{T_n}\langle R \rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n) \rangle$ preserving its leading monomial.

(b) If Definition 2.1.13(aii) holds, then the condition in Definition 2.1.13(ai) may be replaced by $I \subseteq {}_{T_n}\langle R\rangle_{T_n}$. Indeed, assuming Definition 2.1.13(aii) and $I \subseteq {}_{T_n}\langle R\rangle_{T_n}$, we use commutation relations to write $r \in {}_{T_n}\langle R\rangle_{T_n}$ as $r = r' + s$ with $s \in {}_{T_n}\langle S\rangle_{T_n}$, $r' \in {}_{T_n}\langle R\rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ and $\mathrm{lm}(r') \preceq \mathrm{lm}(r)$. Equation (2.1.1) implies now that there is $p \in I$ and $\alpha \in \mathbb{N}^n$ such that $\mathrm{le}(r') = \mathrm{le}(p) + \alpha$. Applying commutation reductions to $\underline{x}^\alpha p$ to reduce it to an element $p' \in {}_{T_n}\langle S \cup I\rangle_{T_n}$ with $\mathrm{lm}(p') = \mathrm{lm}(r')$, we find an expression

$$r' = r'' + cp' + s'$$

with $c \in \mathbb{K}^*$, $s' \in {}_{T_n}\langle S\rangle_{T_n}$ and $r'' \in {}_{T_n}\langle R\rangle_{T_n}$ satisfying $\mathrm{lm}(r'') \prec \mathrm{lm}(r')$. Induction with respect to the well-ordering $\prec$ completes the proof.

(c) Part (a) holds also in the situation of Definition 2.1.13(b) after replacing I and $\prec$ by I' and $\prec''$, respectively. However, the proof Part (b) does not generalize to this setting, because we made use of the fact that $\prec$ is a well-ordering.

Remark 2.1.16. Consider the PBW-reduction-algebra $A = (T_n, S, I, \prec)$.

(a) According to Remark 2.1.15(a) we can write every $p \in {}_{T_n}\langle S \cup I\rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ as

$$p = \sum_{g\in I} a_g g + \sum_{(t,s,t')\in U} tst'$$

for some $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle^I$ and $U \subseteq T_n \times S \times T_n$ finite satisfying

$$\mathrm{le}(a_g) + \mathrm{le}(g) \preceq \mathrm{le}(p) \text{ and } \mathrm{le}^{\mathrm{com}}(t) + \mathrm{le}^{\mathrm{com}}(s) + \mathrm{le}^{\mathrm{com}}(t') \preceq \mathrm{le}(p).$$

Moreover, there is $g \in G$ with equality $\mathrm{le}(a_g) + \mathrm{le}(g) = \mathrm{le}(p)$.

(b) Furthermore, we can determine for an element $p \in T_n$ a finite set $U \subseteq T_n \times S \times T_n$ and $p' \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ such that

$$p = p' + \sum_{(t,s,t')\in U} tst' \text{ and } \mathrm{lm}_{\prec''}(p'), \mathrm{lm}_{\prec''}(tst') \preceq'' \mathrm{lm}_{\prec''}(p)$$

for any ordering $\prec''$ on A.

Lemma 2.1.17. *A PBW-reduction-algebra $(A, \prec)$ admits a PBW-reduction datum $(T_n, S, I, \prec)$ and the residue classes of*

$$B := \{\underline{x}^\alpha \mid \alpha \notin L(I)\}$$

form a $\mathbb{K}$-basis of A. Moreover, the set of irreducible elements of T_n with respect to the ambiguity resolvable reduction system consisting of the $\prec$-monic elements of ${}_{T_n}\langle S, I\rangle_{T_n}$ agrees with the $\mathbb{K}$-span of B and does not depend on the choice of I and S.

Proof. Let S and R be as in Definition 2.1.13. We first observe that the set M of $\prec$-monic elements of ${}_{T_n}\langle R\rangle_{T_n}$ is an ambiguity resolvable reduction system for A by Remark 2.1.10 and hence A can be identified with $(T_n)^{\mathrm{irr}}_M \subseteq {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ as $\mathbb{K}$-algebra by Proposition 2.1.11 and Remark 2.1.8(b).

Consider now the set

$$L := \{\mathrm{le}_\prec(r) \mid 0 \neq r \in {}_{T_n}\langle R\rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle\} \subseteq \mathbb{N}^n .$$

By Dickson's Lemma there is a finite subset $L' \subseteq L$ such that for every $\alpha \in L$ exists an $\alpha' \in L'$ with $\alpha \in \alpha' + \mathbb{N}^n$. Choose for every $\alpha' \in L'$ an $r_{\alpha'} \in {}_{T_n}\langle R\rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ having leading exponent α'. Setting

$$I := \{r_{\alpha'} \mid \alpha' \in L'\},$$

we claim that $(T_n, S, I, \prec)$ is a PBW-reduction datum for A: Indeed, by Remark 2.1.15(a) Condition (aii) in Definition 2.1.13 is satisfied. As by construction $I \subseteq {}_{T_n}\langle R\rangle_{T_n}$, we are done by Remark 2.1.15(b). □

Convention 2.1.18. As orderings in the context of PBW-reduction-algebras are as in Remark 2.1.4(a), we from now assume implicitly that all orderings are of this type.

Notation 2.1.19. Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra and let M denote the $\prec$-monic element of ${}_{T_n}\langle S, I\rangle_{T_n}$. Then $(T_n)^{\mathrm{irr}}_M$ depends only on $A = T_n/\langle S, I\rangle$ and $\prec$. We hence also denote it by $(T_n)^{\mathrm{irr}}_{(A,\prec)}$ and, similarly, we write $\rho_{(A,\prec)}$ and for ρ_M.

The following algorithm evaluates the map $\rho_{(A,\prec)}$:

Algorithm 2.1.20 Given a PBW-reduction-algebra $(A, \prec)$ and $t \in T_n$ this algorithm computes the irreducible representation $\rho_{(A,\prec)}(t)$.

Input: A PBW-reduction-algebra $A = (T_n, S, I, \prec)$ and $t \in T_n$.
Output: An element $u \in T_n$ such that $u = \rho_{(A,\prec)}(t)$.

Initialize $u = 0$.
Replace t by a reduction of t in ${}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ under S.
while $t \neq 0$ **do**
  if $\mathrm{le}(t) \in L(I)$ **then**
    Choose $p \in I$ such that $\gamma := \mathrm{le}(p) - \mathrm{le}(t) \in \mathbb{N}^n$.
    Apply reductions under S to reduce $\underline{x}^\gamma p$ to an element $p' \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ with $\mathrm{le}(p') = \mathrm{le}(t)$.
    Set $t := t - \mathrm{lc}(t)/\mathrm{lc}(p')p'$.
  else
    Set $u := u + \mathrm{lt}(t)$ and $t := \mathrm{tail}(t)$.
return u.

Lemma 2.1.21. *Algorithm 2.1.20 is correct and terminates.*

Proof. Termination is clear, because we replace in each iteration of the while-loop t by an element with smaller leading monomial with respect to the well-ordering $\prec$.

Notice that we have $u - t \in {}_{T_n}\langle I, S\rangle_{T_n}$ and that $u \in (T_n)^{\mathrm{irr}}_{(A,\prec)}$. Hence the correctness follows by Proposition 2.1.11(c). □

A particularly well-behaved case of PBW-reduction-algebras are PBW-algebras:

Definition 2.1.22. A PBW-reduction-algebra $A = (T_n, S, \{0\}, \prec)$ is called a *PBW-algebra*. If the elements in S are of type $x_j x_i - c_{ij} x_i x_j$, we say that A is a *quasi-commutative PBW-algebra*.

Corollary 2.1.23. *[Lev05, Theorem 1.2.3] Let* $S := \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \leq i < j \leq n\} \subseteq T_n$ *be a standard reduction system with respect to a monomial well-ordering* $\prec$ *that induces a monomial ordering on* $\mathrm{SMon}(T_n)$ *by restriction. Then the overlap ambiguities of the* $\mathbb{K}$*-algebra*

$$A := T_n / \langle S \rangle ,$$

read

$$c_{ik} c_{jk} d_{ij} x_k - x_k d_{ij} + c_{jk} x_j d_{ik} - c_{ij} d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} x_i d_{jk}$$

for $1 \leq i < j < k \leq n$ *and* A *is a PBW-algebra if and only if these ambiguities can be reduced to zero under* S*.*

The first part of the following corollary is obvious and a proof of the second assertion can be found for example in [Lev05, Theorem 1.4.7].

Corollary 2.1.24. *The set of standard monomials forms a* $\mathbb{K}$*-basis of a PBW-algebra . Moreover, PBW-algebras are left and right Noetherian rings.*

The following two examples of PBW-algebras are frequently used throughout this thesis:

Example 2.1.25. The polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ in n variables is a PBW-algebra.

Example 2.1.26. The Weyl algebra in the variables $x_1, \dots, x_n$ and derivations $\partial_1, \dots, \partial_n$ defined by

$$D_n := \mathbb{K}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / \langle \{[\partial_j, x_i] - \delta_{ij}, [x_i, x_j], [\partial_i, \partial_j] \mid \text{ for } 1 \leq i, j \leq n\} \rangle$$

is a PBW-algebra (see also Example 1.2.2).

From Corollary 2.1.24 we deduce:

Lemma 2.1.27. *PBW-reduction-algebras are left and right Noetherian rings.*

Proof. Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra with $S := \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \le i < j \le n\}$. We introduce the multi-filtration $F_\bullet^\prec$ on A indexed by $\mathbb{N}^n$ (see [GTL00]) given by

$$F_\alpha^\prec A := \sum_{\underline{x}^\beta \preceq \underline{x}^\alpha} \mathbb{K}\,\overline{\underline{x}^\beta} \subseteq A$$

for $\alpha \in \mathbb{N}^n$. Note that this filtration is indeed exhaustive since A is generated by the standard monomials of T_n as $\mathbb{K}$-algebra. Consider now the associated multi-graded ring

$$\mathrm{Gr}^{F^\prec} A := \bigoplus_{\alpha \in \mathbb{N}^n} F_\alpha^\prec A / F_{\prec\alpha}^\prec A,$$

where $F_{\prec\alpha}^\prec A := \bigcup_{\beta \in \mathbb{N}^n : \underline{x}^\beta \prec \underline{x}^\alpha} F_\beta^\prec A$. The $\mathbb{K}$-algebra $\mathrm{Gr}^{F^\prec} A$ is isomorphic to a factor algebra of a quasi-commutative PBW-algebra via the map

$$\varphi : \mathrm{Gr}^{F^\prec} A \to B := (T_n / \langle \{x_j x_i - c_{ij} x_i x_j \mid 1 \le i < j \le n\} \rangle) / \left\langle \{\overline{\mathrm{lm}_\prec(p)} \mid p \in I\} \right\rangle.$$
$$\mathrm{Gr}_{e_i}^{F^\prec} A \ni \overline{x_i} \mapsto \overline{x_i} + \left\langle \{\overline{\mathrm{lm}_\prec(p)} \mid p \in I\} \right\rangle.$$

The $\mathbb{K}$-algebra B is as a quotient of a PBW-algebra left and right Noetherian (see Corollary 2.1.24). Now [GTL00, Lemma 1.2] implies the claim. □

Lemma 2.1.28. *Consider the $\mathbb{K}$-algebra $\mathbb{K}\langle \underline{x}, \underline{y} \rangle := \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$ and its factor algebra $P := \mathbb{K}\langle \underline{x}, \underline{y} \rangle / \langle S \rangle$, where*

$$\begin{aligned} S :=& \{[x_j, x_i] \mid 1 \le i < j \le n\} \cup \{[y_l, y_k] - d_{kl} \mid 1 \le k < l \le m\} \\ &\cup \{[y_k, x_i] - f_{ik} \mid 1 \le i \le n, 1 \le k \le m\} \end{aligned}$$

with $d_{kl}, f_{ik} \in {}_{\mathbb{K}}\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle) \rangle$. Canonically identifying the ideal $J \subseteq \mathbb{K}[\underline{x}]$ with a subset of ${}_{\mathbb{K}}\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}, \underline{y} \rangle) \rangle$, define the $\mathbb{K}$-algebra

$$A := P / {}_P\langle \overline{J} \rangle_P.$$

Then we have:

(a) There exists a well-ordering such that S is a reduction system with respect to that ordering.

(b) If the surjective $\mathbb{K}$-linear homomorphism

$$\psi : \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]/J) \underline{y}^\beta \to A,\ \overline{\underline{x}^\alpha} \underline{y}^\beta \mapsto \overline{\underline{x}^\alpha \underline{y}^\beta}$$

is injective, then A is isomorphic to a PBW-reduction-algebra. Given any ordering $\prec$ on A, a corresponding PBW-reduction datum is given by $(\mathbb{K}\langle \underline{x}, \underline{y} \rangle, S, J', \prec)$, where J' is a Gröbner basis of $J \subseteq \mathbb{K}[\underline{x}]$ with respect to the ordering induced by $\prec$.

Proof.

(a) The set S is a reduction system with respect to the (well)-ordering $\prec$ on $\mathrm{SMon}(T_n)$ if and only if it satisfies $\mathrm{lm}_\prec(d_{kl}) \prec y_k y_l$ for $1 \leq k < l \leq m$ and $\mathrm{lm}_\prec(f_{ik}) \prec x_i y_k$ for $1 \leq i \leq n$ and $1 \leq k \leq m$. So S is a reduction system with respect to any refinement of the partial ordering $<$ given by

$$\underline{x}^\alpha \underline{y}^\beta < \underline{x}^{\alpha'} \underline{y}^{\beta'} \text{ if and only if } |\beta| < |\beta'|,$$

(with $\alpha, \alpha' \in \mathbb{N}^n$ and $\beta, \beta' \in \mathbb{N}^m$) by a well-ordering.

(b) We first observe that we may identify

$$A = \mathbb{K}\langle \underline{x}, \underline{y}\rangle / \left\langle S \cup J' \right\rangle$$

showing that A is indeed isomorphic to a PBW-reduction-algebra by Part (a). Then Definition 2.1.13(ai) is clearly satisfied with $R = S \cup J'$. According to Remark 2.1.15(a) it suffices to show for Definition 2.1.13(aii) that

$$L(J') \supseteq \{\mathrm{le}(p) \mid 0 \neq p \in {}_{\mathbb{K}\langle \underline{x}, \underline{y}\rangle}\left\langle S \cup J' \right\rangle_{\mathbb{K}\langle \underline{x}, \underline{y}\rangle} \cap {}_{\mathbb{K}}\left\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}, \underline{y}\rangle)\right\rangle\}$$

holds. Consider $p = \sum_{(\alpha,\beta)} p_{(\alpha,\beta)} \underline{x}^\alpha \underline{y}^\beta \in {}_{\mathbb{K}\langle \underline{x}, \underline{y}\rangle}\langle S \cup J' \rangle_{\mathbb{K}\langle \underline{x}, \underline{y}\rangle} \cap {}_{\mathbb{K}}\left\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}, \underline{y}\rangle)\right\rangle$ with $p_{(\alpha,\beta)} \in \mathbb{K}$ not all zero. Note that p is mapped to zero under the composition of the projection $\pi : \mathbb{K}\langle \underline{x}, \underline{y}\rangle \twoheadrightarrow A$ with ψ^{-1}, that is, we have

$$p \in \bigoplus_{\beta \in \mathbb{N}^m} J \underline{y}^\beta.$$

Consequently, it holds for every $\beta \in \mathbb{N}^m$ that $\sum_{\alpha \in \mathbb{N}^n} p_{(\alpha,\beta)} \underline{x}^\alpha \in J$ and hence

$$(\alpha', \beta') := \mathrm{le}_\prec(p) = \mathrm{le}_\prec\Big(\sum_{\alpha \in \mathbb{N}^n} p_{(\alpha,\beta')} \underline{x}^\alpha \underline{y}^{\beta'}\Big) = \Big(\mathrm{le}_\prec\Big(\sum_{\alpha \in \mathbb{N}^n} p_{(\alpha,\beta')} \underline{x}^\alpha\Big), \beta'\Big) \in L(J'),$$

because J' is a Gröbner basis of $J \subseteq \mathbb{K}[\underline{x}]$ with respect to $\prec$.

□

Definition 2.1.29. Keeping the setup and notation of Lemma 2.1.28 and assuming that ψ is injective, we call the PBW-reduction-algebra $A = (\mathbb{K}\langle \underline{x}, \underline{y}\rangle, S, J', \prec)$ *elementary*.

We have seen in Example 2.1.26 that the global sections of the sheaf of differential operators on the affine space $\mathbb{C}^n$ can be represented as a PBW-reduction-algebra. The next example shows that locally a similar statement holds for smooth varieties. More precisely, we represent coordinate system rings as elementary PBW-reduction-algebras:

Example 2.1.30. Let X be a smooth irreducible affine variety defined by the vanishing of the prime ideal $I \subseteq \mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_n]$. Assume that X has a global coordinate system, that is, there exists a coordinate system $(\overline{f_i}, \theta_i)_{1 \leq i \leq m}$ (with $f_i \in \mathbb{C}[\underline{x}]$) on the open neighborhood $X \subseteq X$. Recall that according to Remark 1.2.3 we may assume that $\theta_1, \ldots, \theta_m$ are induced by $\theta_1^l, \ldots, \theta_m^l \in \Theta_{\mathbb{C}^n}(\mathbb{C}^n)$.

(a) We prove that the coordinate system ring $\mathcal{D}_X(X)$ is isomorphic to an elementary PBW-reduction-algebra: By the properties of coordinate systems, we have a $\mathbb{C}$-linear isomorphism

$$\psi : \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{C}[\underline{x}]/I)\underline{\theta}^\beta \to \mathcal{D}_X(X) = \mathbb{C}\langle \overline{x_1}, \ldots, \overline{x_n}, \theta_1, \ldots, \theta_m \rangle \subseteq \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\underline{x}]/I),$$
$$\overline{\underline{x}^\alpha}\underline{\theta}^\beta \mapsto \overline{x_1}^{\alpha_1} \cdots \overline{x_n}^{\alpha_n}\underline{\theta}^\beta$$

and the generators of the $\mathbb{C}$-algebra $\mathcal{D}_X(X)$ satisfy $[\overline{x_j}, \overline{x_i}] = 0$, $[\theta_l, \theta_k] = 0$ and $[\theta_k, x_i] = \overline{\theta_k^l(x_i)}$ for $1 \leq i \leq j \leq n$ and $1 \leq k \leq l \leq m$. Consequently, ψ factors through the quotient algebra

$$T_X := \mathbb{C}\langle \underline{x}, \underline{y} \rangle / \langle S \cup I \rangle \cong (\mathbb{C}\langle \underline{x}, \underline{y} \rangle / \langle S \rangle) / \langle I \rangle$$

of $\mathbb{C}\langle \underline{x}, \underline{y} \rangle := \mathbb{C}\langle \underline{x}, y_1, \ldots, y_m \rangle$, where

$$S := \{[x_j, x_i] \mid 1 \leq i < j \leq n\} \cup \{[y_l, y_k] \mid 1 \leq k < l \leq m\}$$
$$\cup \{[y_k, x_i] - \theta_k^l(x_i) \mid 1 \leq i \leq n, 1 \leq k \leq m\}$$

via the surjective $\mathbb{C}$-linear maps

$$\bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{C}[\underline{x}]/I)\underline{\theta}^\beta \xrightarrow{\psi_1} T_X \xrightarrow{\psi_2} \mathcal{D}_X(X).$$
$$\overline{\underline{x}^\alpha}\theta^\beta \longmapsto \overline{\underline{x}^\alpha \underline{y}^\beta} \longmapsto \overline{x_1}^{\alpha_1} \cdots \overline{x_n}^{\alpha_n}\underline{\theta}^\beta.$$

The injectivity of ψ_1 follows from the injectivity of ψ and the injectivity of ψ_2 from the surjectivity of ψ_1 and the injectivity of ψ. As ψ_2 is a $\mathbb{C}$-algebra homomorphism, the coordinate system ring $\mathcal{D}_X(X)$ is isomorphic to T_X as $\mathbb{C}$-algebra.

Now consider a well-ordering $\prec$ on T_X (for existence see Lemma 2.1.28(a)) and let I' be a Gröbner basis of $I \subseteq \mathbb{C}[\underline{x}]$ with respect to the ordering induced by $\prec$ on $\mathbb{C}[\underline{x}]$. Then we see by Lemma 2.1.28 that T_X is isomorphic to the elementary PBW-reduction-algebra $(\mathbb{C}\langle \underline{x}, \underline{y} \rangle, S, I', \prec)$. In particular, a PBW-reduction datum is effectively computable in this case.

(b) Note that we may assume by Remark 1.2.12 that f_m agrees with some x_i, say x_n, and that $\theta_i^l(x_n) = \delta_{i,m}$. In this case, the $\mathbb{C}$-subalgebra V of $\mathcal{D}_X(X)$ generated by $\overline{x_1}, \ldots, \overline{x_n}$, $\theta_1, \ldots, \theta_{m-1}$ and $\overline{f_m}\theta_m$ can again be represented as an elementary PBW-reduction-algebra as follows: Arguing as for ψ, we have an isomorphism

$$\bigoplus_{\alpha \in \mathbb{N}^m} (\mathbb{C}[\underline{x}]/I)\theta_1^{\alpha_1} \cdots \theta_{m-1}^{\alpha_{m-1}} (\overline{x_n}\theta_m)^{\alpha_m} \cong V$$

and may hence apply Lemma 2.1.28 to identify V with the elementary PBW-reduction-algebra

$$T_X^V := (\mathbb{C}\langle \underline{x}, y_1, \ldots, y_{m-1}, z\rangle, S_V, I', \prec^V)$$

with $\prec^V$ any well-ordering inducing the same ordering as $\prec$ on $\mathbb{C}[\underline{x}]$ (see Part (a)) and

$$S_V := \{[x_j, x_i], [y_l, y_k], [z, y_k], [y_k, x_i] - \theta_k^l(x_i), [z, x_i] - x_n\theta_m^l(x_i) \mid \\ 1 \leq i \leq j \leq n, 1 \leq k \leq l \leq m-1\} \setminus \{0\}.$$

Notice that we may consider T_X^V as a subalgebra of T_X by identifying z with $x_n y_m$.

(c) We remark that in the situation of Part (b), we have

$$\begin{aligned} V/x_nV = V/_V\langle x_n\rangle_V &\cong \bigoplus_{\alpha \in \mathbb{N}^m} (\mathbb{C}[\underline{x}]/\langle I, x_n\rangle)\theta_1^{\alpha_1} \cdots \theta_{m-1}^{\alpha_{m-1}} (\overline{x_n}\theta_m)^{\alpha_m} \\ &\cong \bigoplus_{\alpha \in \mathbb{N}^m} (\mathbb{C}[x_1, \ldots, x_{n-1}]/\phi_{x_n}(I))\theta_1^{\alpha_1} \cdots \theta_{m-1}^{\alpha_{m-1}} (\overline{x_n}\theta_m)^{\alpha_m}, \end{aligned}$$

where ϕ_{x_n} stands for the $\mathbb{C}$-algebra endomorphism of $\mathbb{C}\langle \underline{x}, \underline{y}\rangle$ that maps x_n to 0 and acts on all other variables as identity. By the same arguments as in Part (a), the above algebra can be realized as the elementary PBW-reduction-algebra

$$T_X^{V/x_nV} := (\mathbb{C}\langle x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}, z\rangle, S_{V/x_nV}, I_{V/x_nV}, \prec^{V/x_nV}),$$

with $\prec^{V/x_nV}$ a suitable well-ordering such that

$$S_{V/x_nV} := \{[x_j, x_i], [y_l, y_k], [z, y_k], [z, x_i], [y_k, x_i] - \phi_{x_n}(\theta_k^l(x_i)) \mid \\ 1 \leq i \leq j \leq n-1, 1 \leq k \leq l \leq m-1\} \setminus \{0\},$$

is a reduction system and $I_{V/x_nV} \subseteq \mathbb{C}[x_1, \ldots, x_{m-1}]$ a Gröbner basis of $\phi_{x_n}(I)$ with respect to the ordering induced by $\prec^{V/x_nV}$. Note that the map $T_X^V \to T_X^{V/x_nV}$ induced by the canonical projection $V \to V/x_nV$ sends $\overline{x_n}$ to $\overline{0}$ and the residue classes of the other variables to the corresponding residue classes in T_X^{V/x_nV}.

(d) We keep the assumption of Part (b) and consider the subvariety $X_0 := V(x_n) \cap X \subseteq X$. Then $(\overline{f_i}, \theta_i)_{1 \le i \le m-1}$ is a global coordinate system on X_0 (where we interpret the θ_i as derivations on $\mathcal{O}_{X_0}(X_0)$ by Remark 1.2.3). According to Part (a) the coordinate system ring $\mathcal{D}_{X_0}(X_0)$ is isomorphic to the elementary PBW-reduction-algebra

$$\mathbb{C}\langle \underline{x}, y_1, \ldots, y_{m-1} \rangle / \langle J \cup \{x_n\} \cup S_{X_0} \rangle$$

where S_{X_0} is obtained from S by deleting all equations involving y_m. This algebra is obviously isomorphic to

$$\mathbb{C}\langle x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1} \rangle / \langle \phi_{x_n}(J \cup S_{X_0}) \rangle$$

and a PBW-reduction datum of the latter algebra can be obtained as outlined in Part (a). Note that we have

$$V/x_n V \cong \mathcal{D}_{X_0}(X_0)[z].$$

Remark 2.1.31. Note that there were some attempts by Oaku to deal algorithmically with coordinate system rings [Oak96]. He suggested two methods: Taking X as in the above example, he considers the $\mathbb{C}$-subalgebra of the Weyl-algebra generated by $x_1, \ldots, x_n$ and $\theta_1^l, \ldots, \theta_m^l$. He then claims that this subalgebra equals $\bigoplus_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m} \mathbb{C}\, \underline{x}^\alpha (\theta_1^l)^{\beta_1} \cdots (\theta_m^l)^{\beta_m}$. But this is in general not true: We may assume without loss of generality $f_i = x_i$ and $\theta_i^l = \partial_i + \sum_{m+1 \le k \le n} a_k^i(\underline{x}) \partial_k$ for suitably chosen $a_k^i(\underline{x}) \in \mathbb{C}[\underline{x}]$. Hence the commutator $[\theta_j^l, \theta_i^l]$ for $i \neq j$ is of the form $\sum_{m+1 \le k \le n} b_k^{ij}(\underline{x}) \partial_k$ and only an element of the above direct sum if it equals zero, meaning that the lifted derivations also commute, which is in general not true:

Considering for instance $X = V(x_3) \subseteq \mathbb{C}^3$, we see that x_1 and x_2 are global coordinates on X and that we may choose as lifts of their derivations $\partial_1 + x_2 x_3 \partial_3$ and ∂_2. Now we have $[\partial_2, \partial_1 + x_2 x_3 \partial_3] = x_3 \partial_3$. Obviously, we can resolve the issue in this basic example by choosing different lifts, but in the following example it is not clear how to resolve that problem: Consider the global coordinate neighborhood $X \subseteq \mathbb{C}^5$ defined by the prime ideal $I = \langle x_1^2 x_3 - x_1 x_2 + x_4^2 + 1, x_1^3 x_3 + x_4^2 + x_2 + x_3 + 1, dw - 1 \rangle \subseteq \mathbb{C}[x_1, x_2, x_3, x_4, w]$ for $d = -6x_1^2 x_3 x_4 + 4x_1 x_3 x_4 - 2x_2 x_4$. Proceeding as in Remark 1.2.11(b), we see that the commuting derivations

$$\begin{aligned}
\theta_1 = \partial_2 + d^{-2}((&-12x_1^3 x_3 x_4^2 - 4x_1^2 x_3 x_4^2 - 4x_1 x_2 x_4^2 + 8x_1 x_3 x_4^2 - 4x_2 x_4^2)\partial_1 \\
&+ (18x_1^5 x_3^2 x_4 - 12x_1^4 x_3^2 x_4 + 6x_1^3 x_2 x_3 x_4 + 12x_1^3 x_3^2 x_4 - 6x_1^2 x_2 x_3 x_4 \\
&- 8x_1^2 x_3^2 x_4 + 8x_1 x_2 x_3 x_4 - 2x_2^2 x_4)\partial_4 \\
&+ (-18x_1^5 x_3^2 w + 12x_1^4 x_3^2 w - 6x_1^3 x_2 x_3 w - 12x_1^3 x_3^2 w + 12x_1^2 x_3 x_4^2 w \\
&+ 6x_1^2 x_2 x_3 w + 8x_1^2 x_3^2 w + 24x_1 x_3 x_4^2 w - 8x_1 x_2 x_3 w - 4x_2 x_4^2 w - 8x_3 x_4^2 w \\
&+ 2x_2^2 w)\partial_w)
\end{aligned}$$

and

$$\begin{aligned}\theta_2 = \partial_3 + d^{-2}((&-12x_1^5x_3x_4^2 + 20x_1^4x_3x_4^2 - 4x_1^3x_2x_4^2 - 8x_1^3x_3x_4^2 + 4x_1^2x_2x_4^2 - 12x_1^2x_3x_4^2 \\ &+ 8x_1x_3x_4^2 - 4x_2x_4^2)\partial_1 \\ &+ (-6x_1^6x_3^2x_4 - 6x_1^5x_2x_3x_4 + 4x_1^5x_3^2x_4 + 2x_1^4x_2x_3x_4 - 2x_1^3x_2^2x_4 \\ &+ 12x_1^3x_3^2x_4 - 6x_1^2x_2x_3x48x_1^2x_3^2x_4 + 8x_1x_2x_3x_4 - 2x_2^2x_4)\partial_4 \\ &+ (6x_1^6x_3^2w + 6x_1^5x_2x_3w - 4x_1^5x_3^2w - 12x_1^4x_3x_4^2w - 2x_1^4x_2x_3w \\ &+ 16x_1^3x_3x_4^2w + 2x_1^3x_2^2w - 12x_1^3x_3^2w - 12x_1^2x_2x_4^2w - 8x_1^2x_3x_4^2w \\ &+ 6x_1^2x_2x_3w + 8x_1^2x_3^2w + 8x_1x_2x_4^2w + 24x_1x_3x_4^2w - 8x_1x_2x_3w \\ &- 8x_3x_4^2w + 2x_2^2w)\partial_w)\end{aligned}$$

on $\mathbb{C}[x_1, x_2, x_3, x_4, w]_d$ induce commuting derivations on X that $\mathcal{O}_X(X)$-generate $\Theta_X(X)$. Yet, if we replace d^{-2} by w^2 to obtain derivations on $\mathbb{C}[x_1, x_2, x_3, x_4, w]$, the so obtained derivations fail to commute.

As Oaku's method completely relies on the above direct sum representation, this shows that his method does in general not work.

His second method uses the Leibnitz rule to define a non-associative "multiplication". He bases the proof of correctness of this method on his flawed first method, hence not giving a comprehensive proof of correctness. The underlying method is still correct, because one easily shows that we could replace our multiplication for coordinate system rings by the Leibnitz rule and then one notices that our Algorithm 2.1.45 and his algorithm do basically the same thing.

Also note that our more general setup has the advantage that it deals simultaneously with (factor algebras of) PBW-algebras and coordinate system rings as well as some variants of them (as considered in Example 2.1.30). Moreover, we allow (and need) more general orderings. Using the commutation relations it is easy to see which orderings are actually permitted.

Eventually, we will be interested in implementations of our algorithms. For this we need to be able to present a given PBW-reduction datum by a finite set of data:

Definition 2.1.32. Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra and $\mathbb{K}' \subseteq \mathbb{K}$ a subfield.

(a) We say that $\mathbb{K}'$ is a *computable* subfield of $\mathbb{K}$ if all elements of $\mathbb{K}'$ can be represented by a finite set of data: their sum, product and quotient can be calculated in a finite number of steps, and there is a finite procedure that determines whether a given expression of elements of $\mathbb{K}'$ is zero or not.

(b) We say that $\mathbb{K}'$ is $(A, S, I, \prec)$*-computable* (or $(A, \prec)$*-computable* for short) if it is computable and $S, I \subseteq \mathbb{K}'\langle x_1, \dots, x_n\rangle$. We write

$$A_{\mathbb{K}'} := (T_n, S, I, \prec)_{\mathbb{K}'} := (\mathbb{K}'\langle x_1, \dots, x_n\rangle, S, I, \prec).$$

2.1.2 Gröbner bases for PBW-reduction-algebras

Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra and E a finite set. Given $a \in T_n^E$, we consider $\overline{a}$ as an element of A^E via the canonical isomorphism $A^E \cong T_n^E / \langle S^E \cup I^E \rangle$. Orderings on A^E are now introduced as follows:

Definition 2.1.33. Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra and E a finite set.

(a) We say that the monomial ordering $\prec^E$ on $\mathrm{SMon}(T_n^E)$ is a *ordering* on A^E if it induces an ordering on each factor of A^E. Then we write $A^E = (A^E, \prec^E)$.

(b) If $\prec^E$ is a well-ordering, we call $\prec^E$ a *well-ordering* on A^E. If $(T_n, S_e, I_e, \prec_e^E)$ is moreover a corresponding PBW-reduction datum of A for $e \in E$, we also write $A^E = (A^E, \prec^E) = (T_n, S_e, I_e, \prec_e^E)_{e \in E}$ and say that $(T_n, S_e, I_e, \prec_e^E)_{e \in E}$ is a PBW-reduction datum for $(A^E, \prec^E)$. In this case, we introduce the map

$$\rho_{(A^E, \prec^E)} := \bigoplus_{e \in E} \rho_{(A, \prec_e^E)} : T_n^E \to (T_n^E)^{\mathrm{irr}}_{(A^E, \prec_E)} := \bigoplus_{e \in E} (T_n)^{\mathrm{irr}}_{(A, \prec_e^E)}(e).$$

We also define the map

$$\tau_{(A^E, \prec^E)} : A^E \to (T_n^E)^{\mathrm{irr}}_{(A^E, \prec_E)} \subseteq T_n^E$$

as the inverse of the composed map $(T_n^E)^{\mathrm{irr}}_{(A^E, \prec_E)} \hookrightarrow T_n^E \twoheadrightarrow A^E$. We sometimes also use the notation $\rho_{\prec^E}$ and $\tau_{\prec^E}$ for the above maps if that does not cause any ambiguity.

For $0 \neq a \in A$, we define the data introduced in Definition 2.1.1(d) and (e) by the corresponding data of $\tau_{(A^E, \prec^E)}(a)$ and adapt the convention for the leading exponents and monomials of 0 accordingly.

If $\prec^E$ is a well-ordering on A^E, a PBW-reduction datum $(T_n, S_e, I_e, \prec_e)_{e \in E}$ for $(A^E, \prec^E)$ exists by Lemma 2.1.17. Given such a PBW-reduction datum, the maps $\rho_{(A^E, \prec^E)}$ and $\tau_{(A^E, \prec^E)}$ are computable.

Remark 2.1.34. Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra and E and $E_1, \ldots, E_s$ finite sets. Then we have:

(a) Given a total order $<$ on E and a (well-)ordering $\prec'$ on A, $(\prec')^E_{top,<}$ and $(\prec')^E_{pot,<}$ are (well-)orderings on A^E. If $(T_n, S, I', \prec')$ is a PBW-reduction datum for $(A, \prec')$ then corresponding PBW-reduction data for $(A^E, (\prec')^E_{top,<})$ and $(A^E, (\prec')^E_{pot,<})$ are given by $(T_n, S, I', \prec')_{e \in E}$.

(b) We introduce (well-)orderings on $A^{E_1} \oplus \cdots \oplus A^{E_s}$ via its identification with $A^{E_1 \sqcup \cdots \sqcup E_s}$. In particular, if $\prec_i^{E_i}$ is a (well-)ordering on A^{E_i} for $1 \leq i \leq s$, then $\prec_{1,\ldots,s}^{E_1,\ldots,E_s}$ is a (well-) ordering on $A^{E_1 \sqcup \cdots \sqcup E_s} \cong A^{E_1} \oplus \ldots A^{E_s}$. If $(T_n, S_{e_i}, I_{e_i}, (\prec_i^E)_{e_i})_{e_i \in E_i}$ is PBW-reduction datum for $(A^{E_i}, \prec_i^{E_i})$ for $1 \leq i \leq s$, then $(T_n, S_e, I_e, (\prec_{\phi(e)}^E)_e)_{e \in E_1 \sqcup \cdots \sqcup E_s}$

is PBW-reduction datum for $(A^{E_1\sqcup\cdots\sqcup E_s}, \prec^{E_1,\ldots,E_s}_{1,\ldots,s})$, where $\phi(e) = i$ for $e \in E_i \subseteq E_1 \sqcup \cdots \sqcup E_s$.

Definition 2.1.35. Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec^E_e)_{e\in E}$ and $M \subseteq A^E$ an A-submodule.

(a) We call the finite set $G \subseteq M$ a *Gröbner basis* of M (with respect to $\prec^E$) if every $m \in M$ has a so-called *standard representation*, i.e., there exists $a \in A^G$ such that

$$m = \sum_{g\in G} a_g g \text{ and } \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(a_g) + \mathrm{ele}_{\prec^E}(g) \preceq^E \mathrm{ele}_{\prec_E}(m) \text{ for all } g \in G.$$

(b) If G is a Gröbner basis of M, we say that G is reduced if $0 \notin G$, $\mathrm{lc}_{\prec^E}(g) = 1$ for all $g \in G$, and if we have for all $g \in G$, $e \in E$ and $\alpha \in \mathbb{N}^n$

$$(\tau_{(A^E,\prec^E)}(g))_{e,\alpha} \neq 0 \text{ implies } (\alpha, e) \neq \mathrm{ele}(g') + \gamma \text{ for all } g \neq g' \in G, \gamma \in \mathbb{N}^n\,.$$

We point out that we did not define a standard representation on Definition 2.1.35(a) by requiring only the weakened condition $\mathrm{ele}_{\prec^E}(a_g g) \preceq^E \mathrm{ele}_{\prec^E}(g)$, because such a definition would not allow us to use Gröbner bases to determine syzygy modules.

Remark 2.1.36. Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ an ordering on A^E and $M \subseteq A^E$ an A-submodule. To circumvent the problem that we do in general not have a well-defined notion of leading exponents of elements of A^E with respect to $\prec^E$, we define Gröbner bases in this situation as follows: We say that a finite set $G \subseteq M$ is a *Gröbner basis* of M with respect to $\prec^E$ if there exists $h \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^G)\rangle$ with $\overline{h_g} = g$ for $g \in G$ such that for every $t \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle$ with $\bar{t} \in M$ exists $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^G)\rangle$ such that

$$\bar{t} = \sum_{g\in G} \overline{a_g} g \text{ and } \mathrm{le}_{\prec^E_{\mathrm{lcomp}(h_g)}}(a_g) + \mathrm{ele}_{\prec^E}(h_g) \preceq^E \mathrm{ele}_{\prec_E}(t) \text{ for all } g \in G.$$

We say in that case that $\{h_g \mid g \in G\}$ induces a Gröbner basis of M (with respect to $\prec^E$).

Note that since there exists by definition of PBW-reduction-algebras a well-ordering $\prec'$ on A, every $m \in M$ has a representative $t \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle$. Moreover, this definition is compatible with Definition 2.1.35(a).

Our aim is now to adapt the Buchberger algorithm for well-orderings from the commutative setting to our situation. In order to formulate a suitably modified Buchberger criterion, we first introduce normal forms and s-polynomials:

Definition 2.1.37. Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec^E_e)_{e\in E}$ and let $a, a' \in A^E$ be nonzero.

(a) Given a finite set $G \subseteq A^E$, we call $r \in A^E$ satisfying

(i) there exists some $h \in A^G$ with

$$a = \sum_{g \in G} h_g g + r$$

such that $\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(h_g) + \mathrm{ele}_{\prec^E}(g) \preceq^E \mathrm{ele}_{\prec^E}(a)$ for all $g \in G$ and

(ii) $\mathrm{ele}_{\prec^E}(r) \notin L_{\prec^E}(G)$ if $r \neq 0$

a *(left) normal form* of a with respect to G. We say that r is *reduced* if $(\alpha, e) \notin L_{\prec^E}(G)$ given that $(\tau_{(A^E, \prec^E)}(r))_{e,\alpha} \neq 0$. We define the normal form of $0 \in A^E$ with respect to G to be 0.

(b) The *s-polynomial* of a and a' with $e := \mathrm{lcomp}(a) = \mathrm{lcomp}(a')$ is defined by

$$\mathrm{spoly}(a, a') := \begin{cases} \frac{1}{\mathrm{lc}(\underline{x}^{c_{a,a'}} a)} \underline{x}^{c_{a,a'}} a - \frac{1}{\mathrm{lc}(\underline{x}^{c_{a',a}} a')} \underline{x}^{c_{a',a}} a', & \text{if } \underline{x}^{b_{a,a'}}(e) \in (T_n^E)^{\mathrm{irr}}_{(A^E, \prec^E)} \\ 0, & \text{else,} \end{cases}$$

where $b_{a,a'}, c_{a,a'} \in \mathbb{N}^n$ are given by $(b_{a,a'})_i := \max\{\mathrm{le}(a)_i, \mathrm{le}(a')_i\}$ and $(c_{a,a'})_i := (b_{a,a'})_i - \mathrm{le}(a)_i$ for $1 \leq i \leq n$. If $\mathrm{lcomp}(a) \neq \mathrm{lcomp}(a')$, we set $\mathrm{spoly}(a, a') := 0$.

(c) The *s-polynomial* of a and $p \in I_e$ is defined by

$$\mathrm{spoly}(a, p) := \begin{cases} \underline{x}^{c_{a,p}} a, & \text{if } e = \mathrm{lcomp}(a) \\ 0, & \text{else,} \end{cases}$$

where $b_{a,p}, c_{a,p} \in \mathbb{N}^n$ are given by $(b_{a,p})_i := \max\{\mathrm{le}(a)_i, \mathrm{le}(p)_i\}$ and $(c_{a,p})_i := (b_{a,p})_i - \mathrm{le}(a)_i$ for $1 \leq i \leq n$.

Note that we consider for the definition of the s-polynomial in Definition 2.1.37(c) p as an element of ${}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E) \rangle$ (and not as its class in A^E).

Remark 2.1.38. We keep the notation of Definition 2.1.37. Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec_e^E)_{e \in E}$. Consider $a, a' \in A^E$ satisfying $e := \mathrm{lcomp}(a) = \mathrm{lcomp}(a')$ and $\underline{x}^{b_{a,a'}}(e) \in (T_n^E)^{\mathrm{irr}}_{(A^E, \prec^E)}$. Then

$$\mathrm{ele}(\mathrm{spoly}(a, a')) \prec^E (b_{a,a'}, e) = \mathrm{ele}(\underline{x}^{c_{a,a'}} a) = \mathrm{ele}(\underline{x}^{c_{a',a}} a').$$

Similarly, we have for $p \in I_e$

$$\mathrm{ele}(\mathrm{spoly}(a, p)) \prec^E (b_{a,p}, e) = c_{a,p} + \mathrm{ele}(a).$$

The following algorithm clearly computes a normal form and terminates, hence showing the existence of normal forms:

Algorithm 2.1.39 Given a PBW-reduction-algebra A, a finite set $G \subseteq A^E$, a well-ordering $\prec^E$ and $a \in A^E$, this algorithm computes a normal form of a with respect to G and $\prec^E$.

Input: A PBW-reduction-algebra A, a finite set E, a well-ordering $\prec^E$ on the free module $A^E = (T_n, S_e, I_e, \prec_e^E)_{e \in E}$, $G \subseteq A^E$ finite and $a \in A^E$.
Output: A normal form $b \in A^E$ of a with respect to G.
1: **while** $a \neq 0$ and $\tilde{G} := \{g \in G \mid \mathrm{ele}_{\prec^E}(a) \in L_{\prec^E}(\{g\})\} \neq \emptyset$ **do**
2: Choose $g \in \tilde{G}$.
3: Set $a := \mathrm{lc}_{\prec^E}(a) \cdot \mathrm{spoly}(a, g)$.
4: **return** a.

Remark 2.1.40. Note that the above algorithm can be modified to return a reduced normal form using the same method as in the commutative setting (see e.g. [GP08, Algorithm 1.6.11]).

Remark 2.1.41. Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec_e^E)_{e \in E}$ and $M \subseteq A^E$ an A-submodule. If G is a Gröbner basis of M, then clearly $m \in A^E$ is an element of M if and only if some normal form of m with respect to G is 0. Moreover, assuming $m \in M$ and using induction on $\mathrm{lm}_{\prec^E}(m)$, one easily proves that every normal form of m with respect to G is 0.

Our algorithm for computing Gröbner bases is based on a noncommutative variant of the Buchberger criterion for polynomial rings that takes into account the additional relations:

Proposition 2.1.42. *[Buchberger criterion for PBW-reduction-algebras] Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec_e^E)_{e \in E}$ and $G \subseteq A^E$ a finite set. Then G is a (left) Gröbner basis (with respect to $\prec^E$) of the A-module ${}_A\langle G\rangle$ if and only if*

(a) any (or some) normal form of $\mathrm{spoly}(g, g')$ with respect to G is 0 for all $g, g' \in G$ and

(b) for all $g \in G$ and $p \in I_{\mathrm{lcomp}(g)}$ any (or some) normal form of $\mathrm{spoly}(a, g)$ with respect to G is 0.

For the proof we adapt a standard proof of the commutative Buchberger criterion to our setting. It relies on the following lemma, whose proof from the commutative setting carries over word by word:

Lemma 2.1.43. *Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec_e^E)_{e \in E}$. Let $G \subseteq A^E \setminus \{0\}$ be finite with the property that all its elements possess the same leading monomial. Assume that we have for $m = \sum_{g \in G} a_g g$ with $a \in \mathbb{K}^G$ that $\mathrm{lm}(m) \prec^E \mathrm{lm}(g)$ for $g \in G$. Then there exists $d \in \mathbb{K}^{G \times G}$ such that $m = \sum_{(g,g') \in G \times G} d_{(g,g')} \,\mathrm{spoly}(g, g')$.*

The following remark lists some comparisons of (leading) monomials with respect to $\prec^E$ that are frequently used throughout our proof of Proposition 2.1.42:

Remark 2.1.44. Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec_e^E)_{e\in E}$ and $\prec_o^E$ any ordering on A^E such that S_e is a reduction system with respect to $(\prec_o^E)_e$ for all $e \in E$. Define for $l \in \mathbb{N}$, $1 \leq i_1, \dots, i_l \leq n$ the vector $\alpha := \sum_{1\leq j\leq l} e_{i_j} \in \mathbb{N}^n$ and let $e \in E$.

(a) We have $\underline{x}^\alpha(e) \preceq_o^E x_{i_1} \cdots x_{i_l}(e)$.

(b) Independently of the choice of $\prec_o^E$, we can find $r_{i_1,\dots,i_l} \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ and $f_{i_1,\dots,i_l} \in \mathbb{K}^*$ with $\mathrm{ele}_{\prec_o^E}(r_{i_1,\dots,i_l}(e)) \prec_o^E (\alpha, e)$ such that

$$x_{i_1} \cdots x_{i_l}(e) - f_{i_1,\dots,i_l}\underline{x}^\alpha(e) - r_{i_1,\dots,i_l}(e) \in {}_{T_n}\langle S\rangle_{T_n}$$

and hence

$$\overline{x_{i_1} \cdots x_{i_l}(e)} = \overline{f_{i_1,\dots,i_l}\underline{x}^\alpha(e) + r_{i_1,\dots,i_l}(e)}$$

holds in A^E, because non-trivial reductions with the commutation relations contained in S applied to a monomial decrease its leading monomial for any ordering with respect to which S is a reduction system. In particular, for a permutation $\sigma \in S_l$ exists $t \in {}_{\mathbb{K}}\big\langle \{\underline{x}^\beta(e) \in \mathrm{SMon}(T_n) \mid (\beta, e) \prec_o^E (\alpha, e)\}\big\rangle$ with

$$\bar{t} = \overline{\frac{1}{f_{i_1,\dots,i_l}} x_{i_1} \cdots x_{i_l}(e) - \frac{1}{f_{i_{\sigma(1)},\dots,i_{\sigma(l)}}} x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(l)}}(e)}.$$

For $\prec_o^E = \prec^E$ we have moreover: If $\underline{x}^\alpha(e) \in (T_n^E)^{\mathrm{irr}}_{(A^E,\prec^E)}$ then $f_{i_1,\dots,i_l}$ and $r_{i_1,\dots,i_l}$ can be additionally chosen such that

$$\rho_{(A^E,\prec^E)}(x_{i_1} \cdots x_{i_l}(e)) = f_{i_1,\dots,i_l}\underline{x}^\alpha(e) + r_{i_1,\dots,i_l}(e).$$

Otherwise $\mathrm{ele}_{\prec^E}(\rho_{(A^E,\prec^E)}(x_{i_1} \cdots x_{i_l}(e))) \prec^E (\alpha, e)$.

(c) Let $a \in A$ and $g \in A^E$. Then $\mathrm{ele}_{\prec^E}(ag) \preceq^E \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(a) + \mathrm{ele}_{\prec^E}(g)$ with equality if and only if the monomial with extended leading exponent $\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(a) + \mathrm{ele}_{\prec^E}(g)$ is irreducible.

Proof of Proposition 2.1.42. By Remark 2.1.41 it the clear that if G is a Gröbner basis, then every normal form stated in the criterion is 0. Conversely, consider $0 \neq m \in {}_A\langle G\rangle$ and choose $h \in A^G$ such that

$$m = \sum_{g\in G} h_g g \tag{2.1.2}$$

satisfying additionally that

$$(\alpha, e) := \max\nolimits_{\prec^E}\{\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(h_g) + \mathrm{ele}_{\prec^E}(g) \mid g \in G\}$$

is minimal with respect to $\prec^E$. If $(\alpha, e) \preceq^E \operatorname{ele}_{\prec^E}(m)$ then Equation (2.1.2) is a standard representation and we are finished. Otherwise, setting

$$G' := \{g \in G \mid \operatorname{le}_{\prec^E_{\operatorname{lcomp}(g)}}(h_g) + \operatorname{ele}_{\prec^E}(g) = (\alpha, e)\}$$

and writing

$$m = \sum_{g' \in G'} \operatorname{lt}_{\prec^E_e}(h_{g'})g' + \sum_{g' \in G'} \operatorname{tail}_{\prec^E_e}(h_{g'})g' + \sum_{g \in G \setminus G'} h_g g, \tag{2.1.3}$$

we have for $g' \in G'$ by Remark 2.1.44(c)

$$\begin{aligned} \operatorname{ele}_{\prec^E}(\operatorname{tail}_{\prec^E_e}(h_{g'})g') &\preceq^E \operatorname{le}_{\prec^E_e}(\operatorname{tail}_{\prec^E_e}(h_{g'})) + \operatorname{ele}_{\prec^E}(g') \\ &\prec^E \operatorname{le}_{\prec^E_e}(h_{g'}) + \operatorname{ele}_{\prec^E}(g') = (\alpha, e), \end{aligned} \tag{2.1.4}$$

and by Remark 2.1.44(c) and by choice of G' it holds for $g \in G \setminus G'$

$$\operatorname{ele}_{\prec^E}(h_g g) \preceq^E \operatorname{le}_{\prec^E_{\operatorname{lcomp}(g)}}(h_g) + \operatorname{ele}_{\prec^E}(g) \prec^E (\alpha, e). \tag{2.1.5}$$

Hence the leading monomial of $l := \sum_{g \in G'} \operatorname{lt}_{\prec^E_e}(h_g)g$ is strictly smaller than $\underline{x}^\alpha(e)$. Now we need to distinguish two cases: If $\underline{x}^\alpha(e) \in (T^E_n)^{\operatorname{irr}}_{(A^E, \prec^E)}$ then all summands in the sum expression of l have leading monomial $\underline{x}^\alpha(e)$ according to Remark 2.1.44(c). So we may invoke Lemma 2.1.43 to find an element $d \in \mathbb{K}^{G' \times G'}$ such that

$$l = \sum_{(g,g') \in G' \times G'} d_{(g,g')} \underbrace{\operatorname{spoly}(\operatorname{lm}_{\prec^E_e}(h_g)g, \operatorname{lm}_{\prec^E_e}(h_{g'})g')}_{s_{(g,g')}:=}. \tag{2.1.6}$$

Expanding the s-polynomial, we have for $g, g' \in G'$

$$s_{(g,g')} = \frac{1}{\operatorname{lc}_{\prec^E}(\operatorname{lm}_{\prec^E_e}(h_g)g)} \operatorname{lm}_{\prec^E_e}(h_g)g - \frac{1}{\operatorname{lc}_{\prec^E}(\operatorname{lm}_{\prec^E_e}(h_{g'})g')} \operatorname{lm}_{\prec^E_e}(h_{g'})g'.$$

By definition of $c_{g,g'}$ and $c_{g',g}$ (see Definition 2.1.37(b)) there exists $\beta_{(g,g')} \in \mathbb{N}^n$ such that $c_{g,g'} + \beta_{(g,g')} = \operatorname{le}_{\prec^E_e}(h_g)$ and $c_{g',g} + \beta_{(g,g')} = \operatorname{le}_{\prec^E_e}(h_{g'})$. Applying Remark 2.1.44(b), we obtain

$$\begin{aligned} s_{(g,g')} =& (d_g \underline{x}^{\beta_{(g,g')}} \underline{x}^{c_{g,g'}} + r^{(g,g')})g - (d_{g'} \underline{x}^{\beta_{(g,g')}} \underline{x}^{c_{g',g}} + r'^{(g,g')})g' \\ =& \underline{x}^{\beta_{(g,g')}} \left(d_g \underline{x}^{c_{g,g'}} g - d_{g'} \underline{x}^{c_{g',g}} g'\right) + r^{(g,g')}g + s^{(g,g')}g' \end{aligned}$$

for suitably chosen $d_g, d_{g'} \in \mathbb{K}^*$ and $r^{(g,g')}, r'^{(g,g')} \in A$ with

$$\operatorname{lm}_{\prec^E_e}(r^{(g,g')}) \prec^E_e \operatorname{lm}_{\prec^E_e}(h_g) \text{ and } \operatorname{lm}_{\prec^E_e}(r'^{(g,g')}) \prec^E_e \operatorname{lm}_{\prec^E_e}(h_{g'}). \tag{2.1.7}$$

As $\underline{x}^\alpha(e)$ is irreducible and $(\alpha, e) = c_{g,g'} + \beta_{(g,g')} + \mathrm{lm}_{\prec^E}(g)$, the monomial with extended leading coefficient $c_{g,g'} + \mathrm{lm}_{\prec^E}(g) = c_{g',g} + \mathrm{lm}_{\prec^E}(g')$ is also irreducible, where the latter equality follows from Remark 2.1.38. That remark implies also that $\mathrm{lm}_{\prec^E}(s_{(g,g')}) \prec \underline{x}^\alpha(e)$ and so we deduce that $\mathrm{lt}_{\prec^E}(d_g \underline{x}^{c_{g,g'}} g) = \mathrm{lt}_{\prec^E}(d_{g'} \underline{x}^{c_{g',g}} g')$. Now by the definition of $\mathrm{spoly}(g, g')$ there exists $f_{(g,g')} \in \mathbb{K}^*$ such that

$$s_{(g,g')} = f_{(g,g')} \underline{x}^{\beta_{(g,g')}} \,\mathrm{spoly}(g, g') + r^{(g,g')} g + r'^{(g,g')} g' \tag{2.1.8}$$

and

$$\beta_{(g,g')} + \mathrm{ele}_{\prec^E}(\mathrm{spoly}(g, g')) \prec^E (\alpha, e). \tag{2.1.9}$$

By hypothesis we find an element $k^{(g,g')} \in A^G$ satisfying

$$\mathrm{spoly}(g, g') = \sum_{g'' \in G} k^{(g,g')}_{g''} g'' \tag{2.1.10}$$

and $\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g'')}}(k^{(g,g')}_{g''}) + \mathrm{ele}_{\prec^E}(g'') \preceq^E \mathrm{ele}_{\prec^E}(\mathrm{spoly}(g, g'))$. This yields together with Remark 2.1.44(c) and Equation (2.1.9) the estimate

$$\begin{aligned} \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g'')}}(\underline{x}^{\beta(g,g')} k^{(g,g')}_{g''}) + \mathrm{ele}_{\prec^E}(g'') &\preceq^E \beta_{(g,g')} + \mathrm{le}_{\prec^E_{\mathrm{lcomp}(g'')}}(k^{(g,g')}_{g''}) + \mathrm{ele}_{\prec^E}(g'') \\ &\preceq^E \beta_{(g,g')} + \mathrm{ele}_{\prec^E}(\mathrm{spoly}(g, g')) \\ &\prec^E (\alpha, e). \end{aligned} \tag{2.1.11}$$

Combining Equations (2.1.6), (2.1.8) and (2.1.10) we obtain

$$l = \sum_{(g,g') \in G' \times G'} d_{(g,g')} \left(f_{(g,g')} \sum_{g'' \in G} \underline{x}^{\beta(g,g')} k^{(g,g')}_{g''} g'' + r^{(g,g')} g + r'^{(g,g')} g' \right)$$

and plugging this equation into Equation (2.1.3) contradicts by Equations (2.1.4), (2.1.5), (2.1.7) and (2.1.11) the minimality of (α, e).

In the other case, $\underline{x}^\alpha(e)$ is reducible, say $\alpha = \beta + \mathrm{lm}_{\prec^E_e}(p)$ for some $p \in I_e$ and $\beta \in \mathbb{N}^n$. Then there exists by definition of $\mathrm{spoly}(g, p)$ for $g \in G'$ a vector $\gamma_g \in \mathbb{N}^n$ such that

$$\mathrm{le}_{\prec^E_e}(h_g) + \mathrm{ele}_{\prec^E}(g) = (\alpha, e) = \gamma_g + c_{g,p} + \mathrm{ele}_{\prec_E}(g)$$

(see Definition 2.1.37(c) for the definition $c_{g,p}$). Therefore there is $q_g \in \mathbb{K}^*$

$$\mathrm{lm}_{\prec^E_e}(h_g) g = (q_c \underline{x}^{\gamma_g} \cdot \underline{x}^{c_{g,p}} + t_g) g = q_c \underline{x}^{\gamma_g} \cdot \mathrm{spoly}(g, p) + t_g g$$

with $t_g \in A$ such that $\mathrm{le}_{\prec^E_e}(t_g) \prec^E_e \mathrm{le}_{\prec^E_e}(h_g)$ by Remark 2.1.44(b). Using that

$$\gamma_g + \mathrm{ele}_{\prec^E}(\mathrm{spoly}(p, g)) \prec^E \gamma_g + c_{g,p} + \mathrm{ele}_{\prec_E}(g) = (\alpha, e)$$

by Remark 2.1.38 and that $\mathrm{spoly}(g, p)$ has a normal form that is 0 with respect to G, we may argue as in the first case. This finishes our proof. □

The above lemma yields the following algorithm for computing Gröbner bases:

Algorithm 2.1.45 Given a PBW-reduction-algebra A, a well-ordering $\prec^E$ and a finite set $G \subseteq A^E$, this algorithm computes a Gröbner basis of the module ${}_A\langle G\rangle$ with respect to $\prec^E$.

Input: A PBW-reduction-algebra A, a finite set E, a well-ordering $\prec^E$ on the module $A^E = (T_n, S_e, I_e, \prec_e^E)_{e\in E}$ and $G \subseteq A^E$ finite.
Output: A finite set $H \subseteq A^E$ such that H is a Gröbner basis of ${}_A\langle G\rangle$ with respect to $\prec^E$.

Initialize $H := G \setminus \{0\} := \{g_1, \ldots, g_s\}$.
Set $T := \{(g_i, g_j) \mid 1 \leq i < j \leq s\} \cup \{(g, s(\mathrm{lcomp}(g))) \mid g \in H, s \in I_{\mathrm{lcomp}(g)}\}$.
while $T \neq \emptyset$ **do**
  Choose $(t_1, t_2) \in T$.
  Delete $\{(t_1, t_2)\}$ from T.
  Compute a normal form r of $\mathrm{spoly}(t_1, t_2)$ with respect to H and $\prec^E$ by applying Algorithm 2.1.39.
  if $r \neq 0$ **then**
    Set $T := T \cup \{(r, h) \mid h \in H\} \cup \{(r, s(\mathrm{lcomp}(r))) \mid s \in I_{\mathrm{lcomp}(r)}\}$ and $H := H \cup \{r\}$.
return H.

Lemma 2.1.46. *The above algorithm is correct and terminates.*

Proof. The correctness follows immediately from Proposition 2.1.42. We keep the notation of Algorithm 2.1.45 and denote by H_i the set H at the beginning of the i-th iteration of the while-loop and by r_i the normal form r computed during the i-th run of that loop. For the termination consider now the sets $L(H_i)$ and note that if the normal form r_i is nonzero (and hence added) then $\mathrm{ele}_{\prec^E}(r_i) \notin L(H_i)$. Hence the sets $L(H_i)$ form an increasing sequence in $\mathbb{N}^n \times E$ with a proper inclusion $L(H_i) \subsetneq L(H_{i+1})$ if and only if the inclusion $H_i \subseteq H_{i+1}$ is proper. Notice that there is an inclusion preserving one-to-one correspondence between subsets of $\mathbb{N}^n \times E$ of type $\bigcup_{\gamma\in C}(\gamma + \mathbb{N}^n)$ (with $C \subseteq \mathbb{N}^n \times E$) and monomial $\mathbb{K}[\underline{x}]$-submodules of $\mathbb{K}[\underline{x}]^E$ via

$$\bigcup_{\gamma\in C}(\gamma + \mathbb{N}^n) \mapsto {}_{\mathbb{K}[\underline{x}]}\langle\{\underline{x}^\gamma(e) \mid (\gamma, e) \in C\}\rangle.$$

As the image in $\mathbb{K}[\underline{x}]^E$ of the sequence of the $L(H_i)$ under that one-to-one correspondence gets stationary because $\mathbb{K}[\underline{x}]$ is a Noetherian ring, so does the sequence of the $L(H_i)$ and hence also the sequence of the H_i showing termination. □

Remark 2.1.47. The above algorithm can be modified to compute a reduced Gröbner basis applying the same methods as in the commutative setting (see e.g. [GP08, Remark 1.7.2]).

An algorithm for computing left generators of a two-sided submodule of a free A-module carries over immediately from the setting of PBW-algebras (see e.g. [BGTV03, Algorithm 6] or [Lev05, Algorithm 2.3.1]):

Algorithm 2.1.48 Given a PBW-reduction-algebra A, a well-ordering $\prec^E$ and a finite set $G \subseteq A^E$, this algorithm computes a (left) Gröbner basis of the two-sided module ${}_A\langle G\rangle_A$ with respect to $\prec^E$.

Input: A PBW-reduction-algebra A, a finite set E, a well-ordering $\prec^E$ on the module $A^E = (T_n, S_e, I_e, \prec_e^E)_{e\in E}$ and $G \subseteq A^E$ finite.
Output: A finite set $H \subseteq A^E$ such that H is a Gröbner basis of ${}_A\langle G\rangle_A$ with respect to $\prec^E$.

Initialize an empty set G'.
while $G \neq G'$ **do**
  Set $G' := G$.
  Replace G by a Gröbner basis of the left ideal ${}_A\langle G\rangle$ using Algorithm 2.1.45.
  Set $R := \{gx_i \mid g \in G, 1 \leq i \leq n\}$.
  for $r \in R$ **do**
    Compute a left normal form r' of r with respect to G using Algorithm 2.1.39.
    if $r' \neq 0$ **then** $\triangleright$ r is not in ${}_A\langle G\rangle$ by Remark 2.1.41.
      Set $G := G \cup \{r'\}$.
return G.

Lemma 2.1.49. *The above algorithm is correct and terminates.*

Proof. The correctness is clear. The algorithm terminates as A is by Lemma 2.1.27 a left Noetherian ring and hence every ascending chain of A-submodules of A^E gets stationary. □

Lemma 2.1.46, Remark 2.1.47 and Lemma 2.1.49 imply:

Proposition 2.1.50. *Let A be a PBW-reduction-algebra, E a finite set, $\prec^E$ a well-ordering on $A^E = (T_n, S_e, I_e, \prec_e)_{e\in E}$ and $G \subseteq A^E$ a finite subset. Then (reduced) Gröbner bases of the left A-modules ${}_A\langle G\rangle$ and ${}_A\langle G\rangle_A$ with respect to $\prec^E$ exist.*

These Gröbner bases are computable if a PBW-reduction datum $(T_n, S_e, I_e, \prec_e^E)_{e\in E}$ for $(A, \prec^E)$ is computable and if there exists an $(A^E, \prec^E)$-computable subfield $\mathbb{K}' \subseteq \mathbb{K}$ such that $G \subseteq A^E_{\mathbb{K}'}$.

Definition 2.1.51. Let A be a PBW-reduction-algebra, E a finite set and $\prec^E$ a well-ordering on A^E. We call $\prec^E$ *computable* if a PBW-reduction datum for $(A^E, \prec^E)$ is computable.

Convention 2.1.52. From now on, when we talk about computability or formulate algorithms in the context of a PBW-reduction-algebra A, we always assume that there exists an A-computable subfield $\mathbb{K}' \subseteq \mathbb{K}$ such that it is $(A^E, \prec^E)$-computable for all appearing free A-modules A^E of finite rank and well-orderings $\prec^E$ and that all considered submodules of A^E are generated by subsets which are defined over $A^E_{\mathbb{K}'}$. Similarly, we assume that all other input data is also defined over $A_{\mathbb{K}'}$ or $A^E_{\mathbb{K}'}$.

Our variant of the Buchberger algorithm (Algorithm 2.1.45) requires that the free A-module A^E is given in terms of a PBW-reduction datum. However, we have in general no method to compute a PBW-reduction datum of A^E with respect to a given well-ordering. Yet, in certain situations such a datum is computable: The next corollary explains how to derive from a PBW-reduction datum for a given PBW-reduction-algebra a PBW-reduction datum for a factor algebra of that PBW-reduction-algebra using Gröbner bases:

Corollary 2.1.53. *Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra and $M \subseteq A$. Then*

$$A/{}_A\langle M\rangle_A$$

is canonically isomorphic to the PBW-reduction-algebra

$$B := T_n/\left\langle S \cup I \cup \tau_{(A,\prec)}(G)\right\rangle,$$

where G stands for a left Gröbner basis of ${}_A\langle M\rangle_A$ with respect to $\prec$. Moreover, a PBW-reduction datum of B is given by $(T_n, S, I \cup \tau_{(A,\prec)}(G), \prec)$. In particular, PBW-reduction data for factor algebras of PBW-algebras are computable.

Proof. Clearly the map

$$\psi : T_n \to A,\ t \mapsto \overline{t}$$

induces the claimed isomorphism. For the second claim it is by Remark 2.1.15(a) enough to show that

$$L(I \cup \tau_{(A,\prec)}(G)) \supseteq \{\mathrm{le}(t) \mid 0 \neq t \in {}_{T_n}\left\langle S \cup I \cup \tau_{(A,\prec)}(G)\right\rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle\}.$$

So consider $0 \neq t \in {}_{T_n}\left\langle S \cup I \cup \tau_{(A,\prec)}(G)\right\rangle_{T_n} \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$. If $\mathrm{le}(t) \in L(I)$, we are finished. Otherwise we have according to Definition 2.1.13(aii) that $\mathrm{lm}(t)$ is irreducible with respect to the $\prec$-monic elements of ${}_{T_n}\langle S \cup I\rangle_{T_n}$ and hence $\mathrm{lm}(t) = \mathrm{lm}(\rho_{(A,\prec)}(t)) = \mathrm{lm}(\overline{t}_A)$, where $\overline{t}_A$ and $\overline{t}_B$ denote the residue classes of t in A and B, respectively. We have by choice of t that $\overline{t}_B = 0 \in B$ and hence $\overline{t}_A \in {}_A\langle M\rangle_A$. As G is a Gröbner basis of that ideal there exists $a \in A^G$ satisfying

$$\overline{t}_A = \sum_{g \in G} a_g g \text{ and } \mathrm{le}(a_g) + \mathrm{le}(g) \preceq \mathrm{le}(\overline{t}_A) = \mathrm{le}(t) \text{ for all } g \in F$$

with equality for some $g' \in G$. As $\mathrm{le}(g') = \mathrm{le}(\tau_{(A,\prec)}(g'))$ this concludes the proof. ☐

Corollary 2.1.54. *Let A be a PBW-algebra or a factor algebra thereof. Then PBW-reduction data with respect to well-orderings are computable.*

The following remark outlines how to perform certain Gröbner basics in our setting using the corresponding ideas of the commutative setting:

Remark 2.1.55. Given a PBW-reduction-algebra $A = (T_n, S, I, \prec)$, a finite set E and two A-submodules $M = {}_A\langle M'\rangle, N = {}_A\langle N'\rangle \subseteq A^E$ with M' and N' finite, the following problems are algorithmically solvable:

(a) We can decide whether $N \subseteq M$. For this we fix a well-ordering $\prec^E$ on A^E (e.g. an ordering of type $\prec^E_{pot,<}$ on $A^E = (T_n, S, I, \prec)_{e\in E}$). Then we determine a Gröbner basis G of M by Algorithm 2.1.45 and after that we compute normal forms of n' with respect to G for all $n' \in N'$. By Remark 2.1.41 the module N is an A-submodule of M if and only if all of these normal forms are zero.

(b) Generators of the intersection $M \cap A^{E'}$ for some subset $E' \subseteq E$ are determined by computing a Gröbner basis G of M with respect to an ordering of type $\prec^E_{pot,<}$, where $<$ is a total order satisfying $e' < e$ for all $e' \in E'$ and $e \in E \setminus E'$. Indeed, the intersection is then generated by $\{g \in G \mid \mathrm{lcomp}(g) \in E'\}$.

Another application of Gröbner bases is the computation of so-called syzygies:

Definition 2.1.56. Let A be a ring, E a finite set and $H_1, \dots, H_s \subseteq A^E$ finite subsets. The A-module

$$\mathrm{syz}_A(H_1, \dots, H_s) := \{(a_1, \dots, a_s) \in A^{H_1} \oplus \dots \oplus A^{H_s} \mid \sum_{1\le i\le s} \sum_{h_i \in H_i} (a_i)_{h_i} h_i = 0\}$$

is called the *syzygy-module* of $H_1, \dots, H_s$ (in $A^{H_1} \oplus \dots \oplus A^{H_s}$). Similarly, for $h_1, \dots, h_t \in A^E$ the *syzygy-module* $\mathrm{syz}_A(h_1, \dots, h_t) \subseteq A^t$ is defined by

$$\mathrm{syz}_A(h_1, \dots, h_t) := \mathrm{syz}_A(\{h_1\}, \dots, \{h_t\})$$

under the identification $A^{\{h_1\}} \oplus \dots \oplus A^{\{h_t\}} \cong A^t,\ (a_1(h_1), \dots, a_t(h_t)) \mapsto \sum_{1\le i\le t} a_i(e_i)$.

The following lemma shows that syzygies over PBW-reduction-algebras are computable in the same manner as in the commutative setting (given that we can determine a corresponding PBW-reduction datum).

Lemma 2.1.57. *Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra, E a finite set and $H \subseteq A^E$ finite. Let G be a Gröbner basis of ${}_A\langle\{h + (h) \mid h \in H\}\rangle \subseteq A^{E\sqcup H}$ with respect to the ordering $\prec^{E\sqcup H}_{pot,<}$, where $<$ is a total ordering on $E \sqcup H$ with $h < e$ for $e \in E$ and $h \in H$. Then*

$$\mathrm{syz}_A(H) = {}_A\big\langle \pi_H(G \cap A^H)\big\rangle.$$

Proof. Let $g \in G \cap A^H$. Then $g = \sum_{h\in H} g_h(h + (h)) = \sum_{h\in H} g_h h + \sum_{h\in H} g_h(h) \in A^H$ shows that $\sum_{h\in H} g_h h = 0$ and hence $\pi_H(g) \in \mathrm{syz}_A(H)$.

Conversely, consider $s \in \mathrm{syz}_A(H)$. Then $\sum_{h\in H} s_h h = 0$ implies $s' := \sum_{h\in H}(s_h h + s_h(h)) \in {}_A\langle\{h + (h) \mid h \in H\}\rangle \cap A^H$. As G is a Gröbner basis, there exists $a \in A^G$ satisfying

$$s' = \sum_{g\in G} a_g g \text{ and } \mathrm{le}_{\prec'_{\mathrm{lcomp}(g)}}(a_g) + \mathrm{ele}_{\prec'}(g) \preceq' \mathrm{ele}_{\prec'}(s'),$$

where $\prec'$ stands for $\prec_{pot,<}^{E\sqcup H}$. As $\mathrm{lcomp}(s') \in H$ and by the choice of the ordering $\prec_{pot,<}^{E\sqcup H}$, we must have $a_g = 0$ for all $g \notin A^H$ and hence

$$s = \pi_H(s') = \pi_H(\sum_{g\in G\cap A^H} a_g g) = \sum_{g\in G\cap A^H} a_g \pi_H(g).$$

□

In the situation of Definition 2.1.56, if $A = (T_n, S, I, \prec)$ is a PBW-reduction-algebra and there exists an A^E-computable subfield $\mathbb{K}' \subseteq K$ such that $H_1, \ldots, H_s \subseteq A^E_{\mathbb{K}'}$, then A-generators of $\mathrm{syz}_A(H_1, \ldots, H_s)$ are effectively computable over $A_{\mathbb{K}'}$ via Gröbner bases .

Remark 2.1.58. Given a PBW-reduction-algebra $A = (T_n, S, I, \prec)$, a finite set E and two A-submodules $M = {}_A\langle M'\rangle, N = {}_A\langle N'\rangle \subseteq A^E$ with M' and N' finite, we can determine generators of the intersection $M \cap N$ as in the commutative case (see e.g. [GP08, Section 2.8.3]).

Remark 2.1.59. We point out that given a PBW-reduction-algebra A, the main computational problem is determining a corresponding PBW-reduction datum. If the PBW-reduction datum $A = (T_n, S, I, \prec)$ is given, then a PBW-reduction datum for $(A^E, \prec_{top,<})$ and $(A^E, \prec_{pot,<})$ for any finite set E and any total order on E is known by Remark 2.1.34(a). In summary, we have then algorithms for the following Gröbner basics:

(a) We can solve the module membership problems for submodules of A^E by using by Remark 2.1.55(a).

(b) Projections of submodules of A^E to $A^{E'}$ for a subset of $E' \subseteq E$ are computable (see Remark 2.1.55(b)). More generally, we find A-generators of intersections of submodules of A^E by Remark 2.1.58.

(c) We can determine syzygies of finite subsets of A^E by Lemma 2.1.21.

In the next section, we will explain how to compute Gröbner bases with respect to non-well-orderings.

2.2 Weight filtrations

The subject of study in this section are filtrations of type $F^{\mathbf{u}}_{\bullet}A$ induced by a so-called weight vector $\mathbf{u}$ on the PBW-reduction-algebra A. These filtrations have been studied theoretically and algorithmically for nonnegative weight vectors on PBW-algebras in [BGTV03]. Our first object of investigation is the subalgebra $F^{\mathbf{u}}_0 A$. Combining the methods of [BGTV03] and [OT01], we then develop an algorithm for computing Gröbner bases on A with respect to non-well orderings based on the homogenization of A with respect to a positive weight vector. Using a $\mathbf{u}$-weighted degree ordering this algorithm enables us finally to determine generators of the filtration induced by $F^{\mathbf{u}}_{\bullet}A$ on submodules of free A-modules, hence showing that these filtered modules are always well-filtered.

2.2.1 Weight filtrations on PBW-reduction-algebras

We assume in this subsection that $A = (T_n, S, I, \prec)$ with $S := \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \leq i < j \leq n\}$ is a PBW-reduction-algebra if not stated otherwise. We are particularly interested in filtrations on A induced by so called weight vectors:

Definition 2.2.1. Let $\mathbf{u} \in \mathbb{Z}^n$, E a finite set and $\mathbf{s} \in \mathbb{Z}^E$.

(a) The vector $\mathbf{u}$ induces a grading

$$T_n^E = \bigoplus_{l \in \mathbb{Z}} (T_n^E)_l^{\mathbf{u}},$$

on T_n^E by assigning weight $\mathbf{u}_i$ to x_i, i.e.,

$$(T_n^E)_l^{\mathbf{u}} := {}_{\mathbb{K}}\Big\langle \{x_{i_1} \cdots x_{i_k}(e) \mid e \in E, k \in \mathbb{N}, 1 \leq i_1, \ldots, i_k \leq n, \textstyle\sum_{1 \leq j \leq k} \mathbf{u}_{i_j} = l\} \Big\rangle$$

for $l \in \mathbb{Z}$. So every nonzero $r \in T_n^E$ can be uniquely written as $r = \sum_{s_1 \leq i \leq s_2} r_i$ with $r_i \in (T_n^E)_i^{\mathbf{u}}$ and $r_{s_1}, r_{s_2} \neq 0$. We call s_2 the $\mathbf{u}$-*degree* of r and write $\deg_{\mathbf{u}}(r) = s_2$. If $s_1 = s_2$, we say that r is $\mathbf{u}$-*homogeneous*. We define the $\mathbf{u}$-*leading terms* of r by $\mathrm{lt}_{\mathbf{u}}(r) := r_{s_2}$. The elements $r_{s_1}, \ldots, r_{s_2}$ are called the homogeneous parts of r. We set $\deg_{\mathbf{u}}(0) := -\infty$. We denote the associated filtered ring of $T_n = \bigoplus_{l \in \mathbb{Z}} (T_n)_l^{\mathbf{u}}$ by $(T_n, F_\bullet^{\mathbf{u}})$.

(b) Considering A as a quotient module of T_n, the filtration $F_\bullet^{\mathbf{u}} A$ stands for its quotient filtration (see Remark 1.1.12(c)). We define for $a \in A$

$$\deg_{\mathbf{u}}(a) := \deg_{F^{\mathbf{u}}}(a).$$

Similarly, for $a' \in A^E$, we set

$$\deg_{\mathbf{u}[\mathbf{s}]}(a') := \deg_{F^{\mathbf{u}}[\mathbf{s}]}(a')$$

and suppress $\mathbf{s}$ if it is the zero vector.

(c) We say that $\mathbf{u}$ is a *weight vector* on A if $\deg_{\mathbf{u}}(d_{ij}) \leq \deg_{\mathbf{u}}(x_i x_j)$ for all $1 \leq i < j \leq n$. We call the weight vector $\mathbf{u}$ *good* if for every finite set E, every shift vector $\mathbf{s} \in \mathbb{Z}^E$ and every submodule $M \subseteq A^E$ the filtration $F^{\mathbf{u}}[\mathbf{s}]_\bullet M$ is a good filtration.

