Analysis and modeling of water distribution network in the framework of switched DAEs

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This thesis is dedicated to my father, who taught me that the best kind of knowledge to have is that which is learned for its own sake.

It is also dedicated to my mother, who taught me that even the largest task can be accomplished if it is done one step at a time.

The sad thing is that they taught me all except how to live without them.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification at this, or at any other university. This dissertation is completely my own work and contains nothing which is the outcome of work done in collaboration with others (except as specified in the text and in the acknowledgements)

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Abstract

Various physical phenomenons with sudden transients that results into structrual changes can be modeled via switched nonlinear differential algebraic equations (DAEs) of the type

$$E_\sigma \dot{x} = A_\sigma x + f_\sigma + g_\sigma(x) \tag{1}$$

where $E_p, A_p \in \mathbb{R}^{n\times n}, x \mapsto g_p(x)$ is a mapping, $p \in \{1, \cdots, P\}, P \in \mathbb{N}$ $f \in \mathbb{R} \to \mathbb{R}^n$, $\sigma : \mathbb{R} \to \{1, \cdots, P\}$.

Two related common tasks are:

1. Investigate if (1) has a solution and if it is unique.
2. Find a connection among a solution of (1) and solutions of related partial differential equations.

In the linear case $g(x) \equiv 0$ the task 1 has been tackeled already in a distributional solution framework.

A main goal of the dissertation is to give contribution to task 1 for the nonlinear case $g(x) \not\equiv 0$; also contributions to the task 2 are given for switched nonlinear DAEs arising while modeling sudden transients in water distribution networks. In addition, this thesis contains the following further contributions:

The notion of structured switched nonlinear DAEs has been introduced, allowing also non regular distributions as solutions. This extend

a previous framework that allowed only piecewise smooth functions as solutions. Further six mild conditions were given to ensure existence and uniqueness of the solution within the space of piecewise smooth distribution. The main condition, namely the regularity of the matrix pair (E, A), is interpreted geometrically for those switched nonlinear DAEs arising from water network graphs.

Another contribution is the introduction of these switched nonlinear DAEs as a simplication of the PDE model used classically for modeling water networks. Finally, with the support of numerical simulations of the PDE model it has been illustrated that this switched nonlinear DAE model is a good approximation for the PDE model in case of a small compressibility coefficient.

Zusammenfassung

Verschiedenste physikalische Phänomene, die durch eine plötzliche Änderung von Systemparametern entstehen, können durch geschaltete nichtlineare Differential algebraische Gleigungen (differential algebraic equations, DAEs) der Form

$$E_\sigma \dot{x} = A_\sigma x + f_\sigma + g_\sigma(x) \tag{2}$$

modelliert werden, wobei $E_p, A_p \in \mathbb{R}^{n \times n}, x \mapsto g_p(x)$ eine Abbildung ist, $p \in \{1, \cdots, P\}, P \in \mathbb{N}$ $f \in \mathbb{R} \to \mathbb{R}^n$, $\sigma : \mathbb{R} \to \{1, \cdots, P\}$. Zwei damit einhergehende Aufgaben sind:

1. Untersuche, ob (1) eine Lösung hat und diese eindeutig ist.
2. Finde einen Zusammenhang zwischen einer Lösung von (1) und den Losungen von mit (1) verwandten partiellen Differentialgleichungen.

Im linearen Fall $g_\sigma(x) \equiv 0$ sind die Aufgabe 1 bereits innerhalb eines distributionellem Lösungsrahmen behandelt worden.

Ein Hauptziel dieser Arbeit besteht darin, auch fur den nichtlinearen Fall $g_\sigma(x) \not\equiv 0$; Beitrage zur Aufgaben 1 zu geben; beigetragen wird ebenfalls zur Aufgabe 2, im Rahmen von geschalteten nichtlineare DAEs, die bei der Modellierung von plötzlichen Veränderungen in Wasserverteilungsnetzwerken vorkommen. Im Einzelnen beinhaltet diese Arbeit unter anderem folgenden Beitrage:

Eingeführt wird der Begriff einer strukturierten geschalteten nichtlineare DAE (structured switched nonlinear DAE), welcher auch nichtreguläre Distributionen als Lösungen erlaubt. Dies erweitert einen fruheren Ansatz, der lediglich stükweise glatte Funktionen als Lösungen gestattete. Des Weiteren werden sechs, nicht sehr restriktive, Bedingungen angegeben, die zusammen Existenz einer Lösung und ihre Eindeutigkeit, innerhalb des Raumes der stuckweise glatten Distributionen, garantieren. Die Hauptbedingung, namlich die Regularität des Matrixpaars (E, A), wird geometrisch interpretiert, fur diejenigen geschaltete nichtlineare DAE, die von Wassernetzwerkgraphen herrühren.

Ein weiterer Beitrag besteht in der Einführung von eben diesen geschalteten nichtlineare DAEs, was gegenüber dem klassischen, auf partiellen Differentialgleichungen basierenden, Modell für Wassernetzwerke, eine Vereinfachung darstellt.

Schlielißch wurde mittels nummerischer Simulationen des klassischen Modells aufgezeigt, da solch eine geschaltete nichtlineare DAE eine gute Approximation fur das klassische Modell darstellt,wenn dort der Kompressionskoeffizient klein gewält ist.

Contents

Chapter 1

Introduction

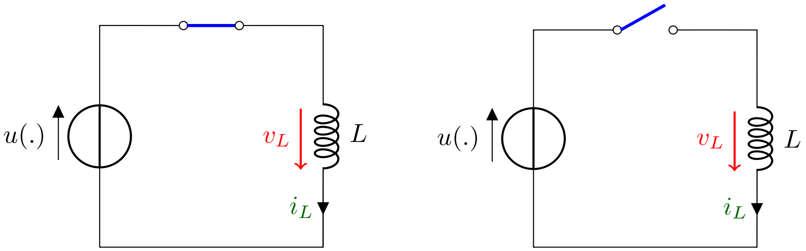
A water distribution network is a arrangement to transport the water to the consumer in a regular manner. The discontinuance in the water supply is troublesome for the consumers. One of the potential reasons of the cutoff may be pipe breakage. It may crop up by the instantaneous change in the current conditions in the network due to some inescapable hard faults. In this dissertation the abrupt events which create such instantaneous changes in flow conditions are scrutinized. Indicatively, this work is about the impacts of of these events. Classically the flow of the fluids into a network is modeled in the framework of a system of hyperbolic balance laws 8], 6] and 37]. In this work these sudden changes are viewed as a shift from current steady state to another steady state, this idea is the catalyst to model and study them in a framework, which is especially useful for such switches.The solution theory for this framework (so called framework of switched differential algebraic equations (switched DAEs)) is founded for the systems whose dynamics are shown by linear DAEs, e.g. linear electrical circuits. Thus, the comportment of the nonlinearities in the mathematical model of the elements of the water network was a difficulty in its modeling in the proposed framework of switched DAEs (due to the presence of Diracs). One aim of this work is to formulate and introduce the notion of solution for the switched nonlinear DAEs, which is applicable for the nonlinearties having a special structure. Further aims includes the introduction of this simple framework as an approximation to the classical modeling framework of system of hyperbolic balance laws to model the transitory events.

1.1 Switched differential algebraic equations

A dynamical system describes the evolution of a state over time. Based on the type of their state, dynamical systems can be classified into:
Continuous : if the state takes values in Euclidean space $\mathbb{R}^n$ for some $n \geq 1$. Then $x \in \mathbb{R}^n$ denotes the state of a continuous dynamical system,
Discrete: if the state takes values in a countable or finite set $\{\mathbf{p}_1, \mathbf{p}_2, \dots\}$, where $\mathbf{p}$ to denote the state of a discrete system. For example, a light switch is a dynamical system whose state takes on two values, $\mathbf{p} \in \{\mathrm{ON}, \mathrm{OFF}\}$.
Hybrid: systems in which these two kinds of dynamics (discrete and continuous) coexist and interact 52].

A hybrid systems may exhibit unusual phenomena, like impulses (discontinuous state evolution) and Zeno behaviour (accumulation of discontinuities in finite time). Whereas the switched systems are special kind of such hybrid systems, where the discrete dynamics are replaced by an external input σ, the switching signal. Such systems are used to describe systems whose (continuous) dynamics are subject to hard faults i.e. abrupt changes. The evolution of a switched system is described by continuous trajectories moreover, Zeno behaviour is excluded by definition 7].

For example the sudden changes occur in electrical circuits with switches on changing the position of the switch. Such circuits are modeled as a switched differential algebraic equation (Ordinary differential equatios (ODEs) with algebraic constraints): the system is modeled as a time-varying whose coefficient matrices are piecewise constant. The time-variance follows from the action of the switches present in the circuit, but can also be induced by faults occurring in the circuit. Consider an electrical circuit modeled with connected and disconnected switch and an inductor shown in the Figure 1.1.



(a) Connected switch (1.1a) (b) Disonnected switch (1.1b)

Figure 1.1: Electrical circuit with connected (a) and disconnected switch (b)

Standard circuit analysis of these two circuits shown in the Figure 1.1(a) and 1.1(b) for the given switch setting (connected and disconnected) yields differential algebraic equations (1.1a) and (1.1b), respectively.

Switch connected

$$L\frac{d}{dt}i_L = v_L,$$

$$v_L = u \tag{1.1a}$$

Switch disconnected

$$L\frac{d}{dt}i_L = v_L,$$

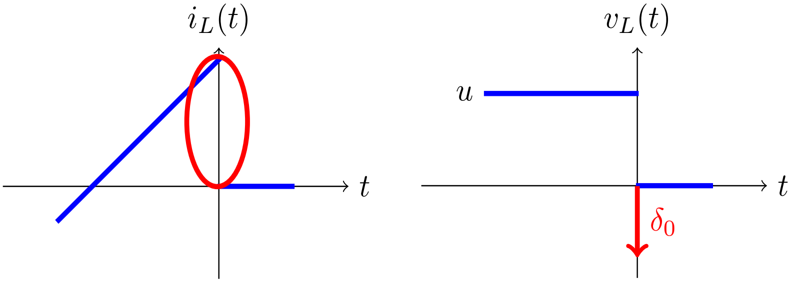
$$i_L = 0 \tag{1.1b}$$

where L, i_L, v_L and u denotes the inductance, current through inductor and voltage through the inductor and constant voltage of the voltage source, respectively.

In general, switches or component faults induce jumps in certain state variables, and it is common to define additional jump maps based on physical arguments 67]. However, it turns out that the appropriate formulation as a switched DAE already implicitly defines these jumps, no additional jump map must be given. Further detail is presented in the Chapter 3.

In the Figure 1.1, when connecting a coil via a switch to a constant voltage source, one can observe a spark when opening the switch, which can be explained by a voltage peak induced by the rapid drop of the current in the coil. The solution of these equations is abstractly shown in the Figure 1.2. The jump in the current through the inductor cause an impulse in the voltage across inductor v_L.

In order to allow for jumps in the solution, the problem is embedded into a distributional solution framework. The solution of the electrical circuit in the Figure 1.1 is shown in the Figure 1.2.



(a) Jump in i_L (1.1).

(b) Dirac in v_L (1.1)

Figure 1.2: Derivative of jump in i_L is a Dirac in v_L.

1.2 Motivation

The framework of switched DAEs is used to model the structural changes in a dynamical system. The occurrence of such structural changes in flow networks, is a motivation to model and to analyse them in the framework of switched DAEs.

The transients induce change from one steady state to another. These outcomes are a short lived and burst of energy in a system caused by a sudden change of state. Such variety of state passes off in the water network and is termed as a hydraulic transient. . Moreover occurrence of hydraulic transients in the procedure of water distribution network is inevitable and may endanger the ability of the distribution system to fulfill its operational requirements depending on the rigor and frequency of the pressure fluctuations that may happen. The typical results are controlled or uncontrolled changes in pump or valve settings. They may cause certain transients in the pressure in response to these sudden changes. When the water distribution system functions normally the flow and pressure is considered steady; they do not vary with time, or when fluctuations are small with respect to mean flow values. Any rapid disturbance in the water which is generated due to a change in the mean flow conditions, will initiate a sequence of pressure waves in the system. The term ‘water hammer’, ‘transient flow’, and ‘surge’, describe the unsteady flow of fluids in the pipes.

A very common sudden change occurs in the water networks is due to the sudden closing of a valve. Due to this instantaneous closure, the transformation of kinetic energy into pressure energy causes significant changes in pressure, which can lead to serious problems in the management of a pressurised network. The phenomenon is very complex, and a large number of different factors influence its course. In fact the change in the momentum of the liquid requires pressure changes results in expansion or compression of the pipe and liquid. The compression and expansion of liquid is given by its compressibility and measured in terms of compressibility coefficient β. This conversion of kinetic energy carried by moving fluid into strain energy in the pipe walls, causing a 'pulse wave' that may of abnormal pressure travel from the disturbance into pipe system 73]. The hammering sound that is sometimes heard as an indication of this conversion and create an impact force. The energy losses such as fluid friction, as well as the reflection and transmission of waves in pipe junctions, cause the transient pressure waves to gradually decay until new steady pressures are established 33]. A pressure surge which frequently generated is shown in the Figure 1.3 along with the possible breakage of pipeline they may cause is shown in the Figure 1.4.

Hydraulic transient analysis has traditionally focused almost exclusively on preventing catastrophic failure in pipe and pump. Less attention has been given to the analysis of pressures that may occur in the distribution system. Furthermore it is very time consuming to analyse the strength of the pressure spike by classical method of the system of hyperbolic conservation laws. Also due to numerical restrictions total compressibility can not be considered. On decreasing the value of β will increase the pressure spike and a Dirac impulse, which is mathematically embedded in the solution space of switched DAE framework. The details are presented in the Chapter 3 and 67]. The pressure transients in response to the sudden structural changes and in a water network can be analysed in this simplified framework of switched DAEs. Another motivation to model water network transients in the switched DAEs framework is the hydroelectric analogy.

A switch is an electrical circuit component which make current flow or stop in the electric circuit, it is equivalent in functionality as a valve in the water flow system which is used to allow or stop the flow.

Classically water flow is modeled by using system of hyperbolic balance laws. Firstly to do the modeling in the framework of switched DAE a simplified system of ODE is formed by assuming incompressibility which

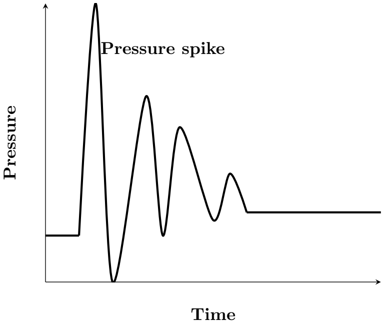


Figure 1.3: Evolution of a surge.

Figure 1.4: The effect of the surge (pipe breakage) published after permission of the owner 59].

results into a nonlinear ODE due to the presence of friction and other nonlinearities in water flow modeling. In modeling water flow other components that are modeled as algebraic constraints, like reservoir at an end of pipe or a junction, which are modeled as assigning a constant pressure or flow balance, respectively. On introducing the switches as algebraic constraints results into a switched nonlinear DAE. Therefore a solution theory framework was needed to model nonlinear switched DAE. In literature 69] the framework is only applicable to the linear switched DAEs. Moreover the further existence and uniqueness theory already is presented for switched nonlinear DAEs 53] excluded impulses. In the case of modeling of hydraulic transient where impulses are expected the previously developed solution theory is non applicable.

In order to compare classical modeling approach with proposed switched DAE model an approximation way is needed in case of PDE modeling. If it can be shown that both modeling approaches are the limiting case of each other then quantitatively the Dirac impulse length can be approximated using a simplified framework of switched DAEs.

Furthermore, to model water network, classical PDE model in general coupling conditions 18] and their wellposedness is given in 17] . Similarly the wellposedness of general water network for all possible network topologies in the framework of switched DAEs is required.

1.3 Outline of the thesis

The thesis is structured as follows:

Chapter 2 provides the fundamentals of fluid flow modeling, and novel approach of hyperbolic balance laws to model water flow.

Chapter 3 preliminaries of the linear switched DAEs are recalled. Further a solution framework for switched nonlinear DAEs in the presence of impulses is established for the structured nonlinearities present in water network.

Chapter 4 presents a water hammer model for a simple network with both hyperbolic balance laws and switched differential algebraic equations is presented . The results of both models are compared quantitatively.

Chapter 5 will provide a general water network structure is presented in the framework of switched DAEs and its existence uniqueness of the solution is presented.

Chapter 6 in this chapter, further networks are presented to demonstrate the pertinence of the hypothesis developed in the former chapters.

Chapter 7 in this chapter, the conclusion drawn from the thesis has been presented along with the limitations of this work.

1.4 Previously published results and joint work

The following parts of this dissertation are already published or submitted for publication. The results of the Chapter 3 have been published in 46] which is joint work with StephanTrenn (Technische Universität Kaiserslautern, now University of Groningen). The results presented in the Chapter 4 is published in 45]. It is joint work with Stephan Trenn (Technische Universität Kaiserslautern, now University of Groningen), and Jochen Kall. The results in the Section 6.2.3 is submitted for publication 47] (without wellposedness of switched DAEs model). This is joint work with Stephan Trenn (Technische Universität Kaiserslautern, now University of Groningen).

Chapter 2

Fluid flow modeling

A water distribution system is a connection of pipelines that distribute water to the consumers. For the analysis (speed, pressure, different dynamics, etc.) of the water flow in this system, it is significant to see the fundamentals of the modeling of flow in pipes.

Many flow problems are modeled by systems of 2×2 hyperbolic balance laws, for instance shallow water equations for hydraulic networks (e.g.36],63]), isothermal Euler equations for gas pipeline networks (e.g.41]), and Aw-Rascle equations for road traffic networks (e.g.19]). This chapter is splited into three parts; in the first part physical variables and fluid properties that are used for the modeling is presented. In the second part basic mathematical theory of hyperbolic balance laws is brought out. The third section is devoted to present modeling of flow in an arbitrary pipe of a water network.

2.1 Review of flow modeling

In order to model and analyse the flow, firstly it is significant to see fluid and flow properties in which flow is assessed in different regards. The change in these properties is responsible for dynamics (instantaneous changes lasts for very short interval of time, so called transients). In this Section those properties will be presented briefly. Moreover, the types of possible flows which may come out as the outcome of changes in the values of these attributes will be stated later in this Section 54] `?john16`] 32].

Fluid properties

1. **Density**: The density of a fluid, denoted by the Greek letter ρ, is defined as mass per unit of volume. Density is typically used to characterise the mass distribution of a fluid system. Its unit is g/cm^3 or kg/m^3.

2. **Pressure**: One of the important variables in fluid flow and particularly for water flow in pipes is pressure. The pressure is an important property to be controlled and observed. The uncontrolled high pressure may result into some catastrophic situation. The pressure is denoted by P and formally defined as follows:

   Definition 2.1.1. Pressure is a normal force per unit area in a fluid 54].

   The pressure may change as the result of the change in the density, but it depends on the type of fluid or on the type of flow. Its unit is Nm^{-2} or Pa.

3. **Velocity** : Suppose that the fluid is contained with a domain $\mathcal{D} \subseteq \mathbb{R}^d$ where $d = 1, 2, 3$, and $x = (x, y, z)^\top \in \mathcal{D}$ is a position in $\mathcal{D}$. Imagine a small fluid particle or a speck of dust moving in a fluid flow field prescribed by the velocity field $\mathbf{u}(t, x) = (u, v, w)^\top$. Suppose the position $(x_0, y_0, z_0)^\top$ of the particle at time t is recorded by the variables $(x(t), y(t), z(t))^\top$. The velocity of the particle at

time t at position $(x(t), y(t), z(t))^\top$ is

$$\frac{d}{dt}x(t) = u(x(t), y(t), z(t), t),$$
$$\frac{d}{dt}y(t) = v(x(t), y(t), z(t), t),$$
$$\frac{d}{dt}z(t) = w(x(t), y(t), z(t), t).$$

Definition 2.1.2. Velocity is defined as a vector quantity that refers to "the rate at which an object changes its position." Also referred to as ‘speed with direction’, and it is denoted by $\mathbf{u}$.

4. **Trajectory**: The particle path or trajectory of a fluid particle is the curve traced out by the particle as the time progresses. If the particle begins at position $(x_0, y_0, z_0)^\top$ then its particle path is the solution to the system of differential equations (the same as those above but here in shorter vector notation)

$$\frac{d}{dt}x(t) = \mathbf{u}(x(t), t),$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$.

5. **Streamline**: Suppose for a given fluid flow $\mathbf{u}(x, t)$ we fix time t. A streamline is an integral curve of $\mathbf{u}(x, t)$ for t fixed, i.e. it is a curve $x = x(s)$ parameterised by a variable s, that satisfied the system of differential equations

$$\frac{d}{ds}x(s) = \mathbf{u}(x(s), t),$$

with t held constant.

Remark 2.1.3. If the velocity field $\mathbf{u}$ is time independent or equivalently $\frac{\partial \mathbf{u}}{\partial t} = 0$, then trajectories and streamlines coincide.

Furthermore, the flows for which $\frac{\partial \mathbf{u}}{\partial t} = 0$ *hold true are said to be steady.*

4. **Mass flow**: The mass flow is defined as the time rate of this mass passing through an area.

$$\text{mass flow} = \dot{m} = \lim_{\triangle t \to 0} \frac{\triangle m}{\triangle t} = \rho \mathbf{u} \mathbf{A}$$

   Where **u** and **A** denotes the velocity and area, respectively.

5. **Mass flux** The mass flux is defined simply as mass flow per unit area.

$$\text{mass flux} = \frac{\dot{m}}{\mathbf{A}} = \rho \mathbf{u}$$

6. **Compressibility coefficient**: Compressibility is defined as the amount of pressure change needed to change a given volume of a fluid. The compressibility of a substance is measured in terms of a coefficient, which is called the compressibility coefficient and denoted by β. It is defined as a measure of the relative density change $d\rho$ of a fluid as a response to a pressure change, defined as

$$\beta = \frac{\frac{d\rho}{dP}}{\rho} = \frac{1}{\rho} \frac{d\rho}{dP} \tag{2.1}$$

   where the leading coefficient $\frac{1}{\rho}$ is due to the fact that; it is change relative to a given density.

7. **Bulk modulus**: It is a measure of the substance's resistance to uniform compression. It is the reciprocal of the compressibility coefficient (β) and similar to the spring factor2], The bulk modulus is denoted by K and mathematically can be written as:

$$K = \rho \frac{dP}{d\rho} \tag{2.2}$$

For water of normal temperature the pressure $K = 2.06 \times 10^9 Pa$ and ρ_a is the atmospheric density of water that is $\rho_a = 1000kg/m^3$

8. **Relationship of compressibility coefficient β and velocity of sound c**: The speed of sound is defined in classical mechanics as:

$$c^2 = \frac{\partial P}{\partial \rho}.$$

and the relationship of β and c is given as

$$\beta = \frac{1}{\rho_a c^2}.$$

The speed of sound is very important while working with numerical simulations.

9. **Flow dimensionality**: In general, all physical flows are three dimensional (3D). Nevertheless, it is often convenient, and sometimes quite accurate to view them as being of a lower dimensionality, e.g., 1D or 2D.

Fluid dimensionality can be formally defined as follows:

Definition 2.1.4. The dimensionality of a flow field corresponds to the number of spatial coordinates needed to describe all properties of the flow `?john16`].

Throughout the thesis, dimensionality of flow is considered to be 1, because flow in pipes can be described by 1D with sufficient accuracy.

10: **Reynold's number Re**: This is an important number used to distinguish between two basic types of flow.

Definition 2.1.5. The Reynolds number is the typical ratio of inertial to viscous forces within the fluid; it is denoted by (Re).

The inertial forces give rise to the dynamic pressure. The ratio of ($\rho\mathbf{u}^2$) and viscous forces ($\frac{\mu\mathbf{u}}{L}$) can be expressed as:

$$\mathrm{Re} = \frac{\rho\mathbf{u}^2}{\frac{\mu\mathbf{u}}{L}} = \frac{\rho\mathbf{u}L}{\mu} = \frac{\mathbf{u}L}{\nu} \tag{2.3}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, L is characteristic dimension of pipe and $\mathbf{u}$ is velocity of the fluid. The viscous forces are the forces due to the friction between the layers of any real fluid. In fluid mechanics the fluid is taken as in the continuum condition, meaning that fluid particles are very closely packed so necessarily there is friction between layers of fluid. The inertial forces are the forces which are due to the particles of fluid resisting any change in momentum.The Reynolds number is a convenient parameter for predicting if a flow condition will be laminar or turbulent. It can be interpreted that when the viscous forces are dominant (slow flow, low (Re)) they are sufficient enough to hold all the fluid particles in line, then the flow is laminar. Even very low (Re) indicates viscous creeping motion, where inertia effects are negligible. When the inertial forces dominate over the viscous forces (when the fluid is flowing faster and (Re) is larger) then the flow is turbulent.

2.2 Flow transitions

The term flow transition means the change of flow from one type flow to another type. In order to deduce the flow transition further, different types of the flows are discussed. To grasp the flow conditions exactly, it is important to understand governing equations and theories. Here, concepts are introduced briefly, more details can be read in

Mathematically, incompressibility is expressed by saying that density ρ of the fluid parcel does not change as it moves in the flow field. This additional constraint simplifies the governing equations, especially in the case when the fluid has a uniform density.

Compressibility is measured by the compressibility coefficient which is defined in section 2.1.

Remark 2.2.1. If the compressibility of fluid in the course of its flow becomes significant, Reynolds number is sufficiently large, and in equation (2.3) inertial forces have to be considered. For incompressible flow inertial force is negligible.

Remark 2.2.2. All these type of flows can exist independently of each other. So any of the four combinations of flows possible:

1. Steady uniform flow
2. Steady non-uniform flow
3. Unsteady compressible flow
4. Unsteady incompressible flow

The above flow regimes are possible for water flow in water pipelines. Some of these regimes include an instantaneous change in the velocity that results into in a rapid modification in the pressure, which is usually termed as ***hydraulic transients*** or simply ***transients***. The occurrence of transients may introduce large pressure forces and rapid fluid accelerations into a water distribution system, which is the main interest of research here.

For the modeling water flow via hyperbolic balance laws, water is considered as slightly compressible hence inertial forces are included to the model. Before introducing the mathematical model for flow in a pipe (compressible and incompressible) the general structure and existence and uniqueness of the solution to the hyperbolic balance laws will be introduced.

2.3 Hyperbolic system of balance laws

A balance law is a mathematical expression of physical principle that the variation of the amount of some extensive quantity over bounded domain is balanced by its flux through the boundaries of the domain and its production consumption inside the domain. Balance laws are therefore used to represent the fundamental dynamics of many physical conservative systems 9]. This is also a frequently used approach for modeling of the fluid flow.

A system of hyperbolic balance laws has two parts, conservation laws and dissipation or a source term. First the system of conservation laws will be introduced.

2.3.1 System of conservation laws

The differential equation of flow are derived by considering a differential volume element of fluid and describing mathematically

a) The conservation of mass of fluid entering and leaving the control volume; the resulting mass balance is called the equation of continuity.

b) The conservation of momentum entering and leaving the control volume; this energy balance is called the equation of motion.

Combining both a) and b) results in a system of conservation laws in one space dimension of the form

$$\partial_t U + \partial_x [F(U)] = 0 \tag{2.4}$$

$U =: \Omega \times \mathbb{R}_+ \to \mathcal{U} \subseteq \mathbb{R}^n$ with $\Omega := [a, b] \subseteq \mathbb{R}^n$ being the domain and the open connected set $\mathcal{U} \subseteq \mathbb{R}^n$ being the range of the problem, $F : \mathcal{U} \to \mathbb{R}^n$ is the flux function. The total amount of conserved quantities inside any given interval $[a, b]$ can change because of the flow across boundary points. A system of the form (2.4) is called conservation laws in continuum physics, where small dissipation effects are neglected. In case of water flow in pipes there exists dissipation due to the friction factor of the pipe walls. Hence it can not be neglected. An addition of a non zero dissipation to the system of conservation laws to a nonzero right hand side of (2.4) is called system of hyperbolic balance laws. In the next Section a general form of hyperbolic balance laws is presented followed by characterisation of existence and uniqueness of their solution.

In the following the U denotes the function and u denotes the function value of $u := U(t,x) \in \mathcal{U}$ for some $x \in \Omega$ and $t \geq 0$.

2.3.2 Hyperbolic balance laws

In general system of balance laws in one space dimension takes the following form

$$\partial_t U + \partial_x [F(U)] = S(U), \qquad \text{on } [a,b] \times \mathbb{R}_+, \tag{2.5a}$$

$$U(0,x) = U_0(x), \qquad x \in \Omega, \tag{2.5b}$$

$$\Psi_1(t, U(t,a)) = 0, \qquad t > 0, \tag{2.5c}$$

$$\Psi_2(t, U(t,b)) = 0, \qquad t > 0,, \tag{2.5d}$$

where $S : \mathcal{U} \to \mathbb{R}^n$ is the source term, $U_0 : \Omega \to \mathcal{U}$ is the initial data and $\Psi_i : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{R}^{b_i}$, $i = 1, 2$, $b_1 + b_2 = n$, are the time-varying, implicit boundary conditions. After the introduction to the general form of balance laws the wellposedness of (2.5a) along with (2.5b), (2.5c) and (2.5d) will be discussed. First the notion of weak solution is introduced, it is because the system (2.5a) does not necessarily admit a classical (i.e. differentiable) solution, even for "well-behaved" initial data U_0. Hence, weak solutions will be considered.

Definition 2.3.1. For any given time $T > 0$, a function $U \in \mathcal{L}^1([a,b] \times \mathbb{R}^+)$ is an entropy solution to the system (2.5) on the domain $\mathbb{D}_T =: \{(t,x)|T \in \mathbb{R}_+, a < x < b\}$ if:

I. The function U is a weak solution in the sense of distribution defined as:

Definition 2.3.2 (Weak solution)**.** *A function* $U : \Omega \times \mathbb{R}_+ \to \mathcal{U}$, *is a* weak solution *of* (2.5) *iff,* U, $F(U)$ *and* $S(U)$ *are locally integrable and for every* $\phi \in \mathcal{C}^1_c(\mathbb{R}_+ \times \Omega \to \mathbb{R})$, *one has*

$$\int\limits_{\mathbb{R}_+ \times \Omega} (U(t,x)\phi_t + F(U(t,x))\phi_x) dxdt + \int\limits_{\Omega} \phi(0,x)U_0(x)dx = \text{-} \int\limits_{\mathbb{R}_+ \times \Omega} S(U(t,x))\phi dxdt. \tag{2.6}$$

where $\mathcal{C}^1_c$ denotes the set of $\mathcal{C}^1$ functions with compact support of $\mathbb{D}_T$;

II. When considering weak solutions of (2.5) uniqueness of solutions cannot be expected in general. Under certain assumptions, uniqueness can be recovered by imposing a so called entropy condition as follows (c.f. 35]).

Definition 2.3.3 (Entropy solution)**.** A differentiable function $\eta \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R})$ is called entropy of the PDE (2.5) if there exists an entropy flux $\zeta \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R})$

$$\mathbf{D}\eta(u) \cdot \mathbf{D}F(u) = \mathbf{D}\zeta(u) \quad \forall u \in \mathcal{U}. \tag{2.7}$$

For a given entropy η with corresponding entropy flux ζ, a weak solution U of (2.5) is called *entropy solution* if it additionally satisfies

$$\eta(U)_t + \zeta(U)_x \leq \mathbf{D}\eta(U) \cdot S(U)$$

in a weak sense. In other words the function U is *entropy admissible* in the sense that there exists an entropy-entropy flux (η, ξ) for the system (2.5) such that for every $\phi \in \mathcal{C}_c^1(\mathbb{R}_+ \times \Omega \to \mathbb{R})$;

$$\begin{aligned} \int_{\mathbb{R}_+ \times \Omega} (\phi_t \eta(U) + \phi_x \xi(U)) dx dt + \int_{\Omega} (\phi(0,x)\eta(U) U_0(x) dx \\ - \int_{\mathbb{R}_+ \times \Omega} \phi \mathbf{D}\eta(U) S(U) dx dt \leq 0 \end{aligned} \tag{2.8}$$

Remark 2.3.4. *For any classical (i.e. differentiable) solution U of (2.5) it is easily seen that (2.7) implies*

$$\eta(U)_t + \zeta(U)_x = \mathbf{D}\eta(U) \cdot S(U).$$

Moreover, if U satisfies the initial condition U_0 for a.e. $x \in [a,b]$ and the boundary conditions,

$$\lim_{x \to a^+} \Psi_1(t, U(t,a)) = 0 \quad \lim_{x \to b^-} \Psi_2(t, U(t,b)) = 0.$$

that is boundary values are assumed to be satisfied in the trace sense, for details see (c.f. 51])24],23]) for more details.

Next the assumptions required to ensure the existence of the weak solution are presented, which are important for the wellposedness.

Wellposedness of (2.5)

We are concerned with the wellposedness of the initial value problem for the system above, in the space of functions with bounded variation. Several results in the literature deal with the wellposedness of balance laws globally in time. The standard approach relies on the requirement that the convective part and the source part be "compatible". This is usually obtained through suitable estimates, the main examples being the following: either the convective part dissipates what the source term increases, or the source term causes a decay in what the convective part produces see 30],5].

Consider conservation laws with boundary

$$\begin{aligned} \partial_t U + \partial_x[F(U)] &= 0, && \text{on } [a,b]\times\mathbb{R}_+, \\ U(0,x) &= U_0(x), && x\in\Omega, \\ \Psi_1(t,U(t,a)) &= 0, && t>0, \\ \Psi_2(t,U(t,b)) &= 0, && t>0, \end{aligned} \tag{2.9}$$

and the source part

$$\begin{aligned} \partial_t U &= S(U), && \text{on } [a,b]\times\mathbb{R}_+, \\ U(0,x) &= U_0(x), && x\in\Omega, \\ \Psi_1(t,U(t,a)) &= 0, && t>0, \\ \Psi_2(t,U(t,b)) &= 0, && t>0, \end{aligned} \tag{2.10}$$

separately. Indeed, the wellposedness of (2.9) is proved below under those assumptions on F , respectively on $S(U)$, that make (2.10), respectively (2.10), wellposed. Besides, a sort of compatibility between the conservation law (2.9) and the ordinary differential equation (2.10) is required. Namely it is required that there exists a domain which is invariant for both(2.9) and (2.10). For existence and uniqueness of weak entropy solutions of the initial/boundary value problem (2.5) the following wellposedness assumptions are usually imposed (c.f. 17] 28] 26]). If the flux F is differentiable, then the PDE (2.5a) can be written in quasi-linear form as

$$U_t + A(U)U_x = 0, \tag{2.11}$$

where $A(u) := \mathbf{D}F(u)$ for $u \in \mathcal{U}$ and $\mathbf{D}F$ denotes the Jacobian of F.

(B) *Connective part*; Consider connective part (2.9) for wellposedness

(B-I) *Bounded variation*
The initial date U_0 is a function of bounded variation with sufficiently small total variation.

(B-II) *Strict hyperbolicity*
The system of conservation laws (2.9) is *strictly hyperbolic*, i.e., for every $u \in \mathcal{U} \subseteq \mathbb{R}^n$, the Jacobian matrix $A(u)$ of the flux function F has n real, distinct eigenvalues denoted by $\lambda_i(u)$, $i = 1, \cdots, n$ and are ordered as follows:

$$\lambda_1(u) < \lambda_2(u) < \lambda_3(u) \cdots < \lambda_n(u).$$

(B-III) *Genuine nonlinearity and linear degeneracy*
For a strictly hyperbolic balance law (2.9) in quasi linear form](2.11), consider the n eigenvalue/eigenvector pairs $(\lambda_j(u), r_j(u))$, $j = 1, \ldots, n$ with differentiable map $u \mapsto \lambda_j(u)$. An assumption is that for each j either $(\lambda_j(\cdot), r_j(\cdot))$ is genuinely nonlinear, i.e.

$$\mathbf{D}\lambda_j(u) \cdot r_j(u) \neq 0, \quad \forall u \in \mathbb{R}^n,$$

or $(\lambda_j(\cdot), r_j(\cdot))$ is linearly degenerate, i.e.

$$\mathbf{D}\lambda_j(u) \cdot r_j(u) = 0, \quad \forall u \in \mathbb{R}^n.$$

(C) *Source part*; Consider source part (2.10) for wellposedness; there exists a positive $\hat{L}$ such that for all $u_1, u_2 \in \mathcal{U}$.

$$\|S(u_1) - S(u_2)\| \leq \hat{L} \cdot \|u_1 - u_2\|$$

(D) *Feasible boundary conditions*
For a strictly hyperbolic balance law (2.5a) assume that

$$\lambda_{b_2}(u) < 0 < \lambda_{b_2+1}(u) \quad \forall u \in \mathcal{U}$$

and denote with $r_+^1(u)$, $r_+^2(u)$, …, $r_+^{b_1}(u)$ the collection of eigenvectors of $A(u)$ corresponding to the positive eigenvalues of $A(u)$. Assume that for all $u \in \mathcal{U}$ the following feasibility assumption for the left boundary condition holds [c.f. 23],28]].

$$\forall u \in \mathcal{U} : \quad \det\left[\mathbf{D}_u \Psi_1(t, u) \cdot R^+(u)\right] \neq 0, \tag{2.12a}$$

where $R^+(u) := [r_+^1(u), \ldots, r_+^{b_1}(u)]$. To formulate a feasibility assumption for the right boundary condition, we can substitute the space variable x by $a + b - x$ in (2.5) with variable $\overline{U} : \mathbb{R}^+ \times \Omega \to \mathcal{U}$ instead of U. For $\overline{u} \in \mathcal{U}$, let $r_-^1(\overline{u}), \ldots, r_-^{b_2}$ be the eigenvectors corresponding to the positive eigenvalues of $\overline{A}(\overline{u}) = -A(\overline{u})$. With $R^-(\overline{u}) := r_-^1(\overline{u}), \ldots, r_-^{b_2}(\overline{u})]$ we assume

$$\forall \overline{u} \in \mathcal{U} : \quad \det\left[\mathbf{D}_{\overline{u}} \Psi_2(t, \overline{u}) \cdot R^-(\overline{u})\right] \neq 0. \tag{2.12b}$$

If (B), (C) , (D) holds by using (c.f. Theorem 2.2 26]) and (c.f. Theorem 2.3 29]) (2.5)is well posed.

In the next section the equations of water flow in a pipeline, used throught this thesis are presented.

2.4 Equation of water flow in pipe

A pipe is the main component for the fluid transportation in a distribution system. The hyperbolic balance laws for the water flow in a pipe will be presented. In order to develop a mathematical model for flow q, the following assumptions are made:

1. The flow is one dimensional, that is mass flow q, density ρ and pressure P depend on space x and time t.
2. Water is considered slightly compressible.
3. Pipe is laid horizontally, and completely filled up with the transported water.
4. The flow is adiabatic and isothermal (i.e., no transfer of thermal energy between fluid and pipeline will be considered, pressure changes of the fluid do not affect its temperature, also temperature changes due to the friction are neglected).
5. Friction effects are described by the Darcy–Weisbach equation (for details see Appendix) with constant value of the friction coefficient c_f 49].

The motion of water flow in a cylindrically pipe is most often described classically by a p-system.

Definition 2.4.1. *A $p-$ system consists of balance laws:*

$$\partial_t \rho + \partial_x(\rho \mathbf{u}) = 0 \tag{2.13a}$$

$$\partial_t(\rho \mathbf{u}) + \partial_x(\rho \mathbf{u}^2 + P(\rho)) + \frac{c_f}{2D\rho}(\rho \mathbf{u}) \mid \rho \mathbf{u} \mid = 0 \tag{2.13b}$$

Assuming, $q = \rho\mathbf{u}$ (2.13) can be written as :

$$\partial_t\rho + \partial_x q = 0 \tag{2.14a}$$

$$\partial_t q + \partial_x(\frac{q^2}{\rho} + P(\rho)) + \frac{c_f}{2D\rho}q \mid q \mid= 0 \tag{2.14b}$$

Furthermore (4.1) needs a closure of the equations as there as three variables ρ, q and P and two equations. Hence an equation of state for the pressure will be derived. In order to model pressure, water is considered slightly compressible.

2.4.1 Equations for slightly compressible materials

To derive the closure in terms of fluid compressiblity β given as:

$$\beta =\frac{1}{\rho}\frac{d\rho}{dP} \tag{2.15}$$

Integrate (2.15), we have

$$\rho =\rho_a e^{\beta(P-P_a)} \tag{2.16}$$

Here ρ_a is the density of fluid at P_a. By Taylor's Theorem,

$$\rho =\rho_a\{1 + \beta(P - P_a) + \frac{\beta^2(P - P_a)^2}{2!} + \cdots\} \tag{2.17}$$

By neglecting the terms of higher power, (2.17) reduced to the following:

$$\rho \simeq\rho_a\{1 + \beta(P - P_a)\} \tag{2.18}$$

rewriting (2.18)

$$p = P(\rho) =\frac{1}{\beta\rho_a}(\rho - \rho_a) + P_a \tag{2.19}$$

Hence, (2.19) is the state equation for slightly compressible fluid for the set of equations (4.1). Hence, the set of equations to describe flow of any arbitrary pipe having variables ρ, P and q can be written as:

$$\partial_t \rho + \partial_x q = 0 \tag{2.20a}$$

$$\partial_t(q) + \partial_x(\frac{q^2}{\rho} + P(\rho)) + \frac{c_f}{2D\rho} q \mid q \mid = 0, \tag{2.20b}$$

$$p = P(\rho) = \frac{1}{\beta \rho_a}(\rho - \rho_a) + P_a. \tag{2.20c}$$

The aim of this work is the modeling of water flow in switched DAE framework. For that purpose, an ODE is needed to model the water flow in pipe. Hence, model (2.20) is simplified by assuming incompressibility $\forall \rho \geq \rho_a$.

