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Bericht Nr. 141

**ON A NEW CONDITION FOR STRICTLY POSITIVE
DEFINITE FUNCTIONS ON SPHERES**

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Juni 1995

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Abstract

Recently, XU and CHENEY (1992) have proved that if all the Legendre coefficients of a zonal function defined on a sphere are positive then the function is strictly positive definite. It will be shown in this paper, that even if finitely many of the Legendre coefficients are zero, the strict positive definiteness can be assured. The results are based on approximation properties of singular integrals, and provide also a completely different proof of the results of Xu and Cheney.

1 Introduction

Let for $m \geq 1$, $S_m \subset \mathbb{R}^{m+1}$ be the m -dimensional unit sphere. A continuous function $K : [-1, 1] \rightarrow \mathbb{R}$ defines for a fixed $\eta \in S_m$ a so-called η -zonal function $\xi \mapsto K(\eta \cdot \xi)$, $\xi \in S_m$, on the sphere, where \cdot denotes the usual inner product in \mathbb{R}^{m+1} . Thus, the function $K(\eta \cdot \cdot)$ depends only on the spherical distance $\arccos(\xi \cdot \eta)$ between ξ and η . Such a continuous function is called *positive definite*, if for any choice of pairwise distinct points $\eta_1, \dots, \eta_N \in S_m$, and any non-zero vector $(a_1, \dots, a_N)^T \in \mathbb{R}^N$,

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j K(\eta_i \cdot \eta_j) \geq 0. \quad (1)$$

SCHOENBERG (1942) has shown that if the function K admits the uniformly convergent series expansion

$$K(t) = \sum_{n=0}^{\infty} k_n P_n(t), \quad t \in [-1, 1], \quad (2)$$

in terms of certain Legendre (or Gegenbauer or ultraspherical) polynomials, a sufficient condition for (1) is that $k_n \geq 0$, $n = 0, 1, \dots$

However, when dealing with problems of interpolation, a stronger condition on the kernel K is useful: K is called *strictly positive definite*, if the quadratic form (1) is strictly positive for any set $\{\eta_1, \dots, \eta_N\} \subset S_m$ of pairwise distinct points and any choice of a non-zero vector $(a_1, \dots, a_N)^T \in \mathbb{R}^N$. XU, CHENEY (1992) have proved, that a sufficient condition for strict positive definiteness is, when all the k_n in the expansion (2) are positive. By a complete different idea we will show in this paper, that even if finitely many k_n are zero, the kernel K is strictly positive definite.

The value of this result can be explained as follows: assume that a function $F : S_m \rightarrow \mathbb{R}$ is known only at finitely many distinct points $\eta_1, \dots, \eta_N \in S_m$ and one looks for an interpolant of the form

$$S(\xi) = \sum_{i=1}^N a_i K(\eta_i \cdot \xi) \quad (3)$$

satisfying the interpolation conditions $S(\eta_i) = F(\eta_i)$, $i = 1, \dots, N$. Then the linear system to be solved is

$$\begin{pmatrix} K(\eta_1 \cdot \eta_1) & \cdots & K(\eta_N \cdot \eta_1) \\ \vdots & \ddots & \vdots \\ K(\eta_1 \cdot \eta_N) & \cdots & K(\eta_N \cdot \eta_N) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} F(\eta_1) \\ \vdots \\ F(\eta_N) \end{pmatrix}.$$

If K is strictly positive definite then the matrix is positive definite, i.e. the interpolation problem is solvable for any choice of pairwise distinct nodal points.

In practice, however, there is often more information of the function F available, e.g. the low order moments of F in terms of its spherical harmonic expansion. In this case, it is desirable to use kernels K in (3) which are orthogonal to these low order spherical harmonics and to perform the interpolation only for the difference between the function values $F(\eta_i)$ and the values of the known projection of F to the span of the spherical harmonics under consideration. But orthogonality of the kernel K to spherical harmonics means that the corresponding coefficients k_n are zero. Thus, the condition given in XU, CHENEY (1992) can not be satisfied. In this case our stronger result is necessary. An example for such a situation is the approximation of the gravitational potential of the earth. From physical properties and measurements of satellite orbits the contribution of the lower order spherical harmonics are known with a

sufficient accuracy, so that e.g. for space borne gradiometry data this approach is very useful, cf. e.g. FREEDEN et al. (1994), RUMMEL (1986), SCHREINER (1994).