Convention 2.2.2. Our definition of a weight vector depends on the PBW-reduction datum of A, or more precisely on S. We could avoid this by only requiring in the definition that there exists some PBW-reduction datum such that Definition 2.2.1(c) holds (with respect to that reduction datum). As we do in practice not consider different sets of commutation relations for a fixed PBW-reduction-algebra and some of our arguments rely on a common set of commutation relations, *we from now on assume that the commutation relations of a given PBW-reduction-algebra are fixed (and hence do not depend on the considered ordering).*

Note that $\mathbf{u}$ being a weight vector on A ensures the compatibility of $F^{\mathbf{u}}_{\bullet}A$ with the commutation relations S of A. Hence we have:

Lemma 2.2.3. *Let $\mathbf{u} \in \mathbb{Z}^n$, E a finite set, $\mathbf{s} \in \mathbb{Z}^E$ and $L \subseteq A^E$ an A-submodule. If $\mathbf{u}$ is a weight vector on A then we have for all $a, a' \in A$*

$$\deg_{\mathbf{u}}(a \cdot a') \leq \deg_{\mathbf{u}}(a) + \deg_{\mathbf{u}}(a')$$

and $F^{\mathbf{u}}_{\bullet}A$ is a filtered $\mathbb{K}$-algebra satisfying

$$F^{\mathbf{u}}_{\bullet}A = {}_{\mathbb{K}}\langle\{\overline{\underline{x}^{\alpha}} \mid \langle \mathbf{u}, \alpha\rangle \leq \bullet\}\rangle.$$

In this case $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}A^E$, $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}L$ and $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}(A^E/L)$ are filtered $F^{\mathbf{u}}_{\bullet}A$-modules.

If $\mathbf{u}$ is a weight vector, we call $F^{\mathbf{u}}_{\bullet}A$ the *weight filtration* associated to $\mathbf{u}$ on A or the $\mathbf{u}$-*weight filtration* on A. If A is moreover naturally isomorphic to its associated graded algebra with respect to $F^{\mathbf{u}}_{\bullet}A$ then we say that A is $\mathbf{u}$-*graded* and we call the homogeneous elements of A with respect to that grading also $\mathbf{u}$-homogeneous. More generally, if A is graded, E a finite set and the shift vector $\mathbf{s} \in \mathbb{Z}^E$ assigns degree $\mathbf{s}_e$ to (e), then we call a homogeneous element of A^E also $\mathbf{u}[\mathbf{s}]$-homogeneous (and similarly for elements of T_n^E). Note that A is $\mathbf{u}$-graded if and only if $\langle S \cup I\rangle$ is $\mathbf{u}$-homogeneous, that is, generated by $\mathbf{u}$-homogeneous elements.

Lemma 2.2.3 implies that $F^{\mathbf{u}}_0 A$ is a $\mathbb{K}$-subalgebra of A if $\mathbf{u}$ is a weight vector on A. We collect some properties of $F^{\mathbf{u}}_0 A$ in this case:

Lemma 2.2.4. *Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A.*

(a) The $\mathbb{K}$-subalgebra $F^{\mathbf{u}}_0 A$ of A is finitely generated and has a finite monomial generating set, that is, a finite generating set consisting of residue classes of standard monomials of T_n. Moreover, such a monomial generating set is computable.

(b) The $\mathbb{K}$-subalgebra $F^{\mathbf{u}}_0 A$ is isomorphic to a PBW-reduction-algebra.

(c) The $F^{\mathbf{u}}_0 A$-modules $F^{\mathbf{u}}_j A$ ($j \in \mathbb{Z}$) are $F^{\mathbf{u}}_0 A$-finitely generated and monomial $F^{\mathbf{u}}_0 A$-generating sets can be computed.

Proof. First note that we have a one-to-one correspondence

$$\varphi : \mathrm{SMon}(T_n) \leftrightarrow \{\alpha \in \mathbb{N}^n\} =: U : \underline{x}^{\alpha} \leftrightarrow \alpha$$

mapping standard monomials to their exponents. We set for $i \in \mathbb{Z}$

$$U_i := \{\alpha \in \mathbb{N}^n \mid \langle \mathbf{u}, \alpha\rangle = i\},\ U^+ := \bigcup_{i \geq 0} U_i \text{ and } U^- := \bigcup_{i \leq 0} U_i.$$

(a) Considering $e_i \in \mathbb{Z}^n$, we have under the above one-to-one correspondence that

$$\varphi(\mathrm{SMon}(T_n) \cap F_0^{\mathbf{u}} T_n) = U^- = \{\alpha \in \mathbb{R}^n \mid \langle \mathbf{u}, \alpha \rangle \leq 0\} \cap \mathbb{N}^n$$

is an intersection of a rational cone and the lattice $\mathbb{Z}^n$, since

$$\mathbb{N}^n = \bigcap_{1 \leq i \leq n} \{\alpha \in \mathbb{R}^n \mid \langle e_i, \alpha \rangle \geq 0\} \cap \mathbb{Z}^n .$$

Hence U^- is by Gordan's lemma (see e.g. [BG09, Lemma 2.9]) a positive affine monoid, and has a computable minimal finite generating set [Koc03, Proposition 3.4.6] [BI10], say $\alpha_1, \ldots, \alpha_s \in \mathbb{Z}^n$. This means that

$$U^- = \{l_1\alpha_1 + \cdots + l_s\alpha_s \mid l \in \mathbb{N}^s\},$$

and if $\alpha_i = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in U^-$ then $\beta_1 = \alpha_i$ or $\beta_2 = \alpha_i$ for $1 \leq i \leq s$.

We claim that $F_0^{\mathbf{u}} A$ is generated by the residue classes of $\underline{x}^{\alpha_1}, \ldots, \underline{x}^{\alpha_s}$ as $\mathbb{K}$-algebra: Clearly, $\overline{\underline{x}^{\alpha_1}}, \ldots, \overline{\underline{x}^{\alpha_s}} \in F_0^{\mathbf{u}} A = \overline{F_0^{\mathbf{u}} T_n}$. As $F_0^{\mathbf{u}} A$ is generated by residue classes of certain standard monomials by Lemma 2.2.3, it suffices to show that $\overline{F_0^{\mathbf{u}} T_n \cap \mathrm{SMon}(T_n)}$ is a subset of the $\mathbb{K}$-algebra generated by $\overline{\underline{x}^{\alpha_1}}, \ldots, \overline{\underline{x}^{\alpha_s}}$. For this we use the well-ordering $\prec$ on A to impose a well-order on the set $F_0^{\mathbf{u}} T_n \cap \mathrm{SMon}(T_n)$ and do induction on this set by this well-order: The induction start is clear as $1 = \min_\prec\{F_0^{\mathbf{u}} T_n \cap \mathrm{SMon}(T_n)\}$. Now assume that $\underline{x}^\alpha \in F_0^{\mathbf{u}} T_n \cap \mathrm{SMon}(T_n)$ and that the claim has been shown for all $\underline{x}^\beta \in F_0^{\mathbf{u}} T_n \cap \mathrm{SMon}(T_n)$ with $\underline{x}^\beta \prec \underline{x}^\alpha$. Since $\alpha \in U^-$, there is $l \in \mathbb{N}^s$ such that $\alpha = \sum_{1 \leq i \leq s} l_i \alpha_i$. By Remark 2.1.44(b) there exists $c \in \mathbb{K}^*$ and $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n) \rangle$ with $\mathrm{lm}(a) \prec \underline{x}^\alpha$ such that

$$\overline{\underline{x}^\alpha} = \overline{\underline{x}^{\sum_{1 \leq i \leq s} l_i \alpha_i}} = c\overline{(\underline{x}^{\alpha_1})^{l_1} \cdots (\underline{x}^{\alpha_s})^{l_s}} + \overline{a}.$$

As $F_0^{\mathbf{u}} A$ is a ring, we have $\overline{a} \in F_0^{\mathbf{u}} A$ and the claim follows now by induction.

(b) We retain the notation of Part (a). Consider the surjective $\mathbb{K}$-algebra map

$$\pi : \mathbb{K}\langle \underline{y} \rangle := \mathbb{K}\langle y_1, \ldots, y_s \rangle \to F_0^{\mathbf{u}} A,\ y_i \mapsto \overline{\underline{x}^{\alpha_i}}.$$

Since A is a PBW-reduction-algebra, there exists by Remark 2.1.44(b) for $1 \leq i < j \leq s$ $f_{ij} \in \mathbb{K}^*$ and $g_{ij} \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n) \rangle$ with $\mathrm{le}_\prec(g_{ij}) \prec \alpha_i + \alpha_j$ such that

$$\underline{x}^{\alpha_j}\underline{x}^{\alpha_i} - f_{ij}\underline{x}^{\alpha_i}\underline{x}^{\alpha_j} - g_{ij} \in {}_{T_n}\langle S \rangle_{T_n} \subseteq {}_{T_n}\langle S, I \rangle_{T_n}.$$

As the weight vector $\mathbf{u}$ is compatible with the commutation relations in S, we may additionally assume by that remark that $\deg_{\mathbf{u}}(g_{ij}) \leq \deg_{\mathbf{u}}(\underline{x}^{\alpha_i}\underline{x}^{\alpha_j}) \leq 0$. By (the proof of) Part (a), we find $g'_{ij}(y_1, \ldots, y_s) \in {}_{\mathbb{K}}\langle \mathrm{SMon}(\mathbb{K}\langle \underline{y} \rangle) \rangle$ such that $g'_{ij}(\overline{\underline{x}^{\alpha_1}}, \ldots, \overline{\underline{x}^{\alpha_s}}) = \overline{g_{ij}} \in A$ and hence

$$S_0 := \{y_j y_i - f_{ij} y_i y_j - g'_{ij} \mid 1 \leq i < j \leq s\} \subseteq \ker(\pi).$$

Define the well-ordering $\prec_0$ on $\mathrm{SMon}(\mathbb{K}\langle \underline{y} \rangle)$ by

$$\underline{y}^\beta \prec_0 \underline{y}^\gamma \text{ if and only if } \sum_{1 \leq k \leq s} \beta_i \alpha_i \prec \sum_{1 \leq k \leq s} \gamma_i \alpha_i$$
$$\text{or } \sum_{1 \leq k \leq s} \beta_i \alpha_i = \sum_{1 \leq k \leq s} \gamma_i \alpha_i \text{ and } \underline{y}^\beta \prec' \underline{y}^\gamma,$$

where $\beta, \gamma \in \mathbb{N}^s$ and $\prec'$ is some well-ordering on $\mathrm{SMon}(\mathbb{K}\langle \underline{y} \rangle)$. By construction, S_0 is a standard reduction system with respect to $\prec_0$. We conclude that $\mathbb{K}\langle \underline{y} \rangle / \ker \pi$ is a PBW-reduction-algebra isomorphic to $F_0^{\mathbf{u}} A$.

(c) We keep the notation of Part (a) and consider first the case $j < 0$. One easily checks that

$$\bigcup_{i \leq j} U_i = U^- + \Delta := \{\alpha + \delta \mid \alpha \in U^-, \delta \in \Delta\},$$

where $\Delta := \{\alpha_i \mid \langle \mathbf{u}, \alpha_i \rangle \leq j\} \cup (\{\sum_{\delta \in \Delta'} l_\delta \delta \mid l \in \mathbb{N}^{\Delta'}, |l| \leq j\} \cap \bigcup_{i \leq j} U_i)$ with $\Delta' := \{\alpha_i \mid j < \langle \mathbf{u}, \alpha_i \rangle < 0\}$. We claim that $\{\overline{\underline{x}^\delta} \mid \delta \in \Delta\}$ is an $F_0^{\mathbf{u}} A$-generating set of $F_j^{\mathbf{u}} A = {}_{\mathbb{K}}\left\langle \overline{\mathrm{SMon}(T_n) \cap F_j^{\mathbf{u}} T_n} \right\rangle$. As in Part (a), we consider the well-ordering $\prec$ on A and proceed by induction with respect to the induced order on $\mathrm{SMon}(T_n) \cap F_j^{\mathbf{u}} T_n$. This set has a minimal element, say $\underline{x}^\beta$. Using the map φ, there exist $\delta \in \Delta$ and $l \in \mathbb{N}^s$ such that $\beta = \delta + \sum_{1 \leq i \leq s} l_i \alpha_i$. From the minimality of $\underline{x}^\beta$ and Definition 2.1.1(c), we deduce that $l = (0)_{1 \leq i \leq s}$. Thus $\underline{x}^\beta = \underline{x}^\delta$ and the inductive step works similar to Part (a).

The case $j = 0$ being clear, we assume now $j > 0$. Arguing as in the proof of Part (a), we can compute a minimal finite set of generators G of U^+. As above, we obtain

$$\bigcup_{i \leq j} U_i = U^- + (\Gamma \cup \{(0)_{1 \leq i \leq n}\}),$$

where $\Gamma := \{\sum_{\gamma \in \Gamma'} l_\gamma \gamma \mid l \in \mathbb{N}^{\Gamma'}, |l| \leq j\} \cap \bigcup_{1 \leq i \leq j} U_i$ with $\Gamma' := G \cap \bigcup_{1 \leq i \leq j} U_i$. The proof that $\{\overline{1}\} \cup \{\overline{\underline{x}^\gamma} \mid \gamma \in \Gamma\}$ is an $F_0^{\mathbf{u}} A$-generating set of $F_j^{\mathbf{u}} A$ is analogous to the proof for the case $j < 0$.

□

We explain now how to represent elements of $F_0^{\mathbf{u}} A$ in terms of a monomial generating set of $F_0^{\mathbf{u}} A$:

Definition and Remark 2.2.5. Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A.

(a) The monomial $\mathbb{K}$-algebra generating set of $F_0^{\mathbf{u}}A$ from (the proof of) Lemma 2.2.4(a) is denoted by $G_A^{\mathbf{u}} := \{\overline{\underline{x}^{\alpha_1}}, \ldots, \overline{\underline{x}^{\alpha_s}}\}$.

(b) We effectively represent an element $\overline{a} \in F_0^{\mathbf{u}}A$ given by $a \in F_0^{\mathbf{u}}T_n \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ as a $\mathbb{K}$-sum of products of elements in $G_A^{\mathbf{u}}$ by constructing a representation by induction on $\mathrm{lm}_{\prec}(a)$ with respect to the well-order $\prec$: As the case $a = 1$ is clear, we may assume that $1 \prec x^{\beta} := \mathrm{lm}_{\prec}(a) \in F_0^{\mathbf{u}}T_n$. Hence there is $i_1 \in \{1, \ldots, s\}$ such that $\beta' := \beta - \alpha_{i_1} \in \mathbb{N}^n$ and $\langle \mathbf{u}, \beta' \rangle \leq 0$. Continuing this way, we write $\beta = \sum_{1 \leq j \leq t} \alpha_{i_j}$ with $1 \leq i_j \leq s$. Using commutation relations (see Remark 2.1.44(b)) we find $f \in \mathbb{K}^*$ and $r \in {}_K\big\langle \mathrm{SMon}(T_n^E)\big\rangle$ with $\mathrm{lm}_{\prec}(r) \prec \mathrm{lm}_{\prec}(a)$ such that $\overline{a} = \overline{f \prod_{i=1,\ldots,t} \underline{x}^{\alpha_{i_t}}} + \overline{r}$. As the commutation relations are compatible with the weight vector $\mathbf{u}$, we may additionally assume $r \in F_0^{\mathbf{u}}T_n$. Induction shows the claim.

(c) We fix now for every $j \in \mathbb{Z}$ a finite set of generators $P_j^{A,\mathbf{u}}$ of the $F_0^{\mathbf{u}}A$-module $F_j^{\mathbf{u}}A$. Note that we may assume by Lemma 2.2.4(c) that this set consists of residue classes of standard monomials in $F_j^{\mathbf{u}}T_n$, say $P_j^{A,\mathbf{u}} = \{\overline{\underline{x}^{\beta_1^j}}, \ldots, \overline{\underline{x}^{\beta_{s_j}^j}}\}$.

(d) A representation $\overline{a} = \sum_{p \in P_j^{A,\mathbf{u}}} g_p p$ with $g \in (F_0^{\mathbf{u}}A)^{P_j^{A,\mathbf{u}}}$ for $a \in F_j^{\mathbf{u}}T_n \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ is also computable by similar methods as in Part (b).

The next remark investigates the interplay for different weight filtrations on A in certain situations:

Remark 2.2.6.

(a) Let $A = (\mathbb{K}\langle \underline{x}, \underline{y}\rangle, S, I, \prec)$ (with $\mathbb{K}\langle \underline{x}, \underline{y}\rangle := \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$) be an elementary PBW-reduction-algebra and $\mathbf{v} \in \mathbb{Z}^{n+m}$ be any weight vector on A. Then we have for the weight vector $\mathbf{w} = ((0)_{1 \leq i \leq n}, (1)_{1 \leq i \leq m})$ on A

$$F_k^{\mathbf{v}} A \cap F_l^{\mathbf{w}} A = \overline{F_k^{\mathbf{v}}\, \mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap F_l^{\mathbf{w}}\, \mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap {}_{\mathbb{K}}\big\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}, \underline{y}\rangle)\big\rangle}$$

for all $k, l \in \mathbb{Z}$: Clearly, it suffices to show that the left hand side is contained in the right hand side. If $a \in F_k^{\mathbf{v}}A \cap F_l^{\mathbf{w}}A$, then there exist representatives $a^{\mathbf{w}} \in F_l^{\mathbf{w}}T_n$ and $a^{\mathbf{v}} = \sum_{(\alpha,\beta) \in \mathbb{N}^{n+m}} a^{\mathbf{v}}_{(\alpha,\beta)} \underline{x}^{\alpha} \underline{y}^{\beta} \in F_k^{\mathbf{v}}T_n$ of a. As reductions with commutation relations do not increase the $\mathbf{v}$- or $\mathbf{w}$-degree of elements of T_n, we may assume that the representatives live in ${}_{\mathbb{K}}\big\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}, \underline{y}\rangle)\big\rangle$. As $\overline{a^{\mathbf{v}} - a^{\mathbf{w}}} = 0$ and as there is a direct sum representation of the form $A = \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{C}[\underline{x}]/J)\underline{y}^{\beta}$, we deduce for $\beta \in \mathbb{N}^m$ with $|\beta| > l$ that $\overline{\sum_{\alpha \in \mathbb{N}^n} a^{\mathbf{v}}_{(\alpha,\beta)} \underline{x}^{\alpha}} = 0 \in A$. Hence $\sum_{(\alpha,\beta) \in \mathbb{N}^{n+m}, |\beta| \leq l} a^{\mathbf{v}}_{(\alpha,\beta)} \underline{x}^{\alpha} \underline{y}^{\beta} \in F_k^{\mathbf{v}}\, \mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap F_l^{\mathbf{w}}\, \mathbb{K}\langle \underline{x}, \underline{y}\rangle \cap {}_{\mathbb{K}}\big\langle \mathrm{SMon}(\mathbb{K}\langle \underline{x}, \underline{y}\rangle)\big\rangle$ is also a representative of a.

(b) Let $\mathbf{v}$ and $\mathbf{w} \in \mathbb{Z}^n$ be weight vectors on $A = (T_n, S, I, \prec)$ such that

$$F_k^{\mathbf{v}} A \cap F_l^{\mathbf{w}} A = \overline{F_k^{\mathbf{v}} T_n \cap F_l^{\mathbf{w}} T_n \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle} \tag{2.2.1}$$

for $k, l \in \mathbb{Z}$ and denote by $P_k^{A,\mathbf{v}} = \{\overline{\underline{x}^{\beta_1^k}}, \dots, \overline{\underline{x}^{\beta_{s_k}^k}}\}$ the $F_0^{\mathbf{v}} A$-generating set of $F_k^{\mathbf{v}} A$ constructed in the proof of Lemma 2.2.4(c) (with the representatives also chosen as in that proof). Then that proof and Equation (2.2.1) imply

$$F_\bullet^{\mathbf{w}} F_k^{\mathbf{v}} A = \sum_{1 \le i \le s_k} (F^{\mathbf{w}}_{\bullet - \langle \beta_i, \mathbf{w}\rangle} F_0^{\mathbf{v}} A) \cdot \overline{\underline{x}^{\beta_i^k}}.$$

Given a weight vector $\mathbf{u}$ on A, we have no general method to determine a PBW-reduction datum (or even a representation as a quotient algebra of a free $\mathbb{K}$-algebra) of $F_0^{\mathbf{u}} A$. Yet, in certain situations such a PBW-reduction datum is computable:

Lemma 2.2.7. *If A is a quasi-commutative PBW-algebra, then $F_0^{\mathbf{u}} A$ is isomorphic to a quotient of a PBW-algebra and a corresponding PBW-reduction datum is computable.*

Proof. According to Lemma 2.2.4(a) a monomial generating set $G_A^{\mathbf{u}} = \{\underline{x}^{\alpha_1}, \dots, \underline{x}^{\alpha^s}\}$ exists and is computable. By the commutation relations of A and by hypothesis, there are $f_{ij} \in \mathbb{K}^*$ such that $\overline{\underline{x}^{\alpha_j} \underline{x}^{\alpha_j}} = \overline{f_{ij} \underline{x}^{\alpha_i} \underline{x}^{\alpha_j}} \in A$ for $1 \le i < j \le s$. Then

$$B := \mathbb{K}\langle y_1, \dots, y_s\rangle / \langle \{y_j y_i - f_{ij} y_i y_j \mid 1 \le i < j \le s\rangle\}\rangle$$

is obviously a quasi-commutative PBW-algebra. The $\mathbb{K}$-algebra homomorphism

$$\psi : B \to A,\ \overline{y_i} \mapsto \overline{\underline{x}^{\alpha_i}}$$

induces now an isomorphism of $\mathbb{K}$-algebras $B / \ker(\psi) \cong F_0^{\mathbf{u}} A$.

We reduce the computation of the kernel of the map ψ to the computation of toric ideals: Consider the commutative $\mathbb{K}$-algebras $A^c := \mathbb{K}[z_1, \dots, z_n]$ and $B^c := \mathbb{K}[v_1, \dots, v_s]$, which are isomorphic to A and B as $\mathbb{K}$-vector spaces, respectively. We denote by τ^A and τ^B the corresponding $\mathbb{K}$-vector spaces isomorphisms given by $\underline{x}^\beta \mapsto \underline{z}^\beta$ and $\underline{y}^\delta \mapsto \underline{v}^\delta$, respectively. By [Stu96, Lemma 4.1 and Algorithm 4.5] there exists for a given well-ordering $\prec$ on B^c a finite computable set $\Gamma \subseteq \mathbb{Z}^s$ such that $\{\underline{v}^{\gamma^+} - \underline{v}^{\gamma^-} \mid \gamma \in \Gamma\}$ is a Gröbner basis of the kernel of the $\mathbb{K}$-algebra homomorphism

$$\psi^c : B^c \to A^c,\ v_i \mapsto \underline{z}^{\alpha_i},$$

where the vectors $\gamma^+, \gamma^- \in \mathbb{N}^s$ are defined by

$$(\gamma^+)_i = \begin{cases} \gamma_i, & \text{if } \gamma_i > 0 \\ 0, & \text{else} \end{cases} \text{ and } (\gamma^-)_i = \begin{cases} -\gamma_i, & \text{if } \gamma_i < 0 \\ 0, & \text{else} \end{cases} \text{ for } 1 \le i \le s.$$

Changing the sign of γ if necessary, we may assume that $\underline{v}^{\gamma^-} \prec \underline{v}^{\gamma^+}$. We define for $\delta \in \mathbb{N}^s$ an element $c_\delta \in \mathbb{K}^*$ by the property $(\underline{x}^{\alpha_1})^{\delta_1} \cdots (\underline{x}^{\alpha_s})^{\delta_s} = c_\delta \underline{x}^{\sum_{1\leq i\leq s} \delta_i \alpha_i} \in A$ and obviously obtain

$$p = \sum_{\delta\in\mathbb{N}^s} p_\delta \underline{y}^\delta \in \ker(\psi) \text{ if and only if } p^c := \sum_{\delta\in\mathbb{N}^s} c_\delta p_\delta \underline{v}^\delta \in \ker(\psi^c)$$

(where $p_\delta \in \mathbb{K}$.) This implies in particular

$$\ker(\psi) \supseteq {}_A\left\langle c_{\gamma^-} \underline{y}^{\gamma^+} - c_{\gamma^+} \underline{y}^{\gamma^-} \mid \gamma \in \Gamma \right\rangle.$$

Denote by $\prec$ also the well-ordering induced by $\prec$ on B under τ_B^{-1} and set $G := \{c_\gamma^- \underline{y}^{\gamma^+} - c_\gamma^+ \underline{y}^{\gamma^-}\}$. We claim that $(\mathbb{K}\langle y_1,\ldots,y_s\rangle, \{y_j y_i - f_{ij} y_i y_j \mid 1 \leq i < j \leq s\}, G, \prec)$ is a PBW-reduction datum for $B/\ker(\psi)$. By Remark 2.1.15(a) and (b) it is enough to show that $\mathrm{le}_\prec(p) \in L_\prec(G)$ for any $p \in \ker(\psi) \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(\mathbb{K}\langle y_1,\ldots,y_s\rangle)\rangle$. As seen above, we have $p^c \in \ker(\psi^c)$ for such p implying that there is $\gamma \in \Gamma$ and $\delta \in \mathbb{N}^s$ such that $\mathrm{lm}_\prec(p^c) = \underline{v}^{\gamma^+ + \delta}$. We deduce $\mathrm{lm}_\prec(p) = \underline{y}^{\gamma^+ + \delta}$ finishing the proof. □

Example 2.2.8. Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A.

(a) If $\mathbf{u} \in \mathbb{Z}^n_{\leq 0}$, then $F_0^{\mathbf{u}} A = A$ and $G_A^{\mathbf{u}} = \{\overline{x_1},\ldots,\overline{x_n}\}$.

(b) Similarly, we have for $\mathbf{u} \in \mathbb{N}^n$ that $G_A^{\mathbf{u}} = \{\overline{x_i} \mid 1 \leq i \leq n, \deg_{\mathbf{u}}(x_i) = 0\}$ and hence there are $1 \leq i_1 < \cdots < i_l \leq n$ such that $G_A^{\mathbf{u}} = \{\overline{x_{i_1}},\ldots,\overline{x_{i_l}}\}$. Then $A_{\mathbf{u}} := \mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle/(\langle S \cup I\rangle \cap \mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle)$ is a PBW-reduction-algebra since $S_{\mathbf{u}} := \{x_{i_k} x_{i_j} - c_{i_j i_k} x_{i_j} x_{i_k} - d_{i_j i_k} \mid 1 \leq j < k \leq l\}$ is a reduction system with respect to the ordering induced by $\prec$ on $\mathrm{Mon}(\mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle)$, which we also denote by $\prec$. (Note that indeed $d_{i_j i_k} \in \mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle$ since $\mathbf{u}$ is a weight vector.) Moreover,

$$\phi_{\mathbf{u}} : A_{\mathbf{u}} \to A,\ \overline{x_{i_j}} \mapsto \overline{x_{i_j}}.$$

is an injective $\mathbb{K}$-algebra homomorphism inducing an isomorphism $A_{\mathbf{u}} \cong F_0^{\mathbf{u}} A$. If $\prec$ is an elimination ordering for $\{x_k \mid 1 \leq k \leq n,\ k \notin \{i_1,\ldots,i_l\}\}$ then we claim that $(\mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle, S_{\mathbf{u}}, I_{\mathbf{u}}, \prec)$ with $I_{\mathbf{u}} := I \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(\mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle)\rangle$ is a PBW-reduction datum for $A_{\mathbf{u}}$: Clearly, Definition 2.1.13(aii) is an immediate consequence of that property for A showing that $(\mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle, S_{\mathbf{u}}, I_{\mathbf{u}}, \prec)$ is a PBW-reduction datum for the PBW-reduction-algebra $A'_{\mathbf{u}} := \mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle / \langle S_{\mathbf{u}} \cup I_{\mathbf{u}}\rangle$. To prove that $A_{\mathbf{u}}$ coincides with $A'_{\mathbf{u}}$ it suffices by Proposition 2.1.11 to prove that the inclusion

$$\mathbb{K}\langle \underline{x}\rangle^{\mathrm{irr}}_{(A,\prec)} \cap \mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle = \mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle^{\mathrm{irr}}_{(A_{\mathbf{u}},\prec)} \subseteq \mathbb{K}\langle x_{i_1},\ldots,x_{i_l}\rangle^{\mathrm{irr}}_{(A'_{\mathbf{u}},\prec)}$$

is in fact an equality. But Definition 2.1.13(aii) for A and $A'_{\mathbf{u}}$ shows by the elimination property of $\prec$ that the module on the right hand side agrees with that on the left hand side, hence proving equality.

(c) Let $n, r \in \mathbb{N}$ with $r \geq 1$ and consider the weight vector $\mathbf{v} := (v_1, -v_2, v_1, v_2)$ defined by $v_1 = (0)_{1 \leq i \leq n} \in \mathbb{Z}^n$ and $v_2 := (1)_{1 \leq i \leq r} \in \mathbb{Z}^r$ on the Weyl algebra D_{n+r}. We have

$$G^{\mathbf{v}}_{D_{n+r}} = \{x_i \mid 1 \leq i \leq n+r\} \cup \{x_i \partial_j \mid n+1 \leq i, j \leq n+r\}.$$

Proceeding as in the proof of Lemma 2.2.4(b) and setting

$$(D_{n+r})_{\mathbf{v}} := \mathbb{K}\langle y_1, \ldots, y_{n+r}, \{z_{ij}\}_{n+1 \leq i,j \leq r+n}\rangle / \langle S_0 \rangle$$

for

$$S_0 := \{y_j y_i - y_i y_j, z_{kl} y_i - y_i z_{kl} - \delta_{il} y_k, z_{pq} z_{kl} - z_{kl} z_{pq} + \delta_{lp} z_{kq} - \delta_{kq} z_{pl} | \\ 1 \leq i \leq j \leq n+r, n+1 \leq k, l, p, q \leq r+n \text{ with } (k,l) \prec_{\text{lex}} (p,q)\}$$

we see by that proof that

$$\phi_{\mathbf{v}} : (D_{n+r})_{\mathbf{v}} \to F^{\mathbf{v}}_0 D_n,\ y_i \mapsto x_i, z_{kl} \mapsto x_k \partial_l$$

is a $\mathbb{K}$-algebra homomorphism and S_0 a standard reduction system with respect to the ordering defined in that proof. One checks using Corollary 2.1.23 that $(D_{n+r})_{\mathbf{v}}$ is even a PBW-algebra. Arguing as in Lemma 2.2.7, we get that $\ker(\phi_{\mathbf{v}})$ is $(D_{n+r})_{\mathbf{v}}$-generated by

$$\{y_i z_{kl} - y_k z_{il}, z_{ij} z_{kl} - z_{kj} z_{il} + \delta_{ij} z_{kl} - \delta_{jk} z_{il} \mid n+1 \leq i, j, k, l \leq n+r\},$$

allowing us to compute PBW-reduction datum for $F^{\mathbf{v}}_0 D_{n+r}$ by Corollary 2.1.53. Moreover, we have

$$P^{D_{n+r},\mathbf{v}}_k = \begin{cases} \{\overline{x^{\beta_{n+1}}_{n+1} \cdots x^{\beta_{n+r}}_{n+r}} \mid \sum_{1 \leq i \leq r} \beta_{n+i} = -k\}, & \text{if } k \leq 0 \\ \{\overline{\partial^{\beta_{n+1}}_{n+1} \cdots \partial^{\beta_{n+r}}_{n+r}} \mid \sum_{1 \leq i \leq r} \beta_{n+i} \leq k\}, & \text{else.} \end{cases}$$

(d) In the situation of Example 2.1.30(b), we have $T^V_X \cong F^{\mathbf{v}}_0 T_X$, where $\mathbf{v}$ is the weight vector assigning weights -1 and 1 to x_n and y_m, respectively, and weight 0 else. Note that the weight vector $\mathbf{w} = ((0)_{1 \leq i \leq n}, (1)_{1 \leq i \leq m})$ on T_X induces the weight vector $\mathbf{w_v} = ((0)_{1 \leq i \leq n}, (1)_{1 \leq i \leq m})$ on T^V_X by Remark 2.2.6.

Moreover, we have

$$P^{T_X,\mathbf{v}}_k = \begin{cases} \{\overline{x^k_n}\}, & \text{if } k \leq 0 \\ \{\overline{y^l_m} \mid 0 \leq l \leq k\}, & \text{else.} \end{cases}$$

2.2.2 Weight filtrations on submodules of free modules

In this subsection, we consider the PBW-reduction-algebra $A = (T_n, S, I, \prec)$ with $S := \{x_j x_i - c_{ij} x_i x_j - d_{ij} \mid 1 \le i < j \le n\}$ and assume that $\mathbf{u} \in \mathbb{Z}^n$ is a weight vector on A. Our aim is to prove that $\mathbf{u}$ is good weight vector on A by giving a computer algebraic proof that explains how to compute for a given set E, an A-submodule $M \subseteq A^E$ and a shift vector $\mathbf{s} \in \mathbb{Z}^E$ a finite set of generators M' of the filtration $F^{\mathbf{u}}_\bullet[\mathbf{s}]M$. Here, we say that a finite set $M' \subseteq M$ generates $F^{\mathbf{u}}_\bullet[\mathbf{s}]M$ (as $F^{\mathbf{u}}_\bullet A$-module) if for every $m \in M$ there exists an $a \in A^{M'}$ such that

$$m = \sum_{m' \in M'} a_{m'} m' \text{ and } \deg_{\mathbf{u}}(a_{m'}) + \deg_{\mathbf{u}[\mathbf{s}]}(m') \le \deg_{\mathbf{u}[\mathbf{s}]}(m) \text{ for all } m' \in M'.$$

We refine the total preorder $\le_{\mathbf{u}[\mathbf{s}]}$ defined by the $\mathbf{u}[\mathbf{s}]$-degree on $\mathrm{SMon}(T_n^E)$ via

$$\underline{x}^{\alpha}(e) \le_{\mathbf{u}[\mathbf{s}]} \underline{x}^{\alpha'}(e') \text{ if and only if } \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha}(e)) \le \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha'}(e')) \tag{2.2.2}$$

for $\alpha, \alpha' \in \mathbb{N}^n$ and $e, e' \in E$ to an ordering on A^E as follows:

Definition 2.2.9. Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A, E a finite set, $\prec^E$ an ordering on A^E and $\mathbf{s} \in \mathbb{Z}^E$ a shift vector. We define the ordering $\prec^E_{\mathbf{u}[\mathbf{s}]}$ on $\mathrm{SMon}(T_n^E)$ by

$$\underline{x}^{\alpha}(e) \prec^E_{\mathbf{u}[\mathbf{s}]} \underline{x}^{\alpha'}(e') \text{ if and only if } \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha}(e)) < \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha'}(e'))$$
$$\text{or } \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha}(e)) = \deg_{\mathbf{u}[\mathbf{s}]}(\underline{x}^{\alpha'}(e')) \text{ and } \underline{x}^{\alpha}(e) \prec^E \underline{x}^{\alpha'}(e')$$

for $\alpha, \alpha' \in \mathbb{N}^n$ and $e, e' \in E$. If $\mathbf{s}$ is the zero vector, we also write $\prec^E_{\mathbf{u}}$. We sometimes use the notation $\prec^E_{\mathbf{u}[\mathbf{s}]}$ without explicitly defining an ordering $\prec^E$ on A^E.

In the situation of Definition 2.2.9 note that $\prec^E_{\mathbf{u}[\mathbf{s}]}$ defines indeed an ordering on A^E since it is compatible with the commutation relations of A. Gröbner bases with respect to orderings of the above type on submodules of free A-modules and generating sets of the filtration $F^{\mathbf{u}}_\bullet[\mathbf{s}]$ on these modules are related as follows:

Lemma 2.2.10. *Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A, E a finite set, $\mathbf{s} \in \mathbb{Z}^E$ a shift vector, $\prec^E$ an ordering and $M \subseteq A^E$ an A-submodule. If G is a Gröbner basis of M with respect to $\prec^E_{\mathbf{u}[\mathbf{s}]}$, then it generates $F^{\mathbf{u}}[\mathbf{s}]_\bullet M$ as $F^{\mathbf{u}}_\bullet A$-module.*

Proof. Let $m \in F^{\mathbf{u}}[\mathbf{s}]_k M$ for some $k \in \mathbb{Z}$. Choose a representative $m' \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle \cap F^{\mathbf{u}}[\mathbf{s}]_k T_n$ of m using Lemma 2.2.3. By assumption there is $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle^G$ and $h \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle^G$ with $\overline{h_g} = g$ satisfying

$$m = \sum_{g \in G} \overline{a_g} g \text{ and } \mathrm{le}(a_g) + \mathrm{ele}(h_g) \preceq^E_{\mathbf{u}[\mathbf{s}]} \mathrm{ele}(m')$$

implying $\deg_{\mathbf{u}}(\overline{a_g}) + \deg_{\mathbf{u}[\mathbf{s}]}(g) \le \deg_{\mathbf{u}}(a_g) + \deg_{\mathbf{u}[\mathbf{s}]}(h_g) \le \deg_{\mathbf{u}}(m') \le k$. Hence $m \in \sum_{g \in G} F^{\mathbf{u}}_{k - \deg_{\mathbf{u}[\mathbf{s}]}(g)} A \cdot g$. □

Note that if $\prec^E$ is a well-ordering, then $\prec^E_{\mathbf{u}[\mathbf{s}]}$ is a well-ordering if and only if $\mathbf{u} \in \mathbb{N}^n$. Since Gröbner bases with respect to well-orderings exist by Proposition 2.1.50, we obtain:

Lemma 2.2.11. *The weight vector* $\mathbf{u} \in \mathbb{N}^n$ *on* A *is a good weight vector.*

If $\mathbf{u}$ is not a positive weight vector, we can still compute Gröbner bases with respect to $\prec^E_{\mathbf{u}[\mathbf{s}]}$ by combining the homogenization methods of [OT01] for the Weyl algebra and those of [BGTV03] for well-orderings on PBW-algebras. For this, we first define the $\mathbf{w}$-homogenized PBW-reduction-algebra of A for a given weight vector $\mathbf{w}$, which is isomorphic to the Rees ring of $F^{\mathbf{w}}_{\bullet} A$ (see also [BGTV03]):

Definition 2.2.12. Let $\mathbf{w} \in \mathbb{N}^n$ be a weight vector on A, E a finite set and $\mathbf{s} \in \mathbb{Z}^E$ a shift vector.

(a) We define the $\mathbf{w}[\mathbf{s}]$-*homogenization* of $p = \sum_{m \in \mathrm{Mon}(T_n^E)} p_m m \in T_n^E$ (with $p_m \in \mathbb{K}$) as

$$h_{\mathbf{w}[\mathbf{s}]}(p) := \sum_{m \in \mathrm{Mon}(T_n^E)} p_m h^{\deg_{\mathbf{w}[\mathbf{s}]}(p) - \deg_{\mathbf{w}[\mathbf{s}]}(m)} m \in (T_n^h)^E := \mathbb{K}\langle h, x_1, \ldots, x_n \rangle^E .$$

For $G \subseteq T_n^E$, we set $h_{\mathbf{w}[\mathbf{s}]}(G) := \{h_{\mathbf{w}[\mathbf{s}]}(g) \mid g \in G\}$. As usual, we suppress $\mathbf{s}$ if it stands for the zero vector.

(b) The $\mathbf{w}$-*homogenized* PBW-reduction-algebra $A^{h(\mathbf{w})}$ is defined as

$$T_n^h / \langle h_{\mathbf{w}}({}_{T_n}\langle S \cup I \rangle_{T_n}) \cup \{hx_i - x_i h \mid 1 \leq i \leq n\} \rangle .$$

(c) We define the ordering $\prec^E_{(1,\mathbf{w})}$ on $\mathrm{SMon}((T_n^h)^E)$ for the ordering $\prec^E$ on A^E by

$$h^\alpha \underline{x}^\beta(e) \prec^E_{(1,\mathbf{w})} h^{\alpha'} \underline{x}^{\beta'}(e') \text{ if and only if } \alpha + \langle \mathbf{w}, \beta \rangle < \alpha' + \langle \mathbf{w}, \beta' \rangle$$
$$\text{or } \alpha + \langle \mathbf{w}, \beta \rangle = \alpha' + \langle \mathbf{w}, \beta' \rangle \text{ and } \underline{x}^\beta(e) \prec^E \underline{x}^{\beta'}(e')$$

for $\alpha, \alpha' \in \mathbb{N}$, $\beta, \beta' \in \mathbb{N}^n$ and $e, e' \in E$.

(d) We call the $\mathbb{K}$-algebra homomorphism given by

$$d_h : T_n^h \to T_n, \; h \mapsto 1, x_i \mapsto x_i$$

dehomogenization map. It induces a map $d_h : A^{h(\mathbf{w})} \to A$. By abuse of notation, we denote the maps d_h^E also by d_h.

Note that the above dehomogenization map of $A^{h(\mathbf{w})}$ is well-defined and that we can indeed identify $A^{h(\mathbf{w})}$ with the Rees algebra $\bigoplus_{k \in \mathbb{Z}} F_k^{\mathbf{w}} A \cdot z^k \subseteq A[z, z^{-1}]$ by sending $h^\alpha \underline{x}^\beta$ to $\underline{x}^\beta z^{\alpha + \langle \mathbf{w}, \beta \rangle}$. Furthermore, homogenized PBW-reduction-algebras are PBW-reduction-algebras:

Lemma 2.2.13. *Let* $\mathbf{w} \in \mathbb{N}^n$ *be a weight vector on* A. *Then*

$$S^{h(\mathbf{w})} := h_{\mathbf{w}}(S) \cup \{hx_i - x_i h \mid 1 \leq i \leq n\}$$

is a standard reduction system with respect to $\prec_{(1,\mathbf{w})}$ *and the* $\mathbb{K}$*-algebra* $A^{h(\mathbf{w})}$ *is a* $(1, \mathbf{w})$*-graded PBW-reduction-algebra. In particular, there is a finite set* $I' \subseteq {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^h) \rangle$ *consisting of* $(1, \mathbf{w})$*-homogeneous elements such that* $(T_n^h, S^{h(\mathbf{w})}, I', \prec_{(1,\mathbf{w})})$ *represents a PBW-reduction datum for* $A^{h(\mathbf{w})}$. *If* A *is a PBW-algebra, then so is* $A^{h(\mathbf{w})}$.

Moreover, if $\prec'$ *is any ordering on* A, *then* $\prec'_{(1,\mathbf{w})}$ *is an ordering on* $A^{h(\mathbf{w})}$. *If* $\mathbf{w}$ *is strictly positive, then there exists a finite set* $I_{\prec'}$ *consisting of* $(1, \mathbf{w})$*-homogeneous elements such that* $(T_n^h, S^{h(\mathbf{w})}, I_{\prec'}, \prec'_{(1,\mathbf{w})})$ *is a PBW-reduction datum.*

Proof. We have for $1 \leq i < j \leq n$ that $h_{\mathbf{w}}(x_j x_i - c_{ij} x_i x_j - d_{ij}) = x_j x_i - c_{ij} x_i x_j - h^{\alpha_{ij}} h_{\mathbf{w}}(d_{ij})$ for some $\alpha_{ij} \in \mathbb{N}$ since $\mathbf{w}$ is a weight vector on A. By definition of the ordering $\prec_{(1,\mathbf{w})}$ we see that $S^{h(\mathbf{w})}$ is indeed a standard reduction system. According to Lemma 2.1.17, there exists some I'' such that $(T_n^{h(\mathbf{w})}, S^{h(\mathbf{w})}, I'', \prec_{(1,\mathbf{w})})$ is a PBW-reduction datum for $A^{h(\mathbf{w})}$. Setting I' to be the set of the $(1, \mathbf{w})$-homogeneous parts of the elements of I'', the particular claim follows as $A^{h(\mathbf{w})}$ is obviously $(1, \mathbf{w})$-graded. Moreover, the claim in the PBW-algebra case is due to Corollary 2.1.23.

Arguing as for $\prec_{(1,\mathbf{w})}$, we see that $S^{h(\mathbf{w})}$ is a standard reduction system for $\prec'_{(1,\mathbf{w})}$. If $\mathbf{w}$ is strictly positive, then the latter ordering is a well-ordering and Lemma 2.1.17 implies the existence of a corresponding PBW-reduction datum. □

The idea is now to homogenize the PBW-reduction-algebra A with respect to a strictly positive weight-vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ and then reduce Gröbner basis computations in A^E with respect to the non-well-ordering $\prec^E$ to Gröbner basis computations in $(A^{h(\mathbf{w})})^E$ with respect to the well-ordering $\prec^E_{(1,\mathbf{w})}$. We first need to ensure that such a strictly positive weight vector exists:

Lemma 2.2.14. *A weight vector* $\mathbf{w} \in \mathbb{N}^n_{>0}$ *on* A *exists and is effectively computable.*

Proof. Consider the set

$$M := \{x_i x_j \mid 1 \leq i < j \leq n\} \cup \{\underline{x}^\alpha \mid \text{ there is } 1 \leq i < j \leq n \text{ with } (d_{ij})_\alpha \neq 0\}$$

of standard monomials appearing with nonzero coefficient in one of the commutation relations in S. According to [GP08, Lemma 1.2.11] there is a strictly positive weight vector $\mathbf{w} \in \mathbb{N}^n$ such that

$$\underline{x}^\alpha \prec \underline{x}^\beta \text{ if and only if } \langle \alpha, \mathbf{w} \rangle < \langle \beta, \mathbf{w} \rangle$$

for all $\underline{x}^\alpha, \underline{x}^\beta \in M$, because $\prec$ is a well-ordering. As $\prec$ is an ordering on A, $\mathbf{w}$ is a weight vector on A. The claim on the computability follows from [GP08, Exercise 1.2.7 and Exercise 1.2.9]. □

If A is an elementary PBW-reduction-algebra, we compute a PBW-reduction datum for the homogenized PBW-reduction-algebra $A^{h(\mathbf{w})}$ with respect to the weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ as follows:

Lemma 2.2.15. *Consider the $\mathbb{K}$-algebra $\mathbb{K}\langle \underline{x}, \underline{y}\rangle := \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$ and the elementary PBW-reduction-algebra*

$$B = \mathbb{K}\langle \underline{x}, \underline{y}\rangle / \langle R\rangle \cong \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]/J)\underline{y}^{\beta}.$$

If $\mathbf{w} \in \mathbb{N}^{n+m}_{>0}$ is a weight vector on B, then $B^{h(\mathbf{w})}$ is also an elementary PBW-reduction-algebra.

In particular, if $\prec$ is an ordering on B, $J' \subseteq \mathbb{K}[\underline{x}]$ a Gröbner basis of J with respect to the ordering induced by $\prec_{\mathbf{w}}$ and $(\mathbb{K}\langle \underline{x}, \underline{y}\rangle, S, J', \prec_{\mathbf{w}})$ a corresponding PBW-reduction datum, then $(\mathbb{K}\langle h, \underline{x}, \underline{y}\rangle, S^{h(\mathbf{w})}, J'', \prec_{(1,\mathbf{w})})$ represents a PBW-reduction datum for $B^{h(\mathbf{w})}$, where J'' is a Gröbner basis of $\langle h_{\mathbf{w}}(J')\rangle \subseteq \mathbb{K}[h, \underline{x}]$ with respect to the ordering induced by $\prec_{(1,\mathbf{w})}$. So a PBW-reduction datum of $B^{h(\mathbf{w})}$ with respect to the ordering $\prec_{(1,\mathbf{w})}$ is computable.

Proof. We denote the canonical isomorphism $\bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]/J)\underline{y}^{\beta} \to B$ by ψ. We first show that the $\mathbb{K}$-linear epimorphism

$$\psi^h : \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[h, \underline{x}]/\langle h_{\mathbf{w}}(J)\rangle)\underline{y}^{\beta} \to B^{h(\mathbf{w})},\ \overline{h^c \underline{x}^{\alpha}}\underline{y}^{\beta} \mapsto \overline{h^c \underline{x}^{\alpha} \underline{y}^{\beta}}$$

is an isomorphism: We consider $p = \sum_{c,\alpha,\beta} \overline{d_{c,\alpha,\beta} h^c \underline{x}^{\alpha}}\underline{y}^{\beta} \in \ker(\psi^h)$ (with $d_{c,\alpha,\beta} \in \mathbb{K}$) and may assume that $d_{c,\alpha,\beta} = 0$ for $c + \langle(\alpha, \beta), \mathbf{w}\rangle \neq k$ for some fixed $k \in \mathbb{Z}$ because $B^{h(\mathbf{w})}$ is $(1, \mathbf{w})$-graded. Defining $d'_h : \bigoplus_{\beta \in \mathbb{N}^m}(\mathbb{K}[h, \underline{x}]/\langle h_{\mathbf{w}}(J)\rangle)\underline{y}^{\beta} \to \bigoplus_{\beta \in \mathbb{N}^m}(\mathbb{K}[\underline{x}]/J)\underline{y}^{\beta}$ by sending $\overline{h^c \underline{x}^{\alpha}}\underline{y}^{\beta}$ to $\overline{\underline{x}^{\alpha}}\underline{y}^{\beta}$, we see that $d_h \circ \psi^h = \psi \circ d'_h$. So we obtain for $\beta \in \mathbb{N}^m$ that $\sum_{c,\alpha} d_{c,\alpha,\beta}\underline{x}^{\alpha} \in J$. We observe that there exists $z \in \mathbb{N}$ with

$$\sum_{c,\alpha} d_{c,\alpha,\beta} h^c \underline{x}^{\alpha} = h^z h_{\mathbf{w}}(\sum_{c,\alpha} d_{c,\alpha,\beta}\underline{x}^{\alpha}) \in \langle h_{\mathbf{w}}(J)\rangle$$

since $\sum_{c,\alpha,\beta} d_{c,\alpha,\beta} h^c \underline{x}^{\alpha}\underline{y}^{\beta}$ and hence also $\sum_{c,\alpha} d_{c,\alpha,\beta} h^c \underline{x}^{\alpha}$ is $(1, \mathbf{w})$-homogeneous. This implies $p = 0$ showing injectivity. Thus $B^{h(\mathbf{w})}$ satisfies the assumptions of Lemma 2.1.28(b). According to [GP08, Exercise 1.7.5] we have $\langle h_{\mathbf{w}}(J)\rangle = \langle h_{\mathbf{w}}(J')\rangle \subseteq \mathbb{K}[h, \underline{x}]$ since J' is a Gröbner basis of J with respect to $\prec_{\mathbf{w}}$. So the claim is an immediate from Lemma 2.1.28. □

We deduce from PBW-reduction data of $A^{h(\mathbf{w})}$ and A a corresponding datum of the $(1, \mathbf{w})$-homogenization of factor algebras of A as explained below:

Lemma 2.2.16. *Let $\mathbf{w} \in \mathbb{N}^n_{>0}$ be a weight vector on A, $M \subseteq A$ be a finite subset, $\prec'$ an ordering on A and $(T^h_n, S^{h(\mathbf{w})}, I'_A, \prec'_{(1,\mathbf{w})})$ a PBW-reduction datum for $A^{h(\mathbf{w})}$. Then $\mathbf{w}$ is* *a weight vector on the PBW-reduction-algebra B, realized as a quotient of T_n, canonically isomorphic to $A/_A\langle M\rangle_A$. We have the representation*

$$B^{h(\mathbf{w})} = (T_n^h, S^{h(\mathbf{w})}, \tau_{(A^{h(\mathbf{w})},\prec'_{(1,\mathbf{w})})}(G') \cup I'_A, \prec'_{(1,\mathbf{w})}),$$

where G' is a Gröbner basis of the left $A^{h(\mathbf{w})}$-ideal generated by the residue classes of $h_{\mathbf{w}}(\tau_{(A,\prec_{\mathbf{w}})}(G))$ with respect to $\prec'_{(1,\mathbf{w})}$ for a left Gröbner basis G of $_A\langle M\rangle_A$ with respect to $\prec_{\mathbf{w}}$. In particular, PBW-reduction data for strictly positively homogenized factor algebras of PBW-algebras are computable.

Proof. Let $A = (T_n, S, I_A, \prec_{\mathbf{w}})$ and $B = (T_n, S, I_B, \prec)$ be PBW-reduction data. We first show that the $\mathbb{K}$-linear morphism

$$\psi : T_n^h/\left\langle h_{\mathbf{w}}({}_{T_n}\langle S \cup I_A\rangle_{T_n}) \cup \{hx_i - x_ih \mid 1 \le i \le n\} \cup h_{\mathbf{w}}(\tau_{(A,\prec_{\mathbf{w}})}(G))\right\rangle \to B^{h(\mathbf{w})},$$
$$\overline{p} \mapsto \overline{p}$$

is an isomorphism. Clearly, ψ is well-defined and surjective. So consider for the injectivity $p \in T_n^h$ with $\psi(\overline{p}) = 0$. This entails by definition of homogenized PBW-reduction-algebras that $p \in {}_{T_n^h}\langle h_{\mathbf{w}}({}_{T_n}\langle S \cup I_B\rangle_{T_n}) \cup \{hx_i - x_ih \mid 1 \le i \le n\}\rangle_{T_n^h}$. As ψ is $(1,\mathbf{w})$-graded, we may assume that p is $(1,\mathbf{w})$-homogeneous. Writing $p = p' + q$ with $p' \in \sum_{k\ge 0} h^k T_n$ and $q \in {}_{T_n^h}\langle\{hx_i - x_ih \mid 1 \le i \le n\}\rangle_{T_n^h}$, we reduce to the case $p \in \sum_{k\ge 0} h^k T_n$. We have now $d_h(p) \in {}_{T_n}\langle S \cup I_B\rangle_{T_n}$ allowing us to consider $\overline{d_h(p)} \in {}_A\langle M\rangle_A \subseteq A$. Hence we find $a \in A^G$ such that

$$\overline{d_h(p)} = \sum_{g\in G} a_g g \text{ and } \mathrm{le}_{\prec_{\mathbf{w}}}(a_g) + \mathrm{le}_{\prec_{\mathbf{w}}}(g) \preceq_{\mathbf{w}} \mathrm{le}_{\prec_{\mathbf{w}}}(\overline{d_h(p)}) \preceq_{\mathbf{w}} \mathrm{le}_{\prec_{\mathbf{w}}}(d_h(p)).$$

Thus there is $r \in {}_{T_n}\langle S \cup I_A\rangle_{T_n}$ satisfying

$$d_h(p) = \sum_{g\in G} \tau_{(A,\prec_{\mathbf{w}})}(a_g)\tau_{(A,\prec_{\mathbf{w}})}(g) + r \text{ and } \mathrm{le}_{\prec_{\mathbf{w}}}(r) \preceq_{\mathbf{w}} \mathrm{le}_{\prec_{\mathbf{w}}}(d_h(p)).$$

Therefore

$$p = h^d h_{\mathbf{w}}(d_h(p)) = \sum_{g\in G} h^{d'_g} h_{\mathbf{w}}(\tau_{(A,\prec_{\mathbf{w}})}(a_g)) h_{\mathbf{w}}(\tau_{(A,\prec_{\mathbf{w}})}(g)) + h^{d_r} h_{\mathbf{w}}(r) \qquad (2.2.3)$$

for suitable $d, d_r \in \mathbb{N}$ and $d' \in \mathbb{N}^G$ proving injectivity.

So $B^{h(\mathbf{w})}$ is canonically isomorphic to

$$A^{h(\mathbf{w})}/_A\left\langle \overline{h_{\mathbf{w}}(\tau_{(A,\prec_{\mathbf{w}})}(G))}\right\rangle_A$$

and thus an application of Corollary 2.1.53 finishes the proof. □

We investigate now the relationship between $\prec^E$ and $\prec^E_{(1,\mathbf{w})}$:

Remark 2.2.17. Let $\mathbf{w} \in \mathbb{N}^n_{>0}$ be a weight vector on A, E a finite set and $\prec^E$ an ordering on A^E. Then there exists for $e \in E$ a set I'_e consisting of $(1,\mathbf{w})$-homogeneous elements such that $\prec^E_{(1,\mathbf{w})}$ is a well-ordering on $(A^{h(\mathbf{w})})^E = (T_n^{h(\mathbf{w})}, S^{h(\mathbf{w})}, I'_e, \prec^E_e)_{e\in E}$ (see Lemma 2.2.13). Furthermore it holds:

(a) The map $\rho_{(A^{h(\mathbf{w})},\prec^E_{(1,\mathbf{w})})}$ preserves $(1,\mathbf{w})$-homogeneity as well as the $(1,\mathbf{w})$-degree since I'_e for $e \in E$ and $S^{h(\mathbf{w})}$ are $(1,\mathbf{w})$-homogeneous.

(b) We have the following relationship between the ordering $\prec^E$ on $\mathrm{SMon}(T_n^E)$ and the ordering $\prec^E_{(1,\mathbf{w})}$ on $\mathrm{SMon}((T_n^{h(\mathbf{w})})^E)$: If $\deg_{(1,\mathbf{w})}(h^\alpha \underline{x}^\beta(e)) = \deg_{(1,\mathbf{w})}(h^{\alpha'}\underline{x}^{\beta'}(e'))$ then
$$\underline{x}^\beta(e) \prec^E \underline{x}^{\beta'}(e') \text{ if and only if } h^\alpha \underline{x}^\beta(e) \prec^E_{(1,\mathbf{w})} h^{\alpha'}\underline{x}^{\beta'}(e')$$
for $\alpha,\alpha' \in \mathbb{N}$, $\beta,\beta' \in \mathbb{N}^n$ and $e,e' \in E$. It holds for a $(1,\mathbf{w})$-homogeneous $a \in {}_{\mathbb{K}}\left\langle \mathrm{SMon}((T_n^{h(\mathbf{w})})^E)\right\rangle$ that
$$d_h(\mathrm{lm}_{\prec^E_{(1,\mathbf{w})}}(\rho_{(A^{h(\mathbf{w})},\prec^E_{(1,\mathbf{w})})}(a))) \preceq^E d_h(\mathrm{lm}_{\prec^E_{(1,\mathbf{w})}}(a)) = \mathrm{lm}_{\prec^E}(d_h(a)),$$
where the inequality is due to Part (a). In particular, $a' \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle$ satisfies
$$d_h(\mathrm{lm}_{\prec^E_{(1,\mathbf{w})}}(\rho_{(A^{h(\mathbf{w})},\prec^E_{(1,\mathbf{w})})}(h_\mathbf{w}(a')))) \preceq^E d_h(\mathrm{lm}_{\prec^E_{(1,\mathbf{w})}}(h_\mathbf{w}(a'))) = \mathrm{lm}_{\prec^E}(a').$$

(c) We point out that $\prec^E_{(1,\mathbf{w})}$ is indeed a well-ordering on the PBW-reduction-algebra $A^{h(\mathbf{w})}$ and hence Gröbner bases with respect to that ordering are computable (see Proposition 2.1.50) given that an underlying PBW-reduction datum is computable. Since the commutation relations as well as the I'_e for $e \in E$ are $(1,\mathbf{w})$-homogeneous, Algorithm 2.1.45 preserves homogeneity: That is, if we apply this algorithm to $(1,\mathbf{w})$-homogeneous elements in $(A^{h(\mathbf{w})})^E$, then the so obtained Gröbner basis consists of $(1,\mathbf{w})$-homogeneous elements. An analogous statement holds for Algorithm 2.1.48.

We explain now the computation of Gröbner bases with respect to non-well-orderings. The existence of these Gröbner bases for orderings of type $\prec^E_{\mathbf{u}[\mathbf{s}]}$ then shows that every weight vector $\mathbf{u}$ on A is good.

Proposition 2.2.18. *Let $\mathbf{w} \in \mathbb{N}^n_{>0}$ be a weight vector on A, E a finite set, $\prec^E$ an ordering on A^E, and $M = {}_A\langle \overline{M'}\rangle \subseteq A^E$ for $M' \subseteq {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle$ finite. If the set $G \subseteq (A^{h(\mathbf{w})})^E$ is a Gröbner basis of ${}_{A^{h(\mathbf{w})}}\left\langle \overline{h_\mathbf{w}(M')}\right\rangle$ with respect to $\prec^E_{(1,\mathbf{w})}$ consisting of $(1,\mathbf{w})$-homogeneous elements, then $d_h(\tau_{\prec^E_{(1,\mathbf{w})}}(G))$ induces a Gröbner basis of M with respect to $\prec^E$. An analogous statement holds for two-sided modules.*

Proof. We first show that $d_h(G) \subseteq M$: As $G \subseteq {}_{A^{h(\mathbf{w})}}\left\langle \overline{h_{\mathbf{w}}(M')} \right\rangle$, there exists for $g \in G$ an $a \in (A^{h(\mathbf{w})})^{M'}$ such that $g = \sum_{m' \in M'} a_{m'} \overline{h_{\mathbf{w}}(m')}$. Hence

$$d_h(g) = \sum_{m' \in M'} d_h(a_{m'}) d_h(\overline{h_{\mathbf{w}}(m')}) = \sum_{m' \in M'} d_h(a_{m'}) \overline{m'} \in M.$$

The second step is proving that $d_h(G)$ is a Gröbner basis of M: For $t \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E) \rangle$ with $\bar{t} \in M$ exists $a \in (T_n)^{M'}$ such that $\bar{t} = \sum_{m' \in M'} \overline{a_{m'}} \overline{m'}$. This implies that there is $r \in {}_{T_n}\langle S^E \cup I^E \rangle_{T_n}$ such that $t = \sum_{m' \in M'} a_{m'} m' + r$ and hence we find $\beta \in \mathbb{N}^{M' \sqcup \{t\} \sqcup \{r\}}$ such that

$$h^{\beta_t} h_{\mathbf{w}}(t) = \sum_{m' \in M'} h^{\beta_{m'}} h_{\mathbf{w}}(a_{m'}) h_{\mathbf{w}}(m') + h^{\beta_r} h_{\mathbf{w}}(r)$$

showing that

$$\overline{h^{\beta_t} h_{\mathbf{w}}(t)} \in {}_{A^{h(\mathbf{w})}}\left\langle \overline{h_{\mathbf{w}}(M')} \right\rangle.$$

As G is a $(1, \mathbf{w})$-homogeneous Gröbner basis and $\overline{h^{\beta_m} h_{\mathbf{w}}(t)}$ is $(1, \mathbf{w})$-homogeneous according to Remark 2.2.17(a), we obtain a $(1, \mathbf{w})[(\deg_{(1,\mathbf{w})}(g))_{g \in G}]$-homogeneous $b \in (A^{h(\mathbf{w})})^G$ such that

$$\overline{h^{\beta_t} h_{\mathbf{w}}(t)} = \sum_{g \in G} b_g g$$

and

$$\mathrm{le}_{(\prec^E_{(1,\mathbf{w})})_{\mathrm{lcomp}(g)}}(b_g) + \mathrm{ele}_{\prec^E_{(1,\mathbf{w})}}(g) \preceq^E_{(1,\mathbf{w})} \mathrm{ele}_{\prec^E_{(1,\mathbf{w})}}(\overline{h^{\beta_t} h_{\mathbf{w}}(t)}) \preceq^E_{(1,\mathbf{w})} \mathrm{ele}_{\prec^E_{(1,\mathbf{w})}}(h^{\beta_t} h_{\mathbf{w}}(t)). \tag{2.2.4}$$

Dehomogenizing we get

$$\bar{t} = \sum_{g \in G} \overline{d_h(\tau_{(\prec^E_{(1,\mathbf{w})})_{\mathrm{lcomp}(g)}}(b_g))} \cdot \overline{d_h(\tau_{\prec^E_{(1,\mathbf{w})}}(g))}. \tag{2.2.5}$$

By Equation (2.2.4) and Remark 2.2.17(b), we have

$$\begin{aligned} &\mathrm{le}_{(\prec^E)_{\mathrm{lcomp}(g)}}(d_h(\tau_{(\prec^E_{(1,\mathbf{w})})_{\mathrm{lcomp}(g)}}(b_g))) + \mathrm{ele}_{\prec^E}(d_h(\tau_{\prec^E_{(1,\mathbf{w})}}(g))) \\ &\preceq^E \mathrm{ele}_{\prec^E}(d_h(h^{\beta_t} h_{\mathbf{w}}(t))) = \mathrm{ele}_{\prec^E}(t) \end{aligned} \tag{2.2.6}$$

concluding the proof. □

Lemma 2.2.14, Proposition 2.2.18 and Remark 2.2.17(c) imply

Corollary 2.2.19. *Let E be a finite set. Gröbner bases with respect to any ordering $\prec^E$ on A^E exist. They are computable if we can compute a weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ on A such that a PBW-reduction datum of $A^{h(\mathbf{w})}$ for the ordering $\prec^E_{(1,\mathbf{w})}$ is computable. In particular, Gröbner bases with respect to orderings of type $\prec_{\mathbf{u}[\mathbf{s}]}$, where $\mathbf{u} \in \mathbb{Z}^n$ is a weight vector and $\mathbf{s} \in \mathbb{Z}^E$ is shift vector, exist.*

We point out that it is possible by Lemma 2.2.14 to compute some weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ on A, but that we have in general no method to determine a suitable weight vector $\mathbf{w}' \in \mathbb{N}^n_{>0}$ on A such that a PBW-reduction datum for $(A^{h(\mathbf{w}')}, \prec^E_{(1,\mathbf{w}')})$ is computable even if some PBW-reduction datum for A is known. However, for PBW-algebras and quotients thereof as well as elementary PBW-reduction-algebras we can determine such a PBW-reduction datum (see Lemma 2.2.16 and Lemma 2.2.15).

Definition 2.2.20. Let A be a PBW-reduction-algebra, E a finite set and $\prec^E$ a non-well-ordering on A^E. We call $\prec^E$ *computable* if a weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ is computable such that the ordering $\prec^E_{(1,\mathbf{w})}$ on $(A^{h(\mathbf{w})})^E$ is computable.

The following algorithm summarizes the computation of such Gröbner bases. *For that notice that when writing algorithms we use $\triangleright$ as comment symbol.*

Algorithm 2.2.21 Given an A-submodule M of a free A-module and an ordering on that free module, this algorithm computes a Gröbner basis of M with respect to that ordering.

Input: A finite set E, an A-module $M = {}_A\langle \overline{M'} \rangle \subseteq A^E$ with $M' \subseteq T^E_n$ finite and a computable ordering $\prec^E$ on A^E.
Output: A finite set $G \subseteq T^E_n$ inducing a Gröbner basis of M with respect to $\prec^E$.

if $\prec^E$ is a well-ordering **then**
  Compute a Gröbner basis G' of M with respect to $\prec^E$ using Algorithm 2.1.45.
  return $\tau_{(A^E, \prec^E)}(G)$.
Determine a suitable weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ on A and a PBW-reduction datum for $((A^{h(\mathbf{w})})^E, \prec^E_{(1,\mathbf{w})})$.
Set $M' := h_{\mathbf{w}}(M')$.
Compute a $(1, \mathbf{w})$-homogeneous Gröbner basis G' of ${}_{A^{h(\mathbf{w})}}\langle \overline{M'} \rangle$ over the ring $A^{h(\mathbf{w})}$ with respect to $\prec^E_{(1,\mathbf{w})}$ using Algorithm 2.1.45. $\triangleright$ Requires corresponding PBW-reduction datum of $A^{h(\mathbf{w})}$.
Set $G := d_h(\tau_{((A^{h(\mathbf{w})})^E, \prec^E_{(1,\mathbf{w})})}(G))$.
return G.

Remark 2.2.22. Note that reduced Gröbner bases with respect to non-well-orderings do in general not exist.

Remark 2.2.23. Our application of the above method is the computation of Gröbner bases with respect to orderings of type $\prec^E_{\mathbf{u}[\mathbf{s}]}$ on A^E, where E is some finite set and $\mathbf{s} \in \mathbb{Z}^E$ a shift vector. We remark that the positive weight vector $\mathbf{w}$ chosen for the homogenization is independent of the weight vector $\mathbf{u}$ and the shift vector $\mathbf{s}$. In some instances, namely when the elements of S are $\mathbf{u}$-homogeneous, we can homogenize in a way depending on $\mathbf{u}[\mathbf{s}]$, which might enhance the computation of Gröbner bases with respect to the ordering $\prec^E_{\mathbf{u}[\mathbf{s}]}$. More precisely, we work over $A^{h(\mathbf{u})}$ and modify Proposition 2.2.18 in this situation as follows:

Noting that $-\mathbf{u}$ is also a weight vector on A, we replace the homogenization $h_{\mathbf{w}}$ by $h_{-\mathbf{u}[-\mathbf{s}]}$ and the ordering $(\prec^E_{\mathbf{u}[\mathbf{s}]})_{(1,\mathbf{w})}$ by the ordering $(\prec^E_{\mathbf{u}[\mathbf{s}]})_h$ defined by

$$h^{\alpha}\underline{x}^{\beta}(e)(\prec^E_{\mathbf{u}[\mathbf{s}]})_h h^{\alpha'}\underline{x}^{\beta'}(e') \text{ if and only if } \alpha < \alpha' \text{ or } \alpha = \alpha' \text{ and } \underline{x}^{\beta}(e) \prec^E_{\mathbf{u}[\mathbf{s}]} \underline{x}^{\beta'}(e')$$

for $\alpha, \alpha' \in \mathbb{N}$, $\beta, \beta' \in \mathbb{N}^n$ and $e, e' \in E$. If we replace $(1, \mathbf{w})$-homogeneous Gröbner basis by a $(1, -\mathbf{u})[-\mathbf{s}]$-homogeneous Gröbner basis, then one can show that Proposition 2.2.18 still holds.

We use Gröbner bases with respect to $\prec^E_{\mathbf{u}[\mathbf{s}]}$ to explicitly find generators of the filtration induced by $F^{\mathbf{u}}[\mathbf{s}]_{\bullet}A^E$ on submodules of A^E under the assumption that we can determine the required PBW-reduction datum:

Proposition 2.2.24. *Let $\mathbf{u} \in \mathbb{Z}^n$ be a weight vector on A, E and E' finite sets, $\mathbf{s} \in \mathbb{Z}^E$ a shift vector, $\prec^E$ an ordering on A^E and $\prec'^{E'}$ an ordering on $A^{E'}$. If $G \subseteq T_n^E \oplus T_n^{E'}$ induces a Gröbner basis of the A-submodule $M \subseteq A^E \oplus A^{E'}$ with respect to $(\prec^E_{\mathbf{u}[\mathbf{s}]}, \prec'^{E'})$ then*

$$M \cap (F^{\mathbf{u}}[\mathbf{s}]_{\bullet}A^E \oplus A^{E'}) = \sum_{g \in G: \pi_E(g) \neq 0} F_{\bullet - \deg_{\mathbf{u}[\mathbf{s}]}(g)}A \cdot \overline{g} + \sum_{g \in G: \pi_E(g) = 0} A \cdot \overline{g}. \tag{2.2.7}$$

In particular, $M \cap (F^{\mathbf{u}}[\mathbf{s}]_k A^E \oplus A^{E'}) = {}_{F_0^{\mathbf{u}}A}\left\langle \{a\overline{g} \mid g \in G, \pi_E(g) \neq 0, a \in P^{A,\mathbf{u}}_{k-\deg_{\mathbf{u}[\mathbf{s}]}(g)}\} \right\rangle + {}_A\langle \{\overline{g} \mid g \in G, \pi_E(g) = 0\} \rangle$ for $k \in \mathbb{Z}$.

Proof. We first observe that the right hand side module of Equation (2.2.7) is obviously contained in the left hand side module of that equation.

Let $m \in M \cap (F^{\mathbf{u}}[\mathbf{s}]_k A^E \oplus A^{E'})$ for fixed $k \in \mathbb{Z}$. By definition of $F^{\mathbf{u}}_{\bullet}[\mathbf{s}]A^E$ there exists a representative $m' \in (F^{\mathbf{u}}_k T_n^E \oplus T_n^{E'}) \cap {}_{\mathbb{K}}\left\langle \mathrm{SMon}(T_n^{E \sqcup E'}) \right\rangle$ of m. Since G induces a Gröbner basis of M, there is $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E) \rangle^G$ such that

$$m = \sum_{g \in G} \overline{a_g g} \text{ and } \mathrm{le}(a_g) + \mathrm{ele}(g) \preceq^{E,E'}_{\mathbf{u}[\mathbf{s}],'} \mathrm{ele}(m'),$$

where we abbreviate $\prec^{E,E'}_{\mathbf{u}[\mathbf{s}],'} := (\prec^E_{\mathbf{u}[\mathbf{s}]}, \prec'^{E'})$. If $\pi_E(g) \neq 0$, this implies that

$$\deg_{\mathbf{u}[\mathbf{s}]}(a_g \pi_E(g)) = \deg_{\mathbf{u}}(a_g) + \deg_{\mathbf{u}[\mathbf{s}]}(\pi_E(g)) \leq \deg_{\mathbf{u}[\mathbf{s}]}(\pi_E(m')) \leq k,$$

hence showing that $\overline{a_g} \in F_{k-\deg_{\mathbf{u}[\mathbf{s}]}(g)}A$. As that $F_0^{\mathbf{u}}A$-module is generated by $P^{A,\mathbf{u}}_{k-\deg_{\mathbf{u}[\mathbf{s}]}(g)}$, the particular claim follows readily. □

Corollary 2.2.19 and Proposition 2.2.24 imply:

Corollary 2.2.25. *Every weight vector on A is a good weight vector.*

Proposition 2.2.24 yields the following algorithms:

Algorithm 2.2.26 Given a weight vector $\mathbf{u}$ and an A-module $M \subseteq A^E \oplus A^{E'}$, this algorithm computes $M \cap (F^{\mathbf{u}}[\mathbf{s}]_\bullet A^E \oplus A^{E'})$.

Input: Two finite sets E, E', a module $M = {}_A\langle M' \rangle \subseteq A^E \oplus A^{E'}$ with M' finite, a weight vector $\mathbf{u} \in \mathbb{Z}^n$ on A, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and computable orderings $\prec^E_{\mathbf{u}[\mathbf{s}]}$ and $\prec'^{E'}$ on A^E and $A^{E'}$, respectively.

Output: Two finite sets $G_1, G_2 \subseteq T_n^E \oplus T_n^{E'}$ with $\pi_E(G_2) = \{0\}$ such that $M \cap (F^{\mathbf{u}}[\mathbf{s}]_\bullet A^E \oplus A^{E'}) = \sum_{g_1 \in G_1} F^{\mathbf{u}}_{\bullet - \deg_{\mathbf{u}[s]}(g_1)} A \cdot \overline{g_1} + {}_A\langle \overline{G_2} \rangle$.

1: Compute a set $G \subseteq T_n^E \oplus T_n^{E'}$ inducing a Gröbner basis of M with respect $(\prec^E_{\mathbf{u}[\mathbf{s}]}, \prec'^{E'})$ by Algorithm 2.2.21.
2: Set $G_1 := \{g \mid g \in G, \pi_E(g) \neq 0\}$.
3: Set $G_2 := \{g \in G \mid \pi_E(g) = 0\}$.
4: **return** G_1, G_2.

Algorithm 2.2.27 Given a weight vector $\mathbf{u}$ and an A-module $M \subseteq A^E \oplus A^{E'}$, this algorithm computes $M \cap (F^{\mathbf{u}}[\mathbf{s}]_k A^E \oplus A^{E'})$ for fixed $k \in \mathbb{Z}$.

Input: Two finite sets E, E', a module $M = {}_A\langle M' \rangle \subseteq A^E \oplus A^{E'}$ with M' finite, a weight vector $\mathbf{u} \in \mathbb{Z}^n$, a shift vector $\mathbf{s} \in \mathbb{Z}^E$, computable orderings $\prec^E_{\mathbf{u}[\mathbf{s}]}$ and $\prec'^{E'}$ on A^E and $A^{E'}$, respectively, and $k \in \mathbb{Z}$.

Output: Two finite sets $G_1, G_2 \subseteq A^E \oplus A^{E'}$ with $\pi_E(G_2) = \{0\}$ such that $M \cap (F^{\mathbf{u}}[\mathbf{s}]_k A^E \oplus A^{E'}) = {}_{F_0^{\mathbf{u}} A}\langle G_1 \rangle + {}_A\langle G_2 \rangle$.

1: Compute a set $G \subseteq T_n^E \oplus T_n^{E'}$ inducing a Gröbner basis of M with respect $(\prec^E_{\mathbf{u}[\mathbf{s}]}, \prec'^{E'})$ by Algorithm 2.2.21.
2: Set $G_1 := \{a\overline{g} \mid g \in G, \pi_E(g) \neq 0, a \in P^{A,\mathbf{u}}_{k - \deg_{\mathbf{u}[\mathbf{s}]}(g)}\}$.
3: Set $G_2 := \{\overline{g} \mid g \in G, \pi_E(g) = 0\}$.
4: **return** G_1, G_2.

Consider now a weight vector $\mathbf{u} \in \mathbb{Z}^n$ on A, a finite set E and a shift vector $\mathbf{s} \in \mathbb{Z}^E$. Abbreviating $\mathrm{Gr}^{F^{\mathbf{u}}[\mathbf{s}]}$ by $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}$ (and similarly for the corresponding symbol maps) and calling the associated graded objects also associated $\mathbf{u}[\mathbf{s}]$-graded objects, we finish this subsection by studying the ring $\mathrm{Gr}^{\mathbf{u}} A$ and explaining how to express $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} M$ for an A-submodule $M \subseteq A^E$ as a $\mathrm{Gr}^{\mathbf{u}} A$-module. (Note that as always we drop the shift vector in the above notation, if it stands for the zero vector.)

Proposition 2.2.28. *Let $\mathbf{u} \in \mathbb{Z}^n$ and $\mathbf{w} \in \mathbb{N}^n_{>0}$ be weight vectors on A, $\prec'$ an ordering on A, and $A^{h(\mathbf{w})} = (T_n^h, S^{h(\mathbf{w})}, I_{\mathbf{w}}, (\prec'_{\mathbf{u}})_{(1,\mathbf{w})})$ a PBW-reduction datum with $(1, \mathbf{w})$-homogeneous $I_{\mathbf{w}}$.*

(a) *We may identify*

$$\operatorname{Gr}^{\mathbf{u}}A = T_n / \langle \operatorname{lt}_{\mathbf{u}}(S) \cup \operatorname{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))\rangle$$

and a PBW-reduction datum of that PBW-reduction-algebra is given by $(T_n, \operatorname{lt}_{\mathbf{u}}(S),$ $\operatorname{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}})), \prec')$.

(b) *If* $\mathbf{u} \in \mathbb{N}^n$ *and* $A = (T_n, S, I_{\mathbf{u}}, \prec_{\mathbf{u}})$*, then a PBW-reduction datum for* $\operatorname{Gr}^{\mathbf{u}}A$ *is given by* $(T_n, \operatorname{lt}_{\mathbf{u}}(S), \operatorname{lt}_{\mathbf{u}}(I_{\mathbf{u}}), \prec)$.

(c) *Consider the finite set* E*, the ordering* $\prec^E$ *on* A^E*, the shift vector* $\mathbf{s} \in \mathbb{Z}^E$ *and the* A*-module* $M \subseteq A^E$*. We have under the identification in Part* (a)

$$\operatorname{Gr}^{\mathbf{u}[\mathbf{s}]}A^E \cong T_n^E / \left\langle \operatorname{lt}_{\mathbf{u}}(S)^E \cup \operatorname{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))^E \right\rangle,$$

where we put $\overline{(e)} \in T_n^E / \left\langle \operatorname{lt}_{\mathbf{u}}(S)^E \cup \operatorname{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))^E \right\rangle$ *in degree* $\mathbf{s}_e$*, and we may consider* $\operatorname{Gr}^{\mathbf{u}[\mathbf{s}]}M$ *as a* $\operatorname{Gr}^{\mathbf{u}}A$*-submodule thereof.*

Furthermore, if $G \subseteq T_n^E$ *induces a Gröbner basis of* M *with respect to* $\prec^E_{\mathbf{u}[\mathbf{s}]}$*, then* $\operatorname{lt}_{\mathbf{u}[\mathbf{s}]}(G) \subseteq T_n^E$ *induces a Gröbner basis of the* $\operatorname{Gr}^{\mathbf{u}}A$*-module* $\operatorname{Gr}^{\mathbf{u}[\mathbf{s}]}M$ *with respect to* $\prec^E$ *under the above isomorphism.*

(d) *We have for* $M \subseteq A$

$$\operatorname{Gr}^{\mathbf{u}}(A/{}_A\langle M\rangle_A) \cong \operatorname{Gr}^{\mathbf{u}}A / \operatorname{Gr}^{\mathbf{u}}{}_A\langle M\rangle_A.$$

If $\prec'$ *is a well-ordering, then a PBW-reduction datum of the above algebra is given by* $(T_n, \operatorname{lt}_{\mathbf{u}}(S), \operatorname{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}})) \cup \rho_{(\operatorname{Gr}^{\mathbf{w}}A, \prec')}(\operatorname{lt}_{\mathbf{u}}(G)), \prec')$*, where* $G \subseteq T_n^E$ *induces a Gröbner basis of* ${}_A\langle M\rangle_A$ *with respect to* $\prec'_{\mathbf{u}}$.