2.4.2 Incompressible flow of water in pipe

Consider an arbitrary pipe in the network. Let $t \in [t_0, T]$ denote the time $t \in \mathbb{R}^+$ and $x \in [a, b] := \Omega$ denote the space. Rewriting flow q and P functions as below:

$$q : [t_0, T] \times \Omega \longrightarrow \mathbb{R} \qquad P : [t_0, T] \times \Omega \longrightarrow \mathbb{R}$$

Now to derive quasi stationary model using following assumptions:

I: The conduit (pipe) walls are rigid, and water is incompressible ($K = \infty$, $\beta = 0$).

II: In case of steady and incompressible flow the Reynolds number (as defined in (2.3)) (Re) $<<$ 1 hence , inertial forces are negligible.

Consider (2.20) and apply assumptions,

$$\begin{aligned}&\frac{\partial \rho}{\partial t} = 0 \quad \text{incompressiblity means } \rho = \rho_a, \\ &q(x,t) = q(t)\end{aligned}$$

On applying assumptions we get:

$$\frac{dq}{dt} + \partial_x(P(x,t)) + \frac{c_f}{2D} q \mid q \mid = 0 \tag{2.21}$$

where, ρ is the water density, $\mathbf{A}, c_f$ and D are constant pipe characteristics. More precisely, for a specific pipe $\mathbf{A}$ denotes the cross sectional area, c_f is the Darcy friction factor and D is the diameter. Substitute $q = \rho_a \mathbf{u}$ to get equation of motion of fluid from (2.21) we get

$$\frac{d\mathbf{u}}{dt} + \frac{1}{\rho_a}\partial_x(P(x,t)) + \rho_a \frac{c_f}{2D} \mathbf{u} \mid \mathbf{u} \mid = 0 \tag{2.22}$$

For practical purposes it is good to write equation (2.22) in terms of two variables pressure P and mass flow Q. This can be achieved by substituting :

$$\mathbf{u} = \frac{Q}{\rho_a \mathbf{A}}$$

By using this (2.22) the equations result into:

$$\frac{dQ}{dt} + \mathbf{A}\partial_x(P(x,t)) + \frac{c_f}{2D\rho_a \mathbf{A}} Q \mid Q \mid = 0 \tag{2.23}$$

Also $\frac{\partial P}{\partial x}(t,x)$ is constant with respect to x as other terms in the equation does not depend on space x. Hence, we approximate $\frac{\partial P}{\partial x}(t,x)$ analytically as:

$$\frac{\partial P}{\partial x}(t,x) = \frac{P(t,x+h) - P(t,x)}{h} \tag{2.24}$$

where $x, x + h \in \Omega$. To find pressure at nodes, formally speaking $P(t, 0)$ and $P(t, L)$. Hence set $x = a$ and $h = b - a$ we obtain

$$\frac{\partial P}{\partial x}(t, a) = \frac{P(t, b) - P(t, a)}{b - a} \tag{2.25}$$

Insert equation (2.25) in (2.23),

$$\frac{dQ}{dt} + \frac{A}{b - a}(P(t, b) - P(t, a)) + \frac{c_f}{2D\rho_a \mathbf{A}} Q \mid Q \mid = 0 \tag{2.26}$$

In particular a domain $[0, L]$ with $x = 0$, $x = L$ as end points of pipe, the length pipe will be L. (2.26) is read as,

$$\frac{dQ}{dt} + \frac{A}{L}(P(t, L) - P(t, 0)) + \frac{c_f}{2D\rho_a \mathbf{A}} Q \mid Q \mid = 0 \tag{2.27}$$

On substituting $\mathbf{A} = \pi \frac{D^2}{4}$ in (2.27),

$$\frac{dQ}{dt} + \frac{\pi D^2}{4L}(P(t, L) - P(t, 0)) + \frac{2c_f}{\pi D^3 \rho_a \mathbf{A}} Q \mid Q \mid = 0 \tag{2.28}$$

where (2.28) is the nonlinear ODE model for incompressible flow of fluid in pipes.

2.5 Summary

To sum up; this chapter introduced the building blocks of the modeling flow of a fluid. Furthermore, the general form of equations for PDE model along with its mathematical theory is introduced. One of the goals of this work is to present two different approaches to model water flow in a pipes (compressible and incompressible). The first approach is used to model fluid flow in a water network via PDEs. The incompressible approach will be used to model a water network in the proposed framework of switched differential algebraic equations (switched DAEs). This chapter provides for the initial keeprations the aforementioned goal.

Chapter 3

Impulses in structured nonlinear switched DAEs

Switched systems are a class of hybrid systems encountered in many practical situations which involve switching between several subsystems depending on various factors 11]. Generally, a switching system consists of a family of continuous-time subsystems and a rule that supervises the switching between them 52]. These subsystems can be modeled via differential algebraic equations (DAEs), which can be linear or nonlinear depending on the dynamics of the system.

In case of linear switched DAEs each subsystem is expressed in terms of linear DAEs, for example linear electric networks with switches. Furthermore, if the system under consideration is nonlinear in nature, then it may not be able to describe using linear equations. For example, if an electrical circuit contains nonlinear elements (e.g., nonlinear resistors, diode, etc.) then each subsystem will turn out to be nonlinear DAEs. A further example is water flow in a pipe, as derived in the Section 2.4.2 which is a nonlinear ODE. In modeling water flow other components that are modeled as algebraic constraints, like reservoir at an end of pipe or a junction, which are modeled as assigning a constant pressure or flow balance, respectively. Hence, as a whole the nonlinear ODE for water flow and algebraic constraints results into a nonlinear differential algebraic equation. Furthermore, on the introduction of control element (e.g., valve, pumps) which behaves like switches in electrical circuit terms, the resulting model of the water flow will be switched nonlinear DAEs. The distributional framework is required for their solutions. The distributional solution theory for the linear switched DAEs is very much establish, but the presence of a nonlinearity makes distributional theory inapplicable. The chapter starts with a general introduction to switched linear DAEs, followed by the extension of linear switched DAEs theory to switched nonlinear DAEs.

3.1 Linear switched DAEs

Consider switched linear DAEs of the following form

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + f_{\sigma(t)}(t). \tag{3.1}$$

where $E_{\mathbf{p}}, A_{\mathbf{p}} \in \mathbb{R}^{n\times n}$, $f_{\mathbf{p}} : \mathbb{R} \to \mathbb{R}^n$ for $p \in \{1, \cdots, \mathbf{P}\}$, is a (time-dependent) inhomogeneity $\mathbf{P} \in \mathbb{N}$, and $\sigma : \mathbb{R} \to \{1, \cdots, \mathbf{P}\}$ is a piecewise constant switching signal, which is assumed to be right continuous and to have locally finite many jumps.

In particular each subsystem is a linear DAE of the form

$$E\dot{x}(t) =Ax(t) + f \tag{3.2}$$

In case of a homogeneous DAE i.e, $f = 0$ in (3.2) takes the form as given below ;

$$E\dot{x}(t) = Ax(t) \tag{3.3}$$

The origin of the theory of DAEs can be traced back to the works of Weierstrass 72] and Kronecker 48], but only in the 1960s the interest in aspects of DAEs, such as computational issues, mathematical theory, and applications has seen an significant growth. The interest for the mathematical and control aspects of DAEs is supported by extensive applications in chemical, electrical and mechanical engineering, as well as in economics, see 50] and the references therein.

It is well known that the solution of each individual DAE (3.3), evolve within a consistency space, defined as:

Definition 3.1.1 (Consistency space for (3.3))**.** The consistency space of (3.3)

$$\left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists x \quad \text{(classical) solution of (3.3)} \\ \text{with} \quad x(0) = x_0 \end{array} \right\}$$

Each x_0 is called consistence initial value.

Definition 3.1.2 (Consistency space for (3.2))**.** The consistency space of (3.2)

$$\left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists f \in \mathcal{C}^\infty \quad \text{classical solution of (3.2)} \\ \text{with} \quad x(0) = x_0 \end{array} \right\}$$

In general, at a switching time $t \in \mathbb{R}$ there does not exist a continuous extension of the solution into the future, because the value $x(t^-)$ immediately before the switch does not have to be within the consistency space corresponding to the DAE after the switch. Therefore, it is necessary to allow for solutions with jumps. However, this leads to difficulties in evaluating the derivative of the solutions. To resolve this problem the distributional framework is adopted.

3.2 Distributions

The presence of inconsistent initial values (or switching) makes it necessary to consider distributional solutions containing in particular Dirac impulses. This is problematic for switched systems as the pre-switch value does not necessarily not suitable ad fit to the post-switch algebraic constraints. To see that a distributional solution theory is required, an example is presented.

Example 3.2.1. Consider the following initial value problem

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \dot{x} = x, \qquad x(0^-) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

From above $x_1 = 0$ and $\dot{x}_1 = x_2$ for $t \geq 0$. These equations are not fit with the initial value. A solution starting from x_0 would require that the state component x_1 jumps from 1 to 0. By $\dot{x}_1 = x_2$, this implies that x_2 has to be the derivative of the jump from 1 to 0 . The classically solution will not exist in this case. Therefore briefly the necessary basic facts about distributions are recalled in the following.

The space of distributions $\mathbb{D}$ consists of all continuous linear maps (functionals) from the space of test functions $\mathcal{C}_0^\infty$ into the real numbers, where $\mathcal{C}_0^\infty$ denotes the space of smooth functions with compact support equipped with a suitable topology for further details see 62] .Distributions are also called generalised functions because any locally integrable function $f : \mathbb{R} \to \mathbb{R}$ induces a distribution via $f_\mathbb{D}$

$$f_\mathbb{D}(\varphi) := \int_\mathbb{R} f\varphi.$$

Every distribution $D \in \mathbb{D}$ can be differentiated via

$$D'(\varphi) := -D(\varphi')$$

and for every differentiable $f : \mathbb{R} \to \mathbb{R}$ it holds

$$(f_{\mathbb{D}})' = (f')_{\mathbb{D}}.$$

The most famous distribution which is not induced by a function is the Dirac delta distribution, given as: (3.4):

$$\delta(\varphi) = \varphi(0), \quad \varphi \in \mathcal{C}_0^\infty \tag{3.4}$$

It is easily seen that the Dirac impulse is the distributional derivative of the Heaviside function

$$\mathbb{1}_{[0,\infty)}(t) := \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

The distributional derivative of Heaviside function is

$$\delta := (\mathbb{1}_{[0,\infty)_{\mathbb{D}}})'$$

Distributions can be multiplied with smooth functions via

$$(\alpha D)(\varphi) := D(\alpha\varphi) \quad D \in \mathbb{D}, \alpha \in \mathcal{C}^\infty \tag{3.5}$$

As shown in 68] the whole space of distributions $\mathbb{D}$ is not a suitable solution space for switched DAEs and it is necessary to introduce a appropriate subspace, namely the space of piecewise-smooth distributions.

Piecewise-smooth distributions

The space of piecewise smooth distributions has been introduced to describe switched linear DAEs. Inconsistent initial values are the reason of the Dirac impulses, for the mathematical description of these impulses distributional framework is required. To understand distribution theory some knowledge about the formalities of distribution theory are needed to understand the concept of piecewise-smooth distribution. First, the space of piecewise-smooth functions is defined as:

$$\mathcal{C}^{\infty}_{\text{pw}} := \left\{ \alpha = \mathbb{1}_{[t_i,t_{i+1})}\alpha_i \;\middle|\; \begin{array}{l} \{t_i \in \mathbb{R} | i \in \mathbb{Z}\} \text{ locally finite} \\ t_i < t_{i+1}, \alpha_i \in \mathcal{C}^{\infty}, i \in \mathbb{Z} \end{array} \right\}.$$

where $\mathbb{1}_{\mathrm{I}}$ denotes the characteristic function of the interval $\mathrm{I} \subseteq \mathbb{R}$. Switched systems can be interpreted as time-varying linear systems with piecewise-constant coefficient matrices. Hence one might aim to generalize the multiplication (3.5) also to piecewise smooth functions. However for the general distributions this is not possible in a consistent way 67]. For this reason the smaller space of *piecewise smooth distributions* instead of the whole space $\mathbb{D}$.

Definition 3.2.2 (Piecewise smooth distribution)**.** *The space of piecewise-smooth distributions is defined as:*

$$\mathbb{D}_{\text{pw}\mathcal{C}^{\infty}} := \left\{ D = f_{\mathbb{D}} + \sum_{\tau \in T} D_{\tau} \;\middle|\; \begin{array}{l} f \in \mathcal{C}^{\infty}_{\text{pw}},\ T \subseteq \mathbb{R} \text{ is discrete} \\ \forall \tau \in T \\ D_{\tau} \in \mathit{span}\{\delta_{\tau}, \delta'_{\tau}, \delta''_{\tau}, \ldots\} \end{array} \right\}.$$

where δ_{τ} is the Dirac impulse located at $\tau \in T$.

In other words a piecewise-smooth distributions is the sum of a piecewise-smooth function and isolated impulses (composed of Dirac-impulses and their derivatives). It is easily seen, that the space of piecewise-smooth distributions is closed under differentiation and therefore recovers the essential property of the space of distributions. Furthermore, on each finite interval, every piecewise-smooth distribution is the finite derivative of a piecewise-smooth function.

In contrast to general distributions, a piecewise-smooth distribution $D = f_{\mathbb{D}} + \sum_{\tau \in T} D_\tau$ can be evaluated at any $t \in \mathbb{R}$ in the following three different ways:

$$\begin{aligned} D(t^+) :=& f(t^+), \quad D(t^-) := f(t^-) \\ D[t] :=& \begin{cases} D_t, & t \in T, \\ 0, & t \notin T. \end{cases} \end{aligned}$$

where $f(t^\pm)$ denotes the left/right limit of the piecewise-smooth function f at $t \in \mathbb{R}$. Furthermore the restriction of a piecewise-smooth distribution $D = f_{\mathbb{D}} + \sum_{\tau \in T} D_\tau$ to any interval $\mathfrak{I} \subseteq \mathbb{R}$ is well defined by

$$D_{\mathfrak{I}} := (f_{\mathfrak{I}})_{\mathbb{D}} + \sum_{\tau \in T \cap \mathfrak{I}} D_\tau$$

where $f_{\mathfrak{I}}(t) = f(t)$ if $t \in \mathfrak{I}$ and $f(t) = 0$ otherwise. Further construction can be seen in 69], as solutions of the switched DAE (3.1) distributions (generalised functions), in particular Dirac impulses, are considered. For this, one have to assume that the switching signal has only a locally finite set of switching times.

3.3 Quasi Weierstrass form

The solutions of the linear DAE (3.3) is strongly related to the concept of regularity of the matrix pencil (E, A) 60]. The regularity is defined as:

Definition 3.3.1. *The matrix pair* (E, A) *is regular, i.e.* $m = n$ *and the polynomial* $\det(sE - A)$ *is not the zero polynomial* $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.

An equvivalence of of the regularity of the matrix pair with quasi Weierstrass form (QWF) is stated as: The matrix pair (E, A) is regular if and only if there exist invertible transformation matrices $S, T \in \mathbb{R}^{n\times n}$ which put (E, A) into quasi Weierstrass form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{3.6}$$

where $N \in \mathbb{R}^{n_2\times n_2}$, with $0 \leq n_2 \leq n$ is a nilpotent matrix, $J \in \mathbb{R}^{n_1\times n_1}$ with $n_1 = n - n_2$ is some matrix and I is the identity matrix of the appropriate size. It can be shown 12] that the transformation matrices can be obtained as $T = [V,\ W]$ and $S = [EV,\ AW]^{-1}$, where $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$ with $\mathcal{V}^*$ and $\mathcal{W}^*$ are obtained via the so called Wong sequences, see Appendix 7:

$$\mathcal{V}_0 := \mathbb{R}^n,\ \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i),\ i \in \mathbb{N},\ \mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i,$$
$$\mathcal{W}_0 := \{0\},\ \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i),\ i \in \mathbb{N},\ \mathcal{W}^* := \bigcap_{i\in\mathbb{N}} \mathcal{W}_i.$$

Here, A^{-1} and E^{-1} do not stand for the corresponding invertible matrices but for the preimage operators and n_1 is the dimension of the space $\mathcal{V}^*$, and n_2 is the dimension of the space $\mathcal{W}^*$. The space $\mathcal{V}^*$ represents the manifold, say it has consistency space $\mathfrak{C}_{(E,A)}$, in which $\mathcal{C}^1$ solutions of the differential algebraic equation evolve, i.e. $\mathfrak{C}_{(E,f)} = \mathcal{V}^*$. x is solution $x(t) \in \mathfrak{C}_{(E,A)}$ for all t. The knowledge of the two limiting spaces $\mathcal{V}^*$, and $\mathcal{W}^*$, used to obtain an explicit solution formula.

3.3.1 Projectors and flow matrices

To formulate the explicit solution formula there is a need to introduce certain projectors and matrices 65]. From the quasi Weierstrass form (3.6) the so called consistency projector can be defined:

$$\varPi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

with the identity matrix $I \in \mathbb{R}^{n_1 \times n_1}$. The consistency projector characterises the space where all solutions evolve within, i.e. $\mathfrak{C}_{(E,A)} = \operatorname{im} \varPi$. Hence it plays a role when considering inconsistent initial values, i.e. when the initial value $x(0^-)$ does not belong to the consistency space.

The differential projector is defined as

$$\varPi^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

with $I \in \mathbb{R}^{n_1 \times n_1}$. Using $\varPi^{\text{diff}}$ the flow matrix A^{diff} is defined as:

$$\begin{aligned} A^{\text{diff}} :=& T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \\ A^{\text{diff}} :=& \varPi^{\text{diff}} A. \end{aligned}$$

where $J \in \mathbb{R}^{n_1 \times n_1}$. Note that

$$A^{\text{diff}} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T^{-1} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} SA = \varPi^{\text{diff}} A.$$

Due to the particular structure of the consistency projector the following property holds

$$A^{\text{diff}} \varPi = \varPi A^{\text{diff}} = A^{\text{diff}}. \tag{3.7}$$

Due to the presence of the so called consistency space the solution of (3.3) may present jumps and also impulses when it is “switched on” at time $t = 0$. To deal with the impulses in the solution formula of DAEs the so called impulsive projectors are introduced

$$\Pi^{\mathrm{imp}} = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

with the matrix $I \in \mathbb{R}^{n_2 \times n_2}$. The projectors do not depend on the specific choice of S and T, furthermore the consistency projector is an idempotent matrix, i.e. $\Pi^2 = \Pi$, in contrast to the the differential and impulse projectors.

3.3.2 Homogeneous DAEs

The regularity of the matrix pair (E, A) and the definition of the corresponding consistency projector and flow matrix, allows to prove that each solution of the homogeneous DAEs solves the following ordinary differential equation

$$\dot{x}(t) = A^{\mathrm{diff}} x(t), \quad t \in \mathbb{R}_+,$$

by assuming a consistent initial value, i.e. $x_0 \in \mathcal{V}^*$. When a DAE is “switched on” at $t = 0$, the initial condition $x(0^-)$ is arbitrary, then the consistency projector is used. Hence the unique solution $x(t)$ on $(0, \infty)$ is given by:

$$x(t) = e^{A^{\mathrm{diff}} t} x(0^+), \quad x(0^+) \in \mathcal{V}^* \tag{3.8}$$

where $x(0^+) = \Pi x(0^-)$, 70].

3.3.3 Initial trajectory problems (ITP) and switched DAEs

In the presence of switches,the initial conditions do not have to be consistent, then no solution with this initial data exists. It is therefore required to produce a precise meaning to the solution to an inconsistent initial value problem. In order to do that, notion of the initial trajectory problem is presented as follows:

Theorem 3.3.2 (68, 69]). *Let* $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$, $f \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *and* (E, A) *be a regular matrix pair. Then the linear initial trajectory problem (ITP)*

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax + f)_{[0,\infty)} \end{aligned} \tag{3.9}$$

has a unique solution $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$. *If* f *is induced by a piecewise-smooth function the unique solution* x *satisfies, for* $t \in (0, \infty)$,

$$\begin{aligned} x(t^+) = e^{A^{\mathrm{diff}}t}\Pi x^0(0^-) + &\int_0^t e^{A^{\mathrm{diff}}(t-s)}\Pi^{\mathrm{diff}} f(s)\mathrm{d}s \\ &- \sum_{i=0}^{n-1} (E^{\mathrm{imp}})^i \Pi^{\mathrm{imp}} f^{(i)}(t^+) \end{aligned}$$

and

$$x[0] = -\sum_{i=0}^{n-1} (E^{\mathrm{imp}})^i x^0(0^-)\delta^{(i)} - \sum_{i=0}^{n-1} (E^{\mathrm{imp}})^i \sum_{j=0}^{i} f^{(i-j)}(0^+)\delta^{(j)} \tag{3.10}$$

where $\delta^{(i)}$ *denotes the* i^{th} *derivative of the Dirac impulse* δ. *In particular, if* $f = 0$, *then*

$$x(0^+) = \Pi x(0^-).$$

By reapplying the ITP at each switching time the following result for switched DAEs immediately established.

Corollary 3.3.3. *The switched system*

$$E_\sigma \dot{x} = A_\sigma x + f$$

with regular matrix pairs $(E_{\mathbf{p}}, A_{\mathbf{p}})$, $\mathbf{p} \in \{1, \cdots, \mathbf{P}\}, \mathbf{P} \in \mathbb{N}$ *has a unique solution for every* $f \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *and every initial trajectory* $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$.

In particular, the jumps and Dirac impulses induced by the switches are uniquely determined.

A brief introduction to the solution theory of linear switched DAEs have been described in the previous section. In the next Section, the extension of this solution theory to the switched nonlinear DAEs is presented.

3.4 Switched nonlinear DAE

In general a switched nonlinear DAEs is of the form:

$$E_{\sigma(t)}(x(t))\dot{x}(t) = g_{\sigma(t)}(t, x(t)). \tag{3.11}$$

where, σ is a switching signal, E_p and A_p describe the flow matrices and g is a nonlinearity. In particular, one assumes that each subsystem can be written in the form

$$E(x)\dot{x} = g(t, x), \tag{3.12}$$

The solution theory for the linear switched DAEs in the distributional framework is very much established. The distributional solution theory background is presented in chapter 2. Rewriting (3.11) in the following form:

$$E_\sigma \dot{x} = A_\sigma x + g_\sigma(t, x) + f_\sigma(t). \tag{3.13}$$

where $E_{\mathbf{p}}, A_{\mathbf{p}} \in \mathbb{R}^{n\times n}$, $g_{\mathbf{p}} : \mathbb{R}^n \to \mathbb{R}^n$ for $\mathbf{p} \in \{1, \cdots, \mathbf{P}\}$, $\mathbf{P} \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a (time-dependent) inhomogeneity and $\sigma : \mathbb{R} \to \{1, \cdots, \mathbf{P}\}$ is a piecewise constant switching signal, which is assumed to be right continuous and to have locally finitely many jumps. In particular each subsystem is a nonlinear DAE of the form

$$E\dot{x} =Ax + g(t, x) + f(t) \tag{3.14}$$

Remark 3.4.1 (Motivation for switched nonlinear DAEs)**.** While modeling hydraulic transients in water distribution systems in the framework of switched DAEs 45], each subsystem turns out to be a nonlinear DAE of the form (3.14), and modeling of transients (e.g., changing valve or pump settings, etc.), give rise to switched nonlinear DAE of the form (3.13). This is our main motivation for studying the solution theory of switched nonlinear DAEs, but certainly these results will likewise be applicable in other areas.

The existing solution theory available for switched nonlinear DAEs in 53] excludes the presence of Dirac impulses by definition; however, when studying e.g. the water hammer effect in water distribution networks these impulses are crucial because if a Dirac impulse occurs in the solution x of (3.13) then it is unclear how $g_\sigma(x)$ has to be evaluated in general (e.g. what is the sine of a Dirac impulse).

Remark 3.4.2 (Idea of the solution concept)**.** If the nonlinearity is not depending on all state variables, this property is called sparsity of nonlinearity. Especially in the state variables in which Dirac is expected may not present as the argument. Meaning that if $x_s \in \mathcal{C}^\infty_{\text{pw}}$ then $g(x_s)$ is nothing but just simple evaluation of function. The formalisation of this concept is shown in the following Section.

3.4.1 Solution concept

The first challenge in studying the nonlinear switched DAE (3.13)

$$E_\sigma \dot{x} = A_\sigma x + g_\sigma(t, x) + f$$

within a distributional solution framework is the nonlinear evaluation $g_\sigma(x)$ for distributional x. Due to the linear nature of the space of distributions it is not possible to have a general nonlinear evaluation of distributions without leaving the space of distributions. The approach to overcome this problem is the assumption that the nonlinearity is sparse in some sense and that g_σ is independent of the possible impulsive parts of x. This is made precise in the following definition.

Definition 3.4.3. *Consider a nonlinear switched DAE of the form (3.13) with $f \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$. We make the following sparsity assumption for all $\mathbf{p} \in \{1, \dots, \mathbf{P}\}$*

($G_\mathbf{p}$) $\exists \overline{g}_\mathbf{p} : \mathbb{R}^{m_\mathbf{p}} \to \mathbb{R}^{n_\mathbf{p}}\ \exists \mathcal{M}_\mathbf{p} \in \mathbb{R}^{m_\mathbf{p} \times n}\ \exists \mathcal{N}_\mathbf{p} \in \mathbb{R}^{n \times n_\mathbf{p}}\ \forall \xi \in \mathbb{R}^n :$ $\boxed{g_\mathbf{p}(\xi) = \mathcal{N}_\mathbf{p} \overline{g}_\mathbf{p}(\mathcal{M}_\mathbf{p}\xi)}$ *with* $m_\mathbf{p} \leq n, \quad n_\mathbf{p} \leq n.$

Then $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ is a solution of (3.13), if

A1: $\mathcal{M}_\sigma x$ is impulse-free, i.e. $(\mathcal{M}_\sigma x)[t] = 0$ for all $t \in \mathbb{R}$ or, in other words, there exists a piecewise-smooth function $\overline{x} : \mathbb{R} \to \mathbb{R}^n$ such that $\mathcal{M}_\sigma x$ is induced by the piecewise-smooth function $\mathcal{M}_\sigma \overline{x}$;

A2: $\mathcal{N}_\sigma \overline{g}_\sigma(\mathcal{M}_\sigma \overline{x})$ is a piecewise-smooth function; and

A3: $E_\sigma \dot{x} = A_\sigma x + G_x + f$ holds as an equality within the space of piecewise-smooth distributions where G_x is the distribution induced by the piecewise-smooth function $\mathcal{N}_\sigma \overline{g}_\sigma(\mathcal{M}_\sigma \overline{x})$.

Remark 3.4.4. The choice of the matrices $\mathcal{M}_\mathbf{p}$ and $\mathcal{N}_\mathbf{p}$ in assumption ($G_\mathbf{p}$) is not unique; in fact, it is always possible to chose $\mathcal{M}_\mathbf{p} = \mathcal{N}_\mathbf{p} = I$ and $\overline{g} = g$. However, this trivial choice will prohibit Dirac impulses in the solution, i.e, in this case $\mathcal{M}_\mathbf{p} x$ will be impulse free if and only if x is itself impulse free. Therefore it is not suitable for our purpose of studying nonlinear switched DAEs in the presence of impulses.

Furthermore, it is actually not correct to just say "x is a solution of (3.13)", because being a solution depends on the choice of $\mathcal{M}_\mathbf{p}$ and $\mathcal{N}_\mathbf{p}$. Consequently, "x is a solution if" and not "x is a solution if, and only if,", because a given x which does not satisfy conditions A1, A2, and A3 may satisfy them for different matrices $\mathcal{M}_\mathbf{p}$ and $\mathcal{N}_\mathbf{p}$ (the suitable choice may actually depend on x). Even if for a given x there does not exist matrices $\mathcal{M}_{\mathbf{p}}$ and $\mathcal{N}_{\mathbf{p}}$ such that A1, A2 and A3 holds, it may still be possible that with a suitably defined nonlinear distributional evaluation x could be seen as a solution of (3.13).

As in linear case inconsistent initial conditions gives rise to initial trajectory problems (ITP) see in the Section 3.3.3, similarly, here nonlinear initial problem (nonITP) is introduced.

Definition 3.4.5 (Solution of nonlinear initial trajectory problem (nonITP))**.** A piecewise smooth distribution x is called a nonITP solution in the sense of definition 3.4.3 with initial trajectory $x^0 \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, and if $f \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$ is induced by a piecewise-smooth function, x fulfills the nonlinear initial trajectory problem (nonITP).

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax + f + g(t,x))_{[0,\infty)} \end{aligned} \tag{3.15}$$

i.e. x is local solution of 3.14 on $[0,\infty)$ which coincides with the initial trajectory x^0 on $(-\infty,0)$.

Next section will establish the existence of uniqueness result for the solution of nonITP 3.15.

3.4.2 Existence and uniqueness of solutions

Similar as in the linear case an existence uniqueness result for nonlinear ITPs will be established:

Theorem 3.4.6. *For $\omega \in (0,\infty]$, consider the local nonlinear (nonITP)*

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\omega)} &= (Ax + g(x) + f)_{[0,\omega)} \end{aligned} \tag{3.16}$$

with initial trajectory $x^0 \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$. We make the following assumptions:

(R): (E, A) *is regular.*

(F): *The inhomogeneity* f *is induced by a piecewise-smooth function* $\overline{f} : \mathbb{R} \to \mathbb{R}^n$, *i.e.* $f = \overline{f}_{\mathbb{D}}$.

(S): $g : \mathbb{R}^n \to \mathbb{R}^n$ *is locally Lipschitz continuous and piecewise-smooth.*

(G): $\exists \overline{g} : \mathbb{R}^{m_g} \to \mathbb{R}^{n_g}\ \exists \mathcal{M} \in \mathbb{R}^{m_g \times n}\ \exists \mathcal{N} \in \mathbb{R}^{n \times n_g}\ \forall \xi \in \mathbb{R}^n : \quad \boxed{g(\xi) = \mathcal{N}\overline{g}(\mathcal{M}\xi)}.$

(M): $\mathcal{M}E^{\mathrm{imp}} = 0$.

(N): $\operatorname{im} \mathcal{N} \subseteq \operatorname{im} E$.

If all these assumptions are satisfied, then there exists $\omega > 0$ *such that the local nonlinear ITP* (3.16), *has a unique solution* $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *(in an analogue sense of Definition 3.4.3) on* $(-\infty, \omega)$.

The proof of this theorem is based on the following lemma.

Lemma 3.4.7 (Modified QWF)**.** Assume the QWF of a regular matrix pair (E, A) has the special form

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N_1 & N_2 \end{bmatrix}, \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right) \tag{3.17}$$

where $[N_1, N_2]$ has full row rank and N_2 is nilpotent. Write $T = [T^v, T_1^w, T_2^w]$ and $S^\top = [S^{v\top}, S_1^{w\top}, S_2^{w\top}]$ corresponding to the block sizes of (3.17). Then for any $\mathcal{M}$ and $\mathcal{N}$ as in assumption (G) the following equivalences hold

$$\begin{aligned} \mathcal{M}E^{\mathrm{imp}} = 0 &\iff \mathcal{M}T_2^w = 0, \\ \operatorname{im} \mathcal{N} \subseteq \operatorname{im} E &\iff S_1^w \mathcal{N} = 0. \end{aligned}$$

Proof. The first equivalence is shown as follows

$$\begin{aligned}
\mathcal{M}E^{\text{imp}} = \mathcal{M}\cdot T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N_1 & N_2 \end{bmatrix} T^{-1} = 0 \\
\iff \mathcal{M}[T^v,\ T_1^w,\ T_2^w] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N_1 & N_2 \end{bmatrix} = 0 \\
&\iff \mathcal{M}[0,\ T_2^w N_1,\ T_2^w N_2] = 0 \\
&\iff \mathcal{M}T_2^w[N_1,\ N_2] = 0 \\
&\overset{*}{\iff} \mathcal{M}T_2^w = 0.
\end{aligned}$$

where equivalence $*$ is a consequence from the full row rank of $[N_1,\ N_2]$. In order to derive the second equivalence, it has been observed first that

$$\operatorname{im} {S_1^w}^\top = \ker E^\top \text{ or, equivalently, } \ker S_1^w = \operatorname{im} E$$

hence the second equivalence follows from

$$\begin{aligned}
& S_1^w \mathcal{N} = 0, \\
\iff & \operatorname{im} \mathcal{N} \subseteq \ker S_1^w, \\
\iff & \operatorname{im} \mathcal{N} \subseteq \operatorname{im} E.
\end{aligned}$$

■

Proof of Theorem 3.4.6. The proof proceeds in several steps. *Step 1:* The matrices S and T are constructed such that (3.17) holds.
Let $\widehat{\mathcal{V}}^*$ and $\widehat{\mathcal{W}}^*$ be the Wong limits of the transposed matrix pair $(E^\top, A^\top)$ and let $n_1 := \dim \widehat{\mathcal{V}}^*$, $n_2^1 := \dim \ker E^\top$, and $n_2^2 := \dim \widehat{\mathcal{W}}^* - n_2^1$. Since by construction $\ker E^\top = \widehat{\mathcal{W}}_1 \subseteq \widehat{\mathcal{W}}^*$ full column rank matrices $\widehat{V}$ and

$\widehat{W} = [\widehat{W}_1, \widehat{W}_2]$ can be chosen such that

$$\operatorname{im} \widehat{V} = \widehat{\mathcal{V}}^*, \ \operatorname{im} \widehat{W} = \widehat{\mathcal{W}}^*, \ \operatorname{im} \widehat{W}_1 = \ker E^\top.$$

With

$$S := [\widehat{V}, \widehat{W}_1, \widehat{W}_2]^\top, \quad T := [E^\top \widehat{V}, A^\top \widehat{W}_1, A^\top \widehat{W}_2]^{-\top}$$

it follows that (SET, SAT) is the transpose of the QWF of $(E^\top, A^\top)$ and hence a QWF itself. Furthermore, by construction $\widehat{W}_1 E = 0$, which shows that (SET, SAT) has the form (3.17) and it remains to be shown that $[N_1, N_2]$ has full row rank. Assume the contrary, then there exists a vector $v \in \mathbb{R}^{n_2^2} \setminus \{0\}$ with $v^\top [N_1, N_2] = 0$ and, therefore,

$$0 = [0, 0, v^\top] \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & N_1 & N_2 \end{bmatrix} = [0, 0, v^\top] SET,$$

which is equivalent to $0 = [0, 0, v^\top][\widehat{V}, \widehat{W}_1, \widehat{W}_2]^\top E = 0$. Hence $v^\top \widehat{W}_2^\top E = 0$, or equivalently, $E^\top \widehat{W}_2 v = 0$ which implies that

$$\{0\} \neq \operatorname{im} \widehat{W}_2 \cap \ker E^\top = \operatorname{im} \widehat{W}_2 \cap \operatorname{im} \widehat{W}_1.$$

This contradicts full rank of $\widehat{W} = [\widehat{W}_1, \widehat{W}_2]$ and Step 1 is complete.
Step 2: Rewriting the nonlinear DAE in coordinates corresponding to the QWF (3.17).
Let $\left(\begin{smallmatrix} v \\ w_1 \\ w_2 \end{smallmatrix}\right) := T^{-1} x$ then $E\dot{x} = Ax + g(x) + f$ is equivalent to

$$SET \left(\begin{smallmatrix} v \\ w_1 \\ w_2 \end{smallmatrix}\right) = SAT \left(\begin{smallmatrix} v \\ w_1 \\ w_2 \end{smallmatrix}\right) + Sg\left(T\left(\begin{smallmatrix} v \\ w_1 \\ w_2 \end{smallmatrix}\right)\right) + Sf$$

Choosing $S^\top = [S^{v\top}, S_1^{w\top}, S_2^{w\top}]$ and $T = [T^v, T_1^w, T_2^w]$ as in Step 1, the nonITP (3.16) is therefore equivalent to

$$\begin{aligned} v_{(-\infty,0)} &= v^0_{(-\infty,0)} \\ \dot v_{[0,\infty)} &= (Jv + \mathcal{N}^v \overline{g}\,(\mathcal{M}x) + f^v)_{[0,\omega)} \end{aligned} \tag{3.18a}$$

$$\begin{aligned} w_{1\,(-\infty,0)} &= w^0_{1\,(-\infty,0)} \\ 0 &= (w_1 + \mathcal{N}_1^w \overline{g}\,(\mathcal{M}x) + f_1^w)_{[0,\omega)} \end{aligned} \tag{3.18b}$$

$$\begin{aligned} w_{2\,(-\infty,0)} &= w^0_{2\,(-\infty,0)} \\ (N_1\dot w_1 + N_2 \dot w_2)_{[0,\infty)} &= (w_2 + \mathcal{N}_2^w \overline{g}\,(\mathcal{M}x) + f_2^w)_{[0,\omega)} \end{aligned} \tag{3.18c}$$

where $\begin{pmatrix} v^0 \\ w_1^0 \\ w_2^0 \end{pmatrix} := T^{-1}x^0$, $\begin{pmatrix} f^v \\ f_1^w \\ f_2^w \end{pmatrix} = Sf$ and $\begin{bmatrix} \mathcal{N}^v \\ \mathcal{N}_1^w \\ \mathcal{N}_2^w \end{bmatrix} = S\mathcal{N}$.

Step 3: Existence and uniqueness of solutions.

Assumption (N) together with Lemma 3.4.7 yields that $\mathcal{N}_1^w = 0$, hence the nonITP (3.18b) simplifies to

$$\begin{aligned} w_{1\,(-\infty,0)} &= w^0_{1\,(-\infty,0)} \\ 0 &= (w_1 + f_1^w)_{[0,\omega)} \end{aligned}$$

which clearly has the unique solution

$$w_1 = w^0_{1\,(-\infty,0)} - f_1^w{}_{[0,\omega)}.$$

Note that w_1 is a piecewise-smooth function (and not a distribution) on $[0,\omega)$. We can plug this solution into (3.18a) and take into account assumption (M) together with Lemma 3.4.7 to obtain

$$\begin{aligned} v_{(-\infty,0)} &= v^0_{(-\infty,0)} \\ \dot v_{[0,\infty)} &= h(\cdot, v)_{[0,\omega)} \end{aligned}$$

where

$$h(t,v) = Jv + \mathcal{N}^v \overline{g}(\mathcal{M}T_1^w v + \mathcal{M}T_1^w w_1(t)) + f_1^w(t),$$

i.e. (3.18a) is a usual ODE where h is smooth in v (in particular, locally Lipschitz) and piecewise-smooth in t (in particular, measurable), hence classical ODE solution theory guarantees existence and uniqueness of a (local) solution v. Note that v is a piecewise-smooth and absolutely continuous function on $[0,\omega)$. Finally, (3.18c) can be written as

$$\begin{aligned} w_{2(-\infty,0)} &= w^0_{2(-\infty,0)} \\ (N_2\dot{w}_2)_{[0,\infty)} &= \left(w_2 + \widetilde{f_2^w}\right)_{[0,\omega)} \end{aligned}$$

where

$$\widetilde{f_2^w} = f_2^w - N_1\dot{w}_1 + \mathcal{N}_2^w \overline{g}(\mathcal{M}T^v v + \mathcal{M}T_1^w w_1).$$

Hence (3.18c) becomes a usual nilpotent DAE nonlinear ITP with (possibly distributional) inhomogeneity $\widetilde{f_2^w}$ and has a unique (distributional) solution on $(-\infty,\omega)$. ■

3.4.3 Further remarks on Theorem 3.4.6

Remarks 3.4.8. We want to discuss in the following how the assumptions of Theorem 3.4.6 may be relaxed.

(R) In linear DAEs the regularity assumption on (E,A) is necessary and sufficient for existence and uniqueness of solutions. Since no strong assumptions on the nonlinearity g is made it is not excluded that g still contains a linear component. In the extreme case $g(x) = Mx$ for some matrix M, the regularity of the matrix pair $(E, A+M)$ is more or less independent of the regularity of (E,A). Hence in a nonlinear setup without further restrictions on g regularity of (E,A) is neither necessary nor sufficient for existence and uniqueness of solutions; however it allows us to use a coordinate transformation which reveals structural aspects.

(F) Requiring that the inhomogeneity does not contain Dirac impulses is important to ensure that the solution w_1 of (3.18b) does not contain Dirac impulses, because w_1 is plugged into the nonlinearity. Furthermore, classical solvability of the nonlinear ODE (3.18a) is only guaranteed for non-impulsive inhomogeneities. However, in the context of impulsive systems one may allow Dirac-impulses in f^v (but not derivatives of Dirac impulses) then the solution exhibit jumps. In (3.18c) the presence of Dirac impulses (and its derivatives) in f_2^w isn't a problem at all.

