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LEARNING AND REPLICATION OF PERIODIC SIGNALS IN NEURAL-LIKE NETWORKS

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Juli 1995

MAT 144/620-143

UNIVERSITÄTSBIBLIOTHEK KAISERSLAUTERN UB

93g 2643(2

Learning and Replication of Periodic Signals in Neural-Like Networks

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Abstract

The paper describes the concepts and background theory for the analysis of a neural-like network for learning and replication of periodic signals containing a finite number of distinct frequency components. The approach is based on the combination of ideas from dynamic neural networks and systems and control theory where concepts of dynamics, adaptive control and tracking of specified time signals are fundamental. The proposed procedure is a two stage process consisting of a learning phase when the network is driven by the required signal followed by a replication phase where the network operates in an autonomous feedback mode whilst continuing to generate the required signal to a desired accuracy for a specified time. The analysis draws on currently available control theory and, in particular, on concepts from model reference adaptive control.

Key Words – Periodic signals, neural-like networks, learning, replication, exponential stability

1 Introduction

The paper presents the results of an ongoing DAAD/British Council sponsored research collaboration between the Centre for Systems and Control Engineering at the University of Exeter and the Arbeitsgruppe Technomathematik at the University of Kaiserslautern. The work lies in the intersection of the areas of systems and control theory and that of dynamic neural networks. It was undertaken in the belief that the combination of ideas from the two areas will enrich the conceptual aspects of both and that their synergy should be capable of producing new and insightfull results. In particular, the rigorous mathematical form of systems and control theory should make possible the development of a proof based foundation for the algorithms. The problem considered here arises out of the work of Doya et al [2], the collaboration leading to the current work and other related studies as described in the PhD thesis [9]. This paper will concentrate on the presentation of basic concepts, modelling procedures, the derivation of suitable learning algorithms, stability analysis and proofs of convergence properties as well as a discussion of the expected performance of the procedure and simulations. A preliminary version of the paper, excluding proofs of the results, was presented at the IEE International Conference CONTROL '94 [10]. This purpose of the current paper is to present a full treatment of the results (including rigorous proofs). This entails the extension of the the results to include the use of appropriate techniques from canonical forms, Lyapunov stability theory, aspects of adaptive control convergence theory (in a modified form), persistency of excitation theory and parameter reduction procedures.

The problem to be considered arises [2] in the development of models of learning of the repetitive or periodic motion of walking. However, in this paper a more general approach and viewpoint is taken to permit application to the development of training algorithms for any periodic action such as is met in control requirements for robotic manufacturing systems involving the repetition of motions associated with assembly or materials transportation. In contrast with previous studies [2], emphasis is placed in this work on the use of system theoretical concepts in the *development of algorithms with provable convergence properties*. It is in this sense in particular that the paper represents an attempt to bring together the two disciplines of neural networks and control theory to provide solutions to a well defined problem in a rigorous mathematical context.

2 Problem Definition

The precise problem to be considered is the construction and training of a dynamic neural-like network, whose aim is to *continously* 'reconfigure' during a learning procedure in order to be able to subsequently replicate an arbitrary periodic signal $r(t)$ of a defined signal class R_N, whose elements are of the form:

$$r(t) \in R_N \Rightarrow r(t) = \sum_{k=1}^{N} A_k \sin(\omega_k t + \phi_k), \ \omega_i \neq \omega_j \ \ for\ i \neq j \tag{2.1}$$

$$1 \leq i, j \leq N, \ \omega_j \in \mathbb{R}^+, \ \phi_k \in [0, 2\pi), \ 0 < A_k \in \mathbb{R} \tag{2.2}$$

The network is initially taken to be described by a state space model of the form

$$\frac{dx(t)}{dt} = A(w(t))x(t) + bu(t) \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$ is a vector of the n states $x_j(t), 1 \leq j$ of the network and $u(t) \in \mathbb{R}$ is a scalar input to be defined subsequently. The dynamic network has the defined output

$$y(t) = c^T x(t) \tag{2.4}$$

where $c \in \mathbb{R}^n$, f^T denotes transpose of the vector f and $w(t) = (w_1(t), \ldots, w_{n_w}(t))^T \in \mathbb{R}^{n_w}$ is a vector of n_w time dependent "weights" $w_j(t)$ to be adjusted continuously to achieve the objective.

The model can be regarded as a "linearization" of models of the form described in [2], for example, where the state dynamics have the typical structure

$$\frac{dx(t)}{dt} = \sigma\left(A(w(t))x(t) + bu(t) \right) \tag{2.5}$$

where $\sigma(\cdot)$ represents a vector-valued sigmoid function. The motivation for the use of the linear version is manyfold and includes:

(a) the potential for the introduction of linear systems theory methods for performance and stability evaluation,

(b) the argument that an algorithm that does not work for the linear system is unlikely to work for the nonlinear case and

(c) the periodic signals (2.1) are solutions of a linear differential equation.

Although not conclusive, these motivations are sufficient to make study of the case described of interest.

The approach consists of two phases, one of *learning* and one of *replication* of the learned responses:

Phase One (THE LEARNING PHASE): The input $u(t)$ is set equal to the desired output $r(t)$ and the dynamics are initiated from any initial state condition. During the transient, the weights are continuously adjusted in such a way that two objectives are achieved i.e. tracking of $r(t)$ by the output occurs in the sense that

$$\lim_{t \to \infty} (r(t) - y(t)) = 0 \tag{2.6}$$

and the system state approaches values that will make Phase Two successful. In control theoretical terms, the network is excited by the external stimulus and the weights are continuously adjusted or adapted to ensure the simultaneous (i) 'identification' of the the state of the (unknown) generating system and (ii) tracking of the input signal $u(t)$ by the network output $y(t)$.

Phase Two (THE REPLICATION PHASE): After a suitable period of time T^*, $u(t)$ is switched/ replaced by the network output $y(t)$, the weight vector $w(t)$ is frozen at its value at $t = T^*$ and the resultant time invariant (positive feedback closed loop) system continues to track the (decoupled) stimulus to the desired accuracy for a desired period T_r, i.e. in the time interval $[T^*, T^* + T_r]$. Mathematically, the requirement for the replication phase is that:

For every desired *replication time period* $T_r > 0$ and *replication accuracy* $\epsilon > 0$ there exists a *switching time* $T^*(\epsilon, T_r) < \infty$ such that the response of the frozen system

$$\dot{x}(t) = \left(A(w(T^*)) + bc^T\right) x(t), \; y(t) = c^T x(t) \tag{2.7}$$

satisfies the ϵ-replication condition:

$$|y(t) - r(t)| < \epsilon \quad \text{for all} \quad t \in [T^*, T^* + T_r].$$

Before analysis of a suitable parameter adaptation law for the weightvector $w(t)$ is undertaken, the problem of answering the principle questions of the existence and properties of a non adaptive linear network that satisfies the requirements of the learning and replication phase, should be addressed.

Lemma 2.1

Consider the asymptotically stable linear time invariant (LTI) dynamical system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bu(t),\ x(0) = x_0 \\ y(t) &= c^T x(t) \\ \sigma(A) &\subset \mathbb{C}^-\end{aligned}$$

with transfer function $G(s) = c^T[sI - A]^{-1}\, b = \frac{n(s)}{p(s)}$, where $n(s)$ and $p(s)$ are coprime. For an input signal $u(t) = \sum_{j=1}^{N} A_j \sin(\omega_j t + \phi_j) \in R_N$, the following property holds true

$$\lim_{t\to\infty} (y(t) - u(t)) = 0 \quad \text{iff} \quad G(\pm i\omega_j) = 1,\ 1 \le j \le N$$

Proof: Due to linearity it is sufficient to consider the case $N = 1$:

Let $y(t)$ have Laplace Transform $\hat{y}(s)$. Then $\hat{y}(s) = \frac{n(s)}{p(s)}\left(\frac{f}{s-i\omega} + \frac{\bar{f}}{s+i\omega}\right) + c^T(sI-A)^{-1}x_0$ with $f = \frac{Ae^{i\varphi}}{2i}$.

The stability condition $\sigma(A) \subset \mathbb{C}^-$ implies $(s \pm i\omega) \mid p(s)$ and hence

$$\begin{aligned}\frac{n(s)}{p(s)(s-i\omega)} &= \frac{n(i\omega)}{p(i\omega)(s-i\omega)} + \frac{n(s)p(i\omega) - n(i\omega)p(s)}{p(s)p(i\omega)(s-i\omega)} \\ &= \frac{n(i\omega)}{p(i\omega)(s-i\omega)} + \frac{q(s)}{p(s)p(i\omega)} \\ q(s) &= \frac{n(s)p(i\omega) - n(i\omega)p(s)}{s-i\omega}\end{aligned}$$

Note that $p(s)$ is Hurwitz and hence $s \mid p(s)$ and as a consequence of the final value theorem for the Laplace transformation there exists a function $h_1(t)$ with $\lim_{t\to\infty} h_1(t) = 0$ such that

$$L^{-1}\left(\frac{n(s)}{p(s)(s-i\omega)}\right) = G(i\omega)e^{i\omega t} + h_1(t).$$

Using this analysis $y(t)$ has the overall form:

$$\begin{aligned}y(t) &= G(i\omega)fe^{i\omega t} + G(-i\omega)\bar{f}e^{-i\omega t} + L^{-1}\left(c^T(sI-A)^{-1}x_0\right) + h_2(t) \\ &= A|G(i\omega)|\sin(\omega t + \varphi + \arg(G(i\omega))) + h(t)\end{aligned}$$

$$\lim_{t\to\infty} h_2(t) = 0, \quad \lim_{t\to\infty} h(t) = 0$$

A simple argument then easily verifies that:

$$\lim_{t\to\infty} (y(t) - u(t)) = 0 \iff G(\pm i\omega) = 1$$

as required. □

Note that as a consequence of this Lemma 2.1 the transfer function of a non adaptive network A, b, c^T solving the problem of the learning phase has to take the value 1 at the frequencies of the reference signal. In general the condition $G(s_0) = 1$ is equivalent to the systems and control theoretical condition that s_0 is a transmission zero of the augmented system $S(A, b, c^T, -1)$. These zeros are identical to the poles of the inverse system (cf. McFarlane [4], p. 174). More precisely:

Lemma 2.2

Given a linear system with matrices $S(A, b, c^T)$ and transfer function $g(s) = c^T(sI - A)^{-1}b$, then the following statements are equivalent:

(i) $g(s_0) = 1$

(ii) $s_0 \in \sigma(A + bc^T)$. □

As an immediate consequence of Lemma 2.1 and 2.2 we obtain:

Proposition 2.3

A necessary condition for exact replication of the signal $r(t)$ is that

(a) the spectrum of $A(w(T^*)) + bc^T$ contains the $2N$ points $\pm i\omega_j$, $j = 1, \ldots, N$ and that

(b) the network state dimension $n \geq 2N$. □

Finally as a further preliminary result we need the following formal statement of the capability of an appropriate dynamical network to replicate an input function $u(t)$ exactly:

Corollary 2.4

Given the LTI system $S(A, b, c^T)$ with $G(\pm i\omega_j) = 1$ for $1 \leq j \leq N$ and any input function $u(t) = \sum_{j=1}^{N} A_j \sin(\omega_j t + \phi_j) \in R_N$, then there exists an initial state $x_0 \in \mathbb{R}^n$, such that $y(t) = u(t)$ for all $t \geq 0$ and hence that the error $y(t) - u(t) = 0 \; \forall t \geq 0$.

Proof: Corollary 2.4 is an immediate consequence of the transmission blocking theorem (cf. McFarlane et al. [4], McFarlane [3]) □

Before further analysis can be undertaken, it is necessary to consider the choice of parameterization and, in particular, the complexity of the parametrization to be used. It is natural to assume the simplest situation i.e. that the network matrix $A(w)$ is *linear* in the weights. The precise form of parameterization chosen is given by

$$A(w(t)) = A_0 + b_0 w^T(t) R \tag{2.8}$$

where the matrix $R \in \mathbb{R}^{n_w \times n}$. The dyadic structure of the parametrization is perhaps the surprising component of this assumption. Its use is justified by the success of the analysis in the remainder of the paper. A simple intuitive justification can be based on the idea that a full parametrization of n^2 weights could be the first consideration but that it can be rejected as the problem is one of spectral matching and hence only around $2N$ weights are intuitively needed to match the real and imaginary parts of the complex numbers defining the frequency content of the signal $r(t) \in R_N$.

In order to structure the network more precisely, the form of A_0 and b_0 needs consideration. The basic building blocks of the network are taken to be a number n of individual dynamic neurons of the first order dynamic form

$$\dot{x}_j(t) = -\alpha x_j(t) + v_j(t), \quad \alpha > 0 \tag{2.9}$$

with output $x_j(t)$ and input $v_j(t)$ generated from an interconnection of the external stimuli and the outputs of the other neurons. The remaining task is to define the interconnection structure. This is done below.