Proof.

(a) The $\mathbb{K}$-linear surjective map

$$\psi : T_n \to \operatorname{Gr}^{\mathbf{u}}A,\ x_{i_1}\cdots x_{i_k} \mapsto \overline{x_{i_1}\cdots x_{i_k}} + F^{\mathbf{u}}_{\deg_{\mathbf{u}}(x_{i_1}\cdots x_{i_k})-1}A \in \operatorname{Gr}^{\mathbf{u}}_{\deg_{\mathbf{u}}(x_{i_1}\cdots x_{i_k})}A$$

with kernel $\langle \operatorname{lt}_{\mathbf{u}}({}_{T_n}\langle S \cup I\rangle_{T_n})\rangle$ induces an isomorphism of $\mathbb{K}$-algebras

$$T_n / \langle \operatorname{lt}_{\mathbf{u}}({}_{T_n}\langle S \cup I\rangle_{T_n})\rangle \cong \operatorname{Gr}^{\mathbf{u}}A.$$

As ${}_{T_n}\langle S \cup I\rangle_{T_n} = {}_{T_n}\langle S \cup d_h(I_{\mathbf{w}})\rangle_{T_n}$, we have clearly

$$\langle \operatorname{lt}_{\mathbf{u}}(S) \cup \operatorname{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))\rangle \subseteq \langle \operatorname{lt}_{\mathbf{u}}({}_{T_n}\langle S \cup I\rangle_{T_n})\rangle.$$

For the converse inclusion consider a $\mathbf{u}$-homogeneous $p \in {}_{T_n}\langle \operatorname{lt}_{\mathbf{u}}({}_{T_n}\langle S \cup I\rangle_{T_n})\rangle_{T_n}$. We may assume that $p \in {}_{\mathbb{K}}\langle \operatorname{SMon}(T_n)\rangle$ and that there exists $p' \in {}_{\mathbb{K}}\langle \operatorname{SMon}(T_n)\rangle$ with $\deg_{\mathbf{u}}(p') < \deg_{\mathbf{u}}(p)$ and $p + p' \in {}_{T_n}\langle S \cup I\rangle_{T_n}$ as $\operatorname{lt}_{\mathbf{u}}(S) \subseteq \langle \operatorname{lt}_{\mathbf{u}}(S) \cup \operatorname{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))\rangle$

and S are standard reduction systems with respect to $\prec'_{\mathbf{u}}$. Now we find $l, l', l'' \in \mathbb{N}$ such that $h^{l''} h_{\mathbf{w}}(p+p') = h^l h_{\mathbf{w}}(p) + h^{l'} h_{\mathbf{w}}(p') \in {}_{T_n^h}\langle S^{h(\mathbf{w})} \cup I_{\mathbf{w}} \rangle_{T_n^h}$. By Remark 2.1.16(a) we write

$$h^{l''} h_{\mathbf{w}}(p+p') = \sum_{g \in I_{\mathbf{w}}} a_g g + \sum_{(t,s,t') \in U} tst' \tag{2.2.8}$$

for some $(1,\mathbf{w})[(\deg_{(1,\mathbf{w})}(g))_{g\in I_{\mathbf{w}}}]$-homogeneous $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^h)\rangle^{I_{\mathbf{w}}}$ and some finite set $U \subseteq T_n^h \setminus \{0\} \times S^{h(\mathbf{w})} \times T_n^h \setminus \{0\}$ satisfying

$$\mathrm{le}(a_g) + \mathrm{le}(g) (\preceq'_{\mathbf{u}})_{(1,\mathbf{w})} \mathrm{le}(h^{l''} h_{\mathbf{w}}(p+p'))$$

and

$$\mathrm{le}^{\mathrm{com}}(t) + \mathrm{le}^{\mathrm{com}}(s) + \mathrm{le}^{\mathrm{com}}(t') (\preceq'_{\mathbf{u}})_{(1,\mathbf{w})} \mathrm{le}(h^{l''} h_{\mathbf{w}}(p+p'))$$

with equality for some $g \in I_{\mathbf{w}}$. Here, we may assume for $(t,s,t') \in U$ that t and t' are $(1,\mathbf{w})$-homogeneous and that all terms appearing in Equation (2.2.8) are $(1,\mathbf{w})$-homogeneous of the same degree. Dehomogenizing we obtain (see Remark 2.2.17(b))

$$p + p' = \sum_{g \in I_{\mathbf{w}}} d_h(a_g) d_h(g) + \sum_{(t,s,t') \in U} d_h(t) d_h(s) d_h(t')$$

with

$$\mathrm{le}(d_h(a_g)) + \mathrm{le}(d_h(g)) \preceq'_{\mathbf{u}} \mathrm{le}(p+p') = \mathrm{le}(p) \tag{2.2.9}$$

and

$$\mathrm{le}^{\mathrm{com}}(d_h(t)) + \mathrm{le}^{\mathrm{com}}(d_h(s)) + \mathrm{le}^{\mathrm{com}}(d_h(t')) \preceq'_{\mathbf{u}} \mathrm{le}(p+p') = \mathrm{le}(p)$$

with equality for some $g \in I_{\mathbf{w}}$. Hence in particular the corresponding inequalities hold also for the $\mathbf{u}$-degree of the considered elements and we obtain by $\mathbf{u}$-homogeneity of p

$$p = \sum_{g \in I'_{\mathbf{w}}} \mathrm{lt}_{\mathbf{u}}(d_h(a_g)) \, \mathrm{lt}_{\mathbf{u}}(d_h(g)) + \sum_{(t,s,t') \in U'} \mathrm{lt}_{\mathbf{u}}(d_h(t)) \, \mathrm{lt}_{\mathbf{u}}(d_h(s)) \, \mathrm{lt}_{\mathbf{u}}(d_h(t'))$$

for some $I'_{\mathbf{w}} \subseteq I_{\mathbf{w}}$ and $U' \subseteq U$. This shows not only $p \in {}_{T_n}\langle \mathrm{lt}_{\mathbf{u}}(S) \cup \mathrm{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))\rangle_{T_n}$, but also that Definition 2.1.13(aii) is fulfilled by Remark 2.1.15(a): For this first note that $\mathrm{le}_{\prec'_{\mathbf{u}}}(r) = \mathrm{le}_{\prec'_{\mathbf{u}}}(\mathrm{lt}_{\mathbf{u}}(r)) = \mathrm{le}_{\prec'}(\mathrm{lt}_{\mathbf{u}}(r))$ holds for $r \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle$ and thus $\mathrm{le}_{\prec'}(p) = \mathrm{le}_{\prec'_{\mathbf{u}}}(p)$ by $\mathbf{u}$-homogeneity of p. Choosing $g \in I_{\mathbf{w}}$ with equality in Equation (2.2.9), we obtain $\mathrm{le}_{\prec'}(p) = \mathrm{le}_{\prec'}(\mathrm{lt}_{\mathbf{u}}(d_h(a_g))) + \mathrm{le}_{\prec'}(\mathrm{lt}_{\mathbf{u}}(d_h(g))) \in L_{\prec'}(\mathrm{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}})))$. As ${}_{T_n}\langle \mathrm{lt}_{\mathbf{u}}({}_{T_n}\langle S \cup I\rangle_{T_n})\rangle_{T_n}$ is a $\mathbf{u}$-homogeneous ideal, it was enough to consider homogeneous p and we are finished.

(b) Follows by similar arguments as Part (a).

(c) We have canonical graded $\mathbb{K}$-algebra isomorphisms

$$\begin{aligned}\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}A^E \cong (\mathrm{Gr}^{\mathbf{u}}A)^E \cong&(T_n/\left\langle \mathrm{lt}_{\mathbf{u}}(S) \cup \mathrm{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))\right\rangle)^E \\ \cong&T_n^E/\left\langle \mathrm{lt}_{\mathbf{u}}(S)^E \cup \mathrm{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))^E\right\rangle,\end{aligned} \tag{2.2.10}$$

where we put $\overline{(e)} \in T_n^E/\left\langle \mathrm{lt}_{\mathbf{u}}(S)^E \cup \mathrm{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))^E\right\rangle$ in degree $\mathbf{s}_e$. Since by the first isomorphism theorem

$$\mathrm{Gr}_k^{\mathbf{u}[\mathbf{s}]}M \cong (F^{\mathbf{u}}[\mathbf{s}]_k M + F^{\mathbf{u}}[\mathbf{s}]_{k-1}A^E)/F^{\mathbf{u}}[\mathbf{s}]_{k-1}A^E \subseteq \mathrm{Gr}_k^{\mathbf{u}[\mathbf{s}]}A^E, \tag{2.2.11}$$

we may identify $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}M$ with a submodule of $T_n^E/\left\langle \mathrm{lt}_{\mathbf{u}}(S)^E \cup \mathrm{lt}_{\mathbf{u}}(d_h(I_{\mathbf{w}}))^E\right\rangle$.

Under the above identification consider $t \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle$ with $0 \neq \bar{t} \in \mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}M$. As that module is $\mathbf{u}[\mathbf{s}]$-graded and the ordering $\prec^E$ is transitive, we reduce to the case that t is $\mathbf{u}[\mathbf{s}]$-homogeneous. Hence there exists $t' \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^E)\rangle$ with $\deg_{\mathbf{u}[\mathbf{s}]}(t') < \deg_{\mathbf{u}[\mathbf{s}]}(t)$ such that $\overline{t+t'} \in M$. So it holds

$$\overline{t+t'} = \sum_{g\in G} \overline{a_g} \cdot \overline{g} \in M$$

and

$$\mathrm{le}_{(\prec^E_{\mathbf{u}[\mathbf{s}]})_{\mathrm{lcomp}(g)}}(a_g) + \mathrm{le}_{\prec^E_{\mathbf{u}[\mathbf{s}]}}(g) \preceq^E_{\mathbf{u}[\mathbf{s}]} \mathrm{le}_{\prec^E_{\mathbf{u}[\mathbf{s}]}}(t+t') = \mathrm{le}_{\prec^E_{\mathbf{u}[\mathbf{s}]}}(t)$$

for some $a \in {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n^G)\rangle$ by assumption. It follows under the above identification

$$\bar{t} = \sum_{g\in G'} \overline{\mathrm{lt}_{\mathbf{u}}(a_g)} \cdot \overline{\mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(g)} \in \mathrm{Gr}^{\mathbf{u}[\mathbf{s}]}M$$

and

$$\mathrm{le}_{\prec^E_{\mathrm{lcomp}(g)}}(\mathrm{lt}_{\mathbf{u}}(a_g)) + \mathrm{le}_{\prec^E}(\mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(g)) \preceq^E \mathrm{le}_{\prec^E}(\mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(t+t')) = \mathrm{le}_{\prec^E}(\mathrm{lt}_{\mathbf{u}[\mathbf{s}]}(t))$$

for $g \in G' := \{g \in G \mid \deg_{\mathbf{u}[\mathbf{s}]}(g) + \deg_{\mathbf{u}}(a_g) = \deg_{\mathbf{u}[\mathbf{s}]}(t)\}$.

(d) The exact sequence

$$0 \to F^{\mathbf{u}}_{\bullet}\, {}_A\langle M\rangle_A \to F^{\mathbf{u}}_{\bullet}A \to F^{\mathbf{u}}_{\bullet}(A/{}_A\langle M\rangle_A) \to 0$$

induces the claimed isomorphism. The other claim follows by Part (c) and Corollary 2.1.53.

□

Corollary 2.2.29. *If A is a PBW-algebra and $\mathbf{u} \in \mathbb{Z}^n$ a weight vector on A, then $\mathrm{Gr}^{\mathbf{u}} A$ is also a PBW-algebra.*

Proof. By Lemma 2.2.14 the exists a weight vector $\mathbf{w} \in \mathbb{N}^n_{>0}$ on A and Lemma 2.2.13 implies that $A^{h(\mathbf{w})}$ is a PBW-algebra. Now the claim is immediate from Proposition 2.2.28(a). □

Corollary 2.2.30. *Consider the $\mathbb{K}$-algebra $\mathbb{K}\langle \underline{x}, \underline{y}\rangle := \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$ and the elementary PBW-reduction-algebra*

$$B = \mathbb{K}\langle \underline{x}, \underline{y}\rangle / \langle R\rangle \cong \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]/J)\underline{y}^\beta$$

with commutation relations S_B. If $\mathbf{u} \in \mathbb{Z}^{n+m}$ is a weight vector on B, then $\operatorname{Gr}^{\mathbf{u}} B$ is also an elementary PBW-reduction-algebra. More precisely,

$$\operatorname{Gr}^{\mathbf{u}} B \cong \mathbb{K}\langle \underline{x}, \underline{y}\rangle / \left\langle \operatorname{lt}_{\mathbf{u}}(S_B) \cup \operatorname{lt}_{\mathbf{u}}(J')\right\rangle \cong \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]) / \left\langle \operatorname{lt}_{\mathbf{u}}(J')\right\rangle \underline{y}^\beta$$

for a Gröbner basis J' of $J \subseteq \mathbb{K}[\underline{x}]$ with respect to the ordering induced by an ordering of type $\prec'_{\mathbf{u}}$. In particular, every ordering on $\operatorname{Gr}^{\mathbf{w}} B$ is computable.

Proof. Lemma 2.2.15 implies that

$$B^{h(\mathbf{w})} \cong \mathbb{K}\langle h, \underline{x}, \underline{y}\rangle / \left\langle S_B^{h(\mathbf{w})} \cup J''\right\rangle \cong \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[h, \underline{x}] / \left\langle J''\right\rangle)\underline{y}^\beta$$

for some $(1, \mathbf{w})$-homogeneous $J'' \subseteq \mathbb{K}[h, \underline{x}]$ such that J'' is a Gröbner basis of ${}_{\mathbb{K}[h,\underline{x}]}\langle h_{\mathbf{w}}(J)\rangle$ with respect to the ordering induced by $(\prec'_{\mathbf{u}})_{(1,\mathbf{w})}$ on $\mathbb{K}[h, \underline{x}]$. So a corresponding PBW-reduction datum of $B^{h(\mathbf{w})}$ is given by $(\mathbb{K}\langle h, \underline{x}, \underline{y}\rangle, S_B^{h(\mathbf{w})}, J'', (\prec'_{\mathbf{u}})_{(1,\mathbf{w})})$. According to Proposition 2.2.28(a) it follows that $(\mathbb{K}\langle \underline{x}, \underline{y}\rangle, \operatorname{lt}_{\mathbf{u}}(S_B), \operatorname{lt}_{\mathbf{u}}(d_h(J'')), \prec'_{\mathbf{u}})$ is a PBW-reduction datum of $\operatorname{Gr}^{\mathbf{u}} B$. By construction of J'' and as J' is a Gröbner basis of J with respect to the ordering induced by $\prec'_{\mathbf{u}}$, we have

$${}_{\mathbb{K}[\underline{x}]}\left\langle \operatorname{lt}_{\mathbf{u}}(d_h(J''))\right\rangle = {}_{\mathbb{K}[\underline{x}]}\langle \operatorname{lt}_{\mathbf{u}}(J)\rangle = {}_{\mathbb{K}[\underline{x}]}\left\langle \operatorname{lt}_{\mathbf{u}}(J')\right\rangle$$

showing

$$\operatorname{Gr}^{\mathbf{u}} B \cong \mathbb{K}\langle \underline{x}, \underline{y}\rangle / \left\langle \operatorname{lt}_{\mathbf{u}}(S) \cup \operatorname{lt}_{\mathbf{u}}(J')\right\rangle.$$

Using the isomorphism $B \cong \bigoplus_{\beta \in \mathbb{N}^m} (\mathbb{K}[\underline{x}]/J)\underline{y}^\beta$, one easily proves the second isomorphism for $\operatorname{Gr}^{\mathbf{u}} B$. The particular claim is now an immediate consequence of Lemma 2.1.28(b). □

Example 2.2.31. Consider the PBW-reduction-algebra T_X introduced in Example 2.1.30 and its weight vector $\mathbf{w} = ((0)_{1 \le i \le n}, (1)_{1 \le i \le m})$. Then $\operatorname{Gr}^{\mathbf{w}} T_X = (\mathbb{K}\langle \underline{x}, \underline{y}\rangle, \operatorname{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, \prec)$ with $\operatorname{lt}_{\mathbf{w}}(S) = \{[x_j, x_i], [y_l, y_k], [y_k, x_i] \mid 1 \le i \le j \le n, 1 \le k \le l \le m\} \setminus \{0\}$, where $I_{\mathbf{w}}$ is a Gröbner basis of I with respect to the ordering induced by $\prec$ on $\mathbb{K}[\underline{x}]$, since $\operatorname{lt}_{\mathbf{w}}(I) = I$ (see Corollary 2.2.30 and Lemma 2.1.28). In particular, $\operatorname{Gr}^{\mathbf{w}} T_X$ is a quotient algebra of the polynomial ring $\mathbb{K}[\underline{x}, \underline{y}]$ and every ordering on it is computable.

Remark 2.2.32. Note that an A-computable field $\mathbb{K}$ is also $\mathrm{Gr}^{\mathbf{u}} A$-computable.

Algorithm 2.2.33 Given a weight vector $\mathbf{u}$ on A and an A-submodule M of a free A-module, this algorithm computes $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} M$.

Input: A weight vector $\mathbf{u} \in \mathbb{Z}^n$ on A, a finite set E, an A-module $M = {}_A\langle \overline{M'} \rangle \subseteq A^E$ with $M' \subseteq T_n^E$ finite, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and a computable ordering $\prec'_{\mathbf{u}}$.

Output: A PBW-reduction datum $(T_n, \mathrm{lt}_{\mathbf{u}}(S), I_{\mathbf{u}}, \prec')$ of $\mathrm{Gr}^{\mathbf{u}} A$ and a finite set $G \subseteq T_n^E$ of $\mathbf{u}[\mathbf{s}]$-homogeneous elements whose residue classes form $\mathrm{Gr}^{\mathbf{u}} A$-generators of $\mathrm{Gr}^{\mathbf{u}[\mathbf{s}]} M \subseteq T_n^E / \langle \mathrm{lt}_{\mathbf{u}}(S)^E \cup I_{\mathbf{u}}^E \rangle$.

```
Compute a finite set G ⊆ T_n^E inducing a Gröbner basis of M with respect to an ordering
  of type (≺'^E_{pot,<})_{u[s]} by Algorithm 2.2.21.
Set G := lt_{u[s]}(G).
if ≺'_u is a non-well-ordering then
  Find a weight vector w ∈ N^n_{>0} such that PBW-reduction datum A^{h(w)} = (T_n, S', I',
  (≺'_u)_{(1,w)}) is computable.
  Replace I' by the set of the (1, w)-homogeneous parts of its elements.
  Set I' := d_h(I').
else
  Compute a PBW-reduction datum (T_n, S, I', ≺'_u) of A.
return (T_n, lt_u(S), lt_{u[s]}(I'), ≺') and G.
```

2.3 Interplay of weight filtrations and submodule structures of a free module over the PBW-reduction-algebra A

In this section, given two weight vectors $\mathbf{v}$ and $\mathbf{w}$ on a PBW-reduction-algebra A satisfying certain assumptions, we study the interplay of the induced weight filtrations on free A-modules with $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodule structures. While this problem is interesting in this own right, it also serves as an intermediate step to treat the corresponding problem for quotients of free A-modules. The assumptions on our weight vectors as well as the concrete choice of problems in this section are motivated by the applications to Hodge theory we have in mind.

Consider now the following situation: Let $A = (T_n, S, I, \prec)$ with $S = \{x_j x_i = c_{ij} x_i x_j + d_{ij} \mid 1 \leq i < j \leq n\}$ be a PBW-reduction-algebra and $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ two weight vectors on A such that $\mathbf{v}$ is a $\mathbf{w}$-*weight* on A, that is, $F_0^{\mathbf{w}}A \subseteq F_0^{\mathbf{v}}A$. Given a finite set E and $V', W' \subseteq T_n^E$ finite subsets, the subjects of our investigation are the submodules $V := {}_{F_0^{\mathbf{v}}A}\langle \overline{V'} \rangle \subseteq A^E$ and $W := {}_{F_0^{\mathbf{w}}A}\langle \overline{W'} \rangle \subseteq A^E$. To simplify notation, we assume that $\overline{v} = \overline{v'} \in A^E$ for $v, v' \in V'$ implies $v = v'$ (and similarly for W').

In view of implementations, we need for our algorithms and for computability the following additional assumptions:

Assumption 2.3.1.

(a) We can determine a computable ordering of type $\prec'_{\mathbf{v}}$ on A.

(b) We can compute a PBW-reduction-datum for $F_0^{\mathbf{v}}A$. More precisely, we can determine the kernel $K_{\mathbf{v}}$ of the surjective $\mathbb{K}$-algebra map

$$\phi_{\mathbf{v}} : A_{\mathbf{v}} := \mathbb{K}\langle\{y_g \mid g \in G_A^{\mathbf{v}}\}\rangle \to F_0^{\mathbf{v}}A,\ y_g \mapsto g$$

and a PBW-reduction datum for $A_{\mathbf{v}}/K_{\mathbf{v}}$ is computable.

(c) Under the assumption made in Part (b), assume additionally that the filtration $F_\bullet^{\mathbf{w}}$ induced by $F_\bullet^{\mathbf{w}}F_0^{\mathbf{v}}A$ on $A_{\mathbf{v}}/K_{\mathbf{v}}$ is given by a weight vector $\mathbf{w}_{\mathbf{v}}$ on $A_{\mathbf{v}}/K_{\mathbf{v}}$ and that we can determine a computable ordering of type $\prec'_{\mathbf{w}_{\mathbf{v}}}$ on $A_{\mathbf{v}}/K_{\mathbf{v}}$.

(d) For any integer $d \in \mathbb{Z}$ we can determine a finite set of $F_0^{\mathbf{v}}A$-generators $P_d^{A,\mathbf{v}}$ of $F_d^{\mathbf{v}}A$ and $\mathbf{t}_d \in \mathbb{Z}^{P_d^{A,\mathbf{v}}}$ such that $F_\bullet^{\mathbf{w}}F_d^{\mathbf{v}}A = \sum_{p\in P_d^{A,\mathbf{v}}} F^{\mathbf{w}}_{\bullet-(\mathbf{t}_d)_p}F_0^{\mathbf{v}}A \cdot p$.

(e) We have $F_0^{\mathbf{v}}F_\bullet^{\mathbf{w}}A = \overline{F_0^{\mathbf{v}}T_n \cap F_\bullet^{\mathbf{w}}T_n \cap {}_{\mathbb{K}}\langle \mathrm{SMon}(T_n)\rangle}$.

(f) We can determine a computable ordering of type $\prec'_{\mathbf{w}}$ for some well-ordering $\prec'$ on A.

Note that Remark 2.2.6(b) states a sufficient condition for Assumption 2.3.1(d). Moreover, we recall that we agreed on Convention 2.1.52.

Remark 2.3.2. We point out that the given PBW-reduction datum of A allows us to compute Gröbner bases with respect to $\prec$ of A-submodules of free A-modules, to solve module membership problems for such submodules, to compute intersections of such submodules and projections to free submodules and to determine syzygies over A (see Remark 2.1.59). Moreover, Assumption 2.3.1 ensures that we can tackle the following problems:

(a) Assumption 2.3.1(a) enables us to compute generators of the filtration $F_\bullet^{\mathbf{v}}M$ for an A-submodule M of a free A-module. So in particular, we can determine $F_0^{\mathbf{v}}A$-generators of $F_k^{\mathbf{v}}M$ for $k \in \mathbb{Z}$.

(b) Assumption 2.3.1(b) ensures that we can perform the Gröbner basics listed above for A also over the ring $F_0^{\mathbf{v}}A$.

(c) A set of $F_\bullet^{\mathbf{w}}F_0^{\mathbf{v}}A$-generators of the filtration induced by $F_\bullet^{\mathbf{w}}A$ on $F_0^{\mathbf{v}}A$-submodules of free $F_0^{\mathbf{v}}A$-modules is computable by Assumption 2.3.1(c). Similarly, we will see that Assumption 2.3.1(e) allows us to solve the corresponding problem for $F_0^{\mathbf{v}}A$-submodules of free A-modules.

(d) A computable ordering of type $\prec'_{\mathbf{w}}$ on A (see Assumption 2.3.1(f)) enables us to realize the algebra $\mathrm{Gr}^{\mathbf{w}} A$ as PBW-reduction-algebra by Algorithm 2.2.33.

The objective of this section is to treat the following problems:

Problem 2.3.3.

(a) Module membership problem: Decide for $a \in A^E$ if $a \in V$ under Assumption 2.3.1(a) and (b).

(b) Find generators of the $F_0^{\mathbf{w}}A$-module $V \cap W$ under Assumption 2.3.1(a)-(c).

(c) Given that a set as in Assumption 2.3.1(d) exits, show that $V \cap F^{\mathbf{w}}[\mathbf{s}]_\bullet A^E$ is a well-filtered $F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}} A$-module and compute a corresponding generating set under Assumption 2.3.1(a)-(d).

(d) Under Assumption 2.3.1 show that $\mathbf{v}$ is a weight on the PBW-reduction-algebra $\operatorname{Gr}^{\mathbf{w}} A$ and represent $\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V$ as $F_0^{\mathbf{v}} \operatorname{Gr}^{\mathbf{w}} A$-module.

Remark 2.3.4. As $F^{(0)_{1\leq i\leq n}} A = A$, the zero vector $(0)_{1\leq i\leq n}$ is obviously a $\mathbf{u}$-weight for any weight vector $\mathbf{u}$ on A. So solving Problem 2.3.3(b) enables us in particular to compute the intersection of an A-submodule M of A^E with a finitely generated $F_0^{\mathbf{u}}A$-submodule of A^E.

Example 2.3.5. With regard to our applications to Hodge theory, we are particularly interested in the situation of Example 2.1.30 in the case

$$\mathbf{v} = ((-\delta_{n,i})_{1\leq i\leq n}, (\delta_{m,i})_{1\leq i\leq m}) \in \mathbb{Z}^{n+m} \text{ and } \mathbf{w} = ((0)_{1\leq i\leq n}, (1)_{1\leq i\leq m}) \in \mathbb{Z}^{n+m}$$

under the condition that x_n is a local coordinate (see Example 2.1.30(b)). In this case, $F_\bullet^{\mathbf{v}} T_X$ is the so-called V-filtration on $D_X(X)$ with respect to the divisor $\{x_n = 0\}$ and $F_\bullet^{\mathbf{w}} A$ is the filtration with respect to the order of differential operators on $D_X(X)$.

Note that we can indeed determine a PBW-reduction datum for T_X by Example 2.1.30(a). Moreover Assumption 2.3.1 is satisfied: Part (a) follows by Lemma 2.2.15 and we have already seen in Example 2.1.30(b) that $F_0^{\mathbf{v}} T_X$ is isomorphic to the PBW-reduction-algebra T_X^V and how to obtain a corresponding PBW-reduction datum. By Example 2.2.8(d) we know that $\mathbf{w}$ induces the weight vector $\mathbf{w}_{\mathbf{v}} = ((0)_{1\leq i\leq n}, (1)_{1\leq i\leq m})$ on T_X^V. Choosing $P_d^{T_X,\mathbf{v}}$ as in that example, that is,

$$P_d^{T_X,\mathbf{v}} = \begin{cases} \{\overline{x_n^d}\}, & \text{if } d \leq 0 \\ \{\overline{y_m^l} \mid 0 \leq l \leq d\}, & \text{else,} \end{cases}$$

we have by Remark 2.2.6(a) and (b) that

$$F_\bullet^{\mathbf{w}} F_d^{\mathbf{v}} T_X = \begin{cases} F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}} T_X \cdot \overline{x_n^d}, & \text{if } d \leq 0 \\ \sum_{0\leq l\leq d} F_{\bullet-l}^{\mathbf{w}} F_0^{\mathbf{v}} T_X \cdot \overline{y_m^l}, & \text{else} \end{cases}$$

implying that Assumption 2.3.1(d) is satisfied. Remark 2.2.6(a) shows also that Assumption 2.3.1(e) holds in this situation. Finally Assumption 2.3.1(f) is an immediate consequence of Lemma 2.1.28.

We remark that part of the difficulty of the above problems stems from the fact that we have to work with a chain of subrings $F_0^{\mathbf{w}}A \subseteq F_0^{\mathbf{v}}A \subseteq A$ and that finitely generated A-modules are in general not finitely generated as $F_0^{\mathbf{v}}A$-modules. Thus we first explain the transformation of these problems into problems involving only the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ and its subalgebra $F_0^{\mathbf{w}}A$.

2.3.1 A one-to-one correspondence for $F_0^{\mathbf{w}}A$-submodules of bounded $\mathbf{v}$-degree of a free A-module

We will see that for the reduction of Problem 2.3.3 into problems not involving the ring A it is sufficient if we can perform the following task: Given a fixed integer $d \in \mathbb{Z}$ and a finite set E, find a free $F_0^{\mathbf{v}}A$-module of finite rank such that all $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of A^E with $\mathbf{v}$-degree bounded by d can be represented via a one-to-one correspondence as $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of that free $F_0^{\mathbf{v}}A$-module, respectively, and make that one-to-one correspondence algorithmic. Hence we construct in this subsection a surjective $F_0^{\mathbf{v}}A$-linear (and hence also $F_0^{\mathbf{w}}A$-linear) map from such a free $F_0^{\mathbf{v}}A$-module to $F_d^{\mathbf{v}}A$. Then we have by the homomorphism theorem a one-to one correspondence between the $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of $F_d^{\mathbf{v}}A$ and the $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodule of the free module containing the kernel of our surjective map.

Note that we do not need for this one-to-one correspondence any assumptions made in Assumption 2.3.1. However, the algorithmic applications of the technique developed here require Assumption 2.3.1(a).

Remark 2.3.6. The inclusion $F_0^{\mathbf{w}}A \subseteq F_0^{\mathbf{v}}A$ implies that for any finite set $N' \subseteq A^E$ and for $N = {}_{F_0^{\mathbf{w}}A}\langle N' \rangle$

$$\deg_{\mathbf{v}}(N) = \deg_{\mathbf{v}}(N') < \infty.$$

Sometimes, we consider the above problem for $F_0^{\mathbf{v}}A$-modules only, and we do this by formally setting $\mathbf{w} := \mathbf{v}$.

The construction of an $F_0^{\mathbf{v}}A$-linear surjective map from a free $F_0^{\mathbf{v}}A$-module to $F_d^{\mathbf{v}}A$ for $d \in \mathbb{Z}$ works as follows: Choose a finite set of $F_0^{\mathbf{v}}A$-generators $P_d^{A,\mathbf{v}}$ of $F_d^{\mathbf{v}}A$ (see Definition and Remark 2.2.5(c)) and define an $F_0^{\mathbf{v}}A$-linear map by

$$\omega_{\mathbf{v},d} : F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}} \to F_d^{\mathbf{v}}A,\ q \mapsto \sum_{p \in P_d^{A,\mathbf{v}}} q_p p. \tag{2.3.1}$$

By choice of $P_d^{A,\mathbf{v}}$ this map is clearly surjective, and $F_0^{\mathbf{v}}A$-generators $K_{\omega_{\mathbf{v},d}}$ of its kernel can be found as described below: We observe that $a \in F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}}$ is in the kernel of $\omega_{\mathbf{v},d}$ if and only if $\sum_{p \in P_d^{A,\mathbf{v}}} a_p p = 0$, that is, if and only if $a \in \operatorname{syz}_A(P_d^{A,\mathbf{v}}) \cap F_0^{\mathbf{v}}A^{P_d^{A,\mathbf{v}}}$. Hence $K_{\omega_{\mathbf{v},d}}$ can be determined by Algorithm 2.2.27 under Assumption 2.3.1(a).

Next, we define a right inverse map of $\omega_{\mathbf{v},d}$

$$\upsilon_{\mathbf{v},d} : F_d^{\mathbf{v}} A \to F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}}$$

by fixing for every $a \in F_d^{\mathbf{v}} A$ a representation

$$a = \sum_{p \in P_d^{A,\mathbf{v}}} q_p^a p \text{ with } q^a \in F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}} \tag{2.3.2}$$

and setting

$$\upsilon_{\mathbf{v},d} : F_d^{\mathbf{v}} A \to F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}},\ a \mapsto q^a. \tag{2.3.3}$$

Remark 2.3.7. Note that we can compute representations as in Equation (2.3.2) by Definition and Remark 2.2.5(d) given that we have a representative of a in $F_d^{\mathbf{v}} T_n$.

We are finally in the position to formulate the one-to-one correspondence:

Lemma 2.3.8. *Let $d \in \mathbb{Z}$. There is an inclusion-, intersection- and sum-preserving one-to-one correspondence*

$$\begin{aligned}\{F_0^{\mathbf{w}} A\text{-modules } K \subseteq (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E \mid \ker(\omega_{\mathbf{v},d}^E) \subseteq K\} &\leftrightarrow \{F_0^{\mathbf{w}} A\text{-modules } J \subseteq F_d^{\mathbf{v}} A^E\} \\ \Omega_{\mathbf{v},d}^E : \quad K &\mapsto \omega_{\mathbf{v},d}^E(K) \\ \upsilon_{\mathbf{v},d}^E(J) + \ker(\omega_{\mathbf{v},d}^E) &\leftarrow J \quad : Y_{\mathbf{v},d}^E.\end{aligned}$$

This correspondence is compatible with $F_0^{\mathbf{v}} A$-module structure, that is, K is an $F_0^{\mathbf{v}} A$-submodule of $(F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$ if and only if $\omega_{\mathbf{v},d}^E(K)$ is one of $F_d^{\mathbf{v}} A^E$. Moreover, if $K' \subseteq F_d^{\mathbf{v}} A^E$ and $\mathbf{u} \in \{\mathbf{v}, \mathbf{w}\}$, then

$$Y_{\mathbf{v},d}^E({}_{F_0^{\mathbf{u}} A}\langle K' \rangle) = {}_{F_0^{\mathbf{u}} A}\langle \upsilon_{\mathbf{v},d}^E(K') \rangle + \ker(\omega_{\mathbf{v},d}^E).$$

Proof. As $F_0^{\mathbf{v}} A$ is naturally an $F_0^{\mathbf{w}} A$-algebra, $F_d^{\mathbf{v}} A^E$ and $\ker(\omega_{\mathbf{v},d}^E)$ have compatible $F_0^{\mathbf{v}} A$- and $F_0^{\mathbf{w}} A$-module structures. Hence there is by the one-to-one correspondence for submodules of a quotient module an inclusion-, intersection- and sum-preserving bijection of $F_0^{\mathbf{w}} A$-modules

$$\begin{aligned}\{K \subseteq (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E \mid \ker(\omega_{\mathbf{v},d}^E) \subseteq K\} &\leftrightarrow \{J \subseteq (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E / \ker(\omega_{\mathbf{v},d}^E)\} \\ K &\mapsto K/\ker(\omega_{\mathbf{v},d}^E)\end{aligned}$$

with K being an $F_0^{\mathbf{v}} A$-submodule if and only if $K/\ker(\omega_{\mathbf{v},d}^E)$ is an $F_0^{\mathbf{v}} A$-submodule. The claim follows now by the isomorphism $F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}} / \ker(\omega_{\mathbf{v},d}) \cong F_d^{\mathbf{v}} A$. □

The following algorithms compute images of $F_0^{\mathbf{v}} A$- and $F_0^{\mathbf{w}} A$-submodules under the one-to-one correspondence of the above lemma.

Algorithm 2.3.9 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{w}}A$-submodule $M \subseteq A^E$, this algorithm computes $\upsilon_{\mathbf{v},d}^E(M)$ for some $d \geq \deg_{\mathbf{v}}(M)$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight, a finite set E, a computable ordering of type $\prec'_{\mathbf{v}}$ on A, a finite set $M \subseteq T_n^E$ and an optional natural number d'.

Output: Two finite subsets $M', K \subseteq (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$, such that $Y_{\mathbf{v},d}^E({}_{F_0^{\mathbf{u}}A}\langle \overline{M} \rangle) = {}_{F_0^{\mathbf{u}}A}\langle M' \rangle + {}_{F_0^{\mathbf{v}}A}\langle K \rangle$ for $\mathbf{u} \in \{\mathbf{v}, \mathbf{w}\}$ and $\ker(\omega_{\mathbf{v},d}^E) = {}_{F_0^{\mathbf{v}}A}\langle K \rangle$, where $d := \max\{\deg_{\mathbf{v}}(M)[, d']\}$.

Set $d := \max\{\deg_{\mathbf{v}}(M)[, d']\}$ and determine $P_d^{A,\mathbf{v}}$.
$M' := \emptyset$.
for $m \in M$ **do**
  Find $q^m \in (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$ such that $\overline{m} = \sum_{e \in E} \sum_{p \in P_d^{A,\mathbf{v}}} q_{e_p}^m p(e)$ as explained in Definition and Remark 2.2.5(d).
  $M' := M' \cup \{q^m\}$.
Compute $F_0^{\mathbf{v}}A$-generators K of $\operatorname{syz}_A(P_d^{A,\mathbf{v}}) \cap F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}}$ by Algorithm 2.2.27 using the ordering $(\prec'_{\mathbf{v}})_{top,<}^{P_d^{A,\mathbf{v}}}$ for some order $<$ on $P_d^{A,\mathbf{v}}$.
return M', K^E.

In the above algorithm, we mean by $\max\{\deg_{\mathbf{v}}(M)[, d']\}$ the value $\max\{\deg_{\mathbf{v}}(M), d'\}$ if d' is defined and $\deg_{\mathbf{v}}(M)$ otherwise.

Algorithm 2.3.10 Given a weight vector $\mathbf{v}$ on A and a subset $M \subseteq (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$, this algorithm computes $\omega_{\mathbf{v},d}^E(M)$.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A, an integer $d \in \mathbb{Z}$, a finite set E and a finite subset $M \subseteq (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$.

Output: A set $M' \subseteq A^E$ such that $\omega_{\mathbf{v},d}^E(M) = M'$.

Set $M' := \emptyset$.
for $m \in M$ **do**
  $M' := M' \cup \{\sum_{e \in E} \sum_{p \in P_d^{A,\mathbf{v}}} m_{e_p} p(e)\}$.
return M'.

2.3.2 Module membership for $F_0^{\mathbf{v}}A$-submodules of a free A-module

In this subsection, it suffices to assume that Assumption 2.3.1(a) and (b) is satisfied. Recall that $V = {}_{F_0^{\mathbf{v}}A}\langle \overline{V'} \rangle \subseteq A^E$ with $V' \subseteq T_n^E$ finite and consider $a \in T_n^E$. We explain how to check whether

$$\overline{a} \in V, \tag{2.3.4}$$

which is equivalent to

$${}_{F_0^{\mathbf{v}}A}\langle \overline{a} \rangle \subseteq V.$$

Since the $\mathbf{v}$-degree of the above ideals is bounded by $d := \max\{\deg_{\mathbf{v}}(V'), \deg_{\mathbf{v}}(a)\}$ and the one-to-one correspondence in Lemma 2.3.8 is inclusion-preserving, our problem reduces to deciding whether

$$_{F_0^{\mathbf{v}}A}\langle \upsilon^E_{\mathbf{v},d}(\overline{a})\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle \subseteq {}_{F_0^{\mathbf{v}}A}\langle \upsilon^E_{\mathbf{v},d}(\overline{V'})\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle,$$

which is in turn equivalent to

$$\upsilon^E_{\mathbf{v},d}(\overline{a}) \in {}_{F_0^{\mathbf{v}}A}\left\langle \upsilon^E_{\mathbf{v},d}(\overline{V'}) \cup K^E_{\omega_{\mathbf{v},d}}\right\rangle.$$

The above module membership problem can be solved over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ by a normal form computation (see also Remark 2.1.55(a) and Assumption 2.3.1(b)).

Remark 2.3.11. In the particular case $\mathbf{v} \in \mathbb{N}^n$, we can solve the module membership problem also over the PBW-reduction-algebra A: Note that $\overline{a} \in V$ if and only if there is $b \in F_0^{\mathbf{v}}A^{V'}$ such that $a = \sum_{v' \in V'} b_{v'}\overline{v}'$. We can test this by computing a reduced Gröbner basis G of $\operatorname{syz}_A(\{\overline{a}\}, \overline{V'})$ with respect to a well-ordering of type $(\prec^{\{\overline{a}\}}, (\prec'_{\mathbf{v}})^{\overline{V'}})$ (see Proposition 2.1.50) under Assumption 2.3.1(a). Namely, we have $\overline{a} \in V$ if and only if there is $b \in F_0^{\mathbf{v}}A^{V'}$ such that $((\overline{a}), b) \in G$.

The following algorithm checks more generally whether ${}_{F_0^{\mathbf{v}}A}\langle P\rangle \subseteq V$ for $P \subseteq A^E$ finite.

Algorithm 2.3.12 Given a weight vector $\mathbf{v}$ on A and two $F_0^{\mathbf{v}}A$-submodules V, P of a free A-module, this algorithm checks if $P \subseteq V$.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A, such that Assumption 2.3.1(a) and (b) is satisfied, a finite set E and submodules $V := {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle$, $P := {}_{F_0^{\mathbf{v}}A}\langle\overline{P'}\rangle \subseteq A^E$ with $V', P' \subseteq T_n^E$ finite.

Output: `true` if $P \subseteq V$ and `false` else.

Set $d := \max\{\deg_{\mathbf{v}}(V'), \deg_{\mathbf{v}}(P')\}$.
Compute $P'' := \upsilon^E_{\mathbf{v},d}(\overline{P'})$, $V'' := \upsilon^E_{\mathbf{v},d}(\overline{V'})$ and $K := K^E_{\omega_{\mathbf{v},d}}$ using Algorithm 2.3.9.
Set $J := {}_{F_0^{\mathbf{v}}A}\langle V'' \cup K\rangle$.
for $p'' \in P''$ **do**
    if $p'' \notin J$ **then** ▷ Decide using Gröbner basis theory over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ (see Remark 2.1.55(a)).
        return `false`.
return `true`.

Remark 2.3.13. With a little extra bookkeeping the above algorithm can be extended to represent $\overline{p'}$ for $p' \in P'$ as an $F_0^{\mathbf{v}}A$-linear combination of $\overline{V'}$ if $\overline{p'} \in V$.

2.3.3 Intersection of $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of a free A-module

Under Assumption 2.3.1(a)-(c) we develop in this subsection a method based on the one-to-one correspondence introduced Subsection 2.3.1 to compute generators the $F_0^{\mathbf{w}}A$-submodule

$$V \cap W \subseteq A^E,$$

where $V = {}_{F_0^{\mathbf{v}}A}\langle \overline{V'}\rangle$ and $W = {}_{F_0^{\mathbf{w}}A}\langle \overline{W'}\rangle$. Setting $d := \max\{\deg_{\mathbf{v}}(V'), \deg_{\mathbf{v}}(W')\} \in \mathbb{Z}$, we get by the one-to-one correspondence in Lemma 2.3.8

$$V \cap W = \omega^E_{\mathbf{v},d}(J_W \cap J_V),$$

where

$$J_W = {}_{F_0^{\mathbf{wv}}F_0^{\mathbf{v}}A}\langle \upsilon^E_{\mathbf{v},d}(\overline{W'})\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle \tag{2.3.5}$$

and

$$J_V = {}_{F_0^{\mathbf{v}}A}\langle \upsilon^E_{\mathbf{v},d}(\overline{V'})\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle. \tag{2.3.6}$$

Now consider the modules

$$R := \operatorname{syz}_{F_0^{\mathbf{v}}A}\left(\upsilon^E_{\mathbf{v},d}(\overline{W'}), \upsilon^E_{\mathbf{v},d}(\overline{V'}), K^E_{\omega_{\mathbf{v},d}}\right),$$

and

$$R' := \pi_{W'}(R) \cap F_0^{\mathbf{wv}}F_0^{\mathbf{v}}A^{W'},$$

where we implicitly identify $F_0^{\mathbf{v}}A^{W'}$ and $F_0^{\mathbf{v}}A^{V'}$ with $F_0^{\mathbf{v}}A^{\upsilon^E_{\mathbf{v},d}(\overline{W'})}$ and $F_0^{\mathbf{v}}A^{\upsilon^E_{\mathbf{v},d}(\overline{V'})}$, respectively. A set of $F_0^{\mathbf{v}}A$-generators of R can be obtained using Gröbner basis theory over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ (see Lemma 2.1.57). Now we determine by Algorithm 2.2.27 a finite set G such that $R' = {}_{F_0^{\mathbf{wv}}F_0^{\mathbf{v}}A}\langle G\rangle$. We claim:

Lemma 2.3.14. *We have*

$$J_W \cap J_V = {}_{F_0^{\mathbf{wv}}F_0^{\mathbf{v}}A}\left\langle \{ \sum_{w'\in W'} g_{w'}\upsilon^E_{\mathbf{v},d}(\overline{w'}) \mid g \in G\} \right\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle. \tag{2.3.7}$$

Proof. For $q \in J_W \cap J_V$ exist $a \in F_0^{\mathbf{wv}}F_0^{\mathbf{v}}A^{W'}$, $b \in F_0^{\mathbf{v}}A^{V'}$ and $c, c' \in F_0^{\mathbf{v}}A^K$ (with $K := K^E_{\omega_{\mathbf{v},d}}$) such that

$$q = \sum_{w'\in W'} a_{w'}\upsilon^E_{\mathbf{v},d}(\overline{w'}) + \sum_{k\in K} c_k k = \sum_{v'\in V'} b_{v'}\upsilon^E_{\mathbf{v},d}(\overline{v'}) + \sum_{k\in K} c'_k k.$$

This implies that $(a, -b, c - c') \in R$. By the choice of G, there is $f \in F_0^{\mathbf{wv}}F_0^{\mathbf{v}}A^G$ such that $a = \sum_{g\in G} f_g g$ and hence $\sum_{w'\in W'} a_{w'}\upsilon^E_{\mathbf{v},d}(\overline{w'}) = \sum_{g\in G} f_g \sum_{w'\in W'} g_{w'}\upsilon^E_{\mathbf{v},d}(\overline{w'})$, which is in the right hand side of Equation (2.3.7). As the other inclusion is obvious, that concludes the proof. □

Since ${}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle = \ker(\omega^E_{\mathbf{v},d})$, we have by Lemma 2.3.8:

Corollary 2.3.15. $V \cap W = {}_{F_0^{\mathbf{w}}A}\langle\{\sum_{w'\in W'} g_{w'}\overline{w'} \mid g \in G\}\rangle$.

Algorithm 2.3.16 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A, an $F_0^{\mathbf{v}}A$-submodule V and an $F_0^{\mathbf{w}}A$-submodule W of a free A-module, this algorithm computes the intersection $V \cap W$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1(a)-(c) is satisfied, a finite set E, submodules $V := {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle, W := {}_{F_0^{\mathbf{w}}A}\langle\overline{W'}\rangle \subseteq A^E$ with $V', W' \subseteq T_n^E$ finite.

Output: A finite set $G \subseteq A^E$ such that ${}_{F_0^{\mathbf{w}}A}\langle G\rangle = V \cap W$.

Set $d := \max\{\deg_{\mathbf{v}}(V'), \{\deg_{\mathbf{v}}(W')\}$.

Compute $V'' := \upsilon^E_{\mathbf{v},d}(\overline{V'})$, $W'' := \upsilon^E_{\mathbf{v},d}(\overline{W'})$ and $K := K^E_{\omega_{\mathbf{v},d}}$ by Algorithm 2.3.9.

Find $R := \operatorname{syz}_{F_0^{\mathbf{v}}A}(W'', V'', K) \subseteq F_0^{\mathbf{v}}A^{W'\sqcup V'\sqcup K}$ (under the above identification) over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ using Gröbner basis theory.

Determine G' such that ${}_{F_0^{\mathbf{w}_\mathbf{v}}F_0^{\mathbf{v}}A}\langle G'\rangle = \pi_{W'}(R) \cap F_0^{\mathbf{w}_\mathbf{v}}F_0^{\mathbf{v}}A^{W'}$ via Algorithm 2.2.27 by working over $F_0^{\mathbf{v}}A$.

Set $G := \{\sum_{w'\in W'} g'_{w'}w' \mid g' \in G'\}$.

return G.

Remark 2.3.17. We remark that similar methods as above can be employed to intersect two finitely generated $F_0^{\mathbf{w}}A$-submodules of a free A-module. However, if an ordering of type $\prec_{\mathbf{w}}$ and a PBW-reduction datum for $F_0^{\mathbf{w}}A$ are computable, it might be preferable to work over the ring $F_0^{\mathbf{w}}A$.

By setting $\mathbf{w} := \mathbf{v}$, Algorithm 2.3.16 enables us to determine the intersection of finitely generated $F_0^{\mathbf{v}}A$-modules. In this case, we do not need to apply Algorithm 2.2.27.

In the case $V = F^{\mathbf{v}}[\mathbf{s}]_k A^E$ for $k \in \mathbb{Z}$, we simplify our method as follows. In view of later applications, we treat a slightly more general case: Namely, assume that $W = {}_{F_0^{\mathbf{w}}A}\langle\overline{W'}\rangle + {}_{F_0^{\mathbf{v}}A}\langle\overline{U'}\rangle \subseteq A^E$ (with $U' \subseteq T_n^E$ finite) is a sum of a finitely generated $F_0^{\mathbf{w}}A$-submodule and a finitely generated $F_0^{\mathbf{v}}A$-submodule of A^E. Replacing d by $\max\{\deg_{\mathbf{v}}(U'), \deg_{\mathbf{v}}(W'), k - \min\{\mathbf{s}_e \mid e \in E\}\}$, assume now that $P_d^{A,\mathbf{v}}$ has been chosen such that $P^{A,\mathbf{v}}_{k-\mathbf{s}_e} \subseteq P_d^{A,\mathbf{v}}$ for $e \in E$. If we keep our other notations, we have to replace Equations (2.3.5) and (2.3.6) by

$$J_W = {}_{F_0^{\mathbf{w}_\mathbf{v}}F_0^{\mathbf{v}}A}\langle\upsilon^E_{\mathbf{v},d}(\overline{W'})\rangle + {}_{F_0^{\mathbf{v}}A}\langle\upsilon^E_{\mathbf{v},d}(\overline{U'})\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle$$

and

$$J_V = \bigoplus_{e\in E} F_0^{\mathbf{v}}A^{P^{A,\mathbf{v}}_{k-\mathbf{s}_e}} + {}_{F_0^{\mathbf{v}}A}\left\langle K^E_{\omega_{\mathbf{v},d}}\right\rangle,$$

where we naturally identify $\bigoplus_{e\in E} F_0^{\mathbf{v}} A^{P_{k-\mathbf{s}_e}^{A,\mathbf{v}}}$ with a free $F_0^{\mathbf{v}} A$-submodule of $(F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$. We denote by

$$\pi^E_{P_d^{A,\mathbf{v}}\setminus P_{k-\mathbf{s}_e}^{A,\mathbf{v}}} : (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E \to \bigoplus_{e\in E} F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}\setminus P_{k-\mathbf{s}_e}^{A,\mathbf{v}}}$$

the projection to the complement of this submodule. Abbreviating $C_e := P_d^{A,\mathbf{v}} \setminus P_{k-\mathbf{s}_e}^{A,\mathbf{v}}$, we consider

$$T := \operatorname{syz}_{F_0^{\mathbf{v}} A}\left(\bigsqcup_{w'\in W'} \{\pi^E_{C_e}(\upsilon^E_{\mathbf{v},d}(\overline{w'}))\}, \bigsqcup_{u'\in U'} \{\pi^E_{C_e}(\upsilon^E_{\mathbf{v},d}(\overline{u'}))\}, \pi^E_{C_e}(K^E_{\omega_{\mathbf{v},d}}) \right),$$

and

$$T' := \pi_{W',U'}(T) \cap (F_0^{\mathbf{w}_{\mathbf{v}}} F_0^{\mathbf{v}} A^{W'} \oplus F_0^{\mathbf{v}} A^{U'}),$$

where we identify $A^{W'}$ and $A^{U'}$ with $A^{\bigsqcup_{w'\in W'}\{\pi^E_{C_e}(\upsilon^E_{\mathbf{v},d}(\overline{w'}))\}}$ and $A^{\bigsqcup_{u'\in U'}\{\pi^E_{C_e}(\upsilon^E_{\mathbf{v},d}(\overline{u'}))\}}$, respectively. Finally, we determine $F_0^{\mathbf{v}} A$-generators of T as well as G and G' such that

$$T' = {}_{F_0^{\mathbf{w}_{\mathbf{v}}} F_0^{\mathbf{v}} A}\langle G\rangle + {}_{F_0^{\mathbf{v}} A}\langle G'\rangle \text{ and } \pi_{W'}(G') = 0$$

by working over the PBW-reduction-algebra $F_0^{\mathbf{v}} A$ and using Algorithm 2.2.27 to compute G and G' and claim:

Lemma 2.3.18. *Identifying $A^{W'\sqcup U'}$ with $A^{W'} \oplus A^{U'}$, we have*

$$V \cap W = {}_{F_0^{\mathbf{w}} A}\left\langle \{ \sum_{w'\in W'} g_{w'}\overline{w'} + \sum_{u'\in U'} g_{u'}\overline{u'} \mid g \in G\} \right\rangle + {}_{F_0^{\mathbf{v}} A}\left\langle \{ \sum_{u'\in U'} g'_{u'}\overline{u'} \mid g' \in G'\} \right\rangle.$$

Proof. We observe that

$$J_W \cap J_V = \left(J_W \cap \bigoplus_{e\in E} F_0^{\mathbf{v}} A^{P_{k-\mathbf{s}_e}^{A,\mathbf{v}}} \right) + {}_{F_0^{\mathbf{v}} A}\left\langle K^E_{\omega_{\mathbf{v},d}} \right\rangle.$$

So consider $q = \sum_{w'\in W'} a_{w'}\overline{w'} + \sum_{u'\in U'} b_{u'}\overline{u'} + \sum_{k\in K} c_k k \in J_W$ with $K := K^E_{\omega_{\mathbf{v},d}}$, $a \in F_0^{\mathbf{w}_{\mathbf{v}}} F_0^{\mathbf{v}} A^{W'}, b \in F_0^{\mathbf{v}} A^{U'}, c \in F_0^{\mathbf{v}} A^K$. We have $q \in \bigoplus_{e\in E} F_0^{\mathbf{v}} A^{P_{k-\mathbf{s}_e}^{A,\mathbf{v}}}$ if and only if $\pi^E_{C_e}(q) = 0$, that is,

$$(a,b,c) \in \operatorname{syz}_{F_0^{\mathbf{v}} A}\left(\bigsqcup_{w'\in W'} \{\pi^E_{C_e}(\upsilon^E_{\mathbf{v},d}(\overline{w'}))\}, \bigsqcup_{u'\in U'} \{\pi^E_{C_e}(\upsilon^E_{\mathbf{v},d}(\overline{u'}))\}, \bigsqcup_{k\in K} \{\pi^E_{C_e}(k)\} \right).$$

This in turn is equivalent to $(a,b) \in \pi_{W',U'}(T) \cap (F_0^{\mathbf{w}} F_0^{\mathbf{v}} A^{W'} \oplus F_0^{\mathbf{v}} A^{U'})$ as claimed. □

This leads to the algorithm below:

Algorithm 2.3.19 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A, a sum $W \subseteq A^E$ of an $F_0^{\mathbf{v}}A$-submodule and an $F_0^{\mathbf{w}}A$-submodule of a free A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{v}}[\mathbf{s}]_k W$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{v}$ is a $\mathbf{w}$-weight on A and such that Assumption 2.3.1(a)-(c) is satisfied, a finite set E, a submodule $W := {}_{F_0^{\mathbf{w}}A}\langle\overline{W'}\rangle + {}_{F_0^{\mathbf{v}}A}\langle\overline{U'}\rangle \subseteq A^E$ with $U', W' \subseteq T_n^E$ finite, $\mathbf{s} \in \mathbb{Z}^E$ a shift vector and $k \in \mathbb{Z}$.

Output: Two finite sets $H, H' \subseteq A^E$ such that $W \cap F^{\mathbf{v}}[\mathbf{s}]_k A^E = {}_{F_0^{\mathbf{w}}A}\langle H\rangle + {}_{F_0^{\mathbf{v}}A}\langle H'\rangle$ and $H' \subseteq {}_{F_0^{\mathbf{v}}A}\langle\overline{U'}\rangle$.

1: Set $d := \max\{\deg_{\mathbf{v}}(U'), \deg_{\mathbf{v}}(W'), k - \min\{\mathbf{s}_e \mid e \in E\}\}$. $\triangleright$ $\deg_{\mathbf{v}}(F^{\mathbf{v}}[\mathbf{s}]_k A^E) = k - \min\{\mathbf{s}_e \mid e \in E\}$.
2: Choose $P_d^{A,\mathbf{v}}$ such that $P_{k-\mathbf{s}_e}^{A,\mathbf{v}} \subseteq P_d^{A,\mathbf{v}}$ for $e \in E$.
3: Apply Algorithm 2.3.9 with the above choice of $P_d^{A,\mathbf{v}}$ to obtain $W'' := \upsilon_{\mathbf{v},d}^E(\overline{W'})$, $U'' := \upsilon_{\mathbf{v},d}^E(\overline{U'})$ and $K := K_{\omega_{\mathbf{v},d}}^E$.
4: Find $T := \operatorname{syz}_{F_0^{\mathbf{v}}A}(\bigsqcup_{w''\in W''}\{\pi_{C_e}^E(w'')\}, \bigsqcup_{u''\in U''}\{\pi_{C_e}^E(u'')\}, \pi_{C_e}^E(K))$ (and identify it with a subset of $F_0^{\mathbf{v}}A^{W'\sqcup U'\sqcup \pi_{C_e}^E(K)}$) via Gröbner basis theory over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$.
5: Determine G, G' such that $\pi_{W',U'}(T) \cap (F_0^{\mathbf{w}_{\mathbf{v}}}F_0^{\mathbf{v}}A^{W'} \oplus F_0^{\mathbf{v}}A^{U'}) = {}_{F_0^{\mathbf{w}_{\mathbf{v}}}F_0^{\mathbf{v}}A}\langle G\rangle + {}_{F_0^{\mathbf{v}}A}\langle G'\rangle$ using Algorithm 2.2.27 by working over $F_0^{\mathbf{v}}A$. $\triangleright$ $\pi_{W'}(G') = 0$.
6: Define $H := \{\sum_{w'\in W'} g_{w'}\overline{w'} + \sum_{u'\in U'} g_{u'}\overline{u'} \mid g \in G\}$ and $H' := \{\sum_{u'\in U'} g'_{u'}\overline{u'} \mid g' \in G'\}$.
7: **return** H, H'.

2.3.4 Induced w-weight filtration on $F_0^{\mathbf{v}}A$-submodules of a free A-module

This subsection is dedicated to computing $F_{\bullet}^{\mathbf{w}}F_0^{\mathbf{v}}A$-generators of the module

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = V \cap F^{\mathbf{w}}[\mathbf{s}]_{\bullet}A^E$$

under Assumption 2.3.1(a)-(d), where $V = {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle$ with $V' \subseteq T_n^E$ finite and $\mathbf{s} \in \mathbb{Z}^E$ stands for a shift vector. Setting $d := \deg_{\mathbf{v}}(V')$, we obtain

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = V \cap F^{\mathbf{w}}[\mathbf{s}]_{\bullet}F_d^{\mathbf{v}}A^E.$$

Since the $\mathbf{v}$-degree of $F^{\mathbf{w}}[\mathbf{s}]_k F_d^{\mathbf{v}}A^E$ for all $k \in \mathbb{Z}$ is bounded by d, we proceed similarly as in Subsection 2.3.3. If we choose $P_d^{A,\mathbf{v}}$ and $\mathbf{t}_d \in \mathbb{Z}^{P_d^{A,\mathbf{v}}}$ as postulated in Assumption 2.3.1(d), that is, with the property $F_{\bullet}^{\mathbf{w}}F_d^{\mathbf{v}}A = \sum_{p\in P_d^{A,\mathbf{v}}} F_{\bullet-(\mathbf{t}_d)_p}^{\mathbf{w}}F_0^{\mathbf{v}}A \cdot p$, then we get under the one-to-one correspondence in Lemma 2.3.8

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = \omega_{\mathbf{v},d}^E(J_V \cap J_{F^{\mathbf{w}}[\mathbf{s}]_{\bullet}}),$$

where

$$J_V = {}_{F_0^{\mathbf{v}}A}\langle \upsilon_{\mathbf{v},d}^E(\overline{V'})\rangle + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E \right\rangle$$

and

$$J_{F^{\mathbf{w}}[\mathbf{s}]_\bullet} = F^{\mathbf{w}_\mathbf{v}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E \right\rangle,$$

with $\mathbf{t}_{e_p} = \mathbf{s}_e + (\mathbf{t}_d)_p$ for $e \in E$, $p \in P_d^{A,\mathbf{v}}$. Consequently, we obtain

$$J_V \cap J_{F^{\mathbf{w}}[\mathbf{s}]_\bullet} = \left(J_V \cap F^{\mathbf{w}_\mathbf{v}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E\right) + {}_{F_0^{\mathbf{v}}A}\left\langle K_{\omega_{\mathbf{v},d}}^E \right\rangle.$$

Applying Algorithm 2.2.26 over the PBW-reduction-algebra $F_0^{\mathbf{v}}A \cong A_{\mathbf{v}}/K_{\mathbf{v}}$, we determine a finite set $G \subseteq (A_{\mathbf{v}}^{P_d^{A,\mathbf{v}}})^E$ such that

$$J_V \cap F^{\mathbf{w}_\mathbf{v}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E = \sum_{g\in G} F^{\mathbf{w}_\mathbf{v}}_{\bullet-\deg_{\mathbf{w}_\mathbf{v}[\mathbf{t}]}(g)} F_0^{\mathbf{v}} A \cdot \overline{g}.$$

This implies that

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}_\mathbf{v}[\mathbf{t}]}(g)} F_0^{\mathbf{v}} A \cdot \omega_{\mathbf{v},d}^E(\overline{g}) = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(\omega_{\mathbf{v},d}^E(g))} F_0^{\mathbf{v}} A \cdot \omega_{\mathbf{v},d}^E(\overline{g})$$

since $\deg_{\mathbf{w}[\mathbf{s}]}(\omega_{\mathbf{v},d}^E(\overline{g})) \leq \deg_{\mathbf{w}_\mathbf{v}[\mathbf{t}]}(g)$ and since the right hand side module of the above equation is obviously contained in the left hand side module of that equation. We summarize the computation:

Algorithm 2.3.20 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a free A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_\bullet V$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1(a)-(d) is satisfied, a finite set E, a submodule $V := {}_{F_0^{\mathbf{v}}A}\langle \overline{V'} \rangle \subseteq A^E$ with $V' \subseteq T_n^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: A finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ such that $F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\mathbf{t}_g} F_0^{\mathbf{v}} A \cdot g = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot g$.

1: Set $d := \deg_{\mathbf{v}}(V')$.
2: Choose $P_d^{A,\mathbf{v}}$ and $\mathbf{t}_d \in \mathbb{Z}^{P_d^{A,\mathbf{v}}}$ such that $F_\bullet^{\mathbf{w}} F_d^{\mathbf{v}} A = \sum_{p\in P_d^{A,\mathbf{v}}} F^{\mathbf{w}}_{\bullet-(\mathbf{t}_d)_p} F_0^{\mathbf{v}} A \cdot p$.
3: Compute $V'' := \upsilon_{\mathbf{v},d}^E(\overline{V'})$ and $K := K_{\omega_{\mathbf{v},d}}^E$ using Algorithm 2.3.9.
4: Define the shift vector $\mathbf{t} \in (\mathbb{Z}^{P_d^{A,\mathbf{v}}})^E$ by $\mathbf{t}_{e_p} = \mathbf{s}_e + (\mathbf{t}_d)_p$ for $e \in E$ and $p \in P_d^{A,\mathbf{v}}$.
5: Find $G' \subseteq (A_{\mathbf{v}}^{P_d^{A,\mathbf{v}}})^E$ such that $\sum_{g'\in G'} F^{\mathbf{w}_\mathbf{v}}_{\bullet-\deg_{\mathbf{w}_\mathbf{v}[\mathbf{t}]}(g')} F_0^{\mathbf{v}} A \cdot \overline{g'} = {}_{F_0^{\mathbf{v}}A}\langle V'' \cup K \rangle \cap F^{\mathbf{w}_\mathbf{v}}[\mathbf{t}]_\bullet (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$ using Algorithm 2.2.26 by working over $F_0^{\mathbf{v}}A$.
6: Define $\mathbf{t}' \in \mathbb{Z}^{G'}$ by $\mathbf{t}'_g := \deg_{\mathbf{w}_\mathbf{v}[\mathbf{t}]}(g')$ for $g \in G'$.

7: Compute $G := \omega^E_{\mathbf{v},d}(\overline{G'})$ by applying Algorithm 2.3.10 and define $\mathbf{t}'' \in \mathbb{Z}^G$ by $\mathbf{t}''_g := \min\{\mathbf{t}'_{g'} \mid g' \in G' \text{ with } \omega^E_{\mathbf{v},d}(\overline{g'}) = g\}$.
8: **return** $G, \mathbf{t}$.

Remark 2.3.21. Note that we can compute for $g \in G$ in the output of the above algorithm a representative $g' \in T_n^E$ with $\deg_{\mathbf{w}[\mathbf{s}]}(g') \leq \mathbf{t}_g$. The same holds also for Algorithm 2.3.22.

Alternatively to Algorithm 2.3.16, we hence compute $V \cap F^{\mathbf{w}}[\mathbf{s}]_k A^E$ as follows:

Algorithm 2.3.22 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a free A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_k V$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1(a)-(d) is satisfied, a finite set E, a submodule $V := {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle \subseteq A^E$ with $V' \subseteq T_n^E$ finite, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and $k \in \mathbb{Z}$.
Output: A finite set $G \subseteq A^E$ such that $V \cap F^{\mathbf{w}}[\mathbf{s}]_k A^E = {}_{F_0^{\mathbf{w}}A}\langle G\rangle$.
1: Set $d := \deg_{\mathbf{v}}(V')$.
2: Choose $P_d^{A,\mathbf{v}}$ and $\mathbf{t}_d \in \mathbb{Z}^{P_d^{A,\mathbf{v}}}$ such that $F^{\mathbf{w}}_\bullet F^{\mathbf{v}}_d A = \sum_{p \in P_d^{A,\mathbf{v}}} F^{\mathbf{w}}_{\bullet-(\mathbf{t}_d)_p} F_0^{\mathbf{v}} A \cdot p$.
3: Compute $V'' := \upsilon^E_{\mathbf{v},d}(\overline{V'})$ and $K := K^E_{\omega_{\mathbf{v},d}}$ using Algorithm 2.3.9.
4: Define the shift vector $\mathbf{t} \in (\mathbb{Z}^{P_d^{A,\mathbf{v}}})^E$ by $\mathbf{t}_{e_p} = \mathbf{s}_e + (\mathbf{t}_d)_p$ for $e \in E$ and $p \in P_d^{A,\mathbf{v}}$.
5: Find $F_0^{\mathbf{w}_\mathbf{v}} F_0^{\mathbf{v}} A$-generators G' of ${}_{F_0^{\mathbf{v}}A}\langle V'' \cup K\rangle \cap F^{\mathbf{w}_\mathbf{v}}[\mathbf{t}]_k (F_0^{\mathbf{v}} A^{P_d^{A,\mathbf{v}}})^E$ over the PBW-reduction-algebra $F_0^{\mathbf{v}}A$ using Algorithm 2.2.27.
6: Compute $G := \omega^E_{\mathbf{v},d}(G')$ by applying Algorithm 2.3.10.
7: **return** G.

While the advantage of the above algorithm over Algorithm 2.3.16 is that we omit the syzygy computation involved in the latter algorithm, the latter algorithm does not require Assumption 2.3.1(d) or any particular choice of $P_d^{A,\mathbf{v}}$.

2.3.5 Associated graded modules to $\mathbf{w}$-weight filtered $F_0^{\mathbf{v}}A$-submodules of a free A-module

We explain how to express $\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V$ for $V = {}_{F_0^{\mathbf{v}}A}\langle V'\rangle$ as a finitely generated $F_0^{\mathbf{v}} \operatorname{Gr}^{\mathbf{w}} A$-module under Assumption 2.3.1.

Proposition 2.3.23. *Let* $\mathbf{s} \in \mathbb{Z}^E$ *be a shift vector and* $\operatorname{Gr}^{\mathbf{w}} A = (T_n, \operatorname{lt}_{\mathbf{w}}(S), J, \prec')$ *under the identification made in Proposition 2.2.28*(a).

(a) The vector $\mathbf{v}$ *is a weight vector on the PBW-reduction-algebra* $(T_n, \operatorname{lt}_{\mathbf{w}}(S), J, \prec')$ *satisfying* $\operatorname{Gr}^{\mathbf{w}} F_0^{\mathbf{v}} A \cong F_0^{\mathbf{v}}(T_n / \langle \operatorname{lt}_{\mathbf{w}}(S) \cup J\rangle)$.

(b) *We may consider* $\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V$ *as an* $F_0^{\mathbf{v}} \operatorname{Gr}^{\mathbf{w}} A$*-submodule of* $T_n^E / \left\langle \operatorname{lt}_{\mathbf{w}}(S)^E \cup J^E \right\rangle$*, where we put* $\overline{(e)}$ *in degree* $\mathbf{s}_e$*. If* $G \subseteq T_n^E$ *is finite with* $F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot \overline{g}$*, then* $\overline{\operatorname{lt}_{\mathbf{w}[\mathbf{s}]}(G)} \subseteq T_n^E / \left\langle \operatorname{lt}_{\mathbf{w}}(S)^E \cup J^E \right\rangle$ *is a set of* $F_0^{\mathbf{v}} \operatorname{Gr}^{\mathbf{w}} A$*-generators of* $\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V$ *under the above identification.*

Proof.

(a) First note that for $k \in \mathbb{Z}$

$$\operatorname{Gr}_k^{\mathbf{w}} F_0^{\mathbf{v}} A = F_k^{\mathbf{w}} F_0^{\mathbf{v}} A / F_{k-1}^{\mathbf{w}} F_0^{\mathbf{v}} A \cong (F_0^{\mathbf{v}} F_k^{\mathbf{w}} A + F_{k-1}^{\mathbf{w}} A)/F_{k-1}^{\mathbf{w}} A = F_0^{\mathbf{v}} \operatorname{Gr}_k^{\mathbf{w}} A$$

and that $\mathbf{v}$ is a weight vector on the PBW-reduction-algebra $(T_n, \operatorname{lt}_{\mathbf{w}}(S), J, \prec')$, because it is one on A. Recall the identification of $T_n / \langle \operatorname{lt}_{\mathbf{w}}(S) \cup J \rangle$ with $\operatorname{Gr}^{\mathbf{w}} A$ is induced by the map

$$\psi : T_n \to \operatorname{Gr}^{\mathbf{w}} A, x_{i_1} \cdots x_{i_k} \mapsto \overline{x_{i_1} \cdots x_{i_k}} + F^{\mathbf{w}}_{\deg_{\mathbf{w}}(x_{i_1} \cdots x_{i_k})-1} A$$

(see the proof of Proposition 2.2.28(a)).Thus the map ψ induces by virtue of $\overline{F_0^{\mathbf{v}} F_\bullet^{\mathbf{w}} A} = \overline{F_0^{\mathbf{v}} T_n \cap F_\bullet^{\mathbf{w}} T_n}$ (see Assumption 2.3.1(e)) the isomorphism

$$F_0^{\mathbf{v}}(T_n / \langle \operatorname{lt}_{\mathbf{w}}(S) \cup J \rangle) \cong F_0^{\mathbf{v}} \operatorname{Gr}^{\mathbf{w}} A.$$

(b) Part (a) allows us to consider

$$\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V \cong \bigoplus_{j \in \mathbb{Z}} \left(F^{\mathbf{w}}[\mathbf{s}]_j V + F^{\mathbf{w}}[\mathbf{s}]_{j-1} A^E \right) / F^{\mathbf{w}}[\mathbf{s}]_{j-1} A^E$$

as an $F_0^{\mathbf{v}} \operatorname{Gr}^{\mathbf{w}} A$-submodule of $T_n^E / \left\langle \operatorname{lt}_{\mathbf{w}}(S)^E \cup J^E \right\rangle$, where $\overline{(e)}$ has degree $\mathbf{s}_e$.

The equality $F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot \overline{g} = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(\overline{g})} F_0^{\mathbf{v}} A \cdot \overline{g}$ implies that the $\sigma^{\mathbf{w}[\mathbf{s}]}(\overline{g})$ for $g \in G$ are $\operatorname{Gr}^{\mathbf{w}} F_0 A$-generators of $\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V$. The claim follows now by the above isomorphism, Part (a) and the identification made in Proposition 2.2.28(a).

□

Note that Assumption 2.3.1(a)-(d) enables us to find G as in the above proposition yielding the following algorithm:

Algorithm 2.3.24 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}} A$-submodule V of a free A-module with shift vector $\mathbf{s}$, this algorithm computes $\operatorname{Gr}^{\mathbf{w}[\mathbf{s}]} V$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1 is satisfied, a finite set E, an $F_0^{\mathbf{v}} A$-module $V = {}_{F_0^{\mathbf{v}} A}\langle \overline{V'} \rangle \subseteq A^E$ with $V' \subseteq T_n^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: A PBW-reduction datum $(T_n, \mathrm{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, \prec')$ of $\mathrm{Gr}^{\mathbf{w}} A$ and a finite $\mathbf{w}[\mathbf{s}]$-homogeneous set $G \subseteq T_n^E$ inducing $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A$-generators of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]} V \subseteq T_n^E / \langle \mathrm{lt}_{\mathbf{w}}(S)^E \cup I_{\mathbf{w}}^E \rangle$.

1: Compute a PBW-reduction datum $(T_n, \mathrm{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, \prec')$ of $\mathrm{Gr}^{\mathbf{w}} A$ via Algorithm 2.2.33.
2: Determine a finite set $G \subseteq T_n^E$ satisfying $F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot \overline{g}$ by Algorithm 2.3.20 and Remark 2.3.21.
3: Set $G := \mathrm{lt}_{\mathbf{w}[\mathbf{s}]}(G)$.
4: **return** $(T_n, \mathrm{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, \prec')$ and G.

Example 2.3.25. In the situation of Example 2.2.31 consider the weight $\mathbf{v} = ((-\delta_{in})_{1 \le i \le n}, (\delta_{im})_{1 \le i \le m})$ on $\mathrm{Gr}^{\mathbf{w}} T_X = (\mathbb{K}\langle \underline{x}, \underline{y} \rangle, \mathrm{lt}_{\mathbf{w}}(S), I_{\mathbf{w}}, \prec)$. Arguing as in Example 2.1.30(b), we see that $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} T_X$ is isomorphic to $(\mathbb{K}\langle \underline{x}, y_1, \dots, y_{m-1}, z \rangle, S_{\mathbf{v}}, I_{\mathbf{w}}, \prec_0)$, where $S_{\mathbf{v}} = \{[x_j, x_i], [y_l, y_k], [y_k, x_i], [z, x_i], [z, y_m] \mid 1 \le i \le j \le n, 1 \le k \le l \le m-1\} \setminus \{0\}$ and $\prec_0$ is any well-ordering such that its restriction to $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle)$ agrees with the restriction of $\prec$ to $\mathrm{SMon}(\mathbb{K}\langle \underline{x} \rangle)$. Here the isomorphism is defined by sending $\underline{x}^\alpha y_1^{\beta_1} \cdots y_{m-1}^{\beta_{m-1}} (x_n y_m)^\gamma$ to $\underline{x}^\alpha y_1^{\beta_1} \cdots y_{m-1}^{\beta_{m-1}} z^\gamma$.

2.4 Interplay of weight filtrations and submodule structures of a module over the PBW-reduction-algebra A

Given two weight vectors $\mathbf{v}$ and $\mathbf{w}$ on a PBW-reduction-algebra A that satisfy certain assumptions, the purpose of this section is to extend the methods from the previous section to quotients of free A-modules. Considering such a quotient A^E/L, the main problem here is that L has in general unbounded $\mathbf{v}$-degree and is hence not compatible with the one-to-one correspondence from Lemma 2.3.8. However, in many cases it suffices to consider $F_d^{\mathbf{v}} L$ for a suitable integer d allowing us to reduce our problems to the setting of the previous section.

We study in this section the following situation: Let $A = (T_n, S, I, \prec)$ be a PBW-reduction-algebra with $S := \{x_j x_i = c_{ij} x_i x_j + d_{ij} \mid 1 \le i < j \le n\}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ two weight vectors on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight. Given a finite set E and $L', V', W' \subseteq A^E$ finite subsets, $L := {}_A\langle L' \rangle$ and $M = A^E/L$, we consider the $F_\bullet^{\mathbf{v}} A$- and $F_\bullet^{\mathbf{w}} A$-submodules

$$V := {}_{F_0^{\mathbf{v}} A}\langle \overline{V'} \rangle \subseteq M \text{ and } W := {}_{F_0^{\mathbf{w}} A}\langle \overline{W'} \rangle \subseteq M,$$

respectively. Note that every finite set $N \subseteq A^E$ can be considered as a residue class of a finite set in T_n^E and similarly every element $a \in A^E$ is the residue class of an element in T_n^E. We denote such a set and element by N_T and a_T, respectively.

In addition to Assumption 2.3.1, we need the following supplementary assumption for one of the problems that we consider in this section:

Assumption 2.4.1. Assumption 2.3.1(a) and (b) holds if we replace A by $\mathrm{Gr}^{\mathbf{w}} A$.

We will develop in this section algorithms that solve the following problems:

Problem 2.4.2.

(a) Represent V as a quotient of a free $F_0^{\mathbf{v}}A$-module under Assumption 2.3.1(a).

(b) Module-membership problem: Check for $m \in A^E$ whether $\overline{m} \in V$ given that Assumption 2.3.1(a) and (b) holds.

(c) Compute the intersection $V \cap W$ if Assumption 2.3.1(a)-(c) is satisfied.

(d) Given that the $F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}}A$-filtration $V \cap F^{\mathbf{w}}[\mathbf{s}]_\bullet M$ is good and that Assumption 2.3.1 and Assumption 2.4.1 are fulfilled, determine generators of that filtration.

Example 2.4.3. We have already seen in Example 2.3.5 that Assumption 2.3.1 is in the setting of Example 2.1.30, under the condition that x_n is a local coordinate, satisfied. Moreover, Assumption 2.4.1 holds in this situation by Example 2.2.31 and Example 2.3.25.