(S) Local Lipschitz continuity is needed to have existence and uniqueness of solutions of the nonlinear ODE (3.18a). Additionally piecewise-smoothness is assumed to ensure that condition A2 in the solution Definition 3.4.3 is satisfied.

(M) The intuition behind this assumption is that due to (3.10) the impulsive parts in the solution x of a (linear) DAE in response to an inconsistent initial value is in the image of E^{imp}. Hence if $\mathcal{M}E^{\mathrm{imp}} = 0$ then the nonlinearity satisfying (G) doesn't "see" the possible Dirac impulses in x and can therefore be evaluated even for distributional x. A convenient consequence of (M) is the ability to solve (3.18a) and (3.18b) independently of (3.18c).

(N) This assumption was used in the proof to show that (3.18b) has a unique solution which then can be plugged into (3.18a) as an inhomogeneity. If (M) holds, one can significantly relax (N) by just requiring that the nonlinear algebraic equation

$$0 = w_1 + \mathcal{N}_1^w \overline{g}(\mathcal{M}T^v v + \mathcal{M}T_1^w w_1) + f_1^w \tag{3.19}$$

is uniquely solvable for w_1 in terms of v and f_1^w or in other words the combined DAE (3.18a), (3.18b) (which due to (M) is independent of w_2) has index one. The problem with this index one assumption is that it is depending on $\overline{g}$ and may be hard to check in the original coordinates.

In the context of switched DAEs usually a global solutions is required, i.e. in order to apply Theorem 3.4.6 to (3.13) for this purpose an additional assumption is needed to exclude the occurrence of finite escape time. From the equivalent representation of each ITP in the form (4.17) it becomes clear that the only source for finite escape time is the nonlinearity in (3.18a). Therefore, it is sufficient to make the following assumption for each $p \in \Sigma$:

(∞_p) All solutions $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ of the nonITP corresponding to mode p do not exhibit finite escape time, i.e. $\omega = \infty$.

Provided all assumptions of Theorem 3.4.6 are satisfied, a sufficient condition for existence of global solutions is *global* Lipschitz continuity of the nonlinear term g. However, in water networks the nonlinearity is quadratic and hence not globally Lipschitz (in that case the nonlinearities are friction terms and hence have a stabilising effect and do not produce finite escape time). In general, it is difficult to formulate non-conservative conditions ensuring global solutions. In the following, the notion is ($\mathrm{R}_\mathbf{p}$), ($\mathrm{S}_\mathbf{p}$), ($\mathrm{M}_\mathbf{p}$), ($\mathrm{N}_\mathbf{p}$) for the corresponding conditions (R), (S), (M), (N) for mode $\mathbf{p} \in \Sigma$. We can now formulate our main existence and uniqueness result for solutions of switched nonlinear DAEs of the form (3.13) as a corollary of Theorem 3.4.6.

Corollary 3.4.9. *Consider the switched DAE* (3.13) *satisfying conditions ($R_\mathbf{p}$), (F), ($S_\mathbf{p}$), ($G_\mathbf{p}$), ($M_\mathbf{p}$), ($N_\mathbf{p}$), ($\infty_\mathbf{p}$) for each mode $\mathbf{p} \in \Sigma$. Then for any initial trajectory $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ on* $(-\infty, 0)$*, there exists a unique distributional solution $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ of* (3.13) *(in the sense of Definition 3.4.3).*

Remark 3.4.10. The assumption (∞_p) is usually too strong because it suffices that the local solution of mode i on $[t_i, t_i + \omega_i)$ covers the (usually finite) interval $[t_i, t_{i+1}]$. Furthermore, not all initial values for mode i have to be considered, only the consistent ones from the previous mode. The advantage of condition (∞_p) is the independence of the switching signal, i.e. existence and uniqueness of solutions can be guaranteed for arbitrary switching signals.

3.5 Examples

In this Section, some examples are discussed which satisfies all the assumptions of Theorem 3.4.6.

3.5.1 Academic example with nontrivial nonlinearity

This academic example is presented to analyse the application of Theorem 3.4.6 to a nonlinear DAE with a non trivial and an interesting nonlinearity. Therefore consider the ITP (3.16) with

$$E=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A=\begin{bmatrix} 0 & c_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 & -c_1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -c_1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$g(x) = \begin{pmatrix} c_2(x_1+x_2+x_4)^2+c_3x_1^2 \\ c_4(x_4-x_3)^3 \\ c_5x_6^4-c_6x_1^2 \\ c_7(x_4-x_6)^3 \\ c_8x_1^2 \\ 0 \\ 0 \end{pmatrix}, \quad f = 0.$$

It is easily verified that conditions (R), (F), (S) are satisfies and S, T, and E^{imp} are calculated as follows:

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ -\alpha & 1-\alpha & 1 & -1 & -2 & -\alpha & \alpha \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & -\alpha & 1-\alpha & 0 & -\alpha \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & c_1+1 & c_1+1 \\ 0 & 1 & -1 & 0 & 0 & c_1+1 & c_1 \\ 0 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -c_1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$E^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha & -\alpha & 0 & -\alpha & 0 & \alpha & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & -\alpha & 0 \end{bmatrix} \quad \text{with} \quad \alpha = \frac{1}{c_1}.$$

To satisfy condition (G) $\mathcal{M}$ and $\mathcal{N}$ are choosen as:

$$\mathcal{M} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{N} = \begin{bmatrix} c_2 & 0 & 0 & 0 & c_3 \\ 0 & c_4 & 0 & 0 & 0 \\ 0 & 0 & c_5 & 0 & -c_6 \\ 0 & 0 & 0 & c_7 & 0 \\ 0 & 0 & 0 & 0 & c_8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{g}(\xi) = \begin{pmatrix} (\xi_1)^2 \\ (\xi_2)^3 \\ (\xi_3)^4 \\ (\xi_4)^3 \\ (\xi_5)^2 \end{pmatrix} \quad \text{with} \quad \xi = \mathcal{M}x = \begin{pmatrix} x_1+x_2+x_4 \\ x_4-x_3 \\ x_6 \\ x_4-x_6 \\ x_1 \end{pmatrix}.$$

for which $g(x) = \mathcal{N}\bar{g}(\mathcal{M}x)$ holds. With this choice it is easily checked that (M) and (N) hold. Altogether, the assumptions of Theorem 3.4.6 hold and conclusion is that for any initial trajectory there is a unique local distributional solution of the nonlinear ITP (3.16).

Remark 3.5.1. The nonzero rows of E^{imp} correspond to state variables containing Dirac impulses. For the previous example, this means x_5 and x_7 in general contain impulses. Hence, if x_5 or x_7 explicitly appear in the nonlinearity, then the assumption (M) will not be satisfiable and Theorem 3.4.6 will not be applicable.

3.5.2 Nonlinear RL electrical circuit

Example 3.5.2. A nonlinear resistor is often used in electrical modeling of physical processes e.g., chaotic processes. The characteristics for nonlinear resistors using nonlinear capacitor are used for further details see 39].

An electric circuit with nonlinear resistor is shown in the Figure 3.1. In order to formulate the mathematical description of the circuit further assume, $x = (i_L, v_R, v_L)^\top$. To show that the Theorem 3.4.6 is applicable also to the nonlinearities which occur in electrical circuit. Therefore consider the nonITP (3.16) with the matrix pairs (E_1, A_1) and (E_2, A_2) for switch closed and switch open dynamics, respectively, and switching signal σ,

$$\sigma = \begin{cases} 1, & \text{switch closed,} \\ 2, & \text{switch open.} \end{cases}$$

1. Switch closed:

   Standard circuit analysis of the circuit in Figure 3.1 yields the following nonlinear differential algebraic equations (DAE).

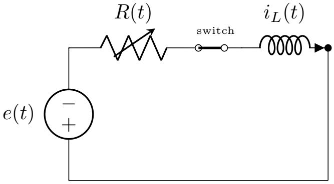


Figure 3.1: Nonlinear RL circuit with switch closed

$$\begin{aligned} v_L &= L\frac{di_L}{dt}, \\ v_R &= \gamma i_L + \alpha(i_L)^3, \\ e &= v_R + v_L. \end{aligned}$$

where, $\gamma, \alpha > 0$ physical constants.

$$E_1 = \begin{bmatrix} L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ -\gamma & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad f_1 = \begin{bmatrix} 0 \\ 0 \\ -e \end{bmatrix}$$

$$g_1(x) = \begin{bmatrix} 0 \\ -\alpha(i_L)^3 \\ 0 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} \frac{1}{L} & 0 & 0 \\ \frac{\gamma}{L} & 1 & 0 \\ -\frac{\gamma}{L} & -1 & 1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1^{imp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Switch open:
   Similarly with circuit analysis of the circuit shown in Figure 3.2 yields the following nonlinear (DAE).

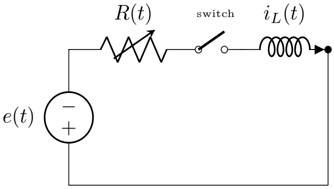


Figure 3.2: Nonlinear RL circuit with switch open

$$\begin{aligned} v_L &= L\frac{di_L}{dt}, \\ v_R &= \gamma i_L + \alpha(i_L)^3, \\ i_L &= 0, \end{aligned}$$

$$E_2 = \begin{bmatrix} L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ -\gamma & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$g_2(x) = \begin{bmatrix} 0 \\ -\alpha(i_L)^3 \\ 0 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\gamma \\ 1 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{imp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ L & 0 & 0 \end{bmatrix}$$

It is easily verified that conditions ($\mathrm{R}_{\mathbf{p}}$), ($\mathrm{F}_{\mathbf{p}}$) ($\mathrm{S}_{\mathbf{p}}$) are satisfies and $S_{\mathbf{p}}$, $T_{\mathbf{p}}$, and $E_{\mathbf{p}}^{\mathrm{imp}}$ are calculated where $\mathbf{p} = 1, 2$. To satisfy condition (G) choose: As $g(x) = g_1(x) = g_2(x)$, consider

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad \mathcal{N} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \bar{g}(i_L) = -\alpha(i_L)^3$$

for which $g(x) = \mathcal{N}\bar{g}(\mathcal{M}x)$ holds. With this choice it is easily checked that (M) and (N) hold. Altogether, the assumptions of Theorem 3.4.6 hold. It leads to the conclusion that for any initial trajectory there is a unique local distributional solution of the nonlinear ITP (3.16).

3.5.3 An example for which the Theorem 3.4.6 is not applicable

Finally, a small academic example is presented for which our approach is not applicable yet and it remains a future research topic, how to treat these kind of equations. Consider ITP (3.16) with

$$E=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}0 & 0 & 1\\ c_1 & 0 & -1\\ -1 & 0 & 0\end{bmatrix},\quad g(x)=\begin{pmatrix}0\\ (x_3)^3\\ 0\end{pmatrix},\quad f=0. \tag{3.20}$$

It is easily verified that conditions (R), (F), (S) are satisfies and S, T, and E^{imp} can be calculated as follows:

$$T=\begin{bmatrix}0 & -1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad S=\begin{bmatrix}1 & 1 & c_1\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix},\quad E^{\text{imp}}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}.$$

With this choice

$$SET=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\quad SAT=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

i.e. that leads to the modified QWF (3.17) with $N_1=[1]$ and $N_2=[0]$. In particular, the ITP (3.16) is equivalent to the ITP given by

$$\dot{v} =(w_2)^3, \tag{3.21a}$$

$$w_1 =0, \tag{3.21b}$$

$$\dot{w_1} =w_2. \tag{3.21c}$$

For a nonzero initial value for $w_1(0^-)$ it is clear that the jump in w_1 (enforced by (3.21b)) results in a Dirac impulse in w_2 (as a consequence from (3.21c)) and the third power of the Dirac impulse enters as an inhomogeneity the ODE (3.21a) for v. As for now, it is not clear how to define a suitable solution concept in this case. Since (3.21) is an equivalent representation of the original nonITP (3.16) given by (3.20) it cannot be solved with our approach (in fact it will not be possible to find matrices

$\mathcal{M}$ and $\mathcal{N}$ such that assumptions (M) and (N) are satisfied). However, our special QWF allows to identify critical inconsistent initial values. In particular, if $w_1(0^-) = 0$ then the ITP (3.21) is solvable and hence for all (possibly inconsistent) initial values $x^0 \in T \operatorname{im} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \operatorname{im} \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ the ITP (3.16),(3.20) will have a solution.

3.6 Summary

In conclusion, in this chapter switched nonlinear DAEs with respect to the existence and uniqueness of solution in the presence of impulses are studied. A theorem with sufficient conditions for the existence of local solution of ITP is presented. Moreover, its extension to switched nonlinear DAEs is presented which is possible under the assumption that no finite escape time occurs between the switches. This notion of solution is also applicable to nonliearities in other dynamical systems.

Chapter 4

Water hammer modeling

The flow of the fluid in a water network exhibit changes in flow regimes for a very short interval of time. Such changes are termed as *transients* in response to the changes in the settings of the control elements (valve or pump, etc.). In particular so called fast transients (instantaneous closure of the valve, sudden pump shutdown) have always created complex conditions, which are capable of causing major problems and damages to the water supply system. The main reason of these possible damages is pressure waves which are produced in response to these transients. These pressure waves (spikes) are dampened or dissipated in a very short interval of time, but they can do enormous damage during that brief period, ranging from catastrophic pipeline bursts to pump defects. Such pressure surges are produced as a consequence of sudden change, in flow rates (for example, with valve closing flow rate tend to zero) in the pipeline 73]. This phenomenon of instantaneous closure of the valve, that generates a pressure surge in water distribution system is called *water hammer* 38].

The aim of this chapter is to compare a model of water hammer via PDEs and switched DAEs.

4.1 Modeling hydraulic transients

The analysis of the most hydraulic transients in pressurised systems are carried out assuming one dimensional flow and are based on the continuity and momentum equations describing the general behavior of fluids in a closed pipe 4]. These transients intrude the steady state conditions at a given point in the pipeline and it starts changing with time and a pressure surge travels along the pipeline starting at the point of the disturbance and will be reflected back from the pipe boundaries (e.g., reservoirs) until a new steady state is reached.

The flow of water in pipes is modeled via a nonlinear system of hyperbolic balance laws. There are two approaches to model the flow in a water network to investigate the of transients and their concussions. These approaches are as follows 20]:

Definition 4.1.1 (Lumped system approach (Mass surge analysis))**.** In this approach the pipe walls are assumed to be rigid and the liquid inside is assumed to be incompressible, i.e, the water inside a pipe is considered to be a solid mass and any flow disturbance is assumed to travel at infinite speed.

Since, the flow variables in water distribution network vary slowly, the compressibility effects are important in rapid transients.

Definition 4.1.2 (Waterhammer analysis approach)**.** It is also called distributive approach, in this approach the liquid is assumed to be slightly compressible 20].

The water hammer analysis approach is used here to model water hammer here in pipes using the pipe flow model presented in the Section 2.4. It is useful to check the aim on a small setup before applying it to the comparatively larger setups. For this reason first a simple setup which exhibit water hammer effect is shown in the following Section.

4.1.1 Simple water setup

To study water hammer consider a simple water setup consisting of an upstream reservoirs with a given pressure P_U, a pipe of length L and a valve conjoined with a downstream reservoir with pressure $P_D < P_U$, see in the Figure 4.1. Here, the most important component is the control component valve, whose open to close setting will create a pressure surge. This setup is used in 21], to study optimal boundary condition to mitigate water hammer, and in 64] for the construction of accurate a finite volume scheme. Here, this setup is used to compare the novel modeling approach via switched DAEs with the more classical PDE models in regard to the water hammer effect.

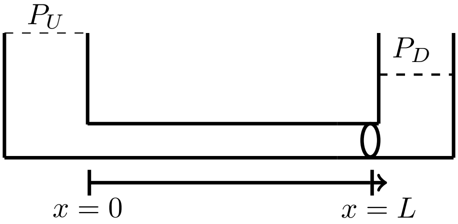


Figure 4.1: Simple set up for water hammer

It is well known that such a configuration will exhibit the water hammer in response to the instantaneous closure of the valve.

4.1.2 PDE solution framework

The balance law for the pipe is given by (2.20), i.e. in terms of (2.5):

$$\Omega = [0, L], \quad \mathcal{U} \subseteq \mathbb{R}_+ \times \mathbb{R}.$$

and, for $u = (\rho, q) \in \mathcal{U}$.

$$\partial_t \rho + \partial_x q = 0 \tag{4.1a}$$

$$\partial_t q + \partial_x (\frac{q^2}{\rho} + P(\rho)) + \frac{c_f}{2D\rho} q \mid q \mid = 0 \tag{4.1b}$$

Consider $U = \begin{pmatrix} \rho \\ q \end{pmatrix}$

$$\partial_t U + \partial_x [F(U)] = S(U) \tag{4.2}$$

where

$$F(u) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + P(\rho), \end{pmatrix}, \quad S(u) = \begin{pmatrix} 0 \\ -c_f \frac{q\,|q|}{2D\rho} \end{pmatrix}. \tag{4.3}$$

The initial condition is given as

$$q(x, 0) = q_0 \quad \rho(x, 0) = \rho_0 \quad P(\rho(x, 0)) = p_0(x). \tag{4.4}$$

some initial flow $q_0 : [0, L] \to \mathbb{R}$ and some initial pressure $p_0 : [0, L] \to \mathbb{R}$. The initial density ρ_0 is induced by the usually considered initial pressure p_0 via the invertible pressure law (2.19). The components in a water network are introduced in this model as boundary conditions. On boundary $x = 0$ there is a reservoir, which will deliver constant pressure P_U at all times. i.e,

$$P(\rho(t, 0)) = P_U \tag{4.5}$$

The invertibility of the pressure function this induces a boundary condition for ρ. On the other end of the pipe a valve is installed in conjunction with a down stream reservoir. Both valve positions on and off are implemented as in (4.6):

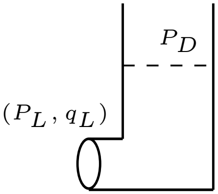


Figure 4.2: Valve setting

$$\begin{aligned} \text{valve open} &: P_L = P_D, \\ \text{valve closed} &: q_L = 0. \end{aligned} \tag{4.6}$$

where P_L, q_L denotes the pressure and flow at $x = L$, respectively i.e, $P(\rho(L,t))$ and $q_L = q(L,t)$.

The valve independent (4.5) and valve dependent (4.6) boundary conditions in terms of (2.5c),(2.5d) are read as:

$$\Psi_1(t,(\rho,q)) = P(\rho) - P_U, \quad \forall t \in \mathbb{R}_+, \tag{4.7}$$

and

$$\Psi_2(t,(\rho,q)) = \begin{cases} P(\rho) - P_D, & t \in (0,t_s), \\ q, & t > t_s, \end{cases} \tag{4.8}$$

where t_s denotes the time for the valve closure. Note that the discontinuity induced by the switch only occurs in the boundary condition, therefore wellposedness can be studied for each mode individually as the PDE can be "restarted" at time $t = t_s$ with the initial value given by the final value of the solution on the time interval $[0,t_s]$; in particular, the wellposedness conditions given in section 2.3.2 can be checked independently of the valve's state 26]:

(B)(B-I) For the numerical simulations constant initial conditions have been imposed, which has zero total variation.

(B-II) The Jacobian of the flux function $F(U)$ is given by:

$$F(U) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + P(\rho), \end{pmatrix}, \tag{4.9}$$

$$A(\rho, q) = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{\rho^2} + P'(\rho) & 2\frac{q}{\rho} \end{pmatrix} \tag{4.10}$$

and invoking the pressure law (2.20c) it can be seen that $P'(\rho) = \frac{K}{\rho_a} > 0$ independently of ρ. Hence the eigenvalues of $A(\rho, q)$ are

$$\lambda_{1/2}(\rho, q) = \frac{q}{\rho} \pm \sqrt{\frac{K}{\rho_a}}.$$

Consequently, $\lambda_1(u) > \lambda_2(u)$ for all $u = (\rho, q) \in \mathcal{U}$ and hence (4.2) with pressure law (2.19) is strictly hyperbolic.

(B-III) The eigenvectors of $A(u)$ are

$$r_1(u) = \begin{pmatrix} 1 \\ \lambda_1(u) \end{pmatrix}, \quad r_2(u) = \begin{pmatrix} 1 \\ \lambda_2(u) \end{pmatrix}.$$

Hence

$$\begin{aligned} \mathbf{D}\lambda_1(u) \cdot r_1(u) &= \left[-\frac{q}{\rho^2}, \; \frac{1}{\rho} \right] \begin{pmatrix} 1 \\ \frac{q}{\rho} + \sqrt{\frac{K}{\rho_a}} \end{pmatrix} \\ &= \frac{1}{\rho} \sqrt{\frac{K}{\rho_a}} \neq 0 \quad \forall u = (\rho, q) \in \mathcal{U} \end{aligned}$$

and, analogously,

$$\mathbf{D}\lambda_2(u) \cdot r_2(u) = -\frac{1}{\rho} \sqrt{\frac{K}{\rho_a}} \neq 0 \quad \forall u = (\rho, q) \in \mathcal{U}.$$

Consequently, genuine nonlinearity is established.

(C) Clearly $S(U)$ is locally Lipschitz continuous.

(D) In order to have sign-definite eigenvalues, i.e.

$$\lambda_1(u) > 0 > \lambda_2(u) \quad \forall u \in \mathcal{U},$$

only subsonic flows are considered here, i.e.

$$\mathcal{U} \subseteq \left\{ (\rho, q) \in \mathbb{R}_+ \times \mathbb{R} \;\middle|\; \frac{q}{\rho} < \sqrt{\frac{K}{\rho_a}} \right\}.$$

and it holds true during in the numerical simulation.

In that case $R^+(u) = [r_1(u)]$ and

$$D_u\Psi_1(t,u) \cdot R^+(u) = \begin{bmatrix} P'(\rho) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_1(u) \end{bmatrix} = P'(\rho) \neq 0,$$

i.e. condition (2.12a) is satisfied for all $u = (\rho, q) \in \mathcal{U}$ and $t > 0$. For the right boundary consider $R^-(\overline{u}) = [r_2(\overline{u})]$ calculated as:

$$D_u\Psi_2(t,u) \cdot R^-(u) = \begin{cases} \begin{bmatrix} P'(\overline{\rho}) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_2(\overline{u}) \end{bmatrix} = P'(\overline{\rho}) \neq 0, \\ \hfill t \in (0, t_s), \\ [0 \;\; 1] \begin{bmatrix} 1 \\ \lambda_2(\overline{u}) \end{bmatrix} = \lambda_2(\overline{u}) \neq 0, \\ \hfill t > t_s, \end{cases}$$

for all $\overline{u} = (\overline{\rho}, \overline{q}) \in \mathcal{U}$; hence, due to the restriction to the subsonic case, condition (2.12b) is satisfied.

The assumptions (B), (C), (D) hold true, hence a reasonable approximations of solutions via numerical simulations is expected to the corresponding initial/boundary value problem (4.2) with the initial condition (4.4) and a valve independent boundary condition (4.7), a valve dependent boundary condition (4.8) as long as the solution remains in the subsonic-case.

In the next Section a switched DAE model which is an approximation to a model via PDE for the simple setup shown in the Figure 4.1 is presented, then the outcome of the models are compared.

4.1.3 Motivation for modeling via switched DAEs

Before presenting the water hammer model via switched DAEs , first a motivation for using this modeling approach is presented, which is based upon the following observations from the PDE simulations:

Observation 4.1.3. After setting up the mathematical model for Figure 4.1 via PDEs, multiple simulations have been produced with the different compressibility coefficients β to see the effect of diluting (decreasing) its value on the pressure profile at the valve. The main observation is that the pressure peaks increased in response to each decrease in compressibility (assigning smaller value of the β)

Observation 4.1.4. Another main observation is shown in the Figure 4.3(b) where two pressure spikes are shown after time $t_s + \varepsilon$. It is observed that for a smaller compressibility coefficient β the pressure spikes dissipate faster as compared to a bigger value of β see the Figure 4.3 (b).

The two above observations provides with the motivation to see effects to consider the exact incompressiblity $\beta = 0$. Due to numerical simulations such restriction may not be possible, but it leads to the intuition that in the case of an exact incompressible case this pressure peak may be an approximated 'Dirac' . To model Dirac, switched DAEs framework

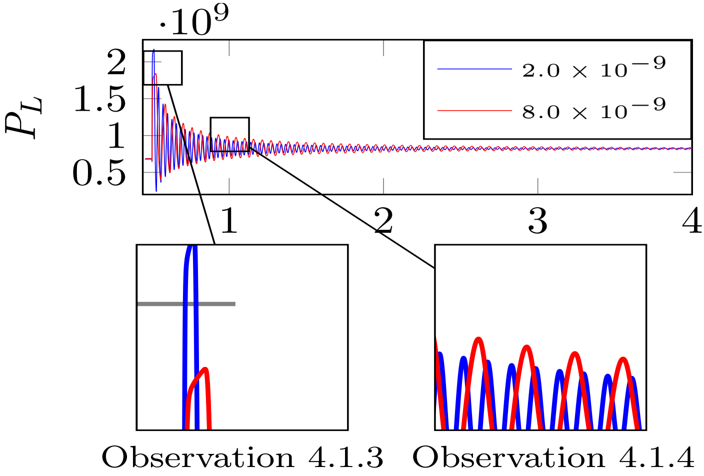


Figure 4.3: Comparison of pressure profile $P_L = P(\rho(t, L))$ with $\beta = 2.0 \times 10^{-9}$ and $\beta = 8.0 \times 10^{-9}$,(Above) with visualisation in zoomed window (below) ; displying the observation 4.1.3 (Left) and observation 4.1.4 (Right). Simulation produced using scheme by the author Jochen Kall in 44]

is used here due to the presence of correct mathematical description of 'Dirac' for details see 69]

4.1.4 Switched DAE framework for the setup 4.1

The quasi-stationary model (2.26) together with the valve-depending boundary conditions (valve open on $[0, t_s)$ and valve closed on $[t_s, \infty)$) for a setup as shown in Figure 4.1 leads to a switched DAE of the form

$$E_\sigma \dot{x} = A_\sigma x + f_\sigma + g_\sigma(x), \tag{4.11}$$

with

$$x = (Q, P_0, P_L)^\top, \sigma(t) = \begin{cases} 1, & t \in [0, t_s), \\ 2, & t \geq t_s, \end{cases}$$

Moreover for both the modes (valve open and valve closed), the non-switched nonlinear DAE is:

$$E_i \dot{x} = A_i x + f_i + g_i(x), \quad \text{for} \quad i = 1, 2. \tag{4.12}$$

$$\mathcal{M} = \mathcal{M}_1 = \mathcal{M}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{N} = \mathcal{N}_1 = \mathcal{N}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$g(x) = \mathcal{N}_1 \overline{g}(\mathcal{M}_1 x), \quad \overline{g}(Q) = -c_2 Q|Q|.$$

1. ***Valve open:*** Consider the equation (4.12) for $i = 1$

$$E_1 x = A_1 x + f_1 + g_1(x) \tag{4.13}$$

where,

$$\left.\begin{aligned} E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ g_1(x) &= \begin{pmatrix} -c_2 Q|Q| \\ 0 \\ 0 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 0 \\ \text{-}P_U \\ \text{-}P_D \end{pmatrix} \end{aligned}\right\} \tag{4.14}$$

where c_1 and c_2 depends on the physical constants of the pipes with :

$$c_1 = \frac{\mathbf{A}}{L} \quad c_2 = \frac{c_f}{2D\rho_a \mathbf{A}}$$

where $c_f, D, \rho_a, \mathbf{A}$ is friction factor of the pipe, diameter, atmospheric density and area of the pipes, respectively.

2. ***Valve close:*** Consider the equation (4.12) for $i = 2$

$$E_2 x = A_2 x + f_2 + g_2(x) \tag{4.15}$$

where,

$$\left.\begin{aligned} E_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & c_1 & -c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ g_2(x) &= \begin{pmatrix} -c_2 Q|Q| \\ 0 \\ 0 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 0 \\ \text{-}P_U \\ 0 \end{pmatrix} \end{aligned}\right\} \tag{4.16}$$

The nonlinearity g is sparse and hence the distributional theory can be from the linear to the nonlinear case as described in the Section 3.4.

Corollary 4.1.5 (to Theorem 3.4.6)**.** Consider the nonlinear initial-trajectory problem (nonITP)

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (A\dot{x} + f + g(x))_{[0,\infty)} \end{aligned}$$

where either $(E, A) = (E_1, A_1)$ *or* $(E, A) = (E_2, A_2)$ *and* $g(x) = g_1(x) = g_2(x)$ *as in* (4.14) *and* (4.16)*.* Then for every initial trajectory $x^0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^3$ and every inhomogeneity f which is either f_1 or f_2 induced by a piecewise-smooth function, there exists a unique solution $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^3$ of the ITP in the sense of Definition 3.4.3.

Proof. Firstly the conditions of the Theorem 3.4.6 will be utillised to show existence and uniqueness checked. Therefore, it has to be checked whether the assumptions of the Theorem 3.4.6 for both (4.14) and (4.16) are satisfied.

1. *Case* $(E, A) = (E_1, A_1)$, first the following transformation matrices are calculated using modified QWF using 3.4.7.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & -c_1 & c_1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$E_1^{imp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S_1 E_1 T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_1 A_1 T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$S_1 f_1 = \begin{bmatrix} c_1 P_U - c_1 P_D \\ \text{-}P_D \\ \text{-}P_U \end{bmatrix},$$

   It is easy to verify that $\det(sE_1 - A_1) \not\equiv 0$, $\Rightarrow$ (E_1, A_1) is regular . Hence, $S_1, T_1, E_1^{\text{imp}}$ can be calculated. Under the assumption that the reservoir pressure only changes smoothly in time (or is constant), then the inhomogeneity f_1 is piecewise-smooth, which implies that the assumption (F) holds. Furthermore the assumption (G) is also satisfied for the given choice of $\mathcal{M}$ and $\mathcal{N}$. The nonlinearity g is locally Lipschitz, which satisfies assumption (S).
   Clearly $\mathcal{M}E_1^{\text{imp}} = 0$ and $\text{im}(\mathcal{N}) = [1, 0, 0]$ and $\text{im}(E) = [1, 0, 0] \Rightarrow$ $\text{im}(\mathcal{N}) \subseteq \text{im}(E)$ which ensure the validity of the assumptions (M) and (N). Hence ITP (4.13), has a unique local solution.

2. *Case* $(E, A) = (E_2, A_2)$.

The matrix pair (E_2, A_2) is regular, i.e. (R) holds. According to Lemma 3.4.7 the follwing is calculated

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{c_1} \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & c_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2^{imp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{c_1} & 0 & 0 \end{bmatrix}$$

$$S_2E_2T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_2A_2T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S_2f_2 = \begin{bmatrix} -P_U \\ 0 \\ 0 \end{bmatrix}.$$

it is immediately seen that with almost the same arguments as for mode 1 (S), (G), (M) and (N) hold mode-2 (valve closed). Hence ITP (4.15) has a unique local solution.

Remark 4.1.6. The nonlinearity g appeared in the ODE part of the differential algebraic equation. The ODE is of the type $\frac{d}{dt}Q = -Q|Q|$ with $Q \geq 0$ the solution of the ODE $\frac{d}{dt}Q = -Q|Q|$ is not exhibiting finite escape time as $\lim_{t\to\infty} Q(t) \to 0$, hence assumption (∞_p) holds true.

Which concludes that all solutions of the corresponding (nonITP) are global, hence the Corollary 3.4.9 ensures existence and uniqueness of distributional solutions of the switched DAE modeling the simple water network in Figure 4.1.

■

Corollary 4.1.7. The switched nonlinear DAE (4.14), (4.16) has for every initial condition $x_{(-\infty,t_0)} = (Q^0, P_0^0, P_L^0) \equiv (Q^0, P_U, P_D) \in \mathbb{R}$ a unique solution in the sense of Definition 3.4.3. In particular, the jump and the Dirac impulse in P_L at t_s are given by:

$$P_L(t_s^+) = P_U, \qquad P_L[t_s] \ = \frac{1}{c_1}Q(t_s^-)\delta_{t_s}.$$

where t_s is the time when valve position is changed from open to close. The jump is $P_L(t_s^-) - P_L(t_s^+) = P_U - P_D$.

Proof. Assume first $(E, A) = (E_1, A_1)$: The state variable x is transformed using transformation matrices T_1, S_1. Also $\dim(\ker(E^\top) = 2$, then $w_1 = \{w_1^1, w_1^2\}$, with $\begin{pmatrix} v \\ w_1^1 \\ w_1^2 \end{pmatrix} := T_1^{-1}x$, result into:

$$v = Q, \qquad w_1^1 = P_L \qquad w_1^2 = P_0.$$

The equivalent initial conditions will be $(v, w_1^1, w_1^2)_{(-\infty,t_0)} = (Q^0, P_D, P_U)$. The nonlinear DAE can be written by using the modified QWF (3.17) as follows

$$S_1 E_1 T_1 \begin{pmatrix} v \\ w_1^1 \\ w_1^2 \end{pmatrix} = S_1 A_1 T_1 \begin{pmatrix} v \\ w_1^1 \\ w_1^2 \end{pmatrix} + S_1 \mathcal{N} g_1 \left(T_1 \begin{pmatrix} v \\ w_1^1 \\ w_1^2 \end{pmatrix} \right) + S_1 f_1$$

The ITP (4.13) is therefore equivalent to

$$\begin{aligned} v_{(-\infty,t_0)} &= v^0_{(-\infty,t_0)} \\ \dot{v}_{[t_0,t_s)} &= \big(c_1(P_U - P_D) + g_1(v)\big)_{[t_0,t_s)} \end{aligned} \tag{4.17a}$$

$$\begin{aligned} {w_1^1}_{(-\infty,t_0)} &= (w_1^1)^0{}_{(-\infty,t_0)} \\ 0 &= \big(w_1^1 - P_D\,\big)_{[t_0,t_s)} \end{aligned} \tag{4.17b}$$

$$\begin{aligned} {w_1^2}_{(-\infty,t_0)} &= (w_1^2)^0{}_{(-\infty,t_0)} \\ &= \big(w_1^2 - P_U\big)_{[t_0,t_s)} \end{aligned} \tag{4.17c}$$

Solving (4.17a), (4.17b) and (4.17c), the following (constant) solution is obtained.

$$(v, w_1^1, w_1^2) = (v(t_s^-), P_D, P_U)$$

and using $\begin{pmatrix} v \\ w_1^1 \\ w_1^2 \end{pmatrix} := T_1^{-1}x$ the result of mode-1 is

$$x_{[t_0,t_s)} = (Q_{[t_0,t_s)}, (P_0)_{[t_0,t_s)}, (P_L)_{[t_0,t_s)}) = (Q(t_s^-), P_U, P_D)$$

Further consider $(E, A) = (E_2, A_2)$, the solution x from previous mode is initial trajectory for this mode

$$x(t_s^-) = (Q(t_s^-), P_0(t_s^-), P_L(t_s^-)) = (Q_{[t_0,t_s)}, P_U, P_D)$$

. Assume $\begin{pmatrix} w_1^1 \\ w_2^1 \\ w_2 \end{pmatrix} := T_2^{-1}x$:

$$w_1^1 = Q, \qquad w_1^2 = P_0 \qquad w_2 = c_1(P_L - w_1^2).$$

The same argument as before the nonlinear DAE in coordinates corresponding to the modified QWF (3.17) is,

$$S_2E_2T_2\begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2 \end{pmatrix} = S_2A_2T_2\begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2 \end{pmatrix} + S_2\mathcal{N}g_2\left(T_1\begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2 \end{pmatrix}\right) + S_2f_2.$$

Componentwise this can be written as:

$$\begin{aligned} (w_1^1)(t_s^-) &= Q_{[t_0,t_s)} \\ 0 &= \left(w_1^1\right)_{[t_s,\infty)} \end{aligned} \tag{4.18a}$$

$$\begin{aligned} w_1^2(t_s^-) &= P_U \\ 0 &= \left(w_1^2 - P_0^0\right)_{[t_s,\infty)} \end{aligned} \tag{4.18b}$$

$$\begin{aligned} w_2(t_s^-) &= c_1(P_U - P_D) \\ \dot{w}_1^1 &= (w_2)_{[t_s,\infty)} \end{aligned} \tag{4.18c}$$

Here $w_1^1(t_s^-) = Q(t_s^-)$ and in mode-2 it becomes $(w_1^1)_{[t_s,\infty)} = 0$, there is a jump in w_1^1 which appeared in (4.18c) as a derivative hence an impulse in w_2 appears. The solution obtained from (4.18a), (4.18b), (4.18c) is:

$$(w_1^1, w_1^2, w_2) = (0, P_U, -Q(t_s^-)\delta_{t_s}) \tag{4.19}$$

which with $\begin{pmatrix} Q \\ P_0 \\ P_L \end{pmatrix} = T_2 \begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2 \end{pmatrix}$ results into

$$Q = w_1^1 \quad P_0 = w_1^2 \quad P_L = w_1^2 - \frac{1}{c_1} w_2.$$

Hence, the solution in the original coordinates is given in (4.19)

$$\begin{aligned} Q &= 0 \\ P_0 &= P_U, \\ P_L &= P_U + \frac{1}{c_1} Q(t_s^-)\delta_{t_s} \end{aligned}$$

In particular, the jump and the Dirac impulse in P_L at switching time $t = t_S$ are given by:

$$P_L[t_s] = \frac{1}{c_1} Q(t_S^-)\delta$$

Further after the valve closure, the time is denoted by $t = t_s^+$, the Dirac will be disappeared, and P_L will become

$$P_L(t_s^+) = P_U.$$

■

4.2 Comparison of both modeling approaches

In this section, quantitative comparison between the PDE model and the switched nonlinear DAE model is presented. The jump and Dirac impulse in the pressure due to closing the valve is the main focus here.

In particular, the initial condition (4.4) is assumed such that the PDE solution on $[0, t_S)$ is stationary, i.e. $q(t, x)$ is constant in time and space (or in other words, when the valve is closed the dynamics in the pipe have settled down). Figure 4.4a shows the results for the pressure profile over the time interval $[0.4s, 3s]$ with initial values

$$q_0(x) \equiv 0, \quad \rho_0(x) \equiv 1.4115 \times 10^3$$

and pipe parameters

$$\begin{aligned} P_a &= 1.01 \times 10^6, & \beta &= \frac{1}{K} = 4 \times 10^{-9}, & \rho_a &= 1000, \\ L &= 5, & D &= 0.5, & c_f &= 0.001, \end{aligned}$$

Clearly, there is a strong pressure spike just after the switching time $t_S = 0.5s$ and then the pressure periodically settles to a new pressure value. The frequency of this periodic behavior is determined by the pipe length L (the larger L the lower the frequency) and the speed of sound (higher for smaller compressibility coefficients β).

4.2.1 Approximation of $P_L = P(\rho(t, L)$ in PDE

In order to compare both results, one has to obtain an approximation of the pressure value $P_L = P(\rho(t, L))$ as t tends to infinity. Pressure at the valve $x = L$ is compared for the time before the valve is closed $t < t_s$, at the time of switching $t = t_s$ and long after the switching time $t > t_s$.

$P_L = P(\rho(t, L))$ for $t < t_s$

To compare the pressure $P(\rho(t, L))$ for $t < t_s$, the PDE model is executed for the time that the pressure will get steady enough. Here, the blue line is showing the pressure before the valve is closed. The calculation from 0 to 0.48 is omitted as it is just the time to get the pressure steady enough.

It is needed for the comparison with the switched DAE model. For the DAE model the pressure at $x = L$, when valve is open is the pressure at downstream reservoir that is P_D. Hence $P_L = P_D$ for $t < t_s$.

$P_L = P(\rho(t, L))$ **for** $t = t_s$

In order to compare the peak in P_L right after the valve is closed with the Dirac impulse $P_L[t_S]$ in response to the switching time, recall that a Dirac impulse δ_{t_s} at $t_s > 0$ can be approximated by a sequence of functions $t \mapsto \delta^{\varepsilon}_{t_s}(t)$ such that $\delta^{\varepsilon}(t) = 0$ for $t \neq [t_s, t_s + \varepsilon]$ and $\int_{t_s}^{t_s+\varepsilon} \delta^{\varepsilon}_{t_s}(t)\mathrm{d}t = 1$. The following Ansatz is made

$$P(\rho(t, L)) \approx \overline{P}^{\text{imp}}_{t_S} \delta^{\varepsilon}(t) + \overline{P}_L, \quad t \in (t_S, T], \tag{4.20}$$

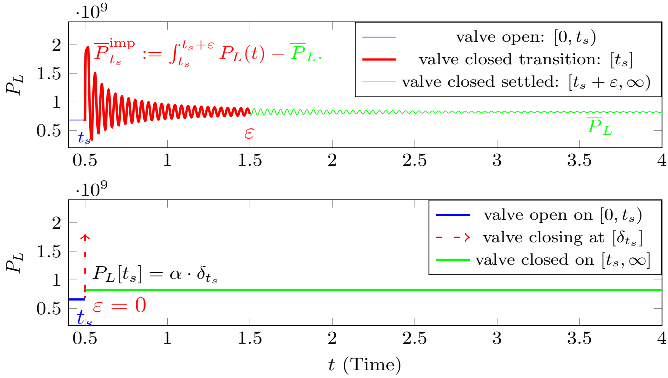
hence the following integral of (4.20) over interval $(t_s, t_s + \epsilon)$ provides with the approximated Dirac impulse occurring in the PDE model written as:

$$\overline{P}^{\text{imp}}_{t_S} := \int_{t_S}^{t_S+\varepsilon} P(\rho(t, L)) - \overline{P}_L \mathrm{d}t.$$

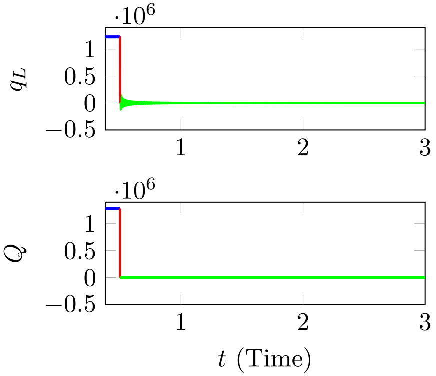
where the interval $(t_s, t_s + \epsilon)$ is the interval of time in which Dirac impulse occurred.