From the above the following network is proposed and defined by the state dimension n, the choice of dimension of the weighted vector to be $n_w := n - 1$ and the interconnections

$$v_j(t) = x_{j+1}, \quad 1 \leq j \leq n-2, \quad v_{n-1}(t) = u(t)$$

and

$$v_n(t) = \sum_{j=1}^{n-1} w_j(t)x_j(t)$$

Alternatively, the parameterization can be defined by the matrices

$$A(w) = -\alpha I_n + F + b_0 w^T R$$

where $F \in \mathbb{R}^{n \times n}$ is a (relatively sparse) matrix with elements

$$F_{i,j} = \begin{cases} 1 : j = i+1, \ 1 \le i \le n-2 \\ 0 : \text{else} \end{cases}$$

and R is an augmented unit matrix of the form

$$R = [I_{n-1}, 0_{n-1,1}]$$

In fuller detail,

$$A(w) = \begin{bmatrix} -\alpha & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & -\alpha & 1 & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & 0 \\ 0 & & & & & -\alpha & 0 \\ w_1 & \dots & \dots & \dots & \dots & w_{n-1} & -\alpha \end{bmatrix}$$

The remaining terms have the form of simple unit basis vectors:

$$b_0 = c = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^T \text{ and } b = \begin{bmatrix} 0 & 0 & \dots & 1 & 0 \end{bmatrix}^T$$

The theoretical advantage of the construction is that the matrix $A(w)+bc^T$ has the form of a (shifted) companion matrix (i.e. in systems theoretical terms, $(A(w) + \alpha I + bc^T, b_0)$ is in controllable canonical form) and hence it is possible to prove the following existence theorem based on the known simple form of the characteristic polynomial of a companion matrix:

Proposition 2.5

(i) The matrix $A(w) + bc^T$ has the characterist c polynomial

$$p_c(s) = \|sI_n - A(w) - bc^T\| = (s+\alpha)^n - w_{n-1}(s+\alpha)^{n-2} - \dots - w_2(s+\alpha) - w_1$$

(ii) If $n \ge 2N+1$ then for each $r(\cdot) \in R_N$, i.e. for each choice of distinct frequencies ω_j, $1 \le j \le N$, there exists a weight vector $w^* \in \mathbb{R}^{n-1}$ such that the spectrum of $A(w^*) + bc^T$ contains the set $\{\pm\omega_j \mid j = 1, \dots, N\}$ of $2N$ points. In particular, if $n = 2N+1$ then w^* is uniquely determined and the remaining eigenvalue is precisely $-(2N+1)\alpha < 0$.

(iii) There exists an initial condition $z(0)$ such that the following system generates the signal $r(t)$ precisely

$$\dot{z}(t) = A(w^*)z(t) + br(t) \tag{2.10}$$

$$r(t) = c^T z(t)$$

Proof:

(i) The proof of this is an easy computation and is hence omitted for brevity.

(ii) If $n \geq 2N+1$ then at least $2N+1$ zeros of the characteristic polynomial which are symmetric with respect to the real axis can be selected arbitrarily. If $n = 2N+1$ then there exists exactly one weight vector $\omega^* \in \mathbb{R}^{n-1}$ such that $\pm i\omega_j$, $j = 1, \ldots, N$, are the zeros of the characteristic polynomial.

(iii) is an immediate consequence of (i) and corollary 2.4.

3 Stability of Learning – A Liapunov Approach

Given the construction in the previous section it is now necessary to develop a weight adaptation or network learning law with the objective of ensuring that convergence of learning occurs in the combined sense that

$$\lim_{t\to\infty} e_1(t) = 0, \qquad \lim_{t\to\infty} e_p(t) = 0$$

where $e_1 := y - r$ denotes the *output error* and $e_p := w - w^*$ the *parameter error.*

Based on corollary 2.4 and proposition 2.5 and the consequent ability to represent the signal $r(t) \in R_N$ by a copy of the network, it is natural to apply a model reference adaptive control approach (cf. Fig. 3.1), where for every $r(\cdot) \in R_N$ the generating system is regarded as being described by the equations (2.10) with fixed ideal parameter w^* and appropriate initial value x_0^*, such that the output of

$$\text{Reference Model} \qquad \dot{\hat{x}} = A(w^*)\hat{x} + bu, \quad \hat{x}(0) = x_0^*, \quad \hat{y} = c^T\hat{x} \tag{3.1}$$

becomes identical to its input if $u(t) = r(t)$ for all $t \geq 0$.

(Note: The ideal parameters w^* and x_0^* are only used for the construction and analysis of the learning rule. **They are not required for their algorithmic realization.**)

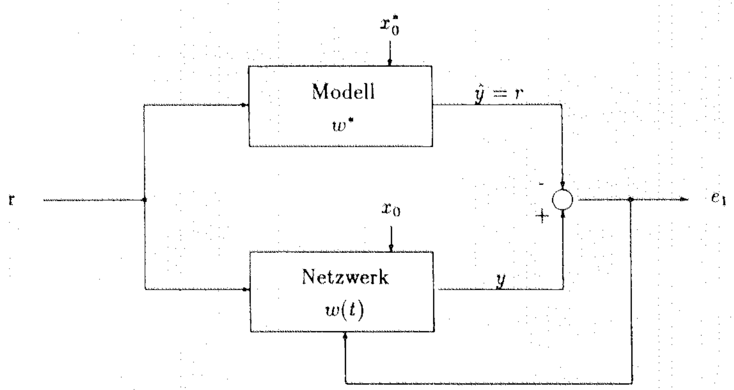

Fig. 3.1: MRAC-approach

The equations for the dynamical network have the general form:

$$\text{Dynamic Neural-like Network} \qquad \dot{x} = A(w(t))x + bu, \; x(0) = x_0, \; y = c^T x. \tag{3.2}$$

Formally the dynamics of the error system for $e := x - \hat{x}$ can be expressed by:

$$\begin{aligned}\dot{e} &= A(w(t))x - A(w^*)\hat{x} \\ &= A(w^*)x + b_0(w - w^*)^T Rx - A(w^*)\hat{x} \\ &= A(w^*)e + b_0 x^T R^T e_p \end{aligned} \tag{3.3}$$

Assume now that the net dimension n is minimal for the given class R_N, i.e. $n = 2N+1$. The approach to the problem uses the Liapunov function candidate

$$V(e, e_p) = e^T Pe + e_p^T Q^{-1} e_p \tag{3.4}$$

where $0 < Q = Q^T \in \mathbb{R}^{2N\times 2N}$ is arbitrary, $P = P^T > 0$ is the unique solution of the Liapunov equation

$$A^T(w^*)P + PA(w^*) = -S \tag{3.5}$$

and $S \in \mathbb{R}^{n\times n}$ is symmetric and positive definite but otherwise arbitrary. Computation of the Liapunov derivative with the adaptation/learning law

$$\dot{w}(t) = \dot{e}_p(t) = -QRx(t)c^T Pe(t) \tag{3.6}$$

leads to the equality

$$\dot{V}(t) = -e^T(t)Se(t) \tag{3.7}$$

and hence that the Liapunov derivative is positive semi-definite. Although simple forms of Liapunov stability results do not resolve the stability problem, application of LaSalles Invariance Principle leads to the following main theorem of this part of the paper.

Theorem 3.1

Using the above adaptive weight evolution law, suppose that $A_k \neq 0$, $1 \leq k \leq N$ and that the frequencies ω_k, $1 \leq k \leq N$ are distinct. Then stability of learning is guaranteed in the strong sense that

$$\lim_{t\to\infty} e(t) = 0, \quad \lim_{t\to\infty} e_p(t) = 0 \tag{3.8}$$

independent of $x(0)$ and $w(0)$.

Proof: The combined error system is

$$\frac{d}{dt}\begin{pmatrix} e(t) \\ e_p(t) \end{pmatrix} = \begin{bmatrix} A(w^*) & cx^T(t)R^T \\ -QRx(t)c^T P & 0 \end{bmatrix}\begin{pmatrix} e(t) \\ e_p(t) \end{pmatrix} \tag{3.9}$$

The derivative of $V(e, e_p)$ along (3.9) is

$$\begin{aligned}\dot{V}(e, e_p) &= \dot{e}^T Pe + e^T P\dot{e} + \dot{e}_p Q^{-1} e_p + e_p^T Q^{-1}\dot{e}_p \\ &= e^T\left(A^T(w^*)P + PA(w^*)\right)e^T + 2e_p\left(Q^{-1}\dot{e}_p + Rxc^T Pe\right)\end{aligned}$$

Applying (3.5) and (3.6) and $\dot{w} = \dot{e}_p$ gives:

$$\dot{V}(e, e_p) = -e^T Se$$

The Liapunov function $V(e, e_p)$ is positive definite, decrescent, radially unbounded and the derivative $\dot{V}$ is negative definite along (3.9). Liapunov's stability theorem then implies that the origin $(e, e_p) = 0$ of (3.9) is globally uniformly stable. In particular $e(\cdot)$ and $e_p(\cdot)$ and hence $w(\cdot)$ are bounded.

$$0 < V(0) - V(t) = \int_0^t e^T(\tau)Se(\tau)d\tau < \infty \quad \text{for } t \geq 0$$

implies that $e(\cdot) \in L^2$. The spectrum $\sigma(A(w(t))) = \{-\alpha\}$ is independent of t and hence $x(\cdot)$ is bounded because $w(\cdot)$ and $r(\cdot)$ are bounded. Finally (3.3) implies that $\dot{e}(\cdot)$ is bounded too, however together with $e \in L^2 \cap L^\infty$ it follows

$$\lim_{t\to\infty} e(t) = 0.$$

Furthermore, w^* is uniquely determined because $n = 2N + 1$. By LaSalle (e, e_p) converges towards the maximal positive invariant set

$$M \subset E = \left\{(e, e_p) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \,|\, \dot{V}(e, e_p) = 0\right\} = \{(e, e_p) \in \mathbb{R}^n \times \mathbb{R}^{n-1} | e = 0\}.$$

The state $x(\cdot)$ of (3.2) is excited by the periodic signal $r(\cdot)$, hence $x(\cdot)$ oscillates and does not converge. This implies together with (3.3) that $M = \{0\}$, because every positive invariant subset $\tilde{M} \neq \{0\}$ of E contains elements $(\hat{e}, \hat{e}_p)$ with $\hat{e}_p \neq 0$ for which the right hand side of (3.3) is nonzero. Any trajectory starting in such a point leaves $\tilde{M}$ contradicting the positive invariance of $\tilde{M}$. □

Theorem 3.1 guarantees asymptotic convergence of the signals and the parameter vector $w(t)$ for a network with state dimension $n = 2N + 1$. This choice of network dimension is regarded as *minimal* in the sense that Proposition 2.3. requires that $n \geq 2N$. If the choice of $n = 2N$ is made then the spectral condition requires that $Trace(A(w) + bc^T) = 0$ which is impossible as it is easily verified that $Trace(A(w) + bc^T) = -(2N + 1)\alpha \neq 0$. It follows that, in order to apply the above result, an explicit exact knowledge of the number of different frequencies in the signals $r(\cdot)$ is required. The following theorem weakens this assumption simply to that of a knowledge of an upper bound on the number of frquencies present in $r(t)$. The mathematical expression of this is the generalization of the convergence result to networks of arbitrary dimension $n \geq 2N + 1$.

The precise statement follows and includes a proof of the *exponential convergence of the learning scheme* i.e. the convergence of e and e_p to zero is bounded from above by a decaying exponential.

Theorem 3.2

Let $Q \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and $S \in \mathbb{R}^{n\times n}$ be positive definite symmetric matrices and $P \in \mathbb{R}^{n\times n}$ be the unique solution of the Liapunov equation (3.5), then the combined error system (3.9) is exponentially stable. In particular for all initial conditions $x(0)$ and $w(0)$ we have:

(i) If $n = 2N + 1$, then

$$\lim_{t\to\infty} e(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} e_p(t) = 0$$

with exponential convergence.

(ii) If $n > 2N + 1$, then

$$\lim_{t\to\infty} e(t) = 0$$

and

$$w(t) \to \Gamma = \left\{ w \in \mathbb{R}^{n-1} | \pm i\omega_j \in \sigma\left(A(w) + bc^T\right), 1 \leq j \leq N \right\} \tag{3.10}$$

where the notation $w(t) \to \Gamma$ is defined to mean

$$w(t) \to \Gamma :\Leftrightarrow \lim_{t \to \infty} \text{dist}(w, \Gamma) = 0$$

□

Although structurally similar to the proof of the previous theorem, this extended version requires a little more technical machinery. In particular, the proof needs two elements of the theory of persistency of excitation that we will restate subsequently for the readers convenience.

Firstly a result obtained by Morgan and Narendra [5] gives an integral condition on the signal $u(\cdot)$ that is a persistent excitation condition such that a system of form (3.9) is uniformly asymptotically stable.

Secondly a result by Boyd and Sastry [1] provides a frequency domain condition on $r(\cdot)$ for $u(\cdot)$ to persistently exciting (p.e.).

The following theorem is central to the development.