2.4.1 $F_0^{\mathbf{v}}A$-presentations of $F_0^{\mathbf{v}}A$-submodules of an A-module

In this subsection, we only require that $\mathbf{v}$ is a weight vector on A and that Assumption 2.3.1(a) holds. To represent V as a quotient of a free $F_0^{\mathbf{v}}A$-module, where $V = {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle \subseteq M = A^E/{}_A\langle L'\rangle$, we proceed as follows: Note that the surjective $F_0^{\mathbf{v}}A$-linear morphism φ given by

$$\varphi : F_0^{\mathbf{v}}A^{V'} \to V,\ (v') \mapsto \overline{v'}$$

induces an isomorphism of $F_0^{\mathbf{v}}A$-modules $V \cong F_0^{\mathbf{v}}A^{V'}/\ker(\varphi)$. We have that $a \in F_0^{\mathbf{v}}A^{V'}$ is in the kernel of φ if and only if $\sum_{v' \in V'} a_{v'}v' \in L$, that is, there exists $b \in A^{L'}$ such that $\sum_{v' \in V'} a_{v'}v' = \sum_{l' \in L'} b_{l'}l'$. This implies that

$$\ker(\varphi) = \pi_{V'}(\operatorname{syz}_A(V', L')) \cap F_0^{\mathbf{v}}A^{V'},$$

where the above intersection is computable by Algorithm 2.2.27. Hence we obtain:

Algorithm 2.4.4 Given a weight vector $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a finitely presented A-module, this algorithm represents V as a quotient of a free $F_0^{\mathbf{v}}A$-module.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A such that Assumption 2.3.1(a) holds, a finite set E, an A-module $M := A^E/{}_A\langle L'\rangle$ and a submodule $V := {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle \subseteq M$ with $L', V' \subseteq A^E$ finite.

Output: A finite set $Q \subseteq F_0^{\mathbf{v}}A^{V'}$ such that $F_0^{\mathbf{v}}A^{V'}/{}_{F_0^{\mathbf{v}}A}\langle Q\rangle \cong V$ via $\overline{a} \mapsto \overline{\sum_{v' \in V'} a_{v'}v'}$.

1: Compute an A-generating set S of $\operatorname{syz}_A(V', L')$ using Gröbner basis theory.
2: Set $S' := \pi_{V'}(S)$.
3: Compute an $F_0^{\mathbf{v}}A$-generating set Q of ${}_A\langle S'\rangle \cap F_0^{\mathbf{v}}A^{V'}$ by Algorithm 2.2.27.
4: **return** Q.

2.4.2 Module membership for $F_0^{\mathbf{v}}A$-submodules of an A-module

Assume in this subsection that Assumption 2.3.1(a) and (b) is satisfied. Recall that $M = A^E/L$ and $V = {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle \subseteq M$. We explain how to check for $a \in A^E$ whether $\overline{a} \in V$, which is equivalent to

$$a \in {}_{F_0^{\mathbf{v}}A}\langle V'\rangle + L.$$

Setting $d := \max\{\deg_{\mathbf{v}}(V'_T), \deg_{\mathbf{v}}(a_T)\}$, we have $\deg_{\mathbf{v}}(a), \deg_{\mathbf{v}}(V) \leq d$ and hence the above condition is in turn equivalent to

$$a \in {}_{F_0^{\mathbf{v}}A}\langle V'\rangle + (L \cap F_d^{\mathbf{v}}A^E). \tag{2.4.1}$$

An $F_0^{\mathbf{v}}A$-generating set L'' of the above intersection can be determined by Algorithm 2.2.27, reducing the problem to deciding whether

$$a \in {}_{F_0^{\mathbf{v}}A}\langle V' \cup L''\rangle.$$

This problem is solvable by Algorithm 2.3.12.

Algorithm 2.4.5 Given a weight vector $\mathbf{v}$ on A and two $F_0^{\mathbf{v}}A$-submodules V and P of a finitely presented A-module, this algorithm checks if $P \subseteq V$.

Input: A weight vector $\mathbf{v} \in \mathbb{Z}^n$ on A such that Assumption 2.3.1(a) and (b) holds, a finite set E, a module $M = A^E/{}_A\langle L'\rangle$ and submodules $V := {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle, P := {}_{F_0^{\mathbf{v}}A}\langle\overline{P'}\rangle \subseteq M$ with $L', V', P' \subseteq A^E$ finite.

Output: `true` if $P \subseteq V$ and `false` else.

1: Set $d := \max\{\deg_{\mathbf{v}}(V'_T), \deg_{\mathbf{v}}(P'_T)\}$.
2: Compute a set L'' of $F_0^{\mathbf{v}}A$-generators of ${}_A\langle L'\rangle \cap F_d^{\mathbf{v}}A^E$ using Algorithm 2.2.27.
3: **if** $P' \subseteq {}_{F_0^{\mathbf{v}}A}\langle V' \cup L''\rangle$ **then** ▷ Decide by Algorithm 2.3.12
4:     **return** `true`.
5: **return** `false`.

Remark 2.4.6. By Remark 2.3.13 the above algorithm can be extended to represent $\overline{p'} \in P'$ as an $F_0^{\mathbf{v}}A$-linear combination of $\overline{V'}$ if $p \in V$..

2.4.3 Intersection of $F_0^{\mathbf{v}}A$- and $F_0^{\mathbf{w}}A$-submodules of an A-module

Considering the A-module $M = A^E/L$ (where $L = {}_A\langle L'\rangle$) and its submodules $V = {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle$ and $W = {}_{F_0^{\mathbf{w}}A}\langle\overline{W'}\rangle$, we explain in this subsection how to compute the $F_0^{\mathbf{w}}A$-submodule

$$W \cap V \subseteq M$$

under Assumption 2.3.1(a)-(c). Since

$$W \cap V = \overline{{}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap \left({}_{F_0^{\mathbf{v}}A}\langle V' \rangle + L\right)} \subseteq M, \tag{2.4.2}$$

the problem of determining $W \cap V$ reduces to the computation of the intersection of the left $F_0^{\mathbf{w}}A$-module ${}_{F_0^{\mathbf{w}}A}\langle W' \rangle$ with the sum of the A-module L and the $F_0^{\mathbf{v}}A$-module ${}_{F_0^{\mathbf{v}}A}\langle V' \rangle$, that is, we have to compute

$$I := {}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap \left({}_{F_0^{\mathbf{v}}A}\langle V' \rangle + L\right).$$

To tackle this task, we transform the above problem into an intersection of a finitely generated $F_0^{\mathbf{w}}A$-module with a finitely generated $F_0^{\mathbf{v}}A$-module this way reducing to the situation in Subsection 2.3.3. Since $F_0^{\mathbf{w}}A^E \subseteq F_0^{\mathbf{v}}A^E$, we have $\deg_{\mathbf{v}}({}_{F_0^{\mathbf{w}}A}\langle W' \rangle) \leq \deg_{\mathbf{v}}(W'_T) < \infty$ by Remark 2.3.6. Setting $d := \max\{\deg_{\mathbf{v}}(V'_T), \deg_{\mathbf{v}}(W'_T)\}$, we obtain that

$$I = {}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap \left({}_{F_0^{\mathbf{v}}A}\langle V' \rangle + (L \cap F_d^{\mathbf{v}}A^E)\right),$$

where we find a finite set of $F_0^{\mathbf{v}}A$-generators L'' of $L \cap F_d^{\mathbf{v}}A$ by Algorithm 2.2.27. Thus

$$I = {}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap {}_{F_0^{\mathbf{v}}A}\langle V' \cup L'' \rangle$$

reduces the problem to Subsection 2.3.3 and we obtain the following algorithm:

Algorithm 2.4.7 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A, an $F_0^{\mathbf{v}}A$-submodule V and an $F_0^{\mathbf{w}}A$-submodule W of a finitely presented A-module, this algorithm computes $V \cap W$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1(a)-(c) is satisfied, a finite set E, an A-module $M := A^E / {}_A\langle L' \rangle$, submodules $V := {}_{F_0^{\mathbf{v}}A}\langle \overline{V'} \rangle$, $W := {}_{F_0^{\mathbf{w}}A}\langle \overline{W'} \rangle \subseteq M$ with $L', V', W' \subseteq A^E$ finite.

Output: A finite set $G \subseteq A^E$ such that $V \cap W = {}_{F_0^{\mathbf{w}}A}\langle \overline{G} \rangle$.

1: Set $d := \max\{\deg_{\mathbf{v}}(V'_T), \deg_{\mathbf{v}}(W'_T)\}$.
2: Determine $F_0^{\mathbf{v}}A$-generators L'' of ${}_A\langle L' \rangle \cap F_d^{\mathbf{v}}A^E$ using Algorithm 2.2.27.
3: Compute a set of $F_0^{\mathbf{w}}A$-generators G of ${}_{F_0^{\mathbf{w}}A}\langle W' \rangle \cap {}_{F_0^{\mathbf{v}}A}\langle V' \cup L'' \rangle$ by Algorithm 2.3.16.
4: **return** G.

In the case $W = F^{\mathbf{w}}[\mathbf{s}]_k M = \overline{F^{\mathbf{w}}[\mathbf{s}]_k A^E}$ (with $\mathbf{s} \in \mathbb{Z}^E$ and $k \in \mathbb{Z}$), we can also replace Algorithm 2.3.16 by Algorithm 2.3.22 if Assumption 2.3.1(d) additionally holds:

Algorithm 2.4.8 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a finitely presented A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_k V$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1(a)-(d) holds, a finite set E, an A-module $M := A^E / {}_A\langle L' \rangle$, a submodule $V := {}_{F_0^{\mathbf{v}}A}\langle \overline{V'} \rangle \subseteq M$ with $L', V' \subseteq A^E$ finite, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and $k \in \mathbb{Z}$.

Output: A finite set $G \subseteq A^E$ with $\deg_{\mathbf{w}[\mathbf{s}]}(G) \leq k$ such that $V \cap F^{\mathbf{w}}[\mathbf{s}]_k M = {}_{F_0^{\mathbf{w}}A}\langle \overline{G} \rangle$.

1: Set $d' := \max\{\deg_{\mathbf{v}}((P^{A,\mathbf{w}}_{k-\mathbf{s}_e})_T) \mid e \in E\}$. $\triangleright$ $\deg_{\mathbf{v}}(F^{\mathbf{w}}[\mathbf{s}]_k A^E) \leq d'$.
2: Set $d := \max\{d', \deg_{\mathbf{v}}(V'_T)\}$.
3: Determine a set of $F_0^{\mathbf{v}}A$-generators L'' of ${}_A\langle L' \rangle \cap F_d^{\mathbf{v}} A^E$ using Algorithm 2.2.27.
4: Find a set of $F_0^{\mathbf{w}}A$-generators G of $F^{\mathbf{w}}[\mathbf{s}]_k A^E \cap {}_{F_0^{\mathbf{v}}A}\langle V' \cup L'' \rangle$ by Algorithm 2.3.22.
5: **return** G.

Remark 2.4.9. While we were able to reduce the computation of $F^{\mathbf{w}}[\mathbf{s}]_k M \cap V$ to Subsection 2.3.4, we cannot use a similar approach to determine $F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}} A$-generators of $F^{\mathbf{w}}[\mathbf{s}]_\bullet M \cap V$ (in fact, we do not even know whether a finite set of generators exists): Our reduction step made use of the fact that the $\mathbf{v}$-degree of V' and $F^{\mathbf{w}}[\mathbf{s}]_k A^E$ is bounded in order to consider only the elements of L up to a fixed $\mathbf{v}$-degree. But the $\mathbf{v}$-degree of $F^{\mathbf{w}}[\mathbf{s}]_\bullet A^E$ is only bounded if $\mathbf{v} \in \mathbb{Z}^n_{\leq 0}$. (In the latter case, we have $F_0^{\mathbf{v}} A = A$ and hence we could solve our problem using Algorithm 2.2.26.)

However, if we replace in the above algorithm Algorithm 2.3.22 by Algorithm 2.3.20, we compute for fixed $k \in \mathbb{Z}$ a finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ such that

$$F^{\mathbf{w}}[\mathbf{s}]_{k'} M \cap V = \sum_{g \in G} F^{\mathbf{w}}_{k' - \mathbf{t}_g} F_0^{\mathbf{v}} A \cdot \overline{g} = \sum_{g \in G} F^{\mathbf{w}}_{k' - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot \overline{g}$$

for $k' \leq k$. We also remark that the output G of Algorithm 2.3.20 satisfies

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet \, {}_{F_0^{\mathbf{v}}A}\langle G \rangle = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \mathbf{t}_g} F_0^{\mathbf{v}} A \cdot g = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot g.$$

Moreover, it is possible to determine a representative g_T of $g \in G$ with $\deg_{\mathbf{w}[\mathbf{s}]}(g) \leq \mathbf{t}_g$.

If a finite set of $F_\bullet^{\mathbf{w}} F_0^{\mathbf{v}} A$-generators of $F^{\mathbf{w}}[\mathbf{s}]_\bullet M \cap V$ exists, it will be eventually contained in $F^{\mathbf{w}}[\mathbf{s}]_k M \cap V$ for k large enough. While we cannot detect if such a set does not exist, we can decide whether it is contained in $F^{\mathbf{w}}[\mathbf{s}]_k M \cap V$ as we will explain in Subsection 2.4.4. For this, we need to modify Algorithm 2.4.8 as explained above:

Algorithm 2.4.10 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a finitely presented A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_k V$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1(a)-(d) holds, a finite set E, an A-module $M := A^E/{}_A\langle L'\rangle$, a submodule $V := {}_{F_0^{\mathbf{v}}A}\langle\overline{V'}\rangle \subseteq M$ with $L', V' \subseteq A^E$ finite, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and $k \in \mathbb{Z}$.

Output: A finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ with $F^{\mathbf{w}}[\mathbf{s}]_{k'}V = \sum_{g\in G} F^{\mathbf{w}}_{k'-\mathbf{t}_g} F_0^{\mathbf{v}}A \cdot \overline{g} = \sum_{g\in G} F^{\mathbf{w}}_{k'-\deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}}A \cdot \overline{g}$ for $k' \leq k$ and $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}\, {}_{F_0^{\mathbf{v}}A}\langle G\rangle = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\mathbf{t}_g} F_0^{\mathbf{v}}A \cdot g = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}}A \cdot g$.

Set $d' := \max\{\deg_{\mathbf{v}}((P^{A,\mathbf{w}}_{k-\mathbf{s}_e})_T) \mid e \in E\}$. $\triangleright \deg_{\mathbf{v}}(F^{\mathbf{w}}[\mathbf{s}]_k A^E) \leq d'$.
Set $d := \max\{d', \deg_{\mathbf{v}}(V'_T)\}$.
Determine a set of $F_0^{\mathbf{v}}A$-generators L'' of ${}_A\langle L'\rangle \cap F_d^{\mathbf{v}}A^E$ using Algorithm 2.2.27.
Compute a finite set $G \subseteq A^E$ and a vector $\mathbf{t} \in \mathbb{Z}^G$ satisfying $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}\, {}_{F_0^{\mathbf{v}}A}\langle V' \cup L''\rangle = \sum_{g\in G} F^{\mathbf{w}}_{\bullet-\mathbf{t}_g} F_0^{\mathbf{v}}A \cdot g$ by Algorithm 2.3.20.
return $G, \mathbf{t}$.

Remark 2.4.11. Note that we have for the output of the above algorithm also

$$F^{\mathbf{w}}[\mathbf{s}]_{k'}V = \sum_{g\in G} F^{\mathbf{w}}_{k'-\deg_{\mathbf{w}[\mathbf{s}]}(\overline{g})} F_0^{\mathbf{v}}A \cdot \overline{g}$$

for $k' \leq k$. Given that $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V$ is separated, we compute $\deg_{\mathbf{w}[\mathbf{s}]}(\overline{g})$, which is bounded from above by $\mathbf{t}_g$, for $g \in G$ under Assumption 2.3.1(a) and (b) for $\mathbf{w}$ (instead of $\mathbf{v}$) as follows: We observe that we can solve the module membership problem $\overline{g} \in F^{\mathbf{w}}[\mathbf{s}]_{k'}V$ for $k' < \mathbf{t}_g$ by Algorithm 2.4.5 (if we replace $\mathbf{v}$ by $\mathbf{w}$ in that algorithm). Thus we test this stepwise for $k' = \mathbf{t}_g - 1, \mathbf{t}_g - 2, \ldots$ until the test fails, hence implying $\deg_{\mathbf{w}[\mathbf{s}]}(\overline{g}) = k' + 1$. Having assumed that the filtration is separated, this process stops eventually. If the filtration were not separated, this process might not terminate and we have no method to detect this.

Now consider the case $V = F_k^{\mathbf{v}}[\mathbf{s}]M$. Rewriting Equation (2.4.2) as

$$W \cap V = \overline{\left({}_{F_0^{\mathbf{w}}A}\langle W'\rangle + {}_{F_0^{\mathbf{v}}A}\langle L''\rangle\right) \cap F^{\mathbf{v}}[\mathbf{s}]_k A^E} \subseteq M,$$

our problem reduces to Algorithm 2.3.19.

Algorithm 2.4.12 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{w}}A$-submodule W of a finitely presented A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{v}}[\mathbf{s}]_k W$.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1(a)-(c) holds, a finite set E, an A-module $M := A^E/{}_A\langle L'\rangle$, a submodule $W := {}_{F_0^{\mathbf{w}}A}\langle\overline{W'}\rangle \subseteq M$ with $L', W' \subseteq A^E$ finite, a shift vector $\mathbf{s} \in \mathbb{Z}^E$ and $k \in \mathbb{Z}$.

Output: A finite set $G \subseteq A^E$ such that $F^{\mathbf{v}}[\mathbf{s}]_k M \cap W = {}_{F_0^{\mathbf{w}}A}\langle\overline{G}\rangle$.

Set $d' := k - \min\{\mathbf{s}_e \mid e \in E\}$. $\triangleright$ $\deg_{\mathbf{v}}(F^{\mathbf{v}}[\mathbf{s}]_k A^E) \leq d'$.
Set $d := \max\{d', \deg_{\mathbf{v}}(W_T')\}$.
Determine $F_0^{\mathbf{v}}A$-generators L'' of ${}_A\langle L'\rangle \cap F_d^{\mathbf{v}}A^E$ by Algorithm 2.2.27.
Compute a set of $F_0^{\mathbf{w}}A$-generators G of $({}_{F_0^{\mathbf{w}}A}\langle W'\rangle + {}_{F_0^{\mathbf{v}}A}\langle L''\rangle) \cap F^{\mathbf{v}}[\mathbf{s}]_k A^E$ using Algorithm 2.3.19.
return G.

2.4.4 Induced $\mathbf{w}$-weight filtration on $F_0^{\mathbf{v}}A$-submodules of an A-module

Recall that $V = {}_{F_0^{\mathbf{v}}A}\langle \overline{V'}\rangle$ is an $F_0^{\mathbf{v}}A$-submodule of $M = A^E/L$ (with $L = {}_A\langle L'\rangle$) and $\mathbf{s} \in \mathbb{Z}^E$ a shift vector. As already mentioned in Remark 2.4.9, we cannot decide whether $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}M \cap V$ has a finite set of $F_{\bullet}^{\mathbf{w}}F_0^{\mathbf{v}}A$-generators. However, given that such a finite set exists and that Assumption 2.3.1 and Assumption 2.4.1 hold, which we assume from now on, such a set is computable.

Our method is based on the idea to approximate

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = F^{\mathbf{w}}[\mathbf{s}]_{\bullet}^{s}\left(({}_{F_0^{\mathbf{v}}A}\langle V'\rangle + L)/L\right)$$

using quotients filtrations $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}^{q(V_k)}V$ (for $k > N$ for some fixed $N \in \mathbb{Z}$) for a certain increasing sequence of finitely generated $F_0^{\mathbf{v}}A$-modules $V_k \subseteq {}_{F_0^{\mathbf{v}}A}\langle V'\rangle + L$ with the property that we have equality $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = F^{\mathbf{w}}[\mathbf{s}]_{\bullet}^{q(V_k)}V$ for k big enough (see Proposition 1.1.15 and the discussion thereafter). The choice of the V_k is based on the fact that if a finite set of $F_{\bullet}^{\mathbf{w}}F_0^{\mathbf{v}}A$-generators of $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V$ exists, then these generators have $\mathbf{w}[\mathbf{s}]$-degrees smaller or equal than k for $k \in \mathbb{Z}$ large enough and are thus contained in $F^{\mathbf{w}}[\mathbf{s}]_kV$. Recall that we can already compute for fixed $k \in \mathbb{Z}$ a set $V_k' \subseteq A^E$ such that

$$F^{\mathbf{w}}[\mathbf{s}]_{k'}V = \sum_{v \in V_k'} F^{\mathbf{w}}_{k'-\deg_{\mathbf{w}[\mathbf{s}]}(v)}F_0^{\mathbf{v}}A \cdot \overline{v} \tag{2.4.3}$$

for $k' \leq k$ and

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet}\, {}_{F_0^{\mathbf{v}}A}\langle V_k'\rangle = \sum_{v \in V_k'} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(v)}F_0^{\mathbf{v}}A \cdot v \tag{2.4.4}$$

(see Remark 2.4.9). If $F^{\mathbf{w}}[\mathbf{s}]_kV$ is a set of $F_0^{\mathbf{v}}A$-generators of V, we choose $V_k = {}_{F_0^{\mathbf{v}}A}\langle V_k'\rangle$ and

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet}^{q(V_k)}V = \sum_{v \in V_k'} F^{\mathbf{w}}_{\bullet-\deg_{\mathbf{w}[\mathbf{s}]}(v)}F_0^{\mathbf{v}}A \cdot \overline{v}$$

is well-defined. While we could check if $F^{\mathbf{w}}[\mathbf{s}]_kV$ (or equivalently V_k') is a such a set of $F_0^{\mathbf{v}}A$-generators via Algorithm 2.4.5, we can also ensure this property by choosing $k \geq \deg_{\mathbf{w}[\mathbf{s}]}(V_T')$. Assuming this is the case, we derive from Proposition 1.1.15 the following criterion for the equality $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}V = F^{\mathbf{w}}[\mathbf{s}]_{\bullet}^{q(V_k)}V$:

Proposition 2.4.13. *Assume that* ${}_{F_0^{\mathbf{v}} A}\left\langle \overline{V_k'} \right\rangle = V$. *Then we have*

$$F^{\mathbf{w}}[\mathbf{s}]_{\bullet} V = \sum_{v \in V_k'} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(v)} F_0^{\mathbf{v}} A \cdot \overline{v} \tag{2.4.5}$$

if and only if

$$\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V_k) \cap \mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(L) = \mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V_k \cap L). \tag{2.4.6}$$

Once we have determined finite $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A$-generating sets of the intersection on the left hand side of Equation (2.4.6) and of the right hand side module of that equation, we can decide whether these module are equal using Algorithm 2.3.12, because a PBW-reduction datum of $\mathrm{Gr}^{\mathbf{w}} A$ is computable by Algorithm 2.2.33 and Assumption 2.3.1(f) and Assumption 2.4.1 is satisfied. We compute $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A$- and $\mathrm{Gr}^{\mathbf{w}} A$-generators of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}({}_{F_0^{\mathbf{v}} A}\langle V_k \rangle) \subseteq (\mathrm{Gr}^{\mathbf{w}} A)^E$ and $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(L) \subseteq (\mathrm{Gr}^{\mathbf{w}} A)^E$ by Algorithm 2.3.24 and Algorithm 2.2.33, respectively. We note that we may skip the second step of Algorithm 2.3.24 for the former generators since V_k is already of the desired form. Then we intersect these two modules by Algorithm 2.3.16 using Remark 2.3.4. On the other hand, we obtain $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}({}_{F_0^{\mathbf{v}} A}\langle V_k \rangle \cap L)$ by first applying Algorithm 2.3.16 and Remark 2.3.4 to get $F_0^{\mathbf{v}} A$-generators of ${}_{F_0^{\mathbf{v}} A}\langle V_k \rangle \cap L$ and then using Algorithm 2.3.24.

This leads to the following algorithm:

Algorithm 2.4.14 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A, an A-submodule L and an $F_0^{\mathbf{v}} A$-submodule V of a free A-module with shift vector $\mathbf{s}$, this algorithm checks whether the quotient and the submodule filtration induced by $F^{\mathbf{w}}[\mathbf{s}]_{\bullet}$ on $(V+L)/L$ agree.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1 and Assumption 2.4.1 are satisfied, a finite set E, submodules $L = {}_A\langle L' \rangle$ and $V = {}_{F_0^{\mathbf{v}} A}\langle V' \rangle \subseteq A^E$ with $L', V' \subseteq A^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: `true` if $F^{\mathbf{w}}[\mathbf{s}]^{s}_{\bullet}(V + L/L) = F^{\mathbf{w}}[\mathbf{s}]^{q(V)}_{\bullet}(V + L/L)$ and `false` else.

Find $\mathrm{Gr}^{\mathbf{w}} A$-generators L'' of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(L)$ by Algorithm 2.2.33.
Compute $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A$-generators V'' of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}(V)$ via Algorithm 2.3.24.
Find $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A$-generators J of the intersection ${}_{F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A}\langle V'' \rangle \cap {}_{\mathrm{Gr}^{\mathbf{w}} A}\langle L'' \rangle$ using Algorithm 2.3.16 and Remark 2.3.4. ▷ $\mathrm{Gr}^{\mathbf{w}} A$ is PBW-reduction-algebra.
Compute $F_0^{\mathbf{v}} A$-generators K of $L \cap V$ by Algorithm 2.3.16 and Remark 2.3.4.
Determine $F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A$-generators K' of $\mathrm{Gr}^{\mathbf{w}[\mathbf{s}]}({}_{F_0^{\mathbf{v}} A}\langle K \rangle)$ via Algorithm 2.3.24.
if $J \subseteq {}_{F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A}\langle K' \rangle$ **then** ▷ Check by Algorithm 2.3.12.
  return `true`. ▷ $K' \subseteq {}_{F_0^{\mathbf{v}} \mathrm{Gr}^{\mathbf{w}} A}\langle J \rangle$ is always satisfied.
return `false`.

Thus given that $F^{\mathbf{w}}[\mathbf{s}]_{\bullet} V$ is a well-filtered $F^{\mathbf{w}}[\mathbf{s}]_{\bullet} F_0^{\mathbf{v}} A$-module, the following algorithm determines generators of this filtration:

Algorithm 2.4.15 Given a $\mathbf{w}$-weight $\mathbf{v}$ on A and an $F_0^{\mathbf{v}}A$-submodule V of a finitely presented A-module with shift vector $\mathbf{s}$, this algorithm computes $F^{\mathbf{w}}[\mathbf{s}]_\bullet V$ if this filtration is a good filtration.

Input: Two weight vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^n$ on A such that $\mathbf{v}$ is a $\mathbf{w}$-weight and such that Assumption 2.3.1 and Assumption 2.4.1 are satisfied, a finite set E, an A-module $M := A^E/L$ with $L = {}_A\langle L' \rangle$, a submodule $V = {}_{F_0^{\mathbf{v}}A}\langle \overline{V'} \rangle \subseteq M$ with $L', V' \subseteq A^E$ finite and a shift vector $\mathbf{s} \in \mathbb{Z}^E$.

Output: A finite set $G \subseteq A^E$ and $\mathbf{t} \in \mathbb{Z}^G$ such that $F^{\mathbf{w}}[\mathbf{s}]_\bullet M \cap V = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \mathbf{t}_g} F_0^{\mathbf{v}} A \cdot \overline{g} = \sum_{g\in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot \overline{g}$ if such a finite set exists.

1: Choose $k \in \mathbb{Z}$ such that $F^{\mathbf{w}}[\mathbf{s}]_k V$ is a set of $F_0^{\mathbf{v}}A$-generators of V. ▷ E.g. take $k = \deg_{\mathbf{w}[\mathbf{s}]}(V'_T)$.
2: Initialize an empty set $G \subseteq A^E$ and a dynamic vector $\mathbf{t} \in \mathbb{Z}^G$.
3: **while** $F^{\mathbf{w}}[\mathbf{s}]_\bullet V \neq \sum_{g\in G} F^{\mathbf{w}}_{\bullet - \mathbf{t}_g} F_0^{\mathbf{v}} A \cdot g$ **do** ▷ Test by Algorithm 2.4.14.
4: Compute a finite set $G' \subseteq A^E$ and $\mathbf{t}' \in \mathbb{Z}^{G'}$ with $F^{\mathbf{w}}[\mathbf{s}]_{k'} V = \sum_{g \in G'} F^{\mathbf{w}}_{k' - \mathbf{t}'_g} F_0^{\mathbf{v}} A \cdot \overline{g} = \sum_{g \in G'} F^{\mathbf{w}}_{k' - \deg_{\mathbf{w}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} A \cdot \overline{g}$ for $k' \leq k$ using Algorithm 2.4.10 and replace G by G' and $\mathbf{t}$ by $\mathbf{t}'$.
5: Increase k.
6: **return** $G, \mathbf{t}$.

Remark 2.4.16. We have a few remarks on the above algorithm:

(a) If $F^{\mathbf{w}}[\mathbf{s}]_\bullet M \cap V$ were not well-filtered, the algorithm would not terminate.

(b) If we apply Algorithm 2.4.14 multiple times during the execution of Algorithm 2.4.15, we only need to perform the first step of Algorithm 2.4.14 once.

(c) The output of Algorithm 2.4.15 also satisfies

$$F^{\mathbf{w}}[\mathbf{s}]_\bullet V = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{\mathbf{w}[\mathbf{s}]}(\overline{g})} F_0^{\mathbf{v}} A \cdot \overline{g}.$$

If $F^{\mathbf{w}}[\mathbf{s}]_\bullet V$ is separated, we can compute $\deg_{\mathbf{w}[\mathbf{s}]}(\overline{g})$ as explained in Remark 2.4.11.

3 (Strictly) specializable $\mathcal{D}$-modules

The (rational) V-filtration on $\mathcal{D}_X$-modules along a smooth pure codimension one subvariety $X_0 \subseteq X$ is an essential ingredient of the theory of mixed Hodge modules and plays a key role in the computation of Hodge theoretic direct images. We call $\mathcal{D}_X$-modules that possess such a filtration X_0-specializable. Hodge $\mathcal{D}_X$-modules do not only admit a V-filtration, but their Hodge filtration also behaves "well" with respect to this V-filtration making them an example of so-called strictly X_0-specialize $\mathcal{D}_X$-modules. V-filtrations are used to define (filtered) localizations and dual localizations of (strictly) X_0-specializable $\mathcal{D}_X$-modules along X_0. Similar concepts for $\mathcal{D}_X(*X_0)$-modules are applied to construct Hodge theoretic direct images under the open embedding defined by the complement of X_0. While these functors agree with the corresponding $\mathcal{D}$-module theoretic functors, the construction of the filtration on the (dual) localizations and the direct images is subtle.

This chapter lays the theoretical foundation for the algorithms that we present in the next chapter. We review many concepts and results involving V-filtrations, localizations and dual localizations mainly due to Saito or Sabbah (see in particular [Sai88] and [SS17]), and apply them to prepare the algorithmic treatment of the these constructions on a sheaf-theoretic level. In the next chapter we then develop actual algorithms for these problems using the computational theory of weight-filtered PBW-reduction-algebras presented in Chapter 2.

More precisely, given a smooth equidimensional variety X and a pure codimension one subvariety X_0, this chapter is dedicated to the following: In Section 3.1 we treat the unfiltered situation, that is, $\mathcal{D}$-modules without an order filtration, by first introducing the V-filtration on coherent $\mathcal{D}_X$ and $\mathcal{D}(*X_0)$-modules along smooth X_0 and reviewing its main properties. After that, in preparation of the algorithmic computation of the V-filtration, we give a local realization of this filtration relying on so-called local b-functions. Next we describe the localization and dual localizations of X_0-specializable $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules using certain parts of the V-filtration. Then we extend the concept of X_0-specializability to singular X_0 by locally considering certain graph embeddings. Such graph embeddings enable us also to reduce the constructions of localizations and dual localizations to the smooth case. Section 3.2 is dedicated to the analogous constructions in a filtered setting. We first establish for smooth X_0 a notation of strict X_0-specializability in the case of $(\mathcal{D}_X, F^\circ_\bullet)$-modules: Loosely speaking, strict X_0-specializability of a well-filtered $(\mathcal{D}_X, F^\circ_\bullet)$-module $(\mathcal{M}, F_\bullet)$ means in particular that the filtration $F_\bullet$ on all part of the V-filtration is already determined by this filtration on certain parts of a the V-filtration. Unlike for X_0-specializability the notation of strict X_0-specializability for $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-modules differs because well-filtered $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-modules are in general not well-filtered as $(\mathcal{D}_X, F^\circ_\bullet)$-modules. After having defined strict

X_0-specializability also for $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-modules, we turn the localization and dual localization of strictly X_0-specializable $(\mathcal{D}_X, F^\circ_\bullet)$- or $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-modules into strictly X_0-specializable $(\mathcal{D}_X, F^\circ_\bullet)$-modules by using their description in terms of the V-filtration. We also prepare the actual computation of these constructions on a sheaf theoretic level in local coordinates. Finally we extend these constructions to singular X_0.

Recall that we work implicitly on the distinguished affine base (see Subsection 1.1.1), as we are dealing with $\mathcal{O}$-quasi-coherent sheaves.

In this chapter X always denotes a smooth equidimensional variety (over $\mathbb{C}$) and $X_0 \subseteq X$ stands for an equidimensional codimension one subvariety with corresponding embedding $\iota : X_0 \hookrightarrow X$ and defining ideal sheaf $\mathcal{I}$. We write $X^* := X \setminus X_0$ with inclusion $j_{X^*} : X^* \hookrightarrow X$. Under the assumption that X_0 is smooth, we agree upon the following convention:

Convention 3.0.1. Assume that X_0 is smooth. Recall that we can find by Proposition 1.2.9 for every point $p \in X_0$ a coordinate neighborhood U of $X_0 \subseteq X$ containing p and local coordinates $(\underline{x}, t) := (x_1, \ldots, x_n, t)$ with differentials $(\underline{\theta}, \partial_t) := (\theta_1, \ldots, \theta_n, \partial_t)$ on U such that $\mathcal{I}_U = {}_{\mathcal{O}_U}\langle t \rangle$. We sometimes call such a U also coordinate neighborhood of p.

In this chapter when writing t, ∂_t or U, we always assume that we work on a coordinate neighborhood U such that t is part of a local coordinate system with corresponding differential ∂_t and $\mathcal{I}_U = {}_{\mathcal{O}_U}\langle t \rangle$. If not stated otherwise, all statements involving t, ∂_t or U are independent of the choice of U (and p) and the local coordinate system.

3.1 The V-filtration and application to localization and dual localization

The subject of study of this section is the (rational) V-filtration. In the analytic setting, the V-filtration along a coordinate showed up first in the work of Malgrange [Mal83] in the special case of $\mathcal{D}$-module theoretic direct images of $\mathcal{O}$ under graph embeddings and Kashiwara extended that concept along submanifolds to regular holonomic $\mathcal{D}$-modules [Kas83]. We review Kashiwara's definition of the (rational) V-filtration for $\mathcal{D}_X$-modules in the codimension one case and extend this concept to coherent $\mathcal{D}_X(*X_0)$-modules following [SS17]. Then we collect important results about this filtration mainly due to Saito (see [Sai88]) and use them to describe certain localizations and dual localizations (see [Sai88], [Sai93] and [SS17]). Based on this, we prepare the algorithmic treatment of these concepts for the next chapter.

3.1.1 Specializability, localization and dual localization along smooth codimension one subvarieties

We assume in this subsection that X_0 is smooth. The *V-filtration along X_0 on $\mathcal{D}_X$* (indexed by $\mathbb{Z}$) is defined by

$$V^{X_0}_\bullet \mathcal{D}_X := \{p \in \mathcal{D}_X \mid p(\mathcal{I}^j) \subseteq \mathcal{I}^{j-\bullet} \text{ for all } j \in \mathbb{Z}\}, \tag{3.1.1}$$

where $\mathcal{I}^j = \mathcal{O}_X$ for $j \leq 0$. If it is clear from the context that we consider the V-filtration along X_0, we drop the upper index X_0 (we use this convention also for the other V-filtrations that we will define). In local coordinates $(\underline{x}, t)$ on U, we have

$$(V_\bullet \mathcal{D}_X)_U = V_\bullet^{V(t)} \mathcal{D}_U = \sum_{\alpha,\beta\in\mathbb{N},\gamma\in\mathbb{N}^n:\beta-\alpha\leq\bullet} p_{\alpha,\beta,\gamma}\underline{\theta}^\gamma t^\alpha \partial_t^\beta \text{ with } p_{\alpha,\beta,\gamma} \in \mathcal{O}_U\,. \tag{3.1.2}$$

On the complement X^*, the V-filtration is given by $V_k^{X_0} \mathcal{D}_{X^*} := (V_k \mathcal{D}_X)_{X^*} = \mathcal{D}_{X^*}$ for all $k \in \mathbb{Z}$.

Following [SS17], we also introduce the *V-filtration along X_0 on $\mathcal{D}_X(*X_0)$*: Considering the $\mathcal{I}$-adic filtration defined by $\mathcal{I}^k := \mathcal{O}_X(-kX_0)$ for $k \in \mathbb{Z}$, we define the V-filtration on $\mathcal{D}_X(*X_0)$ by

$$V_\bullet^{X_0} \mathcal{D}_X(*X_0) := \{p \in \mathcal{D}_X(*X_0) \mid p(\mathcal{I}^j) \subseteq \mathcal{I}^{j-\bullet} \text{ for all } j \in \mathbb{Z}\}. \tag{3.1.3}$$

So $(V_k \mathcal{D}_X(*X_0))_U = V_k \mathcal{D}_X(*X_0)_U = t^{-k} V_0 \mathcal{D}_U$ and $V_k \mathcal{D}_X(*X_0)_{X^*} = \mathcal{D}_X(*X_0)_{X^*} = \mathcal{D}_{X^*}$ for $k \in \mathbb{Z}$. These V-filtrations define a subring $V_0 \mathcal{D}_X = V_0 \mathcal{D}_X(*X_0)$ of $\mathcal{D}_X$ and $\mathcal{D}_X(*X_0)$, which is $\mathcal{O}_X$-quasi-coherent. Moreover, we have:

Lemma 3.1.1. *The sheaf of ring $V_0 \mathcal{D}_X = V_0 \mathcal{D}_X(*X_0)$ is locally Noetherian, so in particular coherent.*

Proof. We induce the filtration $\mathcal{F}^\circ_\bullet V_0 \mathcal{D}_X$ on $V_0 \mathcal{D}_X$ via the order filtration on $\mathcal{D}_X$. On a coordinate neighborhood $U \subseteq X$ with local coordinates $(\underline{x}, t)$ the associated graded ring is

$$(\mathrm{Gr}^{F^\circ} V_0 \mathcal{D}_X)(U) \cong \mathcal{O}_X(U)[\xi_1, \ldots, \xi_n, t\xi_t].$$

Since on affine open neighborhoods $U' \subseteq X^*$ with local coordinates $x'_1, \ldots, x'_{n+1}$

$$(\mathrm{Gr}^{F^\circ} V_0 \mathcal{D}_X)(U') \cong \mathcal{O}_X(U')[\xi'_1, \ldots, \xi'_{n+1}],$$

the claim follows by Proposition 1.1.16. Proposition 1.1.7(b) implies now the particular claim. □

Notation 3.1.2. We denote by $\mathcal{D}'_X$ either $\mathcal{D}_X$ or $\mathcal{D}_X(*X_0)$.

Recall that all our filtrations are by definition increasing, exhaustive and indexed discretely by the rational numbers. The V-filtration on coherent left $\mathcal{D}'_X$-modules is now defined as follows (see below for uniqueness and compatibility of the notions for $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules):

Definition 3.1.3. The *(rational) V-filtration along X_0* on a coherent left $\mathcal{D}'_X$-module $\mathcal{M}$ is a $V_\bullet \mathcal{D}'_X$-filtration $V_\bullet^{X_0} \mathcal{M} = V_\bullet \mathcal{M}$ on $\mathcal{M}$ satisfying

(a) $V_\alpha^{X_0} \mathcal{M}$ is a coherent $V_0^{X_0} \mathcal{D}'_X$-module for any $\alpha \in \mathbb{Q}$,

(b) $V_k^{X_0}\,\mathcal{D}'_X \cdot V_\alpha^{X_0}\,\mathcal{M} \subseteq V_{\alpha+k}^{X_0}\,\mathcal{M}$ for all $\alpha \in \mathbb{Q}$, $k \in \mathbb{Z}$,

(c) $\mathcal{I} \cdot V_\alpha^{X_0}\,\mathcal{M} = V_{\alpha-1}^{X_0}\,\mathcal{M}$ for all $\alpha < 0$,

(d) for every point $p \in X_0$ exists a coordinate neighborhood $U \subseteq X$ of p such that $-\partial_t t - \alpha$ acts nilpotently on $(\mathrm{Gr}_\alpha^{V^{X_0}}\,\mathcal{M})_U$ for any $\alpha \in \mathbb{Q}$.

We say that a $\mathcal{D}'_X$-module $\mathcal{M}$ is $\mathbb{Q}$-*specializable along* X_0 (or $\mathbb{Q}$-X_0-specializable) if the rational V-filtration along X_0 on $\mathcal{M}$ exists.

Note that Condition (b) is only listed for reference purposes and is already implicitly contained in the requirement that $V_\bullet\,\mathcal{M}$ is a $V_\bullet\,\mathcal{D}'_X$-filtration.

We point out that Definition 3.1.3(d) does not depend on the choice of the coordinate neighborhood or of the local coordinates: Indeed, let U' be another coordinate neighborhood of p with local coordinates $(\underline{x}', t')$ and differentials $(\underline{\theta}', \partial_{t'})$. Then there is a regular function $u : U \cap U' \to \mathbb{C}^*$ such that $t' = ut$ and there are $a_1, \ldots, a_\mathrm{n}, b \in \mathcal{O}_{U \cap U'}$ such that $\partial_{t'} = \sum_{1 \le i \le \mathrm{n}} a_i \theta_i + b \partial_t$. Applying that equation to $t = u^{-1} t'$ gives $u^{-1} + t' \partial_{t'}(u^{-1}) = \partial_{t'}(t) = b$. This implies

$$\partial_{t'} t' = \partial_t t + \underbrace{\Big(\sum_{1 \le i \le \mathrm{n}} a_i \theta_i u + t' \partial_{t'}(u^{-1}) \partial_t u + u^{-1} \partial_t(u) \Big) t}_{\in V_0\,\mathcal{D}_{U \cap U'}}$$

showing that $\partial_{t'} t'$ acts as $\partial_t t$ on $\mathrm{Gr}_\alpha\,\mathcal{M}_{U \cap U'}$ for any $\alpha \in \mathbb{Q}$. So in particular, if $\mathcal{M}$ is $\mathbb{Q}$-specializable along X_0, then Condition 3.1.3(d) holds on every coordinate neighborhood and system of local coordinates as in Convention 3.0.1.

Since we only consider $\mathbb{Q}$-specializability, we often drop the $\mathbb{Q}$ and write "X_0-specializable" or "specializable along X_0".

Convention 3.1.4. Our notation of the V-filtration on $\mathcal{D}'_X$-modules conflicts for quotients of free modules with the filtration induced by $V_\bullet\,\mathcal{D}'_X$. As we are rarely and only for computational purposes interested in the latter filtration, we agree upon the following: If $\mathcal{M} = (\mathcal{D}'_X)^E / \mathcal{L}$ (with E finite and $\mathcal{L} \subseteq (\mathcal{D}'_X)^E$ a submodule) is an X_0-specializable $\mathcal{D}_X$-module, we mean by $V_\bullet\,\mathcal{M}$ always its V-filtration in the sense of Definition 3.1.3 and denote the induced filtration by

$$V_\bullet^{\mathrm{ind}}\,\mathcal{M} := V_\bullet^{X_0,\mathrm{ind}}\,\mathcal{M} := ((V_\bullet\,\mathcal{D}'_X)^E + \mathcal{L}) / \mathcal{L}\,.$$

On the other hand, we set $V_\bullet \mathcal{D}'^E := (V_\bullet\,\mathcal{D}'_X)^E$. Note that this last convention does not cause any ambiguity because $\mathcal{D}_X^E$ is not X_0-specializable.

The V-filtration on the complement of X_0 is trivial:

Remark 3.1.5. Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}_X$-module. Then $(V_k\,\mathcal{D}_X)_{X^*} = \mathcal{D}_{X^*}$ for all $k \in \mathbb{Z}$ implies $V_\alpha\,\mathcal{M}_{X^*} := (V_\alpha\,\mathcal{M})_{X^*} = \mathcal{M}_{X^*}$ for all $\alpha \in \mathbb{Q}$.

The V-filtration is in general not separated:

Remark 3.1.6. Following [Bjö93, Section 2.10.22], consider the case $X_0 = \{0\} \subseteq X = \mathbb{C}$ and the $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X \,/\, {}_{\mathcal{D}_X}\langle t^2\partial_t + 1\rangle$. Since $\overline{t^a\partial_t^b} = \overline{t^a\partial_t^b(-t^2\partial_t)^{b+k}} \in V_{-k}^{\mathrm{ind}}\,\mathcal{M}$ for all $a, b \in \mathbb{N}$ and $k \in \mathbb{N}$, we have $\mathcal{M} = V_k^{\mathrm{ind}}\,\mathcal{M}$ for all $k \in \mathbb{Z}$ showing that the V-filtration is constant in this case and hence not separated.

Remark 3.1.7. There are also more general types of V-filtrations.

(a) We can consider V-filtrations indexed discretely by the complex numbers: For this, fix total order $<$ on $\mathbb{C}$ that agrees with the standard order on $\mathbb{R}$ and such that $a < b$ implies $a + c < b + c$ for all $a, b \in \mathbb{C}$ and any $c \in \mathbb{R}$. Replacing $\mathbb{Q}$ by $\mathbb{C}$, the complexly indexed V-filtration is now defined as in Definition 3.1.3.

(b) Another natural generalization of Definition 3.1.3 are V-filtrations along smooth subvarieties of codimension greater than one: If we assume for a moment that X_0 is smooth of pure codimension m with defining ideal sheaf $\mathcal{I}$, we define $V_\bullet^{X_0}\,\mathcal{D}_X$ by Equation (3.1.1) and in Definition 3.1.3 we replace Condition (d) by

(d') for every point $p \in X_0$ exists a coordinate neighborhood $U \subseteq X$ of p with coordinates $(\underline{x}, t_1, \ldots, t_{\mathrm{m}})$ satisfying $X_0 \cap U = V(t_1, \ldots, t_{\mathrm{m}})$ such that the operator $-\sum_{1\leq i\leq \mathrm{m}} \partial_{t_i} t_i - \alpha$ acts nilpotently on $(\mathrm{Gr}_\alpha^{V^{X_0}}\,\mathcal{M})_U$ for any $\alpha \in \mathbb{Q}$.

respectively.

If not stated otherwise, we mean by V-filtration always the rational V-filtration along a smooth codimension one subvariety as in Definition 3.1.3.

V-filtration on $\mathcal{D}_X$-modules

We focus now first on the V-filtration on $\mathcal{D}_X$-modules. Later we show the compatibility of the notions of V-filtrations on $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules and use this to develop corresponding results for V-filtrations on $\mathcal{D}_X(*X_0)$-modules. The next remark explains the structure of the graded parts of the V-filtration on $\mathcal{D}_X$-modules:

Remark 3.1.8.

(a) Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}_X$-module. By definition of the V-filtration, the sheaves $\mathrm{Gr}_\alpha^V\,\mathcal{M}$ and $V_\alpha\,\mathcal{M}\,/V_{\alpha-1}\,\mathcal{M}$ are $\mathrm{Gr}_0^V\,\mathcal{D}_X$-modules with support on X_0. Recalling that $(\overline{x_i}, \theta_i)_{1\leq i\leq \mathrm{n}}$ is a local coordinate system on $U \cap X_0$, we define the map

$$(\iota\,\mathcal{D}_{X_0})_U \to \mathrm{Gr}_0^V\,\mathcal{D}_U$$

by sending θ_i to $\overline{\theta_i}$ and $f \in (\iota\,\mathcal{O}_{X_0})_U$ to the residue class in $\mathrm{Gr}_0^V\,\mathcal{D}_U$ of a representative of f in $\mathcal{O}_U$ under the isomorphism $(\iota\,\mathcal{O}_{X_0})_U \cong \mathcal{O}_U\,/\,\mathcal{I}_U$. One easily checks that the local maps glue to a global map

$$\iota\,\mathcal{D}_{X_0} \to \mathrm{Gr}_0^V\,\mathcal{D}_X\,.$$

This map allows us to regard $\mathrm{Gr}^V_\alpha \mathcal{M}$ and $V_\alpha \mathcal{M} / V_{\alpha-1} \mathcal{M}$ as $\iota \mathcal{D}_{X_0}$-modules. Under the identification $\mathcal{D}_{X_0} \cong \iota^{-1}\iota \mathcal{D}_{X_0}$, we consider $\iota^{-1} \mathrm{Gr}^V_\alpha \mathcal{M}$ and $\iota^{-1}(V_\alpha \mathcal{M} / V_{\alpha-1} \mathcal{M})$ as $\mathcal{D}_{X_0}$-modules. From now on, we drop the ι^{-1} and write by abuse of notation $\mathrm{Gr}^V_\alpha \mathcal{M}$ and $V_\alpha \mathcal{M} / V_{\alpha-1} \mathcal{M}$ for these $\mathcal{D}_{X_0}$-modules.

(b) By Definition 3.1.3(a) and (d) there exist a finite set $B \subseteq \mathbb{Q} \cap (\alpha-1, \alpha]$ and $c \in \mathbb{N}^B$ such that the polynomial $\prod_{\beta \in B}(-\partial_t t - \beta)^{c_\beta}$ annihilates the module $V_\alpha \mathcal{M}_U / V_{\alpha-1} \mathcal{M}_U$. Writing $1 = \sum_{\beta \in B} d_\beta \prod_{\gamma \in B \setminus \{\beta\}} (s - \gamma)^{c_\gamma}$ with $d \in \mathbb{Q}[s]^B$ using Bézout's identity, we see that $V_\alpha \mathcal{M}_U / V_{\alpha-1} \mathcal{M}_U$ decomposes as a direct sum of generalized eigenspaces $\bigoplus_{\beta \in B} \ker((-\partial_t t - \beta)^N)$ with $N \gg 0$ and deduce that

$$V_\alpha \mathcal{M}_U / V_{\alpha-1} \mathcal{M}_U \to \bigoplus_{\beta \in B} \mathrm{Gr}^V_\beta \mathcal{M}_U, \ \overline{m} \mapsto \Big(d_\beta \prod_{\gamma \in B \setminus \{\beta\}} (-\partial_t t - \gamma)^{c_\gamma} \overline{m}\Big)_{\beta \in B}$$

is a $\mathrm{Gr}^V_0 \mathcal{D}_U$- and $(\iota \mathcal{D}_{X_0})_U$-linear isomorphism. We conclude that $V_\alpha \mathcal{M} / V_{\alpha-1} \mathcal{M}$ is globally isomorphic to $\bigoplus_{\beta \in (\alpha-1, \alpha]} \mathrm{Gr}^V_\beta \mathcal{M}$ as $\mathrm{Gr}^V_0 \mathcal{D}_X$- and $\iota \mathcal{D}_{X_0}$-module by similar arguments as for the independence of Definition 3.1.3(d) on the choice of the coordinate neighborhood and the local coordinates.

We review now some of Saito's results concerning the V-filtration along smooth codimension one subvarieties. All these results are only stated for $\mathcal{D}_X$-modules with one exception: We show that Lemma 3.1.10 and its corollaries hold naturally also for $\mathcal{D}_X(*X_0)$-modules and use these results in Lemma 3.1.25 to prove that the two notions of specializability for $\mathcal{D}_X(*X_0)$-modules are compatible.

Lemma 3.1.9. *[Kas83, Theorem 1] The V-filtration on a coherent $\mathcal{D}_X$-modules is unique if it exists.*

The following lemma is a direct consequence of Definition 3.1.3(d):

Lemma 3.1.10. *[Sai88, (3.1.1.4)] Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}'_X$-module. Then the maps*

$$t\cdot : \mathrm{Gr}^V_\alpha \mathcal{M}_U \to \mathrm{Gr}^V_{\alpha-1} \mathcal{M}_U \ \text{ and } \partial_t\cdot : \mathrm{Gr}^V_{\alpha-1} \mathcal{M}_U \to \mathrm{Gr}^V_\alpha \mathcal{M}_U$$

are bijective for $\alpha \neq 0$.

Proof. For $\alpha \neq 0$ and $i \in \mathbb{N}$ set

$$\mathcal{A}^i_\alpha := \ker((-\partial_t t - \alpha)^i \cdot : \mathrm{Gr}^V_\alpha \mathcal{M}_U \to \mathrm{Gr}^V_\alpha \mathcal{M}_U).$$

By Definition 3.1.3(d), we have $\mathrm{Gr}^V_\alpha \mathcal{M}_U = \bigcup_{i \in \mathbb{N}} \mathcal{A}^i_\alpha$.

We first show inductively that $\mathcal{A}^i_\alpha \subseteq \partial_t \cdot \mathrm{Gr}^V_{\alpha-1} \mathcal{M}_U$ which implies that $\partial_t\cdot : \mathrm{Gr}^V_{\alpha-1} \mathcal{M}_U \to \mathrm{Gr}^V_\alpha \mathcal{M}_U$ is surjective. Multiplying $a^i_\alpha \in \mathcal{A}^i_\alpha$ with $(-\partial_t t - \alpha)$, we see that there is some $a^{i-1}_\alpha \in \mathcal{A}^{i-1}_\alpha$ such that

$$\alpha a^i_\alpha = -\partial_t t a^i_\alpha + a^{i-1}_\alpha$$

and hence $a^i_\alpha \in \partial_t \cdot \operatorname{Gr}^V_{\alpha-1} \mathcal{M}_U$ by induction. Writing $(-\partial_t t - (\alpha - 1)) = (-t\partial_t - \alpha)$ and arguing as above gives that $t\cdot : \operatorname{Gr}^V_\alpha \mathcal{M}_U \to \operatorname{Gr}^V_{\alpha-1} \mathcal{M}_U$ is surjective.

For the injectivity of $\partial_t\cdot : \operatorname{Gr}^V_{\alpha-1} \mathcal{M}_U \to \operatorname{Gr}^V_\alpha \mathcal{M}_U$ assume there is $a_{\alpha-1} \in \operatorname{Gr}^V_{\alpha-1} \mathcal{M}_U$ such that $\partial_t a_{\alpha-1} = 0$. As $t\cdot : \operatorname{Gr}^V_\alpha \mathcal{M}_U \to \operatorname{Gr}^V_{\alpha-1} \mathcal{M}_U$ is surjective there is $a_\alpha \in \operatorname{Gr}^V_\alpha \mathcal{M}_U$ satisfying $ta_\alpha = a_{\alpha-1}$. This implies $(-\partial_t t)a_\alpha = 0$ and hence $a_\alpha = 0 = a_{\alpha-1}$ since $\alpha \neq 0$. An analogous argument shows the injectivity of the other map. □

Corollary 3.1.11. *Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}'_X$-module. We have for $\alpha \in [-1, 0]$ and $k \in \mathbb{Z}$ locally that*

$$V_{\alpha+k} \mathcal{M}_U = \begin{cases} t^{-k} V_\alpha \mathcal{M}_U, & \text{if } k \leq 0, \alpha \neq 0 \\ \sum_{i=0}^k \partial_t^i V_\alpha \mathcal{M}_U, & \text{if } k \geq 0, \alpha \neq -1 \end{cases}$$

and hence globally

$$V_{\alpha+k} \mathcal{M} = \begin{cases} \mathcal{I}^{-k} V_\alpha \mathcal{M} = V_k \mathcal{D}'_X \cdot V_\alpha \mathcal{M}, & \text{if } k \leq 0, \alpha \neq 0 \\ V_k \mathcal{D}'_X \cdot V_\alpha \mathcal{M} = V_k \mathcal{D}_X \cdot V_\alpha \mathcal{M}, & \text{if } k \geq 0, \alpha \neq -1. \end{cases}$$

In particular,

$$V_\alpha \mathcal{M} = V_{<\alpha} \mathcal{M} + V_1 \mathcal{D}'_X \cdot V_{\alpha-1} \mathcal{M} \text{ for } \alpha > 0$$

and the V-filtration along X_0 on $\mathcal{M}$ is a good $V_\bullet \mathcal{D}'_X$-filtration (see Definition 1.1.11(d)*).*

Corollary 3.1.12. *If the $\mathcal{D}'_X$-module $\mathcal{M}$ is $\mathbb{Q}$-specializable along X_0, then we have for $m \in \mathcal{M}_U$ that $\partial_t \cdot m \in V_0 \mathcal{M}_U$ implies $m \in V_{-1} \mathcal{M}_U$.*

For $\alpha < 0$, left multiplication with t acts injectively on $V_\alpha \mathcal{M}_U$:

Lemma 3.1.13. *[Sai88, Lemme 3.1.4] Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}_X$-module. The map*

$$t\cdot : V_\alpha \mathcal{M}_U \to V_{\alpha-1} \mathcal{M}_U$$

is bijective for $\alpha < 0$.

We review Saito's proof for the convenience of the reader:

Proof. Note that the $\mathcal{D}_X$-modules $\mathcal{M}' := \Gamma_{[X_0]}(\mathcal{M})$ (see Proposition 1.4.12(c)) and $\mathcal{M}'' := \mathcal{M} / \mathcal{M}'$ are coherent and t acts injectively on $\mathcal{M}''_U$. The V-filtration on $\mathcal{M}$ induces filtrations

$$V'_\bullet \mathcal{M}' := V_\bullet \mathcal{M} \cap \mathcal{M}' \text{ and } V'_\bullet \mathcal{M}'' := (V_\bullet \mathcal{M} + \mathcal{M}')/ \mathcal{M}'$$

on $\mathcal{M}'$ and $\mathcal{M}''$, respectively. One easily checks that $V'_\bullet \mathcal{M}''$ satisfies all conditions of Definition 3.1.3 and is hence the V-filtration of $\mathcal{M}''$ along X_0 by Lemma 3.1.9. Similarly, it is immediate that $V_\bullet \mathcal{M}$ induces all properties of Definition 3.1.3 but Condition (c) on $V'_\bullet \mathcal{M}'$, because the coherence of $V_0\,\mathcal{D}_X$ (see Lemma 3.1.1) implies Condition (a). The missing condition follows locally for $\alpha < 0$ from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V'_\alpha\,\mathcal{M}'_U & \longrightarrow & V_\alpha\,\mathcal{M}_U & \longrightarrow & V_\alpha\,\mathcal{M}''_U & \longrightarrow & 0 \\ & & \downarrow t\cdot & & \downarrow\!\downarrow t\cdot & & \downarrow t\cdot\ \cong & & \\ 0 & \longrightarrow & V'_{\alpha-1}\,\mathcal{M}'_U & \longrightarrow & V_{\alpha-1}\,\mathcal{M}_U & \longrightarrow & V_{\alpha-1}\,\mathcal{M}''_U & \longrightarrow & 0 \end{array}$$

and the Snake Lemma, where the surjectivity of the vertical maps in the middle and on the right is due to Corollary 3.1.11. Hence we have by Lemma 3.1.9 that $V_\bullet\,\mathcal{M}' = V'_\bullet\,\mathcal{M}'$. Since $\mathcal{M}'$ has support on X_0, Lemma 3.1.16 below implies $V_\alpha\,\mathcal{M}' = 0$ for $\alpha < 0$ and another application of the Snake Lemma to the above diagram shows that the vertical arrow in the middle is in fact bijective. □

Similarly, one shows the "only if "-part of the next statement, whereas the converse direction can be proven using so-called local b-functions (see Remark 3.1.19(b)):

Corollary 3.1.14. *[Sai88, Corollaire 3.1.5] Let* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *be an exact sequence of coherent* $\mathcal{D}_X$*-modules. Then* $\mathcal{M}$ *is* $\mathbb{Q}$*-specializable along* X_0 *if and only if* $\mathcal{M}'$ *and* $\mathcal{M}''$ *are so. In this case*

$$0 \to V_\bullet\,\mathcal{M}' \to V_\bullet\,\mathcal{M} \to V_\bullet\,\mathcal{M}'' \to 0$$

is an exact sequence.

From Lemma 3.1.9 and the above corollary follows:

Proposition 3.1.15. *[Sai88, Proposition 3.1.6] If* $\varphi : \mathcal{M} \to \mathcal{M}'$ *is a morphism between* X_0*-specializable* $\mathcal{D}_X$*-modules, then* φ *is strict with respect to the corresponding rational* V*-filtrations. In particular, the category of* X_0*-specializable* $\mathcal{D}_X$*-modules is abelian and its morphisms are always strict.*

The following lemma is a consequence of Kashiwara's equivalence (see Proposition 1.4.12):

Lemma 3.1.16. *[Sai88, Lemme 3.1.3] Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module such that its support is contained in* X_0*. Then* $\mathcal{M}$ *is* $\mathbb{Q}$*-specializable along* X_0*, and we have on a coordinate neighborhood* U

$$\mathcal{M}_U = \mathcal{M}_0 \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] = (\iota_{X_0\cap U})_+\,\mathcal{M}_0 \ \text{ and } V_\alpha\,\mathcal{M}_U = \bigoplus_{0\le i\le\lfloor\alpha\rfloor} \mathcal{M}_0 \otimes \partial_t^i,$$

where $\mathcal{M}_0 := \ker(t\cdot : \mathcal{M}_U \to \mathcal{M}_U)$ *and* $\iota_{X_0\cap U} : X_0 \cap U \to U$ *is the restriction of* ι*. In particular,* $V_{<0}\,\mathcal{M} = 0$ *and*

$$V_\alpha\,\mathcal{M}_U = \ker(t^{\lfloor\alpha\rfloor}\cdot : \mathcal{M}_U \to \mathcal{M}_U) = \bigoplus_{\alpha\ge k\in\mathbb{N}} \ker((-\partial_t t - k)\cdot : \mathcal{M}_U \to \mathcal{M}_U)$$

for $\alpha \ge 0$*.*

Hence the quasi-inverse in Kashiwara's equivalence in the codimension one case is expressed as follows:

Corollary 3.1.17. *Let $\iota : X_0 \hookrightarrow X$ be an embedding of smooth equidimensional varieties of codimension one. Then a quasi-inverse for $\iota_+ : \mathrm{Mod}_{\mathcal{O}_{X_0}\text{-}\mathrm{qcoh}}(\mathcal{D}_{X_0}) \to \mathrm{Mod}^{\iota(X_0)}_{\mathcal{O}_X\text{-}\mathrm{qcoh}}(\mathcal{D}_X)$ is given by*

$$\mathrm{Gr}_0^V = \mathrm{Gr}_0^{V^{\iota(X_0)}} : \mathrm{Mod}^{\iota(X_0)}_{\mathcal{O}_X\text{-}\mathrm{qcoh}}(\mathcal{D}_X) \to \mathrm{Mod}_{\mathcal{O}_{X_0}\text{-}\mathrm{qcoh}}(\mathcal{D}_{X_0}),\ \mathcal{M} \mapsto \mathrm{Gr}_0^V \mathcal{M} = V_0\,\mathcal{M}$$

by considering $\mathrm{Gr}_0^V \mathcal{M} = V_0\,\mathcal{M}$ as an $\mathcal{D}_{X_0}$-module via the isomorphism $X_0 \cong \iota(X_0)$.

The $V_{<0}$-part of an X_0-specializable $\mathcal{D}_X$-module depends only on the restriction of that module to X^*:

Lemma 3.1.18. *[Sai88, Lemme 3.1.7] Let $\varphi : \mathcal{M} \to \mathcal{M}'$ be a morphism of X_0-specializable $\mathcal{D}_X$-modules. If $\varphi_{X^*} : \mathcal{M}_{X^*} \to \mathcal{M}'_{X^*}$ is an isomorphism then*

$$V_\alpha\,\mathcal{M} \cong V_\alpha\,\mathcal{M}' \text{ for } \alpha < 0.$$

We review Saito's proof:

Proof. By Corollary 3.1.14 and Proposition 3.1.15 we have for $\alpha \in \mathbb{Q}$ an exact sequence

$$0 \to V_\alpha \ker(\varphi) \to V_\alpha\,\mathcal{M} \xrightarrow{\varphi} V_\alpha\,\mathcal{M}' \to V_\alpha(\mathcal{M}' / \operatorname{im}\varphi) \to 0.$$

Since $\ker(\varphi)$ and $\mathcal{M}' / \operatorname{im}\varphi$ have support on X_0 by assumption, the modules on the left and on the right of the above sequence are zero for $\alpha < 0$ by Lemma 3.1.16. This shows the claimed isomorphism. □

We collect statements concerning the existence of the V-filtration on $\mathcal{D}_X$-modules:

Remark 3.1.19. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.

(a) Kashiwara proved that the rational V-filtration along X_0 on $\mathcal{M}$ exists if $\mathcal{M}$ is regular holonomic and with quasi-unipotent local monodromy (see e.g. [Meb89, Théorème III.4.10.1]). In particular, Hodge $\mathcal{D}_X$-modules are X_0-specializable. More generally, a holonomic $\mathcal{D}_X$-module admits a unique V-filtration indexed by the complex numbers (with respect to any ordering as in Remark 3.1.7(a)) (see e.g. [Meb89, Théorème III.4.4.2]).

(b) The existence of (not necessarily rationally indexed) V-filtrations is equivalent to existence of certain b-functions: The *b-function of a local section* $m \in \mathcal{M}_U$ is the minimal monic polynomial $b_m(s) \in \mathbb{C}[s] \setminus \{0\}$ such that $b_m(-\partial_t t)m \in V_{-1}\,\mathcal{D}_U \cdot m$ if such a polynomial exists. The b-function exists for every local section of $\mathcal{M}$ if and only if the (complexly indexed) V-filtration exists [Sab87] [Sab01]. In this case we have

$$V_\alpha \mathcal{M}_U = \{m \in \mathcal{M}_U \mid b_m(z) = 0 \text{ for } z \in \mathbb{C} \text{ implies } z \leq \alpha\} \tag{3.1.4}$$

and hence the roots of the local b-functions are rational if and only if the V-filtration is rational.

Eventually, we are interested in an algorithm for the computation of the V-filtration. Our algorithm is based on the observation that the filtered part $V_\alpha \mathcal{M}$ of an X_0-specializable $\mathcal{D}_X$-module can be represented using the filtration described below (see [Kas83]):

Definition 3.1.20. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. For fixed $\alpha \in \mathbb{Q}$, we define the filtration $V^\alpha_\bullet \mathcal{M} = V^{X_0,\alpha}_\bullet \mathcal{M}$ indexed by the integers by the following properties:

(a) $V^\alpha_i \mathcal{M}$ is a coherent $V_0 \mathcal{D}_X$-module for any $i \in \mathbb{Z}$,

(b) $V_k \mathcal{D}_X \cdot V^\alpha_i \mathcal{M} \subseteq V^\alpha_{i+k} \mathcal{M}$ for all $i, k \in \mathbb{Z}$,

(c) $\mathcal{I} \cdot V^\alpha_i \mathcal{M} = V^\alpha_{i-1} \mathcal{M}$ and $\partial_t V^\alpha_{-i} \mathcal{M}_U + V^\alpha_{-i} \mathcal{M}_U = V^\alpha_{-i+1} \mathcal{M}_U$ on any coordinate neighborhood U for $i \ll 0$,

(d) There exists a finite set $A \subseteq \mathbb{Q}$ satisfying the following condition: Every point $p \in X_0$ has a coordinate neighborhood $U \subseteq X$ such that for $A_i := (A + \mathbb{Z}) \cap (\alpha - 1 + i, \alpha + i]$ the operator $\prod_{a \in A_i}(-\partial_t t - a)$ acts nilpotently on $\mathrm{Gr}^{V^\alpha}_i \mathcal{M}_U := V^\alpha_i \mathcal{M}_U / V^\alpha_{i-1} \mathcal{M}_U$ for every $i \in \mathbb{Z}$.

We point out that Definition 3.1.20(d) is independent of the choice of the coordinate neighborhood and of the choice of the local coordinate system. The lemma below shows that the above filtration exists if and only if $\mathcal{M}$ is X_0-specializable, whereas uniqueness can be proven in the same way as the uniqueness of the V-filtration.

Lemma 3.1.21. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and $\alpha \in \mathbb{Q}$ fixed. Then $V^\alpha_\bullet \mathcal{M}$ exists if and only if $\mathcal{M}$ is $\mathbb{Q}$-specializable along X_0 and we have for $k \in \mathbb{Z}$ in this case*

$$V_{\alpha+k} \mathcal{M} = V^\alpha_k \mathcal{M}\,.$$

Proof. Clearly $(V^\alpha_\bullet \mathcal{M})_{X^*} = \mathcal{M}_{X^*}$ and hence both filtrations are uniquely defined by this property and by their restrictions to coordinate neighborhoods. Thus we may assume that X itself is a coordinate neighborhood and that X_0 has defining ideal sheaf generated by t.

Let $\mathcal{M}$ be X_0-specializable. Setting $V^{\alpha\prime}_k \mathcal{M} := V_{\alpha+k} \mathcal{M}$ for $k \in \mathbb{Z}$, we see by Definition 3.1.3 and Corollary 3.1.11 immediately that $V^{\alpha\prime}_\bullet \mathcal{M}$ satisfies all properties of Definition 3.1.20.

Conversely, assume that $V^\alpha_\bullet \mathcal{M}$ exists. We write $\gamma \in \mathbb{Q}$ as $\gamma = \alpha + \beta + k$ with $\beta \in (-1, 0]$ and $k \in \mathbb{Z}$ and set

$$V'_\gamma \mathcal{M} := V^{\alpha,\beta}_k \mathcal{M},$$

where $V_k^{\alpha,\beta}$ is the maximal $V_0\,\mathcal{D}_X$-submodule of $V_k^{\alpha}\,\mathcal{M}$ containing $V_{k-1}^{\alpha}\,\mathcal{M}$ with the property that $b_\gamma(-\partial_t t) := \prod_{a\in A_k\cap(-\infty,\gamma]}(-\partial_t t - a)$ acts nilpotently on $V_k^{\alpha,\beta}\,\mathcal{M}\,/V_{k-1}^{\alpha}\,\mathcal{M}$. Then all conditions of Definition 3.1.3 immediately follow for $V'_\bullet\,\mathcal{M}$ except for Condition (c). We first show that $tV'_\gamma\,\mathcal{M} = V'_{\gamma-1}\,\mathcal{M}$ for $\gamma \ll 0$: Given $m \in V'_{\gamma-1}\,\mathcal{M} \subseteq V_{k-1}^{\alpha}\,\mathcal{M}$, Definition 3.1.20(c) implies the existence of some $m' \in V'_{\alpha+k}\,\mathcal{M}$ such that $tm' = m$. By definition of $V_{\gamma'-1}\,\mathcal{M}$, there is a natural number $l \in \mathbb{N}$ such that $b_{\gamma-1}(-\partial_t t)^l m \in V'_{\alpha+k-2}\,\mathcal{M} = tV'_{\alpha+k-1}\,\mathcal{M}$, where the equality follows from Definition 3.1.20(c). Thus we have $tb_{\gamma-1}(-\partial_t t-1)^l m' \in tV'_{\alpha+k-1}\,\mathcal{M}$. Since we can prove the injectivity of $t\cdot : V_i^{\alpha}\,\mathcal{M} \to V_{i-1}^{\alpha}\,\mathcal{M}$ for $i \ll 0$ along the lines of the proof of Lemma 3.1.13, it follows that $b_\gamma(-\partial_t t)^l m' \in V'_{\alpha+k-1}\,\mathcal{M}$ and hence $m' \in V'_\gamma\,\mathcal{M}$. Note that Lemma 3.1.10 also holds in our situation since Definition 3.1.20(c) was not needed in the proof of that lemma. Thus, if $\delta < 0$ and if $V'_{<\delta-1}\,\mathcal{M} = tV'_{<\delta}\,\mathcal{M}$, the Snake Lemma and the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V'_{<\delta}\,\mathcal{M} & \longrightarrow & V'_{\delta}\,\mathcal{M} & \longrightarrow & \mathrm{Gr}_{\delta}^{V'}\,\mathcal{M} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle t\cdot} & & \big\downarrow{\scriptstyle t\cdot} & & \big\downarrow{\scriptstyle t\cdot} & & \\
0 & \longrightarrow & V'_{<\delta-1}\,\mathcal{M} & \longrightarrow & V'_{\delta-1}\,\mathcal{M} & \longrightarrow & \mathrm{Gr}_{\delta-1}^{V'}\,\mathcal{M} & \longrightarrow & 0
\end{array}$$

imply that the vertical map in the middle is also surjective. This proves Definition 3.1.3(c), because $V'_\bullet\,\mathcal{M}$ is indexed discretely.

The second claim follows now directly from the above construction and by the uniqueness of the V-filtration. □

Locally, we reduce the computation of the V-filtration to that of local b-functions:

Remark 3.1.22. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. According to Kashiwara, we can decide if $\mathcal{M}$ is $\mathbb{Q}$-specializable along X_0 and approach $V_\bullet^{\alpha+k}\,\mathcal{M}$ for fixed $\alpha \in \mathbb{Q}$ and for suitably chosen $k \in \mathbb{Z}$ in this case using an induced V-filtration: If we represent $\mathcal{M}$ locally on a coordinate neighborhood U as $\mathcal{D}_U^E\,/\,\mathcal{N}$ with E finite and $\mathcal{N} \subseteq \mathcal{D}_U^E$, then $V_\bullet\,\mathcal{D}_U$ induces the filtration

$$V_\bullet^{\mathrm{ind}}\,\mathcal{M}_U = ((V_\bullet\,\mathcal{D}_U)^E + \mathcal{N})/\,\mathcal{N}$$

on $\mathcal{M}_U$, which satisfies all properties of Definition 3.1.20 except for Condition (d).

The $V(t)$-specializability of $\mathcal{M}_U$ is equivalent to the existence of the b-function of $\mathcal{M}_U$ with respect to the induced V-filtration, i.e., the monic nonzero polynomial $b^{(0)}(s) \in \mathbb{Q}[s]$ of minimal degree having only rational roots and satisfying

$$b^{(0)}(-\partial_t t - \bullet)V_\bullet^{\mathrm{ind}}\,\mathcal{M}_U \subseteq V_{\bullet-1}^{\mathrm{ind}}\,\mathcal{M}_U :$$

Indeed, if $\mathcal{M}_U$ is $V(t)$-specializable then there exist local b-function $b_{\overline{(e)}}(s) \in \mathbb{Q}[s]$ with rational roots for $e \in E$ by Remark 3.1.19(b). The product $\prod_{e\in E} b_{\overline{(e)}}(s)$ satisfies the above equation and hence there also exists a minimal polynomial, which has rational roots, fulfilling this equation. The converse direction follows from the construction of a filtration satisfying all conditions of Definition 3.1.20 as described below.

So assume that $b^{(0)}(s)$ as above exists. To determine the rational V-filtration along $V(t)$, we shift now the roots of this b-function: Choose $k \in \mathbb{Z}$ such that the minimal root of $b^{(0)}(s)$ lives in $I := (\alpha + k - 1, \alpha + k]$. Setting $W_\bullet^{(0)} \mathcal{M}_U := V_\bullet^{\mathrm{ind}} \mathcal{M}_U$, we may assume that we have a filtration $W_\bullet^{(i)} \mathcal{M}_U$ satisfying Definition 3.1.20(a)-(c) and a polynomial $b^{(i)}(s) \in \mathbb{Q}[s]$ with minimal root in I in such that $b^{(i)}(-\partial_t t - \bullet)$ annihilates $\mathrm{Gr}_\bullet^{W^{(i)}} \mathcal{M}_U$. Write $b^{(i)}(s) = b_1^{(i)}(s) b_2^{(i)}(s)$, where $b_2^{(i)}(s)$ has roots in interval I, while the roots of $b_1^{(i)}(s)$ are strictly greater than $\alpha + k$ and set

$$b^{(i+1)}(s) := b_1^{(i)}(s+1) b_2^{(i)}(s).$$

This decreases the value of the roots not living in I. Considering

$$W_\bullet^{(i+1)} \mathcal{M}_U := W_{\bullet-1}^{(i)} \mathcal{M}_U + b_1^{(i)}(-\partial_t t - \bullet) W_\bullet^{(i)} \mathcal{M}_U$$

the filtration $W_\bullet^{(i)} \mathcal{M}_U$ induces Properties (a)-(c) of Definition 3.1.20 on $W_\bullet^{(i+1)} \mathcal{M}_U$. Since

$$\begin{aligned} b^{(i+1)}(-\partial_t t - \bullet) W_\bullet^{(i+1)} \mathcal{M}_U = & b_2^{(i)}(-\partial_t t - \bullet) \underbrace{b_1^{(i)}(-\partial_t t - \bullet + 1) W_{\bullet-1}^{(i)} \mathcal{M}_U}_{\subseteq W_{\bullet-1}^{(i+1)} \mathcal{M}_U} \\ & \underbrace{+ b_1^{(i)}(-\partial_t t - \bullet + 1) \underbrace{b^{(i)}(-\partial_t t - \bullet) W_\bullet^{(i)} \mathcal{M}_U}_{\subseteq W_{\bullet-1}^{(i)} \mathcal{M}_U}}_{\subseteq W_{\bullet-1}^{(i+1)} \mathcal{M}_U}, \end{aligned}$$

we have $b^{(i+1)}(-\partial_t t - \bullet) \mathrm{Gr}_\bullet^{W^{(i+1)}} \mathcal{M}_U = 0$. Iterating this process until all roots are in the interval I, we obtain $V_\bullet^{\alpha+k} \mathcal{M}_U$.

Remark 3.1.23. Note that $b^{(0)}(s)$ in the last remark agrees with the minimal monic nonzero polynomial $b'(s) \in \mathbb{Q}[s]$ such that

$$b'(-\partial_t t)\overline{(e)} \in V_{-1}^{\mathrm{ind}} \mathcal{M}_U$$

for all $e \in E$ and $b^{(0)}(s)$ exists if and only if $b'(s)$ exists: Namely, consider $v := g \underline{\theta}^\alpha t^a \partial_t^b(e) \in V_{b-a} \mathcal{D}_U^E)$ with $g \in \mathcal{O}_U$. Then

$$(-\partial_t t - (b-a))v = \underbrace{g \underline{\theta}^\alpha t^a \partial_t^b}_{\in V_{b-a} \mathcal{D}_U}(-\partial_t t)(e) - \underbrace{\partial_t(g) t \underline{\theta}^\alpha t^a \partial_t^b(e)}_{\in V_{b-a-1} \mathcal{D}_U^E}$$

shows that $b'(-\partial_t t)\overline{(e)} \in V_{-1}^{\mathrm{ind}} \mathcal{M}_U$ for all $e \in E$ implies $b'(-\partial_t t)\overline{v} \in V_{b-a-1}^{\mathrm{ind}} \mathcal{M}_U$.