The Dirac impulse induced by the switched DAE is (see Corollary 4.1.7):

$$P_L[t_S] = \frac{1}{c_1} Q(t_S^-) \delta_{t_s} =: P^{\text{imp}}_{t_S} \delta_{t_s}.$$



(a) Comparison of pressure profile at valve ($P_L = P(\rho(t, L))$) with PDE models (above) and switched DAE model (below)



(b) Comparison of flow profile at valve (q_L) with PDE models (above) and switched DAE model (Q) (below)

Figure 4.4: Numerical illustration of PDE and switched DAE model comparison for simple setup in Figure 4.1

Similar as for the PDE simulations it is assumed that the DAE is stationary before the switch, i.e. $Q(t_S^-)$ is obtained by solving

$$0 = \dot{Q} = c_1(P_U - P_D) - c_2 Q|Q|$$

Using the parameters above,

$$Q(t_s^-) = 1.2830 \cdot 10^6, \quad P_{t_S}^{\text{imp}} = \frac{1}{c_1} Q(t_s^-) = 3.2670 \cdot 10^7.$$

4.2.2 Impulse length comparison

A comparison between $\overline{P}_{t_S}^{\text{imp}}$ and $P_{t_S}^{\text{imp}}$ for different values of the compressibility coefficient β is presented in following table.

β	$\overline{P}_{t_S}^{\text{imp}}$	$P_{t_S}^{\text{imp}}$	$RE := \frac{\lvert \overline{P}_{t_S}^{\text{imp}} - P_{t_S}^{\text{imp}} \rvert}{P_{t_S}^{\text{imp}}}$
$4.0 \cdot 10^{-9}$	$4.1251 \cdot 10^7$	$3.2670 \cdot 10^7$	0.2626
$2.0 \cdot 10^{-9}$	$3.4758 \cdot 10^7$	$3.2670 \cdot 10^7$	0.0639
$5.0 \cdot 10^{-10}$	$3.1817 \cdot 10^7$	$3.2670 \cdot 10^7$	0.0261
$2.5 \cdot 10^{-10}$	$3.2069 \cdot 10^7$	$3.2670 \cdot 10^7$	0.0184
$1.25 \cdot 10^{-10}$	$3.2398 \cdot 10^7$	$3.2670 \cdot 10^7$	0.0083

Table 4.1: Impulse length comparison

The Table 4.1 shows that for large value of β (more compressibility) the approximation is not very accurate, however, for decreasing compressibility (considering smaller values of β) the accuracy of the approximation drastically improves. Further the relative error of the Dirac impulse length approximated by PDE and switched DAE model, decreasing by assigning less value to the compressibility coefficient β.

$P_L = P(\rho(t, L))$**for** $t > t_s$

Instead of running the simulation for a very long time, a property of the flow is used which is described in the Section 2.2 and shown in Figure **??**, where the mean flow is well defined and its value remained unchanged even though the flow is not completely steady, statistically. Figure 4.5 is showing the same feature of fluid flow. Using this idea a formula is obtained to estimate the pressure $\overline{P}_L$ a long time after the valve is closed at time t_s.

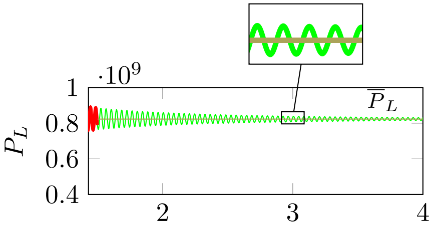


Figure 4.5: Long term $P(\rho(t, L))$ in $t_s + \varepsilon$ to T

For this a settling time $\varepsilon > 0$ is chosen and by taking the average of $P(\rho(t, L))$ on the interval $(t_S + \varepsilon, T]$ where $T > t_S + \varepsilon$ is here the overall simulation time, i.e.

$$\overline{P}_L := \frac{1}{T - (t_S + \varepsilon)} \int_{t_S+\varepsilon}^{T} P(\rho(t, L))\mathrm{d}t. \tag{4.21}$$

with

$$\varepsilon = 1.5, \qquad T = 4.$$

one obtained

$$\overline{P}_L \approx 8.23 \times 10^8$$

The value predicted by the switched DAE is

$$P_L(t_S^+) = P_U \approx 8.23 \times 10^8$$

For large β the approximation is not very accurate, however, for decreasing compressibility the accuracy of the approximation drastically improves. In Table 4.2 the relative error between $\overline{P}_L$ and $P_L(t_S^+)$ is presented for decreasing compressibility coefficients β. (It is important to recall that decreasing compressibility means lowering down or the value of β) Already for the largest value of β, the value $P_L(t_S^+)$ is a very good approximation of $\overline{P}_L$ and the approximation gets better for decreasing β.

β	$\overline{P}_L$	$\frac{\lvert \overline{P}_L - P_L(t_S^+) \rvert}{P_L(t_S^+)}$
$4.0 \cdot 10^{-9}$	$8.2336 \cdot 10^{8}$	$5.4678 \cdot 10^{-04}$
$2.0 \cdot 10^{-9}$	$8.2329 \cdot 10^{8}$	$4.4046 \cdot 10^{-04}$
$5.0 \cdot 10^{-10}$	$8.2305 \cdot 10^{8}$	$7.5942 \cdot 10^{-05}$
$2.5 \cdot 10^{-10}$	$8.2303 \cdot 10^{8}$	$4.5565 \cdot 10^{-05}$
$1.25 \cdot 10^{-10}$	$8.2299 \cdot 10^{8}$	$1.5188 \cdot 10^{-05}$

Table 4.2: Pressure at valve comparison for long after switching.

Choice of settling time ε

Finally, it is worth mentioning that the choice of ε influences the approximation accuracy, see Table 4.3. However, the qualitative behaviour of a decreasing error for decreasing compressibility coefficient remains valid.

	RE	RE	RE	RE
β	$\varepsilon = 1$	$\varepsilon = 1.5$	$\varepsilon = 2$	$\varepsilon = 3$
$4.0 \cdot 10^{-9}$	0.1812	0.2626	0.2616	0.2697
$2.0 \cdot 10^{-9}$	0.0508	0.0639	0.0809	0.0808
$5.0 \cdot 10^{-10}$	0.0438	0.0261	0.0253	0.0233
$2.5 \cdot 10^{-10}$	0.0228	0.0184	0.0016	0.0160
$1.25 \cdot 10^{-10}$	0.0084	0.0083	0.0078	0.0053

Table 4.3: Error comparison with different ε

4.3 Summary

This chapter introduced a switched DAEs model for water hammer on a simple setup, which is compared with a compressible nonlinear system of balance laws. With the support of numerical simulations of the PDE model it is quantitatively illustrated using numerical simulations that a switched DAE model is a good approximation for the PDE model with small compressibility coefficient (less value is assigned to compressibility coefficient β) qualitatively independent of the choice of the ε.

Chapter 5

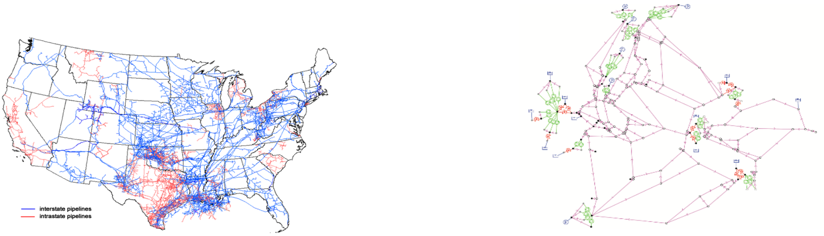
Water network as graph and its solvability

In the previous chapter, a model in the framework of switched nonlinear DAEs is presented as an approximation to the PDE model for hydraulic transient on a simple setup. It turns out that the switched DAEs model is indeed an approximation to the PDE model. Moreover, for further analysis on a larger setup first the general structure of the water network is to be established along with its mathematical wellposedness in both of the modeling frameworks. The valve closures or pump shutdown at different locations create diverse network topologies and it is important to check whether or not all these topologies are solvable in the framework of switched nonlinear DAEs. To formulate this generalisation first the water distribution system is introduced as a network, then the network is expressed as a graph, which results with an improved readability of the model.

This chapter is divided into the following parts; firstly, an overview of the water distribution network is presented. Secondly a brief introduction to the graph theoretical concepts will be revisited. The mathematical structure of the components of the water network is described . Finally, the characterisation of the wellposedness of the solution for the switch nonlinear DAE of a general water network is presented.

5.1 Water distribution system (WDS) as network

Networks can represent almost all sorts of distribution system in the real world. *A network is simply a collection of connected objects* . Some or all of these objects may be connected together by links. The objects and links represents the parts of the network. Therefore it gives a visual picture of how a collection of objects are connected to interact. Based on this reasoning, it follows that many of the things in our everyday lives represent examples of networks, from information networks to social networks through flow networks (water and gas). In the Figure 5.1 a gas and water network is shown.



(a) US Gas network.

(b) Hydraulic network of the Central zone and the East-Pest zone of Budapest.

Figure 5.1: (a) Source: USA energy information (b) Source: 10]

Further to introduce more insight into the water distribution system, a simple fictitious network is depicted in Figure 5.2, composed of a reservoir with a pump station, a storage tank, and number of junctions linked by pipes and valves, etc.

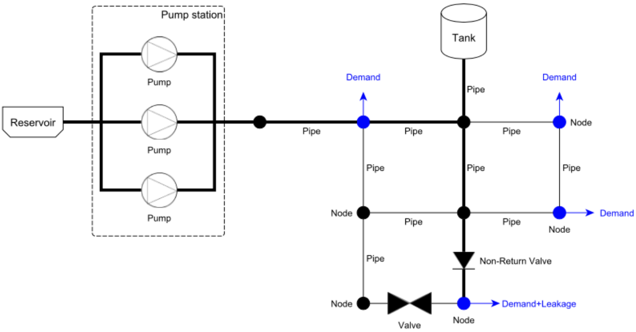


Figure 5.2: Fictitious water distribution network and its components 71] 66].

As illustrated in the Figure 5.2, a water network is predominantly represented as collection of hydraulic elements. Table 2.1 lists typical components of a WDS and their function. The layout of elements maps their topographical interdependencies and is often imposed by the structure of the urban context such as roads, buildings, industrial areas, hospitals, etc.66]. Moreover, there are the two main layouts of distribution networks: branched and looped 71]. Branched networks, or tree networks, are predominantly used to supply water. The loops provide alternative flow pathways, hence, water can be supplied from more than one direction. The presence of loops in the network greatly improve the hydraulics of the distribution system in order to ensure the regularity of the water supply. However, most of the large distribution system essentially is the combination of loop and branches with many interconnected components.

The aim of water network is achieved by means of interconnected elements, given in the Table 5.1. Each of these elements is interrelated

Components	Type	Modeling purpose
Pipe	Link	Conveys water from on object to other
Pump	Link(zero length (node))	Raises the hydraulic pressure to overcome elevation differences
Reservoirs	Object	Provides water to the network
Junction	Object	connection of two or more pipes for inflow and outflow of water
Valve	Link (zero length) or node	Stops or open the flow of water

Table 5.1: Components of water network and their functions. (c.f. 71])

with its neighbors thus the entire WDN behaviour depends on each of its elements. The interrelation of the elements can be more easily represented as a graph. The objects are termed as nodes and links are called edges. They used to delineate the certain properties of components and orientation of their connections. In fact presenting a network as a graph is basically an abstraction of the reality.

5.1.1 Graph preliminaries

Definition 5.1.1 (Graph)**.** A graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a mathematical structure consisting of two sets $\mathcal{N}$ and $\mathcal{E}$. The set $\mathcal{N}$ is a non empty set whose elements are called nodes, and elements of $\mathcal{E}$ are called edges, where $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$. Each edge $e \in \mathcal{E}$ corresponds to a pair $(i, j) \in \mathcal{N} \times \mathcal{N}$, i.e, edge has set of two nodes associated to it, which are its end points. As pair $(i, j) \in \mathcal{E}$ is ordered, the graph $\mathcal{G}$ is called an **oriented or directed graph**. The node i is called starting (initial) node and j is called ending (terminal) node. Graphically, this relation is realized by representing the edges $\{e_1, \cdots, e_m\} \subseteq \mathcal{E}$ as the lines between nodes $\{1, 2, \cdots, n\} \subseteq \mathcal{N}$ 34].

Remark 5.1.2 (Index set of set of edges)**.** The set of edges also can be written as a set of indices $\{1, \ldots, m\}$ such that; $k \in \{1, \ldots, m\}$ iff there exists an edge $e = (i, j) \in \mathcal{E}\ \forall i, j \in \mathcal{N}$, and e is said to be the k^{th} edge in the graph representing the network.

There are different types of graph, used to represent a network depending on the functionality of the network. Some basic types are:

A: In a **simple graph** each edge connects two different nodes and no two edges connect the same pair of nodes.

B: If there exists more than one edge between two same set of nodes, the graph will be called as **multigraph**.

C: A **pseudograph** may include self loops, as well as multiple edges connecting the same pair of nodes.

5.1.2 Connectivity

The connectivity is one of the most important concepts in graph and network theory because it may play an important role in the characterisation of the regularity of a network. A graph may be referred to as (or called) either connected or disconnected in the reference of topological space. Before introducing the notion of connectedness some starting concepts are presented 14].

Definition 5.1.3. An edge $e = (u, v)$ is called **incident** to v for the directed graph. If the direction of the edge is ignored, then the edge e is said to be incident on both u and v. A vertex or node i is said to be *adjacent* to j if there exist a directed edge $e = (i, j)$. On ignoring the direction of edges the nodes i and j are said to adjacent to each other.

The ***degree of a vertex*** v, represented $\deg(v)$ is the number of edges that are connected to it. In a graph with directed edges, the in-degree of a vertex v, denoted $\deg^{+}(v)$, is the number of edges with v as their terminal (ending) vertex. The out-degree of a vertex v, denoted $\deg^{-}(v)$, is the number of edges with v as their initial (starting) vertex.

Definition 5.1.4. A ***path*** is a sequence of edges $e_{k_1}, e_{k_2}, \cdots, e_{k_l}$ between node i to node j, where the end node r of e_{k_m} is incident to the edge $e_{k_{m+1}}$ and the node r is adjacent to the initial vertex s of $e_{k_{m+1}}$ for all $m \in \{1, \cdots, l\}$. Further, i is the initial node of e_{k_1} and j is the terminal node of e_{k_l}.

Definition 5.1.5. A weak path is a path (for definition see 5.1.4) by ignoring the direction of the edges.

Definition 5.1.6 (Connectivity in directed graph)**.** In a directed graph, the connectivity is made symmetric in one of two different ways:

1. A directed graph is said to be strongly connected if given any two vertices u and v, there exists a directed path (defined in 5.1.4) from u to v and a directed path from v to u.
2. A directed graph is said to be weakly connected if it is connected by ignoring the direction.

Definition 5.1.7. A graph is said to be connected if there exists a weak path (defined in 5.1.5) between any two nodes in the graph, meaning that the graph is connected by ignoring the direction of edges.

Remark 5.1.8 (Notion of connectivity in water network)**.** A water network is viewed as a directed graph (to take direction of flow into account). The notion of connectivity is formulated as; A water network is called connected if underlying graph is connected weakly as defined in 5.1.7.

Remark 5.1.9. The direction of an edge from the left to the right node does not mean that the water is always flowing from the left to the right node. Later on, it can be seen that the edge direction tells us how we have to interpret the sign of the flow values. If the flow has a positive sign then it flows from left to right. If the flow has a negative sign then it flows from right to left 42].

The representation of the network is important to analyse how the components of the network may connected and interact. In the next Section various ways to represent a graph are recollected, which are the ways also to represent a network.

5.1.3 Representation of a graph

Toplogy of the graph describes the arrangement of the nodes and edges. It defines how the nodes, within the graph are arranged and connected to each other.

There are different ways of representing the topology of the graph. For the purpose of the water network a term modified incidence matrix is introduced. This term is extended from already existing ways to interpret a graph (c.f. 43]).

Definition 5.1.10 (Incidence Matrix.)**.** The topology of a directed graph is conveniently represented by its incidence matrix. The node-edge incidence matrix $\mathbf{I}^{\text{inc}}(\mathcal{G})$ of a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, is a matrix of order $n \times m$, where n and m denotes number of nodes and edges, respectively. Further each entry $a_{ik} \in \mathbf{I}^{\text{inc}}(\mathcal{G})$ is defined as:

$$a_{ik} = \left\{ \begin{array}{lr} 1, & \text{if } i \text{ is end (terminal) node of the edge } k \\ \text{-1}, & \text{if } i \text{ is starting (initial) node of the edge } k \\ 0, & \text{else} \end{array} \right\}$$

for all $k^{\flat}$ edges and nodes $i \in \mathcal{N}$.

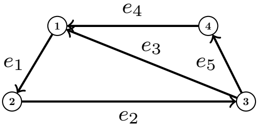
Properties of the incidence matrix

The sum of the columns of $\mathbf{I}^{\mathrm{inc}}(\mathcal{G})$ is zeros. Therefore, rows of $\mathbf{I}^{\mathrm{inc}}(\mathcal{G})$ are linearly dependent. An important result about rank of the incidence matrix, is given. The concept of rank is very useful in this setup to show the regularity of the DAE model of water network. The following Lemma describes the most important property of incidence matrix.

Lemma 5.1.11. *If $\mathcal{G}$ is a connected graph with n nodes, then rank of incidence matrix will be $n-1$.*

The proof is given in the Appendix A.0.10

Example 5.1.12. A connected graph is shown in the Table 5.1.12 with nodes $1, \cdots, 4$ and with edges $e_1, \cdots, e_5$. The incidence matrix of the graph is denoted by $\mathbf{I}^{\mathrm{inc}}(\mathcal{G})$, with order 4×5.



$$\mathbf{I}^{\mathrm{inc}}(\mathcal{G}) = \begin{array}{c} \\ \left(\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \\ -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

From the fact that the sum of the rows of $\mathbf{I}^{\mathrm{inc}}(\mathcal{G})$ is zero it can only be concluded that the rank is at most 3, Lemma 5.1.11 indeed shows that is exactly three because the graph is connected.

There is at least one supply node present in water network, termed as a reference node. While there is a need to make a node as reference node another representation is more suitable. That representation is called a reduced incidence matrix defined as:

Definition 5.1.13 (Reduced incidence matrix)**.** A matrix $\mathbf{I}^{\text{rd}}(\mathcal{G})$ is called reduced incidence matrix, obtained by deleting a row corresponding to the selected reference node from the incidence matrix $\mathbf{I}^{\text{inc}}(\mathcal{G})$. The order of the matrix $\mathbf{I}^{\text{rd}}(\mathcal{G})$ is $(n-1) \times m$ for n nodes and m edges.

Observations about reduced incidence matrix

1. If the graph is connected, $\mathbf{I}^{\text{rd}}(\mathcal{G})$ has full row rank unlike the incidence matrix $\mathbf{I}^{\text{inc}}(\mathcal{G})$ which has the left kernel of dimension 1. Meaning that on removing one row of the incidence matrix, resulting matrix will have trivial left kernel .
2. It has analogy to an electric network. The incidence matrix of an electric network is always reduced by one reference node, which is called the *mass node.* The electric potential at the mass node is fixed. Analogously, the density and the pressure at a supply node is fixed in the gas or water networks .
3. It is important to know that reference node is not deleted form the graph. The row corresponding to the reference node is deleted from the representation. Unlike to incidence matrix by making a node a reference node sum of each column is not zero in reduced incidence matrix.

Example 5.1.14. The reduced incidence matrix obtained after taking node 1 as a reference node is shown in the (5.1).

$$\mathbf{I}^{\text{rd}}(\mathcal{G}) = \begin{array}{c c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} & \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} & \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \end{array} \tag{5.1}$$

In case of the gas and the water network more than one node can be the mass or supply nodes, keeping in view this, the term modified incidence matrix is introduced.

Definition 5.1.15 (Modified incidence matrix)**.** A matrix $\mathbf{I}^{\text{mod}}(\mathcal{G})$ is called the modified incidence matrix, of the complete network if in incidence matrix $\mathbf{I}^{\text{inc}}(\mathcal{G})$ all the rows corresponds to the supply nodes are taken as reference nodes are deleted from the representation matrix of the graph. The resulting matrix after this deletions is called modified incidence matrix and denoted as $\mathbf{I}^{\text{mod}}(\mathcal{G})$. Assume n_s be number of supply nodes, therefore order of $\mathbf{I}^{\text{mod}}(\mathcal{G})$ is $(n - n_s) \times m$.

(5.2) showed the the modified incidence matrix obtained after deleting rows corresponding to node 1 and 4 from incidence matrix.

$$\mathbf{I}^{\text{mod}}(\mathcal{G}) = \begin{array}{cc} \begin{array}{cccc} e_1 & e_2 & e_3 & e_5 \end{array} & \\ \begin{pmatrix} 1 & \text{-1} & 0 & 0 \\ 0 & 1 & \text{-1} & \text{-1} \end{pmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \tag{5.2}$$

Remark 5.1.16. In case of reduced incidence matrix of order $(n-1)\times m$ with one node is taken or considered as a reference node and rank of the matrix will be $n - 1$ if the graph is connected. For modified incidence matrix, there will be more than one reference nodes. For n_s reference (supply) nodes the modified incidence matrix will be of order $(n-n_s)\times m$, $(n - n_s) \leq m$ and of rank of $n - n_s$ if a graph is connected graph. The reduced and modified incidence matrix will have trivial left kernel, hence will have full row rank. In a water network reservoirs are considered as supply or reference nodes.

Remark 5.1.17. In a modified incidence matrix in a column there may be two non zero entries 1 and -1 that shows both ends of the corresponding edge is connected to the junction nodes (other than reference nodes), for example e_2 in (5.2) both ends are connected to node 2 and 3 none of them is a reference node. Furthermore, *exactly one nonzero entry in the column is due to the fact that one end of the corresponding edge is connected to the a reference node which is not the part of the matrix* *representation as modified incidence matrix of the graph.* For example in the modified incidence matrix $\mathbf{I}^{\mathrm{mod}}(\mathcal{G})$ (5.2) the column corresponds to e_1 is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; one end of the e_1 is connected node 2 and other end of e_1 with node 1. The vertex 1 is not the part of the $\mathbf{I}^{\mathrm{mod}}(\mathcal{G})$ in (5.2) . Analogously, this holds for the edge e_3 and e_5. If all entries in the column is zero it means both ends of the corresponding edge is connected to reference nodes (e.g e_4)and ommitted from the modified incidence matrix. Also $(\mathbf{I}^{\mathrm{mod}}(\mathcal{G}))^\top \in \mathbb{R}^{m\times(n-n_s)}$ also can be written as :

$$(\mathbf{I}^{\mathrm{mod}}(\mathcal{G}))^\top =: (\mathbf{I}^{\mathrm{mod}}(\mathcal{G})_l)^\top + (\mathbf{I}^{\mathrm{mod}}(\mathcal{G})_r)^\top$$

Remark 5.1.18. The assignment of the left and right nodes of each branch to the global node numbers may easily described by the incidence matrices $\mathbf{I}^{\mathrm{mod}}(\mathcal{G})_l, \mathbf{I}^{\mathrm{mod}}(\mathcal{G})_r \in \mathbb{R}^{(n-n_s)\times m}$. Consider assignment of the left and right node of k^{th} edge in the graph $\mathcal{G}$.

$$(\mathbf{I}^{\mathrm{mod}}(\mathcal{G})_l)_{ik} = \left\{ \begin{array}{lr} \text{-1}, & \text{if node } i \text{ is the left node of edge } k \\ 0, & \text{otherwise} \end{array} \right\}$$

$$(\mathbf{I}^{\mathrm{mod}}(\mathcal{G})_r)_{ik} = \left\{ \begin{array}{lr} 1, & \text{if node } i \text{ is the right node of edge } k \\ 0, & \text{otherwise} \end{array} \right\}$$

additionally, $\mathbf{I}^{\mathrm{mod}}(\mathcal{G})$ is can be written as:

$$\mathbf{I}^{\mathrm{mod}}(\mathcal{G}) := \mathbf{I}^{\mathrm{mod}}(\mathcal{G})_l + \mathbf{I}^{\mathrm{mod}}(\mathcal{G})_r \in \mathbb{R}^{(n-n_s)\times m}$$

Example 5.1.19. The modified incidence matrix (5.2) can be rewritten as

$$\mathbf{I}^{\text{mod}}(\mathcal{G})_l = \begin{pmatrix} 0 & \text{-1} & 0 & 0 \\ 0 & 0 & \text{-1} & \text{-1} \end{pmatrix}$$

$$\mathbf{I}^{\text{mod}}(\mathcal{G})_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

It is easy to verfiy the following,

$$\mathbf{I}^{\text{mod}}(\mathcal{G}) = \mathbf{I}^{\text{mod}}(\mathcal{G})_l + \mathbf{I}^{\text{mod}}(\mathcal{G})_r$$

Lemma 5.1.20. Consider a modified incidence matrix of a connected graph $\mathbf{I}^{\text{mod}}(\mathcal{G})$. Then it has full row rank iff the graph $\mathcal{G}$ contains at least a reference node.

Remark 5.1.21. Lemma 5.1.20 is a direct consequence if facts that is.

Removing one row of the incidence matrix of a connected graph has full row rank and any subset of linear independent rows is again linear independent.

5.1.4 Relationship of $\mathbf{I}^{\text{inc}}(\mathcal{G})$ and pressure difference ΔP

The pressure difference between two nodes cause a flow through an edge $\forall e \in \mathcal{E}$ connecting those two nodes. Let $P \in \mathbb{R}^n$ be the vector of pressure at the nodes that means, $P_i \in \mathbb{R}$ describes the pressure at the node i for $\forall i \in \mathcal{N}$. The different representations of graph also used to describe the pressure difference $(\Delta P)_k$ for k^{th} edge $e \in \mathcal{E}$, then $(\Delta P)_k$ can be expressed as 13] 42].

$$(\Delta P)_k = P_{r(k)} - P_{l(k)} \tag{5.3}$$

Where $P_{r(k)}$ and $P_{l(k)}$ being the right and left node of the k^{th} edge $e = (i, j)$, with $P_{r(k)} = P_j$ and $P_{l(k)} = P_i$ then (5.3) can be written as

$$(\Delta P)_k = P_j - P_i \quad \forall k \quad \text{corresponding to the edge} \quad e \in \mathcal{E} \tag{5.4}$$

Also can be written in terms of the entries of the incidence matrix,

$$(\Delta P)_k = \sum_{i=1}^{n} a_{ik} P_i \quad \forall k = 1, \cdots, m \tag{5.5}$$

where $(\Delta P)_k$ – pressure drop in branch k; m – number of edges; n – total number of nodes; a_{ik} – element from row i and column k in the incidence matrix (c.f. 1])

$$\Delta P = (\mathbf{I}^{\text{inc}}(\mathcal{G}))^{\top} P \tag{5.6}$$

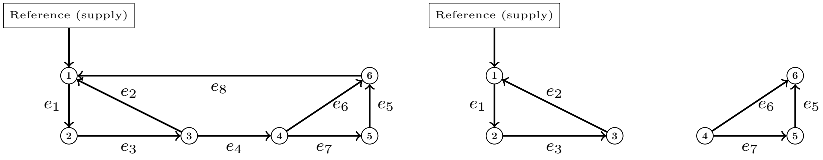
Remark 5.1.22. Consider modified incidence matrix with m edges and n nodes, where n_s is the number if supply nodes. For $P \in \mathbb{R}^{n-n_s}$ and $P_i \in \mathbb{R}$, then following will hold

$$\Delta P = (\mathbf{I}^{mod}(\mathcal{G}))^{\top} P$$

After brief introduction to graph theory, in next Section mathematical formulation of the water network is presented. In a water network some edges are disconnected due to the change in the settings of the control elements (valve, pumps). This disconnection divides a water network into some connected components.

5.1.5 Connected components

A connected component of a directed graph $\mathcal{G}$ is a set of nodes $\mathcal{N}' \subseteq \mathcal{N}$ such that for every pair of nodes $i, j \in \mathcal{N}'$ there is a weak path from i to j .



(a) Connected graph.

(b) Connected components.

Figure 5.3: Connectivity

In the Figure 5.3(a) a connected graph with a reference (supply) is shown. In the the Figure 5.3(b) on removal of the edge e_4 and e_8 results into two components which are internally connected that means there exists a path from any two nodes in both components.Moreover, a connected component lost the connection with the reference (supply) node. Such events in water network may cause inconvenience to the consumer and effects water supply system. In proposition 5.1.23 an assumption on the modified incidence matrix is presented which provides a mathematical framework to avoid such situations.

Lemma 5.1.23. The modified incidence matrix $\mathbf{I}^{\mathrm{mod}}(\mathcal{G})$ has full row rank iff each connected component (see definition in 5.1.7 for connectivity)of the network is connected to at least one reference (supply node).

Proof. Consider a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with n nodes and m edges, also assume n_s be number of supply nodes with $n - n_s \leq m$. The modified incidence matrix $\mathbf{I}^{\mathrm{mod}}(\mathcal{G})$ be of order $(n - n_s) \times m$. Let $\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_k$ be connected components in $\mathcal{G}$. The modified incidence matrix $\mathbf{I}^{\mathrm{mod}}(\mathcal{G})$ can be written as:

$$\mathbf{I}^{\mathrm{mod}}(\mathcal{G}) = \begin{pmatrix} \mathbf{I}^{\mathrm{mod}}(\mathcal{G}_1) & 0 & \dots & 0 \\ 0 & \mathbf{I}^{\mathrm{mod}}(\mathcal{G}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{I}^{\mathrm{mod}}(\mathcal{G}_k) \end{pmatrix} \tag{5.7}$$

(5.7) will have full row rank iff the modified incidence of every connected component $\mathcal{G}_i$ for $i = 1, \cdots, k$ has full row rank. To prove this it is to be shown that the modified incidence matrix of i^{th} connected component $\mathbf{I}^{\text{mod}}(\mathcal{G}_i)$ has a full row rank if and only if it is connected to at least one reference node, which is already shown in 5.1.20. The matrix (5.7) is a block diagonal matrix with all components in the diagonal have full row rank, then (5.7) has full row rank. This concludes the proof. ■

Remark 5.1.24 (Incidence matrix of cycle graph)**.** A simple directed cycle (or cycle) in a directed graph is a closed path where all the nodes $i \in \mathcal{N}$ are different. In the incidence matrix of such graph, there will be two non zero entries in each row 58]. Consider the columns of the directed cycle in a graph is $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ then choose any arbitrary direction to traverse a cycle, say clock wise let the corresponding entry in the column is $a_i = +1$ if the edge e_i has same direction and $a_i = -1$ otherwise, hence it results into the following

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \text{-1} \\ \text{-1} & 1 & \cdots & 0 & 0 \\ 0 & \text{-1} & \ddots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \text{-1} & 1 \end{pmatrix}.$$

The reverse direction will result the same .

Example 5.1.25. Consider the graph in the Figure 5.3 there exists a directed cycle e_1, e_2, e_3, e_1 starting traversing from edge e_1 then follow through e_2 then e_3 to e_1, with incidence matrix:

$$\begin{array}{c c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} & \\ \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

The directed cycle have exactly two non zero entries in each row 1 and -1.

Lemma 5.1.26. Consider an oriented graph $\mathcal{G}$ with modified incidence matrix $\mathbf{I}^{\text{mod}}(\mathcal{G})$. Then the columns of $\mathbf{I}^{\text{mod}}(\mathcal{G})$ are linearly dependent iff corresponding edges form a cycle in the graph.

Proof. Consider $\mathbf{I}^{\text{mod}}(\mathcal{G})$ is the modified incidence matrix of the graph $\mathcal{G}$. Consider $C = \{c_1, \cdots, c_m\}$ be set of columns. Consider $\overline{C} = \{\bar{c}_1, \cdots, \bar{c}_k\} \subseteq C$, with $k \leq m$. It is to be shown that the set of columns $\overline{C}$ are linearly dependent iff subgraph $\mathcal{G}'$ consists of edge set $\mathcal{E}'$ corresponds to the columns $\overline{C}$ is a cycle. "$\Longrightarrow$ " Consider $\overline{C}$ be the set of linearly dependent columns, meaning that

$$\sum_{l=1}^{k} \bar{c} = 0 \tag{5.8}$$

(5.8) in turn implies that $\mathcal{G}'$ formed by the columns of $\overline{C}$ has no row with odd number of non zero entries. Hence the subgraph $\mathcal{G}'$ has no vertex of degree one, therefore using the Lemma A.0.11 from Appendix it must contains a cycle.

"$\Longleftarrow$"

Assume $\mathcal{G}'$ formed by the set of edges $\mathcal{E}'$ formed a simple cycle. Then the columns corresponds to the edge $\mathcal{E}'$ be the incidence matrix of $\mathcal{G}'$, basically incidence matrix of cycle and then (5.8) holds, since each row of the incidence matrix of $\mathcal{G}$ has exactly two non zero entries -1 and 1. This completes the proof. ∎

5.2 Mathematical formulation of water network components

In this Section the components of the water network are described in terms of graph theoretical terms. This description will make general structure of the water network more readable. It leads towards the mathematical formulation of the network properties in a unifying language.

5.2.1 Water components: A map to graph theory

The nodes in a water distribution network are typically grouped by sources (e.g. reservoirs, pipes and storage facilities), control and distribution nodes (e.g. valves, pipe junctions, pumps). Each node and edge has certain pressure $p = P(\rho)$ and mass flow q for modeling via PDEs and for DAE modeling P and Q is used. The components of the water network are as follows:

A: **Pipe**: $\forall e \in \mathcal{E}_{pi}$ where $\mathcal{E}_{\mathrm{pi}}$ denotes set of the edges which are labeled as pipes. In graph theoretical sense it is modeled as an edge between two nodes $e = (i, j)$ with $i, j \in \mathcal{N}$ as its end points with pressure denoted P_i and P_j, respectively.

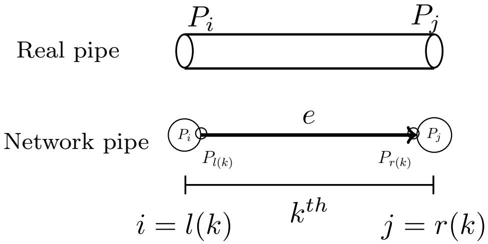


Figure 5.4: View of pipe in a network

For PDE modeling, a system of hyperbolic balance laws is used given in (2.20), the pressure at the end nodes of a pipe are modeled via invertible pressure law, $p(t,x) = P(\rho(t,x))$. The pressure at left is $P_{l(k)}(t) = P_i(t)$ and on right of the k^{th} pipe is $P_{r(k)}(t) = P_j(t)$ for all pipe edge $e \in \mathcal{E}_{\text{pi}}$ as shown in the Figure 5.4. For DAE modeling over a network, there is no space coordinate . The dynamics along each k^{th} pipe edge $e \in \mathcal{E}_{pi}$ modeled via DAE is mathematically written by using Quasi stationary model in network is already presented in the Section 2.4.2. Also each pipe has only one value for the flow $Q_k(t)$ in this model $\forall k$. Furthermore, with the labeled left and right end of each pipe (**??**) can be rewritten as:

$$\frac{dQ_k}{dt} = -c_k(P_{r(k)}(t) + P_{l(k)}(t)) + g(Q_k). \quad \forall e \in \mathcal{E}_{pi} \tag{5.9}$$

where $c_k = \frac{\mathbf{A}_k}{L_k}$, with $\mathbf{A}_k$ and L_k is the area and length, for every k^{th} pipe edge $e \in \mathcal{E}_{\text{pi}}$. The function $g(Q_k)$ is the nonlinear friction of the pipe, and is modeled by the Darcy Weisbach equation see in Appendix **??**.

B: **Reservoir** A reservoir is termed as a supply node. The set contains all reservoir nodes is denoted by $\mathcal{N}_{rs} \subseteq \mathcal{N}$ with an arbitrary mass flux (mass flow) but with fixed pressure. **In the PDE modeling**: A reservoir modeled via boundary condition, for $p(x_i, t)$, where x_i is the location i^{th} reservoir in space. The equation of the reservoir for PDE model is written as:

$$p_i = P(\rho_i(t, x = x_i)) = P_{\text{rs}^i}(t) \quad \forall i \in \mathcal{N}_{\text{rs}} \subseteq \mathcal{N}.$$

DAE modeling

The reservoir equation can be written as an algebraic constraint:

$$P_i(t) = P_{\text{rs}^i}(t) \quad \text{for any} \quad i \in \mathcal{N}_{\text{rs}} \tag{5.10}$$

for every reservoir $i \in \mathcal{N}_{\text{rs}}$ there exists a pressure function $P_{rs^i}(t)$ such that (5.10) holds.

C: **Junction** Formally, junction is defined as a node $i \in \mathcal{N}_{\text{jc}}$, where two or more pipes are connected.

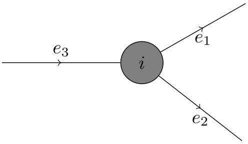


Figure 5.5: Junction of three edges (pump or pipes)

In case of PDE modeling for each edge k, (p_k, q_k) is the pressure and flux along each edge (pipe or pump). For each $i \in \mathcal{N}_{\text{jc}}$ is quipped with wellposed coupling conditions 1] 44].

(a) Mass conservation: First coupling condition expresses the conservation of mass at the junction, that is all the mass from in-flowing pipes leaves through out-flowing edges. Mathematically written as for all $i \in \mathcal{N}_{\text{jc}}$

$$\sum_{j \in \nabla^-(i)} q_j = \sum_{j \in \nabla^+(i)} q_j \tag{5.11}$$

similarly, just for the ease in reading consider (P_k, Q_k) is pressure and flow for DAE modeling, and mass balance will be:

$$\sum_{j \in \nabla^-(i)} Q_j = \sum_{j \in \nabla^+(i)} Q_j \tag{5.12}$$

where ∇^+ is the set of edges coming towards the junction i, and ∇^- is the set of edges going away from the junction i.

(b) Pressure equality: Let there be k edges emerging from a vertex $i \in \mathcal{N}_{\text{jc}}$, equipped with the coupling

$$p_1 = p_2, \cdots, = p_k$$

with $p_j = P(\rho(t, x_j))$ for PDE modeling.

D: **Valve** is a control element which can be opened or closed. In graph methematical terms it is an edge The right side of edge is denoted by $r(k)$ and left end of edge $l(k)$, respectively for each k^{th} pipe edge. In terms of graph theory it is modeled as an edge as shown in the Figure 5.6.

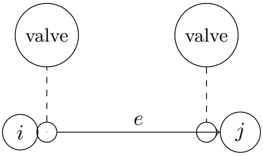


Figure 5.6: Valve at the right and left of the k^{th} edgee

There is a valve installed at each end of the pipe edge. The valve constraints can be written as:

$$(1 - s_{l_k})q_k(t, 0) = s_{l_k}(p_k(t, 0) - P_i(t)) \tag{5.13}$$

$$(1 - s_{r_k})q_k(t, L_k) = s_{r_k}(p_k(t, L_k) - P_j(t)) \tag{5.14}$$

where $p_{l_k}(t) = p_k(t, 0) = P(\rho_k(t, 0))$ and $p_{r_k}(t) = p_k(t, L_k) = P(\rho_k(t, L_k))$ for the k^{th} pipe edge e and $\forall k$. Furthermore, s_{k_l} and s_{k_r} denotes the control variable for the valve at left and valve at right end of the pipe, respectively and can take values as follows:

$$s_{l_k} = \left\{ \begin{array}{ll} 1, & \text{the valve at the left of a } k^{th} \\ & \text{pipe edge is open} \\ 0, & \text{otherwise} \end{array} \right\} \tag{5.15}$$

$$s_{r_k} = \left\{ \begin{array}{ll} 1, & \text{the valve at the right of a } k^{th} \\ & \text{pipe edge is open} \\ 0, & \text{otherwise} \end{array} \right\} \tag{5.16}$$

In case of DAE model, valve is closed only by closing at the right end of the pipe, meaning that $s_{l_k} = 1$ for all cases and for closing valve, s_{r_k} is used as control variable. Hence on closing $s_{r_k} = 0$ which equates $Q_k = 0$ (which is same along whole edge e) and for $s_{r_k} = 1$ which will give $P_{r(k)} = P_j$ and both cases can be combined and written as in (5.17) , for all $e \in \mathcal{E}_{\text{pi}}$ mathematically written as:

$$P_i - P_{l(k)} = 0 \tag{5.17a}$$

$$(1 - s_{r_k})Q_k - s_{r_k}(P_{r(k)} - P_j) = 0 \tag{5.17b}$$

$Q_k(t)$ is the flow along whole k^{th} pipe edge, that is corresponding edge $e \in \mathcal{E}_{\text{pi}}$. Where s_{k_r} is a control variable defined as (5.16). The closure of the valve at right end in this case means the flow though the k^{th} pipe edge will be stopped. In graph theoretical way it means that the k^{th} edge is deleted from the network graph.