The outline of this paper is organized as follows: after some preliminaries, we develop an easy-to-handle equivalent condition for strict positive definiteness. After that, we construct a special strictly positive definite function with vanishing moments, and use afterwards this kernel, which turns out to be a spherical approximate identity, for the proof of our main result.

2 Preliminaries

Assume $m \geq 1$ to be a fixed integer for the further investigations. Let \cdot and $|\cdot|$ denote the usual inner product and the Euclidean norm in \mathbb{R}^{m+1} , respectively, and let $S_m = \{\xi \in \mathbb{R}^{m+1} | |\xi| = 1\}$ be the m -dimensional unit sphere in \mathbb{R}^{m+1} . We write $d\omega_m$ for the induced surface element of S_m . Then it is well-known that the volume of S_m is given by

$$\omega_m = \int_{S_m} d\omega_m = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma((m+1)/2)}.$$

We denote the space of continuous respectively square-integrable functions defined on S_m by $\mathcal{C}(S_m)$ or $\mathcal{L}^2(S_m)$. The standard $\mathcal{L}^2(S_m)$ -inner product is written as

$$(F, G) = \int_{S_m} F(\xi)G(\xi)d\omega_m(\xi), \quad F, G \in S_m.$$

In the following, we repeat some basic facts on spherical harmonics and Legendre polynomials. Details can be found e.g. in MÜLLER (1966). Let Harm_n be the space of all spherical harmonics $Y_n : S_m \rightarrow \mathbb{R}$ of order n , i.e. $Y_n \in \text{Harm}_n$ if and only if $x \mapsto |x|^n Y_n(x/|x|)$ is a homogeneous harmonic polynomial of degree n . The dimension of Harm_n is

$$N(m, n) = \dim \text{Harm}_n = \frac{2n + m - 1}{n} \binom{n + m - 2}{n - 1}.$$

If $Y_p \in \text{Harm}_p$ and $Y_q \in \text{Harm}_q$ with $p \neq q$ then $(Y_p, Y_q) = 0$, i.e. spherical harmonics of different order are orthogonal. We assume that $\{Y_{n,1}, \dots, Y_{n,N(m,n)}\}$ is an orthonormal basis of Harm_n . For $p \in \mathbb{N}_0$, we let $\text{Harm}_{0,\dots,p} = \bigoplus_{n=0}^p \text{Harm}_n$.

The addition theorem for spherical harmonics reads as follows:

$$\sum_{j=0}^{N(m,n)} Y_{n,j}(\xi)Y_{n,j}(\eta) = \frac{N(m,n)}{\omega_m} P_n(\xi \cdot \eta), \quad \xi, \eta \in S_m, \quad (4)$$

where P_n are the (generalized) Legendre polynomials of degree n . They are defined by the requirements

- (i) P_n is a polynomial of degree n
- (ii) $\int_{-1}^1 t^l P_n(t) (1-t^2)^{\frac{m-2}{2}} dt = 0, l = 0, \dots, n-1$
- (iii) $P_n(1) = 1$.

It follows from (4) for fixed $\eta \in S_m$ that $P_n(\cdot \eta) \in \text{Harm}_n$ and that for all $Y_n \in \text{Harm}_n$

$$Y_n(\xi) = (Y_n, \frac{N(m, n)}{\omega_m} P_n(\xi \cdot \eta)), \xi \in S_m,$$

i.e. $\frac{N(m, n)}{\omega_m} P_n(\cdot \eta)$ is the reproducing kernel of Harm_n , cf. ARONSZAJN (1950). Applying the Cauchy-Schwarz inequality to (4), it can be easily deduced that $|P_n(t)| \leq P_n(1) = 1$ for all $t \in [-1, 1]$. Furthermore, the Fourier series of a square-integrable function $F \in \mathcal{L}^2(S_m)$ can be written as

$$F \sim \sum_{n=0}^{\infty} \frac{N(m, n)}{\omega_m} \int_{S_m} F(\eta) P_n(\cdot \eta) d\omega_m(\eta). \quad (5)$$