Remark 3.1.24. Keeping the notation of Remark 3.1.22 and assuming that $\mathcal{M}$ is X_0-specializable, we deduce from $b^{(0)}(s)$ a suitable power p such that $(-\partial_t t-\alpha)^p$ annihilates $\mathrm{Gr}_\alpha^V \mathcal{M}_U$. Namely, take $p = m_{b^{(0)}(s)}(\alpha) := \sum_{z\in\alpha+\mathbb{Z}:b^{(0)}(z)=0} \mathrm{mult}_{b^{(0)}(s)}(z)$, where $\mathrm{mult}_{b^{(0)}(s)}(z)$ denotes the multiplicity of the root z. If we choose $i \in \mathbb{N}$ such that all roots of $b^{(i)}(s)$ live in the interval I, then $b^{(i)}(-\partial_t t + k)$ acts as zero on $\mathrm{Gr}_\alpha^V \mathcal{M}_U$ by construction and the root α has multiplicity p. According to Definition 3.1.3(a) and (d) there is some $l \in \mathbb{N}$ such that $(-\partial_t t - \alpha)^l$ annihilates also that module. By Bézout's identity this implies that our choice of p is valid.

V-filtration on $\mathcal{D}_X(*X_0)$-modules

We study now properties of the V-filtration on $\mathcal{D}_X(*X_0)$-modules. In particular, we will see that the notions of V-filtrations on $\mathcal{D}_X$- and on $\mathcal{D}_X(*X_0)$-modules are compatible:

Lemma 3.1.25. *Let $\mathcal{N}$ be a coherent $\mathcal{D}_X(*X_0)$-module.*

(a) If $\mathcal{N}$ is X_0-specializable as $\mathcal{D}_X(*X_0)$-module, then it is $\mathcal{D}_X$-coherent.

(b) The module $\mathcal{N}$ is X_0-specializable as $\mathcal{D}_X(*X_0)$-module if and only if it is X_0-specializable as $\mathcal{D}_X$-module. In this case, the corresponding V-filtrations agree.

Proof.

(a) We deduce from Corollary 3.1.11 and Definition 3.1.3(a) that $\mathcal{N}$ is as $\mathcal{D}_X$-module on the coordinate neighborhood U generated by the coherent $V_0\,\mathcal{D}_X$-module $V_0\,\mathcal{N}$ and is hence locally $\mathcal{D}_X$-finitely generated.

As $\mathcal{O}_X(*X_0)$ is on U of the form $\mathcal{O}_U[t^{-1}]$ and agrees with $\mathcal{O}_{U'}$ on a neighborhood $U' \subseteq X^*$, it is in particular $\mathcal{O}_X$-quasi-coherent. Because $\mathcal{D}_X(*X_0)$ is $\mathcal{O}_X(*X_0)$-locally free, Proposition 1.1.7(a) shows that $\mathcal{N}$ is $\mathcal{O}_X$-quasi-coherent. Another application of this proposition gives now the $\mathcal{D}_X$-coherence of $\mathcal{N}$.

(b) If $\mathcal{N}$ is X_0-specializable as $\mathcal{D}_X(*X_0)$-module with V-filtration $V_\bullet\,\mathcal{N}$, then it is also X_0-specializable as $\mathcal{D}_X$-module with V-filtration $V_\bullet\,\mathcal{N}$ by Part (a) and definition of the corresponding V-filtrations.

Conversely, we only have to show that if $V_\bullet\,\mathcal{N}$ is the V-filtration on $\mathcal{N}$ considered as $\mathcal{D}_X$-module then $t^{-1}V_\alpha\,\mathcal{N}_U \subseteq V_{\alpha+1}\,\mathcal{N}_U$. By Remark 3.1.19(b), there is for $n \in V_\alpha\,\mathcal{N}_U$ some $v \in V_{-1}\,\mathcal{D}_U$ such that $b_n(-\partial_t t)n = vn$. This implies $b_n(-\partial_t t-1)t^{-1}n = t^{-1}b_n(-\partial_t t)n = t^{-1}vn = v't^{-1}n$ with $v' \in V_{-1}\,\mathcal{D}_U$. Therefore $b_{t^{-1}n}(s)$ divides $b_n(s-1)$ and hence $t^{-1}n \in V_{\alpha+1}\,\mathcal{N}_U$ by this Remark 3.1.19 showing that $V_\bullet\,\mathcal{N}$ is also the V-filtration on $\mathcal{N}$ considered as $\mathcal{D}_X(*X_0)$-module.

□

Lemma 3.1.13 holds for X_0-specializable $\mathcal{D}_X(*X_0)$-modules for all $\alpha \in \mathbb{Q}$:

Lemma 3.1.26. *Let $\mathcal{N}$ be an X_0-specializable $\mathcal{D}_X(*X_0)$-module.*

(a) The maps
$$t\cdot : V_\alpha \mathcal{N}_U \to V_{\alpha-1}\mathcal{N}_U \text{ and } t\cdot : \mathrm{Gr}^V_\alpha \mathcal{N}_U \to \mathrm{Gr}^V_{\alpha-1}\mathcal{N}_U$$
are bijective for all $\alpha \in \mathbb{Q}$. In particular, we have for all $\alpha \in \mathbb{Q}$
$$V_{\alpha-1}\mathcal{N} = \mathcal{I}\cdot V_\alpha \mathcal{N} = V_{-1}\mathcal{D}_X \cdot V_\alpha \mathcal{N}\,.$$

(b) We have $\mathcal{N} \cong \mathcal{D}_X \otimes_{V_0\mathcal{D}_X} V_0\mathcal{N}$.

Proof.

(a) We have $V_\alpha \mathcal{N}_U = t \cdot t^{-1} V_\alpha \mathcal{N}_U$ with $t^{-1}V_\alpha \mathcal{N}_U \subseteq V_{\alpha-1}\mathcal{N}_U$ by Definition 3.1.3(b) showing that Condition (c) in that definition holds for all $\alpha \in \mathbb{Q}$. Thus the claim follows from the injectivity of the action of t on $\mathcal{N}_U$.

(b) According to Corollary 3.1.11 and Lemma 3.1.25(b) the morphism
$$\varphi : \mathcal{D}_X \otimes_{V_0\mathcal{D}_X} V_0\mathcal{N} \twoheadrightarrow \mathcal{N},\ p\otimes n \mapsto pn$$
is surjective. We check the injectivity on the stalks. So consider $q \in U$ and the element $\sum_{0\le i\le s} \partial_t^i \otimes n_i \in \ker(\varphi_q)$ with $n \in V_0\mathcal{N}_q^{\{0,\dots,s\}}$. We may assume that $n_i \notin V_{-1}\mathcal{N}_q = tV_0\mathcal{N}_q$ for $i>0$ if n_i is nonzero, where the last equality holds by Part (a): Namely, if $n_i \in V_{-1}\mathcal{N}_q$, we write $n_i = tn_i'$ with $n_i' \in V_0\mathcal{N}_q$. We choose now $k_i \le i$ maximal such that there is a representation $n_i = t^{k_i}n_i''$ with $n_i'' \in V_0\mathcal{N}_q$. Hence we obtain the representation
$$\sum_{0\le i\le s} \partial_t^i \otimes n_i = 1\otimes n_0 + \sum_{1\le i\le s} \partial_t^i t^{k_i} \otimes n_i'' = 1\otimes n_0 + \sum_{0\le i\le s} (\partial_t^i \otimes \sum_{1\le j\le s: j-k_j = i} \partial_t^{k_j} t^{k_j} n_j'').$$
Applying the same procedure to the right hand side representation if necessary, we obtain after at most s steps the desired representation. Lemma 3.1.10 implies now $\partial_t^i n_i \in V_i\mathcal{N}_q \setminus V_{i-1}\mathcal{N}_q$ for $i>0$ if $n_i \neq 0$ and $n_0 \in V_0\mathcal{N}_U$. As the injectivity on the stalk at $q \in X^*$ is clear, the map φ is an isomorphism.

□

To represent an X_0-specializable $\mathcal{D}_X(*X_0)$-module locally as a quotient of a free $\mathcal{D}_X$-module and to compute its V-filtration, we use that it is a localization of a coherent $\mathcal{D}_X$-module, which is even X_0-specializable as we will see. Hence we study now the V-filtration on localizations of $\mathcal{D}_X$-modules.

Localizations

Recall that we defined the *localization* of the $\mathcal{D}_X$-module $\mathcal{M}$ along X_0 as the $\mathcal{D}_X(*X_0)$-module

$$\mathcal{M}(*X_0) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*X_0)$$

and that this localization comes with the canonical $\mathcal{D}_X$-linear localization map $i_{(*X_0)} : \mathcal{M} \to \mathcal{M}(*X_0)$. The following notation will be useful when considering filtered localizations:

Notation 3.1.27. Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}_X$-module. We write $\mathrm{Loc}_{X_0}(\mathcal{M})$ for the $\mathcal{D}_X$-module $\mathcal{M}(*X_0)$. Similarly, we write $\mathrm{Loc}_{X_0}(\mathcal{N})$ for an X_0-specializable $\mathcal{D}_X(*X_0)$-module $\mathcal{N}$ considered as $\mathcal{D}_X$-module, since $\mathcal{N} \cong \mathcal{N}(*X_0)$.

We study now localizations of X_0-specializable $\mathcal{D}_X$-modules (see also [SS17, Lemma 9.3.1 and Proposition 9.3.4(4)] for the "only if"-part of Part (a) as well as Part (b) of the following lemma):

Lemma 3.1.28. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

(a) The $\mathcal{D}_X$-module $\mathcal{M}$ is X_0-specializable if and only if the $\mathcal{D}_X(*X_0)$-module $\mathcal{M}(*X_0)$ is X_0-specializable.

(b) If $\mathcal{M}$ is X_0-specializable, the natural morphism $i_{(*X_0)} : \mathcal{M} \to \mathcal{M}(*X_0), m \mapsto m \otimes 1$ induces a representation

$$V_0(\mathcal{M}(*X_0)_U) = t^{-1} \cdot (i_{(*X_0)})_U(V_{-1}\,\mathcal{M}_U). \tag{3.1.5}$$

So in particular, $\mathcal{M}(*X_0)_U$ is generated by $t^{-1} \cdot (i_{(*X_0)})_U(V_{-1}\,\mathcal{M}_U)$ as $\mathcal{D}_U$-module.

Proof.

(a) Let $\mathcal{M}(*X_0)$ be $\mathbb{Q}$-specializable along X_0 as $\mathcal{D}_X(*X_0)$-module and hence also as $\mathcal{D}_X$-module by Lemma 3.1.25(b). By the exact sequence

$$0 \to \Gamma_{[X_0]}\,\mathcal{M} \to \mathcal{M} \to \mathcal{M}(*X_0)$$

the natural map $\mathcal{M}\,/\Gamma_{[X_0]}(\mathcal{M}) \to \mathcal{M}(*X_0)$ of $\mathcal{D}_X$-modules is injective and thus the module $\mathcal{M}\,/\Gamma_{[X_0]}\,\mathcal{M}$ is X_0-specializable by Corollary 3.1.14 being isomorphic to a submodule of the X_0-specializable $\mathcal{D}_X$-module $\mathcal{M}(*X_0)$. As $\Gamma_{[X_0]}(\mathcal{M})$ has support on X_0, it is X_0-specializable by Lemma 3.1.16, which implies the $\mathbb{Q}$-specializability of $\mathcal{M}$ by Corollary 3.1.14.

The other implication is [SS17, Lemma 9.3.1].

(b) Since $\mathcal{M}(*X_0)$ is an X_0-specializable $\mathcal{D}_X$-module by Part (a) and Lemma 3.1.25(b), the natural morphism $\mathcal{M} \to \mathcal{M}(*X_0)$ induces by Lemma 3.1.18 isomorphisms

$$V_\alpha \mathcal{M} \cong V_\alpha(\mathcal{M}(*X_0)) \text{ for } \alpha < 0,$$

and hence

$$V_0 \mathcal{M}(*X_0)_U = t^{-1} \cdot (i_{(*X_0)})_U(V_{-1} \mathcal{M}_U)$$

by Lemma 3.1.26(a).

□

The localization of an X_0-specializable $\mathcal{D}_X$-module can be represented as follows:

Lemma 3.1.29. *Let M be an X_0-specializable $\mathcal{D}_X$-module. On a coordinate neighborhood U, there exists a finite set E and a finite subset $P \subseteq \mathcal{D}_U^E(U)$ such that*

(a) $V_{-1} \mathcal{M}_U \cong (V_0 \mathcal{D}_U)^E / {}_{V_0 \mathcal{D}_U}\langle P \rangle$ *and*

(b) $\mathrm{Loc}_{X_0}(\mathcal{M})_U \cong \mathcal{D}_U^E / {}_{\mathcal{D}_U}\langle t^{-1} P t \rangle$.

Proof.

(a) Since $V_{-1} \mathcal{M}_U$ is a finitely generated $V_0 \mathcal{D}_U$-module by Definition 3.1.3(a) and Corollary 1.1.10, there exist a finite $V_0 \mathcal{D}_U$-generating set $E \subseteq V_{-1} \mathcal{M}_U(U)$ of $V_{-1} \mathcal{M}_U$ and a $V_0 \mathcal{D}_U$-linear surjective map

$$\rho : (V_0 \mathcal{D}_U)^E \twoheadrightarrow V_{-1} \mathcal{M}_U, \ (e) \mapsto e, \tag{3.1.6}$$

inducing an isomorphism $V_{-1} \mathcal{M}_U \cong (V_0 \mathcal{D}_U)^E / \ker(\rho)$, where $\ker(\rho)$ is finitely generated as $V_0 \mathcal{D}_U$-module, say by $P \subseteq V_0 \mathcal{D}_U^E(U)$, by Lemma 3.1.1 and Corollary 1.1.10.

(b) Since $\mathcal{M}(*X_0)$ is X_0-specializable by Lemma 3.1.28(a), we obtain by Lemma 3.1.28(b) a surjective $V_0 \mathcal{D}_U$-linear map

$$\rho' : (V_0 \mathcal{D}_U)^E \twoheadrightarrow V_0 \mathcal{M}(*X_0)_U, \ (e) \mapsto t^{-1} e.$$

Its kernel is ${}_{V_0 \mathcal{D}_U}\langle t^{-1} P t \rangle$ by Part (a) as $V_{-1} \mathcal{M} = V_{-1} \mathcal{M}(*X_0)$ by Lemma 3.1.18, as the map $t^{-1} \cdot : V_{-1} \mathcal{M}(*X_0)_U \to V_0 \mathcal{M}(*X_0)_U$ is bijective by Lemma 3.1.26(a) and as $t^{-1} \cdot V_0 \mathcal{D}_U = V_0 \mathcal{D}_U \cdot t^{-1}$. The claim follows now from Lemma 3.1.26(b) and Lemma 3.1.30 below.

□

The next lemma explains how to obtain from a finite $V_0 \mathcal{D}_U$-presentation of $V_0 \mathcal{M}(*X_0)_U$ a finite $\mathcal{D}_U$-presentation of $\mathcal{M}(*X_0)_U$:

Lemma 3.1.30. *Consider a finite set E and a $V_0\,\mathcal{D}_X$-submodule $\mathcal{J} \subseteq V_0\,\mathcal{D}_X^E$. The canonical isomorphism*

$$\mathcal{D}_X \otimes_{V_0\,\mathcal{D}_X} V_0\,\mathcal{D}_X^E \cong \mathcal{D}_X^E$$

induces an isomorphism of $\mathcal{D}_X$-modules

$$\mathcal{D}_X \otimes_{V_0\,\mathcal{D}_X} (V_0\,\mathcal{D}_X^E\,/\mathcal{J}) \cong \mathcal{D}_X^E\,/\,\mathcal{D}_X \cdot \mathcal{J},\ p \otimes \overline{q} \mapsto \overline{pq}.$$

Proof. By the right-exactness of the tensor product, we have a commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{D}_X \otimes_{V_0\,\mathcal{D}_X} \mathcal{J} & \longrightarrow & \mathcal{D}_X \otimes_{V_0\,\mathcal{D}_X} V_0\,\mathcal{D}_X^E & \longrightarrow & \mathcal{D}_X \otimes_{V_0\,\mathcal{D}_X} (V_0\,\mathcal{D}_X^E\,/\mathcal{J}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{D}_X\,\mathcal{J} & \longrightarrow & \mathcal{D}_X^E & \longrightarrow & \mathcal{D}_X^E\,/\,\mathcal{D}_X\,\mathcal{J} & \longrightarrow & 0, \end{array}$$

where the dashed arrow is obtained by the universal property of cokernels and agrees with the map given in the lemma. The assertion follows now by the Snake Lemma. □

Dual localizations

The dual localization along X_0 for $\mathcal{D}_X$-modules is derived from the localization functor along X_0 as its adjoint by the $\mathcal{D}$-module theoretic duality functor. Yet, we follow [SS17, Section 9.4] and give an alternative definition of dual localization functor along X_0 for X_0-specializable $\mathcal{D}_X$-modules using the V-filtration.

Definition 3.1.31. Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}_X$-module. Then

$$\mathrm{DLoc}_{X_0}(\mathcal{M}) := \mathcal{M}(!X_0) := \mathcal{D}_X \otimes_{V_0\,\mathcal{D}_X} V_{<0}\,\mathcal{M}$$

is called the *dual localization of $\mathcal{M}$ along X_0*.

The next proposition collects important results concerning the dual localization:

Proposition 3.1.32. *[SS17, Proposition 9.4.2] Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}_X$-module. Then it holds:*

(a) $\mathcal{M}(!X_0)$ is an X_0-specializable $\mathcal{D}_X$-module.

(b) The natural map $i_{(!X_0)} : \mathcal{M}(!X_0) \to \mathcal{M}, p \otimes m \mapsto pm$ induces isomorphisms

$$V_\alpha\,\mathcal{M}(!X_0) \cong V_\alpha\,\mathcal{M}$$

for $\alpha < 0$. So in particular, $\mathcal{M}(!X_0)_{X^*} \cong \mathcal{M}_{X^*}$. Moreover the kernel and the cokernel of the map

$$\mathrm{Gr}_0^V\, i_{(!X_0)} : \mathrm{Gr}_0^V\,\mathcal{M}(!X_0) \to \mathrm{Gr}_0^V\,\mathcal{M}$$

are isomorphic to the kernel and cokernel of $\partial_t \cdot : V_{-1}\,\mathcal{M} \to V_0\,\mathcal{M}$, respectively.

(c) The map

$$\partial_t \cdot : \mathrm{Gr}^V_{-1} \mathcal{M}(!X_0) \to \mathrm{Gr}^V_0 \mathcal{M}(!X_0)$$

is bijective.

Remark 3.1.33. As every X_0-specializable $\mathcal{D}_X(*X_0)$-module is also an X_0-specializable $\mathcal{D}_X$-module, we use Definition 3.1.31 to define the dual localization of X_0-specializable $\mathcal{D}_X(*X_0)$-modules.

3.1.2 Specializability, localization and dual localization along general codimension one subvarieties

Let $X_0 \subseteq X$ now be an arbitrary equidimensional codimension one subvariety. We first investigate the case that the ideal sheaf $\mathcal{I}$ is globally generated by the regular function $f : X \to \mathbb{C}$ and extend the concept of X_0-specializability to this case using Kashiwara's equivalence for the graph embedding along f. More precisely, considering the $\mathcal{D}_X$-module $\mathcal{M}$ and the embedding

$$i_f : X \hookrightarrow X \times \mathbb{C}_{t'},\ x \mapsto (x, f(x)),$$

we study the V-filtration along $X \times \{0\}$ on the $\mathcal{D}$-module theoretic direct image $(i_f)_+ \mathcal{M}$.

Specializability for $\mathcal{D}_X$-modules

Definition 3.1.34. We say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is $\mathbb{Q}$*-specializable along* f (or f-specializable) if $(i_f)_+ \mathcal{M}$ is $\mathbb{Q}$-specializable along $X \times \{0\}$.

We show that for f being smooth $\mathbb{Q}$-specializability along f and along X_0 are equivalent:

Lemma 3.1.35. *[Sai88, Lemme 3.2.4] Let $\iota : Y \hookrightarrow X$ be a closed embedding of smooth equidimensional varieties and $t : X \to \mathbb{C}$ a smooth regular function such that $t \circ \iota : Y \to \mathbb{C}$ is smooth and nonzero. Setting $X_0 = t^{-1}(0)$ and $Y_0 = \iota^{-1} X_0$, a coherent $\mathcal{D}_Y$-module $\mathcal{M}$ is Y_0-specializable if and only if $\iota_+ \mathcal{M}$ is X_0-specializable. In this case, we have on a coordinate neighborhood U with coordinates $(x_1, \dots, x_{\mathrm{n}}, t)$ and differentials $(\theta_1, \dots, \theta_{\mathrm{n}}, \partial_t)$ such that $\iota Y \cap U = V(x_l, \dots, x_{\mathrm{n}})$*

$$(\iota_+ \mathcal{M})_U = (\iota\, \mathcal{M})_U \otimes_{\mathbb{C}} \mathbb{C}[\theta_l, \dots, \theta_{\mathrm{n}}]$$

and

$$(V^{X_0}_{\bullet} \iota_+ \mathcal{M})_U = (\iota V^{Y_0}_{\bullet} \mathcal{M})_U \otimes_{\mathbb{C}} \mathbb{C}[\theta_l, \dots, \theta_{\mathrm{n}}].$$

We review Saito's proof:

Proof. As the statement is local, we may assume that Y is a codimension one subvariety of X and that $\underline{x}, t$ is a coordinate system on all of X. Hence we have $\iota_+ \mathcal{M} = \iota\, \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\theta_{\mathrm{n}}]$ (see Equation (1.4.2) and the paragraph below for the explicit $\mathcal{D}_X$-structure). Clearly, if $\mathcal{M}$

is Y_0-specializable, then $\iota V^{Y_0}_\bullet \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\theta_{\mathrm{n}}]$ satisfies Definition 3.1.3 showing that $\iota_+ \mathcal{M}$ is X_0-specializable.

Conversely, assume that $\iota_+ \mathcal{M}$ is X_0-specializable. Consider $m = \sum_{0\le i\le s} m_i \otimes \theta_{\mathrm{n}}^i \in V^{X_0}_\alpha \iota_+ \mathcal{M}$ with $m_i \in \iota \mathcal{M}$ and $\alpha \in \mathbb{Q}$. Because we have $x_{\mathrm{n}}, \theta_{\mathrm{n}} \in V^{X_0}_0 \mathcal{D}_X$, left multiplication with $\prod_{0\le i<s}(-\theta_{\mathrm{n}} x_{\mathrm{n}} - i)$ shows that $m_s \otimes \theta_{\mathrm{n}}^s \in V^{X_0}_\alpha \iota_+ \mathcal{M}$. Thus we obtain by multiplying $m_s \otimes \theta_{\mathrm{n}}^s$ with powers of x_{n} or θ_{n} that $m_s \otimes \theta_{\mathrm{n}}^j \in V^{X_0}_\alpha \iota_+ \mathcal{M}$ for all $j \in \mathbb{N}$. Induction implies $V^{X_0}_\alpha \iota_+ \mathcal{M} = \iota V'_\alpha \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\theta_{\mathrm{n}}]$ for some $V'_\alpha \mathcal{M} \subseteq \mathcal{M}$. One easily checks that $V'_\bullet \mathcal{M}$ satisfies Definition 3.1.3, which finishes the proof. □

Corollary 3.1.36. *If X_0 is smooth, then a coherent $\mathcal{D}_X$-module is $\mathbb{Q}$-specializable along X_0 if and only if it is f-specializable.*

Remark 3.1.37. If $\mathcal{M}$ is regular holonomic $\mathcal{D}_X$-module, then its direct images $(i_f)_+ \mathcal{M}$ is also regular holonomic by [HTT08, Theorem 6.1.5]. Hence Remark 3.1.19(a) implies that $\mathcal{M}$ is f-specializable.

Specializability for $\mathcal{D}_X(*X_0)$-modules

Consider now a coherent $\mathcal{D}_X(*X_0)$-module $\mathcal{N}$. Since $\mathcal{N} \cong \mathcal{N}(*X_0)$, the direct image $(i_f)_+ \mathcal{N}$ is by Lemma 1.4.19 a $\mathcal{D}_{X\times\mathbb{C}}(*X \times \{0\})$-module, which is coherent as such. This motivates the following definition:

Definition 3.1.38. Let $\mathcal{N}$ be a coherent $\mathcal{D}_X(*X_0)$-module. We say that $\mathcal{N}$ is $\mathbb{Q}$-*specializable along* f (or f-specializable) if $(i_f)_+ \mathcal{N}$ is $\mathbb{Q}$-specializable along $X\times\{0\}$ as $\mathcal{D}_{X\times\mathbb{C}}(*X\times\{0\})$-module.

Remark 3.1.39. Let $\mathcal{N}$ be a coherent $\mathcal{D}_X(*X_0)$-module.

(a) If $\mathcal{N}$ is f-specializable, then $(i_f)_+ \mathcal{N}$ is $\mathcal{D}_{X\times\mathbb{C}}$-coherent according to Lemma 3.1.25(a). Now Kashiwara's equivalence implies that $\mathcal{N}$ is $\mathcal{D}_X$-coherent and Lemma 3.1.25(b) applied to $(i_f)_+ \mathcal{N}$ for $X\times\{0\} \subseteq X\times\mathbb{C}$ shows that the two notions of f-specializability given in Definition 3.1.34 and Definition 3.1.38 are compatible.

(b) If X_0 is smooth, then $\mathcal{N}$ is X_0-specializable if and only if it is f-specializable by Part (a) and Corollary 3.1.36.

Localization and dual localization

We describe now the localization along X_0 of the f-specializable $\mathcal{D}_X$-module $\mathcal{M}$ in terms of the localization of $(i_f)_+ \mathcal{M}$ along $X \times \{0\}$:

Lemma 3.1.40. *Let $\mathcal{M}$ be an f-specializable $\mathcal{D}_X$-module. Then*

$$\mathcal{M}(*X_0) \cong \mathrm{Gr}_0^{V^{V(t'-f)}} (((i_f)_+ \mathcal{M})(*X \times \{0\})) = V_0^{V(t'-f)} (((i_f)_+ \mathcal{M})(*X \times \{0\}))\,.$$

So in particular, $\mathcal{M}(*X_0)$ is a coherent $\mathcal{D}_X$-module.

Proof. We have by Lemma 1.4.19

$$(i_f)_+(\mathcal{M}(*X_0)) \cong ((i_f)_+\mathcal{M})(*X \times \{0\})$$

and hence Corollary 3.1.17 and Proposition 1.4.12(a) imply the claim. □

Following [SS17, Section 9.4.b], we construct the dual localization of $\mathcal{M}$ along f given that $\mathcal{M}$ is f-specializable:

Definition 3.1.41. Let $\mathcal{M}$ be an f-specializable $\mathcal{D}_X$-module. The $\mathcal{D}_X$-module $\mathcal{M}(!f)$ satisfying

$$(i_f)_+\mathcal{M}(!f) = ((i_f)_+\mathcal{M})(!X \times \{0\})$$

is called the *dual localization of* $\mathcal{M}$ *along* f.

The unique existence of $\mathcal{M}(!f)$ (up to isomorphism) in the above definition relies on Kashiwara's equivalence: We have for $p \in X \times \mathbb{C}$ that

$$((i_f)_+\mathcal{M})(!X \times \{0\})_p = \mathcal{D}_{X,p} \otimes_{(V_0\,\mathcal{D}_X)_p} ((i_f)_+\mathcal{M})_p.$$

As $(i_f)_+\mathcal{M}$ has support on $V(t' - f)$, the above formula shows that the same holds for $((i_f)_+\mathcal{M})(!X \times \{0\})_p$. Now the unique existence of $\mathcal{M}(!f)$ follows from Kashiwara's equivalence.

Remark 3.1.42. [SS17, Corollary 9.4.9] Let $\mathcal{M}$ be an f-specializable $\mathcal{D}_X$-module. Then we have:

(a) By Kashiwara's equivalence there exists a natural morphism $i_{(!f)} : \mathcal{M}(!f) \to \mathcal{M}$ induced by $i_{(!X\times\{0\})} : ((i_f)_+\mathcal{M})(!X \times \{0\}) \to (i_f)_+\mathcal{M}$.

(b) The $\mathcal{D}_X$-module $\mathcal{M}(!f)$ is coherent and f-specializable by Proposition 3.1.32(a).

In order to define the dual localization along X_0, we need to show that the above construction is independent of the choice of f. Similar considerations are also necessary to extend this construction as well as the concept of specializability to the case where $\mathcal{I}$ cannot be generated by a single regular function.

Generalization of the above constructions

The following lemma is essential to generalize our notion of $\mathbb{Q}$-specializability to singular codimension one subvarieties:

Lemma 3.1.43. *[SS17, Section 9.3.c] Let $u : X \to \mathbb{C}^*$ be a regular function and $\mathcal{M}$ a coherent $\mathcal{D}_X$-module.*

(a) The $\mathcal{D}_X$-module $\mathcal{M}$ is $\mathbb{Q}$-specializable along f if and only if it is $\mathbb{Q}$-specializable along uf.

(b) We have $\mathcal{M}(!f) = \mathcal{M}(!uf)$.

Now assume that X_0 is any equidimensional codimension one subvariety of X. As X is smooth, $\mathcal{I}$ is locally generated by a single regular function.

Definition 3.1.44. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$- or $\mathcal{D}_X(*X_0)$-module.

(a) Let $U' \subseteq X$ be an open neighborhood and $f : U' \to \mathbb{C}$ a nonzero regular function such that $\mathcal{I}_U = {}_{\mathcal{O}_U}\langle f \rangle$. We say that $\mathcal{M}$ is $\mathbb{Q}$-*specializable along* f (or f-specializable) if $\mathcal{M}_{U'}$ is f-specializable.

(b) We say that $\mathcal{M}$ is $\mathbb{Q}$-*specializable along* X_0 (or X_0-specializable) if and only if $\mathcal{M}$ is f-specializable along any regular function f as in Part (a).

Remark 3.1.45.

(a) By Lemma 3.1.43, a coherent $\mathcal{D}_X$- or $\mathcal{D}_X(*X_0)$-module $\mathcal{M}$ is $\mathbb{Q}$-specializable along X_0 if and only if every $p \in X_0$ has an affine open neighborhood U' such that $\mathcal{I}_{U'}$ is generated by a regular function $f : U' \to \mathbb{C}$ and $\mathcal{M}_{U'}$ is $\mathbb{Q}$-specializable along f.

(b) Assume that X_0 is smooth. Then Definition 3.1.44 is compatible with Definition 3.1.3 by Corollary 3.1.36 and Lemma 3.1.43.

Lemma 3.1.43(b) enables us to define the dual localization of X_0-specializable $\mathcal{M}$ because local existence implies by uniqueness global existence. In particular this definition will be for smooth X_0 compatible with Definition 3.1.31.

Definition 3.1.46. Let $\mathcal{M}$ be an X_0-specializable $\mathcal{D}_X$-module. The *dual localization* $\mathcal{M}(!X_0)$ *of* $\mathcal{M}$ *along* X_0 is defined by

$$\mathcal{M}(!X_0)_{X^*} = \mathcal{M}_{X^*}$$

and

$$\mathcal{M}(!X_0)_{U'} = \mathcal{M}_U(!f),$$

for open neighborhoods U' such that $\mathcal{I}_{U'}$ is generated by the nonzero regular function $f : U' \to \mathbb{C}$. It comes with the canonical dual localization map $i_{(!X_0)} : \mathcal{M}(!X_0) \to \mathcal{M}$ defined by $(i_{(!X_0)})_{U'} = i_{(!f)}$.

Remark 3.1.47.

(a) If $\mathcal{M}$ is an X_0-specializable $\mathcal{D}_X$-module, then so are $\mathcal{M}(*X_0)$ and $\mathcal{M}(!X_0)$ [SS17, Sections 9.3.c and 9.4.b].

(b) As in Remark 3.1.33, Definition 3.1.46 defines also the dual localization along X_0 of X_0-specializable $\mathcal{D}_X(*X_0)$-modules.

We use a similar notation as in Notation 3.1.27:

Notation 3.1.48. Given that the $\mathcal{D}_X$-module $\mathcal{M}$ and the $\mathcal{D}_X(*X_0)$-module $\mathcal{N}$ are X_0-specializable, we set $\mathrm{Loc}_{X_0}(\mathcal{M}) := \mathcal{M}(*X_0)$, $\mathrm{DLoc}_{X_0}(\mathcal{M}) := \mathcal{M}(!X_0)$, $\mathrm{Loc}_{X_0}(\mathcal{N}) := \mathcal{N}$ and $\mathrm{DLoc}_{X_0}(\mathcal{N}) := \mathcal{N}(!X_0)$ and consider all these modules as (coherent) $\mathcal{D}_X$-modules.

3.2 Compatibility of the V-filtration with the order filtration and application to filtered localization and dual localization

We have studied in Subsection 1.3.2 well-filtered $\mathcal{D}$-modules with respect to the order filtration and have seen in Remark 1.4.18 that endowing the localization of the well-filtered $(\mathcal{D}_X, F_\bullet^\circ)$-module $(\mathcal{M}, F_\bullet)$ with the naive filtration $F_\bullet\,\mathcal{M}(*X_0) := F_\bullet\,\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*X_0)$ can lead to a non well-filtered $(\mathcal{D}_X, F_\bullet^\circ)$-module even if $\mathcal{M}$ is regular holonomic. However, if $F_\bullet\,\mathcal{M}$ or $F_\bullet\,\mathcal{M}(*X_0)$ satisfy compatibility properties with respect to the V-filtration of the underlying module, we can replace the naive filtration in an intrinsic way by a good $F_\bullet^\circ\,\mathcal{D}_X$-filtration. Motivated by this, we study in this section such properties referred to as strict specializability. For this we first review the corresponding material presented in [Sai88] for the $\mathcal{D}_X$-module case and in [SS17] for the $\mathcal{D}_X(*X_0)$-module case and follow then [Sai93] and [SS17] to define filtered localizations and dual localizations. Based on these considerations, we prepare the algorithmic treatment of these localizations on a sheaf theoretic level using local coordinates.

3.2.1 Strict specializability, filtered localization and dual localization along smooth codimension one subvarieties

In this subsection, we assume that $X_0 \subseteq X$ is smooth (with defining ideal sheaf $\mathcal{I}$). Recall that $U \subseteq X$ stands for a coordinate neighborhood with local coordinates $(\underline{x}, t)$ such that $\mathcal{I}_U = {}_{\mathcal{O}_U}\langle t\rangle$. Our aim is now to study certain compatibility conditions for rational V-filtrations and filtrations with respect to the order of differential operators.

Compatibility for $\mathcal{D}_X$-modules

As pointed out in Corollary 3.1.11, the V-filtration along X_0 on a $\mathbb{Q}$-specializable $\mathcal{D}_X$-module $\mathcal{M}$ is completely determined by the $V_\alpha\,\mathcal{M}$ for $\alpha \in [-0, 1]$. Another feature of the V-filtration is that $V_\alpha\,\mathcal{M}$ for $\alpha < 0$ depends only on $\mathcal{M}(*X_0) = j_{X^*} j_{X^*}^{-1}\,\mathcal{M}$ (see Lemma 3.1.18). Thus we are now in particular interested in X_0-specializable well-filtered $(\mathcal{D}_X, F_\bullet)$-modules $(\mathcal{M}, F_\bullet)$ such that $F_\bullet V_\bullet\,\mathcal{M}$ is already determined by the $F_\bullet V_\alpha\,\mathcal{M}$ with $\alpha \in [-1, 0]$ and such that $F_\bullet V_\alpha\,\mathcal{M} = V_\alpha\,\mathcal{M} \cap j_{X^*} j_{X^*}^{-1} F_\bullet\,\mathcal{M}$ for $\alpha < 0$. This motivates the following definition:

Definition 3.2.1. A well-filtered $(\mathcal{D}_X, F_\bullet^\circ)$-module $(\mathcal{M}, F_\bullet)$ is called *quasi-unipotent* along X_0 if

(a) $\mathcal{M}$ is $\mathbb{Q}$-specializable along X_0,

(b) $t\cdot : F_p V_\alpha\,\mathcal{M}_U \to F_p V_{\alpha-1}\,\mathcal{M}_U$ is surjective for $p \in \mathbb{Z}$ and $\alpha < 0$,

(c) $\partial_t \cdot : F_p \operatorname{Gr}_\alpha^V \mathcal{M}_U \to F_{p+1} \operatorname{Gr}_{\alpha+1}^V \mathcal{M}_U$ is surjective for $p \in \mathbb{Z}$ and $\alpha > -1$.

We say that $(\mathcal{M}, F_\bullet)$ is *strictly* $\mathbb{Q}$*-specializable along* X_0 (or strictly X_0-specializable) if it is quasi-unipotent along X_0 and $\mathrm{Gr}^F V_\alpha \mathcal{M}$ is a coherent $\mathrm{Gr}^{F^\circ} V_0 \mathcal{D}_X$-module for all $\alpha \in \mathbb{Q}$.

We point out that $\mathrm{Gr}^F V_\alpha \mathcal{M}$ being $\mathrm{Gr}^{F^\circ} V_0 \mathcal{D}_X$-coherent is by Proposition 1.1.17 equivalent to $F_\bullet V_\alpha \mathcal{M}$ being a well-filtered $F^\circ_\bullet V_0 \mathcal{D}_X$-module.

Example 3.2.2. A Hodge $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet)$ with Hodge filtration $F_\bullet \mathcal{M}$ is by definition strictly X_0-specializable.

Remark 3.2.3.

(a) Note that if $\mathcal{M}$ is $\mathbb{Q}$-specializable along X_0, Definition 3.2.1(c) is equivalent to

$$F_\bullet \mathcal{M}_U = \sum_{i \in \mathbb{N}} \partial_t^i F_{\bullet - i} V_0 \mathcal{M}_U \,. \tag{3.2.1}$$

Indeed, if we denote the filtration on the right hand side by $F'_\bullet \mathcal{M}_U$, then we have

$$F'_\bullet V_\alpha \mathcal{M}_U = \sum_{0 \le i \le \lfloor \alpha \rfloor} \partial_t^i F_{\bullet - i} V_0 \mathcal{M}_U + \partial_t^{\lceil \alpha \rceil} F_{\bullet - \lceil \alpha \rceil} V_{\alpha - \lceil \alpha \rceil} \mathcal{M}_U \text{ for } \alpha \ge 0.$$

Therefore,

$$\mathrm{Gr}^V_\alpha F'_\bullet \mathcal{M}_U = \partial_t^{\lceil \alpha \rceil} (\mathrm{Gr}^V_{\alpha - \lceil \alpha \rceil} F_{\bullet - \lceil \alpha \rceil} \mathcal{M}_U) \text{ for } \alpha > 0$$

and

$$F'_\bullet V_0 \mathcal{M}_U = F_\bullet V_0 \mathcal{M}_U \,.$$

Since we have by definition of $F'_\bullet \mathcal{M}_U$ that $F'_\bullet \mathcal{M}_U \subseteq F_\bullet \mathcal{M}_U$ we have equality if and only if $\mathrm{Gr}^V_\alpha F_\bullet \mathcal{M}_U = \partial_t^{\lceil \alpha \rceil} (\mathrm{Gr}^V_{\alpha - \lceil \alpha \rceil} F_{\bullet - \lceil \alpha \rceil} \mathcal{M}_U)$ for all $\alpha > 0$ or equivalently $\mathrm{Gr}^V_\alpha F_\bullet \mathcal{M}_U = \partial_t (\mathrm{Gr}^V_{\alpha - 1} F_{\bullet - 1} \mathcal{M}_U)$ for all $\alpha > 0$.

(b) Definition 3.2.1(b) is equivalent to $F_\bullet V_\alpha \mathcal{M} = V_\alpha \mathcal{M} \cap j_{X^*} j_{X^*}^{-1} F_\bullet \mathcal{M} := \{m \in V_\alpha \mathcal{M} \mid m \in F_p \mathcal{M}_{X^*}\}$ for $\alpha < 0$.

Remark 3.2.4. Lemma 3.1.10 and Lemma 3.1.13 imply that the maps in Definition 3.2.1(b) and (c) are in fact bijective.

Recall that $\mathrm{Gr}^V_\alpha \mathcal{M}$ can be considered as a $\mathcal{D}_{X_0}$-module by Remark 3.1.8 and that the filtered module $(V_\alpha \mathcal{M}, F_\bullet)$ naturally induces a filtration $F_\bullet \mathrm{Gr}^V_\alpha \mathcal{M}$ on the former module. We sometimes also write $\mathrm{Gr}^V_\alpha (\mathcal{M}, F_\bullet)$ for this filtered module.

Definition 3.2.5. We call a well-filtered X_0-specializable $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet)$ *regular along* X_0 if $\mathrm{Gr}^F \mathrm{Gr}^V_\alpha \mathcal{M}$ is a coherent $\mathrm{Gr}^{F^\circ} \mathcal{D}_{X_0}$-module for each $\alpha \in \mathbb{Q}$.

We will see that regularity in the sense of the above definition implies that the induced filtration on the so-called vanishing and nearby cycles is a good filtration (see Subsection 3.2.3).

Lemma 3.2.6. *[Sai88, Lemme 3.4.6] An X_0-quasi-unipotent $(\mathcal{D}_X, F_\bullet^\circ)$-module $(\mathcal{M}, F_\bullet)$ is regular along X_0 if and only if it is strictly X_0-specializable.*

Proof. Assume that $(\mathcal{M}, F_\bullet)$ is strictly X_0-specializable. Obviously the $\mathrm{Gr}^{F^\circ} V_0 \mathcal{D}_X$-coherence of $\mathrm{Gr}^F V_\alpha \mathcal{M}$ implies the $\mathrm{Gr}^{F^\circ} V_0 \mathcal{D}_X$- and hence the $\mathrm{Gr}^{F^\circ} \mathrm{Gr}_0^V \mathcal{D}_X$-coherence of $\mathrm{Gr}^F \mathrm{Gr}_\alpha^V \mathcal{M}$ by Proposition 1.1.7(a) for $\alpha \in \mathbb{Q}$. Since $-\partial_t t - \alpha$ acts nilpotently on $\mathrm{Gr}_\alpha^V \mathcal{M}_U$,

$$\mathrm{Gr}_\bullet^{F^\circ} \mathrm{Gr}_0^V \mathcal{D}_U \cong \mathrm{Gr}_\bullet^{F^\circ}(\iota|_{X_0 \cap U} \mathcal{D}_{X_0 \cap U}[\partial_t t]) \cong \bigoplus_{i+j=\bullet} \iota|_{X_0 \cap U} \mathrm{Gr}_i^{F^\circ}(\mathcal{D}_{X_0 \cap U})(\partial_t t)^j,$$

(where $\iota|_{X_0 \cap U} : X_0 \cap U \to U$ stands for the restriction of ι) implies that $\mathrm{Gr}^F \mathrm{Gr}_\alpha^V \mathcal{M}$ is even $\mathrm{Gr}^{F^\circ} \mathcal{D}_{X_0}$-coherent.

The other direction is [Sai88, Lemme 3.4.6]. □

The category of strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-modules supported on X_0 can be characterized using a a filtered version of Kashiwara's equivalence (see Proposition 1.4.12) due to Sabbah:

Proposition 3.2.7. *[SS17, Proposition 7.6.2](Filtered Kashiwara's equivalence) Consider a closed embedding $\iota : X_0 \hookrightarrow X$ of smooth equidimensional algebraic varieties of codimension one. The functor*

$$\iota_+ : \mathrm{Mod}_{\mathrm{coh}}(F_\bullet \mathcal{D}_{X_0}) \to \mathrm{Mod}_{\mathrm{coh}}^{X_0, \mathrm{ss}_{X_0}}(F_\bullet \mathcal{D}_X)$$

induces an equivalence of categories between the category $\mathrm{Mod}_{\mathrm{coh}}(F_\bullet \mathcal{D}_{X_0})$ and the full subcategory $\mathrm{Mod}_{\mathrm{coh}}^{X_0, \mathrm{ss}_{X_0}}(F_\bullet \mathcal{D}_X)$ of $\mathrm{Mod}_{\mathrm{coh}}(F_\bullet \mathcal{D}_X)$ whose objects are supported on X_0 and strictly $\mathbb{Q}$-specializable along X_0. Its quasi-inverse is given by

$$(\mathcal{N}, F_\bullet) \mapsto \mathrm{Gr}_0^{V^{X_0}}(\mathcal{N}, F_\bullet)(-1).$$

Compatibility for $\mathcal{D}_X(*X_0)$-modules

Contrary to X_0-specializability, the notions of strict X_0-specializability differ for $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules. We define strict $\mathbb{Q}$-specializability of $\mathcal{D}_X(*X_0)$-modules as follows:

Definition 3.2.8. We say that a well-filtered $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module $(\mathcal{N}, F_\bullet)$ is *strictly $\mathbb{Q}$-specializable* along X_0 (or strictly X_0-specializable) if

(a) $\mathcal{N}$ is $\mathbb{Q}$-specializable along X_0,

(b) $\mathrm{Gr}_\bullet^{F^\circ} V_\alpha \mathcal{N}$ is a coherent $\mathrm{Gr}_\bullet^{F^\circ} V_0 \mathcal{D}_X(*X_0)$-module for all $\alpha \in \mathbb{Q}$.

Example 3.2.9. Consider a Hodge $\mathcal{D}_{X^*}$-module $(\mathcal{N}, F_\bullet)$ with Hodge filtration $F_\bullet \mathcal{N}$. Then the $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module $(j_{X^*} \mathcal{N}, F_\bullet)$ with filtration defined by $F_\bullet j_{X^*} \mathcal{N} := j_{X^*} F_\bullet \mathcal{N}$ is strictly X_0-specializable.

The next remark explains why we do not need conditions as in Definition 3.2.1(b) and (c):

Remark 3.2.10. If $(\mathcal{N}, F_\bullet)$ is an X_0-specializable well-filtered $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module, then $F_p\mathcal{N}$ is an $\mathcal{O}_X(*X_0)$-module and hence we have by Remark 3.1.26(a) that

$$t\cdot : F_pV_\alpha\mathcal{N}_U \to F_pV_{\alpha-1}\mathcal{N}_U \tag{3.2.2}$$

is an isomorphism for all $\alpha \in \mathbb{Q}$, that is, Definition 3.2.1(b) holds for filtered $\mathcal{D}_X(*X_0)$-modules for all $\alpha \in \mathbb{Q}$. So in particular, $F_\bullet\mathcal{N}$ is already determined by the $F_\bullet V_\alpha\mathcal{N}$ for $\alpha \in (-1, 0]$. Note however, that Definition 3.2.1(c) is in general not satisfied.

We point out that a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module is in general not even well-filtered as $(\mathcal{D}_X, F_\bullet^\circ)$-module. However, following [SS17, Proposition 9.3.4], we turn such modules into strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-modules by equipping them with the following filtration:

Definition 3.2.11. Let $(\mathcal{N}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module. We define the $F_\bullet^\circ\mathcal{D}_X$-filtration $F_\bullet^{\mathrm{Loc}}$ on $\mathrm{Loc}_{X_0}(\mathcal{N})$ by

$$F_\bullet^{\mathrm{Loc}}\,\mathrm{Loc}_{X_0}(\mathcal{N}) = \sum_{i\in\mathbb{N}} F_i\mathcal{D}_X \cdot F_{\bullet-i}V_0\mathcal{N}_X$$

and write $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet) := (\mathrm{Loc}_{X_0}(\mathcal{N}), F_\bullet^{\mathrm{Loc}}) = (\mathcal{N}, F_\bullet^{\mathrm{Loc}})$.

Clearly, the above filtration is exhaustive as $F_\bullet\mathcal{N}$ is exhaustive and $V_0\mathcal{N}$ is a set of $\mathcal{D}_X$-generators of $\mathcal{N}$ by Corollary 3.1.11. In particular, we have on a coordinate neighborhood U that

$$F_\bullet^{\mathrm{Loc}}\,\mathrm{Loc}_{X_0}(\mathcal{N})_U = \sum_{i\in\mathbb{N}} \partial_t^i \cdot F_{\bullet-i}V_0\mathcal{N}_U$$

and on the complement of X_0

$$F_\bullet^{\mathrm{Loc}}\,\mathrm{Loc}_{X_0}(\mathcal{N})_{X^*} = F_\bullet\mathcal{N}_{X^*}\,.$$

Before we prove that $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ is indeed a strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module, so in particular well-filtered as such, we state some important properties of this module:

Remark 3.2.12. Let $(\mathcal{N}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module.

(a) We have $F_\bullet^{\mathrm{Loc}}V_\alpha\mathcal{N} \subseteq F_\bullet V_\alpha\mathcal{N}$ for all $\alpha \in \mathbb{Q}$ with equality $F_\bullet^{\mathrm{Loc}}V_\alpha\mathcal{N} = F_\bullet V_\alpha\mathcal{N}$ for $\alpha \leq 0$.

(b) It holds by Part (a) and Remark 3.2.10 that $F_\bullet\mathcal{N} \cong (F_\bullet^{\mathrm{Loc}}\mathcal{N}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(*X_0)$. So we have locally on a coordinate neighborhood U

$$F_\bullet\mathcal{N}_U = \{n \in \mathcal{N}_U \mid t^a n \in F_\bullet^{\mathrm{Loc}}\mathcal{N}_U \text{ for some } a \in \mathbb{N}\}.$$

Lemma 3.2.13. *Let* $(\mathcal{N}, F_\bullet)$ *be a strictly* X_0*-specializable* $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$*-module. Then the* $(\mathcal{D}_X, F^\circ_\bullet)$*-module* $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ *is strictly* X_0*-specializable.*

Proof. We first show that $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ is $F^\circ_\bullet \mathcal{D}_X$-good: The $\mathcal{O}_X$-coherence of $F_p\mathcal{N}$ implies that of $F_pV_0\mathcal{N}$, so say the latter module is locally $\mathcal{O}_U$-generated by the finite set $G_p \subseteq F_pV_0\mathcal{N}(U)$. Then $F^{\mathrm{Loc}}_p \mathcal{N}_U$ is $\mathcal{O}_U$-generated by

$$\bigcup_{j\in\mathbb{N}} \{\partial_t^{\leq j} G_{p-j}\},$$

which is finite since $F_j\mathcal{N} = 0$ for $j \ll 0$. By strict X_0-specializability of $(\mathcal{N}, F_\bullet)$ there exists $p \in \mathbb{Z}$ such that $F_qV_0\mathcal{N} = F^\circ_{q-p}V_0\mathcal{D}_X \cdot F_pV_0\mathcal{N}$ for $q \geq p$. Now we have for $q \geq p$

$$\begin{aligned}
F^\circ_{q-p}\mathcal{D}_U \cdot F^{\mathrm{Loc}}_p\mathcal{N}_U &= \Big(\sum_{j\geq 0, j+k=q-p} \partial_t^j F^\circ_k V_0\mathcal{D}_U\Big) \cdot \Big(\sum_{i\geq 0} \partial_t^i F_{p-i}V_0\mathcal{N}_U\Big) \\
&= \sum_{i,j\geq 0, j+k=q-p} \partial_t^{i+j} F^\circ_k V_0\mathcal{D}_U \cdot F_{p-i}V_0\mathcal{N}_U \\
&= \sum_{j\geq 0} \partial_t^j F_{q-j}V_0\mathcal{N}_U,
\end{aligned}$$

which shows that $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ is indeed a well-filtered $(\mathcal{D}_X, F^\circ_\bullet)$-module.

We show that $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ is strictly X_0-specializable: We know by Lemma 3.1.25(b) that Condition 3.2.1(a) is satisfied. Condition 3.2.1(b) follows from Remark 3.2.10 and Remark 3.2.12(a), while Condition 3.2.1(c) is immediate from the definition of $F^{\mathrm{Loc}}_\bullet\mathcal{N}$ and from Remark 3.2.3(a). Remark 3.2.12(a) and the strict X_0-specializability of $(\mathcal{N}, F_\bullet)$ imply also the $\mathrm{Gr}^{F^\circ}_\bullet V_0\mathcal{D}_X$-coherence of $\mathrm{Gr}^{F^{\mathrm{Loc}}}_\bullet V_\alpha\mathcal{N}$ for $\alpha \leq 0$ and hence Condition 3.2.1(c) entails it for $\alpha \geq 0$ since $\partial_t \cdot \mathrm{Gr}^{F^\circ}_p V_0\mathcal{D}_U = \mathrm{Gr}^{F^\circ}_p V_0\mathcal{D}_U \cdot \partial_t \subseteq \mathrm{Gr}^{F^\circ}_{p+1} V_0\mathcal{D}_U$ for $p \in \mathbb{Z}$. □

Example 3.2.14. Consider the Hodge $\mathcal{D}_{X^*}$-module $(\mathcal{N}, F_\bullet)$ with Hodge filtration $F_\bullet\mathcal{N}$. Its Hodge theoretic direct image $(j_{X^*})_+(\mathcal{N}, F_\bullet)$ agrees with $\mathrm{Loc}_{X_0}(j_{X^*}\mathcal{N}, F_\bullet)$.

As $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ is $(\mathcal{D}_X, F^\circ_\bullet)$-good, we have for p big enough

$$F^{\mathrm{Loc}}_q\mathcal{N} = F^\circ_{q-p}\mathcal{D}_X \cdot F^{\mathrm{Loc}}_p\mathcal{N}$$

for all $q \geq p$. Setting

$$F^{(p)}_q\mathcal{N} := \begin{cases} F^{\mathrm{Loc}}_q\mathcal{N}, & \text{if } q \leq p \\ F^\circ_{q-p}\mathcal{D}_X \cdot F^{\mathrm{Loc}}_p\mathcal{N}, & \text{else,} \end{cases}$$

p is big enough if and only if $F^{(p)}_\bullet\mathcal{N} = F^{\mathrm{Loc}}_\bullet\mathcal{N}$. We develop now a criterion that allows us to check if a given p is big enough. For this note that if $\mathcal{N}'$ is an $\mathcal{O}_X$-submodule of $\mathcal{N}$, we may identify $\mathcal{N}'(*X_0) := \mathcal{N}' \otimes_{\mathcal{O}_X} \mathcal{O}_X(*X_0) \subseteq \mathcal{N}(*X_0) \cong \mathcal{N}$ with an $\mathcal{O}_X$-submodule of $\mathcal{N}$. Part (a) of the following criterion to test the above equality of filtrations is based on results by Saito [Sai88, Proposition 3.2.2 and Remarque 3.2.3]:

Proposition 3.2.15. *Let $(\mathcal{N}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module.*

(a) We have

$$F^{(p)}_\bullet V_0\mathcal{N}_U = (F^{(p)}_\bullet \mathcal{N}_U)(*X_0 \cap U) \cap V_0\mathcal{N}_U \tag{3.2.3}$$

if and only if

$$t\cdot : V_0\mathcal{N}_U \to V_{-1}\mathcal{N}_U \tag{3.2.4}$$

is an $F^{(p)}$-strict isomorphism.

(b) If $F_p\mathcal{N}$ generates $(\mathcal{N}, F_\bullet)$ as $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module, i.e., $F_q\mathcal{N} = F^\circ_{q-p}\mathcal{D}_X(*X_0)\cdot F_p\mathcal{N}$ for $q \geq 0$, then

$$(F^{(p)}_\bullet \mathcal{N})(*X_0) = (F^{\mathrm{Loc}}_\bullet \mathcal{N})(*X_0) = F_\bullet\mathcal{N}\,. \tag{3.2.5}$$

Proof.

(a) If Equation (3.2.3) holds, Map (3.2.4) is clearly $F^{(p)}$-strict as this map is bijective by Lemma 3.1.26(a).

Conversely, assume that Map (3.2.4) is $F^{(p)}$-strict and consider $n \in V_0\mathcal{N}_U$ such that there is $a \in \mathbb{N}$ with $t^a n \in F^{(p)}_q\mathcal{N}_U$. Thus $t^a n \in F^{(p)}_q V_{-a}\mathcal{N}_U$ and hence $n \in F^{(p)}_q V_0\mathcal{N}_U$ by assumption and Lemma 3.1.26(a).

(b) As $F^{(p)}_\bullet\mathcal{N} \subseteq F^{\mathrm{Loc}}_\bullet\mathcal{N}$, the corresponding inclusion is trivial. For the reverse inclusion we work on a coordinate neighborhood U and choose $n \in (F^{\mathrm{Loc}}_q\mathcal{N}_U)(*X_0 \cap U)$. By Definition 3.1.3(b) there exists some $a \in \mathbb{N}$ such that

$$t^a n \in (F^{\mathrm{Loc}}_q\mathcal{N}_U)(*X_0 \cap U) \cap V_0\mathcal{N}_U = F_q V_0\mathcal{N}_U,$$

where the equality follows from Remark 3.2.12(b). We are done if $q \leq p$ since then $F_q V_0\mathcal{N} = F^{\mathrm{Loc}}_q V_0\mathcal{N} = F^{(p)}_q V_0\mathcal{N}$ by Remark 3.2.12(a) and definition of $F^{(p)}_\bullet\mathcal{N}$. Otherwise we have

$$t^a n \in F_q V_0\mathcal{N}_U = (F^\circ_{q-p}\mathcal{D}_X(*X_0)_U\cdot F_p\mathcal{N}_U)\cap V_0\mathcal{N}_U \subseteq F^\circ_{q-p}\mathcal{D}_X(*X_0)_U\cdot F_p V_0\mathcal{N}_U,$$

where the equality holds by assumption. For the inclusion notice that we can write $n' \in V_0\mathcal{N}_U$ as $n' = \sum_{l\in L} b_l l$ with $L \subseteq F_p\mathcal{N}_U$ finite and $b \in (F^\circ_{q-p}D_X(*X_0)_U)^L$ by hypothesis. Choosing $c \in \mathbb{N}^L$ such that $t^{c_l} l \in V_0\mathcal{N}_U$ by Definition 3.1.3(b), Remark 3.2.10 implies that $t^{c_l} l \in F_p V_0\mathcal{N}_U$ and we obtain a representation $n' = \sum_{l\in L}(b_l t^{-c_l})\cdot(t^{c_l} l) \in F^\circ_{q-p}(D_X(*X_0)_U)F_p V_0\mathcal{N}_U$.

Express $t^a n$ as an element of $F^\circ_{q-p}(\mathcal{D}_X(*X_0)_U)\cdot F_p V_0\mathcal{N}_U$ and multiply this expression with a suitable power of t to cancel out denominators. Then we get by Remark 3.2.12(a) and by definition of $F^{(p)}_\bullet\mathcal{N}$

$$t^{a+b} n \in F^\circ_{q-p}\mathcal{D}_U\cdot F_p V_0\mathcal{N}_U = F^\circ_{q-p}\mathcal{D}_U\cdot F^{\mathrm{Loc}}_p V_0\mathcal{N}_U = F^\circ_{q-p}\mathcal{D}_U\cdot F^{(p)}_p V_0\mathcal{N}_U,$$

that is, $n \in (F_q^{(p)}\mathcal{N}_U)(*U \cap X_0)$.

The second equality follows from Remark 3.2.12(a).

□

The following lemma gives a necessary condition for $F_\bullet^{\mathrm{Loc}}\mathcal{N} = F_\bullet^{(p)}\mathcal{N}$:

Lemma 3.2.16. *Consider a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module $(\mathcal{N}, F_\bullet)$. If we have $F_\bullet^{\mathrm{Loc}}\mathcal{N}_U = F_\bullet^{(p)}\mathcal{N}_U$, then $F_p^{(p)}V_0\mathcal{N}_U$ generates $(V_0\mathcal{N}_U, F_\bullet^{(p)})$ as $(V_0\mathcal{D}_U, F_\bullet^\circ)$-module.*

Proof. We have to show

$$F_q^{(p)}V_0\mathcal{N}_U = F_{q-p}^\circ V_0\mathcal{D}_U \cdot F_p^{(p)}V_0\mathcal{N}_U$$

for all $q \geq p$. As this obviously holds for $q = p$, we proceed inductively and may assume that the above equation is satisfied for all $p \leq q < q'$. Note that

$$\begin{aligned} F_{q'}^{(p)}\mathcal{N}_U = F_{q'-p}^\circ\mathcal{D}_U \cdot F_p^{\mathrm{Loc}}\mathcal{N}_U &= F_{q'-p}^\circ\mathcal{D}_U \cdot \sum_{i\in\mathbb{N}} \partial_t^i F_{p-i}^{\mathrm{Loc}}V_0\mathcal{N}_U \\ &= \sum_{i\in\mathbb{N}} F_{q'-p+i}^\circ\mathcal{D}_U \cdot F_{p-i}^{(p)}V_0\mathcal{N}_U \end{aligned}$$

by Remark 3.2.12(a). Hence we may choose for $n \in F_{q'}^{(p)}V_0\mathcal{N}_U$ a minimal integer $r \geq 0$ and for $0 \leq j \leq r$ a finite set $G_j \subseteq F_p^{(p)}V_0\mathcal{N}_U$ and elements $c_g^j \in F_{q'-\deg_{F^{(p)}}(g)-j}^\circ V_0\mathcal{D}_U$ for $g \in G_j$ such that there is a representation

$$n = \sum_{0\leq j\leq r} (\partial_t^j \sum_{g\in G_j} c_g^j g).$$

If $r > 0$,

$$\partial_t \left(\sum_{1\leq j\leq r} (\partial_t^{j-1} \sum_{g\in G_j} c_g^j g) \right) = n - \sum_{g\in G_0} c_g^0 g \in V_0\mathcal{N}_U$$

implies by Corollary 3.1.12 that $\sum_{1\leq j\leq r}(\partial_t^{j-1}\sum_{g\in G_j} c_g^j g) \in V_{-1}\mathcal{N}_U$. Iterating the above argument shows $\sum_{g\in G_r} c_g^r g \in F_{q'-r}^{(p)}V_{-1}\mathcal{N}_U$. According to Remark 3.2.12(a) and assumption this implies the existence of an element $n' \in F_{q'-r}^{(p)}V_0\mathcal{N}$ such that $\sum_{g\in G_r} c_g^r g = tn'$. By induction assumption there exist, given that $q' - r \geq p$, a set $G' \subseteq F_p^{(p)}V_0\mathcal{N}_U$ and $c' \in (F_{q'-p-r}^\circ V_0\mathcal{D}_U)^{G'}$ satisfying $n' = \sum_{g\in G'} c'_{g'}g'$. Setting $G' := \{n'\}$ and $c'_{n'} = 1$ otherwise, we obtain

$$n = \sum_{0\leq j\leq r-1} (\partial_t^j \sum_{g\in G_j} c_g^j g) + \partial_t^{r-1} \sum_{g\in G'} \partial_t t c'_{g'} g'$$

contradicting the minimality of r. □

Lemma 3.2.17. *Let $(\mathcal{N}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module. If $F_p\mathcal{N}_U$ generates $(\mathcal{N}_U, F_\bullet)$ as $(\mathcal{D}_X(*X_0)_U, F^\circ_\bullet)$-module and $F^{(p)}_p V_0\mathcal{N}_U$ generates $(V_0\mathcal{N}_U, F^{(p)}_\bullet)$ as $(V_0\mathcal{D}_U, F^\circ_\bullet)$-module, then $F^{\mathrm{Loc}}_\bullet\mathcal{N}_U = F^{(p)}_\bullet\mathcal{N}_U$ if and only if $t\cdot : V_0\mathcal{N}_U \to V_{-1}\mathcal{N}_U$ is $F^{(p)}$-strict.*

Proof. As $(\mathcal{N}, F_\bullet)$ is strictly X_0-specializable, we have by Remark 3.2.12(a) and (b)

$$F^{\mathrm{Loc}}_\bullet V_0\mathcal{N}_U = F_\bullet V_0\mathcal{N}_U = (F^{\mathrm{Loc}}_\bullet\mathcal{N})(*X_0)_U \cap V_0\mathcal{N}_U\,.$$

This implies by assumption and Proposition 3.2.15(b) that

$$F^{\mathrm{Loc}}_\bullet V_0\mathcal{N}_U = (F^{(p)}_\bullet\mathcal{N})(*X_0)_U \cap V_0\mathcal{N}_U\,. \tag{3.2.6}$$

Note that $F^{\mathrm{Loc}}_\bullet V_0\mathcal{N}$ generates $F^{\mathrm{Loc}}_\bullet\mathcal{N}$ as $F^\circ_\bullet\mathcal{D}_X$-module by definition. On the other hand $F^{(p)}_\bullet V_0\mathcal{N}$ generates $F^{(p)}_\bullet\mathcal{N}$ as $F^\circ_\bullet\mathcal{D}_X$-module: Indeed, since $(\mathcal{N}, F^{\mathrm{Loc}}_\bullet)$ is strictly X_0-specializable as $(\mathcal{D}_X, F_\bullet)$-module and $F^{\mathrm{Loc}}_q\mathcal{N} = F^{(p)}_q\mathcal{N}$ for $q \le p$, we have

$$F^{(p)}_q\mathcal{N} = \begin{cases} F^{\mathrm{Loc}}_q\mathcal{N} = \sum_{i\in\mathbb{N}} F^\circ_i\mathcal{D}_X\cdot F^{(p)}_{q-i}V_0\mathcal{N}, & q \le p\\ F^\circ_{q-p}\mathcal{D}_X\cdot F^{\mathrm{Loc}}_p\mathcal{N} = F^\circ_{q-p}\mathcal{D}_X\cdot\sum_{i\in\mathbb{N}} F^\circ_i\mathcal{D}_X\cdot F^{(p)}_{p-i}V_0\mathcal{N}, & \text{else.}\end{cases}$$

Therefore the condition $F^{(p)}_\bullet\mathcal{N}_U = F^{\mathrm{Loc}}_\bullet\mathcal{N}_U$ is equivalent to

$$F^{(p)}_\bullet V_0\mathcal{N}_U = F^{\mathrm{Loc}}_\bullet V_0\mathcal{N}_U = (F^{(p)}_\bullet\mathcal{N})(*X_0)_U \cap V_0\mathcal{N}_U, \tag{3.2.7}$$

where the last equality is due to Equation (3.2.6). By Proposition 3.2.15(a), Equation (3.2.7) is again equivalent to $t\cdot : V_0\mathcal{N}_U \to V_{-1}\mathcal{N}_U$ being an $F^{(p)}$-strict isomorphism. This finishes the proof. □

This leads to the following criterion for testing $F^{\mathrm{Loc}}_\bullet\mathcal{N} = F^{(p)}_\bullet\mathcal{N}$, which depends only on $F_\bullet\mathcal{N}$ and $F^{(p)}_\bullet\mathcal{N}$:

Corollary 3.2.18. *Let $(\mathcal{N}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module. We have $F^{(p)}_\bullet\mathcal{N}_U = F^{\mathrm{Loc}}_\bullet\mathcal{N}_U$ if only if the following conditions are satisfied:*

(a) $F_p\mathcal{N}_U$ *generates* $(\mathcal{N}_U, F_\bullet)$ *as* $(\mathcal{D}_X(*X_0)_U, F^\circ_\bullet)$*-module.*

(b) $F^{(p)}_p V_0\mathcal{N}_U$ *generates* $V_0\mathcal{N}_U$ *as* $V_0\mathcal{D}_U$*-module.*

(c) $F^{(p)}_p V_0\mathcal{N}_U$ *generates* $(V_0\mathcal{N}_U, F^{(p)}_\bullet)$ *as* $(V_0\mathcal{D}_U, F^\circ_\bullet)$*-module.*

(d) $F^{(p)}_p V_{-1}\mathcal{N}_U$ *generates* $(V_{-1}\mathcal{N}_U, F^{(p)}_\bullet)$ *as* $(V_0\mathcal{D}_U, F^\circ_\bullet)$*-module.*

The corresponding global statement also holds.

Proof. Assume that $F_\bullet^{(p)}\mathcal{N}_U = F_\bullet^{\mathrm{Loc}}\mathcal{N}_U$ holds. Then Lemma 3.2.16 implies Condition (c) and hence also Condition (b), because $F_\bullet^{\mathrm{Loc}} = F_\bullet^{(p)}$ is exhaustive. Moreover, we deduce that $F_p^{\mathrm{Loc}}V_0\mathcal{N}_U$ generates $F_\bullet^{\mathrm{Loc}}V_0\mathcal{N}_U$ as $(V_0\mathcal{D}_U, F_\bullet^\circ)$-module implying that $F_pV_0\mathcal{N}_U$ generates $F_\bullet V_0\mathcal{N}_U$ as $(V_0\mathcal{D}_U, F_\bullet^\circ)$-module by Remark 3.2.12(a). Now Condition (a) follows from Remark 3.2.10. As $t\cdot : V_0\mathcal{N}_U \to V_{-1}\mathcal{N}_U$ is $F^{(p)}$-strict and bijective by Lemma 3.2.17 and Lemma 3.1.26(a), Condition (d) follows from Condition (c) and $t \cdot F_\bullet^\circ V_0\mathcal{D}_U = F_\bullet^\circ V_0\mathcal{D}_U \cdot t$.

Conversely, Conditions (c) and (d) imply that $t\cdot : V_0\mathcal{N}_U \to V_{-1}\mathcal{N}_U$ is $F^{(p)}$-strict: As $F_q^{(p)}\mathcal{N} = F_q^{\mathrm{Loc}}\mathcal{N}$ for $q \le p$, we have according to Remark 3.2.12(a) and Remark 3.2.10 that $t \cdot F_q^{(p)}V_0\mathcal{N}_U = F_q^{(p)}V_{-1}\mathcal{N}_U$ in the case $q \le p$. On the other hand for $q > p$ it holds by Condition (c), the previous case, and Condition (d) that

$$\begin{aligned} t \cdot F_q^{(p)}V_0\mathcal{N}_U &= t \cdot F_{q-p}^\circ V_0\mathcal{D}_U \cdot F_p^{(p)}V_0\mathcal{N}_U = F_{q-p}^\circ V_0\mathcal{D}_U \cdot t \cdot F_p^{(p)}V_0\mathcal{N}_U \\ &= F_{q-p}^\circ V_0\mathcal{D}_U \cdot F_p^{(p)}V_{-1}\mathcal{N}_U = F_q^{(p)}V_{-1}\mathcal{N}_U\,. \end{aligned}$$

So the claim follows by Lemma 3.2.17 since Condition (a) holds. □

Localization and dual localization

Consider a strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module $(\mathcal{M}, F_\bullet)$. We introduce the filtration $F_\bullet$ on the $\mathcal{D}_X(*X_0)$-module $\mathcal{M}(*X_0)$ by

$$F_\bullet(\mathcal{M}(*X_0)) := (F_\bullet\mathcal{M})(*X_0) = F_\bullet\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*X_0), \tag{3.2.8}$$

where the right hand side means the image of $F_\bullet\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*X_0)$ in $\mathcal{M}(*X_0)$.

Remark 3.2.19. We have by Definition 3.2.1(b), Lemma 3.1.18 and the definition of the filtration $F_\bullet\mathcal{M}(*X)$ that the natural map $i_{(*X_0)} : \mathcal{M} \to \mathcal{M}(*X_0)$ induces isomorphisms

$$F_\bullet V_\alpha\mathcal{M} \cong F_\bullet V_\alpha(\mathcal{M}(*X_0))$$

for $\alpha < 0$.

Lemma 3.2.20. *Let $(\mathcal{M}, F_\bullet)$ be a strictly X_0-specializable $\mathcal{D}_X$-module. Then $(\mathcal{M}(*X_0), F_\bullet)$ is strictly X_0-specializable as $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module.*

Proof. Clearly, $(\mathcal{M}, F_\bullet)$ is $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-well-filtered. Its X_0-specializability follows from Lemma 3.1.28(a). By Remark 3.2.10 for $\mathcal{N} = \mathcal{M}(*X_0)$ and since $t \cdot \mathrm{Gr}^{F^\circ} V_0\mathcal{D}_X(*X_0) = \mathrm{Gr}^{F^\circ} V_0\mathcal{D}_X(*X_0) \cdot t$ it suffices to show that Definition 3.2.8(b) holds for $\alpha < 0$. Because $\mathrm{Gr}_\bullet^F V_\alpha\mathcal{M}$ is $\mathrm{Gr}_\bullet^{F^\circ} V_0\mathcal{D}_X$-coherent by assumption and $F_\bullet^\circ V_0\mathcal{D}_X = F_\bullet^\circ V_0\mathcal{D}_X(*X_0)$, Remark 3.2.19 implies that $\mathrm{Gr}_\bullet^F V_\alpha\mathcal{M}(*X_0)$ is coherent as $\mathrm{Gr}_\bullet^{F^\circ} V_0\mathcal{D}_X(*X_0)$-module for $\alpha < 0$. □

The above lemma enables us to endow $\mathcal{M}(*X_0)$ with the filtration $F_\bullet^{\mathrm{Loc}}\,\mathcal{M}(*X_0)$ via Definition 3.2.11 turning

$$\mathrm{Loc}_{X_0}(\mathcal{M}, F_\bullet) := (\mathcal{M}(*X_0), F_\bullet^{\mathrm{Loc}}) \tag{3.2.9}$$

into a strictly X_0-specializable $\mathcal{D}_X$-module. As in Lemma 3.1.28(b) we use the V-filtration on $\mathcal{M}$ to describe $\mathrm{Loc}_{X_0}(\mathcal{M}, F_\bullet)$:

Remark 3.2.21. Let $(\mathcal{M}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X, F_\bullet)$-module. According to Remark 3.2.10 and Remark 3.2.19, the canonical map $i_{(*X_0)} : \mathcal{M} \to \mathcal{M}(*X_0), m \mapsto m \otimes 1$ induces a representation

$$F_\bullet V_0\,\mathcal{M}(*X_0)_U = t^{-1} \cdot i_{(*X_0)}(F_\bullet V_{-1}\,\mathcal{M}_U).$$

Thus, we rewrite $F_\bullet^{\mathrm{Loc}}\,\mathcal{M}(*X_0)_U$ in terms of $F_\bullet V_{-1}\,\mathcal{M}_U$ as

$$F_\bullet^{\mathrm{Loc}}\mathcal{M}(*X_0)_U = \sum_{i\in\mathbb{N}} \partial_t^i t^{-1}(F_{\bullet-i}V_{-1}\,\mathcal{M}_U \otimes_{\mathcal{O}_U} \mathcal{O}_U), \tag{3.2.10}$$

where $F_{\bullet-i}V_{-1}\,\mathcal{M}_U \otimes_{\mathcal{O}_U} \mathcal{O}_U$ stands for its image in $\mathcal{M}_U \otimes \mathcal{O}_X(*X_0)_U$.

Example 3.2.22. Consider a Hodge $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet)$ with Hodge filtration $F_\bullet\,\mathcal{M}$. Then the Hodge theoretic localization $(j_{X^*})_+ j_{X^*}^{-1}(\mathcal{M}, F_\bullet)$ agrees with $\mathrm{Loc}_{X_0}(\mathcal{M}, F_\bullet)$.

Defining the filtration $F_\bullet^{\mathrm{DLoc}}\,\mathcal{M}(!X_0)$ by

$$F_\bullet^{\mathrm{DLoc}}\,\mathcal{M}(!X_0) := \sum_{i\in\mathbb{N}} F_{\bullet-i}V_{<0}^{X_0}\,\mathcal{M} \otimes_{\mathcal{O}_X} F_i^\circ\,\mathcal{D}_X \tag{3.2.11}$$

(interpreted in the same manner as above) we set $\mathrm{DLoc}_{X_0}(\mathcal{M}, F_\bullet) := (\mathcal{M}(!X_0), F_\bullet^{\mathrm{DLoc}})$. Then a filtered version of Proposition 3.1.32 holds:

Lemma 3.2.23. *[SS17, Proposition 9.4.2] Let $(\mathcal{M}, F_\bullet)$ be a strictly X_0-specializable $\mathcal{D}_X$-module.*

(a) Then $\mathrm{DLoc}_{X_0}(\mathcal{M}, F_\bullet)$ is strictly X_0-specializable as $(\mathcal{D}_X, F_\bullet^\circ)$-module.

(b) The isomorphisms in Proposition 3.1.32(b) *and* (c) *are filtered.*

Remark 3.2.24. Given a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module $(\mathcal{N}, F_\bullet)$, we endow $\mathcal{N}(!X_0)$ with the filtration $F_\bullet\,\mathcal{N}(!X_0)$ defined as in Equation (3.2.11) and set

$$\mathrm{DLoc}_{X_0}(\mathcal{N}, F_\bullet) := (\mathcal{N}(!X_0), F_\bullet^{\mathrm{DLoc}}).$$

Since this filtered module agrees with the dual localization of the strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ and as $F_\bullet V_\alpha\,\mathcal{N} = F_\bullet^{\mathrm{Loc}} V_\alpha\,\mathcal{N}$ for $\alpha \leq 0$, Lemma 3.2.23 holds also in this situation.

Example 3.2.25. Given a Hodge $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet)$, the Hodge theoretic dual localization $(j_{X^*})_! j_{X^*}^{-1}(\mathcal{M}, F_\bullet)$ is realized by $\mathrm{DLoc}_{X_0}(\mathcal{M}, F_\bullet)$. Similarly, for a Hodge $\mathcal{D}_{X^*}$-module $(\mathcal{N}, F_\bullet)$, the Hodge theoretic extraordinary direct image $(j_{X^*})_!(\mathcal{N}, F_\bullet)$ is represented by $\mathrm{DLoc}_{X_0}(j_{X^*}\,\mathcal{N}, F_\bullet)$.

3.2.2 Strict specializability, filtered localization and dual localization along general codimension one subvarieties

As for X_0-specializability, we extend strict X_0-specializability to singular codimension one subvarieties by locally considering filtered direct images under certain graph embeddings as in Subsection 3.1.2. So let $X_0 \subseteq X$ now be an arbitrary pure codimension one subvariety. First we assume that its defining ideal sheaf $\mathcal{I}$ is globally generated by the regular function $f : X \to \mathbb{C}$ and consider the corresponding graph embedding $i_f : X \to X \times \mathbb{C}_{t'}$.

Strict specializability for $\mathcal{D}_X$-modules

Mirroring Definition 3.1.34, we define:

Definition 3.2.26. We say that a well-filtered $(\mathcal{D}_X, F^\circ_\bullet)$-module $(\mathcal{M}, F_\bullet)$ is *quasi-unipotent, regular and strictly $\mathbb{Q}$-specializable along f* if $(i_f)_+(\mathcal{M}, F_\bullet)$ is quasi-unipotent, regular and strictly $\mathbb{Q}$-specializable along $X \times \{0\}$, respectively.

As usual we abbreviate strictly $\mathbb{Q}$-specializable along f by f-specializable.

Example 3.2.27. A Hodge $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet)$ with Hodge filtration $F_\bullet \mathcal{M}$ is by definition strictly f-specializable.

Analogously to Lemma 3.1.35, our two notations of strict X_0-specializability are compatible for smooth X_0:

Lemma 3.2.28. *[Sai88, Lemme 3.2.4] Let $\iota : Y \hookrightarrow X$ be a closed embedding of smooth equidimensional varieties and $t : X \to \mathbb{C}$ a smooth regular function such that $t \circ \iota : Y \to \mathbb{C}$ is smooth and nonzero. Setting $X_0 = t^{-1}(0)$ and $Y_0 = \iota^{-1} X_0$, a well-filtered $(\mathcal{D}_Y, F^\circ_\bullet)$-module $(\mathcal{M}, F_\bullet)$ is quasi-unipotent and strictly $\mathbb{Q}$-specializable along Y_0 if and only if $\iota_+(\mathcal{M}, F_\bullet)$ is quasi-unipotent and strictly $\mathbb{Q}$-specializable along X_0, respectively.*

Proof. As in the proof of Lemma 3.1.35, we may assume that Y is of codimension one in X. Keeping the notion of that proof (so in particular assuming that X is a coordinate neighborhood), the claim on the quasi-unipotence follows from that lemma and from the representation

$$F_\bullet \iota_+ \mathcal{M} = \bigoplus_{k \in \mathbb{N}} \iota F_{\bullet - k - 1} \mathcal{M} \otimes \theta_{\mathrm{n}}^k$$

(see Equation (1.4.6)).