E: **Pump** is also a control element and it is modeled as an edge between two pressure nodes in network. When a pump is ON ($\bar{s}_k = 1$), the characteristic curve of the pump describes the relationship between the pressure difference across the pump and the flow rate. When the pump is OFF ($\bar{s}_k = 1$), the flow through the pump is zero. For k^{th} pump edge $e \in \mathcal{E}_{\text{pu}}$ $\forall k$ that connects nodes i and j with flow Q_k through the pump , the pressure difference across the pump at a time step can be represented with a polynomial function $f_{\text{pu}}(Q_{\text{pu}})$

$$P_j - P_i = \left\{ \begin{array}{lll} f_{\text{pu}}(Q_k) & & \bar{s}_k = 1 \\ \text{unspecified} & Q_k = 0 & \bar{s}_k = 0 \end{array} \right\} \tag{5.18}$$

Combining both cases given in (5.18); $\forall e = (i, j) \in \mathcal{E}_{\text{pu}}$ is modeled as:

$$(1 - \bar{s}_k)Q_k + \bar{s}_k(P_j - P_i) = \bar{s}_k f_{pu}(Q_k). \tag{5.19}$$

In comparison with a pipe, pump have a negligible length and thus the matter of constant or variable flow in a pump is not encountered. The flow through a pump is usually restricted in sign that is $Q_k \geq 0$ 31]. $\forall e \in \mathcal{E}_{\text{pu}}, \forall k$.

All above given components described formulated the equations to model a network mathematically. Further, the general structure and acceptable topologies of the water network will be formulated. Usually there are different coupling conditions can be applied at the junction. For pipe network under discussion here we need two types of coupling conditions.

5.2.2 Model structure

Consider network with n nodes and m edges, i.e., $|\mathcal{E}| = m$ and $|\mathcal{N}| = n$. Further assume set of edges are divided into the set $\mathcal{E}_{\text{pi}}$ and $\mathcal{E}_{\text{pu}}$, which denotes set of edges which are pipes and pumps, respectively. Assume $\mathbf{m}$ is the number the edges which are pipes and $\breve{m}$ are number of edges which pumps, i.e., $\mathbf{m} = |\mathcal{E}_{\text{pi}}|$ and $\breve{m} = |\mathcal{E}_{\text{pu}}|$. The set of nodes is partitioned into two parts, $\mathcal{N}_{\text{rs}}$, and $\mathcal{N}_{\text{jc}}$ with number of nodes, n_{rs}, and n_{jc}, respectively and denotes the set contains reservoir and junction nodes, respectively. Further $n = n_{rs} + n_{\text{jc}}$ and $m = \mathbf{m} + \breve{m}$. The overall state vector of this system description is given by

$$x = \begin{bmatrix} Q_{e_{\mathrm{pi}}} \\ Q_{e_{\mathrm{pu}}} \\ P_l \\ P_r \\ P_{rs} \\ P_{\mathrm{jc}} \end{bmatrix}$$

x is of dimension $\Upsilon = 3\mathbf{m} + \breve{m} + n_{rs} + n_{\mathrm{jc}}$. Further the description of the variables are as follows,

$Q_{e_{\mathrm{pi}}}$ =Vector of flows through **m** pipes.

$Q_{e_{\mathrm{pu}}}$ =Vector of flow through $\breve{m}$ pumps.

P_l =Vector of pressure at left end of **m** pipes

P_r =Vector of pressure at right end of **m** pipes.

P_{rs} =Vector of pressure at n_{rs} reservoir nodes.

P_{jc} =Vector of pressure at n_{jc} junctions.

Consider the nonlinear DAE for the water network,

$$E\dot{x} = Ax + f + g(x) \tag{5.20}$$

With the following overall system description can be obtained in terms of matrix pair $E, A \in \mathbb{R}^{\Upsilon \times \Upsilon}$

$$E = \begin{array}{c} \begin{array}{cccccc} Q_{e_{\mathrm{pi}}} & Q_{e_{\mathrm{pu}}} & P_l & P_r & P_{\mathrm{rs}} & P_{\mathrm{jc}} \end{array} \\ \begin{pmatrix} \mathcal{I}_{pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$A = \begin{array}{cc} \begin{array}{cccccc} Q_{e_{pi}} & Q_{e_{pu}} & P_l & P_r & P_{\mathrm{rs}} & P_{\mathrm{jc}} \end{array} & \\ \begin{pmatrix} 0 & 0 & A_l & A_r & 0 & 0 \\ A^{\mathrm{jc}}_{\mathrm{f_{pi}}} & A^{\mathrm{jc}}_{\mathrm{f_{pu}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A^{\mathtt{v}}_l & 0 & A^{\mathrm{rs}}_l & A^{\mathrm{jc}}_l \\ \widetilde{A}^{\mathrm{cv}}_r & 0 & 0 & \widetilde{A}^{\mathtt{v}}_r & \widetilde{A}^{\mathrm{rs}}_r & \widetilde{A}^{\mathrm{jc}}_r \\ 0 & \widetilde{A}^{\mathrm{cv}}_{\mathrm{pu}} & 0 & 0 & \widetilde{A}^{\mathrm{rs}}_{\mathrm{pu}} & \widetilde{A}^{\mathrm{jc}}_{\mathrm{pu}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 \end{pmatrix} & \begin{array}{c} (5.9) \\ (5.12) \\ (5.17a) \\ (5.17b) \\ (5.19) \\ (5.10) \end{array} \end{array} \tag{5.21}$$

$$g(x) = \begin{bmatrix} \text{-}g_{\mathrm{pi}}(Q_{e_{\mathrm{pi}}}) \\ 0 \\ 0 \\ 0 \\ \text{-}K_{\mathrm{pu}} g_{\mathrm{pu}}(Q_{e_{\mathrm{pu}}}) \\ 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0_{(3\mathbf{m}+\breve{m}+n_{\mathrm{jc}})\times 1} \\ \text{-}\bar{f}_{n_{\mathrm{rs}}\times 1} \end{bmatrix} \quad K_{\mathrm{pu}} = (\bar{s_1}, \cdots, \bar{s}_{\mathbf{m}}) \tag{5.22}$$

$$\text{For} \quad e \in \mathcal{E}_{\text{pi}} \quad g_{\text{pi}}(Q_{e_{\text{pi}}}) = \begin{bmatrix} g^{1}_{\text{pi}}(Q_1) \\ \vdots \\ g^{\mathbf{m}}_{\text{pi}}(Q_{\mathbf{m}}) \end{bmatrix}$$

$$\text{For} \quad e \in \mathcal{E}_{\text{pu}} \quad g_{\text{pu}}(Q_{e_{\text{pu}}}) = \begin{bmatrix} g^{1}_{\text{pu}}(Q_1) \\ \vdots \\ g^{\breve{m}}_{\text{pu}}(Q_{\breve{m}}) \end{bmatrix}$$

where $g^{l}_{\text{pi}} : \mathbb{R} \to \mathbb{R}$ is the friction factor l^{th} pipe edge is the nonlinear friction function for the pipe, also $g^{k}_{\text{pu}} : \mathbb{R} \to \mathbb{R}$ denotes the linear or nonlinear pump characteristics k^{th} pump edge $e \in \mathcal{E}_{\text{pu}}$. The vector of function $\bar{f}_{n_{\text{rs}\times 1}}$ denotes a vector with the fixed pressures function at each reservoirs.

Example 5.2.1. This example is given to present the structure of the general matrix structures for the better understanding.

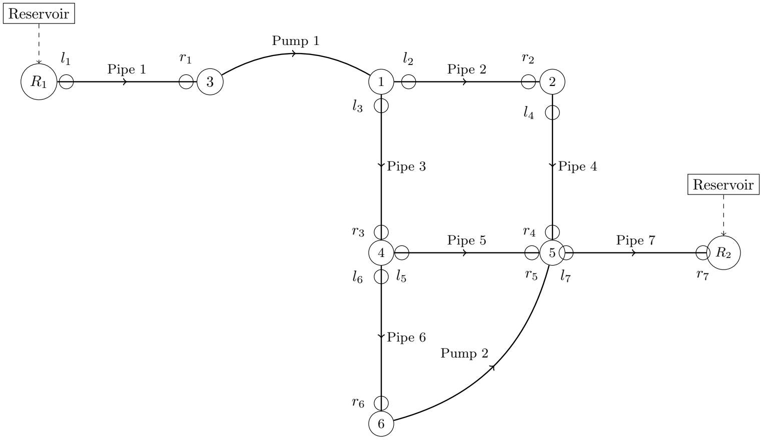


Figure 5.7: Water network c.f. EPANET example 61]

Consider E and A in (5.21) with the following block matrices
For the left end of the pipes following blocked matrices are formulated:

$$\mathcal{I}_{\text{pi}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_l = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_7 \end{bmatrix} \quad A_r = \begin{bmatrix} \text{-}c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{-}c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{-}c_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}c_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{-}c_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{-}c_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{-}c_7 \end{bmatrix}$$

$$A_l^v = \begin{bmatrix} \text{-}1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{-}1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{-}1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{-}1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{-}1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{-}1 \end{bmatrix} \quad A_l^{\text{rs}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_l^{\text{jc}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For the right end of the pipes following blocked matrices are formulated:

$$A_r^v = \begin{bmatrix} \text{-}s_{r_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{-}s_{r_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{-}s_{r_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}s_{r_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{-}s_{r_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{-}s_{r_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{-}s_{r_7} \end{bmatrix}, \quad A_r^{\text{jc}} = \begin{bmatrix} 0 & 0 & s_{r_1} & 0 & 0 & 0 \\ 0 & s_{r_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{r_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{r_4} & 0 \\ 0 & 0 & 0 & 0 & s_{r_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{r_6} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_r^{cv} = \begin{bmatrix} 1\text{-}s_{r_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1\text{-}s_{r_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1\text{-}s_{r_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\text{-}s_{r_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1\text{-}s_{r_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1\text{-}s_{r_6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1\text{-}s_{r_7} \end{bmatrix} \quad A_r^{r^*} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & s_{r_7} \end{bmatrix}$$

The pump is formulates with the following blocked matrices

$$A_{\text{pu}}^{cv} = \begin{bmatrix} 1\text{-}\bar{s}_1 & 0 \\ 0 & 1\text{-}\bar{s}_2 \end{bmatrix} \quad A_{\text{pu}}^{r^*} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_{\text{pu}}^{\text{jc}} = \begin{bmatrix} \bar{s}_1 & 0 & \text{-}\bar{s}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{s}_2 & \text{-}\bar{s}_2 \end{bmatrix}$$

The modified incidence matrix is partitioned as:

$$A^{\mathrm{jc}}_{\mathrm{f_{pi}}} = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad A^{\mathrm{jc}}_{\mathrm{f_{pu}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$A^{\mathrm{jc}}_{\mathrm{f}} = \left[A^{\mathrm{jc}}_{\mathrm{f_{pi}}} \; A^{\mathrm{jc}}_{\mathrm{f_{pu}}} \right] = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

Further the reservoir nodes are given in the following matrix

$$I_{r^*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The structure of matrices E and A is given as below ,

$$\mathrm{E}= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$
A=\left(\begin{array}{ccccccccccccccccccccccccccccccc}
0&0&0&0&0&0&0&0&0 & c_1&0&0&0&0&0&0 & -c_1&0&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&c_2&0&0&0&0&0 & 0&-c_2&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&c_3&0&0&0&0 & 0&0&-c_3&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&c_4&0&0&0 & 0&0&0&-c_4&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&c_5&0&0 & 0&0&0&0&-c_5&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&c_6&0 & 0&0&0&0&0&-c_6&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&c_7 & 0&0&0&0&0&0&-c_7 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & -1&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 1&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&-1&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&-1&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&-1&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&-1&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&-1&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&-1 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&1&0\\
1-s_{r_1}&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & -s_{r_1}&0&0&0&0&0&0 & 0&0 & 0&0&s_{r_1}&0&0&0\\
0&1-s_{r_2}&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&-s_{r_2}&0&0&0&0&0 & 0&0 & 0&-s_{r_2}&0&0&0&0\\
0&0&1-s_{r_3}&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0&-s_{r_3}&0&0&0&0 & 0&0 & 0&0&0&s_{r_3}&0&0\\
0&0&0&1-s_{r_4}&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0&0&-s_{r_4}&0&0&0 & 0&0 & 0&0&0&0&-s_{r_4}&0\\
0&0&0&0&1-s_{r_5}&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0&0&0&-s_{r_5}&0&0 & 0&0 & 0&0&0&0&s_{r_5}&0\\
0&0&0&0&0&1-s_{r_6}&0&0&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&-s_{r_6}&0 & 0&0 & 0&0&0&0&0&s_{r_6}\\
0&0&0&0&0&0&1-s_{r_7}&0&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&-s_{r_7} & 0&s_{r_7} & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&1-\bar{s}_1&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & \bar{s}_1&0&-\bar{s}_1&0&0&0\\
0&0&0&0&0&0&0&0&1-\bar{s}_2 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&\bar{s}_2&-\bar{s}_2\\
0&-1&-1&0&0&0&0&1&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&1&0&-1&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
1&0&0&0&0&0&0&-1&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&1&0&-1&-1&0&1&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&1&1&0&-1&0&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&-1 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 1&0 & 0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&0&0&0&0&0&0 & 0&1 & 0&0&0&0&0&0
\end{array}\right)
$$

5.2.3 Structure of the blocked matrices

The matrices A_l , $A_r \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}$ consists of the coefficients of $P_{l(k)}$ and $P_{r(k)}$ in the pipe equation (5.9), respectively for k^{th} pipe edge e and $e \in \mathcal{E}_{\text{pi}}$. The matrix takes the form:

$$A_l = (c_1, \cdots, c_{\mathbf{m}}) \qquad A_r = \text{-}A_l \quad \forall e \in \mathcal{E}_{\text{pi}}$$

Further, $A_{\text{f}}^{\text{jc}} \in \mathbb{R}^{n_{\text{jc}}\times m}$ is a matrix constructed from mass balance equations (5.12) can be written as:

$$A_{\text{f}}^{\text{jc}} = \begin{bmatrix} A_{\text{f}_{\text{pi}}}^{\text{jc}} & A_{\text{f}_{\text{pu}}}^{\text{jc}} \end{bmatrix}$$

The matrix $A^{\mathrm{jc}}_{\mathrm{f_{pi}}}$ is the matrix of the coefficients of the connection of junction nodes with the pipe edges . The matrix $A^{\mathrm{jc}}_{\mathrm{f_{pu}}}$ represents the coefficients of the connection of junction nodes with the pump edges .

Remark 5.2.2. The reservoir (supply) nodes are removed from the representation of the network topology to get modified incidence matrix.

A^{v}_{l} is an identity matrix of order $\mathbf{m} \times \mathbf{m}$ consists of the coefficients of all the equation (5.17a) of the pressure $P_{l(k)}$, which will be identity, that is

$$A^{\mathrm{v}}_{l} = -\mathcal{I}$$

$\mathcal{I}$ denoted identity matrix . The matrix of coefficients of the P_i in (5.17a) is denoted as three different matrices namely, A^{rs}_{l} and A^{jc}_{l} with following matrices structure:

1. The matrix A^{rs}_{l} of order $\mathbf{m} \times n_{\mathrm{rs}}$ consists of 1 or 0, each entry $a_{l(k)i} \in A^{\mathrm{rs}}_{l}$ is formulated as:

$$a_{l(k)i} = \left\{ \begin{array}{lr} 1, & \text{left end } l(k) \text{ of the } k^{th} \text{ pipe} \\ & \text{is connected to node } i \in \mathcal{N}_{\mathrm{rs}} \\ 0, & \text{otherwise} \end{array} \right\}$$

2. The matrix A^{jc}_{l} of order $\mathbf{m} \times n_{\mathrm{jc}}$ consists of 1 or 0, each entry $a_{l(k)i} \in A^{\mathrm{jc}}_{l}$ formulated as under:

$$a_{l(k)i} = \left\{ \begin{array}{lr} 1, & \text{left end } l(k) \text{ of the } k^{th} \text{ pipe} \\ & \text{is connected to node } i \in \mathcal{N}_{\mathrm{jc}} \\ 0, & \text{otherwise} \end{array} \right\}$$

Similarly, for the matrices $\widetilde{A}^{\mathrm{cv}}_{r}$, $\widetilde{A}^{v}_{r}$, $\widetilde{A}^{\mathrm{rs}}_{r}$ and $\widetilde{A}^{\mathrm{jc}}_{r}$ denotes the coefficients of (5.17b). The matrices are defined as:

1. The matrix $\widetilde{A}_r^{\text{cv}}$ of order $\mathbf{m} \times \mathbf{m}$ consists of matrix of coefficients of Q_k in (5.17b), for the closing of pipe at the right formulated as:
$$\widetilde{A}_r^{\text{cv}} = (1 - s_{r_1}, \cdots, 1 - s_{r_\mathbf{m}})$$
2. $\widetilde{A}_r^{v}$ is a matrix of order $\mathbf{m} \times \mathbf{m}$ consists of the coefficients of $P_{r(k)}$ in (5.17b) formulated as
$$\widetilde{A}_r^{v} = -(s_{r_1}, \cdots, s_{r_\mathbf{m}}) = \mathcal{I} - K_{\text{pi}}^2$$
where $K_{\text{pi}}^2 = (s_{r_1}, \cdots, s_{r_\mathbf{m}})$, contains control variables for each pipe.
3. The matrix $\widetilde{A}_r^{r^*}$ of order $\mathbf{m} \times n_{\text{rs}}$ is the coefficients of P_j, in (5.17b), each entry $a_{r(e)i} \in \widetilde{A}_r^{\text{rs}}$ is characterised as under:
$$a_{r(k)i} = \left\{ \begin{array}{ll} s_{r_k}, & \text{right end } r(k) \text{ of the } k^{th} \text{ pipe} \\ & \text{is connected to node } i \in \mathcal{N}_{\text{rs}} \\ 0, & \text{otherwise} \end{array} \right\}$$
4. The matrix $\widetilde{A}_r^{\text{jc}}$ of order $\mathbf{m} \times n_{\text{jc}}$ is the coefficients of P_j, the entries of the matrix formulated as under:
$$a_{r(k)i} = \left\{ \begin{array}{ll} s_{r_k}, & \text{right end } r(k) \text{ of the } k^{th} \text{ pipe} \\ & \text{is connected to node } i \in \mathcal{N}_{\text{jc}} \\ 0, & \text{otherwise} \end{array} \right\}$$
where control variable s_{r_k} is defined in (5.16).

Furthermore, the coefficient matrices of the pump equation (5.19) are denoted as $\widetilde{A}_{\text{pu}}^{\text{cv}}$, $\widetilde{A}_{\text{pu}}^{\text{rs}}$ and $\widetilde{A}_{\text{pu}}^{\text{jc}}$, for $e_k = (i, j)$, for all $e_k \in \mathcal{E}_{\text{pi}}$ and can be given as:

1. The matrix $\widetilde{A}_{\text{pu}}^{\text{cv}}$ is of order $\breve{m} \times \breve{m}$ and consists of coefficients Q_k showing, the pump on or shutoff equation (5.19) can be written as

a following diagonal matrix

$$\widetilde{A}_{\text{pu}}^{\text{cv}} = (1 - \bar{s}_1, \cdots, 1 - \bar{s}_{\breve{m}})$$

2. The matrix $\widetilde{A}_{\text{pu}}^{\text{rs}}$ is of order $m_{\text{pu}} \times n_{\text{rs}}$ matrix and has structure :

$$a_{ki} = \left\{ \begin{array}{lr} \bar{s}_k, & k^{th} \text{ pump edge is going towards node } i \in \mathcal{N}_{\text{rs}} \\ \text{-}\bar{s}_k, & k^{th} \text{ pump edge is away from node } i \in \mathcal{N}_{\text{rs}} \\ 0, & \text{otherwise} \end{array} \right\}$$

for all $a_{ki} \in \widetilde{A}_{\text{pu}}^{\text{rs}}$.

3. Finally the matrix $\widetilde{A}_{\text{pu}}^{\text{jc}}$ is of order $\breve{m} \times n_{\text{jc}}$ with structure given as:

$$a_{ki} = \left\{ \begin{array}{lr} \bar{s}_k, & k^{th} \text{ pump edge is going towards node } i \in \mathcal{N}_{\text{jc}} \\ \text{-}\bar{s}_k, & k^{th} \text{ pump edge is away from node } i \in \mathcal{N}_{\text{jc}} \\ 0, & \text{otherwise} \end{array} \right\}$$

for every $a_{ki} \in \widetilde{A}_{\text{pu}}^{\text{jc}}$ where $\bar{s}_k$ is defined as follows:

$$\bar{s}_k = \left\{ \begin{array}{lr} 1, & k^{th} \text{ pump is on} \\ 0, & k^{th} \text{ pump is shutdown} \end{array} \right\}$$

5.3 Solvability of water network nonlinear DAE

In the Section 5.2 the structure of matrix pair (E, A) of a general water network is presented. Further to check whether the solution of the general nonlinear DAE (5.20) exist or not . For this purpose first regularity of matrix pair (E, A) is checked, which leads to some further assumptions on the topology of the water network. In order to show the regularity of the matrix pair (E, A) it needs to be shown that, $\det(A - \theta E) \not\equiv 0$. First

$(A - \theta E)$ is calculated as follows;

$$\mathbf{G}[\theta] := A\text{-}\theta E = \begin{bmatrix} \theta\mathcal{I}_{\mathrm{pi}} & 0 & A_l & A_r & 0 & 0 \\ A^{\mathrm{jc}}_{\mathrm{f}_{\mathrm{pi}}} & A^{\mathrm{jc}}_{\mathrm{f}_{pu}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A^{\mathrm{v}}_l & 0 & A^{\mathrm{rs}}_l & A^{\mathrm{jc}}_l \\ \widetilde{A}^{\mathrm{cv}}_r & 0 & 0 & \widetilde{A}^{\mathrm{v}}_r & \widetilde{A}^{\mathrm{rs}}_r & \widetilde{A}^{\mathrm{jc}}_r \\ 0 & \widetilde{A}^{\mathrm{cv}}_{pu} & 0 & 0 & \widetilde{A}^{\mathrm{rs}}_{pu} & \widetilde{A}^{\mathrm{jc}}_{pu} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 \end{bmatrix} \in \mathbb{R}^{\Upsilon\times\Upsilon}[\theta] \quad (5.23)$$

with $\Upsilon = 3\mathbf{m} + \breve{m} + n_{\mathrm{rs}} + n_{\mathrm{jc}}$ some simplifications will be done, for the sake of further investigation on the structure of the matrix.

1. Consider the following invertible transformation matrix,

$$\bar{U}_1 = \begin{bmatrix} \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I}_{n_{\mathrm{jc}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & \text{-}A^{\mathrm{rs}}_l \\ 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & \text{-}A^{\mathrm{rs}}_r \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & \text{-}A^{\mathrm{rs}}_{\mathrm{pu}} \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} \end{bmatrix}$$

such that

$$= \bar{U}_1 \mathbf{G}[\theta]$$

$$= \begin{bmatrix} \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I}_{n_{\text{jc}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & \text{-}A_l^{\text{rs}} \\ 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & \text{-}A_r^{\text{rs}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & \text{-}A_{\text{pu}}^{\text{rs}} \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\text{rs}} \end{bmatrix} \begin{bmatrix} \theta\mathcal{I}_{\text{pi}} & 0 & A_l & A_r & 0 & 0 \\ A_{\text{f}_{\text{pi}}}^{\text{jc}} & A_{\text{f}_{pu}}^{\text{jc}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_l^{\text{v}} & 0 & A_l^{\text{rs}} & A_l^{\text{jc}} \\ \widetilde{A}_r^{\text{cv}} & 0 & 0 & \widetilde{A}_r^{\text{v}} & \widetilde{A}_r^{\text{rs}} & \widetilde{A}_r^{\text{jc}} \\ 0 & \widetilde{A}_{pu}^{\text{cv}} & 0 & 0 & \widetilde{A}_{pu}^{\text{rs}} & \widetilde{A}_{pu}^{\text{jc}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\text{rs}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \theta\mathcal{I}_{\text{pi}} & 0 & A_l & A_r & 0 & 0 \\ A_{\text{f}_{\text{pi}}}^{\text{jc}} & A_{\text{f}_{pu}}^{\text{jc}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_l^{\text{v}} & 0 & 0 & A_l^{\text{jc}} \\ \widetilde{A}_r^{\text{cv}} & 0 & 0 & \widetilde{A}_r^{\text{v}} & 0 & \widetilde{A}_r^{\text{jc}} \\ 0 & \widetilde{A}_{pu}^{\text{cv}} & 0 & 0 & 0 & \widetilde{A}_{pu}^{\text{jc}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\text{rs}} & 0 \end{bmatrix} := \mathbf{G}_1[\theta]$$

2. Assume $K_{\text{pi}}^1 = \text{diag}(c_1, \cdots, c_{\mathbf{m}}) \in \mathbb{R}_+^{\mathbf{m}\times\mathbf{m}}$, where $c_i = \frac{\mathbf{A}_i}{L_i} > 0\ \forall i$, where $\mathbf{A}_i$ and L_i denotes the area and lenght of the pipe, respectively. By construction $A_l = K_{\text{pi}}^1 A_l^v$, discussed in the Section 5.2.3. Consider the invertible transformation matrix $\bar{U}_2$

$$\bar{U}_2 = \begin{bmatrix} \mathcal{I}_{\mathbf{m}} & 0 & \text{-}K_{\text{pi}}^1 & 0 & 0 & 0 \\ 0 & \mathcal{I}_{n_{\text{jc}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\text{rs}} \end{bmatrix}$$

such that

$$
\begin{aligned}
&= \bar{U}_2 \mathbf{G}_1[\theta] \\
&= \begin{bmatrix}
\mathcal{I}_{\mathbf{m}} & 0 & \text{-}K^1_{\mathrm{pi}} & 0 & 0 & 0 \\
0 & \mathcal{I}_{n_{\mathrm{jc}}} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}}
\end{bmatrix}
\begin{bmatrix}
\theta\mathcal{I}_{\mathrm{pi}} & 0 & A_l & A_r & 0 & 0 \\
A^{\mathrm{jc}}_{\mathrm{f}_{\mathrm{pi}}} & A^{\mathrm{jc}}_{\mathrm{f}_{pu}} & 0 & 0 & 0 & 0 \\
0 & 0 & A^{\mathrm{v}}_l & 0 & 0 & A^{\mathrm{jc}}_l \\
\widetilde{A}^{\mathrm{cv}}_r & 0 & 0 & \widetilde{A}^{\mathrm{v}}_r & 0 & \widetilde{A}^{\mathrm{jc}}_r \\
0 & \widetilde{A}^{\mathrm{cv}}_{pu} & 0 & 0 & 0 & \widetilde{A}^{\mathrm{jc}}_{pu} \\
0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0
\end{bmatrix} \\
&= \begin{bmatrix}
\theta\mathcal{I}_{\mathrm{pi}} & 0 & 0 & A_r & 0 & K^1_{\mathrm{pi}} \\
A^{\mathrm{jc}}_{\mathrm{f}_{\mathrm{pi}}} & A^{\mathrm{jc}}_{\mathrm{f}_{pu}} & 0 & 0 & 0 & 0 \\
0 & 0 & A^{\mathrm{v}}_l & 0 & 0 & A^{\mathrm{jc}}_l \\
\widetilde{A}^{\mathrm{cv}}_r & 0 & 0 & \widetilde{A}^{\mathrm{v}}_r & 0 & \widetilde{A}^{\mathrm{jc}}_r \\
0 & \widetilde{A}^{\mathrm{cv}}_{pu} & 0 & 0 & 0 & \widetilde{A}^{\mathrm{jc}}_{pu} \\
0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0
\end{bmatrix} := \mathbf{G}_2[\theta]
\end{aligned}
$$

3. $K^2_{\mathrm{pi}} = (s_1, \cdots, s_{\mathbf{m}}) \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}_{+}$ and by the construction the invertible transformation matrix

$$
\bar{U}_3 = \begin{bmatrix}
\mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{I}_{n_{\mathrm{jc}}} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 \\
K^2_{\mathrm{pi}} & 0 & 0 & \text{-}K^1_{\mathrm{pi}} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}}
\end{bmatrix}
$$

such that

$$
\begin{aligned}
&= \bar{U}_3 \mathbf{G}_2[\theta] \\
&= \begin{bmatrix} \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I}_{n_{\mathrm{jc}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 \\ K_{\mathrm{pi}}^2 & 0 & 0 & \text{-}K_{\mathrm{pi}}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} \end{bmatrix} \begin{bmatrix} \theta\mathcal{I}_{\mathrm{pi}} & 0 & A_l & A_r & 0 & 0 \\ A_{\mathrm{f_{pi}}}^{\mathrm{jc}} & A_{\mathrm{f}_{pu}}^{\mathrm{jc}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_l^{\mathrm{v}} & 0 & 0 & A_l^{\mathrm{jc}} \\ \widetilde{A}_r^{\mathrm{cv}} & 0 & 0 & \widetilde{A}_r^{\mathrm{v}} & 0 & \widetilde{A}_r^{\mathrm{jc}} \\ 0 & \widetilde{A}_{pu}^{\mathrm{cv}} & 0 & 0 & 0 & \widetilde{A}_{pu}^{\mathrm{jc}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \theta\mathcal{I}_{\mathrm{pi}} & 0 & 0 & A_r & 0 & K_{\mathrm{pi}}^1 A_l^{\mathrm{jc}} \\ A_{\mathrm{f_{pi}}}^{\mathrm{jc}} & A_{\mathrm{f}_{pu}}^{\mathrm{jc}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_l^{\mathrm{v}} & 0 & 0 & A_l^{\mathrm{jc}} \\ \bar{\bar{A}}_r^{\mathrm{cv}} & 0 & 0 & 0 & 0 & K_{\mathrm{pi}}^1 K_{\mathrm{pi}}^2 (A_r^{\mathrm{jc}} - A_l^{\mathrm{jc}}) \\ 0 & \widetilde{A}_{pu}^{\mathrm{cv}} & 0 & 0 & 0 & \widetilde{A}_{pu}^{\mathrm{jc}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 \end{bmatrix} := \mathbf{G}_3[\theta]
\end{aligned}
$$

4.

$$
\bar{U}_4 = \begin{bmatrix} 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 \\ 0 & \mathcal{I}_{n_{\mathrm{jc}}} & 0 & 0 & 0 & 0 \\ \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} \end{bmatrix}
$$

such that

$$= \bar{U}_4 \mathbf{G}_3[\theta]$$

$$= \begin{bmatrix} 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 \\ 0 & \mathcal{I}_{n_{\text{jc}}} & 0 & 0 & 0 & 0 \\ \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\mathbf{m}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}_{\text{rs}} \end{bmatrix} \begin{bmatrix} \theta\mathcal{I}_{\text{pi}} & 0 & 0 & A_r & 0 & K^1_{\text{pi}} A^{\text{jc}}_l \\ A^{\text{jc}}_{\text{f}_{\text{pi}}} & A^{\text{jc}}_{\text{f}_{pu}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A^{\text{v}}_l & 0 & 0 & A^{\text{jc}}_l \\ \bar{\bar{A}}^{\text{cv}}_r & 0 & 0 & 0 & 0 & K^1_{\text{pi}} K^2_{\text{pi}} (A^{\text{jc}}_r - A^{\text{jc}}_l) \\ 0 & \widetilde{A}^{\text{cv}}_{pu} & 0 & 0 & 0 & \widetilde{A}^{\text{jc}}_{pu} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\text{rs}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\bar{A}}^{\text{cv}}_r & 0 & 0 & 0 & 0 & K^1_{\text{pi}} K^2_{\text{pi}} (A^{\text{jc}}_r - A^{\text{jc}}_l) \\ 0 & \widetilde{A}^{\text{cv}}_{pu} & 0 & 0 & 0 & \widetilde{A}^{\text{jc}}_{pu} \\ A^{\text{jc}}_{\text{f}_{\text{pi}}} & A^{\text{jc}}_{\text{f}_{pu}} & 0 & 0 & 0 & 0 \\ \theta\mathcal{I}_{\text{pi}} & 0 & 0 & A_r & 0 & K^1_{\text{pi}} A^{\text{jc}}_l \\ 0 & 0 & A^{\text{v}}_l & 0 & 0 & A^{\text{jc}}_l \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\text{rs}} & 0 \end{bmatrix} := \mathbf{G}_4[\theta]$$

Hence the transformation matrix can be written as $\mathbf{U}$ written as

$$\mathbf{U} = \bar{U}_4 \bar{U}_3 \bar{U}_2 \bar{U}_1$$

$\mathbf{U}$ is invertible as it is product of three invertible matrices and further $\mathbf{G}[\theta]$ and $\mathbf{G}_4[\theta]$ are related as follows;

$$\mathbf{U}\mathbf{G}[\theta] = \mathbf{G}_4[\theta] \tag{5.24}$$

Also due to invertibility of $\mathbf{U}$ (5.24) can be written as (5.25) ;

$$\mathbf{G}[\theta] = (\mathbf{U})^{\text{-1}} \mathbf{G}_4[\theta] \tag{5.25}$$

Observation 5.3.1. Using (5.25) ; for the matrix $\mathbf{G}[\theta] \in \mathbb{R}^{\Upsilon \times \Upsilon}[\theta]$ to be invertible it is sufficient to show that $\mathbf{G}_4[\theta] \in \mathbb{R}^{\Upsilon \times \Upsilon}$ is invertible.

Further consider

$$\mathbf{G}_4[\theta] = \begin{bmatrix} \bar{\bar{A}}_r^{\mathrm{cv}}[\theta] & 0 & 0 & 0 & 0 & K_{\mathrm{pi}}^1 K_{\mathrm{pi}}^2 (A_r^{\mathrm{jc}} - A_l^{\mathrm{jc}}) \\ 0 & \widetilde{A}_{pu}^{\mathrm{cv}} & 0 & 0 & 0 & \widetilde{A}_{pu}^{\mathrm{jc}} \\ A_{\mathrm{f_{pi}}}^{\mathrm{jc}} & A_{\mathrm{f}_{pu}}^{\mathrm{jc}} & 0 & 0 & 0 & 0 \\ \theta \mathcal{I}_{\mathrm{pi}} & 0 & 0 & A_r & 0 & K_{\mathrm{pi}}^1 A_l^{\mathrm{jc}} \\ 0 & 0 & A_l^{\mathrm{v}} & 0 & 0 & A_l^{\mathrm{jc}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 \end{bmatrix} \tag{5.26}$$

where,

$$\begin{aligned} \widetilde{A}_r^{\mathrm{jc}} &= K_{\mathrm{pi}}^2 A_r^{\mathrm{jc}} \\ \widetilde{A}_{\mathrm{pu}}^{\mathrm{jc}} &= K_{\mathrm{pu}} A_{\mathrm{pu}}^{\mathrm{jc}} \qquad K_{\mathrm{pu}} = (\bar{s}_1, \cdots, \bar{s}_{m_{\mathrm{pu}}}) \\ \bar{\bar{\mathbf{m}}}_r^{\mathrm{cv}}[\theta] &= (c_1(1 - s_{r_1}) + \theta s_{r_1}, \cdots, c_{\mathbf{m}}(1 - s_{r_{\mathbf{m}}}) + \theta s_{r_{\mathbf{m}}} \\ \widetilde{A}_{\mathrm{pu}}^{\mathrm{cv}} &= (1 - \bar{s}_1, \cdots, 1 - \bar{s}_{\breve{m}}) \end{aligned}$$

On interchanging column 3 and column 6 the resulting matrix is;

$$\mathbf{G}_5[\theta] = \begin{bmatrix} \bar{\bar{A}}_r^{\mathrm{cv}}[\theta] & 0 & K_{\mathrm{pi}}^1 K_{\mathrm{pi}}^2 (A_r^{\mathrm{jc}} - A_l^{\mathrm{jc}}) & 0 & 0 & 0 \\ 0 & \widetilde{A}_{pu}^{\mathrm{cv}} & 0 & \widetilde{A}_{pu}^{\mathrm{jc}} & 0 & 0 \\ A_{\mathrm{f_{pi}}}^{\mathrm{jc}} & A_{\mathrm{f}_{pu}}^{\mathrm{jc}} & 0 & 0 & 0 & 0 \\ \theta \mathcal{I}_{\mathrm{pi}} & 0 & K_{\mathrm{pi}}^1 A_l^{\mathrm{jc}} & A_r & 0 & 0 \\ 0 & 0 & A_l^{\mathrm{jc}} & 0 & 0 & A_l^{\mathrm{v}} \\ 0 & 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 \end{bmatrix}$$

where

$$\begin{aligned} K_{\mathrm{pi}} &= K_{\mathrm{pi}}^1 K_{\mathrm{pi}}^2, \\ &= (c_1, \cdots, c_{\mathbf{m}})(s_{r_1}, \cdots, s_{r_{\mathbf{m}}}), \\ &= (c_1 s_{r_1}, \cdots, c_{\mathbf{m}} s_{r_{\mathbf{m}}}) \end{aligned} \tag{5.27}$$

The following partitioning of (5.26) is obtained

$$\mathbf{G}_5[\theta] = \left[\begin{array}{ccc|ccc} \bar{\bar{A}}^{\mathrm{cv}}_r[\theta] & 0 & K_{\mathrm{pi}}(A^{\mathrm{jc}}_r - A^{\mathrm{jc}}_l) & 0 & 0 & 0 \\ 0 & \widetilde{A}^{\mathrm{cv}}_{pu} & K_{\mathrm{pu}}A^{\mathrm{jc}}_{\mathrm{pu}} & 0 & 0 & 0 \\ A^{\mathrm{jc}}_{\mathrm{f_{pi}}} & A^{\mathrm{jc}}_{\mathrm{f_{pu}}} & 0 & 0 & 0 & 0 \\ \hline \theta\mathcal{I}_{\mathrm{pi}} & 0 & K^1_{\mathrm{pi}}A^{\mathrm{jc}}_l & 0 & 0 & A_r \\ 0 & 0 & A^{\mathrm{jc}}_l & 0 & A^v_l & 0 \\ 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 & 0 \end{array}\right] \tag{5.28}$$

by using (5.3) , following substitutions can be made

$$(A^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top = A^{\mathrm{jc}}_r\text{-}A^{\mathrm{jc}}_l$$

For an edge $e \in \mathcal{E}_{\mathrm{pu}}$, by using (5.4) following can be written

$$K_{\mathrm{pu}}A^{\mathrm{jc}}_{\mathrm{pu}} = K_{\mathrm{pu}}(A^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top$$

it is due to the fact each pump edge also has a left and right node; as it is a directed edge. By summing up the above simplifications, (5.28) can be read as:

$$\mathbf{G}_5[\theta] = \left[\begin{array}{ccc|ccc} \bar{\bar{A}}^{\mathrm{cv}}_r[\theta] & 0 & K_{\mathrm{pi}}(A^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top & 0 & 0 & 0 \\ 0 & \widetilde{A}^{\mathrm{cv}}_{pu} & K_{\mathrm{pu}}(A^{\mathrm{jc}}_{\mathrm{pu}})^\top & 0 & 0 & 0 \\ A^{\mathrm{jc}}_{\mathrm{f_{pi}}} & A^{\mathrm{jc}}_{\mathrm{f_{pu}}} & 0 & 0 & 0 & 0 \\ \hline \theta\mathcal{I}_{\mathrm{pi}} & 0 & K^1_{\mathrm{pi}}A^{\mathrm{jc}}_l & 0 & 0 & A_r \\ 0 & 0 & A^{\mathrm{jc}}_l & 0 & A^v_l & 0 \\ 0 & 0 & 0 & \mathcal{I}_{\mathrm{rs}} & 0 & 0 \end{array}\right] \tag{5.29}$$

Further denote the partitions of the matrix (5.29)

$$\mathbf{G}_5[\theta] = \left[\begin{array}{c|c} \mathbf{G}_{51}[\theta] & 0 \\ \hline \mathbf{G}_{52} & \mathbf{G}_{53} \end{array}\right] \tag{5.30}$$

where

$$\mathbf{G}_{52}[\theta] = \begin{bmatrix} \theta\mathcal{I}_{\mathrm{pi}} & 0 & K_{\mathrm{pi}}^{1}A_{l}^{\mathrm{jc}} \\ 0 & 0 & \widetilde{A}_{l}^{\mathrm{jc}} \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{G}_{53} = \begin{bmatrix} 0 & 0 & A_r \\ 0 & A_l^{\mathrm{v}} & 0 \\ \mathcal{I}_{\mathrm{rs}} & 0 & 0 \end{bmatrix} \tag{5.31}$$

Remark 5.3.2. The matrix $\mathbf{G}_5[\theta]$ is invertible if and only if $\mathbf{G}_{53} \in \mathbb{R}^{(2\mathbf{m}+n_{\mathrm{rs}})\times(2\mathbf{m}+n_{\mathrm{rs}})}$ and $\mathbf{G}_{51}[\theta] \in \mathbb{R}^{(\mathbf{m}+\check{m}+n_{\mathrm{jc}})\times(\mathbf{m}+\check{m}+n_{\mathrm{jc}})}[\theta]$ is invertible.