If $G \in \mathcal{L}^2[-1, 1]$, we obtain for fixed $\eta \in S_m$ for the η -zonal function $\xi \mapsto G(\xi \cdot \eta), \xi \in S_m$,

$$\int_{S_m} G(\xi \cdot \eta) Y_n(\eta) d\omega_m = \omega_{m-1} \int_{-1}^1 G(t) P_n(t) (1-t^2)^{\frac{m-2}{2}} dt Y_n(\xi), \xi \in S_m. \quad (6)$$

This formula of Funk and Hecke allows to obtain the Fourier coefficients of an η -zonal function by a simple one-dimensional integration. Together with the addition theorem we conclude that the Fourier expansion of $G(\cdot \eta)$ is given by

$$G(\cdot \eta) \sim \sum_{n=0}^{\infty} \frac{N(m, n) \omega_{m-1}}{\omega_m} \int_{-1}^1 G(t) P_n(t) (1-t^2)^{\frac{m-2}{2}} dt P_n(\cdot \eta). \quad (7)$$

Note that the Legendre polynomials are also expressible (for $m \geq 2$) by the generating function

$$\frac{1}{(1-2rt+r^2)^{\frac{m-1}{2}}} = \sum_{n=0}^{\infty} \binom{n+m-2}{n} P_n(t) r^n, |r| < 1, t \in [-1, 1].$$

This shows that the Legendre polynomials are up to a constant the Gegenbauer or ultraspherical polynomials C_n^λ with $\lambda = (m-1)/2$, cf. e.g. SZEGÖ (1959).

For $r \in (0, 1)$ the uniform convergent series

$$Q_r(t) = \sum_{n=0}^{\infty} \frac{N(m, n)}{\omega_m} r^n P_n(t), t \in [-1, 1],$$

has the explicit m

which is known a
is described in

Theorem 2.1. A

This theorem sho

Proof. Since k_n
 $\tilde{K} : S_m \times S_m \rightarrow$
kernel of a Hilbert
 $Y_{n,j} | \ n = 0, \dots,$
 $(K(\eta_{i\cdot}), K(\eta_{j\cdot}))$

turns out to be
therefore positive
linearly independent

Since the series of $Q_r^{0,\dots,p\perp}$ in terms of the Legendre polynomials is uniformly convergent, it follows for all $\xi \in S_m$, $n \geq p+1$

$$\begin{aligned} 0 &= \left(\sum_{i=1}^N a_i Q_r^{0,\dots,p\perp}(\eta_{i\cdot}), \frac{N(m,n)}{\omega_m} P_n(\cdot \xi) \right) \\ &= \sum_{i=1}^N a_i r^n P_n(\eta_{i\cdot} \xi) \\ &= r^n \sum_{i=1}^N a_i P_n(\eta_{i\cdot} \xi). \end{aligned}$$

Thus, $\sum_{i=1}^N a_i P_n(\eta_{i\cdot}) = 0$ for all $n \geq p+1$. The same calculation performed in the backward direction implies then

$$\sum_{i=1}^N a_i Q_r^{0,\dots,p\perp}(\eta_{i\cdot}) = 0 \quad (9)$$

for all $r \in (0, 1)$.

Now, let $i^* \in \{1, \dots, N\}$ be fixed, and choose $\varepsilon < \Theta_N$, where the nodal width Θ_N is given by

$$\Theta_N = \max_{j=1,\dots,N} \sup_{j \neq i} |\eta_i - \eta_j|.$$

For this ε and $\eta = \eta_{i^*}$ let $H \in \mathcal{C}(S_m)$ have the properties of Lemma 4.1. Then we conclude from Corollary 2.2 and the fact that $H(\eta_i) = 0$ if $i \neq i^*$,

$$a_{i^*} = \sum_{i=1}^N a_i H(\eta_i) = \lim_{\substack{r \rightarrow 1 \\ r < 1}} \left(\sum_{i=1}^N a_i Q_r^{0,\dots,p\perp}(\eta_{i\cdot}), H \right).$$

On the other hand side it follows from (9) that for all $r \in (0, 1)$

$$\left(\sum_{i=1}^N a_i Q_r^{0,\dots,p\perp}(\eta_{i\cdot}), H \right) = 0.$$

Thus, $a_{i^*} = 0$. Since $i^* \in \{1, \dots, N\}$ was chosen arbitrarily, it follows $a_1 = \dots = a_N = 0$, and hence the functions $Q_r^{0,\dots,p\perp}(\eta_{1\cdot}), \dots, Q_r^{0,\dots,p\perp}(\eta_{N\cdot})$ are linearly independent, as required. \square