<u>**Theorem 3.3**</u> (Morgan and Narendra [5])

Consider the nonautonomous linear system

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^{\bullet} = \begin{bmatrix} A(t) & U^T(t) \\ -U(t)P(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{3.11}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $U(t) \in \mathbb{R}^{m\times n}$ and $P(t) \in \mathbb{R}^{n\times n}$. Let $A(t)$ be bounded piecewise continuous and such that $\dot{z} = A(t)z$ is uniformly asymptotically stable. Let $P(t)$ be a symmetric positive definite matrix of bounded continuous functions such that $\dot{P} + A^T P + PA$ is negative definite[1]. Let $U(t)$ be piecewise bounded continuous. Then the system (3.11) is uniformly asymptotically stable if and only if there exist positive constants t_0, T_0, ϵ_0 and δ_0 such that there exists a $t_2 \in [t, t+T_0]$ such that for any unit vector $w \in \mathbb{R}^m$

$$\left| \int_{t_2}^{t_2+\delta_0} U(\tau)^T w d\tau \right| \geq \epsilon_0 \text{ for all } t \geq t_0 \,. \tag{3.12}$$

□

The implications of this result are that we have to show that given $r(\cdot)$ for $U(\cdot) := Rx(\cdot)c^T$ (3.12) holds true. Condition (3.12) is difficult to verify, but if we restrict the class of functions to which $U(\cdot)$ belongs to $P := \{U : \mathbb{R}^+ \to \mathbb{R}^{m\times n} |$ any component function of $U(\cdot)$ and their derivatives are bounded and piecewise continuous $\}$ then for any $U(\cdot) \in P$ the condition (3.12) is equivalent to the statement:

There exist positive constants T_0, t_0, ϵ_0 such that for any unit vector $w \in \mathbb{R}^m$

$$\int_t^{t+T_0} \left| U^T(\tau) w \right| d\tau \geq \epsilon_0 \text{ for all } t \geq t_0 \tag{3.13}$$

(Narendra and Annaswamy [6])[2]

[1]Such P's exist by Krasovskii's theorem

[2]Functions $U(\cdot) \in P$ that fullfill (3.13) for some T_0, t_0, $\epsilon_0 > 0$ are called p.e. (Narendra and Annaswamy [6])

Clearly the essential step is to show that $Rx(\cdot)c^T$ is p.e. To accomplish this task the following theorem has great value by giving a condition in the frequency domain on the input of a linear time invariant system (LTI) such that its state is p.e..

Theorem 3.4 (Boyd and Sastry [1], Narendra and Annaswamy [6])

Consider the LTI system $\dot{x} = Ax + br$ with $x(t) \in \mathbb{R}^n$, $r(t) \in \mathbb{R}$, A asymptotically stable, and (A, b) controllable. Suppose also that the autocovariance of $r(\cdot)$, $\text{Cov}_r(\tau) := \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} r(s)r(s+\tau)^T ds$ exist and converge uniformly w.r.t. t. Also let $S_r(d\omega)$ denote the spectral measure of $r(\cdot)$[3]. **Then** $x(\cdot)$ is p.e. if and only if $S_r(d\omega)$ is concentrated on at least n points (i.e. iff $(|\text{supp}(S_r(d\omega)|) \geq n$ or, equivalently, it is a measure consisting of at least n point measures). □

For a more detailed discussion of the notions of autocovariance and spectral measure see Appendix A.

Proof of Theorem 3.2: From the above results for proving exponential convergence of e and e_p it is sufficient to show that $u(\cdot)$ is p.e. Let $n = 2N + 1$, then we have $|\text{supp}(S_r(d\omega))| = n - 1$. The LTI subsystem of (3.9) with state $\zeta := R\hat{x}$ fulfils the conditions of theorem 3.4, therefore ζ is p.e. As $Rx = R(\hat{x} + e)$ with $\lim_{t\to\infty} e(t) = 0$, $Rx(\cdot)$ is also p.e. Hence theorem 3.3 ensures uniform asymptotic stability of the error system (3.9), which therefore has exponentially stability because (3.9) is a linear differential equation.
It remains to prove property (ii). For this we need a characterization of the set Γ in theorem 3.2:

Lemma 3.5

Let $u = r = \sum_{j=1}^{N} A_j \sin(\omega_j t + \phi_j) \in R_N$ and $n \geq 2N + 1$. Then the following statements are equivalent:

(i) $w \in \Gamma$

(ii) $\text{Cov}_{Rx}(0)e_p = \text{Cov}_{Rx}(0)(w - w^*) = 0$

where w^* is any fixed element in Γ.

Proof: Define the transfer function (matrices)

$$G_w(s) := c^T(sI - A(w))^{-1}b$$

and

$$\tilde{G}_w(s) := (sI - A(w,)^{-1}b.$$

Lemma 2.2 implies that Γ can be expressed in the form

$$\begin{aligned} \Gamma &= \{w \in \mathbb{R}^{n-1} | \pm i\omega_j \in \sigma(A(w) + bc^T),\ 1 \leq j \leq N\} \\ &= \{w \in \mathbb{R}^{n-1} | G_w(\pm i\omega_j) = 1,\ 1 \leq j \leq N\}. \end{aligned}$$

It is easily verified that

$$\begin{aligned} \tilde{G}_w(s) &= \left[\frac{1}{(s+\alpha)^{n-1}}, \frac{1}{(s+\alpha)^{n-2}}, \ldots, \frac{1}{s+\alpha}, G_w(s)\right]^T \\ G_w(s) &= \frac{w_1 + w_2(s+\alpha) + \ldots + w_{n-1}(s+\alpha)^{n-2}}{(s+\alpha)^n} \end{aligned}$$

[3] defined by $\text{Cov}_r(\tau) =: \int_{-\infty}^{\infty} e^{i\omega\tau} S_r(d\omega)$

and hence $w \in \Gamma$ if and only if

$$\underbrace{\begin{bmatrix} 1 & \dots & (i\omega_1+\alpha)^{n-3} & (i\omega_1+\alpha)^{n-2} \\ 1 & \dots & (-i\omega_1+\alpha)^{n-3} & (-i\omega_1+\alpha)^{n-2} \\ \vdots & & \vdots & \vdots \\ 1 & \dots & (i\omega_k+\alpha)^{n-3} & (i\omega_k+\alpha)^{n-2} \\ 1 & \dots & (-i\omega_k+\alpha)^{n-3} & (-i\omega_k+\alpha)^{n-2} \end{bmatrix}}_{=:\Omega_1} \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ \vdots \\ w_{n-1} \end{bmatrix} = \begin{bmatrix} (i\omega_1+\alpha)^n \\ (-i\omega_1+\alpha)^n \\ \vdots \\ (i\omega_k+\alpha)^n \\ (-i\omega_k+\alpha)^n \end{bmatrix}. \qquad (3.14)$$

This implies that Γ is the translated subspace $\Gamma = w^* + \ker \Omega_1$, or, equivalently,

$$w \in \Gamma \;\Leftrightarrow\; e_p \in \ker \Omega_1. \qquad (3.15)$$

Equation (3.14) is equivalent to

$$\underbrace{\begin{bmatrix} \tilde{G}_w(i\omega_1)^T R^T \\ \tilde{G}_w(-i\omega_1)^T R^T \\ \vdots \\ \tilde{G}_w(i\omega_k)^T R^T \\ \tilde{G}_w(-i\omega_k)^T R^T \end{bmatrix}}_{=:\Omega_2} \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ \vdots \\ w_{n-1} \end{bmatrix} = \begin{bmatrix} i\omega_1+\alpha \\ -i\omega_1+\alpha \\ \vdots \\ i\omega_k+\alpha \\ -i\omega_k+\alpha \end{bmatrix} \qquad (3.16)$$

and hence:

$$w \in \Gamma \;\Leftrightarrow\; e_p \in \ker \Omega_2. \qquad (3.17)$$

$R\tilde{G}_w(s)$ is independent of w, hence we obtain for the Laplace transform of Rx:

$$R\hat{x}(s) = R\tilde{G}_w(s)\hat{r}(s)$$

From the Linear Filter Lemma in Appendix A we next conclude that:

$$S_{Rx}(dv) = R\tilde{G}_w(iv)S_r(dv)\tilde{G}_w^*(iv)R^T,$$

where S_{Rx} denotes the spectral measure of Rx. Furthermore:

$$\begin{aligned} \mathrm{Cov}_{Rx}(0) &= \int_{-\infty}^{\infty} R\tilde{G}_w(iv)S_r(dv)\tilde{G}_w^*(iv)R^T \\ &= \sum_{k=1}^{N} \Big(S_r(\{i\omega_j\})R\tilde{G}_w(i\omega_j)\tilde{G}_w^*(i\omega_j)R^T + S_r(\{-i\omega_j\})R\tilde{G}_w(-i\omega_j)\tilde{G}_w^*(-i\omega_j)R^T \Big). \end{aligned}$$

Noting that $(R\tilde{G}_w(\pm i\omega_j))^T$ are the linear independent rows of the matrix Ω_2, it follows that:

$$\mathrm{Cov}_{Rx}(0)e_p = 0 \;\Leftrightarrow\; \Omega_2 e_p = 0 \;\Leftrightarrow\; w \in \Gamma$$

□

By lemma 3.5 it now suffices for the completion of the proof of theorem 3.2 to show that $\lim_{t\to\infty} \mathrm{Cov}_{Rx}(0)e_p(t) = 0$. For this let $\epsilon > 0$ and consider:

$$|e_p^T(t)\mathrm{Cov}_{Rx}(0)e_p(t)| \le \underbrace{|e_p^T(t)\mathrm{Cov}_{Rx}(0)e_p(t) - e_p^T(t)\frac{1}{T}\int_t^{t+T} Rx(\tau)x^T(\tau)R^T d\tau e_p(t)|}_{=:I_1}$$

$$+\underbrace{|e_p^T(t)\frac{1}{T}\int_t^{t+T} Rx(\tau)x^T(\tau)R^T d\tau e_p(t) - \frac{1}{T}\int_t^{t+T} e_p^T(\tau)Rx(\tau)x^T(\tau)R^T e_p^T(\tau)d\tau|}_{=:I_2}$$

$$+\underbrace{|\frac{1}{T}\int_t^{t+T} e_p^T(\tau)Rx(\tau)x^T(\tau)R^T e_p^T(\tau)d\tau|}_{=:I_3}$$

Rx and e_p are bounded, i.e. there exists a $K > 0$ such that

$$|Rx(t)|,\ |e_p(t)| < K.$$

Because Cov_{Rx} exists we have:

$$\|\mathrm{Cov}_{Rx}(0) - \frac{1}{T}\int_t^{t+T} Rx(\tau)x^T(\tau)R^T d\tau\| < \frac{\epsilon}{3K^2}$$

for some T_0 (independent of t) and all $T > T_0$. Hence

$$|I_1| < \frac{\epsilon}{3}. \tag{3.18}$$

In the first part of the proof it was shown that $\lim_{t\to\infty} e(t) = 0$. (3.3) then implies

$$\lim_{t\to\infty} x^T(t)Re_p(t) = 0$$

and by (3.9) $\lim_{t\to\infty} e_p(t) = 0$. Thus there exists some $t_1 \ge 0$, such that for all $t > t_1$

$$|e_p^T(t)Rx(t)| < \sqrt{\frac{\epsilon}{3}}, \qquad |\dot{e}_p(t)| < \frac{\epsilon}{6K^3T} \tag{3.19}$$

and

$$\begin{aligned} |I_2| &< |\frac{1}{T}\int_t^{t+T} x^T(\tau)R(e_p(t) - e_p(\tau))x^T(\tau)R^T(e_p(t) + e_p(\tau))d\tau| \\ &\le K^2|\frac{1}{T}\int_t^{t+T} (e_p(t) - e_p(\tau))^T(e_p(t) + e_p(\tau))d\tau|. \end{aligned}$$

Using the mean value theorem

$$|I_2| \le K^2\frac{1}{T}\left\{\left[\int_0^1 \|\dot{e}_p(t + s(\tau - t))\|ds\right]|\tau - t|\right\}|e_p(t) + e_p(\tau)|d\tau < \frac{\epsilon}{3}. \tag{3.20}$$

The conclusion that $|I_3| < \frac{\epsilon}{3}$ is a consequence of (3.19). In summary:

$$|e_p^T(t)\mathrm{Cov}_{Rx}(0)e_p(t)| < \epsilon \text{ for all } t > t_1.$$

This completes the proof of theorem 3.2. □

However, we are left with what appears, at first sight, to be a serious and rather damning problem for the learning rule (3.6). More precisely, it appears that the learning rule for the weight adaptation is not suitable for implementation because it needs the unknown state $e(t)$ of the error system and via P the unknown ideal parameter vector w^*. This is not acceptable as a knowledge of w^* is not part of the assumption that $r(t)$ is not known. However, as a consequence of the following lemma it is shown that a special choice of the positive definite matrix S ensure that the relation $Pc = c$ holds. In this situation (3.6) simplifies to the ***implementable*** learning law:

$$\dot{w}(t) = -QRx(t)e_1(t), \quad w(0) \in \mathbb{R}^{n_w} \tag{3.21}$$

Note that this law is implementable as it uses only the available signals x and $e_1 = r - y$ and hence a knowledge of the 'state' $\hat{x}$ of the generating system is not needed.

Lemma 3.1

Let $S \in \mathbb{R}^{(2N+1)\times(2N+1)}$ be partioned as $S = \begin{bmatrix} S_{11} & -w^* \\ -(w^*)^T & 2\alpha \end{bmatrix}$ with $S_{11} \in \mathbb{R}^{2N\times 2N}$ positive definite and symmetric and:

$$(w^*)^T S_{11}^{-1} w^* < 2\alpha \, . \tag{3.22}$$

Then the uniquely determined positive definite solution $P \in \mathbb{R}^{(2N+1)\times(2N+1)}$ of the Liapunov equation (3.5) is of the form $P = \begin{bmatrix} P_{11} & 0_{2N,1} \\ 0_{1,2N} & 1 \end{bmatrix}$. In particular the equation $Pc = c$ holds true.

Proof: Condition (3.22) ensures that S is positive definite. It remains to investigate the properties of P.