In order to show $\mathbf{G}_{53}$ is invertible:
Consider

$$\mathbf{G}_{53} = \begin{bmatrix} 0 & 0 & A_r \\ 0 & A_l^{\mathrm{v}} & 0 \\ \mathcal{I}_{\mathrm{rs}} & 0 & 0 \end{bmatrix}$$

By construction $A_l^{\mathrm{v}} \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}$ is identity matrix, $A_r \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}$ is a diagonal matrix. The matrix $\mathcal{I}_{\mathrm{rs}} \in \mathbb{R}^{n_{\mathrm{rs}}\times n_{\mathrm{rs}}}$ is identity matrix. They are all invertible which results into the invertibility of $\mathbf{G}_{53}$.

Observation 5.3.3. $\mathbf{G}_5[\theta] \in \mathbb{R}^{\Upsilon\times\Upsilon}$ is invertible iff
$\mathbf{G}_{51}[\theta] \in \mathbb{R}^{(\mathbf{m}+\check{m}+n_{\mathrm{jc}})\times(\mathbf{m}+\check{m}+n_{\mathrm{jc}})}[\theta]$ is invertible.

Further consider for some simplifications

$$\mathbf{G}_{51}[\theta] = \begin{bmatrix} \bar{\bar{A}}_{r}^{\mathrm{cv}}[\theta] & 0 & K_{\mathrm{pi}}(A_{\mathrm{f_{pi}}}^{\mathrm{jc}})^{\top} \\ 0 & \widetilde{A}_{pu}^{\mathrm{cv}} & K_{\mathrm{pu}}(A_{f_{\mathrm{pu}}}^{\mathrm{jc}})^{\top} \\ A_{\mathrm{f_{pi}}}^{\mathrm{jc}} & A_{\mathrm{f}_{pu}}^{\mathrm{jc}} & 0 \end{bmatrix} \tag{5.32}$$

$K^1_{\mathrm{pi}} \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}$ and $K_{\mathrm{pi}} \in \mathbb{R}^{\mathbf{m}\times\mathbf{m}}$ diagonal matrices can be partitioned into open and closed valves (edges) as;

$$K^1_{\mathrm{pi}} = \begin{bmatrix} (K^1_{\mathrm{pi}})_{\mathrm{op}} & 0 \\ 0 & (K^1_{\mathrm{pi}})_{\mathrm{cl}} \end{bmatrix} \qquad K_{\mathrm{pi}} = \begin{bmatrix} K^{\mathrm{op}}_{\mathrm{pi}} & 0 \\ 0 & K^{\mathrm{cl}}_{\mathrm{pi}} \end{bmatrix}$$
$$K_{\mathrm{pu}} = \begin{bmatrix} K^{\mathrm{op}}_{\mathrm{pu}} & 0 \\ 0 & K^{\mathrm{cl}}_{\mathrm{pu}} \end{bmatrix}$$

where

$$\begin{aligned} &(K^1_{\mathrm{pi}})_{\mathrm{op}} \in \mathbb{R}_+^{\mathbf{m}_{\mathrm{op}}\times\mathbf{m}_{\mathrm{op}}}, \quad \text{and} \quad (K^1_{\mathrm{pi}})_{\mathrm{cl}} \in \mathbb{R}_+^{\mathbf{m}_{\mathrm{cl}}\times\mathbf{m}_{\mathrm{cl}}} \\ &(K_{\mathrm{pi}})_{\mathrm{op}} \in \mathbb{R}_+^{\mathbf{m}_{\mathrm{op}}\times\mathbf{m}_{\mathrm{op}}} \quad \text{and} \quad (K_{\mathrm{pi}})_{\mathrm{cl}} \in \mathbb{R}_+^{\mathbf{m}_{\mathrm{cl}}\times\mathbf{m}_{\mathrm{cl}}}, \\ &K^{\mathrm{op}}_{\mathrm{pu}} \in \mathbb{R}_+^{\breve{m}_{\mathrm{op}}\times\breve{m}_{\mathrm{op}}} \quad \text{and} \quad K^{\mathrm{cl}}_{\mathrm{pu}} \in \mathbb{R}_+^{\breve{m}_{\mathrm{cl}}\times\breve{m}_{\mathrm{cl}}}. \end{aligned}$$

Split (5.32) into close and open valve and pumps, (5.32) takes the form

$$\mathbf{G}_{51}[\theta] = \begin{bmatrix} (\bar{\bar{A}}^{\mathrm{cv}}_r)_{\mathrm{op}}[\theta] & 0 & 0 & 0 & K^{\mathrm{op}}_{\mathrm{pi}}(A^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top_{\mathrm{op}} \\ 0 & (\bar{\bar{A}}^{\mathrm{cv}}_r)_{\mathrm{cl}} & 0 & 0 & K^{\mathrm{cl}}_{\mathrm{pi}}(A^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top_{\mathrm{cl}} \\ 0 & 0 & (\widetilde{A}^{\mathrm{cv}}_{\mathrm{pu}})_{\mathrm{op}} & 0 & K^{\mathrm{op}}_{\mathrm{pu}}(A^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top_{\mathrm{op}} \\ 0 & 0 & 0 & (\widetilde{A}^{\mathrm{cv}}_{\mathrm{pu}})_{\mathrm{cl}} & K^{\mathrm{op}}_{\mathrm{cl}}(A^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top_{\mathrm{cl}} \\ (A^{\mathrm{jc}}_{\mathrm{f_{pi}}})_{\mathrm{op}} & (A^{\mathrm{jc}}_{\mathrm{f_{pi}}})_{\mathrm{cl}} & (A^{\mathrm{jc}}_{\mathrm{f_{pu}}})_{\mathrm{op}} & (A^{\mathrm{jc}}_{\mathrm{f_{pu}}})_{\mathrm{cl}} & 0 \end{bmatrix} \tag{5.33}$$

Investigating the structure in further detail,

$$\bar{\bar{A}}^{\mathrm{cv}}_r[\theta] = (c_1(1 - s_{r_1}) + \theta s_{r_1}, \cdots, c_{\mathbf{m}}(1 - s_{r_{\mathbf{m}}}) + \theta s_{r_{r_{\mathbf{m}}}}) \tag{5.34}$$

it can be divided into closed $s_{r_k} = 0$ and open valve $s_{r_k} = 1$ for any pipe k for all $e_k \in \mathcal{E}_{\mathrm{pi}}$

$$\bar{\bar{A}}^{\mathrm{cv}}_r[\theta] = \begin{bmatrix} (\bar{\bar{A}}^{\mathrm{cv}}_r)_{\mathrm{op}}[\theta] & 0 \\ 0 & (\bar{\bar{A}}^{\mathrm{cv}}_r)_{\mathrm{cl}} \end{bmatrix}$$

Assume $\mathbf{m}_{\mathrm{op}}$ be number of open valve and $\mathbf{m}_{\mathrm{cl}}$ are closed for all pipe edges $e \in \mathcal{E}_{\mathrm{pi}}$. Relabel and reordering the the open pipe edges as $1, \cdots, \mathbf{m}_{\mathrm{op}}$ and closed pipe edges are $\mathbf{m}_{\mathrm{op}} + 1, \cdots, \mathbf{m}_{\mathrm{cl}}$ in the way that first writing open and closed edges. First substitute $s_{r_k} = 1$ in (5.27) and then (5.34), that will results into the following

$$\begin{aligned} (\bar{\bar{\mathbf{A}}}^{\mathrm{cv}}_r)_{\mathrm{op}} &= (\theta, \cdots, \theta \cdots \theta)_{\mathbf{m}_{\mathrm{op}}} \in \mathbb{R}^{\mathbf{m}_{\mathrm{op}} \times \mathbf{m}_{\mathrm{op}}}[\theta] \\ (K^1_{\mathrm{pi}})_{\mathrm{cl}} &= (c_{m_{\mathrm{op}}+1}, \cdots, c_{\mathbf{m}_{\mathrm{cl}}}) \in \mathbb{R}^{\mathbf{m}_{\mathrm{cl}} \times \mathbf{m}_{\mathrm{cl}}}. \end{aligned} \tag{5.35}$$

where $K^{\mathrm{op}}_{\mathrm{pi}} \in \mathbb{R}^{\mathbf{m}_{\mathrm{op}} \times \mathbf{m}_{\mathrm{op}}}$ is the diagonal matrix with the c_i's corresponding to open pipe edges. On substituting $s_{r_k} = 0$ in (5.27) for closed pipes j the following will be found

$$(\bar{\bar{\mathbf{A}}}^{\mathrm{cv}}_r)_{\mathrm{cl}} = (K^1_{\mathrm{pi}})_{\mathrm{cl}} \quad K^{\mathrm{cl}}_{\mathrm{pi}} = 0_{\mathbf{m}_{\mathrm{cl}} \times \mathbf{m}_{\mathrm{cl}}} \quad K^{\mathrm{cl}}_{\mathrm{pi}}(A^{\mathrm{jc}}_{\mathrm{f_{pi}}})^{\top} = 0_{n_{\mathrm{jc}} \times \mathbf{m}_{\mathrm{cl}}}$$

Similarly for $\widetilde{A}^{\mathrm{cv}}_{\mathrm{pu}}$

$$\widetilde{A}^{\mathrm{cv}}_{\mathrm{pu}} = \begin{bmatrix} K^{\mathrm{op}}_{\mathrm{pu}}(\widetilde{A}^{\mathrm{cv}}_{\mathrm{pu}})_{\mathrm{op}} & 0 \\ 0 & K^{\mathrm{cl}}_{\mathrm{pu}}(A^{\mathrm{cv}}_{\mathrm{pu}})_{\mathrm{cl}} \end{bmatrix}$$

On substituting $\bar{s}_k = 1$ for working pumps k the following will be obtained

$$\begin{aligned} (\widetilde{A}^{\mathrm{cv}}_{\mathrm{pu}})_{\mathrm{op}} &= (0, \cdots, 0) \in \mathbb{R}^{\breve{m}_{\mathrm{op}} \times \breve{m}_{\mathrm{op}}} \\ K^{\mathrm{op}}_{\mathrm{pu}} = \mathcal{I}_{\breve{m}_{\mathrm{op}}}, \quad K^{\mathrm{op}}_{\mathrm{pu}}(A^{\mathrm{jc}}_{\mathrm{f_{pu}}})^{\top}_{\mathrm{op}} &= (A^{\mathrm{jc}}_{\mathrm{f_{pu}}})^{\top}_{\mathrm{op}} \end{aligned} \tag{5.36}$$

On substituting $\bar{s}_k = 0$ for shutdown pumps e_k the following will be found

$$\begin{aligned} (\widetilde{A}^{\text{cv}}_{\text{pu}})_{\text{cl}} &= (1, \cdots, 1) \in \mathbb{R}^{\breve{m}_{\text{cl}} \times \breve{m}_{\text{cl}}} \\ K^{\text{cl}}_{\text{pu}} = 0_{\breve{m}_{\text{cl}} \times \breve{m}_{\text{cl}}} \quad K^{\text{cl}}_{\text{pu}}(A^{\text{jc}}_{\text{f}_{\text{pu}}})^{\top}_{\text{op}} &= 0_{(n_{\text{jc}} \times \breve{m}_{\text{cl}})} \end{aligned} \tag{5.37}$$

where op denotes open, and cl denotes close edges via using valves in pipe or open or closed edges corresponding to the pump settings. That is $\mathbf{m}_{\text{op}}$ and $\mathbf{m}_{\text{cl}}$ denotes the number of pipe edges open and closed, respectively.

$$\mathbf{G}_{31}[\theta] = \begin{bmatrix} \theta\mathcal{I}_{\mathbf{m}_{\text{op}}} & 0 & 0 & 0 & K^{\text{op}}_{\text{pi}}(A^{\text{jc}}_{\text{f}_{\text{pi}}})^{\top}_{\text{op}} \\ 0 & (K^{1}_{\text{pi}})_{\text{cl}}\mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (A^{\text{jc}}_{\text{f}_{\text{pu}}})^{\top}_{\text{op}} \\ 0 & 0 & 0 & \mathcal{I} & 0 \\ (A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{op}} & (A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{cl}} & (A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{op}} & (A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{cl}} & 0 \end{bmatrix} \tag{5.38}$$

further an invertible transformation matrix $\bar{\bar{U}}$

$$\bar{\bar{U}} = \begin{bmatrix} \mathcal{I}_{\mathbf{m}_{\text{op}}} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I}_{\mathbf{m}_{\text{cl}}} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\breve{m}_{\text{cl}}} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I}_{n_{\text{jc}}} & 0 \\ 0 & \text{-}(A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{cl}}((K^{1}_{\text{pi}})_{\text{cl}})^{\text{-}1} & 0 & \text{-}(A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{cl}} & \mathcal{I}_{\breve{m}_{\text{cl}}} \end{bmatrix}$$

such that

$$
\begin{aligned}
&=\bar{\bar{U}}\mathbf{G}_{51}[\theta] \\
&= \begin{bmatrix} \mathcal{I}_{\mathbf{m}_{\text{op}}} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I}_{\mathbf{m}_{\text{cl}}} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_{\breve{m}_{\text{cl}}} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I}_{n_{\text{jc}}} & 0 \\ 0 & \text{-}(A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{cl}}((K^{1}_{\text{pi}})_{\text{cl}})^{\text{-}1} & 0 & \text{-}(A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{cl}} & \mathcal{I}_{\breve{m}_{\text{cl}}} \end{bmatrix} \times \\
&\begin{bmatrix} \theta I_{\mathbf{m}_{\text{op}}} & 0 & 0 & 0 & K^{\text{op}}_{\text{pi}}(A^{\text{jc}}_{\text{f}_{\text{pi}}})^{\top}_{\text{op}} \\ 0 & (K^{1}_{\text{pi}})_{\text{cl}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (A^{\text{jc}}_{\text{f}_{\text{pu}}})^{\top}_{\text{op}} \\ 0 & 0 & 0 & \mathcal{I} & 0 \\ (A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{op}} & (A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{cl}} & (A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{op}} & (A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{cl}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \theta I_{\mathbf{m}_{\text{op}}} & 0 & 0 & 0 & K^{\text{op}}_{\text{pi}}(A^{\text{jc}}_{\text{f}_{\text{pi}}})^{\top}_{\text{op}} \\ 0 & (K^{1}_{\text{pi}})_{\text{cl}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (A^{\text{jc}}_{\text{f}_{\text{pu}}})^{\top}_{\text{op}} \\ 0 & 0 & 0 & \mathcal{I}_{\text{jc}} & 0 \\ (A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{op}} & 0 & (A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{op}} & 0 & 0 \end{bmatrix}
\end{aligned}
$$

By using Laplace formula see row 2 and column 2 and row 4 and column 4 are discarded from (5.33), and it takes the form

$$
\overline{\mathbf{G}}_{51}[\theta] = \begin{bmatrix} \theta\mathcal{I}_{\mathbf{m}_{\text{op}}} & 0 & (K^{\text{op}}_{\text{pi}})(A^{\text{jc}}_{\text{f}_{\text{pi}}})^{\top}_{\text{op}} \\ 0 & 0 & (A^{\text{jc}}_{\text{f}_{\text{pu}}})^{\top}_{\text{op}} \\ (A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{op}} & (A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{op}} & 0 \end{bmatrix} \tag{5.39}
$$

On substitution from (5.35) and (5.36) in (5.39). Only the edges corresponding to which valves are open in pipe edge and pumps are open (working pump) are considered.

$$
\overline{\mathbf{G}}_{51}[\theta] = \begin{bmatrix} \theta\mathcal{I}_{\mathbf{m}_{\text{op}}} & 0 & \widehat{K}_{\text{pi}}(\widehat{A}^{\text{jc}}_{\text{f}_{\text{pi}}})^{\top} \\ 0 & 0 & (\widehat{A}^{\text{jc}}_{\text{f}_{\text{pu}}})^{\top} \\ \widehat{A}^{\text{jc}}_{\text{f}_{\text{pi}}} & \widehat{A}^{\text{jc}}_{\text{f}_{\text{pu}}} & 0 \end{bmatrix} \tag{5.40}
$$

where

$$
\begin{aligned}
\widehat{K}_{\text{pi}} &:= (K^{\text{op}}_{\text{pi}}) \in \mathbb{R}^{\mathbf{m}_{\text{op}} \times \mathbf{m}_{\text{op}}}, \\
\widehat{A}^{\text{jc}}_{\text{f}_{\text{pi}}} &:= (A^{\text{jc}}_{\text{f}_{\text{pi}}})_{\text{op}} \in \mathbb{R}^{\mathbf{n}_{\text{jc}} \times \mathbf{m}_{\text{op}}} \\
\widehat{A}^{\text{jc}}_{\text{f}_{\text{pu}}} &:= (A^{\text{jc}}_{\text{f}_{\text{pu}}})_{\text{op}} \in \mathbb{R}^{\mathbf{n}_{\text{jc}} \times \breve{m}_{\text{op}}}
\end{aligned}
$$

Observation 5.3.4. *The matrix* $\mathbf{G}_{51}[\theta] \in \mathbb{R}^{(\mathbf{m}+\breve{m}+n_{\text{jc}})\times(\mathbf{m}+\breve{m}+n_{\text{jc}})}[\theta]$ *has full rank iff* $\overline{\mathbf{G}}_{51}[\theta] \in \mathbb{R}^{(\mathbf{m}_{op}+\breve{m}_{op}+n_{\text{jc}})\times(\mathbf{m}_{op}+\breve{m}_{op}+n_{\text{jc}})}[\theta]$ *has full rank.*

Now before presenting the main theorem on the characterisation of the regularity of the matrix pair E, A the motivation for these assumptions will be presented. These assumptions are also physically motivated for the regularity along with mathematical importance.

Motivation for the assumptions on the network topologies: Consider (5.40) from the row 3 and column 2 the basic conditions for the regularity are clear and are explained in following Remark 5.3.5.

Remark 5.3.5 (Cycles of pumps)**.** It is clear that if the columns of $\widehat{A}^{\text{jc}}_{\text{f}_{\text{pu}}} \in \mathbb{R}^{n_{\text{jc}} \times \breve{m}_{\text{op}}}$ are linearly dependent that will implies the rank deficiency of (5.40). A Lemma 5.1.26 is presented to make graph topological relation of the full column rank to avoid this situation. By same arguments the full row rank of

$$
\widehat{A}^{\text{jc}}_{\text{f}} = \begin{bmatrix} \widehat{A}^{\text{jc}}_{\text{f}_{\text{pi}}} & \widehat{A}^{\text{jc}}_{\text{f}_{\text{pu}}} \end{bmatrix} \in \mathbb{R}^{n_{\text{jc}} \times m_{\text{op}}}
$$

is required otherwise the matrix will rank deficient. Hence not regular.

Furthermore, the Corollaries 5.3.6 and 5.3.7 are presented to illustrate the topological network for the assumptions to ensure the regularity of the water network.

Corollary 5.3.6 (Corollary to the Lemma 5.1.26)**.** *The matrix* $\widehat{A}^{\text{jc}}_{f_{\text{pu}}} \in \mathbb{R}^{n_{\text{jc}} \times \breve{m}_{op}}$ *have full column rank iff there exists no pumps working in a cycle.*

Proof. The columns of $\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}$ represents the edges which are pumps. Then the linear dependence of the column of $\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}$ will imply the existence of the cycle (loop) of the edges representing pumps. Using Lemma 5.1.26 the proof concluded. ■

Corollary 5.3.7 (Corollary to the Lemma 5.1.23)**.** *The modified incidence matrix* $\widehat{A}^{\mathrm{jc}}_{f} \in \mathbb{R}^{n_{\mathrm{jc}} \times m_{op}}$ *has full row rank iff each connected component is connected to a supply node (reservoir).*

Proof. By the definition of modified incidence matrix 5.1.15, all supply nodes (reference nodes) are deleted from the incidence matrix. The reservoir nodes are omitted from the description given in the Section 5.2.3. Then every column in modified incidence matrix representing the edge whose one end is connected to a reservoir node contains only one non zero entry. The matrix $\widehat{A}^{\mathrm{jc}}_{\mathrm{f}}$ is the modified incidence matrix will be of full row rank of it is connected to at least one reservoir (reference) node. By the same arguments used in the Lemma 5.1.23 concludes the proof. ■

In the next theorem the characterisation of the regularity of the general water network DAE is presented.

Theorem 5.3.8 (**Regularity characterisation of water network DAE** (5.20))**.** *Consider the general structure of water network described by nonlinear DAE* (5.20) *with*

(1) $n = n_{\mathrm{jc}} + n_{\mathrm{rs}}$*, where* n *are total number of nodes with* n_{jc} *junction and* n_{rs} *reservoir nodes,*

(2) $m = \mathbf{m} + \breve{m}$ *with* m *edges out of which* $\mathbf{m}$ *are pipes and* $\breve{m}$ *pumps,*

(3) $m_{op} = \mathbf{m}_{op} + \breve{m}_{op}$*, where* $\mathbf{m}_{op}, \breve{m}_{op}$ *denotes open pipes (not closed via valve) and open pumps (not shutdown), respectively.*

Then the matrix pairs (E, A) *of the system* (5.21) *is regular iff the following assumptions hold true;*

(C-I): *The matrix* $\widehat{A}^{\mathrm{jc}}_{f_{\mathrm{pu}}} \in \mathbb{R}^{n_{\mathrm{jc}} \times \breve{m}_{op}}$ *have full column rank; that is there exists no cycle of pumps in the network.*

(C-II): *The modified incidence matrix* $\widehat{A}^{\mathrm{jc}}_{f} = \begin{bmatrix} \widehat{A}^{\mathrm{jc}}_{\mathrm{pi}} & \widehat{A}^{\mathrm{jc}}_{\mathrm{pu}} \end{bmatrix} \in \mathbb{R}^{n_{\mathrm{jc}} \times m_{op}}$ *have full row rank meaning that, each connected component of the network is connected to a supply node (reservoir).*

Proof. By the observations 5.3.1, 5.3.3 and 5.3.4 it is sufficient to show that $\overline{\mathbf{G}}_{51}[\theta]$ has full rank iff assumptions C-I and C-II holds true.

First assume that the matrix pair (E, A) is regular. Then it is to be shown that the assumptions C-I and C-II holds true.

By the regularity of the matrix pair (E, A) it means that the matrix $\overline{\mathbf{G}}_{51}[\theta]$ has full rank. On contrary assume the assumptions C-I and C-II do not hold true that is ; there exists a cycle of pump in the network ; $A^{\mathrm{jc}}_{\mathrm{pu}}$ does not have full column rank. It implies the column

$$\begin{bmatrix} 0 \\ 0 \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{pu}} \end{bmatrix}$$

is the blocked columns which are linearly dependent. Hence $\overline{\mathbf{G}}_{51}[\theta]$ does not has full rank. The matrix pair (E, A) is not regular which is contradiction to the assumptions. Hence C-I holds true. For analogous reason also Condition C-II is fulfilled.

Assume now, that Condition C-I and C-II are fulfilled. Schur's complement theorem ensures that $\overline{\mathbf{G}}_{51}[\theta]$ given as follows;

$$\overline{\mathbf{G}}_{51}[\theta] = \begin{bmatrix} \theta \mathcal{I}_{\mathbf{m}_{\mathrm{op}}} & 0 & \widehat{K}_{\mathrm{pi}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^{\top} \\ 0 & 0 & (A^{\mathrm{jc}}_{\mathrm{f_{pu}}})^{\top} \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}} & A^{\mathrm{jc}}_{\mathrm{f_{pu}}} & 0 \end{bmatrix} \in \mathbb{R}^{\Upsilon_{\mathrm{pi}} \times \Upsilon_{\mathrm{pi}}}[\theta]$$

with $\Upsilon_{\mathrm{pi}} = \mathbf{m}_{\mathrm{op}} + \breve{m}_{\mathrm{op}} + n_{\mathrm{jc}}$ is invertible iff the schur complement of $\overline{\mathbf{G}}_{31}[\theta]$ with respect to $\theta\mathcal{I}_{\mathrm{op}}$ defined as follows is invertible.

$$\begin{aligned}&\begin{bmatrix} 0 & (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & 0 \end{bmatrix} - \frac{1}{\theta}\begin{bmatrix} 0 \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}} \end{bmatrix}\begin{bmatrix} 0 & \widehat{K}_{\mathrm{pi}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top \end{bmatrix} \\ &= \begin{bmatrix} 0 & (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & \frac{1}{\theta}\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}}\widehat{K}_{\mathrm{pi}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top \end{bmatrix}\end{aligned} \tag{5.41}$$

Now it remained to show $\begin{bmatrix} 0 & (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & \frac{1}{\theta}\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}}\widehat{K}_{\mathrm{pi}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top \end{bmatrix}$ have full rank. Denote

$$\mathbf{M}(\vartheta) = \begin{bmatrix} 0 & (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & \frac{1}{\vartheta}\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}}\bar{K}_{\mathrm{pi}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top \end{bmatrix} \quad \vartheta \in (0, \infty)$$

As $\mathbf{M}$ has full rank iff $\overline{\mathbf{U}}\mathbf{M}$ have full rank, where $\overline{\mathbf{U}}$ is the invertible transformation matrix.

For the matrix $\overline{\mathbf{U}}$ give as;

$$\overline{\mathbf{U}} = \begin{bmatrix} \mathcal{I}_{\breve{m}_{\mathrm{op}}} & 0 \\ \frac{1}{\vartheta}\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & \mathcal{I}_{n_{\mathrm{jc}} \times n_{\mathrm{jc}}} \end{bmatrix}$$

Such that

$$\overline{\mathbf{U}}\mathbf{M} = \begin{bmatrix} 0 & (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & \frac{1}{\vartheta}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}}\widehat{K}_{\mathrm{pi}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top + \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top) \end{bmatrix} \tag{5.42}$$

Further the following is true as well;

$$\begin{aligned}&\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}}\bar{K}_{\mathrm{pi}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top + \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ &= \begin{bmatrix} \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}} & \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} \end{bmatrix}\begin{bmatrix} \widehat{K}_{\mathrm{pi}} & 0 \\ 0 & \mathcal{I}_{\breve{m}_{\mathrm{op}}} \end{bmatrix}\begin{bmatrix} (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}})^\top \\ (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \end{bmatrix}\end{aligned} \tag{5.43}$$

where $L_K = \begin{bmatrix} \widehat{K}_{\mathrm{pi}} & 0 \\ 0 & \mathcal{I}_{\breve{m}_{\mathrm{op}}} \end{bmatrix}$ which is a diagonal matrix with positive entries, hence positive definite by using the Lemma A.0.15.
Further by assumption (C-II) $\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} = \begin{bmatrix} \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}} & \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} \end{bmatrix}$ have full row rank. Hence (5.43) have full rank by using A.0.16 $\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top$ have full rank since L_K is positive definite diagonal matrix and B has full rank (row rank) by assumption C-II. Hence (5.42) can be rewritten as;

$$\overline{\mathbf{U}}\mathbf{M} = \begin{bmatrix} 0 & (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & \frac{1}{\vartheta} \widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top \end{bmatrix}$$

$\overline{\mathbf{U}}\mathbf{M}$ have full rank if and only of $\overline{\mathbf{U}}\mathbf{M}$ have trivial kernel.

Indeed

Consider $x \in \mathbb{R}^{\breve{m}\times 1} \neq 0, y \in \mathbb{R}^{n_{\mathrm{jc}}\times 1} \neq 0$ and $\overline{\mathbf{U}}\mathbf{M}z = 0$ where $z = \begin{bmatrix} x \\ y \end{bmatrix}$ then

$$\begin{bmatrix} 0 & (\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top \\ \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} & \frac{1}{\vartheta} \widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0,$$

$$(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top y = 0, \tag{5.44}$$

$$\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} x + \frac{1}{\vartheta} \widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top y = 0 \tag{5.45}$$

From (5.45)

$$\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} x = \frac{1}{\vartheta} \widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top y$$

$$y = \frac{1}{\vartheta} (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top)^{-1} \widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} x \tag{5.46}$$

substitute (5.46) into (5.44) this result into the following

$$\frac{1}{\vartheta}(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top(\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top)^{\text{-}1}\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}} x = 0, \tag{5.47}$$

Further $\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top$ is positive definite hence invertible. Further inverse of a positive definite matrix is positive definite. Also using the Lemma A.0.16 $(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top(\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top)^{\text{-}1}\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}$ is positive definite hence invertible. Since $(\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top)^{\text{-}1}$ is positive definite, being inverse of a positive definite matrix. Also $\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}$ has full rank (column rank) by assumption C-I. Also $\mathbf{M}$ is invertible the for invertible transformation matrix $\bar{U}$; $\bar{U}\mathbf{M}$ is invertible, as product of two invertible matrices is again invertible.
Hence from (5.47)

$$x = 0$$

due to the positive definiteness. Hencethe invertibility of the matrix $(\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}})^\top(\widehat{A}^{\mathrm{jc}}_{\mathrm{f}} L_K (\widehat{A}^{\mathrm{jc}}_{\mathrm{f}})^\top)^{\text{-}1}\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}$ and $\vartheta \neq 0$. Further from (5.46)

$$y = 0$$

Hence the matrix $\overline{\mathbf{U}}\mathbf{M}$ have full rank as $z = 0$ shows it has trivial kernel. Hence $\mathbf{M}$ is invertible for all $\vartheta \in (0, \infty)$ for invertible transformation matrix $\bar{U}$. The matrix $\overline{\mathbf{G}}_{51}[\theta]$ have full rank. If there doest not exist a cycle of pumps in the network and each connected component is connected to the supply node.

Hence $\overline{\mathbf{G}}[\theta]$ is invertible that. Hence the matrix pair (E, A) is regular.

■

The matrix $\mathbf{G}[\theta]$ has full rank. Hence the system (5.21) is regular.

Example 5.3.9. This example is presented to show in which cases assumption C-II will not hold.

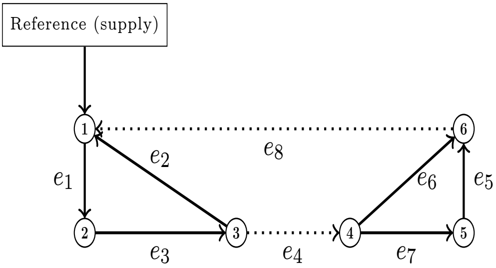


Figure 5.8: The matrix $\mathcal{G}$ violates C-II

$$\hat{A}_{\mathrm{f}}^{\mathrm{jc}}(\mathcal{G}) = \begin{array}{cccccc} e_1 & e_2 & e_3 & e_5 & e_6 & e_7 & \\ \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{array} & \begin{array}{c} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{array}\right) & \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Consider the edge e_4 and e_8 are shown dotted to represent that the flow through them is zero because of closing of valves at the right ends of both of them. The matrix does not has full row rank as the component does not have full row rank. Hence C-II will not hold.

Example 5.3.10. This example is shown to give a clear view of the assumption C-I

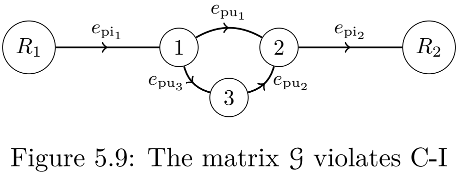


Figure 5.9: The matrix $\mathcal{G}$ violates C-I

$$\hat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}(\mathcal{G}) = \begin{array}{c} \\ \left(\begin{array}{ccc} e_{\mathrm{pu}_1} & e_{\mathrm{pu}_2} & e_{\mathrm{pu}_3} \\ \text{-1} & \text{-1} & 0 \\ 0 & \text{-1} & 1 \\ 0 & 1 & \text{-1} \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

The matrix for open (working) pumps is denoted as $\hat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}(\mathcal{G})$. The rank of the matrix $\hat{A}^{\mathrm{jc}}_{\mathrm{f_{pu}}}(\mathcal{G})$ is not full. Hence C-I does not hold.

Existence and uniqueness of the solution of (5.20)

(R) The regularity assumption of the Theorem 3.4.6 holds true if the network topology satisfies the assumption in the Theorem 5.3.8.

(G) Consider following choice of $\mathcal{M}$ and cN given as:

$$\mathcal{M} = \begin{bmatrix} \mathcal{I}_{m_{\mathrm{pi}}} & 0_{m_{\mathrm{pi}} \times \breve{m}} & 0_{m_{\mathrm{pi}} \times m_{\mathrm{pi}}} & 0_{m_{\mathrm{pi}} \times n_{r^*}} & 0_{m_{\mathrm{pi}} \times n_{\mathrm{jc}}} \\ 0_{\breve{m} \times m_{\mathrm{pi}}} & \mathcal{I}_{\breve{m}} & 0_{\breve{m} \times m_{\mathrm{pi}}} & 0_{\breve{m} \times n_{\mathrm{rs}}} & 0_{\breve{m} \times n_{\mathrm{jc}}} \end{bmatrix} \qquad \mathcal{N} = \begin{bmatrix} \mathcal{I}_{\mathbf{m}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \mathcal{I}_{\breve{m}} \\ 0 & 0 \end{bmatrix}$$

with $\mathcal{M} \in \mathbb{R}^{(\mathbf{m}+\breve{m}) \times (3\mathbf{m}+\breve{m}+n_{r^*}+n_{\mathrm{jc}})}$

and $\mathcal{N} \in \mathbb{R}^{(\mathbf{m}+\breve{m}) \times (3\mathbf{m}+\breve{m}+n_{r^*}+n_{\mathrm{jc}})}$

To check the assumption (G): it is clear for the above choice of $\mathcal{M}, \mathcal{N}$ the following holds

$$g(x) = \begin{bmatrix} \text{-}g_{\text{pi}}(Q_{e_{\text{pi}}}) \\ 0 \\ 0 \\ 0 \\ \text{-}K_{\text{pu}} g_{\text{pu}}(Q_{e_{\text{pu}}}) \\ 0 \end{bmatrix} \qquad \mathcal{M}x = \begin{bmatrix} Q_{e_{\text{pi}}} \\ Q_{e_{\text{pu}}} \end{bmatrix}$$

$$\bar{g}(\mathcal{M}x) = \begin{bmatrix} \text{-}g_{\text{pi}}(Q_{e_{\text{pi}}}) \\ \text{-}K_{\text{pu}} g_{\text{pu}}(Q_{e_{\text{pu}}}) \end{bmatrix} \qquad g(x) = \mathcal{N}\bar{g}(\mathcal{M}x)$$

for above choice of $\mathcal{M}$ and $\mathcal{N}$ assumption (G) holds.

(M) Further assumption (M) of the Theorem 3.4.6 is check. That is

$$\mathcal{M}E^{\text{imp}} = 0$$

hold true. Where $E^{\text{imp}} = T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{\text{-}1}$. Where T is a transformatio matrix.

(N) Further assumption (N) of the Theorem 3.4.6 will be checked. The following condition hold true when there is no nonlinearity of the pump that is $K_{\text{pu}} = 0$

$$\operatorname{im}(\mathcal{N}) \subseteq \operatorname{im}(E)$$

On the other hand if $K_{\text{pu}} \neq 0$ then there may exist a pump nonlinearity (in case of nonlinear pump characteristics) then the assumption $\operatorname{im}(\mathcal{N}) \subseteq \operatorname{im}(E)$ can be relaxed and Remark (N) in 3.4.8 can be utilised.

(F) f is induced by piecewise-smooth function. As usually $\bar{f}$ usually denotes constant and pressure or smoothly changed pressure or demand, that is $\bar{f} \in_{\mathrm{pw}\mathcal{C}^{\infty}}$.

(S) In order to show that the function $g(x)$ is locally Lipschitz. As -$g_{\mathrm{pi}}(Q_{e_{\mathrm{pi}}})$ is locally Lipschitz.

Hence for there exist a local solution for (5.21) . For the global solution the condition ∞_p should be checked, which extend this to the switched nonlinear DAEs. This will hold just by restricting the nonlinearities for the pump characteristics. In case of linear pump characteristic it will hold true. For the pipe friction nonlinearity ∞_p will hold true.

5.4 Summary

In nut shell in this chapter a general structure of a water network is constructed. Further a theorem is presented according to which under two assumptions on the network topology and networ connectivity the matrix pair for general water network is regular. Further more the physical relavence of these assumptions is presented.

Chapter 6

Application to further water networks

A water network consist of multiple pipes, pumps and reservoirs. The closure of valves installed at different locations may generate different kind of network topologies due to origination of the connected components. Therefore it is important to testify the mathematical theory of wellposedness presented in the Chapter 3 and in the Chapter 5. For this purpose some sample networks are presented. These networks are taken from EPANET sample networks with some modifications, and are used for modeling and testing of water networks. Furthermore the resulting Figures found via simulating PDE model of the respective network and results got by its modeling via proposed framework of switched DAEs are compared quantitatively or quantitatively (in some cases) .

6.1 Transient: Pump shutdown

Instantaneous pump shutdown in a water network can cause hydraulic transients (pressure surge) and may cause ruptures in the pump casings (entrance segment of the pump device) 3]. This transient is presented on a simple example of a water network shown in Figure 6.1 and 6.2 for PDE and switched DAE modeling, respectively. It is important to recall that both of the network setups are same in all aspects and just drawn twice to show how this network may look in real, and realised as a graph (network) . The motivations behind constructing this example network with pump are :

1. To present a qualitative comparison of the PDE and switched DAE modeling for water hammer caused by the sudden pump shutdown.
2. In the solution theory of switched nonlinear DAE condition (N) of Theorem 3.4.6 does not hold for a mode in this example, but solution can be found by using the relaxed condition in the Remark (N) in Remarks 3.4.8.

The pump edge of negligible length $e = (f, b)$ is modeled via the following nonlinear expression, see e.g. 42] or 55, Sec. 3.2]:

$$\text{Mode-1:} \quad P_b - P_\text{f} = g_\text{pu}(Q_\text{pu}), \tag{6.1}$$

$$\text{Mode-2:} \quad Q_{pu} = 0, \tag{6.2}$$

where Q_{pu} denotes the flow through the pump and $g_\text{pu}(Q_\text{pu})$ is the linear or nonlinear pump characteristics. The equations modeling both modes (6.1) and (6.2) are combined in via control variable s_pu as (6.3).

$$(1 - s_\text{pu})Q_\text{pu} + s_\text{pu}(P_b - P_f - aQ_\text{pu}^d) = 0 \tag{6.3}$$

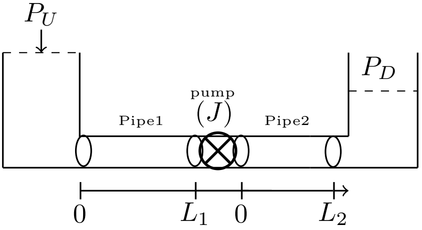


Figure 6.1: Pump setup with two pipes PDE.

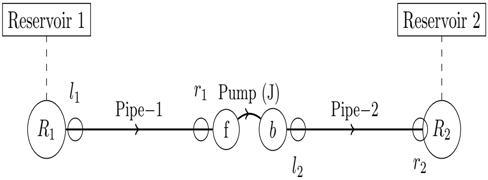


Figure 6.2: Realisation of network with two pipe and pump.

where the decision variable s_{pu} is defined as

$$s_{\text{pu}} = \left\{ \begin{array}{ll} 1, & \text{pump is working} \\ 0, & \text{pump shutdown} \end{array} \right\}$$

6.1.1 PDE model

Consider the network of two pipes and, a pump is installed at the junction J of two pipes. The PDE model of the network shown in the Figure 6.1 is modeled as follows:

1. Pipe 1 Assume $x \in [0, L_1]$ and $t \in [0, T]$. Further assume p_1, q_1, ρ_1 is the pressure, density and flow in pipe 1 with $p_1(t, x) = P(\rho_1(t, x))$.

$$\partial_t \rho_1 + \partial_x q_1 = 0 \tag{6.4a}$$

$$\partial_t q_1 + \partial_x \left(\frac{q_1^2}{\rho_1} + P(\rho_1)\right) + \frac{c_f}{2D_1\rho_1} q_1 \mid q_1 \mid = 0, \tag{6.4b}$$

$$\tag{6.4c}$$

boundary conditions:

$$p_1(t,0) = P(\rho_1(t,0)) = P_U \tag{6.4d}$$

coupling condition for the junction J of pipes with pump

$$(1 - s_{\mathrm{pu}})q_1(t,L_1) + s_{\mathrm{pu}}(p_1(t,L_1) - p_2(t,0) + f_{\mathrm{pu}}q_1(t,L_1)) = 0, \tag{6.5}$$

$$q_2(t,0) = q_1(t,L_1) \tag{6.6}$$

2. Pipe 2 Assume $x \in [0, L_2]$ and $t \in [0, T]$. Further assume p_2, q_2, ρ_2 is the pressure and flow in pipe 2. Then it is modeled as

$$\partial_t \rho_2 + \partial_x q_2 = 0 \tag{6.7a}$$

$$\partial_t q_2 + \partial_x \left(\frac{q_2^2}{\rho_2} + P(\rho_2)\right) + \frac{c_f}{2D_2\rho_2} q_2 \mid q_2 \mid = 0, \tag{6.7b}$$

$$p_2(t,L_2) = P(\rho_2(t,L_2)) = P_D \tag{6.7c}$$

with initial conditions $\forall x \in [0, L_1] \cup [0, L_2]$

$$\begin{aligned} q_1(0,x) &= q_2(0,x) = q_0 \\ p_1(0,x) &= p_2(0,x) = P(\rho(0,x)) = p_0(x). \end{aligned} \tag{6.8}$$

The pressure in the whole network is modeled via invertible pressure function 2.20c.