Assuming now that $(\mathcal{M}, F_\bullet)$ and hence $\iota_+(\mathcal{M}, F_\bullet)$ are quasi-unipotent along X_0, we show that $\mathrm{Gr}^F V_\alpha^{Y_0} \mathcal{M}$ is $\mathrm{Gr}^{F^\circ} V_0^{Y_0} \mathcal{D}_Y$-coherent if and only if $\mathrm{Gr}^F V_\alpha^{X_0} \iota_+ \mathcal{M}$ is $\mathrm{Gr}^{F^\circ} V_0^{X_0} \mathcal{D}_X$-coherent, which then implies the claim on the strict specializability. We proof this by applying the equivalence in Proposition 1.1.17: Note that by Lemma 3.1.35

$$F_\bullet V_\alpha^{X_0} \iota_+ \mathcal{M} = \bigoplus_{k \in \mathbb{N}} \iota F_{\bullet - k - 1} V_\alpha^{Y_0} \mathcal{M} \otimes \theta_{\mathrm{n}}^k. \tag{3.2.12}$$

Using $F_0^\circ V_0^{X_0}\,\mathcal{D}_X = \mathcal{O}_X$ and $F_0^\circ V_0^{Y_0}\,\mathcal{D}_Y = \mathcal{O}_Y$ one checks that the $F_0^\circ V_0^{Y_0}\,\mathcal{D}_Y$-coherence of $F_q V_\alpha^{Y_0}\,\mathcal{M}$ for all $q < p$ and the $F_0^\circ V_0^{X_0}\,\mathcal{D}_X$-coherence of $F_p V_\alpha^{X_0}\iota_+\,\mathcal{M}$ are equivalent. Now assume that the $F_\bullet^\circ V_0^{Y_0}\,\mathcal{D}_Y$-module $F_\bullet V_\alpha^{Y_0}\,\mathcal{M}$ is generated by $F_p V_\alpha^{Y_0}\,\mathcal{M}$. Then $F_\bullet V_\alpha^{X_0}\iota_+\,\mathcal{M}$ is generated by $F_{p+1} V_\alpha^{X_0}\iota_+\,\mathcal{M}$ as $F_\bullet^\circ V_0^{X_0}\,\mathcal{D}_X$-module: Namely, we have for $q > p+1$

$$\begin{aligned}
F_q V_\alpha^{X_0}\iota_+\,\mathcal{M} &= \bigoplus_{k\in\mathbb{N}} \theta_{\mathrm{n}}^k(\iota F_{q-k-1} V_\alpha^{Y_0}\,\mathcal{M}\otimes 1)\\
&= \bigoplus_{k\in\mathbb{N}:q-k-1\le p} \theta_{\mathrm{n}}^k(\iota F_{q-k-1} V_\alpha^{Y_0}\,\mathcal{M}\otimes 1)\\
&+ \bigoplus_{k\in\mathbb{N}:q-k-1> p} \theta_{\mathrm{n}}^k(\iota F_{q-k-1-p} V_0^{Y_0}\,\mathcal{D}_Y\,\iota F_p V_\alpha^{Y_0}\,\mathcal{M}\otimes 1)\\
&\subseteq \bigoplus_{k\in\mathbb{N}:q-k-1\le p} F_k V_0^{X_0}\,\mathcal{D}_X(\iota F_{q-k-1} V_\alpha^{Y_0}\,\mathcal{M}\otimes 1)\\
&+ \bigoplus_{k\in\mathbb{N}:q-k-1> p} \underbrace{\theta_{\mathrm{n}}^k F_{q-k-1-p} V_0^{Y_0\,\mathcal{M}}}_{F_{q-(p+1)} V_0^{Y_0}\,\mathcal{D}_Y}(\iota F_p V_\alpha^{Y_0}\,\mathcal{M}\otimes 1).
\end{aligned}$$

Similarly, if $F_\bullet V_\alpha^{X_0}\iota_+\,\mathcal{M}$ is generated by $F_p V_\alpha^{X_0}\iota_+\,\mathcal{M}$ as $F_\bullet^\circ V_0^{X_0}\,\mathcal{D}_X$-module, $F_\bullet V_\alpha^{Y_0}\,\mathcal{M}$ is generated by $F_{p-1} V_\alpha^{Y_0}\,\mathcal{M}$ as $F_\bullet^\circ V_0^{Y_0}\,\mathcal{D}_Y$-module. □

Corollary 3.2.29. *If f is smooth, then $\mathcal{M}$ is quasi-unipotent and strictly $\mathbb{Q}$-specializable along X_0 if and only if it is quasi-unipotent and strictly $\mathbb{Q}$-specializable along f, respectively.*

Recall that ι stands to the embedding $X_0 \hookrightarrow X$ with defined ideal sheaf $\mathcal{I} = {}_{\mathcal{O}_X}\langle f\rangle$.

Corollary 3.2.30. *[Sai88, Corollaire 3.2.5] Let X_0 be smooth and $(\mathcal{M}, F_\bullet)$ be strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module. Then we have for $\alpha\in\mathbb{Q}$*

$$\iota_+(\mathrm{Gr}_\alpha^{V^{X_0}}\,\mathcal{M}, F_\bullet) \cong \mathrm{Gr}_\alpha^{V^{X\times\{0\}}}(i_f)_+(\mathcal{M}, F_\bullet)$$

as $(\mathcal{D}_X, F_\bullet^\circ)$-modules.

The subcategory $\mathrm{Mod}_{\mathrm{coh}}^{X_0,\mathrm{ss}_{X_0}}(F_\bullet\,\mathcal{D}_X)$ of the category of well-filtered and strictly X_0-specializable $F_\bullet^\circ\,\mathcal{D}_X$-modules supported on X_0 plays an important role in filtered Kashiwara's equivalence (see Proposition 3.2.7) if X_0 is smooth. It can be characterized as follows:

Lemma 3.2.31. *([Sai88, Lemma 3.2.6]) Let $(\mathcal{M}, F_\bullet)$ be a well-filtered $(\mathcal{D}_X, F_\bullet^\circ)$-module such that $\mathcal{M}$ is supported on $f^{-1}(0)$. Then the following are equivalent:*

(a) $(\mathcal{M}, F_\bullet)$ is quasi-unipotent and regular along f,

(b) $f\cdot F_\bullet\,\mathcal{M}\subseteq F_{\bullet-1}\,\mathcal{M}$,

(c) *there exists a canonical isomorphism* $(i_f)_+(\mathcal{M}, F_\bullet) \cong (i_0)_+(\mathcal{M}, F_\bullet)$ *of* $(\mathcal{D}_{X\times\mathbb{C}}, F^\circ_\bullet)$*-modules.*

If X_0 *is smooth, then the above conditions are equivalent to*

(d) $(\mathcal{M}, F_\bullet)$ *is strictly* $\mathbb{Q}$*-specializable along* X_0*.*

Proof. The first part of the lemma is [Sai88, Lemma 3.2.6]. The additional condition for X_0 smooth follows from Lemma 3.2.6 and Corollary 3.2.29. □

Strict specializability of $\mathcal{D}_X(*X_0)$-modules

By Remark 1.4.22, we may define strict X_0-specializability for $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-modules as follows (see also Lemma 1.4.19):

Definition 3.2.32. A well-filtered $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module $(\mathcal{N}, F_\bullet)$ is called *strictly* $\mathbb{Q}$*-specializable along* f if $(i_f)_+(\mathcal{N}, F_\bullet)$ is strictly $\mathbb{Q}$-specializable along $X \times \{0\}$ considered as $(\mathcal{D}_{X\times\mathbb{C}_{t'}}(*X \times \{0\}), F^\circ_\bullet)$-module.

Remark 3.2.33. Analogous to Corollary 3.2.29 we have for smooth X_0 that the well-filtered $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module $(\mathcal{N}, F_\bullet)$ is strictly $\mathbb{Q}$-specializable along X_0 if and only if it is strictly $\mathbb{Q}$-specializable along f.

As for smooth X_0 we want to endow the strictly X_0-specializable $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module $(\mathcal{N}, F_\bullet)$ with a good filtration that makes it strictly f-specializable as $(\mathcal{D}_X, F^\circ_\bullet)$-module. We use for this our standard trick of considering the direct image under the graph embedding i_f. As the $(\mathcal{D}_{X\times\mathbb{C}}(*X \times \{0\}), F^\circ_\bullet)$-module $(i_f)_+(\mathcal{N}, F_\bullet)$ is strictly $X \times \{0\}$-specializable, the $(\mathcal{D}_{X\times\mathbb{C}}, F^\circ_\bullet)$-module $\mathrm{Loc}_{X\times\{0\}}((i_f)_+(\mathcal{N}, F_\bullet))$ is well-defined and strictly $X \times \{0\}$-specializable. If the latter module is strictly $t' - f$-specializable, we may apply filtered Kashiwara's equivalence (Proposition 3.2.7), that is, induce a filtration $F^{\mathrm{Loc}}_\bullet$ on $\mathcal{N}$ via

$$(\mathcal{N}, F^{\mathrm{Loc}}_\bullet) :\cong (\mathrm{Gr}_0^{V^{V(t'-f)}}(\mathrm{Loc}_{X\times\{0\}}((i_f)_+(\mathcal{N}, F_\bullet))))(-1) \tag{3.2.13}$$

to get a good filtration on $\mathcal{N}$ as $\mathcal{N}$ is isomorphic to $\mathrm{Gr}_0^{V^{V(t'-f)}}(\mathrm{Loc}_{X\times\{0\}}((i_f)_+\mathcal{N}))$ by Kashiwara's equivalence. We write $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet) := (\mathcal{N}, F^{\mathrm{Loc}}_\bullet)$ and we will see in Lemma 3.2.39 that the definition of the filtration does not depend on the choice of f. While such an approach is not possible in Sabbah's more general situation (see [SS17, Section 9.3.c]), we show that our setting allows the application of filtered Kashiwara's equivalence:

Proposition 3.2.34. *Let* $(\mathcal{N}, F_\bullet)$ *be a strictly* f*-specializable* $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$*-module. Then the* $(\mathcal{D}_{X\times\mathbb{C}}, F_\bullet)$*-module* $\mathrm{Loc}_{X\times\{0\}}((i_f)_+(\mathcal{N}, F_\bullet))$ *is strictly* $\mathbb{Q}$*-specializable along* $t' - f$*.*

Proof. By Lemma 3.2.31 it is equivalent to show that $(t'-f)\cdot F_\bullet^{\mathrm{Loc}}(i_f)_+\mathcal{N} \subseteq F_{\bullet-1}^{\mathrm{Loc}}(i_f)_+\mathcal{N}$ holds locally on an affine open cover of $X\times\mathbb{C}_{t'}$. Choosing an affine open cover $\mathcal{U}$ of X, the $U'\times\mathbb{C}$ for $U'\in\mathcal{U}$ form an affine open cover of $X\times\mathbb{C}$ and we have

$$((i_f)_+\mathcal{N})_{U'\times\mathbb{C}} \cong (i_f|_{U'})_+\mathcal{N}_{U'},$$

where $i_f|_{U'}: U'\to U'\times\mathbb{C}$ denotes the corresponding restriction of i_f. Hence we may assume that X is affine.

Since $(\mathcal{N},F_\bullet)$ is by assumption a well-filtered $(\mathcal{D}_X(*X_0),F_\bullet^\circ)$-module, there exists a finite set $H\subseteq\mathcal{N}(X)$ and $\mathbf{d}\in\mathbb{Z}^H$ such that $F_\bullet\mathcal{N}=\sum_{h\in H}F^\circ_{\bullet-\mathbf{d}_h}\mathcal{D}_X(*X_0)\cdot h$. Consider now the $\mathcal{D}_X$-submodule $\mathcal{N}':=\sum_{h\in H}\mathcal{D}_X\cdot h$ of $\mathcal{N}$ with filtration

$$G_\bullet\mathcal{N}' := \sum_{h\in H}(F^\circ_{\bullet-\mathbf{d}_h}\mathcal{D}_X)\cdot h.$$

Then $(\mathcal{N}',G_\bullet)$ is a well-filtered $(\mathcal{D}_X,F_\bullet^\circ)$-module such that we may canonically identify

$$(\mathcal{N},F_\bullet) = (\mathcal{N}'(*X_0),G_\bullet), \tag{3.2.14}$$

where $G_\bullet\mathcal{N}'(*X_0)$ is defined as in Equation (3.2.8). This leads by Remark 1.4.21 to a natural identification

$$(i_f)_+(\mathcal{N},F_\bullet) = (i_f)_+(\mathcal{N}',G_\bullet)(*X\times\{0\}). \tag{3.2.15}$$

According to filtered Kashiwara's equivalence (Proposition 3.2.7), $(i_f)_+(\mathcal{N}',G_\bullet)$ is strictly $(t'-f)$-specializable and Lemma 3.2.31 implies $(t'-f)\cdot G_\bullet(i_f)_+\mathcal{N}'\subseteq G_{\bullet-1}(i_f)_+\mathcal{N}'$. It follows from Equation (3.2.15) that

$$(t'-f)\cdot F_\bullet(i_f)_+(\mathcal{N}) \subseteq F_{\bullet-1}(i_f)_+\mathcal{N}.$$

As $(i_f)_+(\mathcal{N},F_\bullet)$ is strictly $X\times\{0\}$-specializable by assumption, the preceding inclusion induces for $\alpha\in\mathbb{Q}$ an inclusion

$$(t-f)\cdot F_\bullet V_\alpha^{X\times\{0\}}(i_f)_+\mathcal{N} \subseteq F_{\bullet-1}(i_f)_+\mathcal{N}\cap V_\alpha^{X\times\{0\}}(i_f)_+\mathcal{N} = F_{\bullet-1}V_\alpha^{X\times\{0\}}(i_f)_+\mathcal{N}.$$

By Remark 3.2.12(a) this shows

$$(t'-f)\cdot F_\bullet^{\mathrm{Loc}}V_\alpha^{X\times\{0\}}(i_f)_+\mathcal{N} \subseteq F_{\bullet-1}^{\mathrm{Loc}}V_\alpha^{X\times\{0\}}(i_f)_+\mathcal{N},$$

for $\alpha\le 0$, where $((i_f)_+\mathcal{N},F_\bullet^{\mathrm{Loc}}) = \mathrm{Loc}_{X\times\{0\}}((i_f)_+(\mathcal{N},F_\bullet))$. Since $F_\bullet^{\mathrm{Loc}}(i_f)_+(\mathcal{N}) = \sum_{i\in\mathbb{N}}\partial_{t'}^iF^{\mathrm{Loc}}_{\bullet-i}V_0^{X\times\{0\}}(i_f)_+\mathcal{N}$ and $(t'-f)\cdot\partial_{t'}^i = \partial_{t'}^i(t'-f) - i\partial_{t'}^{i-1}$, we obtain

$$(t'-f)\cdot F_\bullet^{\mathrm{Loc}}(i_f)_+\mathcal{N} \subseteq \sum_{i\in\mathbb{N}}(\partial_{t'}^i\underbrace{(t'-f)F^{\mathrm{Loc}}_{\bullet-i}V_0^{X\times\{0\}}(i_f)_+\mathcal{N}}_{\subseteq F^{\mathrm{Loc}}_{\bullet-i-1}V_0^{X\times\{0\}}(i_f)_+\mathcal{N}} - i\partial_{t'}^{i-1}F^{\mathrm{Loc}}_{\bullet-i}V_0^{X\times\{0\}}(i_f)_+\mathcal{N})$$

and hence

$$(t'-f)\cdot F_\bullet^{\mathrm{Loc}}(i_f)_+\mathcal{N} \subseteq F_{\bullet-1}^{\mathrm{Loc}}(i_f)_+\mathcal{N}$$

as desired. The claim follows now by Lemma 3.2.31. □

The module $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ is indeed strictly f-specializable:

Proposition 3.2.35. *([SS17, Corollary 9.3.6 and Remark 9.3.8]) Let $(\mathcal{N}, F_\bullet)$ be strictly f-specializable as $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module. Then $\mathrm{Loc}_{X_0}(\mathcal{N}, F_\bullet)$ is strictly f-specializable as $(\mathcal{D}_X, F_\bullet^\circ)$-module.*

Localization and dual localization

Consider a strictly f-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module $(\mathcal{M}, F_\bullet)$. The definition of a good filtration $F_\bullet^{\mathrm{Loc}}$ on $\mathcal{M}(*X_0)$ which makes $\mathcal{M}(*X_0)$ strictly f-specializable reduces to the above case as $(\mathcal{M}(*X_0), F_\bullet)$ is a strictly f-specializable $(\mathcal{D}_X(*X_0), F_\bullet)$-module according to Remark 1.4.21 and Lemma 3.2.20. Namely, we define

$$\mathrm{Loc}_{X_0}(\mathcal{M}, F_\bullet) := \mathrm{Loc}_{X_0}(\mathcal{M}(*X_0), F_\bullet) \cong (\mathrm{Gr}_0^{V^{V(t'-f)}}(\mathrm{Loc}_{X\times\{0\}}((i_f)_+(\mathcal{M}, F_\bullet))))(-1), \tag{3.2.16}$$

where the isomorphism follows from Remark 1.4.21 and Equation (3.2.9). We denote the filtration on $\mathrm{Loc}_{X_0}(\mathcal{M}, F_\bullet)$ also by $F_\bullet^{\mathrm{Loc}}$.

On the other hand, we introduce a good $(\mathcal{D}_X, F_\bullet^\circ)$-filtration $F_\bullet^{\mathrm{DLoc}}$ on $M(!f)$ by applying the same method as for defining the filtration $F_\bullet^{\mathrm{Loc}}$ on strictly f-specializable $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-modules: Using that $(i_f)_+(\mathcal{M}(!f)) = ((i_f) + \mathcal{M})(!X \times \{0\})$ (see Definition 3.1.41), we consider the filtration

$$F_\bullet^{\mathrm{DLoc}}(((i_f)_+ \mathcal{M})(!X \times \{0\}))$$

defined by Equation (3.2.11). If the above filtered $(\mathcal{D}_{X\times\mathbb{C}}, F_\bullet^\circ)$-module is strictly specializable along $t' - f$, we induce a good filtration $F_\bullet^{\mathrm{DLoc}}$ on $\mathcal{M}(!f)$ via

$$(\mathcal{M}(!f), F_\bullet^{\mathrm{DLoc}}) :\cong \mathrm{Gr}_0^{V^{V(t'-f)}}(\mathrm{DLoc}_{X\times\{0\}}((i_f)_+(\mathcal{M}, F_\bullet))(-1). \tag{3.2.17}$$

The next proposition justifies our approach:

Proposition 3.2.36. *Let $(\mathcal{M}, F_\bullet)$ be a strictly f-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module. Then the $(\mathcal{D}_{X\times\mathbb{C}}, F_\bullet^\circ)$-module $\mathrm{DLoc}_{X\times\{0\}}((i_f)_+(\mathcal{M}, F_\bullet))$ is strictly $\mathbb{Q}$-specializable along $t' - f$.*

Proof. Arguing as in the proof of Proposition 3.2.34 and using Lemma 3.2.23(b), we have

$$(t' - f)F_\bullet^{\mathrm{DLoc}} V_{<0}^{X\times\{0\}}((i_f)_+ \mathcal{M})(!X \times \{0\}) \subseteq F_{\bullet-1}^{\mathrm{DLoc}} V_{<0}^{X\times\{0\}}((i_f)_+ \mathcal{M})(!X \times \{0\}).$$

Considering $m \in F_p V_{<0}^{X\times\{0\}}(i_f)_+ \mathcal{M}$, we obtain for $\partial_{t'}^i \otimes m \in F_{p+i}^{\mathrm{DLoc}}((i_f)_+ \mathcal{M})(!X \times \{0\})$ that

$$(t' - f)(\partial_{t'}^i \otimes m) = \partial_{t'}^i \otimes (t' - f)m - \partial_{t'}^{i-1} \otimes im \in F_{p+i-1}^{\mathrm{DLoc}}((i_f)_+ \mathcal{M})(!X \times \{0\})$$

and the claim follows now as in the proof of Proposition 3.2.34. □

Corollary 3.2.37. *[SS17, Corollary 9.4.9] If $(\mathcal{M}, F_\bullet)$ is a strictly f-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module, then so is $(\mathcal{M}(!f), F_\bullet^{\mathrm{DLoc}})$.*

Remark 3.2.38. Remark 3.2.24 allows us to extend the above construction of dual localizations along f to strictly X_0-specializable $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-modules.

Generalization of the above constructions

The following lemma is needed to generalize the notion of strict X_0-specializability to arbitrary codimension one subvarieties:

Lemma 3.2.39. *[SS17, Section 9.4.b]Let $u : X \to \mathbb{C}^*$ be a regular function. Then a well-filtered $(\mathcal{D}_X, F_\bullet^\circ)$-module is strictly $\mathbb{Q}$-specializable along f if and only if it is strictly $\mathbb{Q}$-specializable along uf. An analogous statement holds for well-filtered $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-modules. Moreover all constructions in this subsection yield the same results if we replace f by uf.*

Now assume that X_0 is any pure codimension one subvariety of X. Note that locally $\mathcal{I}$ is generated by a single regular function. This motivates the following definition (recall that $\mathcal{D}'_X$ stands either for $\mathcal{D}_X$ or $\mathcal{D}_X(*X_0)$):

Definition 3.2.40. Let $(\mathcal{M}, F_\bullet)$ be a well-filtered $(\mathcal{D}'_X, F_\bullet^\circ)$-module.

(a) Let $U' \subseteq X$ be an open subset and $f : U' \to \mathbb{C}$ a nonzero regular function such that $\mathcal{I}_{U'} = \mathcal{O}_{U'}\langle f \rangle$. We say that $(\mathcal{M}, F_\bullet)$ is *strictly $\mathbb{Q}$-specializable along f* (or strictly f-specializable) if $(\mathcal{M}_{U'}, F_\bullet)$ is a strictly f-specializable $(\mathcal{D}'_{U'}, F_\bullet^\circ)$-module.

(b) We call $(\mathcal{M}, F_\bullet)$ strictly *$\mathbb{Q}$-specializable along X_0* (or strictly X_0-specializable) if the $(\mathcal{D}'_X, F_\bullet^\circ)$-module $(\mathcal{M}, F_\bullet)$ is strictly f-specializable along any regular function f as in Part (a).

Remark 3.2.41. We have in the situation of Definition 3.2.40:

(a) Assume that X_0 is smooth. Then Definition 3.2.40 is compatible with Definition 3.2.1 by Lemma 3.2.39.

(b) In Definition 3.2.40(b) it is enough to require that every point $p \in X_0$ has an open neighborhood $U' \subseteq X$ with a regular function $f : U' \to \mathbb{C}$ as in Part (a) such that $(\mathcal{M}, F_\bullet)$ is strictly f-specializable.

As in Subsection 3.1.2, Lemma 3.2.39 allows us to introduce a filtration on the dual localization of X_0-specializable $\mathcal{M}$ because local existence implies by uniqueness global existence. In particular this definition will be for smooth X_0 compatible with Definition 3.1.46 and Equation (3.2.11).

Definition 3.2.42. Let $(\mathcal{M}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$- or $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module. The *dual localization* $\mathrm{DLoc}_{X_0}(M, F_\bullet)$ *of* $(\mathcal{M}, F_\bullet)$ *along* X_0 is defined by

$$\mathrm{DLoc}_{X_0}(M, F_\bullet)_{X^*} = (\mathcal{M}_{X^*}, F_\bullet),$$

and

$$\mathrm{DLoc}_{X_0}(M, F_\bullet)_{U'} = (\mathcal{M}_U(!f), F_\bullet^{\mathrm{DLoc}}),$$

where U' is an open neighborhood such that $\mathcal{I}_{U'}$ is generated by the nonzero function $f : U' \to \mathbb{C}$. We denote the filtration on $\mathrm{DLoc}_{X_0}(M, F_\bullet)$ also by $F_\bullet^{\mathrm{DLoc}}$.

The filtration $F_\bullet^{\mathrm{Loc}}$ on $\mathrm{Loc}_{X_0}(\mathcal{M})$ is defined analogously and we write $\mathrm{Loc}_{X_0}(M, F_\bullet) := (\mathrm{Loc}_{X_0}(\mathcal{M}), F_\bullet^{\mathrm{Loc}})$.

Remark 3.2.43. Let $(\mathcal{M}, F_\bullet)$ be a strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$- or $(\mathcal{D}_X(*X_0), F_\bullet^\circ)$-module. The $\mathrm{Loc}_{X_0}(M, F_\bullet)$ and $\mathrm{DLoc}_{X_0}(M, F_\bullet)$ is strictly X_0-specializable [SS17, Sections 9.3.c and 9.4.b].

Example 3.2.44. Examples 3.2.14, 3.2.22 and 3.2.25 generalize to the filtered setting.

3.2.3 Vanishing and nearby cycles

We finish this section by introducing the so-called vanishing and nearby cycle functors. Let $U' \subseteq X$ be an open subset, $f : U' \to \mathbb{C}$ a regular function with $\mathcal{I}_{U'} = {}_{\mathcal{O}_U}\langle f \rangle$ and $(\mathcal{M}, F_\bullet)$ a good $(\mathcal{D}_X, F_\bullet^\circ)$-module. We set

$$(\widetilde{\mathcal{M}}, F_\bullet) := (i_f)_+(\mathcal{M}_{U'}, F_\bullet),$$

where $i_f : U' \hookrightarrow U' \times \mathbb{C}_t$ stands for the graph embedding. Recall that if $\mathcal{M}$ is f-specializable then $\mathrm{Gr}_\alpha^{V^{U'\times\{0\}}} \widetilde{\mathcal{M}}$ for $\alpha \in \mathbb{Q}$ is naturally endowed with a filtration $F_\bullet$ defined by

$$\left(F_\bullet V_\alpha^{U'\times\{0\}} \widetilde{\mathcal{M}} + V_{<\alpha}^{U'\times\{0\}} \widetilde{\mathcal{M}}\right) / V_{<\alpha}^{U'\times\{0\}} \widetilde{\mathcal{M}} \cong F_\bullet V_\alpha^{U'\times\{0\}} \widetilde{\mathcal{M}} / F_\bullet V_{<\alpha}^{U'\times\{0\}} \widetilde{\mathcal{M}}.$$

Definition 3.2.45. Let $(\mathcal{M}, F_\bullet)$ be an f-specializable well-filtered $(\mathcal{D}_X, F_\bullet^\circ)$-module. We define for $\alpha \in [-1, 0)$ and $\lambda = \exp(2\pi i \alpha)$

$$\psi_{f,\lambda}(\mathcal{M}, F_\bullet) := (\psi_{f,\lambda}\, \mathcal{M}, F_\bullet) := (\mathrm{Gr}_\alpha^{V^{U'\times\{0\}}} \widetilde{\mathcal{M}}, F_\bullet)$$

and call $(\psi_f\, \mathcal{M}, F_\bullet) := \bigoplus_{-1 \le \alpha < 0} (\mathrm{Gr}_\alpha^{V^{U'\times\{0\}}} \widetilde{\mathcal{M}}, F_\bullet)$ the *nearby cycles* and $(\psi_{f,1}\, \mathcal{M}, F_\bullet)$ the *unipotent nearby cycles*.

Similarly, we define for $\alpha \in (-1, 0]$ and $\lambda = \exp(2\pi i \alpha)$

$$\phi_{f,\lambda}(\mathcal{M}, F_\bullet) := (\phi_{f,\lambda}\, \mathcal{M}, F_\bullet) := (\mathrm{Gr}_\alpha^{V^{U'\times\{0\}}} \widetilde{\mathcal{M}}, F_{\bullet+1})$$

and say that $(\phi_f\, \mathcal{M}, F_\bullet) := \bigoplus_{-1 < \alpha \le 0} (\mathrm{Gr}_\alpha^{V^{U'\times\{0\}}} \widetilde{\mathcal{M}}, F_{\bullet+1})$ are the *vanishing cycles* and that $(\phi_{f,1}\, \mathcal{M}, F_\bullet)$ are the *unipotent vanishing cycles*.

By Lemma 3.2.31 and filtered Kashiwara's equivalence we have for $f : X \to \mathbb{C}$ such that $(\mathcal{M}, F_\bullet)$ is strictly f-specializable and supported on $V(f)$ that

$$\phi_{f,1}(\mathcal{M}, F_\bullet) \cong \mathrm{Gr}_0^{V^{X\times\{0\}}}(i_0)_+(\mathcal{M}, F_\bullet)(-1) \cong (\mathcal{M}, F_\bullet)$$

motivating the shift in the definition of the filtration on $\phi_{f,\lambda}\,\mathcal{M}$.

Remark 3.2.46. Forgetting the filtrations in Definition 3.2.45, we define the corresponding notations in a non-filtered situation. Notice that while $\phi_f\,\mathcal{M} \cong V_0^{U'\times\{0\}}\widetilde{\mathcal{M}}/V_{-1}^{U'\times\{0\}}\widetilde{\mathcal{M}}$ as $\mathcal{D}_{U'}$-module by Remark 3.1.8, this isomorphism is not compatible with the $F^\circ_\bullet\,\mathcal{D}_{U'}$-structures of these modules.

Remark 3.2.47. We point out that by Remark 3.1.8 the (unipotent) vanishing and nearby cycles can be considered as $(\mathcal{D}_{U'}, F^\circ_\bullet)$-modules supported on $V(f)$ and the $\psi_{f,\lambda}(\mathcal{M}, F_\bullet)$ and $\phi_{f,\lambda}(\mathcal{M}, F_\bullet)$ are equipped with $(\mathcal{D}_{U'}, F^\circ_\bullet)$-linear filtered nilpotent endomorphisms

$$\mathrm{N} = -\partial_t t - \alpha : \psi_{f,\lambda}(\mathcal{M}, F_\bullet) \to \psi_{f,\lambda}(\mathcal{M}, F_\bullet)(-1)$$
$$\mathrm{N} = -\partial_t t - \alpha : \phi_{f,\lambda}(\mathcal{M}, F_\bullet) \to \phi_{f,\lambda}(\mathcal{M}, F_\bullet)(-1)$$

for $\lambda = \exp(2\pi i\alpha)$. The unipotent vanishing and nearby cycles come with $(\mathcal{D}_{U'}, F^\circ_\bullet)$-linear filtered morphisms

$$\psi_{f,1}(\mathcal{M}, F_\bullet) \underset{\mathrm{var}=t\cdot,\ (-1)}{\overset{\mathrm{can}=-\partial_t\cdot}{\rightleftarrows}} \phi_{f,1}(\mathcal{M}, F_\bullet)$$

such that $\mathrm{can}\circ\mathrm{var} = \mathrm{N}$ on $\phi_{f,1}(\mathcal{M}, F_\bullet)$ and $\mathrm{var}\circ\mathrm{can} = \mathrm{N}$ on $\psi_{f,1}(\mathcal{M}, F_\bullet)$, where the (-1) on the lower arrow indicates the corresponding shift in filtration on $\psi_{f,1}(\mathcal{M}, F_\bullet)$.

Remark 3.2.48. The above considerations can be generalized to arbitrary non-zero functions $f : U' \to \mathbb{C}$.

4 Algorithms for (strictly) specializable $\mathcal{D}$-modules

The purpose of this chapter is to develop algorithms for the computation of the constructions from the previous chapter by combining the theory established in that chapter and the computational methods for (bi)-weight-filtered PBW-reduction-algebras from Chapter 2. More precisely, given a smooth equidimensional variety X with a pure codimension one subvariety X_0 and assuming that X_0 is smooth, we develop algorithms for the (filtered) V-filtration along X_0 on $\mathcal{D}_X$-and $\mathcal{D}_X(*X_0)$-modules as well as the corresponding graded parts. Based on this we establish methods for the computation of vanishing and nearby cycles and their attached morphisms var, can and N. Moreover, we give new algorithms for the localizations and dual localizations along (not necessarily smooth) X_0 of (strictly) X_0-specializable $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules relying on the V-filtration and extend them to the filtered situation.

The outline of this chapter is as follows: In Section 4.1 we justify our passage to global sections for affine X and investigate the ring $\mathcal{D}_X$ and, if X_0 is smooth, also the V-filtration on $\mathcal{D}_X$ along X_0. As a first step to solve the problems outlined above, we then consider in Section 4.2 the case that X is a global coordinate neighborhood of X_0 and use that the global sections $\mathcal{D}_X(X)$ have a realization as PBW-reduction-algebra with $V_\bullet^{X_0}$- and $F_\bullet^\circ$-filtrations induced by weight vectors permitting us to apply the algorithms from Chapter 2. Building on this we develop techniques to compute (filtered) V-filtrations and their graded parts as well as localizations and dual localizations along X_0. Next, we consider in Section 4.3 computations in local coordinates for not necessarily smooth X_0 by reducing them to the previous section via a graph embedding and a coordinate change. Finally, we extend in Section 4.4 the results of the previous two sections to general affine varieties via an algorithm that glues filtered presentations given on an affine open cover of X. Moreover, we indicate how to generalize these methods to non-affine X.

We keep the notation of the previous chapter. So in particular X stands for a smooth equidimensional variety and $X_0 \subseteq X$ is a pure codimension one subvariety with embedding $\iota : X_0 \hookrightarrow X$ and defining ideal sheaf $\mathcal{I}$. We write $X^* = X \setminus X_0$ for the complement and j_{X^*} for the corresponding inclusion into X.

Algorithmically the following questions arise in the context of this chapter: Given coherent $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules $\mathcal{M}$ and $\mathcal{N}$ with optional good $F_\bullet^\circ \mathcal{D}_X$- and $F_\bullet^\circ \mathcal{D}_X(*X_0)$-filtrations $F_\bullet \mathcal{M}$ and $F_\bullet \mathcal{N}$, respectively, find algorithms that perform the following tasks:

- Decide if $\mathcal{M}$ and $\mathcal{N}$ are (strictly) X_0-specializable.

- If $(\mathcal{M},(F_\bullet))$ and $(\mathcal{N},(F_\bullet))$ are (strictly) X_0-specializable and X_0 is smooth, compute $(V_\alpha\,\mathcal{M},(F_\bullet))$ and $(V_\bullet\,\mathcal{N},(F_\bullet))$, respectively for all $\alpha\in\mathbb{Q}$.
- If $\mathcal{M}$ and $\mathcal{N}$ are (strictly) X_0-specializable, compute presentations of the (dual) localizations $\mathrm{Loc}_{X_0}(\mathcal{M},(F_\bullet))$, $\mathrm{Loc}_{X_0}(\mathcal{N},(F_\bullet))$, $\mathrm{DLoc}_{X_0}(\mathcal{M},(F_\bullet))$ and $\mathrm{DLoc}_{X_0}(\mathcal{N},(F_\bullet))$ as $(\mathcal{D}_X,(F^\circ_\bullet))$-modules.
- Given that $U'\subseteq X$ is open and $f:U'\to\mathbb{C}$ is a regular function such that $\mathcal{M}$ is (strictly) f-specializable, find representations of the vanishing and nearby cycle functors $\phi_f(\mathcal{M},(F_\bullet))$ and $\psi_f(\mathcal{M},(F_\bullet))$, of their unipotent equivalents $\phi_{f,1}(\mathcal{M},(F_\bullet))$ and $\psi_{f,1}(\mathcal{M},(F_\bullet))$ and of the maps can and var.

Here, we mean for instance by $(\mathcal{M},(F_\bullet))$ the pair $(\mathcal{M},F_\bullet)$ if $\mathcal{M}$ is equipped with the optional good filtration $F_\bullet\,\mathcal{M}$ and the module $\mathcal{M}$ otherwise. We solve in this chapter all problems expect for checking if a given modules is strictly X_0-specializable. In addition to that, we indicate how to make the quasi-inverse in Kashiwara's equivalence for mixed Hodge modules computationally accessible.

4.1 Reducing the problem to a global section situation

As every smooth equidimensional variety has a finite cover by smooth irreducible affine varieties of the same dimension and a sheaf is uniquely determined by its restrictions to such a cover and the gluing data, it suffices to explain how to do the computations on elements of such a cover and how to patch the so obtained objects together. *Hence we assume in this chapter if not stated otherwise that X is a (smooth) irreducible affine variety and identify it with a closed set of $\mathbb{C}^{\mathrm{n}}$ for a suitable natural number $\mathrm{n}\in\mathbb{N}$.*

For our computations, we wish to pass to the global sections, requiring equivalences of categories

$$\Gamma(X,\bullet):\mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_X)\to\mathrm{Mod}_{\mathrm{fg}}(\Gamma(X,\mathcal{A}_X)) \tag{4.1.1}$$

and

$$\Gamma(X,\bullet):\mathrm{Mod}_{\mathrm{coh}}(F^\circ_\bullet\mathcal{A}_X)\to\mathrm{Mod}_{\mathrm{fg}}(\Gamma(X,F^\circ_\bullet\mathcal{A}_X)), \tag{4.1.2}$$

where $\mathcal{A}_X$ stands for $\mathcal{D}_X,\mathcal{D}_X(*X_0)$ or (if X_0 is smooth) $V_0^{X_0}\,\mathcal{D}_X=V_0^{X_0}\,\mathcal{D}_X(*X_0)$. The sheaf of rings $\mathcal{A}_X$ being $\mathcal{O}_X$-quasi-coherent and locally Noetherian (see Proposition 1.2.18, Remark 1.4.17 and Lemma 3.1.1), the equivalence of categories in the unfiltered situation is immediate by Corollary 1.1.10. Since the ring $\mathrm{Gr}^F\,\mathcal{A}_X$ is locally left Noetherian and $\mathcal{O}_X$- or $\mathcal{O}_X(*X_0)$-locally free by Lemma 1.2.17, Remark 1.4.17 and the proof of Lemma 3.1.1, we have according to Corollary 1.1.10 that

$$\Gamma(X,\bullet):\mathrm{Mod}_{\mathrm{coh}}(\mathrm{Gr}^{F^\circ}\,\mathcal{A}_X)\to\mathrm{Mod}_{\mathrm{fg}}(\Gamma(X,\mathrm{Gr}^{F^\circ}\,\mathcal{A}_X))$$

is an equivalence of categories. By Proposition 1.1.17 and [HTT08, Proposition D.1.1] the functor in Equation (4.1.2) is hence essentially surjective with an essentially surjective inverse.

One easily checks that filtered morphisms are indeed preserved under this functor and that this functor is fully faithful. This allows us to replace all sheaves involved by their global sections. All notations and results carry over to the global section case by applying the above equivalences of categories. Replacing X by X_0 (if X_0 is smooth), we see that similar considerations hold also in this case.

Let now O_X, D_X, $D_X(*X_0)$, $V_0 D_X = V_0 D_X(*X_0)$ (for smooth X_0), D_{X_0} (for smooth X_0), M, $M(*X_0)$, N and I denote the global sections of $\mathcal{O}_X$, $\mathcal{D}_X$, $\mathcal{D}_X(*X_0)$, $V_0\,\mathcal{D}_X = V_0\,\mathcal{D}_X(*X_0)$, $\mathcal{D}_{X_0}$, $\mathcal{M}$, $\mathcal{M}(*X_0)$, $\mathcal{N}$ and $\mathcal{I}$, respectively. As M and N are finitely generated D_X- and $D_X(*X_0)$-modules with optional good $F^\circ_\bullet D_X$- and $F^\circ_\bullet D_X(*X_0)$-filtrations, respectively, we may assume

$$(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet) \text{ and } (N, F_\bullet) = (D_X(*X_0)^E/L, F^\circ[\mathbf{s}]_\bullet)$$

with E some finite set, $\mathbf{s} \in \mathbb{Z}^E$ a shift vector and $K \subseteq D_X^E$ and $L \subseteq D_X(*X_0)^E$ submodules, respectively.

Before we start with developing actual algorithms, we need to understand the structure and computational properties of D_X and, if X_0 is smooth, also of $V_0^{X_0} D_X$: While we can represent D_X as a $\mathbb{C}$-algebra in terms of generators and relations and consider it is a PBW-reduction-algebra (see [Bav10, Theorem 1.2]), we do not know how to determine a corresponding PBW-reduction datum and hence how to solve Gröbner basics over such a type of ring. However, we have seen in Chapter 2 that a PBW-reduction datum of D_X is computable for certain X: For instance the global sections of $\mathcal{D}_{\mathbb{C}^n}$ coincide with the Weyl algebra D_n allowing us to apply our considerations of Chapter 2 (see Example 1.2.2 and Example 2.1.26). More generally, if X has a global coordinate system then D_X is by Example 2.1.30 a PBW-reduction-algebra with computable PBW-reduction datum and similarly good properties. So our approach will be to do the computations locally using local coordinate systems and then glue the so obtained objects. Before we begin with the local computations, we assume now for a moment that X_0 is smooth and describe the V-filtration on D_X along X_0:

Lemma 4.1.1. *The $\mathbb{C}$-subalgebra $V_0^{X_0} D_X$ of D_X is generated by O_X and $\operatorname{Der}_I(O_X) := \operatorname{Der}_{\mathcal{I}}(\mathcal{O}_X)(X)$. Moreover, it holds*

$$V_k^{X_0} D_X = \begin{cases} I^{-k} V_0^{X_0} D_X, & \text{if } k \leq 0 \\ V_{k-1}^{X_0} D_X + \Theta_X(X) \cdot V_{k-1}^{X_0} D_X, & \text{else} \end{cases}$$

and

$$V_k^{X_0}\,\mathcal{D}_X(*X_0) = I^{-k} V_0^{X_0}\,\mathcal{D}_X \text{ for } k \in \mathbb{Z}\,.$$

Proof. Denoting our claimed V-filtration by $V'_\bullet D_X$, we obviously have $V'_\bullet D_X \subseteq V_\bullet D_X$. For the converse inclusion it suffices to show that for some affine open cover $\{D(g)\}_{g \in G}$ of X with $G \subseteq O_X$ finite

$$V_\bullet D_{D(g)} \subseteq (V'_\bullet D_X) \otimes_{O_X} O_X[g^{-1}] \text{ for all } g \in G,$$

under the identification $D_{D(g)} = D_X \otimes_{O_X} O_X[g^{-1}]$. This is clearly the case for $D(h) \subseteq X^*$ as $h \in I$ implies $h \cdot \Theta_X(X) \subseteq I \cdot \Theta_X(X) \subseteq \mathrm{Der}_I(O_X)$. Thus $\mathrm{Der}_I(O_X) \otimes_{O_X} O_X[h^{-1}] = \Theta_X(X) \otimes_{O_X} O_X[h^{-1}]$ and hence $V'_k D_X \otimes_{O_X} O_X[h^{-1}] = D_X \otimes_{O_X} O_X[h^{-1}] = V_k D_X \otimes_{O_X} O_X[h^{-1}]$ for all $k \in \mathbb{Z}$. This reduces the problem to the case that $D(g)$ is a coordinate neighborhood of X_0 with coordinates $x_1, \dots, x_{\dim X - 1}, t$ and derivations $\theta_1, \dots, \theta_{\dim X-1}, \partial_t$ such that $\mathcal{I}_{D(g)}$ is generated by t. By the definition of local coordinate systems, we have $\theta_i, t\partial_t \in \mathrm{Der}_{O_{D(g)}I}(\mathcal{O}_{D(g)}) = \mathrm{Der}_I(O_X) \otimes_{O_X} O_X[g^{-1}]$, where the equality is due to Remark 1.2.15. The claim follows now by the representations of $V_\bullet \mathcal{D}_{D(g)}$ in Equation (3.1.2) and by a similar representation of $V_\bullet \mathcal{D}_X(*X_0)_{D(g)}$. □

Hence it remains to describe $\mathrm{Der}_I(O_X)$.

Lemma 4.1.2. *Let $X_0 = \bigsqcup_{j \in J} V(I_j)$ be the decomposition of X_0 into irreducible components with $I_j \subseteq \mathbb{C}[x_1, \dots, x_\mathrm{n}]$ prime and generated by $I_{(j)} = \{f_1^j, \dots, f_{s_j}^j\}$. Then we have:*

(a) The O_X-module $\mathrm{Der}_{I_j}(O_X)$ is generated by the operators induced from $I_{(j)}\Theta_{\mathbb{C}^\mathrm{n}}(\mathbb{C}^\mathrm{n})$ and

$$\det \begin{pmatrix} (\partial_{l_m}(f_{k_i}^j))_{\substack{1 \le i \le r, \\ 1 \le m \le r+1}} \\ (\partial_{l_m})_{1 \le m \le r+1} \end{pmatrix}$$

for $1 \le k_1 < \dots \le k_r \le s_j$ and $1 \le l_1 < \dots \le l_{r+1} \le \mathrm{n}$, where $r = \mathrm{n} - \dim X_0$.

(b) The O_X-module $\mathrm{Der}_I(O_X)$ is O_X-generated by $\bigcup_{j \in J} \left(\prod_{j' \in J \setminus \{j\}} I_{(j')} \right) \mathrm{Der}_{I_j}(O_X)$.

Proof.

(a) Follows immediately from [Bav10, Theorem 1.1].

(b) Write $\mathrm{Der}_I(O_X)'$ for the O_X-module generated by $\bigcup_{j \in J} \left(\prod_{j' \in J \setminus \{j'\}} I_{(j)} \right) \mathrm{Der}_{I_j}(O_X)$. It clearly holds that this module is contained in $\mathrm{Der}_I(O_X)$. It is now enough to show

$$\mathrm{Der}_I(O_X) \otimes_{O_X} O_X[g^{-1}] \subseteq \mathrm{Der}_I(O_X)' \otimes_{O_X} O_X[g^{-1}]$$

for a finite affine open cover $\{D(g)\}_{g \in G}$ of X with $G \subseteq O_X$. Arguing as in the proof of Lemma 4.1.1, we may restrict ourselves to those g such that $X_0 \cap D(g) \neq \emptyset$. So in particular it suffices to consider $g = \prod_{j \in J \setminus \{j'\}} f_{k_j}^j$ with $1 \le k_j \le s_j$ and $j' \in J$. But now we have $\mathrm{Der}_I(O_X)' \otimes_{O_X} O_X[g^{-1}] = \mathrm{Der}_{I_{j'}}(O_X) \otimes_{O_X} O_X[g^{-1}] = \mathrm{Der}_I(O_X) \otimes_{O_X} O_X[g^{-1}]$ finishing the proof.

□

As for D_X we do in general not know how to realize $V_0 D_X$ in terms of a PBW-reduction datum. However, on a coordinate neighborhood of X_0 in X, Example 2.1.30 explains how to obtain such a presentation. Hence we first consider the case that X_0 is smooth and X is a global coordinate neighborhood of X_0 and develop algorithms for the problems outlined at the beginning of this chapter using the methods from Chapter 2. Then we generalize this in two directions. Via gluing we consider the case of smooth X_0 and general X and via graph embeddings and gluing we treat the case that X_0 is singular. Before we start, we agree upon the following convention:

Convention 4.1.3. In this chapter when formulating algorithms, we assume that there exists a computable subfield $\mathbb{K} \subseteq \mathbb{C}$ containing $\mathbb{Q}$ such that we can decide whether all (complex) zeros of a given polynomial $p(s) \in \mathbb{K}[s]$ are rational and such that $X_0 \subseteq X \subseteq \mathbb{C}^{\mathrm{n}}$ are defined by the vanishing of polynomials in $\mathbb{K}[x_1, \ldots, x_{\mathrm{n}}]$. We also assume that all appearing input data (such as generators of modules) is defined over $\mathbb{K}$.

For readability of our algorithms, when writing $(D_X^E/K, F^\circ[\mathrm{s}]_\bullet)$, we implicitly assume that E is a finite set, $K \subseteq D_X^E$ a submodule given by a finite set of generators and $\mathbf{s} \in \mathbb{Z}^E$ (and likewise for finitely presented $(D_X(*X_0), F_\bullet^\circ)$-modules).

4.2 Computations using global coordinate systems for smooth codimension one subvarieties

Consider the affine $\mathrm{n}+1$-space $\mathbb{C}^{\mathrm{n}+1}$ with coordinates $x_1, \ldots, x_{\mathrm{n}}, t$ and the smooth, irreducible subvariety $X = V(J) \subseteq \mathbb{C}^{\mathrm{n}+1}$ of dimension $\mathrm{m}+1$, defined by the prime ideal $J \subseteq \mathbb{C}[\underline{x}, t] := \mathbb{C}[x_1, \ldots, x_{\mathrm{n}}, t]$, with the property that it is a global coordinate neighborhood of its smooth pure codimension one subvariety X_0. By Remark 1.2.12, we may assume that a set of global coordinates is given by the residue classes of $f_1, \ldots, f_{\mathrm{m}}, t \in \mathbb{C}[\underline{x}, t]$, that $X_0 = V(J \cup t)$ and that corresponding derivations are induced by derivations $\theta_1^l, \ldots, \theta_{\mathrm{m}}^l, \theta_{\mathrm{m}+1}^l \in \mathrm{Der}(\mathbb{C}[\underline{x}, t])$ of the form $\theta_i^l = \sum_{1 \leq j \leq \mathrm{n}} a_j^i(\underline{x})\partial_i + \delta_{i(\mathrm{m}+1)}\partial_t$ (for $a_j^i \in \mathbb{C}[\underline{x}]$). So it holds in particular $\theta_i^l(t) = \delta_{i,(\mathrm{m}+1)}$. According to Example 2.1.30 D_X is realized as the PBW-reduction-algebra

$$T_X := (\mathbb{C}\langle \underline{x}, t, \theta_1, \ldots, \theta_{\mathrm{m}}, \partial_t \rangle, S, J', \prec)$$

with

$$S = \{[x_j, x_i], [t, x_i], [\theta_p, \theta_k], [\theta_k, t], [\theta_k, x_i] - \theta_k^l(x_i), [\partial_t, \theta_k], [\partial_t, x_i] - \theta_{\mathrm{m}+1}^l(x_i), [\partial_t, t] - 1 | \text{ for } 1 \leq i \leq j \leq \mathrm{n}, 1 \leq k \leq p \leq \mathrm{m}\} \setminus \{0\},$$

$\prec$ any well-order such that S is a standard reduction system with respect to $\prec$ (for instance a well-ordering satisfying $\underline{x}^\alpha t^\beta \underline{\theta}^\gamma \partial_t^\delta \prec \underline{x}^{\alpha'} t^{\beta'} \underline{\theta}^{\gamma'} \partial_t^{\delta'}$ if $|\gamma| + \delta < |\gamma'| + \delta'$ using usual multi-index notation) and $J' \subseteq \mathbb{C}[\underline{x}, t]$ a Gröbner basis of J with respect to the ordering induced by $\prec$. Obviously, the isomorphism between D_X and T_X is given by sending $\underline{x}, t, \theta_1, \ldots, \theta_{\mathrm{m}}$ and $\theta_{\mathrm{m}+1}$ to $\underline{x}, t, \theta_1, \ldots, \theta_{\mathrm{m}}$ and ∂_t, respectively. Denoting by $\mathbf{v} \in \mathbb{Z}^{\mathrm{n}+\mathrm{m}+2}$ the weight vector on T_X that assigns weight 1 to ∂_t, weight -1 to t and weight 0 else, that isomorphism induces isomorphisms $V_\bullet D_X \cong F_\bullet^{\mathbf{v}} T_X$. Similarly, writing $\mathbf{w} \in \mathbb{Z}^{\mathrm{n}+\mathrm{m}+2}$ for the weight vector that gives ∂_t and θ_i ($1 \leq i \leq \mathrm{m}$) weight 1 and the other variables weight 0, we obtain $F_\bullet^\circ D_X \cong F_\bullet^{\mathbf{w}} T_X$. Note that by Example 2.3.5 and Example 2.4.3 all assumptions of Section 2.3 and Section 2.4 are satisfied and we may hence apply the methods developed in Chapter 2. We point out that PBW-reduction data of the subalgebra $F_0^{\mathbf{v}} D_X$ and the subquotient algebras $\mathrm{Gr}_0^{\mathbf{v}} D_X \cong D_{X_0}[t\partial_t]$ and D_{X_0} of D_X are computable by Example 2.1.30.

From now on we identify D_X with T_X and use also the notation $F_\bullet^{\mathbf{u}} D_X$ for a weight vector $\mathbf{u}$ on T_X. We usually write D_X, but we represent its elements as elements of T_X, which are in turn given as residue classes of elements of $\mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle := \mathbb{C}\langle \underline{x}, t, \theta_1, \ldots, \theta_m, \partial_t\rangle$. We usually omit the residue class notation when its clear from the context that we interpret elements of the latter $\mathbb{C}$-algebra as elements of T_X by taking residue classes. We use analogous conventions also for other PBW-reduction-algebras considered in this section.

Remark 4.2.1. In view of Convention 4.1.3 we may assume that some generating set of J is defined over $\mathbb{K}[x_1, \ldots, x_\mathrm{n}]$. Hence our system of global coordinates can be realized as residue classes of polynomials in $\mathbb{K}[x_1, \ldots, x_\mathrm{n}]$ implying that we may assume that $\mathbb{K}$ is a T_X- and T_{X_0}-computable field.

Equipped with the tools from Chapter 2, we start by developing an algorithm for the V-filtration:

4.2.1 The V-filtration on D_X-modules

We want to check whether $M = D_X^E/K$ is $\mathbb{Q}$-specializable along X_0 and compute the V-filtration in this case. As X is a global coordinate system, we may apply Lemma 3.1.21 and Remark 3.1.22 globally. This reduces the computation of $V_\alpha M$ to the computation the b-function with respect to the induced V-filtration along X_0 on D_X^E/K. Recall that by Remark 3.1.23 the polynomial $b(s) \in \mathbb{K}[s]$ is the induced b-function on that module if and only if $b(s)$ is the minimal nonzero monic polynomial satisfying

$$b(-\partial_t t)\overline{(e)} \subseteq F_{-1}^{\mathbf{v}}(D_X^E/K) \tag{4.2.1}$$

for all $e \in E$. Hence it suffices to give an algorithm for the computation of a minimal polynomial as in Equation (4.2.1) on finitely presented D_X-modules. We call this polynomial also the induced b-function with respect to $\mathbf{v}$. For this purpose we adapt the methods of Oaku and Takayama (see [OT01]) to our situation:

We identify by Proposition 2.2.28 $\mathrm{Gr}^{\mathbf{v}} K$ with a submodule of $(\mathrm{Gr}^{\mathbf{v}} D_X)^E$ and $\mathrm{Gr}^{\mathbf{v}} D_X = \mathrm{Gr}^{\mathbf{v}} T_X$ with an elementary PBW-reduction-algebra of type

$$(\mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle, \mathrm{lt}_{\mathbf{v}}(S), J'_{\mathbf{v}}, \prec),$$

where $\mathbf{v}$-homogeneous $J'_{\mathbf{v}} \subseteq \mathbb{C}[\underline{x}, t]$ is determined using Corollary 2.2.30.

Remark 4.2.2. We point out that we may consider $\mathbb{C}[\partial_t t]$ as a $\mathbb{C}$-subalgebra of $\operatorname{Gr}^{\mathbf{v}} D_X$: Note that $\mathbb{C}[\partial_t t] \subseteq \mathbb{C}\langle t, \partial_t\rangle / \langle [\partial_t, t] - 1\rangle$ has $\mathbb{C}$-basis $\{t^k \partial_t^k \mid k \in \mathbb{N}\}$. As $\operatorname{Gr}^{\mathbf{v}} D_X$ is an elementary PBW-reduction-algebra we have $\sum_{k\in\mathbb{N}} a_k t^k \partial_t^k = 0$ (with $a_k \in \mathbb{C}$) in $\operatorname{Gr}^{\mathbf{v}} D_X$ if and only if $a_k t^k = 0$ for all $k \in \mathbb{N}$. If there is $k \in \mathbb{N}$ with $a_k \neq 0$, then it follows that there exists $f \in \mathbb{C}[\underline{x}, t]$ with $t^k + t^{k+1} f \in J$ implying $t \in J$ or $1 + tf \in J$ as J is prime. In both cases that is a contraction to $X_0 = V(J, t)$ being a codimension one subvariety of $X = V(J)$.

Lemma 4.2.3. *The b-function with respect to* $\mathbf{v}$ *on* D_X^E/K *corresponds under the substitution of* s *by* $-\partial_t t$ *to the monic generator of the* $\mathbb{C}[-\partial_t t]$*-ideal*

$$\bigcap_{e\in E} (\mathbb{C}[-\partial_t t] \cap K_e),$$

where $K_e := \{\pi_e(k) \mid k \in \operatorname{Gr}^{\mathbf{v}} K, \pi_{e'}(k) = 0$ *for all* $e' \in E \setminus \{e\}\}$ *for* $e \in E$*. In particular, the b-function with respect to* $\mathbf{v}$ *exists if and only if that ideal is nonzero.*

Proof. If $b(s)$ is the b-function with respect to $\mathbf{v}$ then $b(-\partial_t t)(e) \in (K + F_{-1}^{\mathbf{v}} D_X^E) \cap F_0^{\mathbf{v}} D_X^E$ implies $b(-\partial_t t)(e) \in \operatorname{Gr}^{\mathbf{v}} K$. Hence $b(-\partial_t t)$ is an element of $\bigcap_{e\in E}(\mathbb{C}[-\partial_t t] \cap K_e)$.

Conversely, let $b'(-\partial_t t)$ be the monic generator of the ideal $\bigcap_{e\in E}(\mathbb{C}[-\partial_t t] \cap K_e)$. We see that $b'(-\partial_t t)(e) \in K + F_{-1}^{\mathbf{v}} D_X^E$ for $e \in E$ and hence

$$b'(-\partial_t t)\overline{(e)} \in F_{-1}^{\mathbf{v}}(D_X^E/K).$$

Consequently, $b'(s)$ must agree with the b-function $b(s)$. □

Recall that a $\mathbf{v}$-homogeneous $\operatorname{Gr}^{\mathbf{v}} D_X$-generating set G of $\operatorname{Gr}^{\mathbf{v}} K$ can be determined by Algorithm 2.2.33. From G we obtain $\operatorname{Gr}^{\mathbf{v}} D_X$-generators G_e of K_e by computing a Gröbner basis G'_e of G with respect to an ordering of type $\prec_{pot,<}$, where $<$ is an order on E such that e is the minimal element, and setting $G_e := \pi_e(G'_e \cap D_X(e))$. To compute $\mathbb{C}[-\partial_t t] \cap K_e$ we first eliminate $\underline{x}, \underline{\theta}$ from K_e by computing $G''_e \subseteq \mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle$ inducing a Gröbner basis of K_e with respect to an elimination ordering for these variables. Note that for instance the well-ordering

$$\underline{x}^\alpha t^\beta \underline{\theta}^\gamma \partial_t^\delta \prec^{\text{elim}} \underline{x}^{\alpha'} t^{\beta'} \underline{\theta}^{\gamma'} \partial_t^{\delta'} \text{ if and only if } \underline{x}^\alpha \underline{\theta}^\gamma \prec \underline{x}^{\alpha'} \underline{\theta}^{\gamma'}$$
$$\text{or } \underline{x}^\alpha \underline{\theta}^\gamma = \underline{x}^{\alpha'} \underline{\theta}^{\gamma'} \text{ and } t^\beta \partial_t^\delta \prec t^{\beta'} \partial_t^{\delta'}$$

for $\alpha, \alpha' \in \mathbb{N}^{\mathrm{n}}, \beta, \beta', \delta, \delta' \in \mathbb{N}$ and $\gamma, \gamma' \in \mathbb{N}^{\mathrm{m}}$ is indeed an ordering on $\operatorname{Gr}^{\mathbf{v}} D_X$ of desired type. We observe that the elements of G''_e are $\mathbf{v}$-homogeneous because Gröbner basis computations over the PBW-reduction-algebra $\operatorname{Gr}^{\mathbf{v}} D_X$ preserve $\mathbf{v}$-homogeneity since $\operatorname{lt}_{\mathbf{v}}(S)$ and $J'_{\mathbf{v}}$ are $\mathbf{v}$-homogeneous. Then

$$\{t^{\max\{\deg_{\mathbf{v}}(g),0\}} \partial_t^{\max\{-\deg_{\mathbf{v}}(g),0\}} g \mid g \in G''_e \cap \mathbb{C}\langle t, \partial_t\rangle\}$$

is a set of $\mathbb{C}[-\partial_t t]$-generators of $\mathbb{C}[-\partial_t t] \cap K_e$. Substituting $-\partial_t t$ by s and performing a greatest common divisor computation in $\mathbb{C}[s]$ of that set of generators, gives a principal generator of the latter ideal. A principal generator of $\bigcap_{e\in E}(\mathbb{C}[-\partial_t t] \cap K_e)$ is now given by the least common multiple of these principal generators of the $\mathbb{C}[-\partial_t t] \cap K_e$.

Algorithm 4.2.4 Given global coordinate neighborhood X of X_0 and a D_X-module M, this algorithm computes the induced b-function along X_0 on M.

Input: A D_X-module $M := D_X^E/K$.
Output: A polynomial $b(s) \in \mathbb{K}[s]$ such that $b(s)$ is the induced b-function along X_0 on M if $b(s)$ is nonzero. Otherwise that b-function does not exist.

```
Compute a set G ⊆ K⟨x̲, t, θ̲, ∂_t⟩^E inducing Gr^v D_X-generators G' of Gr^v K ⊆ Gr^v D_X^E
  by Algorithm 2.2.33.  ▷ Gr^v D_X is a PBW-reduction-algebra.
for e ∈ E do
  Compute a Gröbner basis G'_e of G' with respect to an ordering of type ≺_{pot,<}, where <
    is an order on E such that e is minimal.
  Set G_e := π_e(G'_e ∩ D_X(e)).
  Compute a set G''_e ⊆ K⟨x̲, t, θ̲, ∂_t⟩ inducing a Gröbner basis of _{Gr^v D_x}⟨G_e⟩ with respect
    to an elimination ordering for x̲, θ̲.
  Consider H_e := {t^{max{deg_v(g),0}} ∂_t^{max{−deg_v(g),0}} g | g ∈ G''_e ∩ K⟨t, ∂_t⟩} as a subset of
    K⟨t, ∂_t⟩/⟨[∂_t, t] − 1⟩.
  if H_e = ∅ then
    return 0.
  Substitute −∂_t t by s in H_e.
  Compute the monic greatest common divisor b_e(s) ∈ K[s] of the elements in H_e.
Set b(s) to be the monic least common multiple of the b_e(s) for e ∈ E.
return b(s).
```

We derive now from Lemma 3.1.21 and Remark 3.1.22 the following algorithm for the computation of the V-filtration.

Algorithm 4.2.5 Given a global coordinate neighborhood X of X_0 and a D_X-module M, this algorithm tests whether M is X_0-specializable and computes $V_\alpha^{X_0} M$ in this case.

Input: A D_X-module $M := D_X^E/K$ and $\alpha \in \mathbb{Q}$.
Output: If M is X_0-specializable, a finite set $V \subseteq D_X^E$ such that $V_\alpha^{X_0} M = {}_{F_0^{\mathrm{v}} D_X}\langle \overline{V} \rangle \subseteq D_X^E/K$. Otherwise a notification that M is not X_0-specializable.

```
Compute the induced b-function b(s) ∈ K[s] along X_0 of M using Algorithm 4.2.4.
if b(s) = 0 then
  return Module is not specializable along X_0.
Compute the roots Z := {z ∈ C | b(z) = 0}.
if Z ⊄ Q then
  return Module is not Q-specializable along X_0.
if Z = ∅ then
  return 1.
Set k := ⌈min Z − α⌉ and l := ⌈max Z − (α + k)⌉.  ▷ Minimal root of b(s) lives in the
  interval (α + k − 1, α + k] and maximal root is in the interval (α + k + l − 1, α + k + l].
```

for $i = 0, \dots, l$ **do**
  $Z_i := \{z \in Z \mid \alpha + k + i - 1 < z \leq \alpha + k + i\}$.
  $V_{-i-k} := \{t^{\max\{0,i+k\}} \partial_t^{\leq \max\{0,-i-k\}}(e) \mid e \in E\}$. ▷ Residue classes are generators of $V^{\text{ind}}_{-i-k}M$.
for $i = 1, \dots, l$ **do**
  for $j = 0, \dots, l - i$ **do**
    $V_{-j-k} := V_{-j-k-1} \cup (\prod_{i \leq r \leq l} \prod_{z \in Z_r} (-\partial_t t - z + j + k + i - 1)^{\text{mult}_{b(s)}(z)}) \cdot V_{-j-k}$.
    ▷ Residue classes form generators of $W^{(i)}_{-j-k}M$ (see Remark 3.1.22).
return V_{-k}.

The V-filtration is computable if it exists:

Remark 4.2.6. Assume that $M = D_X^E/K$ is X_0-specializable. Consider the set $R := \{-1 \leq z \leq 0 \mid \text{there is } k \in \mathbb{Z} : b(z+k) = 0\}$, where $b(s)$ stands for the induced b-function with respect to $\mathbf{v}$ on M, and assume moreover that the residue classes of $V_\alpha \subseteq D_X^E$ form a finite set of $V_0 D_X$-generators of $V_\alpha M$ for $\alpha \in R$.

(a) $V_\bullet M$ is already determined by the $V_\alpha M$ with $\alpha \in R$ by Corollary 3.1.11: For $\beta = \alpha + k$ with $\alpha \in R$ and $k \in \mathbb{Z}$ we have

$$V_\beta M = \begin{cases} {}_{F_0^{\mathbf{v}} D_X}\left\langle \overline{t^{-k} V_\alpha} \right\rangle, & \text{if } k \leq 0, \alpha \neq 0 \\ {}_{F_0^{\mathbf{v}} D_X}\left\langle \overline{\partial_t^{\leq k} V^\alpha} \right\rangle, & \text{if } k \geq 0, \alpha \neq -1. \end{cases}$$

As $V_\bullet M$ is discretely indexed by $R + \mathbb{Z}$, it is completely computable.

(b) Assume we have computed finite sets V_{-1} and $V_0 \subseteq D_X^E$ such that their residue classes $F_0^{\mathbf{v}} D_X$-generate $V_{-1}M$ and $V_0 M$, respectively. According to Definition 3.1.3(b), there are $b \in (F_0^{\mathbf{v}} D_X^{V_{-1}})^{V_0}$ and $c \in (F_0^{\mathbf{v}} D_X^{V_0})^{V_{-1}}$ such that $t\overline{v_0} = \sum_{v_{-1} \in V_{-1}} (b_{v_0})_{v_{-1}} \overline{v_{-1}}$ and $\partial_t \overline{v_{-1}} = \sum_{v_0 \in V_0} (b_{v_{-1}})_{v_0} \overline{v_0}$ for $v_0 \in V_0$ and $v_{-1} \in V_{-1}$. Such representations are determined by Algorithm 4.2.5 on the fly without additional Gröbner basis computations: Recall that Lemma 3.1.21 and Remark 3.1.22 enable us to find $F_0^{\mathbf{v}} D_X$-generators of $V_0 M$ and $V_{-1}M$ by computing such generators of $V^k_{-k}M$ and $V^k_{-k-1}M$, respectively, for a suitably fixed k. For the computation of $V^k_\bullet M$, we first pick sets $G^0_j \subseteq D_X^E$ for $j \in \mathbb{Z}$ such that their residue classes $V_0 D_X$-generate $V^{\text{ind}}_j M$, namely we set

$$G^0_j := \{t^{\max\{0,-j\}} \partial_t^{\leq \max\{0,j\}}(e) \mid e \in E\}. \tag{4.2.2}$$

We easily read off of $g \in G^0_j$ an $F_0^{\mathbf{v}} D_X$-linear combination of tg and $\partial_t g$ in terms of G^0_{j-1} and G^0_{j+1}, respectively. Using the notation of Remark 3.1.22, we then compute iteratively generators $G^i_\bullet$ of $W^{(i)}_\bullet M$ by setting

$$G^{i+1}_j = G^i_{j-1} \cup b^{(i)}_1(-\partial_t t - \bullet) G^i_j.$$

Now we express tg and $\partial_t g$ for $g \in G_j^{i+1}$ as $F_0^{\mathbf{v}} D_X$-linear combinations of G_{j-1}^{i+1} and G_{j+1}^{i+1}, respectively, by using the corresponding combinations for the elements of G_{j-1}^{i}, G_j^i and G_{j+1}^i and the commutation relation $[\partial_t, t] = 1$.

Hence the V-filtration along X_0 is determined by the following algorithm:

Algorithm 4.2.7 Given a global coordinate neighborhood X of X_0 and a finitely generated D_X-module M, this algorithm tests whether M is X_0-specializable and computes $V_\bullet^{X_0} M$ in this case.

Input: A D_X-module $M := D_X^E / K$.

Output: If M is X_0-specializable, a finite set $V \subseteq D_X^E$ and a vector $\mathbf{d} \in [-1, 0]^V$ such that $V_\bullet M$ is discretely indexed by $\{\mathbf{d}_v \mid v \in V\} + \mathbb{Z}$ and $V_\bullet M = \sum_{v \in V} F^{\mathbf{v}}_{\lfloor \bullet - \mathbf{d}_v \rfloor} D_X \cdot \overline{v}$ and $V_{\mathbf{d}_v} M = \sum_{v' \in V : \mathbf{d}_{v'} = \mathbf{d}_v} F_0^{\mathbf{v}} D_X \cdot \overline{v}$ for $v \in V$. Otherwise a notification that M is not X_0-specializable.

Compute the induced b-function $b(s) \in \mathbb{K}[s]$ of M along X_0 using Algorithm 4.2.4.
if $b(s) = 0$ **then**
    return `Module is not specializable along` X_0.
Compute the roots $Z := \{z \in \mathbb{C} \mid b(z) = 0\}$.
if $Z \not\subset \mathbb{Q}$ **then**
    return `Module is not` $\mathbb{Q}$`-specializable along` X_0.
Initialize an empty set V and a (dynamic) vector $\mathbf{d} \in \mathbb{Z}^V$.
Set $R := (Z + \mathbb{Z}) \cap [-1, 0]$.
if $R = \emptyset$ **then**
    Set $V = \{1\}$ and $\mathbf{d}_1 = -1$.
for $\alpha \in R$ **do**
    Compute a finite set $V' \subseteq D_X^E$ such that $V_\alpha M = {}_{F_0^{\mathbf{v}} D_X}\langle \overline{V'} \rangle$ using Algorithm 4.2.5.
    Set $V := V \sqcup V'$ and define $\mathbf{d}_{v'} := \alpha$ for $v' \in V'$.
return $V, \mathbf{d}$.

Remark 4.2.8. The above algorithm can be modified to compute the not necessarily rationally indexed V-filtration and the V-filtration along smooth equidimensional subvarieties of higher codimension if this subvarity is defined by the vanishing of a subset of global coordinates: The above algorithm relies only Lemma 3.1.21 and Remark 3.1.22 as well as the computability of the induced b-function, which can be generalized to such a situation. We remark that the computation of the b-function in the higher codimension case is a bit more complicated, because in Lemma 4.2.3 we do not have to intersect with $\mathbb{C}[-\partial_t t]$, but with a $\mathbb{C}$-algebra of the form $\mathbb{C}[-\sum_i \partial_{t_i} t_i]$. This can be done by adapting the methods of Oaku and Takayama [OT01] to our situation.

4.2.2 The V-filtration on strictly X_0-specializable $(D_X, F^\circ_\bullet)$-modules

If $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ is strictly X_0-specializable, then we can also compute $F_\bullet V_\alpha M$ for fixed $\alpha \in \mathbb{Q}$. Since the filtrations $F^\circ_\bullet D_X$ and $V_\bullet D_X$ are induced by the weight vectors $\mathbf{w}$ and $\mathbf{v}$ on T_X, respectively, the problem reduces by Example 2.3.5 and Example 2.4.3 to Algorithm 2.4.15. More generally, we have:

Algorithm 4.2.9 Given a global coordinate neighborhood X of X_0 and an X_0-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet)$ such that $(V_\alpha^{X_0} M, F_\bullet)$ is $(V_0^{X_0} D_X, F^\circ_\bullet)$-good, this algorithm computes the latter filtered module.

Input: An X_0-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ and $\alpha \in \mathbb{Q}$ such that $(V_\alpha M, F_\bullet)$ is a good $(V_0 D_X, F^\circ_\bullet)$-module.

Output: A finite set $G \subseteq D_X^E$ and $\mathbf{d} \in \mathbb{Z}^G$ such that $F_\bullet V_\alpha M = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \mathbf{d}_g} F_0^{\mathbf{v}} D_X \cdot \overline{g} = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{F^{\mathbf{w}}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} D_X \cdot \overline{g}$.

1: Determine a finite set $V \subseteq D_X^E$ satisfying $V_\alpha M = {}_{F_0^{\mathbf{v}} D_X}\langle \overline{V} \rangle$ by Algorithm 4.2.5.

2: Find $G \subseteq D_X^E$ and $\mathbf{d} \in \mathbb{Z}^G$ such that $F^{\mathbf{w}}[\mathbf{s}]_\bullet \, {}_{F_0^{\mathbf{v}} D_X}\langle \overline{V} \rangle = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \mathbf{d}_g} F_0^{\mathbf{v}} D_X \cdot \overline{g}$ using Algorithm 2.4.15.

3: **return** $G, \mathbf{d}$.

Remark 4.2.10.

(a) With regard to the output G of the above algorithm, we note that for $g \in G$ a representative $g' \in \mathbb{K}\langle \underline{x}, t, \underline{\theta}, \partial_t \rangle$ with $F_\bullet V_\alpha M = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{F^{\mathbf{w}}[\mathbf{s}]}(g')} F_0^{\mathbf{v}} D_X \cdot \overline{g}$ is computable.

(b) The above algorithm does not detect if the $(V_0 D_X, F^\circ_\bullet)$-module $(V_\alpha M, F_\bullet)$ is not well-filtered. In such a case it does not terminate because neither does Algorithm 2.4.15 (see Remark 2.4.16(a)). We also remark that we have no method to check whether a well-filtered X_0-specializable $(D_X, F^\circ_\bullet)$-module is X_0-regular.

(c) If $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ is strictly X_0-specializable, then a filtered analogue of Remark 4.2.6(a) holds: Consider the set $R := \{-1 \le z \le 0 \mid \text{there is } k \in \mathbb{Z} : b(z+k) = 0\}$, where $b(s)$ stands for the induced b-function with respect to $\mathbf{v}$ on M and let $V_\alpha \subseteq D_X^E$ be such that $F_\bullet V_\alpha M = \sum_{v \in V_\alpha} F^{\mathbf{w}}_{\bullet - \deg_{F^{\mathbf{w}}[\mathbf{s}]}(v)} F_0^{\mathbf{v}} D_X \cdot \overline{v}$ for $\alpha \in R$. Then $F_\bullet V_\bullet M$ is already determined by the $F_\bullet V_\alpha M$ for $\alpha \in R$ by a filtered version of Corollary 3.1.11: We have for $\beta = \alpha + k$ with $\alpha \in R$ and $k \in \mathbb{Z}$

$$F_\bullet V_\beta M = \begin{cases} t^{-k} F_\bullet V_\alpha M, & \text{if } k \le 0, \alpha \neq 0 \\ \sum\limits_{0 \le i \le k} \partial_t^i F_{\bullet - i} V_\alpha M, & \text{if } k \ge 0, \alpha \neq -1. \end{cases}$$

As $V_\bullet M$ is discretely indexed by $R + \mathbb{Z}$, $(V_\bullet M, F_\bullet)$ is completely computable.

4.2.3 The V-filtration on $D_X(*X_0)$-modules

Notice that for $N = D_X[\bar{t}^{-1}]^E/L$ exists some D_X-submodule $L' \subseteq D_X^E$ such that

$$N = (D_X^E/L')[\bar{t}^{-1}].$$

According to Lemma 3.1.28(a) the module N is X_0-specializable if and only if D_X^E/L' is X_0-specializable and hence we reduce the computation of the V-filtration on N to that of the V-filtration on D_X^E/L' as follows: By Lemma 3.1.18 and Lemma 3.1.28(a) we have

$$V_\alpha N \cong V_\alpha D_X^E/L'$$

for $\alpha < 0$ given that N is X_0-specializable. As

$$V_{\alpha+k}N = \bar{t}^{-k}V_\alpha N$$

for any $k \in \mathbb{Z}$ by Lemma 3.1.26(a), this completely determines the V-filtration leading to the following algorithm:

Algorithm 4.2.11 Given a global coordinate neighborhood X of X_0 and a finitely generated $D_X[\bar{t}^{-1}]$-module N, this algorithm tests whether N is $\mathbb{Q}$-specializable along X_0 and computes $V_\bullet^{X_0}N$ in this case.

Input: A $D_X[\bar{t}^{-1}]$-module $N := D_X[\bar{t}^{-1}]^E/L$ with $L = {}_{D_X[\bar{t}^{-1}]}\langle L'\rangle$ and $L' \subseteq D_X^E$.

Output: If N is X_0-specializable, a finite set $V \subseteq D_X^E$ and a vector $\mathbf{d} \in \mathbb{Q}^V$ such that $V_{\mathbf{d}_v+k}N = \sum_{v'\in V:\mathbf{d}_v=\mathbf{d}_{v'}} \bar{t}^{-k}F_0^{\mathbf{v}}D_X \cdot \overline{v'}$ for $v \in V$ and $k \in \mathbb{Z}$, and such that $V_\bullet N$ is discretely indexed by $\{\mathbf{d}_v \mid v \in V\} + \mathbb{Z}$. Otherwise a notification that N is not X_0-specializable

1: **if** $D_X^E/{}_{D_X}\langle L'\rangle$ is not $\mathbb{Q}$-specializable along X_0 **then** ▷ Test by Algorithm 4.2.7
2: **return** `Module is not` $\mathbb{Q}$`-specializable along` X_0.
3: Determine $V \subseteq D_X^E$ and $\mathbf{d} \in \mathbb{Q}^V$ as in Algorithm 4.2.7 for $D_X^E/{}_{D_X}\langle L'\rangle$. ▷ Compute $V_\bullet(D_X^E/{}_{D_X}\langle L'\rangle)$.
4: Set $V' := \{v \in V \mid \mathbf{d}_v \neq 0\}$ and define $\mathbf{d}' \in \mathbb{Q}^{V'}$ by $\mathbf{d}'_{v'} := \mathbf{d}_{v'}$ for $v' \in V'$.
5: **return** $V', \mathbf{d}'$.

Remark 4.2.12. While it was relatively easy to reduce the computation of the V-filtration of finitely presented X_0-specializable $D_X[\bar{t}^{-1}]$-modules to that of D_X-modules, the filtered case is more subtle. The problem stems for the fact that if $(N, F_\bullet) = (D_X[\bar{t}^{-1}]^E/L, F^\circ[\mathbf{s}]_\bullet)$ is a strictly X_0-specializable $(D_X[\bar{t}^{-1}], F_\bullet^\circ)$-module with $N \cong (D_X^E/L')[\bar{t}^{-1}]$, then in general $F[\mathbf{s}]_\bullet^\circ V_\alpha N \neq F^\circ[\mathbf{s}]_\bullet V_\alpha D_X^E/L'$ for $\alpha < 0$. We will explain in Subsection 4.2.6 (see in particular Remark 4.2.31) how to solve this problem.