Let J_1 be the stable matrix $J_1 := \begin{bmatrix} -\alpha & 1 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & 0 & -\alpha \end{bmatrix} \in \mathbb{R}^{\ell\times\ell}$, then $A := A(w^*) = \begin{bmatrix} J_{2N} & 0_{2N,1} \\ (w^*)^T & -1 \end{bmatrix}$.

Now let P be partioned in the form $P = \begin{bmatrix} P_{11} & p_{12} \\ p_{12}^T & p_{22} \end{bmatrix}$ with $p_{12} \in \mathbb{R}^{2N}$ and note that, as a consequence,

$$A^T P + PA = \left[\begin{array}{c|c} J_{2N}^T P_{11} + w^* p_{12}^T + P_{11} J_{2N} + p_{12}(w^*)^T & J_{2N}^T p_{12} + w^* p_{22} - \alpha p_{12} \\ \hline p_{12}^T J_{2N} + p_{22}(w^*)^T - \alpha p_{12}^T & -2\alpha p_{22} \end{array}\right]$$

$$= \begin{bmatrix} -S_{11} & w^* \\ (w^*)^T & -2\alpha \end{bmatrix}$$

and we obtain $p_{22} = 1$, $p_{12} = 0$ and P_{11} is the uniquely determined, positive definite solution of the Liapunov equation $J_{2N}^T P_{11} + P_{11} J_{2N} = -S_{11}$. □

4 The Replication Phase

At no stage in the above analysis of the dynamics in the learning phase is the desired stability property of the system in the replication phase required by the analysis. However, the proof of the following theorem shows that the adaptation law (3.21) is compatible with the requirements for the replication phase given in the original problem definition.

Theorem 4.1

Given the proposed network structure, the adaptation law (3.21) and a reference signal r of the form (2.1), then there exists, for every time period $T_r > 0$ and replication accuracy $\epsilon > 0$, a switching time

$T^*(\epsilon, T_r) < \infty$ such that the response in the replication phase (assumed v_0 begin at $t = T^*$) satisfies the desired inequality:

$$|y(t) - r(t)| < \epsilon \text{ for all } t \in [T^*, T^* + T_r]. \tag{4.1}$$

Proof: By Proposition 2.5 (iii) there exists an initial condition z_0^* such that $r(t) = c^T z(t)$, where

$$\dot{z}(t) = (A(w^*)z(t) + bc^T)z(t), \quad z(0) = z_0^*. \tag{4.2}$$

Let $T^* < \infty$ be arbitrary for the present and $J = [T^*, T^* + T_r]$. Let S_β be a β-neighbourhood ($\beta > 0$) of the integral curve (4.2), i.e. $S_\beta = \{(t,x) \| |x - z(t)| \le \beta$ and $t \in J\}$ and $K = \max_{t \ge 0} r(t)$. Then there exists a constant $\tilde{c} < 0$ such that $|z(t)| < \tilde{c}K$ for all $t \ge 0$, hence $|x| \le \tilde{c}K + \beta$ in S_β. This implies

$$|A(w^*)x - A(w(T^*))x| = |\sum_{j=1}^{2N}(w_j^* - w_j(T^*))x_j| \le |w(T^*) - w^*|(cK + \beta) \tag{4.3}$$

for all $t \ge 0$ in S_β. Because the right hand side of (4.2) fullfills a Lipschitz condition on S_β, we have continuous dependence of the solution, therefore it is guaranteed that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|y(t) - r(t)| \le |x(t) - z(t)| < \epsilon \text{ on } S_\beta,$$

if $|A(w^*)x - A(w(T^*))x| < \delta$ on S_β and $|x(T^*) - z(T^*)| < \delta$. (4.2) and (2.10) are autonomous systems, hence δ depends on T_r and ϵ but not on T^*. By theorem 3.2 we have $|(e(t), e_p(t))| \overset{t\to\infty}{\longrightarrow} 0$, hence for $\delta' = \frac{\delta}{cK+\beta}$ there exists a $T^* < \infty$ such that

$$|w(T^*) - w^*| < \delta' \text{ and } |x(T^*) - z(T^*)| < \delta' \le \delta.$$

□

The theorem indicates that the problem is solved by the proposed algorithm but it has not been found to be possible to explicitly characterize T^* in terms of T_r and ϵ.

5 Learning on the Imaginary Axis

In section 4 it has been shown that the proposed solution of the learning phase automatically sets up the conditions that solve the replication phase. The problem examined in this section is based on the observation that, as, in general, $w(T^*) \neq w^*$, there is no guarantee, however, that the $2N$ eigenvalues of the frozen system (2.7) at $t = T^*$ will lie on the imaginary axis. As a consequence it should be expected that at least one eigenvalue will lie in the open right half plane, i.e. the system is likely to be unstable in the replication phase and this instability will cause unbounded network responses beyond the interval $[0, T^* + T_\mu]$. This is clearly an undesirable property!

This problem of instability in the replication phase can be avoided by a fairly simple and easily implemented reparametrization of the network with only N adjustable parameters. This is possible, because the pole assignment problem for the N frequencies w_j, $1 \le j \le N$, of a signal r satisfying (2.1) requires only N free parameters. The procedure can be outined as follows. Without any loss of generality we assume that $\alpha = 1$ in order to simplify the form of the equations. The corressponding equations are easily derived in the more general case of $\alpha > 0$ but the details are omitted for brevity. It then follows that, by an appropriate selection of a projection operator, it can be ensured that for all $w \in \mathbb{R}^{n_w}$, the spectrum of $A(w) + bc^T$ is precisely $-2N - 1$ together with $2N$ symmetrically placed points placed exactly on the imaginary axis of the complex plane. As an added bonus this parameter reduction also has the added advantage (as observed in simulations) that it typically leads to a faster convergence of the adaptation algorithm (c.f. section 7) in the learning phase.

To begin the discussion, note that the characteristic polynomial of $A(w^*) + bc^T$ is of the form

$$q(s) = (s + 2N + 1) \prod_{j=1}^{N} (s^2 + \omega_j^2) = (s + 2N + 1) \left(s^{2N} + \sum_{j=1}^{N} \vartheta_j s^{2j-2} \right) \tag{5.1}$$

with

$$\vartheta_j = \sum_{\substack{i_1, i_2, \ldots, i_{N-j+1} = 1 \\ i_1 < i_2 < \ldots < i_{N-j+1}}}^{N} \prod_{\ell=1}^{N-j+1} \omega_{i_\ell}^2 \text{ for } 1 \leq j \leq N. \tag{5.2}$$

Comparing the coefficients of $q(s)$ with those of the representation

$$\rho_c(s) = \sum_{j=0}^{2N-1} \left(\binom{2N+1}{j} - \sum_{\ell=j}^{2N-1} \binom{\ell}{j} w_{\ell+1} \right) s^j + \binom{2N+1}{2N} s^{2N} + s^{2N+1} \tag{5.3}$$

leads to the following system of equations

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \binom{2}{1} & \binom{3}{1} & \cdots & \cdots & \binom{2N-1}{1} \\ 0 & 0 & 1 & \binom{3}{2} & \cdots & \cdots & \binom{2N-1}{2} \\ 0 & 0 & 0 & 1 & \cdots & \cdots & \binom{2N-1}{3} \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & 1 & \binom{2N-1}{2N-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{=:T \in \mathbb{R}^{2N \times 2N}} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ \vdots \\ w_{2N-1} \\ w_{2N} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -(2N+1) & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -(2N+1) & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & & \ddots & 0 & -(2N+1) \\ 0 & 0 & 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}}_{=:U \in \mathbb{R}^{2N \times N}} \underbrace{\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vdots \\ \vdots \\ \vartheta_{N-1} \\ \vartheta_N \end{bmatrix}}_{=:\vartheta} + \underbrace{\begin{bmatrix} \binom{2N+1}{0} \\ \binom{2N+1}{1} \\ \vdots \\ \vdots \\ \binom{2N+1}{2N-2} \\ \binom{2N+1}{2N-1} \end{bmatrix}}_{=:\zeta \in \mathbb{R}^{2N}}$$

these can be expressed in the more compact form:

$$w = T^{-1}U\vartheta + T^{-1}\zeta \tag{5.4}$$

Replacing in section 3 w^* by $T^{-1}U\vartheta^* + T^{-1}\zeta$, $w(t)$ by $T^{-1}U\vartheta(t) + T^{-1}\zeta$ and $e_p(t)$ by

$$e_p(t) := \vartheta(t) - \vartheta^*$$

leads (via the Liapunov function approach) to the new learning rule:

$$\dot{\vartheta}(t) = Q \begin{bmatrix} T^{-1}U \\ 0_{1,N} \end{bmatrix}^T x(t)e_1(t). \tag{5.5}$$

Using the same arguments and with only an increase in algebraic complexity, it is easily proved that theorem 3.2 remains true if (3.21) is replaced by (5.5).

6 Robustness of Network Dynamics during the Learning Phase

An issue both in dynamical systems but, in particular, in control systems is the need for insensitivity of the form of the solution trajectories/state trajectories in the learning phase to classes of unknown and unmeasurable disturbances. This *robustness* problem is vital in learning as it must be guaranteed that, even if the dynamics of learning is disturbed, the network is still capable of extracting useful information out of the stimulating signal. It is a complex matter to attempt an analysis of arbitrary forms of disturbance so, for the purposes of this paper, attention is focussed on disturbances to the signal $r(t)$.

The robustness analysis of the learning scheme is essentially based on the already proved exponential stability of the undisturbed error system:

$$\frac{d}{dt}\begin{bmatrix} e(t) \\ e_p(t) \end{bmatrix} = \begin{bmatrix} A(w^*) & cx^T(t)R^T \\ -QRx(t)c^T & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ e_p(t) \end{bmatrix} \tag{6.1}$$

We consider additive disturbances d on the reference signal $r(\cdot)$ by replacing r by $r + d$ as the driving term in the learning phase:

$$\begin{aligned} \dot{x}_d &= A(w(t))x_d + b(r+d),\ x_d(0) = x_0 \\ y_d &= c^T x_d \end{aligned}$$

The model dynamics remain unchanged:

$$\begin{aligned} \dot{\hat{x}} &= A(w^*)\hat{x} + br,\ \hat{x}(0) = x_0^* \text{ fixed} \\ \hat{y} &= c^T\hat{x} = r \end{aligned}$$

For the state error $e = x_d - \hat{x}$ we obtain:

$$\dot{e} = A(w^*)e + cx_d^T R^T e_p + bd \tag{6.2}$$

Replacing x by x_d and y by y_d in the learning rule (3.6) the overall disturbed error system is described by:

$$\begin{aligned} \tfrac{d}{dt}\begin{bmatrix} e(t) \\ e_p(t) \end{bmatrix} = & \begin{bmatrix} A(w^*) & cx^T(t)R^T \\ -QRx(t)c^T & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ e_p(t) \end{bmatrix} \\ & + \begin{bmatrix} c(x_d(t) - x(t))^T R^T e_p(t) \\ -QR(x_d(t) - x(t))c^T e(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} d(t) \end{aligned} \tag{6.3}$$

Comparing the right hand sides of the disturbed and undisturbed system (6.1) the extra disturbance terms are:

$$f(t,e,e_p) := \begin{bmatrix} c(x_d(t)-x(t))^T Re_p(t) \\ -QR^T(x_d(t)-x(t))c^T e(t) \end{bmatrix} \tag{6.4}$$

$$g(t) := \begin{bmatrix} b \\ 0 \end{bmatrix} d(t) \tag{6.5}$$

As (6.1) is exponentially stable, there exist constants $\gamma > 0$ and $k \geq 0$ such that the *undisturbed* errors satisfy the inequality

$$|(e(t), e_p(t))| \leq K|e(0), e_p(0)|e^{-\gamma t}.$$

The following theorem is the main result of this section and is useful in providing bounds on the effect of the disturbance on behaviour of the error.