Also consider the notation $P_f := P_1(L_1) = p_1(t, L_1)$ and $P_b := P_2(0) = p_2(t, 0)$ for the pressure infront (entrance segment of the pump device) and at the back of the pump (pressurised side of the pump).

The PDE model given in the Section 6.1.1 is well-posed 22]. For a rigorous mathematical study of existence and uniqueness of the Riemann problem solution at a junction under the assumption of subcritical flows see (25], 27]). In the next Section the approximated switched DAE model is proposed.

6.1.2 Switched DAE model

The DAE model of the water network consists of two reservoir, pipe 1 and pipe 2 and a pump is shown in the Figure 6.2, with mode 1 and mode 2 described by (E_1, A_1, f_1, g_1) and (E_2, A_2, f_2, g_2), respectively is an nonITP

$$E_{\mathbf{p}}\dot{x} = A_{\mathbf{p}}x + f_{\mathbf{p}} + g_{\mathbf{p}}(x) \quad for \quad \mathbf{p} = 1, 2$$

$$x = (Q_1, Q_2, Q_{\mathrm{pu}}, P_{l_1}, P_{l_2}, P_{r_1}, P_{r_2}, P_{R_1}, P_{R_2}, P_{\mathrm{f}}, P_b)^\top$$

and general $E_{\mathbf{p}}$ and $A_{\mathbf{p}}$.

$$E_{\mathbf{p}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{\mathbf{p}} = \begin{bmatrix} 0 & 0 & 0 & \text{-}c_1 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{-}c_1 & 0 & 0 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{-}1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1-s_1 & 0 & 0 & 0 & 0 & \text{-}1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1-s_2 & 0 & 0 & 0 & 0 & \text{-}1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1-s_{\mathrm{pu}} & 0 & 0 & 0 & 0 & 0 & 0 & s_{\mathrm{pu}} & \text{-}s_{\mathrm{pu}} \\ 1 & 0 & \text{-}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{-}1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$f_{\mathbf{p}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \text{-}P_U \\ \text{-}P_D \end{bmatrix} \qquad g_{\mathbf{p}} = \begin{bmatrix} \text{-}c_2 Q_1 |Q_1| \\ \text{-}c_2 Q_2 |Q_2| \\ 0 \\ 0 \\ 0 \\ 0 \\ \text{-}s_{\mathrm{pu}} g_{\mathrm{pu}}(Q_{\mathrm{pu}}) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Where , P_{R_1} and P_{R_2} denotes the pressure at the upstream and downstream reservoir, respectively, with $P_{R_1} = P_U$ and $P_{R_2} = P_D$. The variable $Q_1, Q_2, Q_{\mathrm{pu}}$ denotes the flow through pipe 1, pipe 2 and the flow through the pump, respectively. Furthermore, $P_{l_1}, P_{r_1}, P_{l_2}, P_{r_2}$ denotes the pressure at the left end, right end of the pipe 1, and pressure at the left end, right end of the pipe 2, respectively. And s_{pu} is the control variable for pump on ($s_{\mathrm{pu}} = 1$) and shut down ($s_{\mathrm{pu}} = 0$) ,respectively. The area and length of both pipes are chosen the same that is and $c_1 = \frac{\mathbf{A}}{L}$ with $L_1 = L$ and $L_2 = L$.

Lemma 6.1.1. Consider the nonlinear initial-trajectory problem (nonITP)

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}$$
$$(E\dot{x})_{[0,\infty)} = (A\dot{x} + f + g(x))_{[0,\infty)}$$

where either $(E, A) = (E_1, A_1), g(x) = g_1(x)$ or

$(E, A) = (E_2, A_2), g(x) = g_2(x)$ as in (6.1.2). Then for every initial trajectory $x^0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{11}$ and every inhomogeneity f induced by a piecewise-smooth function, there exists a unique solution $x \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{11}$ of the (nonITP) in the sense of Definition 3.4.3 if assumptions of the Theorem 3.4.6 and 5.3.8 holds true. In particular the pressure at t_s (time when pump shutdown) is

$$P_{\mathrm{f}}[t_s] = P_U + \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}, \quad P_b[t_s] = P_D - \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}$$

Proof. **Pump on**

Consider the pump is working for $s_{\mathrm{pu}} = 1$ defined by matrices (E_1, A_1)

$$E_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0&0&0&-c_1&0&c_1&0&0&0&0&0 \\ 0&0&0&0&-c_1&0&0&c_1&0&0&0 \\ 0&0&0&-1&0&0&0&1&0&0&0 \\ 0&0&0&0&-1&0&0&0&0&0&1 \\ 1-s_1&0&0&0&0&-1&0&0&0&1&0 \\ 0&1-s_2&0&0&0&0&-1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&-1 \\ 1&0&-1&0&0&0&0&0&0&0&0 \\ 0&-1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P_U \\ -P_D \end{bmatrix} \quad g_1 = \begin{pmatrix} -c_2 Q_1|Q_1| \\ -c_2 Q_2|Q_2| \\ 0 \\ 0 \\ 0 \\ 0 \\ -g_{\mathrm{pu}}(Q_{\mathrm{pu}}) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Firstly , to ensure regularity the assumptions of the Theorem 5.3.8 namely C-I and C-II is checked.As it is clear from the Figure 6.2 there is no loops of pump as there is only one pump present. Further when pump is on all connected componets are connected to the reservoir. Hence the assumptions of C-I and C-II of Theorem 5.3.8 holds it in turn means (R) holds and the network is regular. The transformation matrices S_1, T_1 are calculated as;

$$T_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2c_1} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2c_1} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2c_1} \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2c_1} \end{bmatrix} \quad S_1 = \begin{bmatrix} 1 & 1 & -c_1 & -c_1 & c_1 & c_1 & c_1 & 0 & 0 & -c_1 & c_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

where the impulsive projector is denoted as E_1^{imp} and is given as:

$$E_1^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2c_1} & \frac{1}{2c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2c_1} & \frac{1}{2c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2c_1} & \frac{1}{2c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2c_1} & \frac{1}{2c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S_1E_1T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad S_1A_1T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_1f_1 = \begin{bmatrix} c_1P_D - c_1P_U \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P_U \\ -P_D \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrices $\mathcal{M}_1$ and $\mathcal{N}_1$ can be selected for which assumption (G) is true

$$\mathcal{M}_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \end{bmatrix} \quad \mathcal{N}_1 = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&1 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$$

The assumption (F) is true as reservoir pressures are constant. It is easy to see that assumption (M) of the Theorem 3.4.6 also hold, but assumption (N) does not hold as $\mathrm{im}(\mathcal{N}_1) \not\subseteq \mathrm{im}(E_1)$ and Theorem 3.4.6 does not directly applicable. In order to utilise the Remark 3.4.8(N) first rewrite the nonlinear DAE using the modified QWF (3.17):

$$\begin{aligned} x &= T_1 \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}, \\ v &= \{v\}, \\ w_1 &= \{w_1^1, \cdots, w_1^9\} \qquad w_2 = \{w_2\}. \end{aligned}$$

DAE will take the following form in transformed variables,

$$\begin{aligned} \dot{v} &= c_1 P_U - c_1 P_D + g_1(v + \frac{w_1^4}{2} + \frac{w_1^5}{2}) + g_2(v - \frac{w_1^4}{2} + \frac{w_1^5}{2}), \\ w_1^i &= 0, \qquad\qquad\qquad\qquad i = \{1, 2, 3, 5, 8, 9\} \\ w_1^4 &= g_{\mathrm{pu}}(v + \frac{w_1^4}{2} + \frac{w_1^5}{2}), \\ w_1^6 &= P_U, \\ w_1^7 &= P_D, \\ \dot{w}_1^4 + \dot{w}_1^5 &= w_2. \end{aligned}$$

By solving above equations

$$
\begin{aligned}
x(t_s^-) = &(\frac{Q_1(t_s^-)}{2}, \frac{Q_1(t_s^-)}{2}, \frac{Q_1(t_s^-)}{2}, P_U, \frac{(P_U + P_D)}{2} \\
&- g_{\mathrm{pu}}(Q_{\mathrm{pu}}(t_s^-)),) \frac{(P_U + P_D)}{2} + g_{\mathrm{pu}}(Q_{\mathrm{pu}}(t_s^-)) \\
&P_U, P_U, P_D, \frac{(P_U + P_D - g_{\mathrm{pu}}(Q_{\mathrm{pu}}(t_s^-)))}{2}, \frac{(P_U + P_D + g_{\mathrm{pu}}(Q_{\mathrm{pu}}(t_s^-))}{2})
\end{aligned}
$$

Pump shutdown $t = t_s$

Consider the pump is instantaneously shut down for $s_{\mathrm{pu}} = 0$ defined by matrices (E_2, A_2), the initial conditions of mode 2 is $x(t_s^-)$.

$$
E_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, A_2 = \begin{bmatrix}
0 & 0 & 0 & -c_1 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -c_1 & 0 & 0 & c_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1-s_1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1-s_2 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

$$
f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P_U \\ -P_D \end{bmatrix} \qquad g_2 = \begin{pmatrix} -c_2 Q_1 |Q_1| \\ -c_2 Q_2 |Q_2| \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

The assumptions of the Theorem will be checked, clearly (E_1, A_1) is regular,

Indeed

$$
\widehat{A}^{\mathrm{jc}}_{\mathrm{f}_{\mathrm{pi}}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

The matrix $A^{\mathrm{jc}}_{\mathrm{f}}$ also have full row rank as it is connected to two reservoirs, hence using the Theorem 5.3.8 holds it in turn means (R) holds and the netowk is regular. The transformation matrices S_2, T_2 and $E^{\mathrm{imp}}_2$are calculated as;

$$T_2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{c_1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -\frac{1}{c_1} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -\frac{1}{c_1} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \frac{1}{c_1} \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the impulsive projector is deonted as E_2^{imp} and is given as:

$$E_2^{\text{imp}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S_2E_2T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad S_2A_2T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_2f_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P_U \\ P_D \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrices $\mathcal{M}_2$ and $\mathcal{N}_2$ can be selected for which G is true,. The assumption (M) and (N) is also true.

$$\mathcal{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{N}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Writing nonlinear DAE using the modified QWF (3.17):

$$\begin{aligned} x &= T_1 \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}, \\ v &= \{\}, \\ w_1 &= \{w_1^1, \cdots, w_1^9\} \qquad w_2 = \{w_2^1, w_2^2\}. \end{aligned}$$

DAE will take the following form in transformed variables,

$$\begin{aligned} w_i^1 &= 0, \qquad i = \{1,2,3,4,5,8,9\} \\ w_1^6 &= P_U, \\ w_1^7 &= P_D, \\ \dot{w}_3 + \dot{w}_4 &= w_2^1 + g_1(w_1^3 + w_1^4) \\ \dot{w}_3 - \dot{w}_5 &= w_2^2 + g_2(w_1^3 + w_1^5) \end{aligned}$$

where $g_1(0) = 0, g_2(0) = 0$.

The solution is read as

$$\begin{aligned} x =& (0,0,0,P_U, P_D - \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-), P_U + \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-), P_D, P_U, P_D, \\ & P_U + \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}, P_D - \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}) \end{aligned} \tag{6.9}$$

In particular, the pressure infront of pump P_f(entrance segment of the pump device) and after the pump is P_b (pressurised segment of pump) and its solution is given as ;

$$P_{\mathrm{f}} = P_U + \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}, \qquad P_b = P_D - \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}).$$

Further at $t = t_s$ it is given as

$$P_{\mathrm{f}}[t_s] = \frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}, \qquad P_b[t_s] = -\frac{1}{c_1} Q_{\mathrm{pu}}(t_s^-)\delta_{t_s}$$

After the pump shutdown all is settled to the following values.

$$P_{\mathrm{f}}(t_s^+) = P_U \qquad P_b((t_s^+) = P_D.$$

■

6.1.3 Qualitative comparison

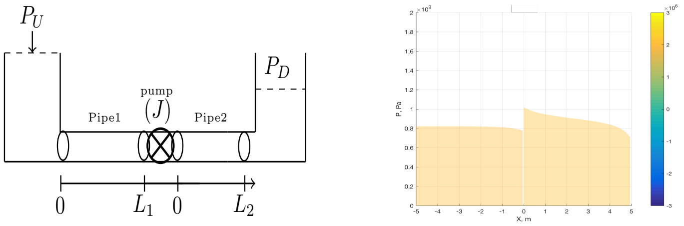
Assume that the PDE solution on $[0, t_S)$ is stationary, i.e. $q_i(t,x)$ $i = 1, 2$ is approximately constant in time and space (or in other words, when the valves are closed the dynamics in all pipe have approximately settled down). For numerical simulations Flux-Corrected Transport (FCT) scheme with artificial viscosity (< 0.25) is used.

$$q_i(0,x) \equiv 0, \quad \rho_i(0,x) \equiv 1 \times 10^3$$

and pipes parameters:

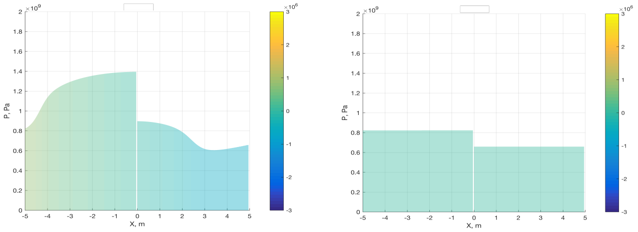
$$P_a = 1.01 \times 10^6, \quad \beta = \frac{1}{K} = 4 \times 10^{-9}, \quad \rho_a = 1000,$$
$$P_U = 8.231 \times 10^8 \quad P_D = 6.5081 \times 10^8$$
$$L_1 = 5, \quad L_2 = 5 \quad D_1 = 0.5, \quad D_2 = 0.5 \quad c_f = 0.02 \quad g_{\mathrm{pu}}(Q_{\mathrm{pu}}) = a(Q_{\mathrm{pu}})^3.$$

The parameters P_a, ρ_a and β are physical parameters and c_f is chosen via the so-called moody chart denoting friction of the pipe, see in the Appendix the Figure 7.1



(a) Pump network setup

(b) Pressure in the network before pump shutdown ($t < t_s$)

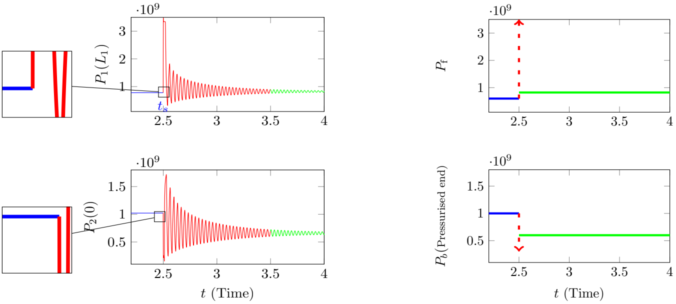


(c) Pressure in the network at pump shutdown($t = t_s$)

(d) Pressure in the network 'long after'pump shutdown($t = t_s$)

Figure 6.3: PDE simulations of the network in the Figure 6.1 x vs P where 0 is the position of pump in space coordinate

The Figure shown in 6.3 shows the PDE simulation of the network . The simulation firstly is run till an approximated steady state is achieved in the network and is shown in the Figure 6.3(b) for the time $t < t_s$ which is the time before switch. In the Figure 6.3, the location of pump in space is at 0 along x axis. The pump shutdown at time $t = t_s$ which is modeled by $s_{\text{pu}} = 0$. The pressure profile in both pipes along with the pressure across the pump is shown in the Figure 6.3(c) the end of the pipe 1 is the space variable infront of the pump (entrance segment of the pump) and the start of pipe 2 represents the pressure at the back of the pump (the pressurised side of pump). The Figure 6.3(d) showing when after shutdown the network reached to other steadty state. The pressure infront of the pump rises showing a positive pressure impulse at the time $t = t_s$ as shown in the Figure 6.3(c) at $x = 0$(infront of the pump) and same can be seen in the Figure 6.4(a) above. The pressure at the back of pump will decrease due to no further flow across the pump showing a negative "Dirac impulse". The negative pressure impulse at pressurised side of pump is shown in the Figure [4-18 33]] is qualitatively same as in the Figure shown in 6.4(a) (below).



(a) PDE simulation for pressure $P_f = p_{1(L_1)}$ infront of pump (above) and $P_b = p_{2(0)}$ (below)

(b) Switched DAE pressure P_f infront of pump and P_b after the pump

Figure 6.4: Comparison of the pressure profiles of PDE of the network in 6.1 and switched DAE model of the network in 6.2 with time ($t = t_s$) is time at which pump shutdown

In the Figure 6.4 PDE and switched DAE plots of pressure against time are shown. These pressure profile are plotted against time. The positive impulse infront of the pump (entrance segment of these devices) and negative impulse at the back of the pump shown. It is to be noted the direction of the respective Dirac impulse of the both modeling techniques is same. Hence the qualitatively model via switched DAEs is an approximation for the PDE model. The Figure shown 6.4(a)(above) is the main interest as this positive pressure spike will get higher and higher on decreasing compressibility (lower down the value of β), hence approaching to the same length as the length of the Dirac in switched DAE model.

Hence the approximated switched DAE model on a network agrees the result by PDE model. In conclusion, switched nonlinear DAEs framework provided with simple framework to analyse the impacts of transients in a water work. Furthermore, location of the positive Dirac which is located by simplified framework of switched DAE may be the indication where potentially a pipe or pump shut down may cause a breakage.

6.2 A water network with six pipes

Consider a network of six pipes and two reservoirs (c.f. `?Epanet`]). The topology of the network is shown in the Figure 6.5. Assume the length of each pipe is L_k for arbitrary k^{th} pipe edge, where $k \in \{1, \cdots, 6\}$. The space domain for each pipe is $[0, L_k]$. Define the following sets,

$$\begin{aligned}\mathcal{N}_{\text{jc}} &= \{1, 2, 3, 4\}\\ \mathcal{N}_{\text{rs}} &= \{R_1, R_2\},\\ \mathcal{E}_{\text{pi}} &= \{e_1, e_2, e_3, e_4, e_5, e_6\}.\end{aligned}$$

Where, $\mathcal{E}_{\text{pi}}, \mathcal{N}_{\text{rs}}$ and $\mathcal{N}_{\text{jc}}$, denotes the set of pipe edges, reservoir nodes and the junction nodes, respectively. There is a valve installed at left and right end of each k^{th} pipe edge e. Further assume $|\mathcal{N}_{\text{rs}}| = n_{r^*} = 2$, $|\mathcal{E}_{\text{pi}}| = \mathbf{m} = 6$ and $|\mathcal{N}_{\text{jc}}| = n_{\text{jc}} = 4$.

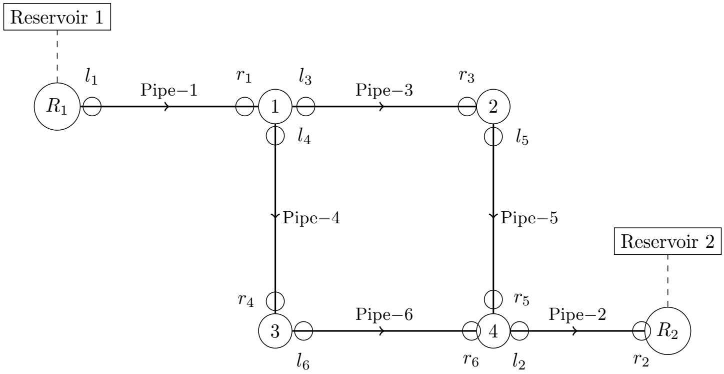


Figure 6.5: A network of six pipes

The next Section is presenting a PDE model for the network shown in the Figure 6.5.

6.2.1 PDE model

The model of water flow in the arbitrary k^{th} pipe edge e in the network is based on (2.20). The pipe edge is cosidered within space interval $[0, L_k]$ $\forall k$. Furthermore two or more than two pipes are joined at junction, and each junction is modeled via two coupling conditions, namely pressure equality and flow balance equation for all pipe edges emerging from or to the junction node $i \in \mathcal{N}_{\text{jc}}$ as given in (6.11). These coupling conditions are wellposed, for detailed investigation on wellposedness of coupling conditions see, 16] 26] 29] and 40]. For a pipe edge ρ_k, p_k, q_k are the state variables used to describe the water flow in k^{th} edge $\forall k$. Hence the PDE model can be drwan up as:

For each pipe,

$$
\begin{aligned}
&\partial_t \rho_k + \partial_x q_k = 0, \\
&\partial_t (q_k) + \partial_x (\frac{q_k^2}{\rho_i} + P(\rho_k)) + \frac{c_f}{2D_i \rho_k} q_k \mid q_k \mid = 0.
\end{aligned} \tag{6.10}
$$

The coupling conditions on the each junction is given as follows:

$$
\begin{aligned}
&p_l(t, L_l) = p_k(t, 0), \quad \forall \quad i \in \mathcal{N}_{jc}, \quad \forall k \in \nabla^-(i) \quad \text{and} \quad \forall l \in \nabla^+(i) \\
&\sum_{k \in \nabla^-(i)} q_k = \sum_{l \in \nabla^+(i)} q_l. \quad \forall i \in \mathcal{N}_{\mathrm{jc}}
\end{aligned} \tag{6.11}
$$

There are two reservoir one R_1 is located at the left of pipe 1 and other R_2 at the right end of pipe 6 . The pressures at the reservoir R_j is modeled as under for $j = 1, 2$:

$$
\begin{aligned}
&(R_1) \qquad p_1(t, 0) = P(\rho_1(t, 0)) = P_U, \\
&(R_2) \qquad p_6(t, L_6) = P(\rho_6(t, L_6)) = P_D.
\end{aligned} \tag{6.12}
$$

A valve is installed at each pipe at left and right end of the $k^\flat$ pipe $e = (i, j)$ and modeled as:

$$
\begin{aligned}
(1 - s_{l_k}) q_k(t, 0) &= s_{l_k}(p_k(t, 0) - P_i(t)), \\
(1 - s_{r_k}) q_k(t, L_k) &= s_{r_k}(p_k(t, L_k) - P_j(t)). \qquad i, j \in \mathcal{N}
\end{aligned} \tag{6.13}
$$

where the pressure at left of k^{th} edge e, can be written as $p_{l_k}(t) = p(t, 0) = P(\rho_k(t, 0))$ and $p_{r_k}(t) = p(t, L_k) = P(\rho_k(t, L_k))$ where L_k is length of the k^{th} pipe edge for all $k = \{1, \cdots, 6\}$.

In the next Section a switched DAE model of setup used for the PDE model shown in the Figure 6.5 is presented.

6.2.2 Switched DAE model

Consider the state variable x is expression the dynamics in the netwok.

$$
\begin{aligned}
x &= \{Q, P_l, P_r, P_{\text{rs}}, P_{\text{jc}}\},\\
Q_{\text{pi}} &= \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\},\\
P_l &= \{P_{l_1}, P_{l_2}, P_{l_3}, P_{l_4}, P_{l_5}, P_{l_6}\},\\
P_r &= \{P_{r_1}, P_{r_2}, P_{r_3}, P_{r_4}, P_{r_5}, P_{r_6}\},\\
P_i &= \{P_1, P_2, P_3, P_4\},\\
P_{\text{rs}} &= \{P_{R_1}, P_{R_2}\}.
\end{aligned}
$$

The nonlinear DAE of the network for each mode $\mathbf{p}$ can be written as:

$$E_{\mathbf{p}}x = A_{\mathbf{p}}x + f_{\mathbf{p}} + g_{\mathbf{p}}(x). \tag{6.14}$$

with $\mathbf{p} = 1, 2, 3$ and the matrices $E, A \in \mathbb{R}^{(3\mathbf{m}+n_{\text{jc}}+n_{r*})\times(3\mathbf{m}+n_{\text{jc}}+n_{r*})}$;

$$
\left.
\begin{aligned}
&E_{\mathbf{p}}=\begin{bmatrix} \mathcal{I}_{\text{pi}} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\;
A_{\mathbf{p}}=\begin{bmatrix} 0 & A_l & A_r & 0 & 0\\ A^{\text{jc}}_{f_{\text{pi}}} & 0 & 0 & 0 & 0\\ 0 & A^v_l & 0 & A^r_l & A^{\text{jc}}_l\\ \widetilde{A}^{cv}_r & 0 & \widetilde{A}^v_r & \widetilde{A}^{r^*}_r & \widetilde{A}^{\text{jc}}_r\\ 0 & 0 & 0 & I_{r*} & 0 \end{bmatrix}
\quad f_{\mathbf{p}} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ \text{-}f_{n^*_r} \end{bmatrix}\\
&g_{\mathbf{p}}(x) = \begin{pmatrix} \text{-}g_{\text{pi}}(Q_{\text{pi}})\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}
\quad g_{\text{pi}}(Q_{\text{pi}}) = \begin{bmatrix} g_1(Q_1)\\ g_2(Q_2)\\ g_3(Q_3)\\ g_4(Q_4)\\ g_5(Q_5)\\ g_6(Q_6) \end{bmatrix}
\quad f_{n^*_r} = \begin{bmatrix} P_U\\ P_D \end{bmatrix}
\end{aligned}
\right\}
\tag{6.15}
$$

with matrices description is given as

$$
\begin{aligned}
&\mathcal{I}_{\text{pi}} = I_{6\times 6} \quad A_l = (c_1, c_2, c_3, c_4, c_5, c_6) \quad A_r = -A_l \quad A_l^v = -\mathcal{I}_{6\times 6} \\
&A_r^{cv} = (1 - s_{r_1}, 1 - s_{r_2}, 1 - s_{r_3}, 1 - s_{r_4}, 1 - s_{r_5}, 1 - s_{r_6}) \\
&A_r^v = (\text{-}s_{r_1}, \text{-}s_{r_2}, \text{-}s_{r_3}, \text{-}s_{r_4}, \text{-}s_{r_5}, \text{-}s_{r_6}) \\
&A_l^{\text{rs}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_l^{\text{jc}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad g_i(Q_i) = Q_i|Q_i| \quad \forall i = 1, \cdots, 6 \\
&A_r^{\text{rs}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & s_2 \end{bmatrix} \quad A_r^{\text{jc}} = \begin{bmatrix} s_{r_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & s_{r_3} & 0 & 0 \\ 0 & 0 & s_{r_4} & 0 \\ 0 & 0 & 0 & s_{r_5} \\ 0 & 0 & 0 & s_{r_6} \end{bmatrix}
\end{aligned}
\tag{6.16}
$$

In the next Section the wellposedness of the switched nonlinear ITP (nonITP) (6.15) for each $\mathbf{p}$ will be check which will ensure the existence and uniquness of the solution. This will be accomplished by checking the assumptions of the Theorem 3.4.6 and the assumption on the topology of the network established in 5.3.8. It is important to recall that the these assumptions is checked on the open edges in the network (only those edge be the part of modified incidence matrix which are not closed by the valve at the right end and pump which are working. In this work those pumps are termed as open pumps).

6.2.3 Wellposedness of the model

For the wellposedness of the model conditions of the theorem 3.4.6 is checked. For the following choice of $\mathcal{M}$ and $\mathcal{N}$ condition (G) will hold;

$$\mathcal{M} = \mathcal{M}_1 = \mathcal{M}_2 = \begin{bmatrix} I_{\{6\times 6\}} & O_{\{6\times 18\}} \end{bmatrix}, \quad \mathcal{N} = \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{M}^\top.$$

that is

$$g(x) = \mathcal{N}\overline{g}(\mathcal{M}x), \quad \text{where} \quad \overline{g}(\mathcal{M}x) = g(Q)$$

Also

$$\operatorname{im}(\mathcal{N}) \subseteq \operatorname{im}(E_1)$$

which in turns means assumption (N) is satisfied. A constant pressure is assigned to both of the reservoir hence assumption (F) is trivially true. The assumption (S) is also true as $g_{\mathbf{p}}(x)$ is locally Lipchitz, because each component $Q_i|Q_i|$ is locally Lipchitz hence whole vector of function is Locally Lipchitz by the Remark A.0.4.

The assumption (R) is mode dependent, beacause closing of valve will change the network topology and generate connected components, hence modified incidence matrix. The fulll row rank modified incidence matrix will ensure the regularity of the network.

Mode 1: All valves are open

To check assumption(R), in the Theorem 5.3.8. First consider all valve are open this is modeled by substituting, $s_{r_j} = 1$ for all $j = 1, \cdots, 6$ in (6.16) .The modified incidence matrix for open edges $\widehat{A}_{\mathrm{f}}^{\mathrm{jc}} := (A_{\mathrm{f}}^{\mathrm{jc}})_{\mathrm{op}} = [\widehat{A}_{\mathrm{f}_{\mathrm{pi}}}^{\mathrm{jc}}]$, as there is no pump in the network so the matrix $\widehat{A}_{\mathrm{f}_{\mathrm{pu}}}^{\mathrm{jc}}$ will not appear in the modified incidence matrix $\hat{A}_{\mathrm{f}}^{\mathrm{jc}}$.

$$\widehat{A}_{\mathrm{f}}^{\mathrm{jc}} = \widehat{A}_{\mathrm{f}_{\mathrm{pi}}}^{\mathrm{jc}} = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}, \tag{6.17}$$

For regularity the assumptions in the Theorem (C-I) and (C-II) in 5.3.8 is checked. It has no pump, therefore no need to check the assumption (C-I). For (C-II) consider that is $\widehat{A}_{\mathrm{f}_{\mathrm{pi}}}^{\mathrm{jc}}$ in (6.17) have full row rank for all open edges, hence assumption (R) holds. The matrix pair (E_1, A_1) is regular.

Therefore the transformation matrices S_1, T_1 and impulsive projector E_1^{imp} can be calculated, the matrix E_1^{imp} is given in (6.18).

$$E_1^{\text{imp}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\alpha & 4\alpha & -\alpha & -\alpha & -\alpha & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4\alpha & 2\alpha & \alpha & \alpha & \alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4\alpha & 2\alpha & \alpha & \alpha & \alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3\alpha & 3\alpha & -3\alpha & 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\alpha & 4\alpha & -\alpha & -\alpha & -\alpha & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3\alpha & 3\alpha & 0 & -3\alpha & 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\alpha & 4\alpha & -\alpha & -\alpha & -\alpha & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\alpha & 4\alpha & -\alpha & -\alpha & -\alpha & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\alpha & 4\alpha & -\alpha & -\alpha & -\alpha & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\alpha & 4\alpha & -\alpha & -\alpha & -\alpha & -2\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4\alpha & 2\alpha & \alpha & \alpha & \alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3\alpha & 3\alpha & -3\alpha & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3\alpha & 3\alpha & 0 & -3\alpha & 0 & -3\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\alpha & 4\alpha & -\alpha & -\alpha & -\alpha & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{6.18}$$

Where, $\alpha = \frac{1}{6c_1}$. The non zero rows in E_1^{imp} are the rows corresponding to the state variables can have impulses. Further the assumption (M) of the Theorem 3.4.6 can be checked and its easy to see

$$\mathcal{M}E_1^{\text{imp}} = 0$$

Hence all assumptions of are satisfied and there exist a unique solution for (6.14).

Consider the transformed variables using modified QWF using the Lemma 3.4.7

$$\dot{v}_1 = v_1 + c_1 P_U - c_1 P_D + g_1(Q_1) + g_2(Q_2) - g_4(Q_4) + g_6(Q_6)$$

$$\dot{v}_2 = v_2 + g_3(Q_3) - g_4(Q_4) + g_5(Q_5) + g_6(Q_6)$$

$$w_1^i = \left\{ \begin{array}{ll} 0 & \text{for } i = 1, \cdots, 10 \text{ and for } i = 13, \cdots, 18 \\ P_U, & \text{for } i = 11 \\ & \\ P_D, & \text{for } i = 12 \end{array} \right\}$$

$$\begin{aligned}
\dot{w}_1^7 + \dot{w}_1^8 + \dot{w}_1^9 &= w_2^1 + g_1(Q_1) - g_5(Q_5) - g_6(Q_6), \\
-\dot{w}_1^{10} &= w_2^2 + +g_2(Q_2) - g_5(Q_5) - g_6(Q_6), \\
\dot{w}_1^8 &= w_2^3, \\
\dot{w}_1^9 &= w_2^4.
\end{aligned}$$

Where all Q_i can be written in terms of v_1,v_2,w_1^i by using transformation matrix T_1 found by the modified QWF (3.17) and algorithm given in the Remark A.0.7 and matlab code 7.

$$\begin{aligned}
x &= T_1 \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}, \\
v &= \{v_1, v_2\}, \\
w_1 &= \{w_1^1, \cdots, w_1^{18}\}, \\
w_2 &= \{w_2^1, \cdots, w_2^4\}.
\end{aligned}$$

For this mode all consistent intital values in real variable x, the initial values of transformed variables can be found by using

$$T_1^{\text{-}1} x = \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}.$$

upon solving above equations, the initial values for the next mode are found that are $x(t_s^-)$ by using the transformation matrices. Hence $x(t_s^-)$ is as follows:

$$
\begin{aligned}
&Q_i(t_s^-) = \frac{1}{3}(-c_1P_U + c_1P_D). \quad i = 1, 2\\
&Q_i(t_s^-) = \frac{1}{6}(-c_1P_U + c_1P_D). \quad i = 3, 4, 5, 6\\
&P_{l_1}(t_s^-) = P_U,\\
&P_{l_2}(t_s^-) = \frac{1}{3}P_U + \frac{2}{3}P_D,\\
&P_{l_3}(t_s^-) = P_{l_4}(t_s^-) = \frac{2}{3}P_U + \frac{1}{3}P_D,\\
&P_{l_5}(t_s^-) = P_{l_6}(t_s^-) = \frac{1}{2}P_U + \frac{1}{2}P_D,\\
&P_{r_1}(t_s^-) = \frac{2}{3}P_U + \frac{1}{3}P_D, \quad P_{r_2} = P_D,\\
&P_{r_3}(t_s^-) = P_{r_4}(t_s^-) = \frac{1}{2}P_U + \frac{1}{2}P_D,\\
&P_{r_5}(t_s^-) = P_{r_6}(t_s^-) = \frac{2}{3}P_U + \frac{1}{3}P_D,\\
&P_{R_1} = P_U, \qquad P_{R_2} = P_D,\\
&P_1(t_s^-) = \frac{2}{3}P_U + \frac{1}{3}P_D,\\
&P_2(t_s^-) = P_3(t_s^-) = \frac{1}{2}P_U + \frac{1}{2}P_D,\\
&P_4(t_s^-) = \frac{2}{3}P_D + \frac{1}{3}P_U.
\end{aligned}
$$

It shows when all valves are open there is no Dirac appeared as the consistent initial conditions have been choosed for this mode .

Mode 2 :Pipe 5 and pipe 6 are closed

For this consider $s_1 = 1, s_2 = 1, s_3 = 1, s_4 = 1, s_5 = 0, s_6 = 0$, assumptions of the 3.4.6 and 5.3.8 are true. Hence the following matrices are calculated. In this case

Figure 6.6: A pipe network six pipes with two closed pipes

$$A^{\mathrm{jc}}_{f_{\mathrm{pi}}} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

For regularity the assumptions in the Theorem (C-I) will be checked. Consider that is $A^{\mathrm{jc}}_{\mathrm{f_{pi}}}$ in (6.17) have full row rank for all open edges, hence assumption (R) holds. The matrix pair (E_2, A_2) is regular. Hence the follwoing matrices S_2, T_2 and E_2^{imp} are calculated:

$$E_2^{\mathrm{imp}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & -6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & -6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

The non zero rows in E_2^{imp} are the rows corresponding to the state variables can have impulses. Further condition (M) of the Theorem 3.4.6 can be checked and its easy to see

$$\mathcal{M}E_2^{\mathrm{imp}} = 0$$

Hence all assumptions of are satisfied and there exist a unique solution. Transformed the nonlinear DAE into transformed variable using the modified QWF with variable as:

$$
\begin{aligned}
x &= T_1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \\
w_1 &= \{w_1^1, \cdots, w_1^{18}\}, \\
w_2 &= \{w_2^1, \cdots, w_2^{6}\}.
\end{aligned}
$$

Writing (6.14) in transformed variables

$$
w_1^i = \left\{ \begin{array}{ll} 0 \quad \text{for } i = 1, \cdots, 10 \text{ and for } i = 13, \cdots, 18 & \\ P_U, & \text{for } i = 11 \\ & \\ P_D, & \text{for } i = 12 \end{array} \right\}
$$

$$
\begin{aligned}
&\dot{w}_1^5 + \dot{w}_1^6 + \dot{w}_1^7 + \dot{w}_1^8 + \dot{w}_1^9 = w_2^1, \\
&\dot{w}_1^5 + \dot{w}_1^6 - \dot{w}_1^{10} = w_2^2, \\
&\dot{w}_1^5 + \dot{w}_1^8 = w_2^3, \\
&\dot{w}_1^6 + \dot{w}_1^9 = w_2^4, \\
&\dot{w}_1^5 = w_2^5, \\
&\dot{w}_1^6 = w_2^6.
\end{aligned}
$$

On solving for transformed variable above equations and transforming into original state variables for $t \geq t_s$. Here, only the pressure at the right end of pipe5 and pipe6 are given. The main fact is there will be Dirac in all right ends of pipe as closing of valve at pipe 5 and 6 completely dopped flow in whole network hence is again a case of water hammer.

$$\begin{aligned} Q_i &= 0, \qquad i = 1, 2, 3, 4, 5, 6, \\ P_{r_5}(t) &= P_U + \frac{2}{c_1}(Q_5(t_s^-) + Q_6(t_s^-))\delta_{t_s}, \\ P_{r_6}(t) &= P_U + \frac{2}{c_1}(Q_5(t_s^-) + Q_6(t_s^-))\delta_{t_s}. \end{aligned}$$

The Dirac impulse at $t = t_s$ is given as :

$$\begin{aligned} P_{r_5}[t_s] &= \frac{2}{c_1}(Q_5(t_s^-) + Q_6(t_s^-))\delta_{t_s}, \\ P_{r_6}[t_s] &= \frac{2}{c_1}(Q_5(t_s^-) + Q_6(t_s^-))\delta_{t_s}. \end{aligned}$$

For the value ‘long after’ $t = t_s$:

$$\begin{aligned} P_{r_5}(t_s^+) &= P_U, \\ P_{r_6}(t_s^+) &= P_U. \end{aligned}$$

6.2.4 Numerical result (mode 1 to mode 2)

The parameters P_a, ρ_a and β are physical parameters and c_f is chosen via the so-called moody chart, see e.g. 7.1.