Remark 4.3. *It follows by similar (and even easier) arguments that Q_r is strictly positive definite. We omit the details.*

5 The Main Result

The considerations of the last chapters allow us to prove the main result:

Theorem 5.1. *Let $K : [-1, 1] \rightarrow \mathbb{R}$ be a continuous kernel with uniformly convergent series expansion*

$$K(t) = \sum_{n=0}^{\infty} k_n P_n(t). \quad (10)$$

Assume that $k_n \geq 0$, $n \in \mathbb{N}$, and only finitely many k_n are zero. Then K is strictly positive definite.

Proof. Let p denote the index of the largest integer n for that $k_n = 0$. Then $k_n > 0$ for all $n > p$. Assume that $\eta_1, \dots, \eta_N \in S_m$ are pairwise distinct. We shall show that $K(\eta_1 \cdot), \dots, K(\eta_N \cdot)$ are linearly independent.

If $\sum_{i=1}^N a_i K(\eta_i \cdot) = 0$, similar arguments as in the proof of Theorem 4.2 provide that

$$\sum_{i=1}^N a_i P_n(\eta_i \cdot) = 0 \quad (11)$$

for all n for which $k_n > 0$. In particular, (11) is true for all $n > p$. But then it follows that for every $r \in (0, 1)$

$$\sum_{i=1}^N a_i Q_r^{0, \dots, p \perp}(\eta_i \cdot) = 0.$$

We know from Theorem 4.2 that $Q_r^{0, \dots, p \perp}$ is strictly positive definite, therefore $a_1 = \dots = a_N = 0$. Hence, K is strictly positive definite. \square

Since all the Legendre coefficients of Q_r are positive, similar arguments as above together with Remark 4.3 prove the result of XU, CHENEY (1992):

Corollary 5.2. *If all the k_n in (10) are positive then K is strictly positive definite.*

The question whether a weaker condition than the one of Theorem 4.2 is sufficient for strict positive definiteness is still open. We show by an easy example that the condition that finitely many k_n are greater than zero is not sufficient for strict positive definiteness: let $m = 2$ and assume that the k_n of the uniformly convergent series expansion of a kernel K are zero for odd

n and greater than zero for even n . The kernel K is then an even function. If we choose η_1 to be the North Pole and η_2 to be the South Pole, K satisfies $K(\eta_1 \cdot \xi) = K(\eta_2 \cdot \xi)$, $\xi \in S_m$. Thus, $K(\eta_1 \cdot \cdot)$ and $K(\eta_2 \cdot \cdot)$ are linearly dependent, and so K is not strictly positive definite.

Appendix: Proof of Lemma 4.1

Lemma 4.1 follows immediately, if we can construct for any $\beta > 0$ and all $p \in \mathbb{N}_0$ a continuous function $L : [-1, 1] \rightarrow \mathbb{R}$ with the properties

- (i) $L(1) = 1$
- (ii) $\text{supp } L \subset [1 - \beta, 1]$
- (iii) $\int_{-1}^1 L(t) P_n(t) (1 - t^2)^{\frac{m-2}{2}} dt = 0$, $n = 0, \dots, p$,

since then the function $H \in \mathcal{C}(S_m)$ defined by $H(\xi) = L(\eta \cdot \xi)$, $\xi \in S_m$ satisfies all requirements of Lemma 4.1 with $\varepsilon = (2\beta)^{1/2}$.