Theorem 6.1

Let

$$c_1 = \max\{\|Q\|_2, 1\} \begin{cases} \min\{\frac{1}{\sqrt{\alpha^2-1}}, \frac{\sqrt{n-1}}{\alpha}\}, & \text{for } \alpha \geq 1 \\ \min\{\frac{1}{\sqrt{\alpha^{2n-2}(1-\alpha^2)}}, \frac{\sqrt{n-1}}{\alpha^{n-1}}\}, & \text{for } 0 < \alpha \leq 1 \end{cases} \quad \text{and } 0 < h < \infty$$

be arbitrary. If, also, the disturbance is bounded pointwise by the relation $|d(t)| \leq \delta$ where the bound δ satisfies the condition

$$\delta < \frac{\gamma}{K}\frac{1}{c_1 + \frac{1}{h}}, \tag{6.6}$$

and the following inequality is satisfied

$$|(e(0), e_p(0))| \leq \frac{h}{K}, \tag{6.7}$$

then there exists a unique solution $(e(t), e_p(t))$ of the disturbed error system (6.3) such that for all $t \geq 0$

$$|(e(t), e_p(t))| \leq Ke^{-(\gamma - c_1\delta K)t}|(e(0), e_p(0))| + \frac{K\delta}{\gamma - c_1\delta K}\left(1 - e^{-(\gamma - c_1\delta K)t}\right) \leq h. \tag{6.8}$$

In particular, convergence of the state and parameter errors is achieved in the sense that

$$\limsup_{t\to\infty} |(e(t), e_p(t))| \leq \frac{K\delta}{\gamma - c_1\delta K}. \tag{6.9}$$

Proof: Let $\mathbb{R}^{n-1} \ni z_d := R(x_d - x)$. Then analogous to the proof of theorem 3.2

$$\dot{z}_d = RA(w(t))R^T z_d + Rbd$$

with

$$RA(w(t))R^T = \begin{bmatrix} -\alpha & 1 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & 0 & -\alpha \end{bmatrix}.$$

Also $|d(t)| < \delta$ implies for the components of z_d

$$|z_{d_i}| \leq \frac{\delta}{\alpha^{n-i}}, \; 1 \leq i \leq n-1,$$

and hence $|z_d|_2 \leq c_2\delta$ with

$$c_2 = \begin{cases} \min\{\frac{1}{\sqrt{\alpha^2-1}}, \frac{\sqrt{n-1}}{\alpha}\}, & \alpha \geq 1 \\ \min\{\frac{1}{\sqrt{\alpha^{2n-2}(1-\alpha^2)}}, \frac{\sqrt{n-1}}{\alpha^{n-1}}, & 0 < \alpha \leq 1 \end{cases}$$

The Jacobian of f with respect to e and e_p is then bounded as follows:

$$\|\frac{\partial}{\partial(e, e_p)} f(t, e, e_p)\|_2 = \| \begin{bmatrix} 0 & cz_d^T \\ -Qz_d c^T & 0 \end{bmatrix} \|_2 \leq \max\{\|Q\|, 1\}c_2\delta = c_1\delta =: \beta_1.$$

Now let $\beta_2 := \frac{\delta}{h}$ so that:

$$\begin{aligned} f(t, 0, 0) &= 0, \text{ for all } t \geq 0 \\ |f(t, e_1, e_{p1}) - f(t, e_2, e_{p2})|_2 &\leq \beta_1 |(e_1 - e_2, e_{p1} - e_{p2})|_2 \\ |g(t)| &\leq \beta_2 h \end{aligned}$$

Noting that (6.6) implies that, for the constants γ and K,

$$\delta\left(c_1 + \frac{1}{h}\right)\frac{K}{\gamma} = (\beta_1 + \beta_2)\frac{K}{\gamma} < 1.$$

the total stability theorem then yields the result that, if

$$|(e(0), e_p(0))| \leq \frac{h}{K},$$

then there exists a unique solution $(e(t), e_p(t))$ of (6.3) such that for all $t \geq 0$

$$|(e(t), e_p(t))| \leq Ke^{-(\gamma-\beta_1 K)t}|(e(0), e_p(0))| + \frac{K\beta_2 h}{\gamma - \beta_1 K}\left(1 - e^{-(\gamma-\beta_1 K)t}\right) \leq h.$$

□

We can interpret this result in the following way: If the error system (3.9) with $Pc = c$ is disturbed by a pointwise bounded signal $d(\cdot)$ satisfying the smallness condition (6.6), then the origin is no longer an equilibrium point. Instead (6.3) has a domain of attraction with radius $\frac{K\delta}{\gamma - c_1\delta K}$ for arbitrary bounded initial values and $e(\cdot)$ and $e_p(\cdot)$ remain bounded. This statement remains true for the situation as $h \to \infty$. In this case only the estimate (6.8) for the error system is lost.

7 Simulation Results

In order to demonstrate the convergence of the algorithms and the typical form of results obtained in practice, consider the problem of teaching a network to replicate the simple illustrative signal

$$r(t) = \sin t + \sin 2t \tag{7.1}$$

consisting of $N = 2$ distinct frequency components of roughly equal significance. The network chosen consists of the model in $\mathbb{R}^5$ with network parameter $\alpha = 1$:

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ w_1 & w_2 & w_3 & w_4 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) \tag{7.2}$$

$$y(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.3}$$

7.1 The learning phase

We have $n_w = 4$ weights with corresponding ideal weight $w^* = (-40, 46, , -30, 5)^T$. The number of weights reduces effectively to 2 weights with ideal weight $\vartheta^* = (4,5)^T$ if the projection of the learning rule (5.5) is used. The initial state $x(0)$ is specified by (in the absence of any other information) $x(0) = 0$.

Simulation 7.1

The subject of this simulation is the application of the base algorithm (3.6) on the above example. It shows the influence of the learning rate parameter Q on the speed of convergence and demonstrates exponential convergence of the net output and of the weights towards the ideal weights. The result of a simulation of the learning phase with $Q = 100\ I$ are shown in figures 1 a-c.

(i) Practical convergence is achieved at the output in effectively $t = 400(s)$(i.e. in 130 periods of r) (Fig. 1a) and of the weights in $t = 700(s)$(Fig. 1b).

(ii) The linear decrease of the output and the parameter error e_1 and e_p in a logarithmic scale substantiates the analytical result of exponential convergence given in theorem 3.6 (Fig. 1c). Note that the errors stop decreasing at $t = 1600(s)$ due only to the fixed precision of the numerical calculation.

The result of a simulation of the learning phase with $Q = I$ is shown in figure 2. Again the convergence is exponential, however the rate of convergence is distinctly less in this case, hence confirming Q as a parameter for influencing convergence rates in a systematic way.

Simulation 7.2

This simulation considers the projected learning rule (5.5) and demonstrates an acceleration of the speed of convergence in comparison with the base algorithm. The convergence speed of the projected rule is influenced by the learning rate Q in a similar manner to the base rule.
The figures 3a and b present the result of the simulation of the learning phase using $Q = 100\ I$.

(i) Practical convergence at the net output is achieved in about $t = 20$(s) (ca. 3 periods of r) and of the weights in about $t = 25$(s) (fig. 3a). The convergence speed of the projected learning rule is significantly faster than that of the base algorithm.

(ii) Figure 3b verifies, by comparison with figure 1c and simulation 7.1a, a considerably accelerated exponential decrease in the error.

The result of a simulation of the learning phase using $Q = I$ is contained in figures 4a and b.

(i) As expected the velocity of convergence is slower in comparison to the simulation using $Q = 100\ I$. Practical convergence at the output is achieved in about $t = 55$(s) (9 periods of r) and of the weights in about $t = 100$(s) (fig. 4a).

(ii) The convergence is exponential and its speed is (even for $Q = I$) considerably faster than the convergence of the base algorithm for $Q = 100\ I$. Figure 4b verifies, by comparison with figure 1c and simulation 7.1, a considerably faster exponential decrease of the error.

The results of simulations of the learning phase of both algorithms clearly demonstrate learning convergence of both outputs and parameters. The convergence of the base algorithm is significantly slower by comparison to the speed of the projected version.

Finally we note an interesting observed phenomemon. The parameters/weights are subject to regular but still (exponentially) decaying bursts of activity following periods of apparent quiessence. This so-called 'bursting phenomenon' is clearly due to the nonlinear nature of the learning system and learning law and is the subject of current study.

7.2 The reproduction phase

In the following we present a numerical investigation into the ability of the proposed network to reproduce the signal prescribed in the learning phase. This depends essentially on the accuracy, at the time of switching, of the computed weights and the state of the net which was achieved in the learning phase. Furthermore the ability is affected by the parametrization of the net itself. The principal ability of the net structure for reproduction was shown by Proposition 2.5. For a numerical investigation it is necessary to distinguish the two learning algorithms due to their different parametrization, i.e. either full or reduced by projection. It is important, for an interpretation of the following simulation results, that the accuracy of the weights achieved in the learning phase is decisive, because the error in the reproduction phase appears to be dominated by the weight errors. Clearly, the length of the learning phase is decisive for the achieved accuracy of the weights. This accuracy is the main criterion for deciding the switching time into the reproduction phase. Thus the precision of the reproduction is determined by the length of the learning phase. This statement is supported by the following simulations.

Simulation 7.3

We begin the discussion with a simulation of the reproduction phase of the base algorithm with $Q = 100\ I$ using the data obtained by simulation 7.1.

(i) Figure 5 shows the time evolution of the net output after switching into the reproduction phase at $t_s = 800(s)$. The weight vector w at the switching time is
$w(800) = (-40.0376, 45.8420, -29.9097, 4.9214)^T$, which therefore it has absolute precision of about 10^{-1}. Starting with these values the network continues to reproduce the reference signal for about 12(s) (2 periods of r) at a precision of $\epsilon = 10^{-1}$.

(ii) The result of a simulation using a switching time of $t_s = 1200(s)$ is given in figure 6. At this time the achieved weight vector is $w(1200) = (-39.9972, 46.0033,$
$-30,0075, 5.0039)^T$, which has a precision of about 10^{-2}. This increase of accuracy magnifies the time for which the network is capable of reproducing the signal r to more than 200(s) (30 periods of r). Not before $t = 1500(s)$ can the net output and the reference signal be distinguished distinctly.

Additionally this simulation shows the disadvantage of the base algorithm in comparison to the projected learning rule. In simulation after switching at $t_s = 800$ the autonomous system of the reproduction phase is exponential stable, but its eigenvalues do not lie on the imaginary axis, but in the left half plane (real parts near -0.004), the net output is therefore vanishing exponentially. Against that in the simulation $t_s = 1200$ the system of the reproduction phase is unstable, its eigenvalues lie in the right half plane (real parts near 0.0004), the net output thus grows slowly exponentially.

Due to its construction the projected learning rule assures that the eigenvalues of the system of the reproduction phase are true imaginary and hence guarantees that oscillation is conserved in the reproduction phase and that the net output remains bounded. This behaviour is illustrated by the following simulation of the reproduction phase of the projected learning rule.

Simulation 7.4

This simulation uses the data of the simulation 7.2 for the projected learning rule using $Q = 100\ I$.

(i) The time evolution of the net output during the reproduction phase after a learning time of $t_s = 25$(s) is shown figure 7. The weight at the switching time is $\vartheta(25) = (3.9859, 5.0028)^T$ and has a precision of about 10^{-1}. The reference signal can be reproduced to the desired accuracy for about 12(s) (2 periods of r).

(ii) The result of an extension of the learning time by 5(s) gives figure 8. At time $t_s = 30$ the weight $\vartheta(30) = (4,0034, 5.0018)^T$ was achieved. It has a precision of about 10^{-3}. Through this increase in accuracy the reference signal is seen to be reproduced for about 400(s) (60 periods of r). A distinction of net output and reference signal is not possible before $t = 800$.

At this place we emphasize that, in contrast to the base algorithm, the projected algorithm conserves the property of oscillation because the eigenvalues lie *exactly* on the imaginary axis by construction. In the above example the computed eigenvalues $\{\pm 1.0003i, \pm 2.0003i\}$ correspond to $\vartheta(30)$.

8 Conclusions

The paper has demonstrated the potential for the combination of the ideas of dynamic neural networks and control and systems theory and, in particular, the methods of model reference adaptive control. A linearized version of the original problem posed by Doya et al and analysed through the use of simulaton methods has been shown to be amenable to a rigorous theoretical analysis with provable Liapunov and exponential stability and robustness properties. The algorithm is parameterised by a single parameter that has been demonstrated to have a systematic and useful effect on improving convergence rates in the learning phase.

The analysis is based on stability concepts on an infinite time interval. However learning inevitably takes place on a finite time interval and hence learning is not complete when the learning phase is terminated. The effects of this on the ability of the network to accurately replicate the desired behaviour have been investigated numerically and the solutions seen to be viable. In particular, an accelerated form of the algorithm has been seen to have benefits in ensuring rapid convergence combined with long term stability. The technical construction of this algorithm uses a projected learning rule to ensure that the network, under positive feedback, has an appropriate number of eigenvalues on the imaginary axis.

Finally, the analysis benefits substantially from available control theory in the construction of the network. The analysis is based essentially on the theory of pole allocation which is used successfully not only to prove existence results but also to provide an exact characterization of the number of neurons (network states) required to solve the problem in terms of the number of distinct frequencies in the specified signal to be replicated or learned.

A Results from Generalized Harmonic Analysis

The Generalized Harmonic Analysis theory is known since the beginning of this century. A thorough theoretical basis was provided in Wiener's famous paper Generalized Harmonic Analysis [7]. An important notion is the concept of autocovariance, which usually is defined in a stochastic context. Here we describe a deterministic version:

<u>**Definition A.1**</u>

(i) A function $u : \mathbb{R}^+ \to \mathbb{R}^u$ is called stationary if the limit

$$\mathrm{Cov}_u(\tau) := \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} u(s)u^T(\tau + s)ds \qquad (A.1)$$

exists uniformly with respect to t. $\mathrm{Cov}_u(\tau)$ is called autocovariance of $u(\cdot)$.

(ii) If $\mathrm{Cov}_u(\cdot)$ is continuous, then

$$\mathrm{Cov}_u(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} S_u(d\omega), \qquad (A.2)$$

where $S_u(d\omega)$ is called the spectral measure of $u(\cdot)$. The integral (A.2) exists by Bochner's theorem (cf. Bochner [8]), because $\mathrm{Cov}_u(\cdot)$ is a positive semidefinite function.