Alternatively, we compute the V-filtration along X_0 on N by representing N as a quotient of a free D_X-module and then applying Algorithm 4.2.7 to this representation. Such a representation is determined as explained below:

4.2.4 Localizations of X_0-specializable D_X- and $D_X(*X_0)$-modules

We want to finitely present $\mathrm{Loc}_{X_0}(M) = M \otimes_{O_X} O_X[\overline{t}^{-1}]$ and $\mathrm{Loc}_{X_0}(N) = N$ as D_X-modules given that M and N are X_0-specializable. As every finitely presented D_X-module N' with $N = \mathrm{Loc}_{X_0}(N')$ is X_0-specializable if and only if N is so (see Lemma 3.1.28(a)), we may restrict ourselves to computing $\mathrm{Loc}_{X_0}(M)$. Now Lemma 3.1.29 yields the following algorithm:

Algorithm 4.2.13 Given a coordinate neighborhood X of X_0 and an X_0-specializable D_X-module M, this algorithm represents the localization $\mathrm{Loc}_{X_0}(M)$ as a quotient of a free D_X-module.

Input: An X_0-specializable D_X-module $M = D_X^E/K$.
Output: A finite set E' and a finite set $L \subseteq F_0^{\mathbf{v}} D_X^{E'}$ that satisfy $\mathrm{Loc}_{X_0}(M) \cong D_X^{E'}/{}_{D_X}\langle L\rangle$, $V_k(D_X^{E'}/{}_{D_X}\langle L\rangle) = F_k^{\mathbf{v}}(D_X^{E'}/{}_{D_X}\langle L\rangle)$ (for all $k \in \mathbb{Z}$) and $F_0^{\mathbf{v}}\,{}_{D_X}\langle L\rangle = {}_{F_0^{\mathbf{v}} D_X}\langle L\rangle$.

1: Compute $E' \subseteq D_X^E$ finite such that $\overline{E'}$ is a set of $F_0^{\mathbf{v}} D_X$-generators of $V_{-1}M$ by Algorithm 4.2.5.
2: Represent $V_{-1}M$ as a quotient $F_0^{\mathbf{v}} D_X^{E'}/{}_{F_0^{\mathbf{v}} D_X}\langle L\rangle$ with $L \subseteq F_0^{\mathbf{v}} T_X^{E'}$ finite using Algorithm 2.4.4. ▷ ${}_{F_0^{\mathbf{v}} D_X}\langle L\rangle = \ker(F_0^{\mathbf{v}} D_X^{E'} \to V_{-1}M, (e') \mapsto \overline{e'})$.
3: Set $L := t^{-1} \cdot L \cdot t \subseteq D_X^{E'}$.
4: **return** E', L.

Remark 4.2.14. Assume $M = D_X^E/K$ is X_0-specializable and that we have computed a representation $\mathrm{Loc}_{X_0}(M) \cong D_X^{E'}/L'$ by the above algorithm.

(a) Keeping the notation of that algorithm, we want to make the natural D_X-linear localization map $i_{(*X_0)} : M \to D_X^{E'}/L'$ explicit. As $V_0 M$ generates M as D_X-module, it suffices to compute the images of a finite set of $F_0^{\mathbf{v}} D_X$-generators of $V_0 M$ represented by $V_0 \subseteq D_X^E$ under this map. If we choose V_0 as in Remark 4.2.6(b) and write $tv_0 = \sum_{e' \in E'} (q_{v_0})_{e'} e'$ for $v_0 \in V_0$ with $q \in (F_0^{\mathbf{v}} D_X^{E'})^{V_0}$ using that remark, then

$$\overline{v_0} \otimes 1 = \overline{tv_0} \otimes \overline{t}^{-1} = \left(\sum_{e' \in E'} (q_{v_0})_{e'} \overline{e'} \right) \otimes \overline{t}^{-1} = \sum_{e' \in E'} t^{-1} (q_{v_0})_{e'} t \left(\overline{e'} \otimes \overline{t}^{-1} \right)$$

implies that $i_{(*X_0)}(\overline{v_0}) = \sum_{e' \in E'} t^{-1}(q_{v_0})_{e'} \overline{t(e')}$. Hence we extend Algorithm 4.2.13 as described in Algorithm 4.2.15 below.

(b) To patch our local computations together, we also need to be able to compute the image of $m \otimes \overline{t}^{-k}$ with $k \in \mathbb{N}$ under the isomorphism $M[\overline{t}^{-1}] \cong D_X^{E'}/L'$. This amounts to finding $p \in D_X^{E'}$ such that $t^k \overline{p} = i_{(*X_0)}(m)$. For that assume that $i_{(*X_0)}(m)$ is the residue class of $r \in D_X^{E'}$ and $\deg_{\mathbf{v}}(r) \leq l$. Then $i_{(*X_0)}(m) \in V_l(D_X^{E'}/L') =$

$t^k V_{k+l}(D_X^{E'}/L') = t^k F^{\mathrm{v}}_{k+l}(D_X^{E'}/L')$ by Algorithm 4.2.13 and Lemma 3.1.26(a). As the latter module is $F_0^{\mathrm{v}} D_X$-generated by the residue classes of

$$V := \{t^{k+\max\{0,-k-l\}}\partial_t^{\leq \max\{0,k+l\}}(e') \mid e' \in E'\},$$

we compute $a \in F_0^{\mathrm{v}} D_X^V$ such that $\overline{r} = \sum_{v \in V} a_v \overline{v}$ by Algorithm 2.4.5 and Remark 2.4.6. Now we set $p := \sum_{v \in V} t^{-k} a_v v \in D_X^E$ and obtain $t^k \overline{p} = i_{(*X_0)}(m)$ and hence $m \otimes t^{-k}$ is mapped to $\overline{p}$ under the above isomorphism.

We point out that the converse task of finding the image of $\overline{m} \in D_X^{E'}/L'$ for $m \in D_X^{E'}$ under the isomorphism $D_X^{E'}/L' \cong M[\overline{t}^{-1}]$ is easy. Namely, that image is given by $\sum_{e' \in E'} m_{e'} \cdot (e' \otimes t^{-1})$.

Algorithm 4.2.15 Given a coordinate neighborhood X of X_0 and an X_0-specializable D_X-module M, this algorithm represents the localization $\mathrm{Loc}_{X_0}(M)$ as a quotient of a free D_X-module and computes the natural map $i_{(*X_0)} : M \to \mathrm{Loc}_{X_0}(M)$.

Input: An X_0-specializable D_X-module $M = D_X^E/K$.

Output: A finite set E', a finite subset $L \subseteq F_0^{\mathrm{v}} D_X^{E'}$ and $q \in (D_X^{E'})^E$ such that $\mathrm{Loc}_{X_0}(M) \cong D_X^{E'}/{}_{D_X}\langle L\rangle$ as D_X-modules, $V_k(D_X^{E'}/{}_{D_X}\langle L\rangle) = F_k^{\mathrm{v}}(D_X^{E'}/{}_{D_X}\langle L\rangle)$ for all $k \in \mathbb{Z}$, $F_0^{\mathrm{v}}\,{}_{D_X}\langle L\rangle = {}_{F_0^{\mathrm{v}} D_X}\langle L\rangle$ and the natural map $M \to D_X^{E'}/{}_{D_X}\langle L\rangle$ is given by $(e) \mapsto \overline{q_e}$ for $e \in E$.

1: Compute by Algorithm 4.2.5 finite sets $E', V_0 \subseteq D_X^E$ such that $\overline{E'}$ and $\overline{V_0}$ are $F_0^{\mathrm{v}} D_X$-generators of $V_{-1}M$ and $V_0 M$, respectively.
2: Represent $V_{-1}M$ as a quotient $F_0^{\mathrm{v}} D_X^{E'}/{}_{F_0^{\mathrm{v}} D_X}\langle L\rangle$ with L finite via Algorithm 2.4.4. ▷ ${}_{F_0^{\mathrm{v}} D_X}\langle L\rangle = \ker(F_0^{\mathrm{v}} D_X^{E'} \to V_{-1}M, (e') \mapsto \overline{e'})$.
3: Set $L := t^{-1} \cdot L \cdot t$.
4: Find $c \in (D_X^{V_0})^E$ such that $\overline{(e)} = \sum_{v \in V_0} (c_e)_v \overline{v} \in D_X^E/K$ for $e \in E$ using Gröbner basis theory. ▷ Use that $\overline{V_0}$ is a set of D_X-generators of D_X^E/K.
5: Apply Remark 4.2.14(a) to determine $d \in (F_0^{\mathrm{v}} D_X^{E'})^{V_0}$ such that $t\overline{v} = \sum_{e' \in E'} (d_v)_{e'} \overline{e'}$ for $v \in V_0$.
6: Define $q \in (D_X^{E'})^E$ by $(q_e)_{e'} = \sum_{v \in V_0} (c_e)_v t^{-1} (d_v)_{e'} t$.
7: **return** E', L, q.

Remark 4.2.16. If $X = \mathbb{C}^n$, the localization $\mathrm{Loc}_{X_0}(M)$ can in many cases also be computed via various algorithms developed by Oaku, Takayama and Walther (see [Oak97, Section 7] for M being f-saturated, [OT01, Algorithm 6.4] for M holonomic, [OTW00, Algorithm 3] for $\mathrm{Loc}_{X_0}(M)$ holonomic). Note that unlike our algorithm these algorithms do not require that f is part of a global coordinate system. As Algorithm 4.2.13, these algorithms rely on some kind of b-function (or Bernstein-Sato polynomial) computation. However, our method is advantageous if we are also interested in the V-filtration along X_0 on $\mathrm{Loc}_{X_0}(M)$: Our approach allows the determination of $V_\bullet \mathrm{Loc}_{X_0}(M)$ without an additional b-function computation, whereas the other approaches need an extra b-function computation, namely that of the induced b-function of $\mathrm{Loc}_{X_0}(M)$, to compute $V_\bullet \mathrm{Loc}_{X_0}(M)$.

4.2.5 Localizations of strictly X_0-specializable $(D_X, F^\circ_\bullet)$-modules

Unlike in the previous subsection we consider here only the case of strictly X_0-specializable $(D_X, F^\circ_\bullet)$-modules and treat the case of strictly X_0-specializable $(D_X(*X_0), F^\circ_\bullet)$-modules separately later. The reason for this is that while is was trivial to represent an X_0-specializable $D_X(*X_0)$-module as a localization of an X_0-specializable D_X-module, this is not that easy for strictly X_0-specializable modules and involves additional algorithms and theory because we also have to take the $F_\bullet$-filtration into account.

So assume that $(M, F_\bullet) = (D_X^E/L, F^\circ[\mathbf{s}]_\bullet)$ is a strictly X_0-specializable $(D_X, F^\circ_\bullet)$-module. We base our computation of $\mathrm{Loc}_{X_0}(M, F_\bullet)$ on Equation (3.2.10), which states that

$$F_\bullet^{\mathrm{Loc}} M[t^{-1}] = \sum_{i\in\mathbb{N}} \{\partial_t^i t^{-1}(m \otimes 1) \mid m \in F_{\bullet-i}(V_{-1}M)\}.$$

So we may proceed as in Algorithm 4.2.13 if we additionally assume that the set $E' \subseteq D_X^E$ inducing a set of $F_0^{\mathbf{v}} D_X$-generators of $V_{-1}M$ satisfies

$$F_\bullet V_{-1}M = \sum_{e'\in E'} F^{\mathbf{v}}_{\bullet-\deg_{\mathbf{v}}(e')} D_X \cdot \overline{e'}.$$

Such a set is determined by Algorithm 4.2.9 and we can even find for $e' \in E$ a representative $e'_r \in \mathbb{K}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle$ such that the above equality holds if we replace $\deg_{\mathbf{v}}(e')$ by $\deg_{\mathbf{v}}(e'_r)$. To represent the localization map, we need to modify Algorithm 4.2.15, because we are not in the position to apply Remark 4.2.6(b) as we did in Remark 4.2.14(a). But we can replace that method by Algorithm 2.4.5 and Remark 2.4.6 (or by suitably tracing our computations in Algorithm 4.2.9), yielding the following algorithm:

Algorithm 4.2.17 Given a coordinate neighborhood X of X_0 and a strictly X_0-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet)$, this algorithm represents $\mathrm{Loc}_{X_0}(M, F_\bullet)$ as $(D_X, F^\circ_\bullet)$-module and computes the natural map $i_{(X_0)} : M \to \mathrm{Loc}_{X_0}(M)$.

Input: A strictly X_0-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$.

Output: A finite set E', a finite subset $L \subseteq F_0^{\mathbf{v}} D_X^{E'}$, $\mathbf{d} \in \mathbb{Z}^L$ and $q \in (D_X^{E'})^E$ that satisfy $\mathrm{Loc}_{X_0}(M, F_\bullet) \cong (D_X^{E'} / {}_{D_X}\langle L\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet)$ as $(D_X, F^\circ_\bullet)$-modules, $V_k(D_X^{E'} / {}_{D_X}\langle L\rangle) = F_k^{\mathbf{v}}(D_X^{E'} / {}_{D_X}\langle L\rangle)$ for all $k \in \mathbb{Z}$, $F_0^{\mathbf{v}}\, {}_{D_X}\langle L\rangle = {}_{F_0^{\mathbf{v}} D_X}\langle L\rangle$ and represent the natural localization map $M \to D_X^{E'} / {}_{D_X}\langle L\rangle$ by $(e) \mapsto \overline{q_e}$ for $e \in E$.

1: Find a finite set $E' \subseteq D_X^E$ and $\mathbf{d} \in \mathbb{Z}^{E'}$ that satisfy $F^\circ_\bullet V_{-1}M = \sum_{e'\in E'} F^{\mathbf{w}}_{\bullet-\mathbf{d}_{e'}} F_0^{\mathbf{v}} D_X \cdot \overline{e'}$ by applying Algorithm 4.2.9.

2: Use Algorithm 4.2.5 to compute a finite set $V_0 \subseteq D_X^E$ such that $\overline{V_0}$ is a set of $F_0^{\mathbf{v}} D_X$-generators of $V_0 M$.
3: Represent $V_{-1}M$ as a quotient $F_0^{\mathbf{v}} D_X^{E'} / {}_{F_0^{\mathbf{v}} D_X}\langle L \rangle$ with L finite via Algorithm 2.4.4. $\triangleright$ ${}_{F_0^{\mathbf{v}} D_X}\langle L \rangle = \ker(F_0^{\mathbf{v}} D_X^{E'} \to V_{-1}M, (e') \mapsto \overline{e'})$.
4: Set $L := t^{-1} \cdot L \cdot t$.
5: Determine $c \in (D_X^{V_0})^E$ such that $\overline{(e)} = \sum_{v \in V_0} (c_e)_v \overline{v}$ for $e \in E$ using Gröbner basis theory. $\triangleright$ Use that $\overline{V_0}$ is a set of D_X-generators of D_X^E / K.
6: Compute $d \in (F_0^{\mathbf{v}} D_X^{E'})^{V_0}$ such that $t\overline{v} = \sum_{e' \in E'} (d_v)_{e'} \overline{e'}$ for $v \in V_0$ by Algorithm 2.4.5 and Remark 2.4.6.
7: Define $q \in (D_X^{E'})^E$ by $(q_e)_{e'} = \sum_{v \in V_0} (c_e)_v t^{-1} (d_v)_{e'} t$.
8: **return** $E', L, \mathbf{d}, q$.

4.2.6 Localizations of strictly X_0-specializable $(D_X(*X_0), F_\bullet^\circ)$-modules

Now consider the strictly X_0-specializable $(D_X(*X_0), F_\bullet^\circ)$-module

$$(N, F_\bullet) = (D_X[\overline{t}^{-1}]^E / L, F^\circ[\mathbf{s}]_\bullet).$$

The basic framework for our algorithm to determine the $(D_X, F_\bullet^\circ)$-module $\mathrm{Loc}_{X_0}(N, F_\bullet) = (N, F_\bullet^{\mathrm{Loc}})$ with filtration $F_\bullet^{\mathrm{Loc}}$ given by

$$F_\bullet^{\mathrm{Loc}} N = \sum_{i \in \mathbb{N}} \partial_t^i F_{\bullet - i} V_0 N$$

is as follows: We first represent N as a quotient N_X of a free D_X-module such that $V_0 N_X = F_0^{\mathbf{v}} N_X$ and compute the image $F_\bullet N_X$ of $F_\bullet N$ under this representation. Then we find $p \in \mathbb{Z}$ such that $F_p N = 0$ which implies $F_p^{\mathrm{Loc}} N_X = 0$, where $F_\bullet^{\mathrm{Loc}} N_X$ is induced by the corresponding filtration on N. While $F_p V_0 N_X$ does not generate $F_\bullet^{\mathrm{Loc}} N_X$ (see below), we increase p by 1 and compute $F_p^{\mathrm{Loc}} V_0 N_X = F_p V_0 N_X$. Finally, we use our interim results from the computation of the various $F_p V_0 N_X$ to explicitly give generators of the filtration $F_\bullet^{\mathrm{Loc}} N_X$. Hence there are three main algorithmic tasks:

- Represent N as a quotient N_X of a free D_X-module and transfer $F_\bullet N$ to this setting.
- For a fixed $p \in \mathbb{Z}$, use the above D_X-representation to compute $F_q^{\mathrm{Loc}} V_0 N_X = F_q V_0 N_X$ and $F_q^{\mathrm{Loc}} N_X$ for all $q \leq p$.
- Check for fixed $p \in \mathbb{Z}$ if $F_p^{\mathrm{Loc}} V_0 N_X$ generates $F_\bullet^{\mathrm{Loc}} N_X$, that is, if $F_q^{\mathrm{Loc}} N_X = F_{q-p}^\circ D_X \cdot F_p^{\mathrm{Loc}} V_0 N_X$ for $q > p$.

Before we explain how to tackle these tasks, we fix some notation:

Notation 4.2.18. Let $A' \leq A$ be $\mathbb{C}[t]$-modules. We define for $b \in \mathbb{N}$ the quotient $A' :_A t^b$ and the saturation of A' by t in A by

$$A' :_A t^b := \{a \in A \mid t^b a \in A'\} \leq A \text{ and } A' :_A t^\infty := \bigcup_{b \in \mathbb{N}} A' :_A t^b \leq A,$$

respectively. If t acts bijectively on A, we identify the localization $A'[t^{-1}] := A' \otimes_{\mathbb{C}[t]} \mathbb{C}[t][t^{-1}] \leq A[t^{-1}] \cong A$ with the saturation $A' :_A t^\infty$. In this case, we write $A'[t^{-1}] = A' :_A t^\infty$ and consider this module as a submodule of A.

Representing N as a quotient of a free $D_X$module

Using Algorithm 4.2.15 we compute a D_X-linear isomorphism

$$\rho : N \to N_X := D_X^{E'}/L'$$

and determine the images of $\overline{(e)}$ for $e \in E$ under this isomorphism. Recall that we may also assume that $V_0 N_X = F_0^{\mathbf{v}}(D_X^{E'}/L')$ and that L' and $F_0^{\mathbf{v}} L'$ are D_X- and $F_0^{\mathbf{v}} D_X$-generated by the finite set $L'' \subseteq F_0^{\mathbf{v}} D_X^{E'}$, respectively. So in particular $V_0 N_X = F_0^{\mathbf{v}} D_X^{E'} / {}_{F_0^{\mathbf{v}} D_X^{E'}}\langle L'' \rangle$. We need to describe the image of $F_\bullet N$ under ρ, which we denote by $F_\bullet N_X$:

Lemma 4.2.19. *We have*

$$F_\bullet N_X = \left(\sum_{e \in E} (F^\circ_{\bullet - \mathbf{s}_e} D_X) \cdot \rho(\overline{(e)}) \right) [t^{-1}] \leq D_X^{E'}/L'.$$

Proof. We write $m \in F_p N$ as $m = \sum_{e \in E} \bar{t}^{-a_e} m_e \overline{(e)}$ with $a \in \mathbb{N}^E$ and $m_e \in F^\circ_{p - \mathbf{s}_e} D_X$. Setting $a' := \max\{a_e \mid e \in E\}$, we obtain by D_X-linearity

$$t^{a'} \rho(m) = \rho(t^{a'} m) = \sum_{e \in E} t^{a' - a_e} m_e \rho(\overline{(e)}) \in \sum_{e \in E} (F^\circ_{\bullet - \mathbf{s}_e} D_X) \cdot \rho(\overline{(e)}).$$

Conversely, if $m' \in (\sum_{e \in E} (F^\circ_{\bullet - \mathbf{s}_e} D_X) \cdot \rho(\overline{(e)}))[t^{-1}]$ then there exist $m'' \in N$ such that $m' = \rho(m'')$ and $b \in \mathbb{N}$ such that

$$\rho(t^b m'') = t^b \rho(m'') \in \sum_{e \in E} (F^\circ_{\bullet - \mathbf{s}_e} D_X) \cdot \rho(\overline{(e)}) \subseteq \rho(F_p N).$$

As ρ is an isomorphism this implies $t^b m'' \in F_p N$ and hence $m'' \in F_p N$ because $(N, F_\bullet)$ is a filtered $(D_X(*X_0), F_\bullet)$-module. This shows $m' \in \rho(F_p N)$. □

Computation of $F_p^{\mathrm{Loc}} V_0 N_X$ for fixed p

The computation of $F_p^{\mathrm{Loc}} V_0 N_X$ for fixed $p \in \mathbb{Z}$ is based on the following lemma.

Lemma 4.2.20. *Let $D'_X \in \{D_X, F_0^{\mathbf{v}} D_X\}$. Then for any $a \in \mathbb{N}^E$ we have*

$$F_\bullet N_X = \left(\sum_{e \in E} F^{\mathbf{w}}_{\bullet - \mathbf{s}_e} D'_X \cdot t^{a_e} \rho(\overline{(e)}) \right) [t^{-1}]$$

and

$$F_\bullet^{\mathrm{Loc}} V_0 N_X = \left(V_0 N_X \cap \sum_{e \in E} F^{\mathbf{w}}_{\bullet - \mathbf{s}_e} D'_X \cdot t^{a_e} \rho(\overline{(e)}) \right) :_{V_0 N_X} t^\infty .$$

Proof. Since $t^{a_e + \bullet} F_p^{\mathbf{w}} F_\bullet^{\mathbf{v}} D_X \subseteq F_p^{\mathbf{w}} (F_0^{\mathbf{v}} D_X) t^{a_e} \subseteq F_p^{\mathbf{w}} D_X t^{a_e} \subseteq F_p^{\mathbf{w}} D_X$ for any $p \in \mathbb{Z}$ and $a_e \in \mathbb{N}$, the first claim follows by Lemma 4.2.19. This finishes the proof as the filtration $F_\bullet^{\mathrm{Loc}}$ on $V_0 N_X$ agrees with the filtration induced by $F_\bullet N_X$ according to Remark 3.2.12(a). ☐

Retaining the notation of the previous lemma, we calculate $F_p^{\mathrm{Loc}} V_0 N_X$ by first intersecting

$$P := F_0^{\mathbf{v}} N_X \cap \sum_{e \in E} F^{\mathbf{w}}_{p - \mathbf{s}_e} D'_X \cdot t^{a_e} \rho(\overline{(e)}) \tag{4.2.3}$$

and then using a saturation technique to obtain $F_p^{\mathrm{Loc}} V_0 N_X = P :_{V_0 N_X} t^\infty$. While P can be determined by Algorithm 2.4.7, we can even avoid having to compute such an intersection by setting $D'_X = F_0^{\mathbf{v}} D_X$ and choosing a_e big enough such that $\sum_{e \in E} F^{\mathbf{w}}_{p - \mathbf{s}_e} F_0^{\mathbf{v}} D_X \cdot t^{a_e} \rho(\overline{(e)}) \subseteq F_0^{\mathbf{v}} N_X$: More precisely, if $q_e \in \mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t \rangle^{E'}$ is a representative of $\rho(\overline{(e)})$, a suitable choice is $a_e := \max\{0, \deg_{\mathbf{v}}(q_e)\}$ by Definition 3.1.3(b) since $V_0 N_X = F_0^{\mathbf{v}} N_X$. The drawback of taking $D'_X = F_0^{\mathbf{v}} D_X$ and picking $a_e > 0$ is that the inclusion

$$F_0^{\mathbf{v}} N_X \cap \sum_{e \in E} F^{\mathbf{w}}_{p - \mathbf{s}_e} F_0^{\mathbf{v}} D_X \cdot t^{a_e} \rho(\overline{(e)}) \subseteq F_0^{\mathbf{v}} N_X \cap \sum_{e \in E} F^{\mathbf{w}}_{p - \mathbf{s}_e} D_X \cdot \rho(\overline{(e)}),$$

is in general proper. So we might not start with the largest possible choice of P, which could lead to a more expensive saturation process.

Next, we reduce the computation of $P : t^\infty := P :_{V_0 N_X} t^\infty$ to that of $P : t^a := P :_{V_0 N_X} t^a$ for increasing $a \in \mathbb{N}$:

Lemma 4.2.21. *The equality $P : t^a = P : t^{a+1}$ for $a \in \mathbb{N}$ implies that $P : t^\infty = P : t^a$.*

Proof. Assume that $P : t^\infty \neq P : t^a$. Then there exists $b > a + 1$ such that $P : t^a = P : t^{b-1} \subsetneq P : t^b$. Choose $m \in P : t^b \setminus P : t^{b-1}$. Since $tm \in V_0 N_X$ and $t^{b-1}(tm) \in P$ it follows that $tm \in P : t^{b-1} = P : t^{b-2}$. This implies that $t^{b-2}(tm) = t^{b-1} m \in P$ and hence $m \in P : t^{b-1}$ contradicting our assumption. ☐

We explain now an inductive method that computes $P : t^a$ for a given nonnegative integer a. We may assume that we have computed a finite set $G_j \subseteq F_0^{\mathrm{v}}\, \mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle^{E'}$ inducing a set of $\mathbb{C}[\underline{x}, t]$-generators of $P : t^j$ for all $j < a$, since $P : t^j$ is a finitely generated $\mathbb{C}[\underline{x}, t]$-module being a $\mathbb{C}[\underline{x}, t]$-submodule of the finitely generated $\mathbb{C}[\underline{x}, t]$-module $F_p^{\mathrm{Loc}} F_0^{\mathrm{v}} N_X$.

Lemma 4.2.22. *If we set* $P_a := \ker({}_{\mathbb{C}[\underline{x}]}\langle\overline{G_{a-1}}\rangle \to \mathrm{Gr}_0^{\mathrm{v}} N_X = F_0^{\mathrm{v}} N_X / tF_0^{\mathrm{v}} N_X)$*, we have that*

$$P : t^a = P : t^{a-1} + \mathbb{C}[\underline{x}, t] \cdot \{t^{-1}n \mid n \in P_a\}.$$

Proof. First note that $t^{-1}n$ for $n \in P_a$ is uniquely defined since t acts bijectively on $N \cong N_X$. So in particular $t^{-1}n \in F_0^{\mathrm{v}} N_X = V_0 N_X$ and hence $t^{-1}n \in P : t^a$. The claim follows now since $n \in P : t^a$ implies that $tn \in P : t^{a-1}$. □

For the computation of P_a, we represent $F_0^{\mathrm{v}} N_X / tF_0^{\mathrm{v}} N_X$ as a quotient of a free $\mathrm{Gr}_0^{\mathrm{v}} D_X$-module and realize $\mathrm{Gr}_0^{\mathrm{v}} D_X$ as PBW-reduction-algebra

$$(\mathbb{C}\langle \underline{x}, \underline{\theta}, z\rangle, S^{t,0}, J^{t,0}, \prec^{t,0})$$

as explained in Example 2.1.30(c). The corresponding isomorphism

$$\mathrm{Gr}_0^{\mathrm{v}} D_X \cong \mathbb{C}\langle \underline{x}, \underline{\theta}, z\rangle / \left\langle S^{t,0} \cup J^{t,0} \right\rangle \tag{4.2.4}$$

is induced by the $\mathbb{C}$-linear map

$$\nu : F_0^{\mathrm{v}} D_X \to \mathbb{C}\langle \underline{x}, \underline{\theta}, z\rangle / \left\langle S^{t,0} \cup J^{t,0} \right\rangle : \underline{x}^\alpha t^\beta \underline{\theta}^\gamma (t\partial_t)^\delta \mapsto \underline{x}^\alpha 0^\beta \underline{\theta}^\gamma z^\delta \tag{4.2.5}$$

(where we define $0^0 = 1$). To simplify notation we identify the above algebras and we often write $t\partial_t$ instead of z.

Lemma 4.2.23. *Consider the* $F_0^{\mathrm{v}} D_X$*-module* $Q = F_0^{\mathrm{v}} D_X^E / {}_{F_0^{\mathrm{v}} D_X}\langle R\rangle$*. Then* Q/tQ *can be realized under the above isomorphism as*

$$\mathrm{Gr}_0^{\mathrm{v}} D_X^E / {}_{\mathrm{Gr}_0^{\mathrm{v}} D_X}\langle \nu^E(R)\rangle.$$

Recalling that $V_0 N_X = F_0^{\mathrm{v}} D_X^{E'} / {}_{F_0^{\mathrm{v}} D_X}\langle L''\rangle$, we now obtain

$$F_0^{\mathrm{v}} N_X / tF_0^{\mathrm{v}} N_X \cong \mathrm{Gr}_0^{\mathrm{v}} D_X^{E'} / {}_{\mathrm{Gr}_0^{\mathrm{v}} D_X}\left\langle \nu^{E'}(L'')\right\rangle.$$

We represent $p \in F_0^{\mathrm{v}} D_X^{E'}$ as $p = p_0 + tp' \in F_0^{\mathrm{v}} D_X^{E'}$ with $p' \in F_0^{\mathrm{v}}\, \mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle^{E'}$ and $p_0 \in {}_{\mathbb{C}}\left\langle\{\underline{x}^\alpha \underline{\theta}^\beta (t\partial_t)^\gamma \mid \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m, \gamma \in \mathbb{N}\}\right\rangle^{E'}$ and get for its image $\nu^{E'}(p) = p_0$. Going back to the problem of determining P_a, we have for $c \in \mathbb{C}[\underline{x}]^{G_{a-1}}$

$$\sum_{g \in G_{a-1}} c_g \overline{g} \in P_a \text{ if and only if } \sum_{g \in G_{a-1}} c_g g_0 \in {}_{\mathrm{Gr}_0^{\mathrm{v}} D_X}\left\langle \nu^{E'}(L'')\right\rangle.$$

Thus our problem reduces to a syzygy computation of $G^0_{a-1} := \bigsqcup_{g\in G_{a-1}}\{g_0\}$ and $L''_0 = \bigsqcup_{l\in L''}\{l_0\}$ in $\mathrm{Gr}^{\mathrm{V}}_0\, D_X$ with respect to the block ordering $\prec_b := ((\prec^{t,0})^{G^0_{a-1}}_{\mathbf{u}}, (\prec^{t,0})^{L''_0})$, where $\mathbf{u}$ denotes the weight vector assigning weight 1 to z and $\theta_1, \ldots, \theta_{\mathrm{m}}$ and 0 else. Notice that $\prec_b$ is indeed an ordering on the elementary PBW-reduction-algebra $\mathrm{Gr}^{\mathrm{V}}_0\, D_X$, which implies by Lemma 2.1.28 that a corresponding PBW-reduction datum and thus also a Gröbner basis R of the above syzygy module are computable. Consequently,

$$R' := \{ \sum_{g\in G_{a-1}} \tau(r)_{g_0} g \mid r \in R, \pi_{G^0_{a-1}}(\tau(r)) \in \mathbb{C}[\underline{x}]^{G^0_{a-1}}\}$$

$\mathbb{C}[\underline{x}]$-generates P_a, where we abbreviate $\tau_{(\mathrm{Gr}^{\mathrm{V}}_0\, D_X^{G^0_{a-1}\sqcup L''_0}, \prec_b)}$ by τ (see Definition 2.1.33(b)). It remains to find representatives the elements of $t^{-1}R'$ in $F^{\mathrm{V}}_0 D_X^{E'}$, which is achieved as follows: Suitably tracing our Gröbner basis computations (or by using a normal form computation), we write for $r \in R$ in the $\mathbb{C}$-algebra $\mathbb{C}\langle \underline{x}, \underline{\theta}, t\partial_t\rangle$

$$\sum_{g\in G_{a-1}} \tau(r)_{g_0} g_0 = -\sum_{l\in L''} \tau(r)_{l_0} l_0 + \sum_{(q,j,q')\in Q} qjq'$$

with $Q \subseteq \mathbb{C}\langle \underline{x}, \underline{\theta}, t\partial_t\rangle \times (S^{t,0} \cup J^{t,0})^{E'} \times \mathbb{C}\langle \underline{x}, \underline{\theta}, t\partial_t\rangle$ finite. By construction of the sets $S^{t,0}$ and $J^{t,0}$ we then determine for $j \in (S^{t,0} \cup J^{t,0})^{E'}$ an element $j' \in \mathbb{C}\langle \underline{x}, t, \underline{\theta}, t\partial_t\rangle^{E'}$ such that $j + tj' = 0 \in F^{\mathrm{V}}_0 D_X^{E'}$ and therefore

$$\sum_{g\in G_{a-1}} \tau(r)_{g_0} g_0 = -\sum_{l\in L''} \tau(r)_{l_0} l_0 - \sum_{(q,j,q')\in Q} qtj'q' \in F^{\mathrm{V}}_0 D_X^{E'}.$$

We conclude that in $F^{\mathrm{V}}_0 D_X^{E'}$

$$\{\sum_{g\in G_{a-1}} \tau(r)_{g_0} g' + t^{-1}(\sum_{l\in L''} \tau(r)_{l_0} tl' - \sum_{(q,j,q')\in Q} qtj'q') \mid r \in R, \pi_{G^0_{a-1}}(\tau(r)) \in \mathbb{C}[\underline{x}]^{G^0_{a-1}}\}$$

induces a set of $\mathbb{C}[\underline{x}]$-generators of $\{t^{-1}n \mid n \in P_a\}$.

Remark 4.2.24. We outline some alternative approaches for the computation of $P : t^a$.

(a) Writing $P : t^a = P : t^{a-1} + t^{-1} \cdot ({}_{\mathbb{C}[\underline{x},t]}\langle \overline{G_{a-1}}\rangle \cap F^{\mathrm{V}}_{-1} N_X)$, we could also apply Algorithm 2.4.12 instead of the above method. However this approach seems to be computationally more involved as it requires multiple Gröbner basis computations.

(b) We can use for the computation of P_a that $V_0 N_X / tV_0 N_X = V_0 N_X / V_{-1} N_X$ is even a finitely represented D_{X_0}-module (see Remark 4.2.37 in Subsection 4.2.8) and reduce the problem to a syzygy computation over D_{X_0}. However, the computation of such a $\mathcal{D}_{X_0}$-representation is quite involved. So it should be advantageous to use the method we suggested above.

Algorithm 4.2.25 Auxiliary procedure for Algorithm 4.2.26

Input: A D_X-module $N = D_X^E/{}_{D_X}\langle L\rangle$ with $L \subseteq F_0^{\mathbf{v}}\,\mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle^E$ finite and ${}_{F_0^{\mathbf{v}}D_X}\langle L\rangle = {}_{D_X}\langle L\rangle \cap F_0^{\mathbf{v}} D_X^E$ such that $t\cdot$ acts bijectively on N and a finite set $G \subseteq F_0^{\mathbf{v}}\,\mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle^E$.

Output: A finite set $R \subseteq F_0^{\mathbf{v}} D_X^E$ such that ${}_{\mathbb{C}[\underline{x},t]}\langle \overline{R}\rangle = {}_{\mathbb{C}[\underline{x},t]}\langle \overline{G}\rangle :_{F_0^{\mathbf{v}}N} t \subseteq F_0^{\mathbf{v}} N$.

1. Write $g \in G$ and $l \in L$ in D_X as $g = g_0 + tg'$ and $l = l_0 + tl'$ with $g', l' \in \mathbb{C}\langle \underline{x}, t, \underline{\theta}, t\partial_t\rangle$ and $g_0, l_0 \in {}_{\mathbb{C}}\big\langle \{\underline{x}^\alpha \underline{\theta}^\beta (t\partial_t)^\gamma \mid \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m, \gamma \in \mathbb{N}\}\big\rangle$.
2. Set $L_0 := \bigsqcup_{l\in L}\{l_0\}$ and $G_0 := \bigsqcup_{g\in G}\{g_0\}$.
3. Compute a Gröbner basis R of $\operatorname{syz}_{\operatorname{Gr}_0^{\mathbf{v}} D_X}(G_0, L_0)$ with respect to an ordering of type $(\prec_{\mathbf{u}}^{G_0}, \prec^{L_0})$, where $\mathbf{u}$ is a weight vector on $\operatorname{Gr}_0^{\mathbf{v}} D_X$ assigning weight 1 to $\theta_1, \ldots, \theta_{\mathrm{m}}$ and $t\partial_t$ and 0 else. ▷ Identify $\operatorname{Gr}_0^{\mathbf{v}} D_X$ with a PBW-reduction-algebra as above.
4. **for** $r \in R$ with $\pi_{G_0}(\tau(r)) \in \mathbb{C}[\underline{x}]^{G_0}$ **do**
5. Determine $k_r \in \big\langle S^{t,0} \cup J^{t,0}\big\rangle$ and $k'_r \in \mathbb{C}\langle \underline{x}, t, \underline{\theta}, t\partial_t\rangle$ with $k_r + tk'_r = 0 \in F_0^{\mathbf{v}} D_X$ and $\sum_{g\in G} \tau(r)_{g_0} g_0 = -\sum_{l\in L} \tau(r)_{l_0} l_0 + k_r \in \mathbb{C}\langle \underline{x}, \underline{\theta}, t\partial_t\rangle$.
6. Set $R' := \{\sum_{g\in G} \tau(r)_{g_0} g' + \sum_{l\in L'} t^{-1}\tau(r)_{l_0} tl' - k'_s \mid r \in R, \pi_G(\tau(r)) \in \mathbb{C}[\underline{x}]^G\}$.
7. **return** $G \cup R'$.

Note that the output of the above algorithm can be effectively represented by elements in $F_0^{\mathbf{v}}\,\mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle^E$.

We remark that eventually $P : t^{a-1} = P : t^a$, because N_X is strictly X_0-specializable and hence $F_p V_0 N_X$ is a finitely generated $\mathbb{C}[\underline{x}, t]$-module. We check this equality by Algorithm 2.4.5. Thus the algorithm below correctly computes $F_p V_0 N_X$ and terminates if we take N_X and $C := \sum_{e\in E} F^{\mathbf{w}}_{p-\mathbf{s}_e} D'_X \cdot t^{a_e}\rho(\overline{(e)})$ as input:

Algorithm 4.2.26 Given a $D_X[\overline{t}^{-1}]$-module N and a $\mathbb{C}[\underline{x}, t]$-submodule C, this algorithm computes the saturation $(F_0^{\mathbf{v}} N \cap C) :_{F_0^{\mathbf{v}}N} t^\infty$.

Input: A D_X-module $N = D_X^E/{}_{D_X}\langle L\rangle$ with $L \subseteq F_0^{\mathbf{v}}\,\mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle^E$ finite and ${}_{F_0^{\mathbf{v}}D_X}\langle L\rangle = {}_{D_X}\langle L\rangle \cap F_0^{\mathbf{v}} D_X^E$ such that $t\cdot$ acts bijectively on N, and a finite set $C \subseteq D_X^E$.

Output: A set $G \subseteq F_0^{\mathbf{v}} D_X^E$ inducing a set of $\mathbb{C}[\underline{x}, t]$-generators of $(F_0^{\mathbf{v}} N \cap {}_{\mathbb{C}[\underline{x},t]}\langle C\rangle) :_{F_0^{\mathbf{v}}N_X} t^\infty$ if this $\mathbb{C}[\underline{x}, t]$-module is finitely generated.

1. Set $P := t^j C$, where $j \geq \max\{0, \deg_{\mathbf{v}}(c) \mid c \in C\}$.
2. Find a finite set $G \subseteq F_0^{\mathbf{v}} D_X^E$ inducing $\mathbb{C}[\underline{x}, t]$-generators of $P : t$ by Algorithm 4.2.25.
3. **while** $P \neq P : t$ **do** ▷ Check with Algorithm 2.4.5
4. $P := P : t$.
5. Compute a finite set $G \subseteq F_0^{\mathbf{v}} D_X^E$ inducing $\mathbb{C}[\underline{x}, t]$-generators of $P : t$ using Algorithm 4.2.25.
6. **return** G.

Combining the above algorithm with the methods from the previous subsection, we formulate the following algorithm for the computation of $F_p^{\mathrm{Loc}} V_0 N_X$:

Algorithm 4.2.27 Given a coordinate neighborhood X of X_0 and a strictly X_0-specializable $(D_X(*X_0), F_\bullet^\circ)$-module $(N, F_\bullet)$, this algorithm represents N as a quotient of a D_X-module and computes $F_p^{\mathrm{Loc}} V_0 N$.

Input: A strictly X_0-specializable $(D_X[\overline{t}^{-1}], F_\bullet^\circ)$-module $(N, F_\bullet) = (D_X[\overline{t}^{-1}]^E / L, F^\circ[\mathbf{s}]_\bullet)$ with $L' \subseteq D_X^E$ finite such that $L = {}_{D_X[\overline{t}^{-1}]}\langle \overline{L'} \rangle$ and an integer $p \in \mathbb{Z}$.

Output: A finite set E' and finite subsets $L'', G \subseteq D_X^{E'}$ such that $N \cong D_X^{E'} / {}_{D_X}\langle L'' \rangle$ as D_X-module and $F_p^{\mathrm{Loc}} V_0 N \cong {}_{\mathbb{C}[\underline{x},t]}\langle \overline{G} \rangle \subseteq D_X^{E'} / {}_{D_X}\langle L'' \rangle$.

Use Algorithm 4.2.15 to determine a representation $N_X := D_X^{E'} / {}_{D_X}\langle L'' \rangle$ of $N = D_X^E / {}_{D_X}\langle \overline{L'} \rangle \otimes_{O_X} O_X[\overline{t}^{-1}]$ as D_X-module and $b \subseteq (D_X^{E'})^E$ such that $\overline{(e)} \otimes 1$ is represented by $\overline{b_e} \in N_X$ (for $e \in E$).
Set $j := \min\{\mathbf{s}_e \mid e \in E\}$. ▷ $F_q^{\mathrm{Loc}} N_X = 0$ for $q < j$.
if $p < j$ **then**
  return $E', L'', \{0\}$.
Compute $G \subseteq D_X^{E'}$ inducing generators of $F_p^{\mathrm{Loc}} F_0^{\mathbf{v}} N_X$ by Algorithm 4.2.26 with input N_X and $\{\underline{\theta}^\alpha \partial_t^\beta b_e \mid e \in E, \alpha \in \mathbb{N}^{\mathrm{m}}, \beta \in \mathbb{N}, |\alpha| + \beta + s_e \leq p\}$. ▷ See Lemma 4.2.20.
return E', L'', G.

Computation of $F_p^{\mathrm{Loc}} N_X$ for fixed p

Recall that

$$F_p^{\mathrm{Loc}} N = \sum_{i \in \mathbb{N}} \partial_t^i F_{p-i} V_0 N \cong \sum_{i \in \mathbb{N}} \partial_t^i F_{p-i}^{\mathrm{Loc}} F_0^{\mathbf{v}} N_X.$$

Since $F_q^{\mathrm{Loc}} V_0 N_X = 0$ for $q < j := \min\{\mathbf{s}_e \mid e \in E\}$ by definition, $\{F_q^{\mathrm{Loc}} N_X\}_{q \leq p}$ is determined by $\mathbb{C}[\underline{x}, t]$-generators $G_q \subseteq D_X^{E'}$ of $F_q^{\mathrm{Loc}} F_0^{\mathbf{v}} N_X$ (which can be found by Algorithm 4.2.27) for $q = j, \dots, p$. Namely, we have

$$F_q^{\mathrm{Loc}} N_X = \sum_{j \leq i \leq p} \sum_{g \in G_i} (F_{q-i}^{\mathbf{w}} D_X) \cdot \overline{g} \tag{4.2.6}$$

for all $q \leq p$. The above equation shows that it is even sufficient to determine generators of $F_q^{\mathrm{Loc}} F_0^{\mathbf{v}} N_X / F_1^{\mathbf{w}} F_0^{\mathbf{v}} D_X \cdot F_{q-1}^{\mathrm{Loc}} F_0^{\mathbf{v}} N_X$ for $q = j+1, \dots, p$. Hence we reduce the number of generators by dropping members of G_q that have residue class 0 in that module. So we modify Algorithm 4.2.27 as follows:

Algorithm 4.2.28 Given a coordinate neighborhood X of X_0 and a strictly X_0-specializable $(D_X(*X_0), F^\circ_\bullet)$-module $(N, F_\bullet)$, this algorithm represents N as a quotient of a D_X-module and computes $F^{\mathrm{Loc}}_p N$.

Input: A strictly X_0-specializable $(D_X[\bar{t}^{-1}], F^\circ_\bullet)$-module $(N, F_\bullet) = (D_X[\bar{t}^{-1}]^E/L, F^\circ[\mathbf{s}]_\bullet)$ with $L' \subseteq D_X^E$ finite such that $L = {}_{D_X[\bar{t}^{-1}]}\langle L'\rangle$ and an integer $p \in \mathbb{Z}$.

Output: A finite set E', finite sets $P, G \subseteq D_X^{E'}$ and $\mathbf{d} \in \mathbb{Z}^G$ such that $N \cong D_X^{E'}/\,{}_{D_X}\langle P\rangle$ as D_X-module and $F^{\mathrm{Loc}}_q N \cong \sum_{g\in G} F^{\mathbf{w}}_{q-\mathbf{d}_g} D_X \cdot \overline{g} \subseteq D_X^{E'}/\,{}_{D_X}\langle P\rangle$ for $q \leq p$.

Use Algorithm 4.2.15 to determine a representation $N_X := D_X^{E'}/\,{}_{D_X}\langle P\rangle$ of $N = D_X^E/\,{}_{D_X}\langle \overline{L'}\rangle \otimes_{O_X} O_X[\bar{t}^{-1}]$ as D_X-module and $b \in (D_X^{E'})^E$ such that $\overline{(e)} \otimes 1$ is represented by $\overline{b_e} \in N_X$ (for $e \in E$).
Set $j := \min\{\mathbf{s}_e \mid e \in E\}$. ▷ $F^{\mathrm{Loc}}_q N_X = 0$ for $q < j$.
Initialize an empty set $G \subseteq D_X^{E'}$ and a (dynamic) vector $\mathbf{d} \in \mathbb{Z}^G$.
for $q = j, j+1, \ldots, p$ **do**
  Compute a finite set $G' \subseteq D_X^{E'}$ inducing $\mathbb{C}[\underline{x}, t]$-generators of $F^{\mathrm{Loc}}_q F^{\mathbf{v}}_0 N_X$ by using Algorithm 4.2.26 with input N_X and $\{\underline{\theta}^\alpha \partial_t^\beta b_e \mid e \in E, \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}, |\alpha| + \beta + s_e \leq q\}$. ▷ See Lemma 4.2.20.
  for $g' \in G'$ **do** ▷ Check if generator is needed and add to G if necessary.
    if $g' \notin \sum_{g\in G}(F^{\mathbf{w}}_{q-\mathbf{d}_g} F^{\mathbf{v}}_0 D_X) \cdot \overline{g}$ **then** ▷ Check by Algorithm 2.4.5.
      Set $G := G \cup \{g'\}$.
      Set $\mathbf{d}_{g'} := q$.
return $E', P, G, \mathbf{d}$.

By the above algorithm we compute for a fixed integer $p \in \mathbb{Z}$ a set $G \subseteq F^{\mathbf{v}}_0 D_X^{E'}$ and a vecor $\mathbf{d} \in \mathbb{Z}^G$ such that

$$F^{\mathrm{Loc}}_q V_0 N_X = \sum_{g\in G} F^{\mathbf{w}}_{q-\mathbf{d}_g} F^{\mathbf{v}}_0 D_X \cdot \overline{g} \tag{4.2.7}$$

and

$$F^{\mathrm{Loc}}_q N_X = \sum_{g\in G} F^{\mathbf{w}}_{q-\mathbf{d}_g} D_X \cdot \overline{g} \tag{4.2.8}$$

for $q \leq p$. The next step is now to check whether the latter Equation holds for all $q \in \mathbb{Z}$.

Finding generators of $F^{\mathrm{Loc}}_\bullet N_X$

Recall that the filtration $F^{(p)}_\bullet$ on N_X is defined by

$$F^{(p)}_q N_X = \begin{cases} F^{\mathrm{Loc}}_q N_X, & \text{if } q \leq p \\ F^\circ_{q-p} D_X \cdot F^{\mathrm{Loc}}_p N_X, & \text{if } q > p \end{cases}$$

and hence agrees with

$$\sum_{g\in G} F^{\mathbf{w}}_{\bullet-\mathbf{d}_g} D_X \cdot \overline{g}$$

for G as above.

We apply Corollary 3.2.18 to test whether $F^{\mathrm{Loc}}_{\bullet} N_X = F^{(p)}_{\bullet} N_X$. Criterion 3.2.18(a) is satisfied if we choose $p \geq \max\{\mathbf{s}_e \mid e \in E\}$, because we have $F_{\bullet}N = F^{\circ}[\mathbf{s}]_{\bullet} D_X[\overline{t}^{-1}]^E/L$. We check Part (b) of that criterion by testing $F^{\mathbf{v}}_0 N_X = \sum_{g\in G} F^{\mathbf{v}}_0 D_X \cdot \overline{g}$ via Algorithm 2.4.5 since $V_0 N_X = F^{\mathbf{v}}_0 N_X$ by construction. Assuming that the former conditions are fulfilled, our verification of the remaining two conditions is based on Algorithm 2.4.14, which tests whether certain submodule and quotient filtrations agree. For that, and in preparation of expressing the filtration $F^{\mathrm{Loc}}_{\bullet} N$ on a suitable isomorphic module by a shift vector, we compute the kernel of the surjective D_X-linear map

$$\kappa : D^G_X \to N_X,\ (g) \mapsto \overline{g}$$

using Gröbner basis theory to obtain an isomorphism $(D^G_X/\ker(\kappa), F^{\mathbf{w}}[\mathbf{d}]_{\bullet}) \cong (N_X, F^{(p)}_{\bullet})$. Note in particular that $V_k(D^G_X/\ker(\kappa)) = F^{\mathbf{v}}_k(D^G_X/\ker(\kappa))$ for $k \in \mathbb{Z}$. This implies that Conditions 3.2.18(c) and (d) are equivalent to

$$F^{\mathbf{w}}[\mathbf{d}]_{\bullet} F^{\mathbf{v}}_0(D^G_X/\ker(\kappa)) = \sum_{g\in G} (F^{\mathbf{w}}_{\bullet-\mathbf{d}_g} F^{\mathbf{v}}_0 D_X) \cdot \overline{(g)}$$

and

$$F^{\mathbf{w}}[\mathbf{d}]_{\bullet} F^{\mathbf{v}}_{-1}(D^G_X/\ker(\kappa)) = \sum_{g\in G} (F^{\mathbf{w}}_{\bullet-\mathbf{d}_g} F^{\mathbf{v}}_0 D_X) \cdot \overline{t(g)},$$

that is, the submodule and the quotient filtrations induced by $F^{\mathbf{w}}[\mathbf{d}]_{\bullet}$ on

$$\left({}_{F^{\mathbf{v}}_0 D_X}\langle (g) \mid g \in G\rangle + \ker(\kappa)\right) / \ker(\kappa)$$

and

$$\left({}_{F^{\mathbf{v}}_0 D_X}\langle t(g) \mid g \in G\rangle + \ker(\kappa)\right) / \ker(\kappa)$$

agree, respectively. The latter equivalent conditions can be decided by Algorithm 2.4.14. This leads to the following algorithm:

Algorithm 4.2.29 Given a coordinate neighborhood X of X_0 and a strictly X_0-specializable $(D_X(*X_0), F^{\circ}_{\bullet})$-module $(N, F_{\bullet})$, this algorithm computes a representation of the localization $\mathrm{Loc}_{X_0}(N, F_{\bullet})$ as $(D_X, F^{\circ}_{\bullet})$-module.

Input: A strictly X_0-specializable $(D_X[\overline{t}^{-1}], F^{\circ}_{\bullet})$-module $(N, F_{\bullet}) = (D_X[\overline{t}^{-1}]^E/L, F^{\circ}[\mathbf{s}]_{\bullet})$ with $L' \subseteq D^E_X$ finite such that $L = {}_{D_X[\overline{f}^{-1}]}\langle \overline{L'}\rangle$.

Output: A finite set G, a finite set $K \subseteq D_X^G$, and $\mathbf{d} \in \mathbb{Z}^G$ such that we have $\mathrm{Loc}_{X_0}(N, F_\bullet) \cong (D_X^G / {}_{D_X}\langle K \rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet)$.

Compute a representation $N_X = D_X^{E'}/P$ of $N = D_X^E / {}_{D_X}\langle \overline{L'} \rangle \otimes_{O_X} O_X[\overline{t}^{-1}]$ as D_X-module and $b \in (D_X^{E'})^E$ such that $\overline{(e)} \otimes 1$ is represented by $\overline{b_e} \in N_X$ (for $e \in E$) using Algorithm 4.2.15.
Set $j := \min\{\mathbf{s}_e \mid e \in E\}$. ▷ $F_q^{\mathrm{Loc}} N_X = 0$ for $q < j$.
Set $k := \max\{\mathbf{s}_e \mid e \in E\}$. ▷ $F_k N$ generates $F_\bullet N$ as $F_\bullet^\circ D_X$-module.
Initialize an empty set $G \subseteq D_X^{E'}$ and a (dynamic) vector $\mathbf{d} \in \mathbb{Z}^G$. ▷ Save generators of the filtration in G and corresponding degrees in $\mathbf{d}$.
for $q = j, j+1, \ldots$ **do**
  Compute a finite set $G' \subseteq D_X^{E'}$ inducing $\mathbb{C}[\underline{x}, t]$-generators of $F_q^{\mathrm{Loc}} F_0^{\mathbf{v}} N_X$ by applying Algorithm 4.2.26 with input N_X and $\{\underline{\theta}^\alpha \partial_t^\beta b_e \mid e \in E, \alpha \in \mathbb{N}^{\mathrm{m}}, \beta \in \mathbb{N}, |\alpha| + \beta + s_e \leq q\}$. ▷ See Lemma 4.2.20.
  for $g' \in G'$ **do** ▷ Check if generator is needed and add to G if necessary.
    if $g' \notin \sum_{g \in G} (F^{\mathbf{w}}_{q - \mathbf{d}_g} F_0^{\mathbf{v}} D_X) \cdot \overline{g}$ **then** ▷ Check by Algorithm 2.4.5.
      Set $G := G \cup \{g'\}$.
      Set $\mathbf{d}_{g'} := q$.
  if $q \geq k$ **then** ▷ Condition 3.2.18(a) is satisfied.
    if $F_0^{\mathbf{v}} N_X = \sum_{g \in G} F_0^{\mathbf{v}} D_X \cdot \overline{g}$ **then** ▷ Check using Algorithm 2.4.5.
      Find D_X-generators K of the kernel of the D_X-linear map $\kappa : D_X^E \to N_X, (g) \mapsto \overline{g}$ using Gröbner basis theory.
      if $F^{\mathbf{w}}[\mathbf{d}]_\bullet F_0^{\mathbf{v}} (D_X^G / \ker(\kappa)) = \sum_{g \in G} (F^{\mathbf{w}}_{\bullet - \mathbf{d}_g} F_0^{\mathbf{v}} D_X) \cdot \overline{(g)}$ **then** ▷ Check by Algorithm 2.4.14.
        if $F^{\mathbf{w}}[\mathbf{d}]_\bullet F_{-1}^{\mathbf{v}} (D_X^G / \ker(\kappa)) = \sum_{g \in G} (F^{\mathbf{w}}_{\bullet - \mathbf{d}_g} F_0^{\mathbf{v}} D_X) \cdot \overline{t(g)}$ **then** ▷ Check by Algorithm 2.4.14.
          return $G, K, \mathbf{d}$.

Remark 4.2.30. We remark that the isomorphism in the above algorithm is traceable in analogy with Remark 4.2.14(b).

Remark 4.2.31. Recall that given a strictly X_0-specializable $(D_X[\overline{t}^{-1}], F_\bullet^\circ)$-module $(N, F_\bullet)$ the problem of computing $F_\bullet V_\alpha N$ for $\alpha \in \mathbb{Q}$ is still open. As we have $F_\bullet^{\mathrm{Loc}} V_\alpha N = F_\bullet V_\alpha N$ for $\alpha \leq 0$ (see Remark 3.2.12(a)), the above algorithm enables us to describe $F_\bullet V_\alpha N$ for $\alpha \leq 0$. By Remark 3.2.10 this completely determines $F_\bullet V_\bullet N$.

4.2.7 Dual localization of (strictly) X_0-specializable $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules

Given an X_0-specializable $\mathcal{D}_X$-module $M = D_X^E / K$ and an optional filtration $F_\bullet M = F^{\mathbf{w}}[\mathbf{s}]_\bullet (D_X^E / K)$ making this module strictly X_0-specializable, we explain how to compute

$\mathrm{DLoc}_{X_0}(M,(F_\bullet))$. As we have by definition

$$\mathrm{DLoc}_{X_0}(M) = M(!X_0) = D_X \otimes_{V_0 D_X} V_{<0}\,\mathcal{M}$$

and the V_0D_X-module on the right hand side can be represented as a quotient of a free V_0D_X-module by Algorithms 4.2.5 and 2.4.4, Lemma 3.1.30 allows us to represent $\mathrm{DLoc}_{X_0}(M)$ as a quotient of a free $\mathcal{D}_X$-module. In the filtered case, replacing Algorithm 4.2.5 by Algorithm 4.2.9, the filtration $F_\bullet$ on $V_{<0}M$ will be given by a shift vector on the computed quotient of a V_0D_X-module. Hence, by definition, the filtration on $\mathrm{DLoc}_{X_0}(M)$ is also given by the same shift vector on its representation as a quotient of a free $\mathcal{D}_X$-module obtained by Lemma 3.1.30. This leads to the following algorithm, which in addition represents the natural map $i_{(!X_0)} : \mathrm{DLoc}_{X_0}(M) \to M$:

Algorithm 4.2.32 Given a coordinate neighborhood X of X_0 and a strictly X_0-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet)$, this algorithm represents $\mathrm{DLoc}_{X_0}(M, F_\bullet)$ as $(D_X, F^\circ_\bullet)$-module and computes the natural map $i_{(!X_0)} : \mathrm{DLoc}_{X_0}(M) \to M$.

Input: A strictly X_0-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$.

Output: A finite set E', a finite subset $L \subseteq D_X^{E'}$, $\mathbf{d} \in \mathbb{Z}^L$ and $q \in (D_X^E)^{E'}$ that satisfy $\mathrm{DLoc}_{X_0}(M, F_\bullet) \cong (D_X^{E'} / {}_{D_X}\langle L\rangle, F^\circ[\mathbf{d}]_\bullet)$ as $(D_X, F^\circ_\bullet)$-modules, $V_{<0}(D_X^{E'} / {}_{D_X}\langle L\rangle) = F_0^{\mathbf{v}}(D_X^{E'} / {}_{D_X}\langle L\rangle)$ and induce the natural map $i_{(*X_0)} : D_X^{E'} / {}_{D_X}\langle L\rangle \to M$ via $\overline{(e')} \mapsto \overline{q_{e'}}$ for $e' \in E'$.

1: Compute the induced b-function $b(s) \in \mathbb{Q}[s]$ along X_0 on M by Algorithm 4.2.4 and set $\alpha := \max\{r + z \mid r \in \mathbb{Q}, b(r) = 0, z \in \mathbb{Z}, r + z < 0\}$. ▷ $V_\alpha M = V_{<0}M$.
2: Find a finite set $E' \subseteq \mathbb{K}\langle \underline{x}, t, \underline{\theta}, \partial_t\rangle^E$ that satisfies $F^\circ_\bullet V_\alpha M = \sum_{e' \in E'} F^{\mathbf{w}}_{\bullet - \deg_{F^{\mathbf{w}}[\mathbf{s}]}(e')} D_X \cdot \overline{e'}$ by Algorithm 4.2.9 and Remark 4.2.10(a).
3: Define $\mathbf{d} \in \mathbb{Z}^{E'}$ by $\mathbf{d}_{e'} = \deg_{F^{\mathbf{w}}[\mathbf{s}]}(e')$ for $e' \in E'$.
4: Represent $V_\alpha M$ as a quotient $F_0^{\mathbf{v}} D_X^{E'} / {}_{F_0^{\mathbf{v}} D_X}\langle L\rangle$ with L finite via Algorithm 2.4.4. ▷ ${}_{F_0^{\mathbf{v}} D_X}\langle L\rangle = \ker(F_0^{\mathbf{v}} D_X^{E'} \to V_{<0}M, (e') \mapsto \overline{e'})$.
5: Define $q \in (D_X^E)^{E'}$ by $q_{e'} = e'$ for $e' \in E'$.
6: **return** E', L, q.

Remark 4.2.33. The dual localization of a strictly X_0-specializable $(\mathcal{D}_X(*X_0), F^\circ_\bullet)$-module $(N, F_\bullet)$ is computed by using

$$\mathrm{DLoc}_{X_0}(N, F_\bullet) = \mathrm{DLoc}_{X_0}(\mathrm{Loc}_{X_0}(N, F_\bullet))$$

(see Remark 3.2.24), where the localization and dual localization on the right hand side are determined by Algorithm 4.2.29 and Algorithm 4.2.32, respectively.

4.2.8 Graded parts of V-filtrations

In view of the computations of the vanishing and nearby cycle functors (see Subsection 4.3.3), we explain how to represent the graded parts of the V-filtration along X_0 on D_X-modules as D_{X_0}-modules. Recall that $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ and denote by K' a finite set of D_X-generators of K. Assuming that $(M, F_\bullet)$ is strictly X_0-specializable, or more generally that M is X_0-specializable and $F_\bullet V_\alpha M$ is a good $(V_0 D_X, F_\bullet^\circ)$-filtration, we give a method to represent $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ as a well-filtered $(D_{X_0}, F_\bullet^\circ)$-module for fixed α. For that we first write $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ as a quotient of a free $(V_0 D_X, F_\bullet^\circ)$-module, then we derive from this a free $(\operatorname{Gr}_0^V D_X, F_\bullet^\circ)$-presentation of $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ and finally we use the nilpotence of $(-\partial_t t - \alpha)$ on $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ to obtain the desired $(D_{X_0}, F_\bullet^\circ)$-representation.

Note that since $(V_\alpha M, F_\bullet)$ is a well-filtered $(V_0 D_X, F_\bullet^\circ)$-module, $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ is a well-filtered $(V_0 D_X, F_\bullet^\circ)$-module generated by the residue classes of a set of $(V_0 D_X, F_\bullet^\circ)$-generators of $(V_\alpha M, F_\bullet)$. It is represented as a quotient of a free $(V_0 D_X, F_\bullet^\circ)$-module as follows: First compute a finite set $G \subseteq D_X^E$ and a shift vector $\mathbf{d} \in \mathbb{Z}^G$ such that

$$F_\bullet V_\alpha M = \sum_{g\in G} F^{\mathbf{w}}_{\bullet - \mathbf{d}_g} F_0^{\mathbf{v}} D_X \cdot \overline{g} = \sum_{g \in G} F^{\mathbf{w}}_{\bullet - \deg_{F^{\mathbf{w}}[\mathbf{s}]}(g)} F_0^{\mathbf{v}} D_X \cdot \overline{g}.$$

Then there is a surjective strict $F_0^{\mathbf{v}} D_X$-linear map

$$\rho : (F_0^{\mathbf{v}} D_X^G, F^{\mathbf{w}_{\mathbf{v}}}[\mathbf{d}]_\bullet) \twoheadrightarrow (\operatorname{Gr}_\alpha^V M, F_\bullet),\ (g) \mapsto \overline{g} + V_{<\alpha} M,$$

where $\mathbf{w}_{\mathbf{v}}$ denotes the weight vector induced by $\mathbf{w}$ on the PBW-reduction-algebra $F_0^{\mathbf{v}} D_X$ (see Example 2.2.8(d)). To determine its kernel, we first find a set of $F_0^{\mathbf{v}} D_X$-generators of $V_{<\alpha} M$: Setting

$$\beta := \max\{r + z \mid r \in \mathbb{Q}, b(r) = 0, z \in \mathbb{Z}, r + z < \alpha\},$$

where $b(s)$ denotes the induced b-function along X_0 on M, we get that $V_{<\alpha} M = V_\beta M$. If $G' \subseteq D_X^E$ is finite such that $\overline{G'}$ is a set of $F_0^{\mathbf{v}} D_X$-generators of $V_\beta M$, then $a \in F_0^{\mathbf{v}} D_X^G$ is in the kernel of ρ if and only if $\sum_{g\in G} a_g \overline{g} \in V_\beta M$, that is, if and only if

$$\sum_{g \in G} a_g g \in {}_{F_0^{\mathbf{v}} D_X}\langle G' \rangle + K. \tag{4.2.9}$$

Hence

$$\ker(\rho) = \pi_G \left(\operatorname{syz}_{D_X}(G, G', K') \cap (F_0^{\mathbf{v}} D_X^{G \sqcup G'} \oplus D_X^{K'}) \right)$$

and generators of the above intersection are obtained as outlined in Algorithm 2.2.27. Consequently, we have

$$F_\bullet \operatorname{Gr}_\alpha^V M \cong F^{\mathbf{w}_{\mathbf{v}}}[\mathbf{d}]_\bullet((F_0^{\mathbf{v}} D_X^G)/\ker(\rho)).$$

Algorithm 4.2.34 Given a coordinate neighborhood X of X_0 and an X_0-specializable good $(D_X, F_\bullet^\circ)$-module $(M, F_\bullet)$ such that $(V_\alpha M, F_\bullet)$ is $(V_0 D_X, F_\bullet^\circ)$-good, this algorithm computes a representation of $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ as $(V_0 D_X, F_\bullet^\circ)$-module.

Input: An X_0-specializable good $(D_X, F_\bullet^\circ)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ with $K = {}_{D_X}\langle K' \rangle$ for $K' \subseteq D_X^E$ finite and $\alpha \in \mathbb{Q}$ such that $(V_\alpha M, F_\bullet)$ is $(V_0 D_X, F_\bullet^\circ)$-good.
Output: A finite set G, a finite set $J \subseteq F_0^{\mathbf{v}} D_X^G$ and $\mathbf{d} \in \mathbb{Z}^G$ such that $(\operatorname{Gr}_\alpha^V M, F_\bullet) \cong ((F_0^{\mathbf{v}} D_X^G)/{}_{F_0^{\mathbf{v}} D_X}\langle J \rangle, F^\circ[\mathbf{d}]_\bullet)$ as $(V_0 D_X, F_\bullet^\circ)$-module.

Compute the induced b-function $b(s)$ along X_0 on M, a set $G \subseteq D_X^E$ and $\mathbf{d} \in \mathbb{Z}^E$ such that $F_\bullet V_\alpha M = \sum_{g \in G} F_{\bullet - \mathbf{d}_g}^{\mathbf{w}} F_0^{\mathbf{v}} D_X \cdot \overline{g}$ using Algorithm 4.2.9.
Set $\beta := \max\{r + z \mid r \in \mathbb{Q}, b(r) = 0, z \in \mathbb{Z}, r + z < \alpha\}$. $\triangleright\, V_\beta M = V_{<\alpha} M$.
Determine a finite set $G' \subseteq D_X^E$ such that $\overline{G'}$ is a set of $F_0^{\mathbf{v}} D_X$-generators of $V_\beta M$ by Algorithm 4.2.5.
Compute a finite set S of D_X-generators of $\pi_{G,G'}(\operatorname{syz}_{D_X}(G, G', K'))$ using Gröbner basis theory.
Find $F_0^{\mathbf{v}} D_X$-generators J of ${}_{D_X}\langle S \rangle \cap F_0^{\mathbf{v}} T_X^{G \sqcup G'}$ by Algorithm 2.2.27.
Replace J by $\pi_G(J)$.
return $G, J, \mathbf{d}$.

As in Remark 3.1.8, if $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ is $(V_0 D_X, F_\bullet^\circ)$-well-filtered, then it is also well-filtered as $(\operatorname{Gr}_0^V D_X, F_\bullet^\circ)$-module and $(D_{X_0}, F_\bullet^\circ)$-module, because t acts by zero on $\operatorname{Gr}_\alpha^V M$ and the action of $-\partial_t t - \alpha$ on that module is nilpotent. Hence given that

$$(\operatorname{Gr}_\alpha^V M, F_\bullet) \cong ((F_0^{\mathbf{v}} D_X^G)/J, F^{\mathbf{w}_\mathbf{v}}[\mathbf{d}]_\bullet)$$

with $J = {}_{F_0^{\mathbf{v}} D_X}\langle J' \rangle$ for J' finite and $\mathbf{d} \in \mathbb{Z}^G$, we have $t F_0^{\mathbf{v}} D_X^G = F_{-1}^{\mathbf{v}} D_X^G \subseteq J$ and $(-\partial_t t - \alpha)^{m_{b(s)}(\alpha)} \cdot (g) \in J$ (for $g \in G$) according to Remark 3.1.24. By Lemma 4.2.23 we hence write

$$(\operatorname{Gr}_\alpha^V M, F_\bullet) \cong ((\operatorname{Gr}_0^{\mathbf{v}} D_X^G)/\nu^G(J), F^{\mathbf{w}'_\mathbf{v}}[\mathbf{d}]_\bullet),$$

where $\mathbf{w}'_\mathbf{v}$ stands for the weight vector induced by $\mathbf{w}_\mathbf{v}$ on the realization of $\operatorname{Gr}_0^{\mathbf{v}} D_X$ in Equation (4.2.4), that is, the weight vector assigning weight 1 to θ_i $(1 \leq i \leq m)$ and $t\partial_t$ and weight 0 else. Noting that $\operatorname{Gr}_0^V D_X = D_{X_0}[t\partial_t]$ according to Example 2.1.30(d), the residue classes of

$$G'' := \{(t\partial_t)^j(g) \mid g \in G, 0 \leq j < m_{b(s)}(\alpha)\} \subseteq \operatorname{Gr}_0^{\mathbf{v}} D_X^G$$

D_{X_0}-generate $(\operatorname{Gr}_0^{\mathbf{v}} D_X^G)/\nu^G(J)$. So we get a surjective D_{X_0}-linear morphism

$$\mu : D_{X_0}^{G''} \twoheadrightarrow (\operatorname{Gr}_0^{\mathbf{v}} D_X^G)/\nu^G(J),\ (g'') \mapsto \overline{g''}$$

inducing an isomorphism of $(D_{X_0}, F_\bullet^\circ)$-modules

$$(\operatorname{Gr}_\alpha^V M, F_\bullet) \cong (D_{X_0}^{G''}/\ker(\mu), F^{\mathbf{w}_0}[\mathbf{d}']_\bullet),$$

where $\mathbf{d}'_{g''} := \deg_{F^{\mathbf{w_v}}[\mathbf{d}]}(g'') \leq \mathbf{d}_g + j$ for $g'' = (t\partial_t)^j(g) \in G''$. Note that $\mathbf{d}'_{g''}$ is computable by checking if $\overline{g''} \in F_k^{\mathbf{w}'_\mathbf{v}}((\operatorname{Gr}_0^{\mathbf{v}} D_X^G)/\nu^G(J))$ for $k = \mathbf{d}_g + j - 1, \mathbf{d}_g + j - 2, \ldots$ using Algorithm 2.4.5 until this test fails, because the filtration under consideration is separated.

To compute generators of $\ker(\mu)$ note that $a \in D_{X_0}^{G''}$ is in the kernel of μ is and only if $\sum_{g \in G''} a_{g''} g'' \in \nu^G(J)$. Since $\operatorname{Gr}_0^{\mathbf{v}} D_X \cong D_{X_0}[t\partial_t]$, a set of $F_0^{\mathbf{u}} \operatorname{Gr}_0^V D_X$-generators of

$$\pi_{G''}(\operatorname{syz}(G'', \nu^G(J))) \cap F_0^{\mathbf{u}} \operatorname{Gr}_0^{\mathbf{v}} D_X^{G''},$$

where $\mathbf{u}$ stands for the weight vector on $\operatorname{Gr}_0^{\mathbf{v}} D_X$ assigning weight 1 to $t\partial_t$ and weight 0 else, D_{X_0}-generates also $\ker(\mu)$.

Remark 4.2.35. The isomorphism $\operatorname{Gr}_\alpha^V M \cong D_{X_0}^{G''}/\ker(\mu)$ is traceable: Namely, write $m \in V_\alpha M$ as an $F_0^{\mathbf{v}} D_X$-linear combination of $\overline{G}$ by Algorithm 2.4.5 and Remark 2.4.6. Since t acts as zero on $\operatorname{Gr}_\alpha^V M$, we may even assume that the coefficients of the linear combination live in $D_{X_0}[t\partial_t]$. Noting that $(-\partial_t t - \alpha)^{m_{b(s)}(\alpha)}$ annihilates $\operatorname{Gr}_\alpha^V M$, and writing $(t\partial_t)^{m_{b(s)}(\alpha)} - (\partial_t t + \alpha)^{m_{b(s)}(\alpha)} = \sum_{0 \leq i < m_{b(s)}(\alpha)} a_i (t\partial_t)^i$, we replace recursively any $(t\partial_t)^{m_{b(s)}(\alpha)}$ appearing in the coefficients by $\sum_{0 \leq i < m_{b(s)}(\alpha)} a_i (t\partial_t)^i$. From this we derive a D_{X_0}-linear combination of $\overline{m} \in \operatorname{Gr}_\alpha^V M$ in terms of $\overline{G''}$ from which we can read off the representation of $\overline{m}$ in $D_{X_0}^{G''}/\ker(\mu)$. Tracing the isomorphism in the converse direction is trivial.

We summarize our method:

Algorithm 4.2.36 Given a coordinate neighborhood X of X_0 and an X_0-specializable good $(D_X, F_\bullet^\circ)$-module $(M, F_\bullet)$ such that $(V_\alpha M, F_\bullet)$ is $(V_0 D_X, F_\bullet^\circ)$-good, this algorithm computes a representation of $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ as $(D_{X_0}, F_\bullet^\circ)$-module.

Input: An X_0-specializable good $(D_X, F_\bullet^\circ)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ and $\alpha \in \mathbb{Q}$ such that $(V_\alpha M, F_\bullet)$ is a good $(V_0 D_X, F_\bullet^\circ)$-module.

Output: A finite set G, a finite set $J \subseteq D_{X_0}^G$ and $\mathbf{d} \in \mathbb{Z}^G$ such that we have isomorphisms $(\operatorname{Gr}_\alpha^V M, F_\bullet) \cong (D_{X_0}^G/{}_{D_{X_0}}\langle \overline{J} \rangle, F^\circ[\mathbf{d}]_\bullet)$ as $(D_{X_0}, F_\bullet^\circ)$-modules.

Find by Algorithm 4.2.34 a representation $(F_0^{\mathbf{v}} D_X^{E'}/{}_{F_0^{\mathbf{v}} D_X}\langle J' \rangle, F^{\mathbf{w}}[\mathbf{c}]_\bullet)$ of $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ (with J' finite).
Set $J' := \nu^{E'}(J) \subseteq \operatorname{Gr}_V^0 D_X^{E'}$.
Determine $m_\alpha := m_{b(s)}(\alpha)$, where $b(s)$ is the induced b-function along X_0 on M. ▷ See Remark 3.1.24.
Set $G := \{(\partial_t t)^i(e') \mid 0 \leq i < m_\alpha, e' \in E'\} \subseteq \operatorname{Gr}_V^0 D_X$
for $e' \in E'$ **do**
  for $i = 0, \ldots, m_\alpha - 1$ **do**
    Set $j := \mathbf{d}_{e'} + i - 1$.
    while $\overline{(t\partial_t)^i(e')} \in F_{j-1}^{\mathbf{w}'_\mathbf{v}}(\operatorname{Gr}_0^V D_X^{E'}/{}_{\operatorname{Gr}_0^V D_X}\langle J'' \rangle)$ **do** ▷ Check by Algorithm 2.4.5.
      Set $j := j$.

Set $\mathbf{d}_{(t\partial_t)^j(e')} := j$.

Find a set J of $F_0^{\mathbf{u}}\operatorname{Gr}_0^V D_X$-generators of $\pi_G(\operatorname{syz}(G, \nu^G(J))) \cap F_0^{\mathbf{u}}\operatorname{Gr}_0^{\mathbf{v}} D_X^G$ by Algorithm 2.2.27, where $\mathbf{u}$ stands for the weight vector on $\operatorname{Gr}_0^{\mathbf{v}} D_X$ assigning weight 1 to $t\partial_t$ and weight 0 else.

Write $h \in H$ as $h = \sum_{0 \leq j < m_\alpha, e' \in E'} h_{(t\partial_t)^j(e')}(t\partial_t)^j(e')$ with $_{(t\partial_t)^j(e')} \in D_{X_0}$.

Return $G, J, \mathbf{d}$.

Remark 4.2.37.

(a) Algorithms 4.2.34 and 4.2.36 can be modified to represent $(V_\alpha M/V_{\alpha-1}M, F_\bullet)$ by replacing β by $\alpha - 1$ in Algorithm 4.2.34 and $m_{b(s)}(\alpha)$ by $\deg b(s)$ in Algorithm 4.2.36.

(b) Given an X_0-specializable (unfiltered) D_X-module M, we adapt Algorithms 4.2.34 and 4.2.36 to this situation by computing in Algorithm 4.2.34 just any set of $F_0^{\mathbf{v}} D_X$-generators of $V_\alpha M$ and forgetting all the shift vectors involved.

The following remark is needed to realize the morphisms can and var later on:

Remark 4.2.38. Recall that $(\operatorname{Gr}_\alpha^V M, F_\bullet)$ is endowed with a nilpotent D_{X_0}-linear endomorphism $\mathrm{N} = -\partial_t t - \alpha = -t\partial_t - (\alpha + 1)$. We make this morphism under the isomorphism $(\operatorname{Gr}_\alpha^V M, F_\bullet) \cong (D_{X_0}^{G''}/\ker(\mu), F^{\mathbf{w}_0}[\mathbf{d}']_\bullet)$ explicit : Using the notation of Remark 4.2.35, we obtain

$$t\partial_t \cdot : (D_{X_0}^{G''}/\ker(\mu), F^{\mathbf{w}_0}[\mathbf{d}']_\bullet) \to (D_{X_0}^{G''}/\ker(\mu), F^{\mathbf{w}_0}[\mathbf{d}']_{\bullet+1});$$
$$\overline{(t(\partial_t)^j(g))} \mapsto \begin{cases} \overline{((t\partial_t)^{j+1}(g))}, & \text{if } j < m_{b(s)}(\alpha) - 1 \\ \overline{\sum_{0 \leq i < m_{b(s)}(\alpha)} a_i((t\partial_t)^i(g))}, & \text{if } j = m_{b(s)}(\alpha) - 1. \end{cases}$$

We also represent the $(D_{X_0}, F_\bullet^\circ)$-linear morphisms

$$\partial_t \cdot : (\operatorname{Gr}_\alpha^V M, F_\bullet) \to (\operatorname{Gr}_{\alpha+1}^V M, F_{\bullet+1}) \text{ and } t \cdot : (\operatorname{Gr}_{\alpha+1}^V M, F_\bullet) \to (\operatorname{Gr}_\alpha^V M, F_\bullet) :$$

Since these maps involve not only the module $\operatorname{Gr}_\alpha^V M$ but also $\operatorname{Gr}_{\alpha+1}^V M$, we adapt our notation by additionally using the lower indexes α and $\alpha - 1$ (e.g. we write G_α instead of G for the set whose residue class $V_0 D_X$-generates $V_\alpha M$ and so on). We find by Algorithm 2.4.5 and Remark 2.4.6 elements $b \in (F_0^{\mathbf{v}} D_X^{G_{\alpha+1}})^{G_\alpha}$ and $c \in (F_0^{\mathbf{v}} D_X^{G_\alpha})^{G_{\alpha+1}}$ such that $\partial_t \overline{g} = \sum_{g' \in G_{\alpha+1}} (b_g)_{g'} \overline{g'}$ and $t\overline{g'} = \sum_{g \in G_\alpha} (c_{g'})_g \overline{g}$ for $g \in G_\alpha$ and $g' \in G_{\alpha+1}$. Hence these morphisms are given by

$$\partial_t \cdot : D_{X_0}^{G''_\alpha}/\ker(\rho_\alpha) \to D_{X_0}^{G''_{\alpha+1}}/\ker(\rho_{\alpha+1}),\ \overline{((t\partial_t)^j(g))} \mapsto (t\partial_t + 1)^j \sum_{g' \in G_{\alpha+1}} (b_g)_{g'} \cdot \overline{((g'))}$$

and

$$t \cdot : D_{X_0}^{G''_{\alpha+1}}/\ker(\rho_{\alpha+1}) \to D_{X_0}^{G''_\alpha}/\ker(\rho_\alpha),\ \overline{((t\partial_t)^j(g'))} \mapsto (t\partial_t - 1)^j \sum_{g \in G_\alpha} (c_{g'})_g \cdot \overline{((g))}.$$

To evaluate the above actions on the right hand sides, note that the action of $F_0^{\mathbf{v}} D_X$ on the above modules is given by letting t act as zero on them and that $t\partial_t$ operates as described above.