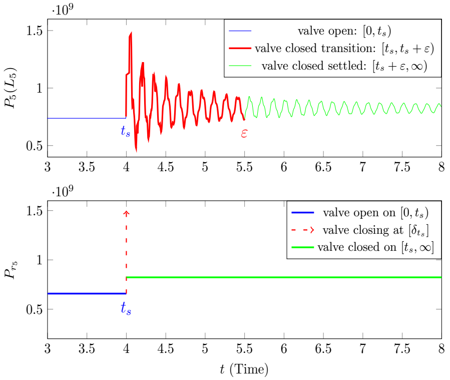


Figure 6.7: Comparison of pressure profile PDE models ($P_5(L_5)$) (above) and switched DAE model (P_{r_5}) (below), profile for $P(r_6)$ is approximately symmetrical. Data for simulation; $P_a = 1.01 \times 10^6$, $\beta = \frac{1}{K} = 4 \times 10^{-9}$, $\rho_a = 1000$ $L_k = 5$, $D_k = 0.5$, $c_f = 0.02\ \forall\ k$

Figure 6.7 clearly shows a strong pressure spike just after the switching time $t_S = 4s$, the pressure oscillatory settles to a new pressure value say $\overline{P}^1_R$. The same behavior occurs for P_{r_5} which settles to $\overline{P}^1_R$. Instead of running the simulation for a very long time, we just chose a settling time $\varepsilon > 0$ and take the average of the pressures on the interval $(t_S + \varepsilon, T]$ where $T > t_S + \varepsilon$ is our overall simulation time, i.e.

$$\overline{P}^1_R := \frac{1}{T - (t_S + \varepsilon)} \int_{t_S+\varepsilon}^{T} p_5(t, L_5)\mathrm{d}t.$$

$$\overline{P}^2_R := \frac{1}{T - (t_S + \varepsilon)} \int_{t_S+\varepsilon}^{T} p_6(t, L_6)\mathrm{d}t.$$

With

$$\varepsilon = 5.5, \quad T = 8$$

we obtain

$$\overline{P}^1_R \approx \overline{P}^2_R \approx 8.23 \times 10^8.$$

The value predicted by the switched DAE solution for $t > t_s$ is,

$$P_{r_5}(t_S^+) = P_{R_1} \approx 8.23 \times 10^8.$$

In Table 6.1 the relative error between $\overline{P}_R^i$, $i = \{1, 2\}$ and $P_{r_5}(t_S^+)$ is presented for decreasing compressibility coefficients β. In order to compare the peak in P_{r_5}, P_{r_6} just after the valve is closed with the Dirac impulse $P_{r_5}[t_s]$ and $P_{r_5}[t_s]$ in response to the switching time, we recall that a Dirac impulse δ_{t_s} at $t_s > 0$ can be approximated by a sequence of functions $t \mapsto \delta_{t_s}^\varepsilon(t)$ such that $\delta^\varepsilon(t) = 0$ for $t \neq [t_s, t_s + \varepsilon]$ and $\int_{t_s}^{t_s+\varepsilon} \delta_{t_s}^\varepsilon(t)\mathrm{d}t = 1$. We therefore make the Ansatz for p_{r_5} and P_{r_6},

β	$\overline{P}_R^1$	$\overline{P}_R^2$	$\frac{\lvert\overline{P}_R^1 - P_{r_5}(t_S^+)\rvert}{P_{r_5}(t_S^+)}$	$\frac{\lvert\overline{P}_R^2 - P_{r_6}(t_S^+)\rvert}{P_{r_6}(t_S^+)}$
$15.0 \cdot 10^{-9}$	$8.1613 \cdot 10^8$	$8.2494 \cdot 10^8$	$8.3 \cdot 10^{-03}$	$2.4 \cdot 10^{-03}$
$9.0 \cdot 10^{-9}$	$8.2644 \cdot 10^8$	$8.2419 \cdot 10^8$	$4.2 \cdot 10^{-03}$	$1.4 \cdot 10^{-03}$
$4.0 \cdot 10^{-9}$	$8.2401 \cdot 10^8$	$8.2408 \cdot 10^8$	$1.2 \cdot 10^{-03}$	$1.3 \cdot 10^{-03}$
$5.0 \cdot 10^{-10}$	$8.2329 \cdot 10^8$	$8.2352 \cdot 10^8$	$3.5 \cdot 10^{-04}$	$6.3 \cdot 10^{-04}$
$2.0 \cdot 10^{-9}$	$8.2317 \cdot 10^8$	$8.2348 \cdot 10^8$	$2.6 \cdot 10^{-04}$	$5.8 \cdot 10^{-04}$

Table 6.1: Comparison of pressure at valves r_5 and r_6 for PDE and switched DAE model.

$$p_{r_5} \approx \overline{P}^{\mathrm{imp}^1_{t_S}} \delta^\varepsilon(t) + \overline{P}_R^1, \qquad p_{r_6} \approx \overline{P}^{\mathrm{imp}^2_{t_S}} \delta^\varepsilon(t) + \overline{P}_R^1 \quad t \in (t_S, T].$$

hence we can approximate the magnitude of the “smoothed-out” Dirac impulse occurring in the PDE model as follows:

$$\overline{P}^{\mathrm{imp}^1_{t_S}} := \int_{t_S}^{t_S+\varepsilon} p_{r_5} - \overline{P}_R^1 \mathrm{d}t.$$

analogously for p_{r_6},

$$\overline{P}^{\text{imp}^2_{t_S}} := \int_{t_S}^{t_S+\varepsilon} p_{r_6} - \overline{P}^1_R \mathrm{d}t.$$

The Dirac impulse induced by the switched DAE are defined i.e.,

$$\begin{aligned} P_{r_5}[t_S] &= \frac{2}{c_1}(Q_5(t_s^-) + Q_6(t_s^-))\delta_{t_s} =: P^{\text{imp}^1_{t_S}}\delta_{t_s}, \\ P_{r_6}[t_S] &= \frac{2}{c_1}(Q_5(t_s^-) + Q_6(t_s^-))\delta_{t_s} =: P^{\text{imp}^2_{t_S}}\delta_{t_s}. \end{aligned}$$

A comparison between $\overline{P}^{\text{imp}^1_{t_S}}$ with $P^{\text{imp}^1_{t_S}}$ and $\overline{P}^{\text{imp}^2_{t_S}}$ with $P^{\text{imp}^2_{t_S}}$ for different values of the compressibility coefficient β is presented in Table 6.2. For large β the approximation is not very accurate, however, for decreasing compressibility the accuracy of the approximation improves.

β	$\overline{P}^{\text{imp}^1_{t_S}}$	$\overline{P}^{\text{imp}^2_{t_S}}$	$P^{\text{imp}^1_{t_S}}$	$P^{\text{imp}^2_{t_S}}$	Γ_1	Γ_2
$15.0 \cdot 10^{-9}$	$5.7821 \cdot 10^7$	$5.7831 \cdot 10^7$	$5.1137 \cdot 10^7$	$5.1137 \cdot 10^7$	0.1307	0.1309
$9.0 \cdot 10^{-9}$	$3.3944 \cdot 10^7$	$3.3951 \cdot 10^7$	$3.8590 \cdot 10^7$	$3.8590 \cdot 10^7$	0.1204	0.1202
$4.0 \cdot 10^{-9}$	$3.0906 \cdot 10^7$	$3.0918 \cdot 10^7$	$2.8407 \cdot 10^7$	$2.8407 \cdot 10^7$	0.0880	0.0884
$5.0 \cdot 10^{-10}$	$2.0299 \cdot 10^7$	$2.0292 \cdot 10^7$	$2.1096 \cdot 10^7$	$2.1096 \cdot 10^7$	0.0378	0.0381
$2.0 \cdot 10^{-10}$	$1.8450 \cdot 10^7$	$1.8457 \cdot 10^7$	$1.8482 \cdot 10^7$	$1.8482 \cdot 10^7$	0.0017	0.0014

Table 6.2: Impulse length comparison over a network of six pipes: with $\Gamma_1 = \frac{\left|\overline{P}^{\text{imp}^1_{t_S}} - P^{\text{imp}^1_{t_S}}\right|}{P^{\text{imp}^1_{t_S}}}$ and $\Gamma_2 = \frac{\left|\overline{P}^{\text{imp}^2_{t_S}} - P^{\text{imp}^2_{t_S}}\right|}{P^{\text{imp}^2_{t_S}}}$

Similar as for the PDE simulations we assume that the DAE is stationary before we switch, i.e. $Q_i(t_S^-)$ for $i \in \{1, \cdots, 6\}$ before closing of the valve. It should be noted that although the compressibility coefficient β does not effect the parameters of the switched DAE model, it does effect the initial value q_0 (and hence via $Q_i(t_S^-)$, because this is chosen to match the stationary solution of the balance law in the Section 6.2.1 considered on $[0, t_S)$ which depends on β. A switched DAE model for water hammer on a network of six pipes shown in the Figure 6.6, which is compared with a compressible nonlinear system of balance laws. With the support of numerical simulations of the PDE model it is illustrated that a switched DAE model is a good approximation for the PDE model with small compressibility coefficient.

Mode 3: Pipe 3 is closed

Firstly the PDE model is simulated the case; the left end valve of pipe 3 is closed. It is done by substituting $s_{l_3} = 0$ in (6.13).

In order to model the same case for DAE model; the control variable $s_{r_3} = 0$ in (6.16). In this network left valve of the pipe edge 3 is closed but the edge is again closed by the . Consider that the mode 1 before switching to mode 3 substitute $p = 3$ in (6.14). The equations of mode 3 are achieved by substituting $s_3 = 0$ in the model in (6.16).

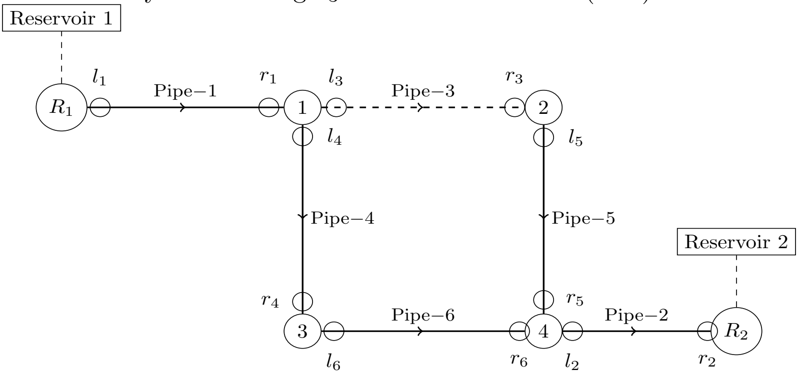


Figure 6.8: A pipe network six connected pipes with one closed edge

$$\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 \end{pmatrix}$$

For regularity the assumptions in the Theorem (C-I) and (C-II) in 5.3.8 is checked. It has no pump, therefore no need to check the assumption (C-I). For (C-II) consider that is $\widehat{A}^{\mathrm{jc}}_{\mathrm{f_{pi}}}$ in (6.17) have full row rank for all open edges, hence assumption (R) holds. The matrix pair (E_3, A_3) is regular. Hence the tranformation matrices T_3, S_3 and impulsive projector E_3^{imp} are calculated:

$$E_3^{\mathrm{imp}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{9}{4}\alpha & \frac{3}{2}\alpha & 0 & \frac{3}{2}\alpha & 0 & \frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2}\alpha & \frac{9}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & -\frac{3}{2}\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

The initial conditions for mode 3 are taken from the solution of mode 1. Rewrite nonITP (6.14)in the modified QWF (3.17) it will have the following form:

$$\begin{aligned}
\dot{v} &= v + c_1 P_U - c_1 P_D + g_1(Q_1) + g_2(Q_2) + g_4(Q_4) + g_6(Q_6) \\
w_i^1 &= 0, \qquad i = \{1,2,3,4,5,8,9,10,13,14,15,16,17,18\} \\
w_1^{11} &= P_U, \\
w_1^{12} &= P_D, \\
\dot{w}_1^3 + \dot{w}_1^7 + \dot{w}_1^9 &= w_2^1 + g_1(Q_1) + g_6(Q_6), \\
\dot{w}_1^3 - \dot{w}_1^8 - \dot{w}_1^{10} &= w_2^2 + g_2(Q_2) - g_6(Q_6), \\
\dot{w}_1^3 &= w_2^3 + g_3(Q_3), \\
\dot{w}_1^9 &= w_2^4 + g_4(Q_4) - g_6(Q_6), \\
\dot{w}_1^3 - \dot{w}_1^8 &= w_2^5 + g_5(Q_5).
\end{aligned}$$

Where all $Q_i = f_1(v, , w_1^i)$ by using transformation matrix T_1 :

$$\begin{aligned}
x &= T_3 \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}, \\
v &= \{v\}, \\
w_1 &= \{w_1^1, \cdots, w_1^{18}\}, \\
w_2 &= \{w_2^1, \cdots, w_2^5\}.
\end{aligned}$$

On solving the set of equations and finding solution in the original coordinates, solution is;

$$
\begin{aligned}
&Q_i(t) = c_1P_U - c_1P_D, \quad i = 1,2,4,5,6 \quad \forall t \geq t_s\\
&Q_3(t) = 0, \qquad P_{l_1}(t) = P_U,\\
&P_{l_2}(t) = \frac{3}{4}P_D + \frac{1}{4}P_U + (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
&P_{l_3}(t) = P_{l_4}(t_s^-) = \frac{3}{4}P_U + \frac{1}{4}P_D - \frac{1}{2c_1}Q_3(t_s^-)\delta_{t_s},\\
&P_{l_5}(t) = \frac{1}{4}P_U + \frac{3}{4}P_D + (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
&P_{l_6}(t) = \frac{1}{2}P_U + \frac{1}{2}P_D + \frac{1}{2c_1}(g_1(\frac{v}{4}) + g_6(\frac{v}{4})),\\
&P_{r_1}(t) = \frac{3}{4}P_U + \frac{1}{4}P_D + (\frac{3}{4c_1} - \frac{1}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
&P_{r_2}(t) = P_D,\\
&P_{r_3}(t) = \frac{3}{4}P_U + \frac{1}{4}P_D + (\frac{3}{4c_1} - \frac{1}{4c_1} + \frac{1}{c_1})Q_3(t_s^-)\delta_{t_s},\\
&P_{r_4}(t) = \frac{1}{2}(P_U + P_D) + \frac{1}{2c_1}(g_1(\frac{v}{4}) + g_6(\frac{v}{4})),\\
&P_{r_5}(t) = P_{r_6}(t) = \frac{1}{4}P_U + \frac{3}{4}P_D + (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
&P_{R_1}(t) = P_U, \quad P_{R_2}(t) = P_D,\\
&P_1(t) = \frac{3}{4}P_U + \frac{1}{4}P_D + (\frac{3}{4c_1} - \frac{1}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
&P_2(t) = \frac{1}{4}P_U + \frac{3}{4}P_D + (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
&P_3(t) = \frac{1}{2}(P_U + P_D) + \frac{1}{2c_1}(g_1(\frac{v}{4}) + g_6(\frac{v}{4})),\\
&P_4(t) = \frac{1}{4}P_U + \frac{3}{4}P_D + (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s}.
\end{aligned}
$$

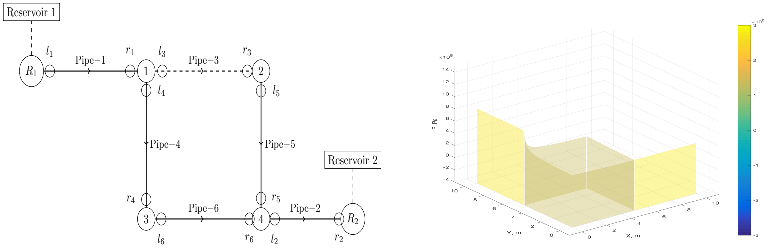
The Diracs are present in the following state variables ;

$$
\begin{aligned}
P_{l_2}[t_s] &= (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
P_{l_3}[t_s] = P_{l_4}[t_s] &= (\frac{3}{4c_1} - \frac{1}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
P_{l_5}[t_s] &= (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
P_{r_1}[t_s] &= (\frac{3}{4c_1} - \frac{1}{4c_1})Q_3(t_s^-)\delta_{t_s},
\end{aligned}
$$

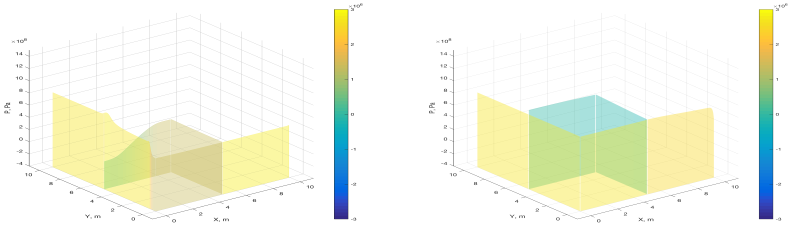
$$
\begin{aligned}
P_{r_3}[t_s] &= (\frac{3}{4c_1} - \frac{1}{4c_1} + \frac{1}{c_1})Q_3(t_s^-)\delta_{t_s},\\
P_{r_5}[t_s] &= (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
P_1[t_s] &= (\frac{3}{4c_1} - \frac{1}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
P_2[t_s] &= (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s},\\
P_4[t_s] &= (\frac{1}{4c_1} - \frac{3}{4c_1})Q_3(t_s^-)\delta_{t_s}.
\end{aligned}
$$

6.2.5 Numerical results (mode1 to mode 3) impulse

In this Section numerical simulations of the pipe network given in the Figure 6.9(a). Before t_s; when $t < t_s$ all the valve both at right and left edge of each of six pipes are open, the steady state is shown in the Figure 6.9(b). At the time $t = t_s$ the valve at the left of pipe 3 is closed the simulation of the network a that time is shown in the Figure 6.9(c).



(a) Pipe network (b) All pipes are open $t < t_s$ approximately steady state



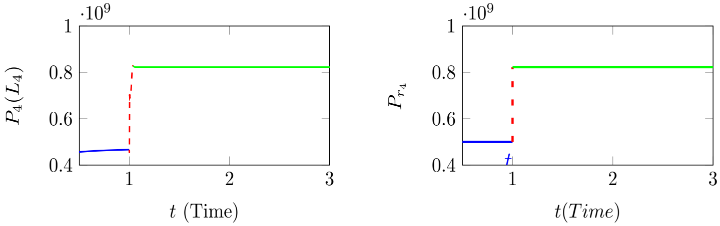
(c) At the switching time $t = t_s$ (d) Steady state after switching time $t > t_s$

Figure 6.9: Numerical illustration of PDE model of six pipes of the topology shown in Figure 6.8

The pressure impulse at the left end of pipe 3 go down first then travel in the whole network until a new steady state is approached in the Figure 6.9(d).

6.2.6 Numerical result (mode1 to mode 3) jumps

In the Figure 6.10 numerical simulation is shown of the network shown in the Figure 6.8, before after and at the closing time t_s. Further it is calculated that there is no Dirac impulse in the state variable P_{r_4} and P_{l_6} this agrees to the results found in the results via PDE and shown in



(a) Pressure at the right end of pipe 4 via PDE

(b) pressure at the right end of pipe 4 via swDAE.

Figure 6.10: Comparison of $(P_4(t, L_4)$ and (P_{r_4})

Similar pressure profile is observed at the left end of the pipe 6. It also agreed in both modeling frameworks.

6.2.7 Numerical result (mode1 to mode 3) comparison

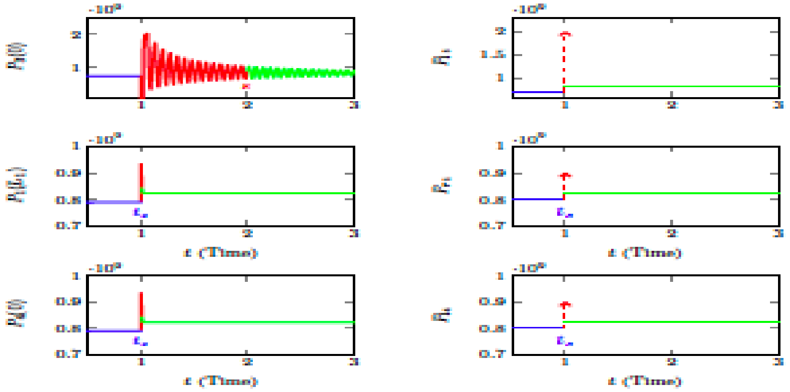


Figure 6.11: Left: Modeling of 6.8 via PDE and Modeling via switched DAE (Dirac comparison)

In the Figure 6.11 the pressure vs time plots are presented in PDE and switched nonlinear DAES modeling framework. The pressure at $P_1(L_1) := p_1(t,0)$ via PDE model is compared with the its corresponding to the pressure (P_{r_1}) plot via switched nonlinear DAE model. The valve at the left of pipe 3 is closed at $t_s = 1s$ the pressure after the valve firstly lower down and then due to the oscillation and flow from other pipe a positive pressure hit the valve again. The integral over the fluctuation from $t = 1$ to $t = 2$ is taken and on average this pressure in PDE model agree in the direction to the Dirac length via switched DAE model. As far as other pressure profiles are concerned they have very strong agreement in both of the frameworks.

6.3 Summary

In summary two water networks are presented with different topologies and hydraulic components in this chapter. These networks are motivated by EPANET sample netoworks (used for the analysis of water modeling in engineering). It is shown that both are well posed under the assumptions of the Theorem 3.4.6 and 5.3.8. Both are solvable with the notion of solution introduced in the Chapter 3. It is shown in the section 6.1 that the modeling via switched differential algebraic equations is a good approximation to the PDE model of this network. The pressure spike behaviour at the entrance segment of pump and the pressurised side of the pump can be guesstimated via simplified framework of switched DAE reasonably. A bit more complex network with junctions, pipes and supply nodes is presented in the Section 6.2, the valve closure effect near the location where the suddenly closed valve is installed. Relatively far nodes have less effect or only exhibit jump and these behaviours are again rationally approximated via the framework of switched DAEs.

Chapter 7

Conclusion

The framework of switched differential algebraic equation is applied to model the dynamics subject to sharp changes in electrical circuits. The water network is a compounding of various constituents, and the change in their settings (e.g., valve closure and pump shutdown etc.) induce indeed a sudden structural change. The classical hyperbolic system of balance laws is simplified to a nonlinear ODE; assuming incompressibility, and the changes in setting of the components by changing the algebraic constraint. Hence the hydraulic transients can be modeled in the framework of switched DAEs.

Before starting the modeling in the proposed framework, the classical method of modeling these component's settings as a boundary condition in the system of hyperbolic partial differential equations is recalled and implemented numerically. For instance the sudden closure of the valve is modeled with the classical approach as a foremost measure. It works out that the pressure spike expected in response to the sudden end of valve getting higher and higher on decreasing compressibility and other steady state after the sudden transient is also approached faster with this decrease in compressibility (it is lessened by decreasing the value of

β; the compressibility coefficient). Hence on considering for $\beta = 0$ the pressure spike looks like a Dirac. The framework of switched DAEs is indeed suitable and a good approximation to the classical modeling in which due to numerical limitations a true incompressibility $\beta = 0$ can not be considered.

In order to model the water network transients, in the new proposed framework first the existence and uniqueness of the solution of the switched nonlinear DAEs is established as a starting point. Fortunately on analysing the nonlinearities present in the water network it is demonstrated that these nonlinearities present are having special kind of sparse structure. This is basically utillised to determine a novel notion of solution. Equally it was required and mentioned previously considering the true incompressibility the pressure spike is a Dirac, existence and uniqueness of the solution to the switched nonlinear DAEs is ensured in the presence of impulses. A theorem with sufficient conditions for the existence of local solution of ITP is presented. Moreover, its extension to switched nonlinear DAEs is presented which is possible under the supposition that no finite escape time occurs between the switches.

It is established with the aid of numerical simulations of PDE that the approximated model of water hammer with switched nonlinear DAEs is a good approximation to the one modeled with a classical PDE model with small compressibility coefficient (considering water incompressible). This approximated model is implemented to a simple setup that is oftentimes applied to study water hammer and it is illustrated by using mathematical simulations that proposed approximated model is indeed suitable.

Furthermore, the pertinence of the whimsy of the solution on a general water network with arbitrary bit of hydraulic components for various hydraulic transients is investigated. Firstly matrix pair (E, A) for describing the kinetics of the general water network is constructed. It is shown that two further assumptions are needed to ensure the applicability of the solution theory developed for the structured nonlinearities in the presence of impulses for all possible sudden changes in the settings of water components. These assumptions are essentially asked to ensure regularity of the matrix pair of the general network structures.

Finally, this notion of solution and general network setup introduced applies to the examples water networks. These networks are modeled using hyperbolic system of balance laws and then in the framework of switched nonlinear DAEs. IIt is noted that the new proposed modeling approach is a good approximation with the simplification of incompressibility. Hence this simple framework is useful to study the impulsive effects in the solutions of a water network. It is depicted by the solution of switched nonlinear DAEs that Dirac appeared in whole network at switching time in reaction to the change in the context of the component at any position in the network, but with different signs and approximated length of Dirac (coefficient of the Dirac delta δ in the solution).

Hence this proposed framework provided with a simple way to examine the impact of the transients as high low or negative pressure surges on the pipe walls. The limitation of this approach is all the comparisons are performed on the end point of every pipe as these components are installed in the network at the junctions (which are end points of the pipes). This is very practical as the components may cause there transients are installed at junctions at thr ends of pipes. The study of the impact at other locations in the pipe is a direction of future work in the framework of switched nonlinear DAEs.

Appendix

In this appendix we introduce several mathematical concepts and lemmas that are used throughout the thesis.

Basic concepts

Definition A.0.1 (Kernel of a linear map)**.** Let V, W be vector spaces over $\mathbb{R}^n$, and let $F : V \to W$ be a linear map. We define the kernel of F ($\ker F$) to be the set of elements $v \in V$such that $F(v) = 0$.

Definition A.0.2 (Image of a linear map)**.** *Let* $F : V \to W$ *be a linear map. The image of* F*,* $\operatorname{im} F$*, is the set of elements* $w \in W$ such that there exists an element of v of V such that $F(v) = w$.

An important result is given by the following:

$$dimV = dim \ker F + dim \operatorname{im} F.$$

Consider a linear map represented as a $m \times n$ matrix M with coefficients in $\mathbb{R}$ and operating on column vectors x with n components over $\mathbb{R}$. The kernel of this linear map is the set of solutions to the equation $Mx = 0$. The dimension of the kernel of M is called the nullity of M.

The image of M is defined as $\operatorname{im} M : \{y : x \in \mathbb{R}^n \text{ such that } Mx = y\}$. The dimension of the image is called rank of a matrix and corresponds to the number of linearly independent rows or columns of the matrix.

Definition A.0.3 (Lipschitz function)**.** A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is if there exists a positive constant L_f such that for all ξ_1, ξ_2 the following inequality

$$\|f(\xi_1) - f(\xi_2)\| \leq L_f(\|\xi_1 - \xi_2\|)$$

holds.

Remark A.0.4. It is worth noting that the property of a vector valued function f of being locally Lipchitz continuous in ξ can equivalently defined by means of seperate Lipchitz conditions for the scalar components f_i of the function. Thus f is locally Lipchitz continuous.

The state of an unstable linear system can go to infinity as time approaches infinity. A nonlinear system's state, however, can go to infinity in finite time.

Definition A.0.5 (Finite escape time)**.** A solution $x(t)$ with the property that $x(t) \to \infty$ as t approaches some finite time is said to exhibit finite escape time .

Local Lipschitz continuity suffices to have uniqueness. What it does not suffice to is global existence. That is, if f is local but not global Lipschitz, then the (unique) solution to the Cauchy problem might cease to exist (blow up) in finite time. Hence local Lipchitz function not exhibiting finite escape time said to be global Lipchitz.

Projectors

Recall that a matrix $\Pi \in \mathbb{R}^{n \times n}$ (or its associated linear map) is a projector by definition if, and only if, it is idempotent, i.e. $\Pi^2 = \Pi$. There is a one-to-one correspondence between projectors in $\mathbb{R}^n$ and direct sums $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W}$, via

$$\operatorname{im} \varPi = \mathcal{V}, \quad \ker \varPi = \mathcal{W};$$

the projector is then said to map onto $\mathcal{V}$ along $\mathcal{W}$.

Lemma A.0.6. *Let $\varPi \in \mathbb{R}^{n \times n}$ be a projector and $M \in \mathbb{R}^{n \times n}$ then*

$$\begin{aligned} \operatorname{im} M \subseteq \operatorname{im} \varPi &\quad \Leftrightarrow \quad \varPi M = M, \\ \ker M \supseteq ker\varPi &\quad \Leftrightarrow \quad M\varPi = M. \end{aligned}$$

Proof. Necessity in both cases is trivial. Since $\varPi$ is the identity on $\operatorname{im} \varPi$ sufficiency for the first case is also clear. Considering the transpose and orthogonal complements, sufficiency of the second case follows with analogous arguments. ■

Matlab codes

To calculate the matrices V and W of the modified Quasi Weierstrass form introducd in the Lemma 3.4.7, we use the following Matlab codes, where the build-in Matlab functions 'colspace" and 'null' are used, 69]. The matrix V is obtained by the following E,A

```
function V = getVspace (E, A)
E = E' , A = A'
[m,n]= size (E);
if (m==n) & size (E )== size (A)
V= eye (n,n);
oldsize =n;
newsize =n;
finished =0;
while finished ==0;
EV= colspace (E · V);
V= getPreImage (A, E · V );
```

```
oldsize = newsize ;
newsize = rank (V);
finished =( newsize == oldsize );
end ;
else
error ('Matrices $E$ and $A$ must be square and of the same size ');
end ;
```

The matrix W is obtained by the following

```
function W = getWspace ($E$,$A$)
$E = E^{'}$ , $A = A^{'}$
[m,n]= size ($E$);
if (m==n) & size $E$ )== size ($A$)
W= zeros (n ,1);
oldsize =0;
newsize =0;
finished =0;
while finished ==0;
$A^{\top}$W= colspace ($A \cdot W$);
W= getPreImage ($E^{\top}$,$A \cdot W$);
oldsize = newsize ;
newsize = rank (W);
finished =( newsize == oldsize );
end ;
else
error ('Matrices $E$ and $A$ must be square and of the same size ');
end ;
```

where the preimage in the Wong sequences is obtained by

```
function $V$= getPreImage ($E, S$)
$E = E^{'}$ , $A = A^{'}$
[m1 ,n1 ]= size ($A$)$; [m2, n2] = size(S); if m1 == m2|m2 == 0 H = null($[A
```

Remark A.0.7. After finiding V and W using the above matlab codes for $E^\top$ and $A^\top$. Following steps are perfomed.

Fluid flow

In this Section some background concept used for the modeling of fluid flows are presented, which have been used in the thesis.

Moody chart

The Moody diagram is a plot of the Darcy friction factor as a function of Reynolds number and relative roughness. The Moody diagram shows both the laminar and turbulent regimes as well as a transition zone between laminar and turbulent flow.

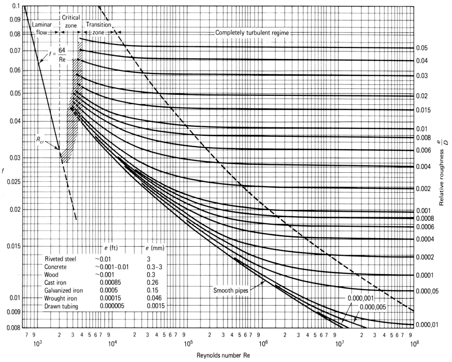


Figure 7.13 Moody diagram. (From L. F. Moody, *Trans. ASME*, Vol. 66, 1944.)

Figure 7.1: Moody chart used to select friction coefficient

A note on numerics

Consistent coupling conditions

For coupling conditions to make sense, they need to fulfil the following consistency requirement.

Definition A.0.8. A set of coupling conditions is called consistent with a given conservation law, if they revert to equal state coupling :

$$\breve{u}^1 = \breve{u}^2$$

for one to one coupling situation.

For coupling conditions to be considered consistent with a conservation law, splitting the domain of an edge into two coupled by a vertex can not change the solution.

Details on numerical scheme

Unless stated otherwise in the respective section, numerical simulations use the following default configuration.

Flux-corrected transport (FCT) is a conservative shock-capturing scheme for solving Euler equations and other hyperbolic equations which occur in gas dynamics, aerodynamics, and magnetohydrodynamics. It is especially useful for solving problems involving shock or contact discontinuities. An FCT algorithm consists of two stages, a transport stage and a flux-corrected anti-diffusion stage. The numerical errors introduced in the first stage (i.e., the transport stage) are corrected in the anti-diffusion stage 56] 57].

Numerical scheme

Flux corrected transport scheme (FCT) is used to simulate the system (2.20). First the scheme is presented, and (2.20) can be rewritten as:

$$\partial_t(QC) + \partial_x(BC) = 0$$

with

$$Q = \begin{pmatrix} \rho_i \\ \frac{q_i}{\rho_i}, \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ \rho_i \end{pmatrix} \quad C = \begin{pmatrix} q_i \\ \frac{q_i^2}{\rho_i^2} + \frac{K}{\rho_a} \end{pmatrix}$$

where A is conserving integral, C are values which should be 'smooth'. The flux is calculated in the following way.

1. The flux with low order scheme is calculated

$$F_{\mathrm{low},i+\frac{1}{2}} = \frac{1}{2dt}((BC)_{i+1} + (BC)_i) + \frac{\mathrm{CFL}}{2}c(Q_{i+1} - Q_i)(C_{i+1} - C_i)$$

   Here, the disspipation term is $\frac{\mathrm{CFL}}{2}c(Q_{i+1} - Q_i)(C_{i+1} - C_i)$.

2. Then the high order flux is calculated

$$F_{hi,i+\frac{1}{2}} = \frac{1}{2dt}((BC)_{i+1} + (BC)_i)$$

By using FCT algorithm following formulas are usied to combine low order and high order scheme:

$$A_{i+\frac{1}{2}} = F_{hi,i+\frac{1}{2}} + F_{\mathrm{low},i+\frac{1}{2}}$$

Then further is calculated as :

$$Q_{dt,i} = Q_i + \frac{1}{B_i dx}(F_{\text{low},i+\frac{1}{2}} - F_{\text{low},i-\frac{1}{2}})$$

$$l_{i+\frac{1}{2}} = \max((0, \min(|A_{i+\frac{1}{2}}|)), dxS_{i+\frac{1}{2}}(Q_{dt,i+2} - Q_{dt,i+1}),$$
$$dxS_{i+\frac{1}{2}}(Q_{dt,i} - Q_{ft,i-1}))$$

where

$$S_{i+\frac{1}{2}} \equiv (A_{i+\frac{1}{2}})$$

$$AC_{i+\frac{1}{2}} = l_{i+\frac{1}{2}} S_{i+\frac{1}{2}}$$

The new result for new time step is calculated as;

$$Q_{\text{new},i} = Qdt, i - \frac{1}{B_i dx}(AC_{i+\frac{1}{2}} - AC_{i-\frac{1}{2}})$$

Discretization

The scheme of discretization is applied to each pipe show in the Figure 7.2. The boundary points are on pipe ends, and the ghost points with values necessary for the coupling conditions.

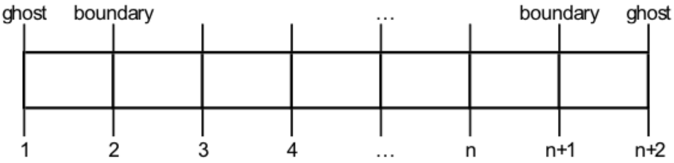


Figure 7.2: scheme discretisation

where in the scheme

∇x = space discretisation

∇t = time step which is same within one time step

The time steps are calculated on density and flux in all pipes by the following formula

$$\nabla t = \frac{\nabla x \cdot \mathrm{CFL}}{\max(c + \frac{q_i}{\rho_i})}$$

Further introduce following notation

$$()_b = \text{boundary points} \qquad ()_g = \text{ghost points}$$

Each pipe have two boundary and two ghost points thus in formulas below those indecis represents related to boundary taking part in coupling. The following types of coupling conditions are considered. The type of condition change over time. For example for open and closed valves.

Remark A.0.9. The scheme used is flux controlled scheme (FCT), which is 2^{nd} order scheme of second order at areas where the solution is smooth solution and 1^{st} order at the areas with fast changing values due to the change in the boundary conditions. The scheme is designed to provide mass and flow conservation despite of the discretization error. ; however dissipation causes small mechanical energy dissipation. It does impact results until till the point where the dynamics are calculated at the scope where flux would equal to zero in steady case 15].

Pipe ends intersection

When one ore more pipes intersects means connected together following condition

$$\begin{aligned}(\rho_i)_g &= \frac{\sum_{k=1}^{n}(1-\chi_{k,i}+\epsilon\chi_{k,i})(\rho_k)_b}{\sum_{k=1}^{n}(1-\chi_{k,i}+\epsilon\chi_{k,i})},\\ (q_i)_g &= \frac{\sum_{k=1}^{n}((\text{-}1)^{s_a}(1-\chi_{k,i})(q_k)_b}{\min(\sum_{k=1}^{n}(1-\chi_{k,i}+\epsilon\chi_{k,i}),1)}\end{aligned}$$

where

$$\chi_{a,i} - \text{Kronecker delta},$$
$$s_k - \text{side of the pipe, connected } 0- \text{ left end } 1- \text{ right end}$$

If the pipe end is with closed valve, this pipe end is not considered in this coupling condition.

CFL condition

The time step in the complete network is always synchronized. The strongest time step restriction on the edges dictates the time step restriction on the whole network. The CFL condition is chosen to be 0.95.

Closed valve

$$\begin{aligned}(\rho_k)_g &= (\rho_k)_b,\\ (q_k)_g &= 0\end{aligned}$$

Fixed pressure

The fixed pressure boundary represents outliet or reservoir depending on the pressure value. Pressure conditions applies in the case is following

$$\begin{aligned}(P_k)_g &= res_p,\\ (q_k)_g &= (q_k)_b\end{aligned}$$

Graphs

In this section some imporant Lemmata about the graph theory are presented.

Lemma A.0.10. If G is a connected graph on n vertices, then rank $Q(G) = n - 1$.

Proof. Suppose x is a vector in the left null space of $Q := Q(G)$, that is, $x^{=}0$. Then $x_i - x_j = 0$ whenever i is connected to j It follows that $x_i = x_j$ whenever there is an a path between i and j. Since G is connected, x must have all components equal. Thus, the left null space of Q is at most one-dimensional and therefore the rank of Q is at least $n - 1$. Also, as observed earlier, the rows of Q are linearly dependent and therefore rank $Q \leq n - 1$. Hence, rank of $Q = n - 1$. ■

Lemma A.0.11. Consider a graph $G = (\mathcal{N}, \mathcal{E})$, where each node $v \in V$ has degree atleast 2. Then G has a cycle.

Proof. Assume, for contradiction, that G has no cycle, and consider the longest path P' in G (one must exist, since the graph is finite). Let v be the final vertex in P', since v has degree 2, it must have two edges e_1 and e_2 incident on it, of which one, say e_1, is the last edge of the path P'. Then e_2 cannot be incident on any other vertex of P' since that would create a cycle (v, e_2, [section of P' ending in e_1], v). So e_2 and its other endpoint are not part of P' , and can be appended to P' to give a strictly longer path, which contradicts our choice of P'. Hence G must contain a cycle. ■

Matrices

Matrices background is very important for the proving the rank properties used to prove the regualrity of matrix pair.

Block diagonal matrices

Consider a matrix L of the form:

$$L = \begin{pmatrix} L_{11} & L_{12} & \cdots & A_{1r} \\ L_{21} & L_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix}$$

(for which the number of rows of blocks equals the number of columns of blocks), the ij^{th} block L_{ij} of L is called a *diagonal block* if $j = i$ and an *off diagonal* block if $j \neq i$. If all of the off diagonal blocks of L are null matrices, that is if

$$L = \begin{pmatrix} L_{11} & 0 & \cdots & 0 \\ 0 & L_{22} & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & L_{rr} \end{pmatrix}$$

then it is called blocked diagonal matrix.

Important results about definite block diagonal matrix

Lemma A.0.12. Let $\mathcal{D} = \{d_i\}$ represents $n \times n$ diagonal matrix. Then

(1) $\mathcal{D}$ is non negative definite if and only if $d_1, \cdots, d_n$ are nonnegative.

(2) $\mathcal{D}$ is positive definite if and only if $d_1, \cdots, d_n$ are positive.

(3) $\mathcal{D}$ is positive semidefinite if and only if $d_i \geq 0$ for $i = 1, \cdots, n$ with equality holding for more than one value of i.

It is trivial that, For any matrices L_1 and L_2,

$$\text{rank}\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} = \text{rank}(L_1) + \text{rank}(L_2)$$

Lemma A.0.13. A blocked diagonal matrix of the form

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ \mathcal{B}_2 & \mathcal{B}_3 \end{pmatrix}$$

as full rank iff $\mathcal{B}_1$ and $\mathcal{B}_3$ have full rank.

Lemma A.0.14. Consider A is a matrix of order $m \times n$ further A has full rank. Then $AA^\top$, $A^\top A$ for $m \neq n$ is symmetric positive definite matrix.

Proof. If A is full rank, then rank(A)= $\min\{m, n\}$. If $m \geq n$, then rank(A)= n. By the rank nulity theorem

$$\text{nulity}(A) = n - \text{rank}(A) = 0.$$

If $x^\top A^\top Ax = 0$, then $Ax_2^2 = 0$; so $Ax = 0$; so $x \in \text{nulity}(A)$; so $x = 0$. Hence $A^\top A$ is positive definite.

On the other hand, if $m < n$, then $\text{rank}(A) = m$; so $\text{rank}(A^\top) = n$. By the rank nulity thoerem ,

$$\text{nulity}(A^\top) = n - \text{rank}(A^\top) = 0$$

If $y^\top AA^\top y = 0$, then $Ay_2^2 = 0$; so $A^\top y = 0$; so $y \in \text{nulity}(A^\top)$; so $y = 0$. Hence $AA^\top$ is positive definite. ■

Lemma A.0.15. A diagonal matrix $Q \in \mathbb{R}^m$ with all diagonal entries $q_{ii} > 0$ is positive definite.

Lemma A.0.16. If M is positive definite and Q has full row rank, then $QMQ^\top$ is positive definite.

Proof. M is positive definite means for all non zero y $y^\top My > 0$. Moreover Q has full row rank implies it has full column rank, means

$$Q^\top x = 0 \implies x = 0.$$

To show that $QMQ^\top$ is positive definite consider for all $x \neq 0$

$$\begin{aligned}
&= x^\top (QMQ^\top)x, \\
&= (x^\top Q)M(Q^\top x), \qquad Q^\top x \neq 0, \\
&= (Q^\top x)^\top M(Q^\top x) > 0 \qquad M \text{ is positive definte}
\end{aligned}$$

■

Lemma A.0.17. Every positive definite matrix have full rank.

Proof. Consider a matrix M which is positive definite that means, $x^\top Mx > 0$ and $x \neq 0$. On contrary suppose that M does not have full rank. There must be a non-zero vector x such that $Mx = 0$ which implies $x^\top Mx = 0$ which is contradiction to the assumption that M is positive definite . ■

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