(iii) If $u : \mathbb{R}^+ \to \mathbb{R}^m$ and $y : \mathbb{R}^+ \to \mathbb{R}^p$ are stationary, then the limit

$$\mathrm{Cov}_{y,u}(\tau) := \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} y(s)u^T(\tau + s)ds$$

exists uniformly in t and is called the covariance of u and y. □

The relation between the auto- and covariance of the input and output signals of a stable time-invariant linear system is described by the linear filter lemma of Boyd and Sastry [1]:

<u>**Lemma A.1**</u>

Let $H(s)$ be a stable, strict proper rational transfer matrix with impuls response $h(t) \in \mathbb{R}^{p\times m}$, $\hat{y}(s) = H(s)\hat{u}(s)$ and let $u(\cdot)$ be stationary. Then

(i) y is stationary with autocovariance

$$\mathrm{Cov}_y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\mathrm{Cov}_u(t + \tau_1 - \tau_2)h^T(\tau_2)d\tau_1 d\tau_2$$

and spectral measure

$$S_y(d\omega) = H(i\omega)S_u(d\omega)H^*(i\omega).$$

(ii)

$$\mathrm{Cov}_{y,u}(t) = \int_{-\infty}^{+\infty} h(\tau_1)\mathrm{Cov}_u(t + \tau_1)d\tau_1$$

$$S_{yu}(d\omega) = H(i\omega)S_u(d\omega)$$

□

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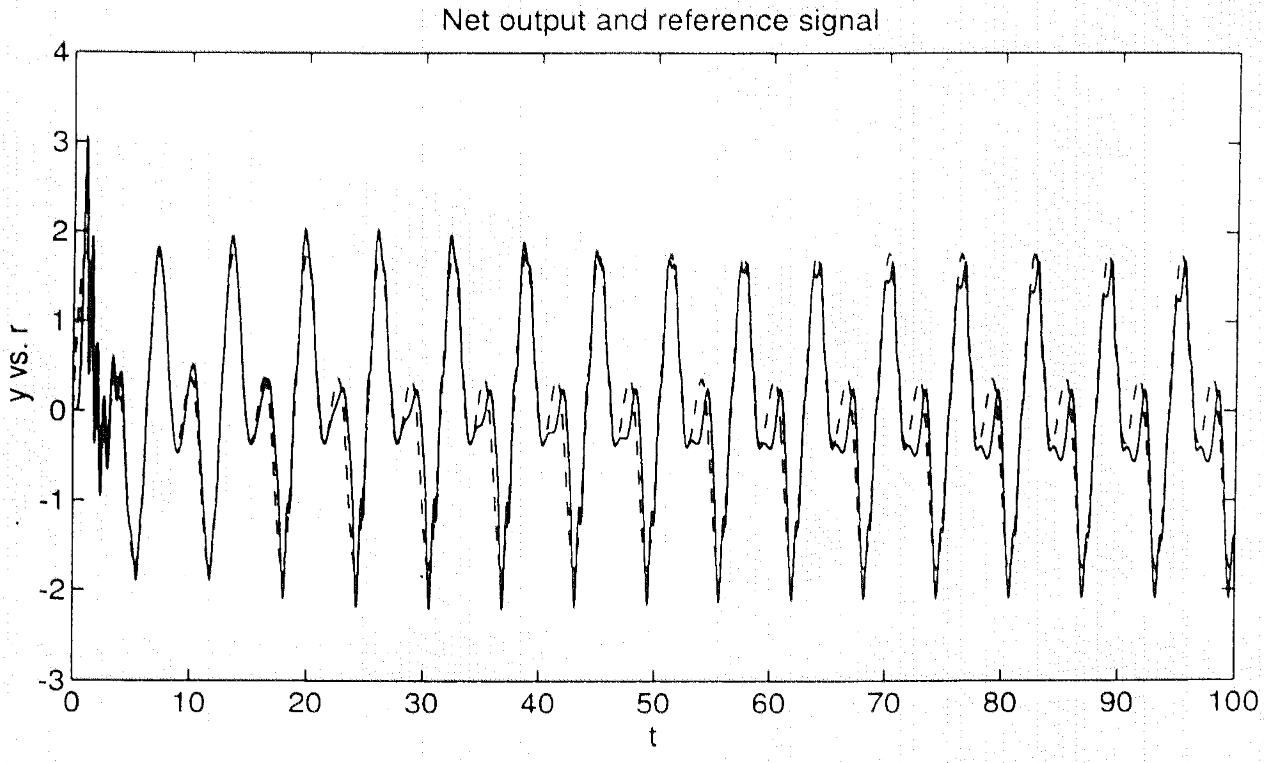

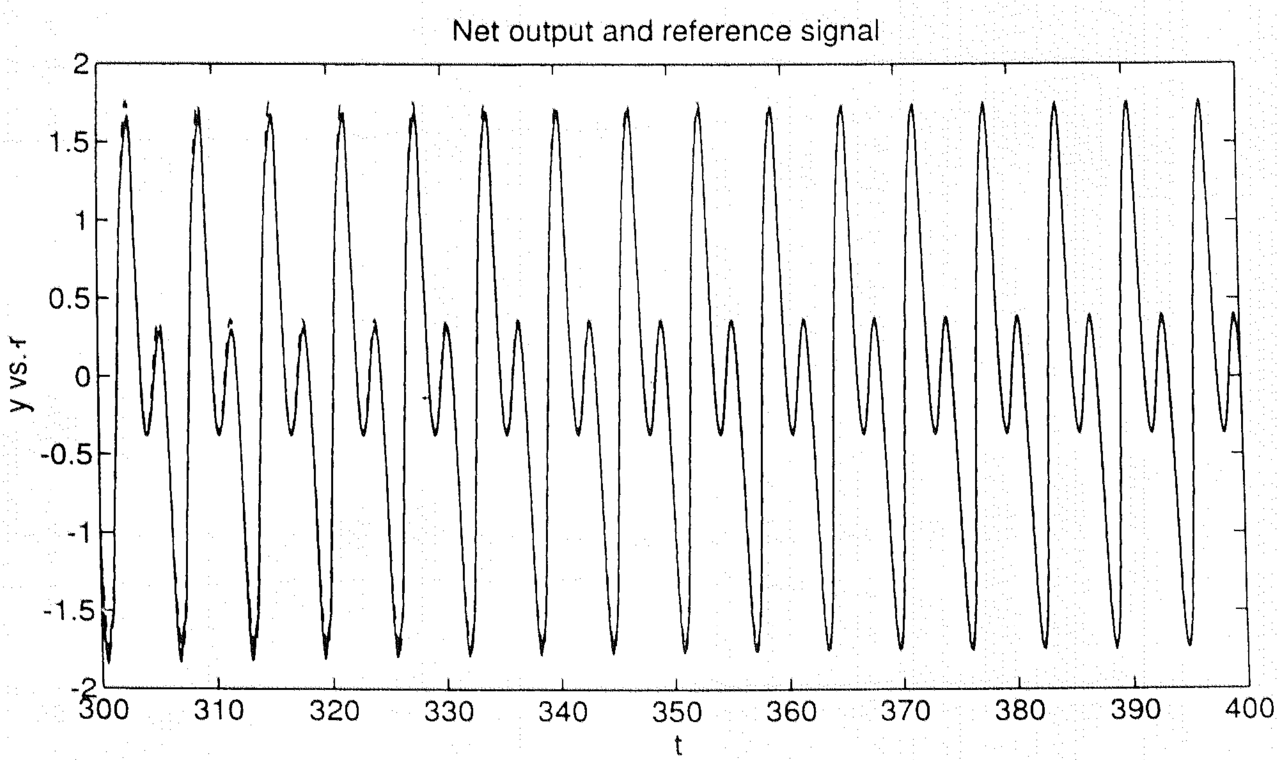

Figure 1a: Basic learning rule with $Q = 100I$

— $y(t)$, - - - $r(t)$

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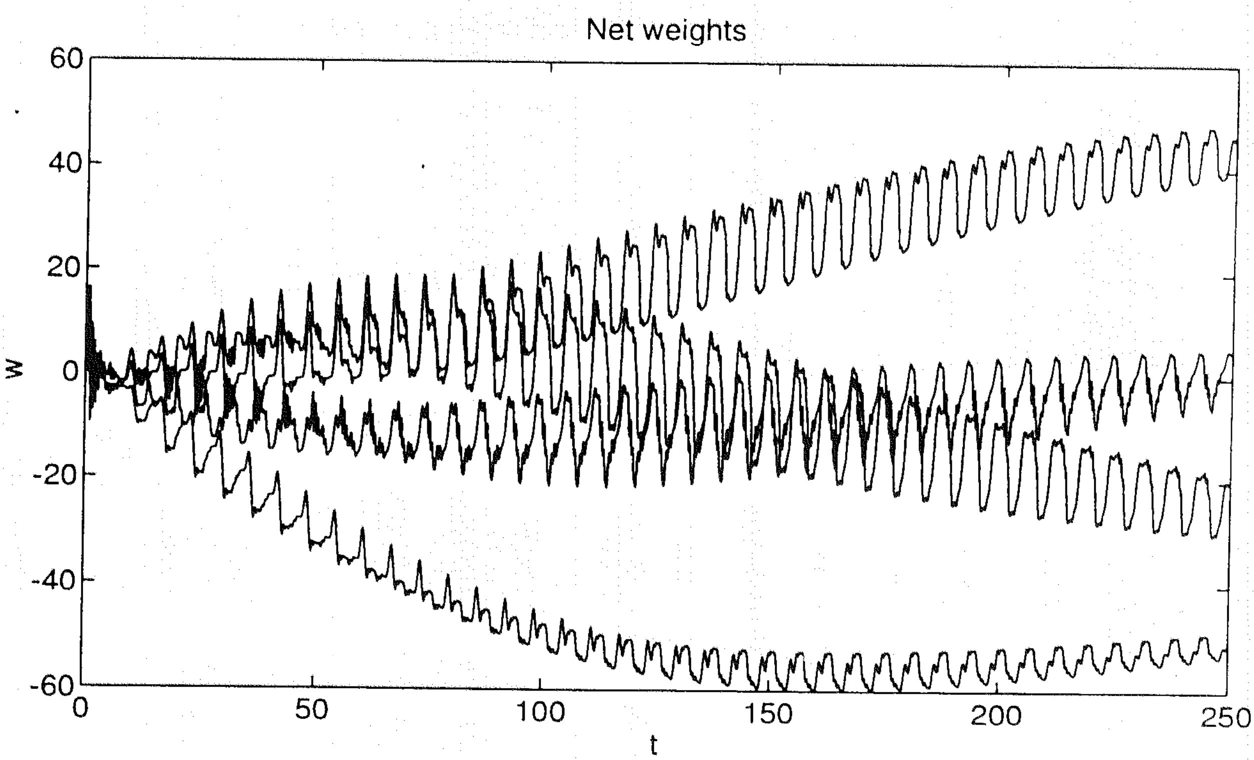