4.3 Computations using global coordinate systems for general codimension one subvarieties

Let $X = V(J) \subseteq \mathbb{C}^{\mathrm{n}}$ with $J \subseteq \mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_{\mathrm{n}}]$ prime be a smooth irreducible affine variety of dimension m with global coordinate system and X_0 a codimension one subvariety with defining ideal sheaf generated by $\overline{f}$ for $f \in \mathbb{C}[\overline{x}]$. The main aim of this section is to represent the localizations $\mathrm{Loc}_{X_0}(M, (F_\bullet))$, $\mathrm{Loc}_{X_0}(N, (F_\bullet))$, $\mathrm{DLoc}_{X_0}(M, (F_\bullet))$ and $\mathrm{DLoc}_{X_0}(N, (F_\bullet))$ as well as the vanishing cycle functors along $\overline{f} \in O_X$. We point out that if X_0 is smooth then the localizations along X_0 is locally computable by the methods of the last section (i.e., shrink X such that $\overline{f}$ is part of a global coordinate system) and that it is possible to glue them together using the method outlined in Section 4.4. However, the advantage of the method in this section, which relies on the graph embedding along the graph of f, is that we do not need to work locally and to glue our local results.

By assumption there exist local coordinates given by the residue classes of $f_1, \ldots, f_{\mathrm{m}} \in \mathbb{C}[\underline{x}]$ and corresponding commuting derivations $\theta_1, \ldots, \theta_{\mathrm{m}} \in \Theta_X(X)$ induced by derivations $\theta_1^l, \ldots, \theta_{\mathrm{m}}^l \in \mathrm{Der}(\mathbb{C}[\underline{x}])$.

Remark 4.3.1. In view of Convention 4.1.3 we may assume that f and some generating set of J are defined over $\mathbb{K}[x_1, \ldots, x_{\mathrm{n}}]$. Hence the derivation $\theta_1^l, \ldots, \theta_{\mathrm{m}}^l$ can be realized over the field $\mathbb{K}$ implying that we may assume that $\mathbb{K}$ is a T_X-computable field.

Remark 4.3.2. According to Example 2.1.30(a) D_X is realized as the PBW-reduction-algebra

$$T_X := (\mathbb{C}\langle \underline{x}, \theta_1, \ldots, \theta_{\mathrm{m}} \rangle, S, J', \prec)$$

with

$$S := \{[x_j, x_i], [\theta_p, \theta_k], [\theta_k, x_i] - \theta_k^l(x_i) \mid \text{for } 1 \le i \le j \le \mathrm{n}, 1 \le k \le p \le \mathrm{m}\} \setminus \{0\},$$

$\prec$ any well-order such that S is a standard reduction system with respect to $\prec$ (for instance a well-ordering satisfying $\underline{x}^\alpha \underline{\theta}^\beta \prec \underline{x}^{\alpha'} \underline{\theta}^{\beta'}$ if $|\beta| < |\beta'|$ using usual multi-index notation) and $J' \subseteq \mathbb{C}[\underline{x}]$ a Gröbner basis of J with respect to the ordering induced by $\prec$. From now on, we identify D_X and T_X. Denoting by $\mathbf{w} \in \mathbb{Z}^{\mathrm{n}+\mathrm{m}}$ the weight vector assigning weight 1 to θ_k ($1 \le k \le \mathrm{m}$) and weight 0 else on T_X, we have under the above identification

$$F_\bullet^\circ D_X = F_\bullet^{\mathbf{w}} T_X.$$

However, we will not perform our computations over this PBW-reduction-algebra but rather over a certain tensor product of this algebra.

All our algorithms rely on taking direct images under the graph embedding

$$i_f : X \to Y := X \times \mathbb{C}_t,\ x \mapsto (x, f(x))$$

and translating the corresponding computations to computations on Y fitting in the situation of Section 4.2. Note that Y has a global coordinate system consisting of t and of the global coordinates of X with corresponding differentials induced by ∂_t and $\theta_1^l, \ldots, \theta_{\mathrm{m}}^l$. Therefore, D_Y is isomorphic to the PBW-reduction-algebra

$$T_Y = (\mathbb{C}\langle \underline{x}, t, \underline{\theta}, \partial_t \rangle, S_Y, J_Y, \prec_Y),$$

where $S_Y = S \cup \{[t, x_i], [\partial_t, x_i], [\theta_j, t], [\theta_j, \partial_t], [\partial_t, t] - 1 \mid 1 \le i \le \mathrm{n}, 1 \le j \le \mathrm{m}\}$, $\prec_Y$ any well-order such that S_Y is a reduction system with respect to $\prec_Y$ and J_Y a Gröbner basis of $\langle J \cup \{t\} \rangle \subseteq \mathbb{C}[\underline{x}, t]$. In particular, T_Y satisfies all properties needed to apply the algorithms of Section 4.2 for the embedding $Y_0 := X \times \{0\} \subseteq Y$. We denote by $\mathbf{v} \in \mathbb{Z}^{\mathrm{n}+\mathrm{m}+2}$ the weight vector assigning weight 1 to ∂_t, weight -1 to t and weight 0 else. By abuse of notation, the weight vector $\mathbf{w} \in \mathbb{Z}^{\mathrm{n}+\mathrm{m}+2}$ stands also for the weight vector on T_Y assigning weight 1 to θ_k and ∂_t and weight 0 else. *As in Section 4.2 we identify D_Y with T_Y, represent its elements in the same manner and use the notation $F_\bullet^{\mathbf{u}} D_Y$ for a weight vector $\mathbf{u}$ on T_Y.*

To represent direct images of finitely presented D_X-modules under the graph embedding i_f, we factor i_f via

$$\begin{array}{ccc} X & \xrightarrow{\quad i_f \quad} & X \times \mathbb{C}_t \\ & {\scriptstyle i_0 : x \mapsto (x,0)} \searrow \quad \nearrow {\scriptstyle \lambda : (x,t) \mapsto (x, t + f(x))} & \\ & X \times \mathbb{C}_t, & \end{array} \tag{4.3.1}$$

and then Example 1.4.10 implies that we have an identification

$$(i_f)_+(D_X^E/_{D_X}\langle Q \rangle) = D_Y^E/_{D_Y}\langle \Lambda^E(Q), t - f \rangle, \tag{4.3.2}$$

where we regard Q as a subset of $D_Y = D_X \otimes_{\mathbb{C}} D_{\mathbb{C}}$ in order to apply Λ^E. When writing $(i_f)_+P$ for a finitely presented D_X-module, we always assume that $(i_f)_+P$ is presented as above. To simplify notation, we often write Λ for Λ^E and similarly for the inverse λ' of λ.

4.3.1 Specializable $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules

Our aim is to decide if $M = D_X^E/K$ and $N = D_X[\overline{f}^{-1}]^E/L$ are X_0-specializable. By definition M is X_0-specializable if and only if $(i_f)_+M$ is Y_0-specializable, which can be checked by Algorithm 4.2.5. Similarly, writing $N = N'[\overline{f}^{-1}]$ with N' a finitely presented D_X-module, we have $(i_f)_+N \cong ((i_f)_+N')[\overline{t}^{-1}]$ (see Lemma 1.4.19). Hence N is X_0-specializable if and only if $((i_f)_+N')[\overline{t}^{-1}]$ is Y_0-specializable which is equivalent to $(i_f)_+N'$ being Y_0-specializable according to Lemma 3.1.28(a). As above we test the latter condition by Algorithm 4.2.5.

Remark 4.3.3. If X_0 is smooth, it is also possible to compute the filtration along X_0 on M: We briefly outline two methods for this. One of them uses the methods from Subsection 4.2.1, while the other relies on a graph embedding as above. The first method determines locally on coordinate neighborhoods of X_0 the V-filtration by Algorithm 4.2.7. The gluing process presented in Subsection 4.4.4 patches then the local results together.

The other method uses the graph embedding i_f and computes the V-filtration on $(i_f)_+M$ along Y_0 by Algorithm 4.2.7. Applying Lemma 3.1.35, that locally links $V_\bullet^{Y_0}(i_f)_+M$ and $V_\bullet^{X_0}M$, allows us also to describe $V_\alpha^{X_0}M$ on coordinate neighborhoods reducing the problem again to a gluing process as above. The advantage of this method is that it requires only one b-function computation to determine the V-filtration along X_0 on all coordinate neighborhoods that we have to consider, whereas the first method needs one b-function computation per coordinate neighborhood.

Remark 4.3.4. We point out that we have no method to check in the filtered situation if $(M, F_\bullet)$ and $(N, F_\bullet)$ are strictly X_0-specializable. However, if they are strictly X_0-specializable, we can compute for smooth X_0 the filtrations $F_\bullet V_\alpha M$ and $F_\bullet V_\alpha N$ for $\alpha \in \mathbb{Q}$ by adapting the methods in the above remark.

4.3.2 Localizations and dual localizations of (strictly) X_0-specializable $\mathcal{D}_X$- and $\mathcal{D}_X(*X_0)$-modules

Considering strictly X_0-specializable $(D_X, F_\bullet^\circ)$ and $(D_X[\overline{f}^{-1}], F_\bullet^\circ)$-modules $(M, F_\bullet)$ and $(N, F_\bullet)$, respectively, the objective of this subsection is to finitely present the $(D_X, F_\bullet^\circ)$-modules $\mathrm{Loc}_{X_0}(M, F_\bullet)$, $\mathrm{Loc}_{X_0}(N, F_\bullet)$, $\mathrm{DLoc}_{X_0}(M, F_\bullet)$ and $\mathrm{DLoc}_{X_0}(N, F_\bullet)$. All computations are based on the same method of taking direct images under the graph embedding i_f, then doing the corresponding computations for $Y_0 \subseteq Y$ and finally using strict Kashiwara's equivalence to derive the results. More precisely, we obtain by Equation (3.2.16) and Equation (3.2.17)

$$\begin{aligned}\mathrm{(D)Loc}_{X_0}(M, F_\bullet) &= \mathrm{Gr}_0^{V^{V(t-\overline{f})}}((\mathrm{D})\mathrm{Loc}_{Y_0}((i_f)_+M, F^\circ[\mathbf{s}]_{\bullet-1}))(-1)\\ &= V_0^{V(t-\overline{f})}((\mathrm{D})\mathrm{Loc}_{Y_0}((i_f)_+M, F^\circ[\mathbf{s}]_{\bullet-1}))(-1).\end{aligned}$$

Recall that a representation of $\mathrm{(D)Loc}_{Y_0}(M_f, (F^\circ[\mathbf{s}]_{\bullet-1})$ in terms of a quotient of a free D_Y-module with a corresponding shift vector inducing the filtration is computable by Algorithm 4.2.17 in the localization case and by Algorithm 4.2.32 in the dual case.

Choosing a finitely presented D_X-module $N' = D_X^E/L'$ satisfying $N = N'[\overline{f}^{-1}]$ and setting $F_\bullet N' = F^\circ[\mathbf{s}]_\bullet N'$, we have $(N, F_\bullet) = (N'(*X_0), F_\bullet)$ and hence Equation (3.2.13) and Remark 1.4.21 imply

$$\begin{aligned}\mathrm{Loc}_{X_0}(N, F_\bullet) &= \mathrm{Gr}_0^{V^{V(t-\overline{f})}}(\mathrm{Loc}_{Y_0}(((i_f)_+N')(*Y_0), F^\circ[\mathbf{s}]_{\bullet-1}))(-1)\\ &= V_0^{V(t-\overline{f})}(\mathrm{Loc}_{Y_0}(((i_f)_+N')(*Y_0), F^\circ[\mathbf{s}]_{\bullet-1}))(-1).\end{aligned}$$

Similarly, by Remark 3.2.38 and Equation (3.2.17)

$$\begin{aligned}\mathrm{DLoc}_{X_0}(N,F_\bullet) &= \mathrm{Gr}_0^{V^{V(t-\overline{f})}}(\mathrm{DLoc}_{Y_0}(\mathrm{Loc}_{Y_0}(((i_f)_+N')(*Y_0),F^\circ[\mathbf{s}]_{\bullet-1})))(-1)\\ &= V_0^{V(t-\overline{f})}(\mathrm{DLoc}_{Y_0}(\mathrm{Loc}_{Y_0}(((i_f)_+N')(*Y_0),F^\circ[\mathbf{s}]_{\bullet-1})))(-1).\end{aligned}$$

The $(D_Y,F^\circ_\bullet)$-module $\mathrm{Loc}_{Y_0}(((i_f)_+N')(*Y_0),F^\circ[\mathbf{s}]_{\bullet-1})$ as well as its dual localization along Y_0 can be written as quotients of free D_Y-modules with filtration induced by shift vectors using Algorithm 4.2.29 and Algorithm 4.2.32. and Remark 4.2.33.

It remains now the following task: Given a strictly $V(t-\overline{f})$-specializable $(D_Y,F^\circ_\bullet)$-module $(D_Y^{E'}/K'',F^\circ[\mathbf{s}']_\bullet)$ supported on $V(t-\overline{f})$, determine a finite presentation of the $(D_X,F^\circ_\bullet)$-module $\mathrm{Gr}_0^{V(t-\overline{f})}(D_Y^{E'}/K'',F^\circ[\mathbf{s}']_\bullet)$. Factorizing i_f as in Equation (4.3.1) via i_0 and λ and applying the inverse λ' of the coordinate change λ yields by Proposition 3.2.7, Proposition 1.4.7 and Example 1.4.9(c) that the latter module is isomorphic to

$$\mathrm{Gr}_0^{V^{V(t)}}(D_Y^{E'}/\Lambda'(K''))=V_0^{V(t)}(D_Y^{E'}/\Lambda'(K''))$$

as $(\mathcal{D}_X,F^\circ_\bullet)$-module reducing the problem to Algorithm 4.2.36.

The remark below explains how to represent the corresponding (dual) localization maps and how to trace our isomorphisms. The latter task is crucial for patching local results together as will be done Section 4.4.

Remark 4.3.5.

(a) We compute the canonical map $M\to\mathrm{Loc}_{X_0}(M)$ as follows in the above situation: Assume that $\mathrm{Loc}_{Y_0}((i_f)_+M)$ is represented by $D_Y^{E'}/K''$ and that $q\in(D_Y^{E'})^E$ defines the natural localization morphism $(i_f)_+M\to D_Y^{E'}/K''$ via $\overline{(e)}\mapsto\overline{q_e}$. (Note that q is computable by Algorithm 4.2.17.) Then $\Lambda'(q)$ defines in the same manner the natural morphism $(i_0)_+M\to D_Y^{E'}/\Lambda'(K'')$. Since both D_Y-modules appearing in the latter morphism are Y_0-specializable by Lemma 3.1.16 and $\overline{(e)}\in V_0^{Y_0}(i_0)_+M=\ker(t\cdot:(i_0)_+M\to(i_0)_+M)$, Proposition 3.1.15 implies that $\overline{\Lambda'(q)_e}\in V_0^{Y_0}(D_Y^{E'}/\Lambda'(K''))$. Representing the latter module as $D_X^{G''}/J$ via Algorithm 4.2.36, Remark 4.2.35 allows us to determine the image $q'_e\in D_X^{G''}/J$ of $\overline{\Lambda'(q)_e}$. Now the localization map $M\to D_X^{G''}/J$ is given by $\overline{(e)}\mapsto q'_e$.

(b) We keep the notation of Part (a). As in Remark 4.2.14(b) we also need to be able to compute the image of $\overline{m}\otimes\overline{f}^{-k}\in M\otimes_{O_X}O_X[\overline{f}^{-1}]$ for $m\in D_X^E$ and $k\in\mathbb{N}$ under the isomorphism $\mathrm{Loc}_{X_0}(M)\cong D_X^{G''}/J$. Regarding $\overline{m}\otimes\overline{f}^{-k}\otimes 1$ as an element of $(i_f)_+(M\otimes_{O_X}O_X[\overline{f}^{-1}])=i_f(M\otimes_{O_X}O_X[\overline{f}^{-1}])\otimes_{\mathbb{C}}\mathbb{C}[\partial_t]=(M\otimes_{O_X}O_X[\overline{f}^{-1}])\otimes_{\mathbb{C}}\mathbb{C}[\partial_t]$, Remark 1.4.20, Equation (4.3.2) and Remark 4.2.14(b) enable us to compute its image under the isomorphisms $(i_f)_+(M\otimes_{O_X}O_X[\overline{f}^{-1}])\cong(i_f)_+M\otimes_{O_X}O_X[t^{-1}]\cong$

D_Y^E/K''. By construction this element is in the $V_0^{t-\overline{f}}$-part of the latter module and we continue as in Part (a).

On the other hand consider the element $\overline{m} \in D_X^{G''}/J$ for $m \in D_X^{G''}$. By construction of $D_X^{G''}/J$ in Algorithm 4.2.36 (see Remark 4.2.35), $\overline{m}$ corresponds to an computationally accessible element $\overline{m'} \in D_Y^{E'}/\Lambda'(K'')$. Rewriting the latter element as an element of $(i_0)_+(M \otimes_{O_X} O_X[\overline{f}^{-1}]) = (M \otimes_{O_X} O_X[\overline{f}^{-1}]) \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ using the above remarks, we have by construction that this element can be considered as an element of $M \otimes_{O_X} O_X[\overline{f}^{-1}]$.

(c) Using a similar argument as in Part (a), we can also make to dual map $\mathrm{DLoc}_{Y_0}(M) \to M$ explicit. Similar remarks apply for the localization and dual localization of strictly X_0-specializable $(D_X[\overline{f}^{-1}], F_\bullet^\circ)$-modules.

(d) We point out that all steps involved in the computation of the (dual) localization of M are traceable by the previous parts of this remark. In particular, we can trace for $\star \in \{*, !\}$ the isomorphism $V_0^{V(t-\overline{f})}(((i_f)_+M)(\star Y_0)) \cong M(\star X_0)$ (in both directions). Moreover, we can decide if $m \in ((i_f)_+M)(\star Y_0)$ is in the $V_0^{V(t-f)}$-part of this module (by applying the coordinate change λ') and or if it is in a certain layer of the $F_\bullet$-filtration.

The following algorithm summarizes the computation of $\mathrm{Loc}_{X_0}(M, F_\bullet)$.

Algorithm 4.3.6 Given a variety X with a global coordinate system and a strictly $V(\overline{f})$-specializable $(D_X, F_\bullet^\circ)$-module $(M, F_\bullet)$, this algorithm computes the localization of this module along $V(\overline{f})$.

Input: A strictly $V(\overline{f})$-specializable $(D_X, F_\bullet^\circ)$-module $(M, F_\bullet) = (D_X^E/_{D_X}\langle K\rangle, F^\circ[\mathbf{s}]_\bullet)$ with $K \subseteq D_X^E$ finite.

Output: A finite set E', a finite set $L \subseteq D_X^{E'}$ and a shift vector $\mathbf{d} \in \mathbb{Z}^{E'}$ such that there is an isomorphism $\mathrm{Loc}_{V(\overline{f})}(M, F_\bullet) \cong (D_X^{E'}/_{D_X}\langle L\rangle, F^\circ[\mathbf{d}]_\bullet)$.

1: Set $K' := \Lambda(K) \cup \{(t-f)(e) \mid e \in E\} \subseteq D_Y$. $\triangleright$ $(i_f)_+M = D_Y^E/_{D_Y}\langle K'\rangle$.
2: Apply Algorithm 4.2.17 to determine a finite set E', a finite subset $L \subseteq D_Y^{E'}$ and a shift vector $\mathbf{d} \in \mathbb{Z}^{E'}$ such that $\mathrm{Loc}_{V(t)}((i_f)_+M, F^\circ[\mathbf{s}]_{\bullet-1}) \cong (D_Y^{E'}/_{D_Y}\langle L\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet)$.
3: Set $L' := \Lambda'(L)$. $\triangleright$ $(i_0)_+(\mathrm{Loc}_{V(\overline{f})}(M)) \cong D_Y^{E'}/_{D_Y}\langle L'\rangle$.
4: Determine finite sets $E'' \subseteq D_Y^{E'}$, $L'' \subseteq D_X^{E''}$ and $\mathbf{d}' \in \mathbb{Z}^{E''}$ such that there is an isomorphism $\mathrm{Gr}_0^{V^{V(t)}}((D_Y^{E'}/_{D_Y}\langle L'\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet))(-1) \cong (D_X^{E''}/_{D_X}\langle L''\rangle, F^{\mathbf{w}}[\mathbf{d}']_\bullet)$ by Algorithm 4.2.36.
5: **return** $E'', L'', \mathbf{d}'$.

Remark 4.3.7. As in Remark 4.2.16, if $X = \mathbb{C}^n$ the localization can in many cases also be computed by the methods of Oaku, Takayama and Walther. If one is only interested in the localized module, it seems advantageous to use their method because in contrast to their algorithms we have to compute two b-functions.

For completeness, we state the algorithms for the dual localization $\mathrm{DLoc}(M, F_\bullet)$ as well as the localization and dual localization of $(N, F_\bullet)$:

Algorithm 4.3.8 Given a variety X with a global coordinate system and a strictly $V(\overline{f})$-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet)$, this algorithm computes the dual localization of this module along $V(\overline{f})$.

Input: A strictly $V(\overline{f})$-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet) = (D_X^E/{}_{D_X}\langle K\rangle, F^\circ[\mathbf{s}]_\bullet)$ with $K \subseteq D_X^E$ finite.

Output: A finite set E', a finite set $L \subseteq D_X^{E'}$ and a shift vector $\mathbf{d} \in \mathbb{Z}^{E'}$ such that we have $\mathrm{DLoc}_{V(\overline{f})}(M, F_\bullet) \cong (D_X^{E'}/{}_{D_X}\langle L\rangle, F^\circ[\mathbf{d}]_\bullet)$.

1: Set $K' := \Lambda(K) \cup \{(t-f)(e) \mid e \in E\} \subseteq D_Y$. $\triangleright$ $(i_f)_+M = D_Y^E/{}_{D_Y}\langle K'\rangle$.

2: Apply Algorithm 4.2.32 to determine a finite set E', a finite subset $L \subseteq D_Y^{E'}$ and a shift vector $\mathbf{d} \in \mathbb{Z}^{E'}$ such that $\mathrm{DLoc}_{V(t)}((i_f)_+M, F^\circ[\mathbf{s}]_{\bullet-1}) \cong (D_Y^{E'}/{}_{D_Y}\langle L\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet)$.

3: Set $L' := \Lambda'(L)$. $\triangleright$ $(i_0)_+(\mathrm{Loc}_{X_0}(M)) \cong D_Y^{E'}/{}_{D_Y}\langle L'\rangle$.

4: Determine by Algorithm 4.2.36 finite sets $E'' \subseteq D_Y^{E'}$ and $L'' \subseteq D_X^{E''}$ and $\mathbf{d}' \in \mathbb{Z}^{E''}$ such that $\mathrm{Gr}_0^{V^{V(t)}}((D_Y^{E'}/{}_{D_Y}\langle L'\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet))(-1) \cong (D_X^{E''}/{}_{D_X}\langle L''\rangle, F^{\mathbf{w}}[\mathbf{d}']_\bullet)$.

5: **return** $E'', L'', \mathbf{d}'$.

Algorithm 4.3.9 Given a variety X with a global coordinate system and a strictly $V(\overline{f})$-specializable $(D_X(*V(\overline{f})), F^\circ_\bullet)$-module $(N, F_\bullet)$, this algorithm computes the localization of this module along $V(\overline{f})$.

Input: A $(D_X[\overline{f}^{-1}], F^\circ_\bullet)$-module $(N, F_\bullet) = (D_X[\overline{f}^{-1}]^E/{}_{D_X[\overline{f}^{-1}]}\langle L\rangle, F^\circ[\mathbf{s}]_\bullet)$ with $L \subseteq D_X^E$ finite that is strictly $V(\overline{f})$-specializable.

Output: A finite set E', a finite set $P \subseteq D_X^{E'}$ and a shift vector $\mathbf{d} \in \mathbb{Z}^{E'}$ such that we have $\mathrm{Loc}_{V(\overline{f})}(N, F_\bullet) \cong (D_X^{E'}/{}_{D_X}\langle P\rangle, F^\circ[\mathbf{d}]_\bullet)$.

1: Set $L' := \Lambda(L') \cup \{(t-f)(e) \mid e \in E\} \subseteq D_Y$.

2: Apply Algorithm 4.2.29 to determine a finite set E', a finite subset $P \subseteq D_Y^{E'}$ and $\mathbf{d} \in \mathbb{Z}^{E'}$ such that $\mathrm{Loc}_{V(t)}((D_Y^E/{}_{D_Y}\langle L'\rangle)(*V(t)), F^\circ[\mathbf{s}]_{\bullet-1}) \cong (D_Y^{E'}/{}_{D_Y}\langle P\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet)$.

3: Set $P' := \Lambda'(P)$.

4: Determine by Algorithm 4.2.36 finite sets $E'' \subseteq D_Y^{E'}$ and $P'' \subseteq D_X^{E''}$ and $\mathbf{d}' \in \mathbb{Z}^{E''}$ such that $\mathrm{Gr}_0^{V^{V(t)}}((D_Y^{E'}/{}_{D_Y}\langle P'\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet))(-1) \cong (D_X^{E''}/{}_{D_X}\langle P''\rangle, F^{\mathbf{w}}[\mathbf{d}']_\bullet)$.

5: **return** $E'', P'', \mathbf{d}'$.

Algorithm 4.3.10 Given a variety X with a global coordinate system and a strictly $V(\overline{f})$-specializable $(D_X(*V(\overline{f})), F^\circ_\bullet)$-module $(N, F_\bullet)$, this algorithm computes the dual localization of this module along $V(\overline{f})$.

Input: A $(D_X[\overline{f}^{-1}], F^\circ_\bullet)$-module $(N, F_\bullet) = (D_X[\overline{f}^{-1}]^E / {}_{D_X[\overline{f}^{-1}]}\langle L\rangle, F^\circ[\mathbf{s}]_\bullet)$ with $L \subseteq D_X^E$ finite that is strictly $V(\overline{f})$-specializable.

Output: A finite set E', a finite set $P \subseteq D_X^{E'}$ and a shift vector $\mathbf{d} \in \mathbb{Z}^{E'}$ such that we have $\mathrm{DLoc}_{V(\overline{f})}(N, F_\bullet) \cong (D_X^{E'} / {}_{D_X}\langle P\rangle, F^\circ[\mathbf{d}]_\bullet)$.

1: Set $L' := \Lambda(L') \cup \{(t - f)(e) \mid e \in E\} \subseteq D_Y$.
2: Apply Algorithm 4.2.29 to determine a finite set E', a finite subset $P \subseteq D_Y^{E'}$ and $\mathbf{d} \in \mathbb{Z}^{E'}$ such that $\mathrm{Loc}_{V(t)}((D_Y^E / {}_{D_Y}\langle L'\rangle)(*V(t)), F^\circ[\mathbf{s}]_{\bullet-1}) \cong (D_Y^{E'} / {}_{D_Y}\langle P\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet)$.
3: Use Algorithm 4.2.32 to determine a finite set E'', a finite subset $P' \subseteq D_Y^{E''}$ and a shift vector $\mathbf{d}' \in \mathbb{Z}^{E'}$ with $\mathrm{DLoc}_{V(t)}(D_Y^{E'} / {}_{D_Y}\langle P\rangle, F^{\mathbf{w}}[\mathbf{d}]_\bullet) \cong (D_Y^{E''} / {}_{D_Y}\langle P'\rangle, F^{\mathbf{w}}[\mathbf{d}']_\bullet)$.
4: Set $P'' := \Lambda'(P')$.
5: Determine by Algorithm 4.2.36 finite sets $E''' \subseteq D_Y^{E''}$ and $P''' \subseteq D_X^{E'''}$ and $\mathbf{d}'' \in \mathbb{Z}^{E'''}$ such that $\mathrm{Gr}_0^{V^{V(t)}}((D_Y^{E''} / {}_{D_Y}\langle P''\rangle, F^{\mathbf{w}}[\mathbf{d}']_\bullet))(-1) \cong (D_X^{E'''} / {}_{D_X}\langle P'''\rangle, F^{\mathbf{w}}[\mathbf{d}'']_\bullet)$.
6: **return** $E''', P''', \mathbf{d}''$.

Remark 4.3.11. Forgetting the filtrations involved in the above algorithms, the algorithms compute localizations and dual localizations of X_0-specializable D_X- and $D_X(*X_0)$-modules.

4.3.3 Vanishing and nearby cycles

The representation of the vanishing and nearby cycles of $(M, F_\bullet)$ as well as of the morphisms var and can follows immediately from Algorithm 4.2.36 and Remark 4.2.38.

4.4 Computations on (affine) varieties via gluing

Assume now that X is a smooth irreducible affine variety and $X_0 \subseteq X$ is a pure codimension one subvariety defined by the ideal sheaf $\mathcal{I}$. The purpose of this section is to develop algorithms for the computation of localizations, dual localizations and nearby and vanishing cycles and the corresponding maps in this more general situation. Our method for this is based on covering X with open neighborhoods that fit into the setting of the last two sections, doing the computations locally on the elements of this cover and then gluing the local results. As all local results are finitely presented D-modules with a filtration, the main task in this section is to devolop an algorithm that glues (filtered) free presentations.

Moreover, the above method can also be employed to make a quasi-inverse for Kashiwara's equivalence for Hodge $\mathcal{D}$-modules explicit.

Before we start, we fix some notation: Let $X = V(J) \subseteq \mathbb{C}^{\mathrm{n}}$ be defined by the vanishing of the prime ideal $J \subseteq \mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_{\mathrm{n}}]$ and $X_0 = V(I')$ defined by the radical ideal

$I' \subseteq \mathbb{C}[\underline{x}]$. Choose $\{f_b \mid b \in B\} \subseteq \mathbb{C}[\underline{x}]$ for a finite index set B such that the residue classes of the f_b generate the ideal $I = I'/J$. This in particular implies $\mathcal{I} = {}_{\mathcal{O}_X}\langle\{\overline{f_b} \mid b \in B\}\rangle$. We set $U_g := D(g) \cap X$ for $g \in \mathbb{C}[\underline{x}]$.

4.4.1 Constructing a gluing cover

First we explain how to construct a cover of X by affine principal open neighborhoods suited for our local computations. Since we may omit considering certain graph embeddings if we choose for smooth X_0 these neighborhoods carefully, we treat this case separately:

Gluing cover in the smooth subvariety case

So assume that X_0 is smooth. We cover X by two different types of affine open subsets, namely coordinate neighborhoods of X_0 and affine open subsets that cover $X^* = X \setminus X_0$. Recall that we have by Remark 1.2.13(a) a method to determine a partial cover of X by coordinate neighborhoods that covers all of X_0. More precisely, we can compute a finite set $C^0 \subseteq \mathbb{C}[\underline{x}]$ and $a^0 \in B^{C^0}$ such that U_c for $c \in C^0$ is a coordinate neighborhood of X_0 with $\mathcal{I}_{U_c} = {}_{\mathcal{O}_{U_c}}\langle f_{a_c^0}\rangle$ and such that $X_0 \subseteq \bigcup_{c \in C} U_c$.

On the other hand, X^* forms an affine open cover of itself. However this cover is for computational purposes often too coarse. Therefore we refine it in two steps: The $U'_b := D(f_b) \cap X$ are an affine open cover of X^* such that on U'_b the empty set $U'_b \cap X_0$ is defined by the vanishing of f_b. To perform actual computations we refine this cover by covering the U'_b themselves by an affine principal open cover such that element of this cover have a global coordinate system. Such a cover is given by open sets U_c corresponding to $c \in C^*$ for a suitable finite set $C^* \subseteq \mathbb{C}[\underline{x}]$, which can be found as outlined in Remark 1.2.13(a). Hence there exists in particular $a^* \in B^{C^*}$ such that $U_c \cap X_0 = U_c \cap V(f_{a_c^*})$ for $c \in C^*$. Moreover, we may assume that C^* was chosen such that for $c \in C^*$ there is a $c' \in \mathbb{C}[\underline{x}]$ such that $c = c' f_{a_c^*}$

To unify our notation, we set $C := C^0 \cup C^*$ and define $a \in B^C$ by $a_c = a_c^0$ for $c \in C^0$ and by $a_c = a_c^*$ for $c \in C^*$.

Gluing cover in the general subvariety case

We drop now the assumption that X_0 is smooth. As for smooth X_0, we cover X again by two different types of subsets. The cover of X^* is constructed as in the smooth case and we keep the corresponding notation. We complete this cover by open patches of the form U_c for $c \in C^0$ for some finite set $C \subseteq \mathbb{C}[\underline{x}]$ with the property that $\mathcal{I}_{U_c}$ generated by single regular function. Note that such a cover exists indeed because the defining ideal sheaf of a pure codimension one subvariety of a smooth equidimensional variety is locally generated by one equation. So for $x \in X_0$ exists by Nakayama's Lemma $b \in B$ such that $\mathcal{I}_x = {}_{\mathcal{O}_{X,x}}\langle f_b\rangle$. This holds then also on an open neighborhood U_x of x in X, that is, $\mathcal{I}_{U_x}$ is $\mathcal{O}_{U_x}$-generated by f_b. Therefore it is enough to find for given $b \in B$ the maximal open set of X such that the restriction of $\mathcal{I}$ to that set is generated by f_b and cover this set by affine opens. Algorithmically this is achieved by computing for all $b' \in B \setminus \{b\}$ a $\mathbb{C}[\underline{x}]$-generating set $S_{b'}$ of $\mathrm{syz}_{\mathbb{C}[\underline{x}]}(f_b, f_{b'}, J)$ via Gröbner basis theory and setting $S'_{b'} := \{s_{f_{b'}} \in S_{b'} \mid \overline{s_{f'_b}} \neq 0 \in \mathbb{C}[\underline{x}]/J\}$. Then

$$\{U_{\prod_{b' \in B \setminus \{b\}} s_{b'}} \mid s_{b'} \in S'_{b'}\}$$

is a cover of that maximal open set. By covering the $U_{\prod_{b' \in B'} s'_b}$ by affine principal opens having a global coordinate system, we may assume that we have constructed a finite set $C^0 \subseteq \mathbb{C}[\underline{x}]$ and an element $a^0 \in B^{C^0}$ such that U_c has a global coordinate system and $\mathcal{I}_{U_c}$ is generated by $f_{a^0_c}$ for $c \in C^0$. The set C and the element a are now defined as in the smooth case.

Representing the ring of differential operators on elements of the cover

Consider $g \in \mathbb{C}[\underline{x}]$ such that U_g has a global coordinate system with corresponding derivations $\theta_1, \dots, \theta_{\mathrm{m}}$ induced by $\theta_1^l, \dots, \theta_{\mathrm{m}}^l \in \mathrm{Der}_J(\mathbb{C}[\underline{x}])[g^{-1}]$ and obtained by Remark 1.2.11(b). In this situation we have an isomorphism

$$\eta_g : U_g \cong V_g := V(J, x_{\mathrm{n}+1}g - 1) \subseteq \mathbb{C}^{\mathrm{n}+1},\ (x_1, \dots, x_{\mathrm{n}}) \mapsto (x_1, \dots, x_{\mathrm{n}}, \frac{1}{g(\underline{x})})$$

with isomorphism of the corresponding rings of differential operators defined via sending the derivation $\theta \in \Theta_{U_g}(U_g) \cong \Theta_X(X)[\overline{g}^{-1}]$ represented by $\overline{\theta^l} \otimes g^{-k}$ for $\theta^l \in \mathrm{Der}(\mathbb{C}[\underline{x}])$ (see Remark 1.2.3) to $\overline{x_{\mathrm{n}+1}^k(\theta^l - x_{\mathrm{n}+1}^2\theta^l(g)\partial_{\mathrm{n}+1})} \in \Theta_{V_g}(V_g)$. The inverse maps $\tilde{\theta} \in \Theta_{V_g}(V_g)$ represented by $\tilde{\theta}^l = \sum_{1 \leq i \leq \mathrm{n}+1} a_i(\underline{x}, x_{\mathrm{n}+1})\partial_i \in \mathrm{Der}(\mathbb{C}[\underline{x}, x_{\mathrm{n}+1}])$ to $\overline{\sum_{1 \leq i \leq \mathrm{n}} a_i(\underline{x}, g^{-1})\partial_i}$ interpreted in the canonical way as an element of $D_X[\overline{g}^{-1}]$.

We point out that the $\eta_c(X_0) \cap V_c \subseteq V_c$ for $c \in C^0$ fit in the situation of Section 4.3. If X_0 is moreover smooth, we may by Remark 1.2.12 further assume that the $\eta_c(X_0) \cap V_c \subseteq V_c$ fulfill the assumptions of Section 4.2. On the other hand, for $c \in C^*$, we represent D_{V_c} as a PBW-reduction-algebra following Example 2.1.30(a) such that the filtration $F^{\circ}_{\bullet}D_{V_c}$ is induced by a weight vector. From now on, we implicitly identify D_{U_c} with the corresponding representation of D_{V_c} as PBW-reduction-algebra for $c \in C$.

4.4.2 General principle of the gluing process

Recall that we are interested in (filtered) localizations and dual localizations of D_X- and $D_X(*X_0)$ together with the natural (dual) localization maps as well as in the (unipotent) vanishing and nearby cycle functors together with the morphisms can and var. Our aim is to represent these objects as quotients of free D_X-modules with filtrations given by shift vectors. Before we explain in detail how to glue the various constructions from certain local data, we outline a gluing process for locally given filtered free presentations on which the patching of these constructions is based. Our method relies from a categorical point of view on the construction of a certain inverse limit and does not depend on the underlying ring. Hence we explain the gluing in the following more general setting:

Let $\{U_a\}_{a\in A}$ be a cover of X for a finite subset $A \subseteq \mathbb{C}[\underline{x}]$ of cardinality greater than one. Consider a Noetherian filtered ring $(S, F_\bullet)$ with the property that S is an O_X-module inducing an O_X-structure on its filtered parts F_kS ($k \in \mathbb{Z}$). Then the filtration $F_\bullet S$ defines a filtration on $S[\overline{a}^{-1}]$ for $a \in A$ via $F_\bullet(S[\overline{a}^{-1}]) := (F_\bullet S)[\overline{a}^{-1}]$ (and similarly for $S[\overline{a}^{-1}][\overline{a'}^{-1}]$ for $a, a' \in A$). We equip the set

$$D := A \cup (A \times A \setminus \{(a, a) \mid a \in A\})$$

with a partial order $\leq$ defined by $a \geq a$, $a \geq (a, a')$, $a \geq (a', a)$ and $(a, a') \geq (a, a')$ for all $a, a' \in A$ with $a \neq a'$. Given for every $a \in A$ a well-filtered $(S[\overline{a}^{-1}], F_\bullet)$-module $(P_a, F_\bullet)$ and $S[\overline{aa'}^{-1}]$-linear filtered isomorphisms $\tau_{a,a'} : F_\bullet P_{(a,a')} := (F_\bullet P_a)[\overline{a'}^{-1}] \cong F_\bullet P_{(a',a)}$ for all $a' \in A \setminus \{a\}$, the $(P_d)_{d\in D}$ define inverse systems in the categories of S-modules and of O_X-modules if we take as bonding maps $P_a \to P_{(a,a')}$ the usual localization maps, denoted by $\rho^a_{a'}$, and as bonding maps $P_a \to P_{(a',a)}$ the map $\rho^a_{a'}$ composed with $\tau_{a,a'}$. Then there exists an S-module P representing the inverse limit of this inverse system (in the category of S-modules). Noting that the inverse limit in the category of S-modules is compatible with the inverse limit in the category of O_X-modules and that the inverse limit functor for abelian categories is left exact, we also obtain O_X-submodules F_kP of P by considering the inverse system (of $\mathcal{O}_X$-modules) with the bonding maps $F_kP_a \to P_{(a,a')}$ and $F_kP_a \to P_{(a',a)}$ defined by restriction of the bonding maps of $(P_d)_{d\in D}$ for $k \in \mathbb{Z}$. By construction this endows P with the $(S, F_\bullet)$-filtration $F_\bullet P$ and we obviously have:

Lemma 4.4.1. *The projection maps $\pi_a : P \to P_a$ of the inverse limit induce canonical isomorphisms*

$$(F_\bullet P)[\overline{a}^{-1}] \cong F_\bullet P_a$$

for all $a \in A$.

Our aim is to compute a free $(S, F_\bullet)$-representation of $(P, F_\bullet)$ under the assumption that we can perform the following tasks and are given our inverse system in the following form:

Assumption 4.4.2. For $(a, a') \in D$ we assume:

(a) We are given P_a, a finite set $G_a \subseteq P_a$ and $\mathrm{s}^a \in \mathbb{Z}^{G_a}$ with the property that $F_\bullet P_a = \sum_{g\in G_a} F_{\bullet-\mathrm{s}^a_g} S[\overline{a}^{-1}] \cdot g_a$.

(b) The module membership problem $p \in F_kP_a$ is solvable for $p \in P_a$ and $k \in \mathbb{Z}$.

(c) We are given the isomorphism $\tau_{a,a'}$ and are able to compute images under this map.

(d) We can decide if $p \in P_{(a,a')}$ is 0.

(e) We can compute the $S[\overline{a}^{-1}]$-syzygy module of elements of P_a.

Example 4.4.3. In our applications, the $(P_a, F_\bullet)$ for $a \in A$ are given in form of filtered presentations. More precisely:

(a) We have for every $a \in A$ a presentation of P_a as a quotient $S[\overline{a}^{-1}]^{E_a}/K_a$ with $K_a = {}_{S[\overline{a}^{-1}]}\langle K'_a \rangle$ for a finite set E_a and a finite subset $K'_a \subseteq S[\overline{a}^{-1}]^{E_a}$.

(b) For every $a \in A$ we are given a finite set $G_a \subseteq S[\overline{a}^{-1}]^{E_a}$ and a shift vector $\mathrm{s}^a \in \mathbb{Z}^{G_a}$ such that $F_\bullet P_a = \sum_{g \in G_a} F_{\bullet - \mathrm{s}^a_g} S[\overline{a}^{-1}] \cdot \overline{g_a}$ and we have moreover a method to test for $s \in S[\overline{a}^{-1}]^{E_a}$ if $\overline{s} \in F_k P_a[\overline{a}^{-1}]$ for $k \in \mathbb{Z}$.

(c) We are given the isomorphism $\tau_{a,a'} : (S[\overline{a}^{-1}]^{E_a}/K_a)[\overline{a'}^{-1}] \to (S[\overline{a'}^{-1}]^{E_{a'}}/K_{a'})[\overline{a}^{-1}]$ for all $(a, a') \in D$.

(d) We can compute the $S[\overline{b}^{-1}]$-syzygy module of elements of an $S[\overline{b}^{-1}]$-free module for all $b \in A \cup \{aa' \mid (a, a') \in D\}$.

Note that hence Assumption 4.4.2 is fulfilled, because Assumption 4.4.2(d) can be reduced to a module membership problem for ${}_{S[\overline{aa'}^{-1}]}\langle K'_a \rangle$, which is solvable by a syzygy computation. Similarly, the task in Assumption 4.4.2(e) can be performed by a syzygy computation over $S[\overline{a}^{-1}]$.

Under Assumption 4.4.2, we compute a filtered free presentation of the inverse limit $F_\bullet P$ based on the above lemma and the observation that this limit can be realized as the kernel of the map

$$\Delta : \prod_{a \in A} F_\bullet P_a \to \prod_{(a,a') \in D} P_{(a,a')},\ (p_a)_{a \in A} \mapsto (\rho^a_{a'}(p_a) - \tau_{a',a}(\rho^{a'}_a(p_{a'})))_{(a,a') \in D}$$

as outlined below: First we construct a finite set G, a shift vector $\mathrm{s} \in \mathbb{Z}^G$ and strict surjective maps $\alpha_a : (S[\overline{a}^{-1}]^G, F[\mathrm{s}]_\bullet) \to (P_a, F_\bullet)$ and $\alpha_{(a,a')} : (S[\overline{a}^{-1}][\overline{a'}^{-1}]^G, F[\mathrm{s}]_\bullet) \to (P_{(a,a)}, F_\bullet)$ inducing an morphism of inverse systems by regarding the $S[\overline{a}^{-1}]^G$ and $S[\overline{a}^{-1}][\overline{a'}^{-1}]^G$ for $(a, a') \in D$ as an inverse system indexed by D with bonding maps induced by the natural localization maps (and analogously for the filtered parts). As the Mittag-Leffler condition is satisfied we then obtain a surjective strict map

$$\alpha : (S^G, F[\mathrm{s}]_\bullet) \to (P, F_\bullet).$$

Let us now explain how to find the above data: To determine maps α_a, we observe that for $p \in F_k P_a$ (with $a \in A$) exists by Lemma 4.4.1 a natural number $l \in \mathbb{N}$ such that $(p_{a'})_{a' \in A} \in \ker(\Delta)$ with $p_a = \overline{a}^l p$ and $p_{a'} \in F_k P_{a'}$ for $a \neq a' \in A$ suitably chosen. The number l and the elements p'_a for $a' \in A$ are constructed as follows from Assumption 4.4.2: We first find $l \in \mathbb{N}$ such that $\overline{a}^l \tau_{a,a'}(\rho^a_{a'}(p)) = p_{a'} \otimes 1 \in \rho^{a'}_a(P_{a'})$ with $p_{a'} \in P_{a'}$ for all $a' \in A \setminus \{a\}$ and set $p_a := \overline{a}^l p$. Using the method from Assumption 4.4.2(b) we increase l until $p_{a'} \in F_k P_{a'}$ for all $a' \in A$ and adapt $(p_{a'})_{a' \in A}$ accordingly. This process terminates as $p_{a'} \otimes 1 \in \tau_{a,a'}((F_k P_a)[\overline{a'}^{-1}]) = (F_k P_{a'})[\overline{a}^{-1}]$. By design we have for all $(a', a'') \in D$ that $\overline{a}^k(\tau_{a',a''}(\rho^{a'}_{a''}(p_{a'})) - \rho^{a''}_{a'}(p_{a''})) = 0$ for k big enough, which is tested by Assumption 4.4.2(d). Replacing l by $l + k$ for suitably chosen k and changing $p_{a'}$ for $a' \in A$ accordingly, we obtain that $(p_{a'})_{a' \in A}$ is in the kernel of Δ. We summarize the computation of $(p_{a'})_{a' \in A}$:

Algorithm 4.4.4 Auxiliary procedure for Algorithm 4.4.7

Input: A cover $\{U_a\}_{a \in A}$ of X with $A \subseteq \mathbb{C}[\underline{x}]$ finite, a Noetherian filtered ring $(S, F_\bullet)$ such that S is an O_X-module inducing an O_X-structure on $F_k S$ (for $k \in \mathbb{Z}$). Moreover, assume we are given the data and methods of Assumption 4.4.2 and an element $p \in F_k P_a$ (with $k \in \mathbb{Z}$ and $a \in A$ also given).

Output: An element $(p_{a'})_{a' \in A} \in \ker(\Delta) \cap \prod_{a' \in A} F_k P_{a'}$ with Δ defined as above such that $p_a = \overline{a}^l p$ for some $l \in \mathbb{N}$.

1: Choose $l \in \mathbb{N}$ and $p_{a'} \in P_{a'}$ with $\overline{a}^l \tau_{a,a'}(\rho^a_{a'}(p)) = p_{a'} \otimes 1$ for all $a' \in A \setminus \{a\}$.
2: Set $p_a := \overline{a}^l p$.
3: Initialize $i := 0$.
4: **while** $\overline{a}^i p_{a'} \otimes 1 \notin F_k P_{a'}$ for all $a' \in A$ **do** ▷ Test by the method in Assumption 4.4.2(b).
5: Set $i := i + 1$.
6: **while** $\overline{a}^i(\tau_{a',a''}(\rho^{a'}_{a''}(p_{a'})) - \rho^{a''}_{a'}(p_{a''})) \neq 0$ for all $(a', a'') \in D$ **do** ▷ Test by Assumption 4.4.2(d)
7: Set $i := i + 1$.
8: Replace $p_{a'} := \overline{a}^i p_{a'}$ for all $a' \in A$.
9: **return** $(p_{a'})_{a' \in A}$.

Remark 4.4.5. In the unfiltered situation, we do not need Assumption 4.4.2(b). Hence we simply drop Lines 4 and 5 in the above algorithm.

Remark 4.4.6. The above procedure requires many tests to make sure that the constructed element $(p_{a' \in A})_{a \in A}$ is in $\ker(\Delta)$. In certain situations, we do not need to perform all these tests, and we can also avoid establishing the isomorphisms in Assumption 4.4.2(c) or performing the task in Assumption 4.4.2(d). Namely, we can sometimes consider the inverse limit P as a subquotient of an already explicitly given object. More precisely, assume additionally that we are (explicitly) given an S-module R such that P_a is isomorphic to R'_a / R''_a with $R''_a \subseteq R'_a \subseteq R_a := R[\overline{a}^{-1}]$ satisfying the following properties: Using the same notation as for P, the canonical isomorphism $R_{(a,a')} \cong R_{(a',a)}$ induces isomorphisms $R'_{(a,a')} \cong R'_{(a',a)}$ and $R''_{(a,a')} \cong R''_{(a',a)}$ compatible with the isomorphism $\tau_{a,a'}$ for $(a, a') \in D$. Moreover the R'_d and R''_d for $d \in D$ with bonding maps induced by the localization maps and the isomorphisms $R_{(a,a')} \cong R_{(a',a)}$ for $(a, a') \in D$ form an inverse system. Then we may replace Assumption 4.4.2(c) and (d) by the assumption below:

(cd') For every $a \in A$, we are given the submodule $R'_a \subseteq R_a$ and methods to decide for $r \in R_a$ if $r \in R'_a$ and to compute images and an element in the preimage of a given element under the surjective map $\mu_a : R'_a \twoheadrightarrow P_a$ lifting the isomorphism $R'_a/R''_a \cong P_a$.

In this situation, keeping the notation as above, we construct the element $(p_{a'})_{a' \in A}$ as outlined below: We first compute a preimage $p' \in R'_a \subseteq R_a$ of p under the map $R'_a \twoheadrightarrow P_a$. Canceling negative powers of $\overline{a}$, we find $l \in \mathbb{N}$ and $p'' \in R$ such that $\overline{a}^l p' = p'' \otimes \overline{a}^0 \in R[\overline{a}^{-1}]$. By the above criterion and by the method from Assumption 4.4.2(b) we can decide if $p'' \otimes \overline{a'}^0 \in R'_{a'}$ and if $\mu_{a'}(p'' \otimes \overline{a'}^0) \in F_k P_{a'}$ for all $a' \in A$. If not, we increase l and adjust p'' accordingly until this is eventually the case. Arguing similarly as above, we see that this process terminates.

Setting $G := \bigsqcup_{a \in A} G_a$ and defining $\mathbf{s} \in \mathbb{Z}^G$ by $\mathbf{s}_{g_a} = \mathbf{s}^a_{g_a}$ for $g_a \in G_a$, we obtain the strict surjective maps

$$\alpha_a : (S[\overline{a}^{-1}]^G, F[\mathbf{s}]_\bullet) \to (P_a, F_\bullet),\ (g) \mapsto g_a$$

for $a \in A$ inducing maps $\alpha_{(a,a')}$ by localization. By construction this defines a morphism of inverse systems giving rise to the strict surjective map

$$\alpha : (S^G, F[\mathbf{s}]_\bullet) \to (\ker(\Delta), F_\bullet),\ (g) \mapsto (g_a)_{a \in A}.$$

We extend this map to a free presentation by iterating the above process as follows: Note that $(\ker(\alpha_d))_{d \in D}$ with induced bonding maps is also an inverse system. Moreover, we have by Lemma 4.4.1 and exactness of localization that $\ker(\alpha_a) \cong \ker(\alpha)[\overline{a}^{-1}]$ and $\ker(\alpha_{(a,a')}) \cong \ker(\alpha)[\overline{a}^{-1}][\overline{a'}^{-1}]$ inducing isomorphisms $\ker(\alpha_{(a,a')}) \cong \ker(\alpha_{(a',a)})$ for $(a, a') \in D$. By left exactness of the inverse limit functor, the inverse limit of the inverse system $(\ker(\alpha_d))_{d \in D}$ agrees with $\ker(\alpha)$. So we repeat the above process (forgetting any filtrations) with the inverse system $(P_d)_{d \in D}$ replaced by $(\ker(\alpha_d))_{d \in D}$ to obtain a map

$$\beta : S^T \to S^G$$

surjecting on $\ker(\alpha)$. Notice that $\ker(\alpha_a)$ is computable by Assumption 4.4.2(e) for $(P_d)_{d \in D}$ showing that $(\ker(\alpha_d))_{d \in D}$ satisfies Assumption 4.4.2(a). Assumption 4.4.2(b) is not needed because we do not have to consider filtrations. Condition (cd') is fulfilled with $R = S^G$ and $R' = \ker(\alpha)$, because we can check for $r \in R_a$ if $r \in R'_a$ by testing if $\sum_{g \in G} r_g g_a = 0 \in P_a$ via Assumption 4.4.2(e) for $(P_d)_{d \in D}$. Note that Assumption 4.4.2(e) is not required, because this condition was only assumed to compute $\ker(\alpha_a)$. So the above process is indeed applicable and we obtain:

Algorithm 4.4.7 Blueprint for the gluing process of filtered finitely presented modules from local data.

Input: A cover $\{U_a\}_{a\in A}$ of X with $A \subseteq \mathbb{C}[\underline{x}]$ finite, a Noetherian filtered ring $(S, F_\bullet)$ such that S is an O_X-module inducing an O_X-structure on $F_k S$ (for $k \in \mathbb{Z}$). Moreover, assume we are given the data and methods of Assumption 4.4.2.

Output: A finite set G, a finite set $T \subseteq S^G$ and $\mathbf{s} \in \mathbb{Z}^G$ such that $(S^G/_S\langle T\rangle, F[\mathbf{s}]_\bullet)$ represents the inverse limit of $(F_\bullet P_d)_{d\in D}$ defined in Assumption 4.4.2.

```
for a ∈ A do
   for g ∈ G_a do
      Apply Algorithm 4.4.4 to g ∈ F_{s^a_g} P_a to obtain the output (g_{a'})_{a'∈A}.
Set G := ⊔_{a∈A} G_a and define s ∈ Z^G by s_g := s^a_g for g ∈ G_a.
Initialize an empty set T ⊆ S^G.
for a ∈ A do
   Determine a set T' ⊆ S[ā^{-1}]^G of S[ā^{-1}]-generators of the S[ā^{-1}]-syzygy module of
   ⊔_{g∈G}{g_a} ⊆ P_a.  ▷ Use Assumption 4.4.2(e).
   for t ∈ T' do
      Find l ∈ N such that ā^l t ∈ S^G.
      for a' ∈ A \ {a} do
         while Σ_{g∈G} a^l t_g g_{a'} ≠ 0 ∈ P_{a'} do  ▷ Checks if ā^l t ∈ ker(α_{a'}) by Assumption 4.4.2(e).
            l := l + 1.
      Set T := T ∪ {ā^l t}.
return G, T, s.
```

Remark 4.4.8. A problem appearing naturally in this context is the following: Consider another inverse system $(P'_d)_{d\in D}$ satisfying the same properties as $(P_d)_{d\in P}$ with inverse limit $P' = S^{G'}/L'$ and projection maps $P' \to P'_a, (g') \mapsto g'_a$ (for $g' \in G'$). Given $S[\overline{a}^{-1}]$-linear maps

$$\nu_a : P'_a \to P_a$$

for $a \in A$ inducing a morphism of inverse system by taking as maps $\nu_{(a,a')} : P'_{(a,a')} \to P_{(a,a')}$ the localization of ν_a at $\overline{a'}$, determine the limit map $\nu : P' \to P$.

To solve this problem, we use the standard gluing method for this sort of situation (see e.g. [Har77, Proof of Proposition II.5.6]): As $\pi_a(\nu((g'))) = \nu_a(g'_a)$ for $g \in G'$ and $a \in A$ and $\nu_a(g'_a)$ can be expressed as an $S[\overline{a}^{-1}]$-linear combination of the $(g_a)_{g\in G}$, we derive a representation $\nu((g')) = \overline{q^{g'}_a} \otimes \overline{a}^{k^{g'}_a} \in P[\overline{a}^1]$ with $q^{g'}_a \in S^G$ and $k^{g'}_a \in \mathbb{N}$. So there exists for $(a, a') \in D$ a natural number $l \in \mathbb{N}$ such that

$$\overline{aa'}^l(\overline{a'}^{k^{g'}_{a'}}\overline{q^{g'}_a} - \overline{a}^{k^{g'}_a}\overline{q^{g'}_{a'}}) = 0 \in S^G/_S\langle T\rangle,$$

which is equivalent to $\overline{aa'}^{l}\pi_{a''}(\overline{a'}^{k_{a'}^{g'}}\overline{q_a^{g'}} - \overline{a}^{k_a^{g'}}\overline{q_{a'}^{g'}}) = 0$ for all $a'' \in A$ and can hence be tested by Assumption 4.4.2(e). Choosing one l that works for all possible choices of a and a', we replace now $q_a^{g'}$ by $\overline{a}^l q_a^{g'}$ and $k_a^{g'}$ by $k_a^{g'} + l$ for all $a \in A$ implying $\overline{a'}^{k_{a'}^{g'}}\overline{q_a^{g'}} - \overline{a}^{k_a^{g'}}\overline{q_{a'}^{g'}} = 0 \in S^G/_S\langle T\rangle = P$ for all $a, a' \in A$. As $\{U_a\}_{a\in A}$ is a cover of X, we compute via Gröbner basis theory a representation $1 = \sum_{a\in A} h_a a^{k_a^{g'}} + j$ with $h \in \mathbb{C}[\underline{x}]^A$ and $j \in J$. It follows that $\mu(\overline{(g)}) = \sum_{a\in A}\overline{h_a q_a^{g'}}$ since $\overline{a'}^{k_{a'}^{g'}}\sum_{a\in A} h_a\overline{q_a^{g'}} = \sum_{a\in A} h_a\overline{a'}^{k_{a'}^{g'}}\overline{q_a^{g'}} = \sum_{a\in A} h_a\overline{a}^{k_a^{g'}}\overline{q_{a'}^{g'}} = \overline{q_{a'}^{g'}}$ for all $a' \in A$.

4.4.3 Localizations of strictly specializable D_X- and $D_X(*X_0)$-modules

We apply the gluing principle presented in the previous subsection to represent the localization of the strictly X_0-specializable $(\mathcal{D}_X, F_\bullet^\circ)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ along X_0 as quotient of a free $(D_X, F_\bullet^\circ)$-module. Considering $\mathrm{Loc}_{X_0}(M) = M \otimes_{O_X} O_X(*X_0)$ as a subquotient of itself, it suffices to show that Example 4.4.3(a), (b) and (d) as well as (cd') are satisfied in this situation. Hence, before we actually glue, we investigate the O_X-module $O_X(*X_0)$ and its localizations:

Remark 4.4.9.

(a) By definition of $O_X(*X_0)$, we can write $\overline{\frac{q}{f_b^k}} \in O_X(*X_0)$ (where $q \in \mathbb{C}[\underline{x}]$ and $k \in \mathbb{N}$) of the form $\overline{\frac{q'}{f_{b'}^i}} \in O_X(*X_0)$ for $b, b' \in B$. We construct q' and i as follows: As J is a prime ideal, we check for increasing i whether $f_{b'}^i q \in {}_{\mathbb{C}[\underline{x}]}\langle f_b^k, J\rangle$ using Gröbner basis theory until this test is positive. From the corresponding representation $f_{b'}^i q = q' f_b^k + j$ with $j \in J$, we read off $q' \in \mathbb{C}[\underline{x}]$.

(b) Let $g \in \mathbb{C}[\underline{x}]$ such that on U_g that $\mathcal{I}_{U_g}$ is $\mathcal{O}_{U_g}$-generated by f_b for some $b \in B$. Then there is an isomorphism

$$\nu_g : O_X(*X_0) \otimes_{O_X} O_X[\overline{g}^{-1}] \cong O_X[\overline{f_b}^{-1}] \otimes_{O_X} O_X[\overline{g}^{-1}],\ \frac{q}{\overline{f_b}^i} \otimes \frac{p}{\overline{g}^k} \mapsto \frac{q}{\overline{f_b}^i} \otimes \frac{p}{\overline{g}^k},$$

where representations of elements of the module on the left hand side as above are determined by Part (a). Moreover there exists some $l \in \mathbb{N}$ such that $\frac{\overline{g}^l}{\overline{f_b}} \in O_X(*X_0)$ making the inverse map explicit. The exponent l is determined by testing $g^l \in {}_{\mathbb{C}[\underline{x}]}\langle J, f_b\rangle : f_{b'}^\infty$ for all $b' \in B$ for increasing l, where the saturation as well as the ideal membership problem are computable via Gröbner bases.

We cover X as in Subsection 4.4.1 and describe first the localization of $(M, F_\bullet)$ on the open subsets covering X^*. We have

$$\mathrm{Loc}_{X_0}(M, F_\bullet)_{U_c} \cong (M_{U_c}, F_\bullet) = (D_{U_c}^E/K_{U_c}, F^\circ[\mathbf{s}]_\bullet)$$

for $c \in C^*$, since $\mathcal{O}_X(*X_0)_{U_c} \cong O_{U_c}$ shows $M(*X_0)_{U_c} \cong M_{U_c}$ and Remark 3.2.19 and Remark 3.1.5 imply the claim on the filtration. Notice that the isomorphism $M(*X_0) \otimes_{O_X} O_X[\overline{c}^{-1}] \to D^E_{U_c}/K_{U_c}$ is given by sending $(\overline{m} \otimes \frac{q}{f_b{}^k}) \otimes \frac{p}{\overline{c}^l}$ to $\overline{m \otimes \frac{c'^i q' p}{c^{i+l}}}$ with $\frac{q}{f_b{}^k} = \frac{q'}{f_{a_c^*}{}^i} \in O_X(*X_0)$, where $i \in \mathbb{N}$ and $q' \in O_X$ are computed as outlined in Remark 4.4.9(a).

Next, we explain how to obtain a presentation as above on U_c for $c \in C^0$. By Algorithm 4.2.15 (if X_0 is smooth) or Algorithm 4.3.6 we get

$$\mathrm{Loc}_{X_0}(M, F_\bullet)_{U_c} \cong \mathrm{Loc}_{V(\overline{f_{a_c}})}(M_{U_c}, F_\bullet) \cong (D^{E_c}_{U_c}/K_c, F^\circ[\mathbf{s}^c]_\bullet).$$

Note images and preimages under the isomorphism $M_{U_c} \otimes_{O_{U_c}} O_{U_c}[\overline{f_{a_c}}^{-1}] \cong D^{E_c}_{U_c}/K_c$ can be determined by Remark 4.2.14(b) or Remark 4.3.5(b) (depending on whether X_0 is smooth). On the other hand, the first isomorphism is induced by ν_c (see Remark 4.4.9(b)) and is made explicit by the considerations in that remark.

Moreover, for $c \in C$, we are able to solve the module membership and the syzygy problem over the PBW-reduction-algebra D_{U_c}. As Algorithm 2.4.5 can be taken as the method in Example 4.4.3(b) and Condition (cd') is satisfied as seen above, all assumption for Algorithm 4.4.7 are fulfilled and we may apply this algorithm to represent $\mathrm{Loc}_{X_0}(M, F_\bullet)$ as a quotient of a free D_X-module with filtration induced by a weight vector.

Remark 4.4.10.

(a) The localization map is constructed as explained in Remark 4.4.8.

(b) We adapt the above gluing process to localizations of well-filtered $(D_X(*X_0), F^\circ_\bullet)$-modules by replacing Algorithm 4.2.15 and Algorithm 4.3.6 by Algorithm 4.2.29 and Algorithm 4.3.9, respectively,

4.4.4 Dual localizations of strictly specializable D_X- and $D_X(*X_0)$-modules along a smooth subvariety

We consider at the moment only the dual localization along smooth X_0, because - unlike for singular X_0 - we may use in this situation the simpler Condition (cd'). Recall that for smooth X_0 (the underlying module of) the dual localization of $(M, F_\bullet) = (D^E_X/K, F^\circ[\mathbf{s}]_\bullet)$ is given by $D_X \otimes_{V_0 D_X} V_{<0}M$ with $V_{<0}M$ being a subobject of M. By definition of the filtration on $\mathrm{DLoc}_{X_0}(M)$ and Lemma 3.1.30 it is now sufficient to present $(V_{<0}M, F_\bullet)$ as a quotient of the form $(V_0 D^{E'}_X/L, F^\circ[\mathbf{s}'])$ since this implies $\mathrm{DLoc}_{X_0}(M, F_\bullet) \cong (D^{E'}_X/D_X L, F^\circ[\mathbf{s}'])$.

Therefore, we explain more generally how to glue $(V_\alpha M, F_\bullet)$ for $\alpha \in \mathbb{Q}$ from local data. By Remark 3.1.5, we have

$$(V_\alpha M, F_\bullet)_{U_c} \cong (M_{U_c}, F_\bullet) = (D^E_{U_c}/K_{U_c}, F^\circ[\mathbf{s}]_\bullet)$$

for $c \in C^*$. Hence we use Algorithm 2.4.5 for the method in Example 4.4.3(b) on U_c. On the other hand, on the open subsets of type U_c with $c \in C^0$ we compute by Algorithm 4.2.9 a representation

$$(V_\alpha M, F_\bullet)_{U_c} \cong ((V_0 D_{U_c}^{E_c})/K_c, F^\circ[\mathbf{s}^c]_\bullet),$$

where the above isomorphism is already explicit by construction. Moreover, we test for $m \in M_{U_c}$ whether $m \in V_\alpha M_{U_c}$ by Algorithm 2.4.5 and explicitly represent it in terms of given generators of $V_\alpha M_{U_c}$ by Remark 2.4.6 if the test is positive. Thus Condition (cd') is satisfied on our cover. While the data in Example 4.4.3(a) and the filtration in Example 4.4.3(b) are given by the above representation, we take Algorithm 2.4.5 for the method in Example 4.4.3(b). As we can solve syzygy problems over D_{U_c}, we may use Algorithm 4.4.7 to construct the desired representation of $\mathrm{DLoc}_{X_0}(M, F_\bullet)$.

Remark 4.4.11.

(a) The dual localization map can be constructed as explained in Remark 4.4.8.

(b) To glue dual localizations of $D_X(*X_0)$-modules we use Remark 4.2.33 and the material presented in this subsection as well as the previous subsection.

4.4.5 Vanishing and nearby cycles

We want to compute the vanishing and nearby cycles of the $(D_X, F_\bullet)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$ along the regular function $f : X \to \mathbb{C}$ given that this modules is strictly f-specializable. Setting $Y := X \times \mathbb{C}_t$ and $Y_0 := V(t) \subseteq Y$ our problem reduces to computing the graded parts $\mathrm{Gr}_\alpha^{V^{Y_0}}((i_f)_+(M, F_\bullet))$.

We only briefly sketch the gluing process and leave the details to the reader: Representing $\mathrm{Gr}_\alpha^{V^{Y_0}}((i_f)_+(M, F^\circ))$ as a quotient of a $V_0^{Y_0} D_Y$-module with filtration induced by a weight vector works in analogy to Subsection 4.4.4 by considering $\mathrm{Gr}_\alpha^{V^{Y_0}} M$ as a subquotient of $(i_f)_+ M$. Regarding now $\mathrm{Gr}_\alpha^{V^{Y_0}}(i_f)_+ M$ as the trivial subquotient of this representation, using Algorithm 4.4.7 with Condition (cd') for the gluing and Algorithm 4.2.36 for the required local representations, we express $\mathrm{Gr}_\alpha^{V^{Y_0}}(i_f)_+ M$ as a quotient of a free D_X-module with filtration induced by a shift vector.

The representation of the morphisms can and var relies now again on gluing the corresponding local maps using the principle outlined in Remark 4.4.8.

Remark 4.4.12. To compute the nearby and vanishing cycles of $(M, F_\bullet)$ along the regular function $f : W \to \mathbb{C}$ with W being a proper open subset of X, we shrink X such that we may assume $X = W$. If W is affine, we continue now as above. Otherwise we refer to Subsection 4.4.8.

4.4.6 Dual localizations of strictly specializable D_X- and $D_X(*X_0)$-modules along singular subvarieties

Assume now that X_0 is singular. We are interested in computing the dual localization along X_0 of the strictly X_0-specializable $(D_X, F^\circ_\bullet)$-module $(M, F_\bullet) = (D_X^E/K, F^\circ[\mathbf{s}]_\bullet)$. Covering X as explained in Subsection 4.4.1, we first describe the dual localization of $(M, F_\bullet)$ on the open subsets of the cover of X^*. We have by definition

$$\mathrm{DLoc}_{X_0}(M, F_\bullet)_{U_c} \cong (M_{U_c}, F_\bullet) = (D_{U_c}^E/K_{U_c}, F^\circ[\mathbf{s}]_\bullet)$$

for $c \in C^*$. Note that on U_c the empty set $U_c \cap X_0$ is defined by the vanishing of $f_{a_c^*}$. In particular, we make the isomorphism $(D_{U_c}^E/K_{U_c}, F^\circ[\mathbf{s}]_\bullet) \cong V_0^{V(t-\overline{f_{a_c^*}})}(i_{f_{a_c^*}})_+ M_{U_c}(!U_c \times \{0\})$ explicit by proceeding as in Algorithm 4.3.8 and Remark 4.3.5(d), where we write by abuse of notation $i_{f_{a_c^*}}$ for the map $U_c \to U_c \times \mathbb{C}, u \mapsto (u, f_{a_c^*}(u))$.

On the other hand, on open sets of type U_c with $c \in C^0$ Algorithm 4.3.8 computes a representation

$$\mathrm{DLoc}_{X_0}(M, F_\bullet)_{U_c} \cong V_0^{V(t-\overline{f_{a_c^0}})}(i_{f_{a_c^0}})_+(M_{U_c}, F_\bullet)(!U_c \times \{0\}) \cong (D_{U_c}^{E_c}/K_c, F^\circ[\mathbf{s}^c]_\bullet)$$

with computable images and preimages under the second isomorphism (see Remark 4.3.5(d)) (here $i_{f_{a_c^0}}$ is to be understood in the same sense as above). This shows that Example 4.4.3(a) and (b) are satisfied on our cover (for the method in the latter part use Algorithm 2.4.5).

Considering $c, c' \in C$ there exists on $U_{cc'} = U_c \cap U_{c'}$ an invertible regular function $u_{c,c'} : U_{cc'} \to \mathbb{C}$ such that $f_{a_{c'}} = u_{c,c'} f_{a_c}$ inducing a coordinate change $\lambda_{c,c'} : U_{cc'} \times \mathbb{C}_t \to U_{cc'} \times \mathbb{C}_t : (\underline{x}, t) \mapsto (\underline{x}, u_{c,c'}(\underline{x})t)$. According to Lemma 3.2.39

$$V_0^{V(t-\overline{f_{a_c}})}((i_{f_{a_c}})_+ M_{U_{cc'}})(!V(t)) \cong V_0^{V(t-\overline{f_{a_{c'}}})}((i_{f_{a_{c'}}})_+ M_{U_{cc'}})(!V(t))$$

with morphism induced by

$$D_{U_{cc'} \times \mathbb{C}} \otimes_{V_0^{V(t)} D_{U_{cc'} \times \mathbb{C}}} V_{<0}^{V(t)}(i_{f_{a_c}})_+ M_{U_{c'}} \to D_{U_{cc'} \times \mathbb{C}} \otimes_{V_0^{V(t)} D_{U_{cc'} \times \mathbb{C}}} V_{<0}^{V(t)}(i_{f_{a_{c'}}})_+ M_{U_{cc'}},$$
$$p \otimes \overline{m} \mapsto \Lambda_{c,c'}(p) \otimes \overline{\Lambda_{c,c'}(m)}$$

with $\Lambda_{c,c'} : D_{U_{cc'} \times \mathbb{C}} \to D_{U_{cc'} \times \mathbb{C}}$ defined as in Example 1.4.9. This establishes together with the above isomorphisms in Example 4.4.3(b).

Remark 4.4.13.

(a) The computation of the dual localization map is again based on Remark 4.4.8.

(b) Using similar methods as in Remark 4.4.11(b) allows us to glue dual localizations of strictly X_0-specializable $(D_X(*X_0), F^\circ_\bullet)$-modules.

4.4.7 A quasi-inverse for Kashiwara's equivalence

Given a closed embedding $\iota' : X \subseteq Y$ of affine smooth pure dimensional varieties of dimensions m and n, there is by Proposition 1.4.12 an equivalence of categories

$$\iota'_+ : \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X) \to \mathrm{Mod}^X_{\mathrm{coh}}(\mathcal{D}_Y). \tag{4.4.1}$$

Moreover, we have a functor

$$\iota'_+ : \mathrm{Mod}_{\mathrm{coh}}(F^\circ_\bullet \mathcal{D}_X) \to \mathrm{Mod}^X_{\mathrm{coh}}(F^\circ_\bullet \mathcal{D}_Y),$$

where the right hand side category denotes the category of well-filtered $(\mathcal{D}_Y, F^\circ_\bullet)$-modules supported on X. Direct images as above are easily computable by Equation (1.4.1) and Definition 1.4.13. However, the computation of a quasi-inverse of the functor in Equation (4.4.1) is more involved.

We directly consider a similar question in the setting of well-filtered modules: A module $(\mathcal{M}, F_\bullet) \in \mathrm{Mod}_{\mathrm{coh}}(F^\circ_\bullet \mathcal{D}_X)$ is (up to isomorphism) uniquely determined by $\iota'_+(\mathcal{M}, F_\bullet)$ and is recovered from a representation $(P, F_\bullet) := (D^{E'}_Y/Q, F^\circ[\mathbf{t}]_\bullet)$ of $\iota'_+(\mathcal{M}, F_\bullet)$ as follows: Compute a partial affine open cover $\mathcal{U}$ of Y covering X with the following property: The set $U \in \mathcal{U}$ is a coordinate neighborhood of X with local coordinates $f_1, \dots, f_{\mathrm{n}}$ such that $f_1, \dots, f_k$ are global coordinates on $U_k := U \cap V(f_{k+1}, \dots, f_{\mathrm{n}})$ and such that $U_k \subseteq U$ has defining ideal sheaf generated by $f_{k+1}, \dots, f_{\mathrm{n}}$ for $\mathrm{m} \leq k \leq \mathrm{n}$ (see Remark 1.2.13(b)). By filtered Kashiwara's equivalence (Proposition 3.2.7) we see that $(\iota'_{U_k})_+(\mathcal{M}_{U_{\mathrm{m}}}, F_\bullet)$ is strictly f_k-specializable if $k > \mathrm{m}$, where ι_{U_k} stands for the inclusion $U_{\mathrm{m}} \subseteq U_k$. As $(\iota'_+(\mathcal{M}, F_\bullet))_U = (\iota'_{U_{\mathrm{n}}})_+(\mathcal{M}_{U_{\mathrm{m}}}, F_\bullet)$, we can stepwise compute a filtered $(\mathcal{D}_{U_k}, F^\circ_\bullet)$-presentation of

$$(\iota'_{U_k})_+(\mathcal{M}_{U_{\mathrm{m}}}, F_\bullet) = \mathrm{Gr}_0^{V^{U_k}} \left((\iota'_{U_{k+1}})_+(\mathcal{M}_{U_{\mathrm{m}}}, F_\bullet) \right) = V_0^{U_k} \left((\iota'_{U_{k+1}})_+(\mathcal{M}_{U_{\mathrm{m}}}, F_\bullet) \right)$$

using Algorithm 4.2.36 (after applying Remark 1.2.12) for $k = \mathrm{n} - 1, \cdots, \mathrm{m}$. This way we determine a presentation $(\mathcal{D}^{E_U}_{X \cap U}/L_U, F^\circ[\mathbf{w}_U]_\bullet)$ of $(\mathcal{M}, F_\bullet)_{U \cap X}$. As $\mathcal{D}^{E_U}_{X \cap U}/L_U$ is identified with a subset of P (and this identification can be made explicit by the methods in Subsection 4.2.8), it is possible to establish the necessary gluing isomorphism (see Assumption 4.4.2(c)). Since the other assumptions for Algorithm 4.4.7 are obviously satisfied because we work over coordinate rings, we may apply this algorithm to compute a representation of $(M, F^\circ_\bullet)$.

Remark 4.4.14. Note that we cannot check whether a well-filtered $(D_Y, F^\circ_\bullet)$-module $(P, F_\bullet)$ supported on X is the direct image of some $(D_X, F^\circ_\bullet)$-module. Yet, for Hodge D_Y-modules this is always the case due to Kashiwara's equivalence for mixed Hodge modules.

4.4.8 Computations on arbitrary varieties

Let X be an (arbitrary) smooth equidimensional algebraic variety, X_0 a pure codimension one subvariety and $\{U\}_{U \in \mathcal{U}}$ a finite affine open cover of X. A well-filtered $(\mathcal{D}_X, F^\circ_\bullet)$-module

$(\mathcal{P}, F_\bullet^\circ)$ is uniquely defined by $(\mathcal{P}(U), F_\bullet^\circ)$ and $(\mathcal{P}(U \cap U'), F_\bullet^\circ)$ as well as the restriction morphisms $\mathcal{P}(U) \to \mathcal{P}(U \cap U')$ for all $U, U' \in \mathcal{U}$.

If $(\mathcal{M}, F_\bullet)$ and $(\mathcal{N}, F_\bullet)$ are given by the data as above, we compute their localizations and dual localizations along X_0 and the vanishing and nearby cycles on the cover $\{U\}_{U\in\mathcal{U}}$ as well as on intersections of this cover by the methods presented in the previous subsections. Moreover it is possible to extend these methods to represent also the restriction maps by keeping track of the corresponding restrictions of $\mathcal{M}$ and $\mathcal{N}$ throughout all algorithms of this chapter.

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