We will show now, how a function L satisfying (i)–(iii) can be constructed for a given $\beta > 0$ and $p \in \mathbb{N}_0$. Assume first that m is even. Choose real numbers h_i with $1 - \beta \leq h_0 < h_1 < \dots < h_{m+p+3} < 1$. We define for $t \in [-1, 1]$

$$A_0(t) = \frac{(t - h_0)_+}{1 - h_0},$$

and for any given $h \in (-1, 1)$

$$B_h(t) = (1 - t^2)(t - h)_+,$$

where t_+ means, as usual,

$$t_+ = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

Setting especially $h = h_i$, $i = 1, \dots, m + p + 3$, it follows easily that the functions A_0, B_{h_i} , $i = 1, \dots, m + p + 3$ are linearly independent. Furthermore, they are Lipschitz-continuous functions with $\text{supp } A_0 = [h_0, 1]$ and $\text{supp } B_{h_i} = [h_i, 1]$, $i = 1, \dots, m + p + 3$. It can be deduced from the Lipschitz continuity (cf. GRONWALL (1914)) that the Fourier series (7) of A_0 and B_{h_i} are uniformly convergent, i.e.

$$B_{h_i}(t) = \sum_{n=0}^{\infty} \frac{N(m, n)}{\omega_m} B_{h_i}^{\wedge}(n) P_n(t),$$

with

$$B_{h_i}^\wedge(n) = \omega_{m-1} \int_{-1}^1 B_{h_i}(t) P_n(t) (1-t^2)^{\frac{m-2}{2}} dt.$$

(A similar result holds for A_0 .)

We are looking for a function L of the form

$$L(t) = A_0(t) - \sum_{i=1}^{m+p+3} b_i B_{h_i}(t), \quad t \in [-1, 1],$$

with parameters b_i to be determined. It can be easily deduced that the conditions (i) and (ii) are fulfilled for any choice of b_1, \dots, b_{m+p+3} . Condition (iii) is equivalent to the equations

$$A_0^\wedge(n) - \sum_{i=1}^{m+p+3} b_i B_{h_i}^\wedge(n) = 0, \quad n = 0, \dots, p. \quad (12)$$

In order to study these equations, we see first that since $A_0 \geq 0$ and $A_0(1) = 1$ it holds $A_0^\wedge(0) > 0$. Thus, (12) is the linear system of equations

$$\begin{pmatrix} B_{h_1}^\wedge(0) & \cdots & B_{h_{m+p+3}}^\wedge(0) \\ \vdots & \ddots & \vdots \\ B_{h_1}^\wedge(p) & \cdots & B_{h_{m+p+3}}^\wedge(p) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{m+p+3} \end{pmatrix} = \begin{pmatrix} A_0^\wedge(0) \\ \vdots \\ A_0^\wedge(p) \end{pmatrix}$$

with non-vanishing right hand side. To get more information on the matrix entries, we see that

$$\begin{aligned} B_h^\wedge(n) &= \int_{-1}^1 (t-h)_+(1-t^2) P_n(t) (1-t^2)^{\frac{m-2}{2}} dt \\ &= \int_h^1 (t-h) P_n(t) (1-t^2)^{\frac{m}{2}} dt. \end{aligned}$$

is a polynomial in the variable h of degree $m+n+2$. In order to ensure the solvability of the above system, we enlarge it to

$$\begin{pmatrix} 1 & \cdots & 1 \\ h_1 & \cdots & h_{m+p+3} \\ \vdots & \ddots & \vdots \\ h_1^{m+1} & \cdots & h_{m+p+3}^{m+1} \\ B_{h_1}^\wedge(0) & \cdots & B_{h_{m+p+3}}^\wedge(0) \\ \vdots & \ddots & \vdots \\ B_{h_1}^\wedge(p) & \cdots & B_{h_{m+p+3}}^\wedge(p) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{m+p+3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ A_0^\wedge(0) \\ \vdots \\ A_0^\wedge(p) \end{pmatrix}. \quad (13)$$

This linear system, now, can be seen to come from the one-dimensional polynomial interpolation problem with nodal points h_1, \dots, h_{m+p+3} , data values $0, \dots, 0, A_0^\wedge(0), \dots, A_0^\wedge(p)$, and trial functions $\{1, h, \dots, h^{m+1}, B_h^\wedge(0), \dots, B_h^\wedge(p)\}$ being polynomials of degrees $0, \dots, m+p+2$. Since we know from e.g. DAVIS

(1963) that this system is unisolvent, there exists a unique solution of (13) which also solves then (12), and hence fulfils the third requirement.

If m is odd, we can apply the same arguments, but to achieve that the $B_h^\wedge(n)$ are polynomials in h , one has to modify the defining equation of the function B_h . For odd m the definition

$$B_h(t) = (1 - t^2)^{\frac{3}{2}}(t - h)_+$$

will ensure that the above arguments can be applied.

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