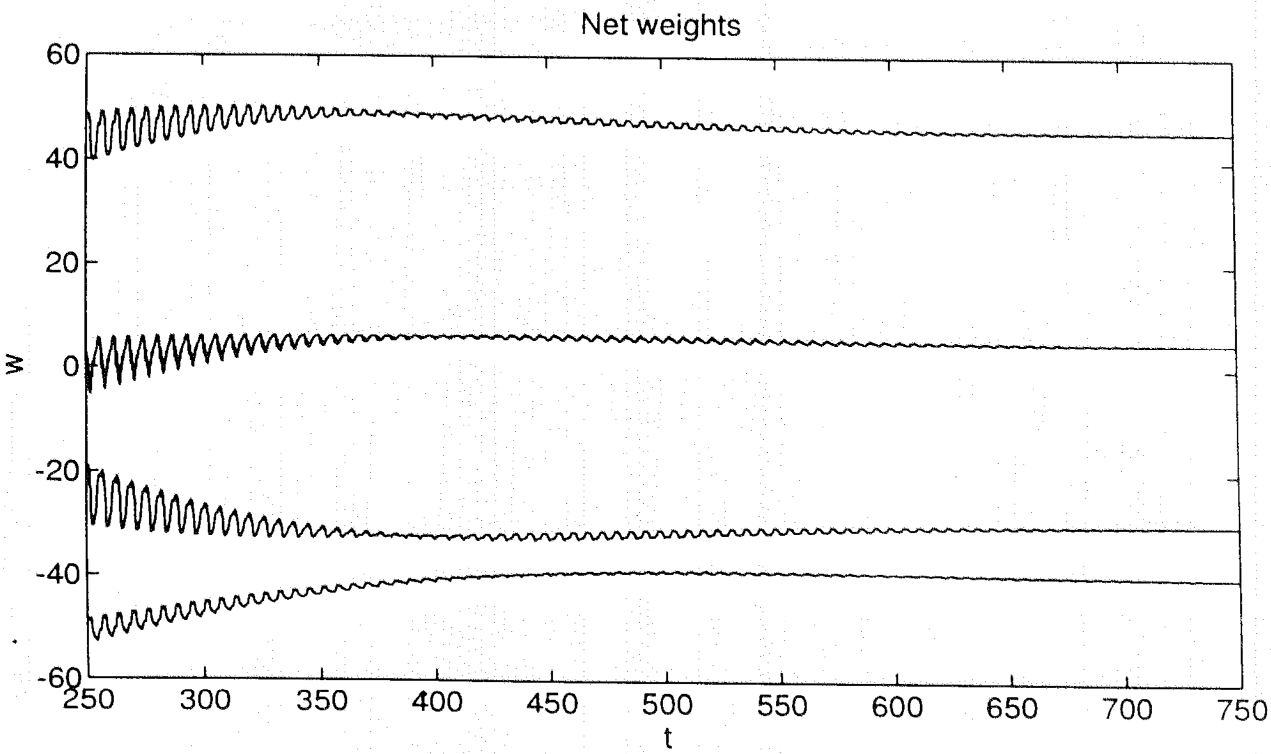
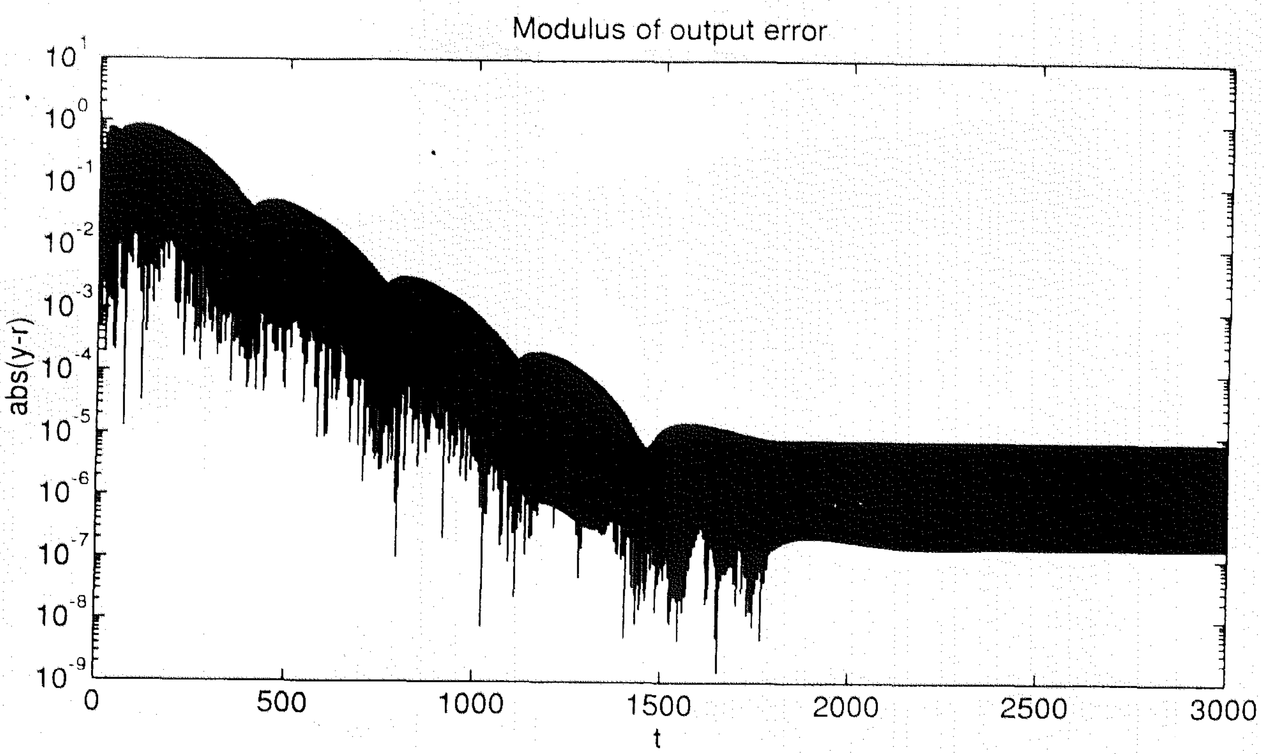


Figure 1b: Basic learning rule with $Q = 100I$
$w^* = (-40, 46, -30, 5)^T$



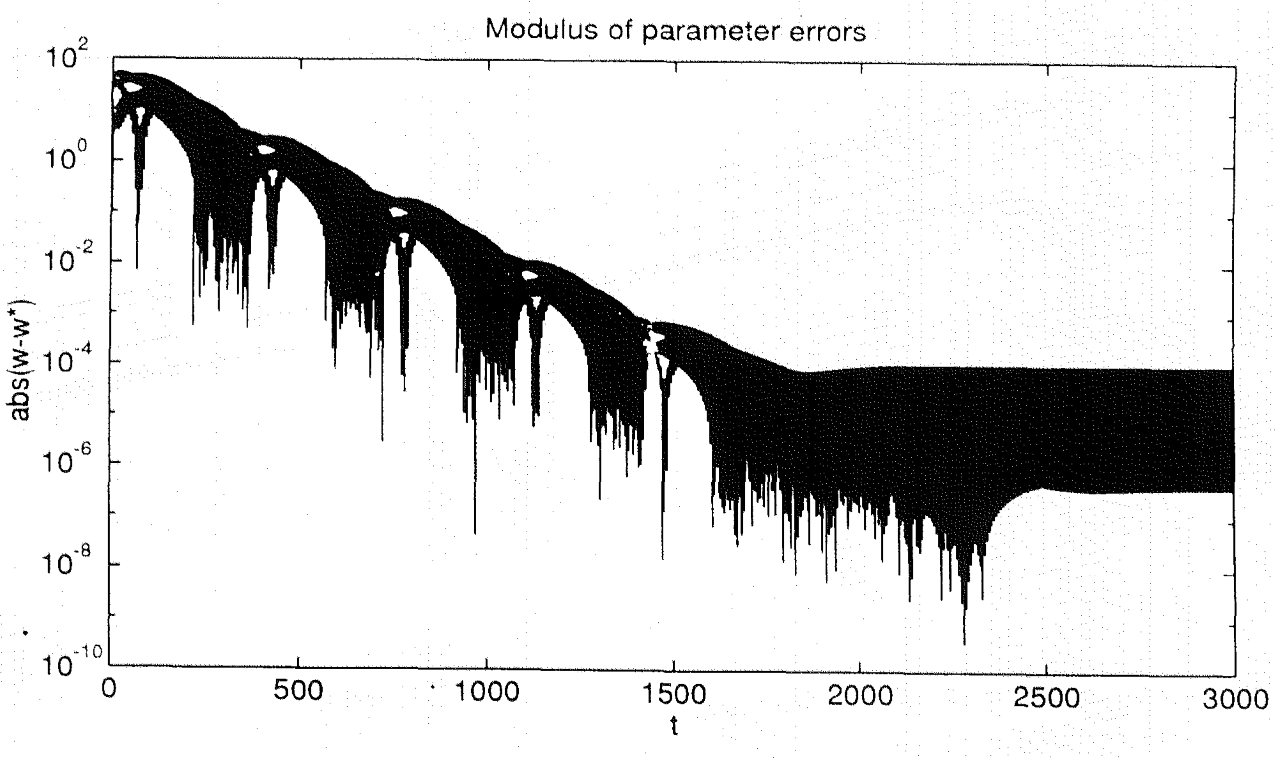


Figure 1c: Basic learning rule with $Q = 100I$
above: $|e_1(t)|$, below: $|e_{p_i}(t)|$

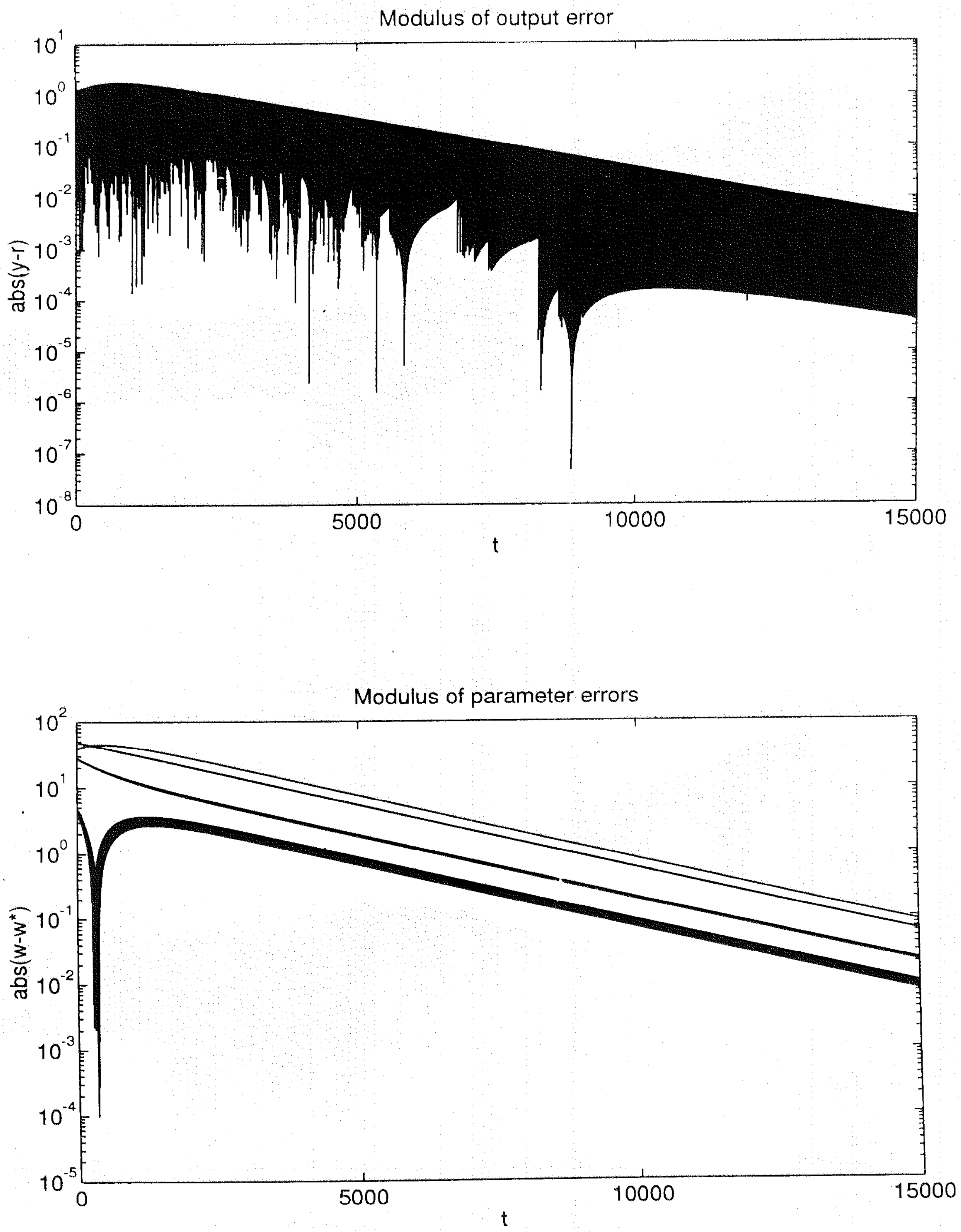
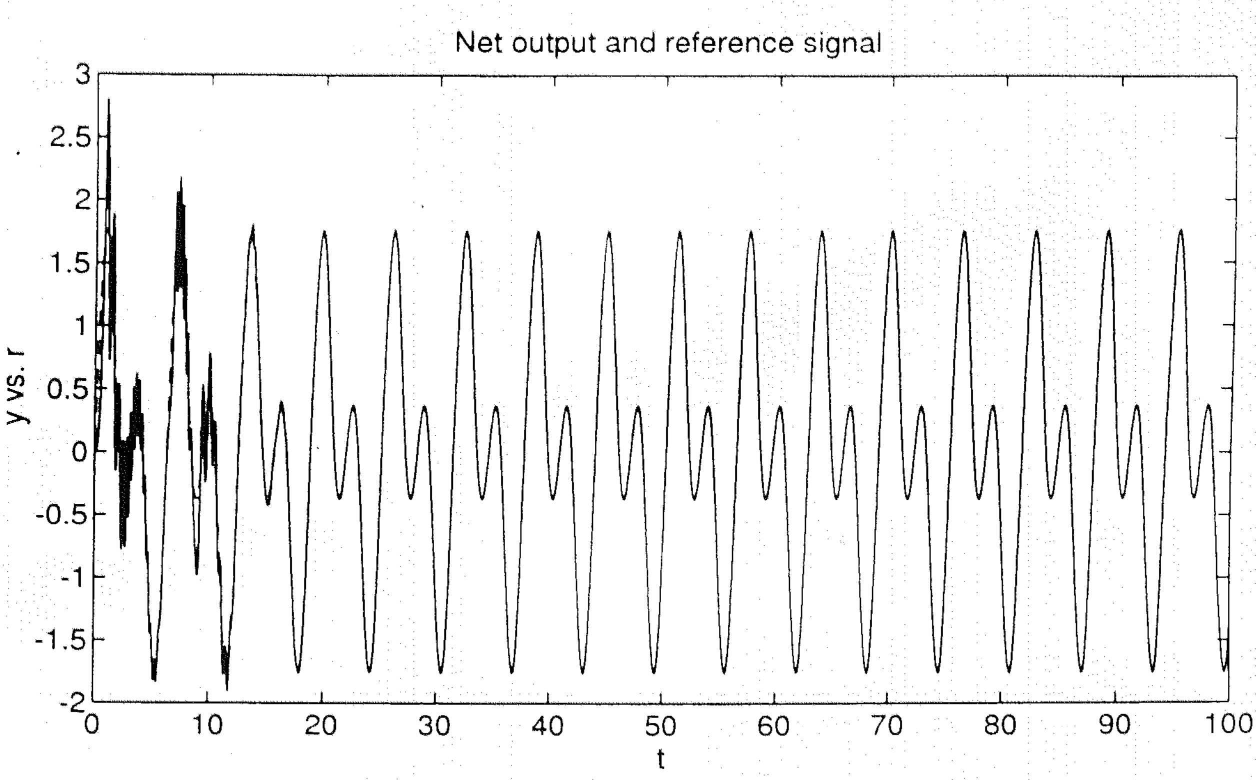


Figure 2: Basic learning rule with $Q = I$
above: $|e_1(t)|$, below: $|e_{p_i}(t)|$



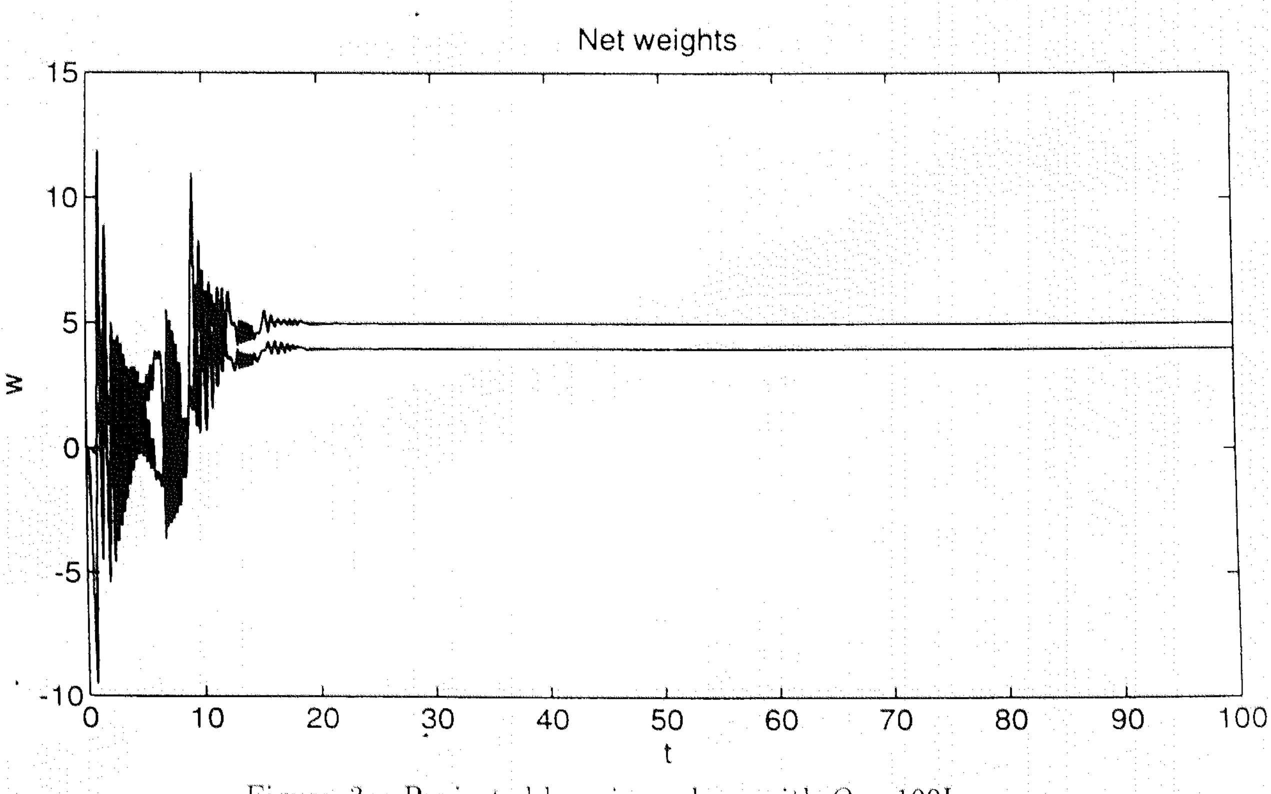
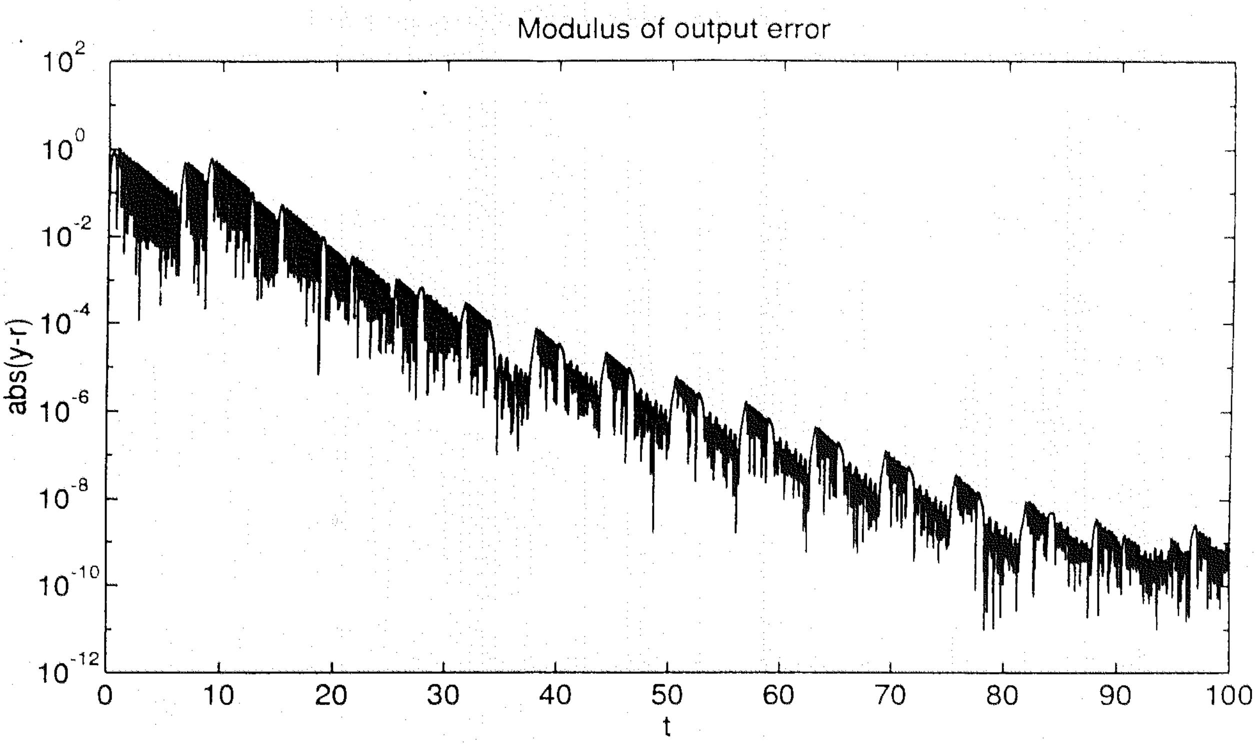


Figure 3a: Projected learning rule with $Q = 100I$
above: — $y(t)$, - - - $r(t)$, below: $\vartheta_i(t)$, $\vartheta^* = (5,4)^T$



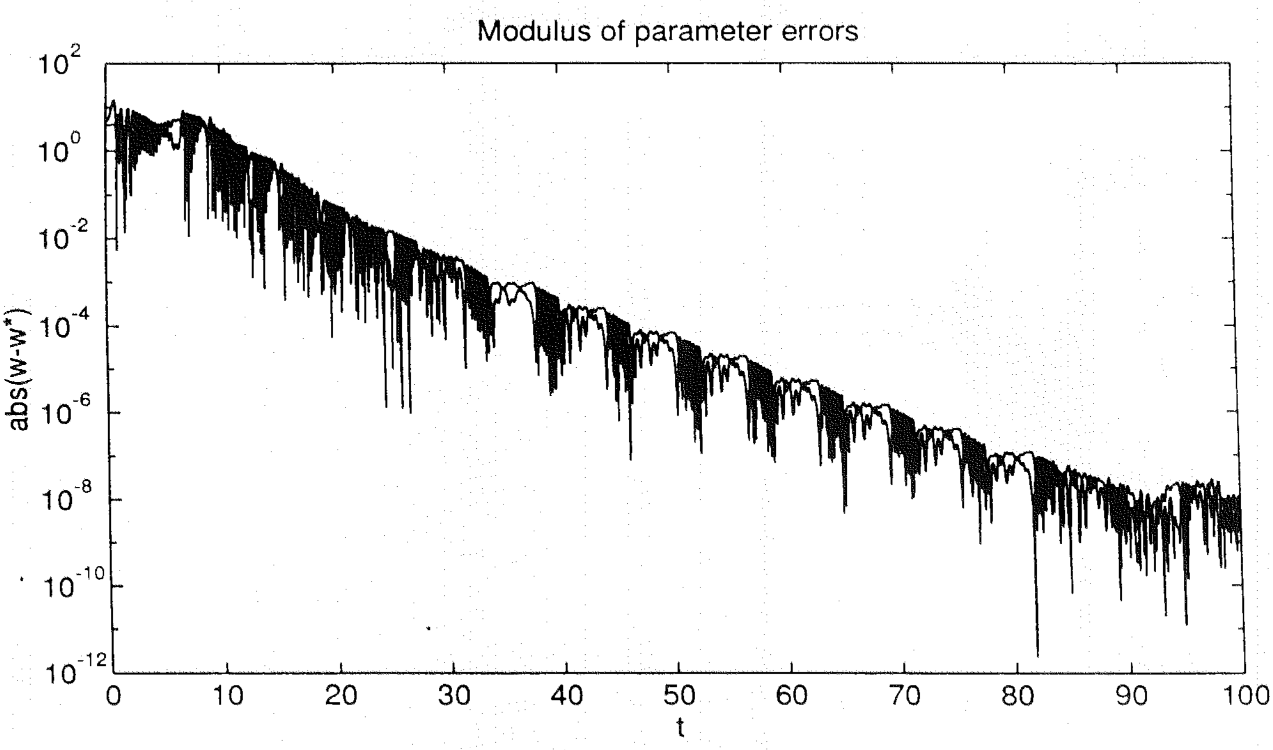


Figure 3b: Projected learning rule with $Q = 1001$
above: $|e_1(t)|$, below: $|e_{p_i}(t)|$

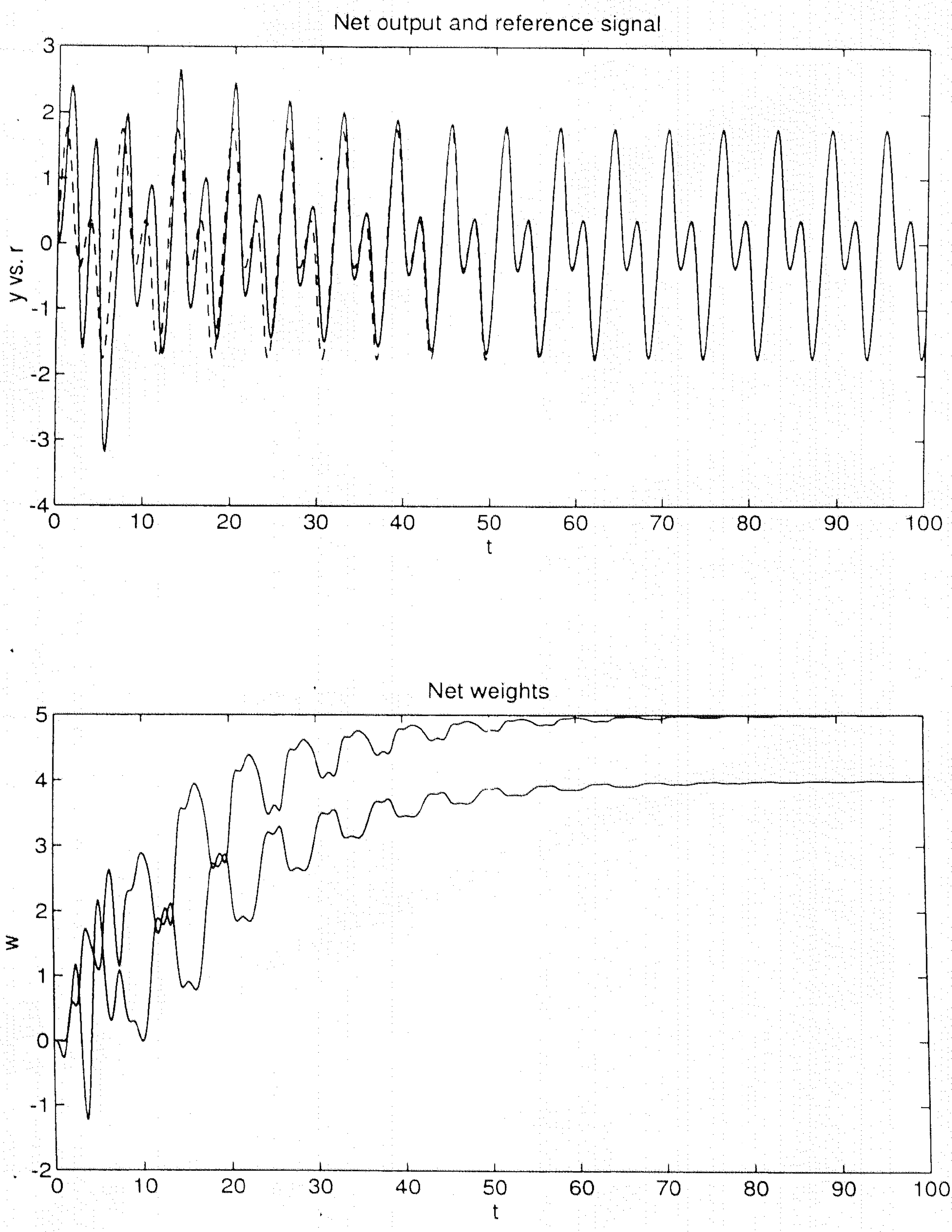
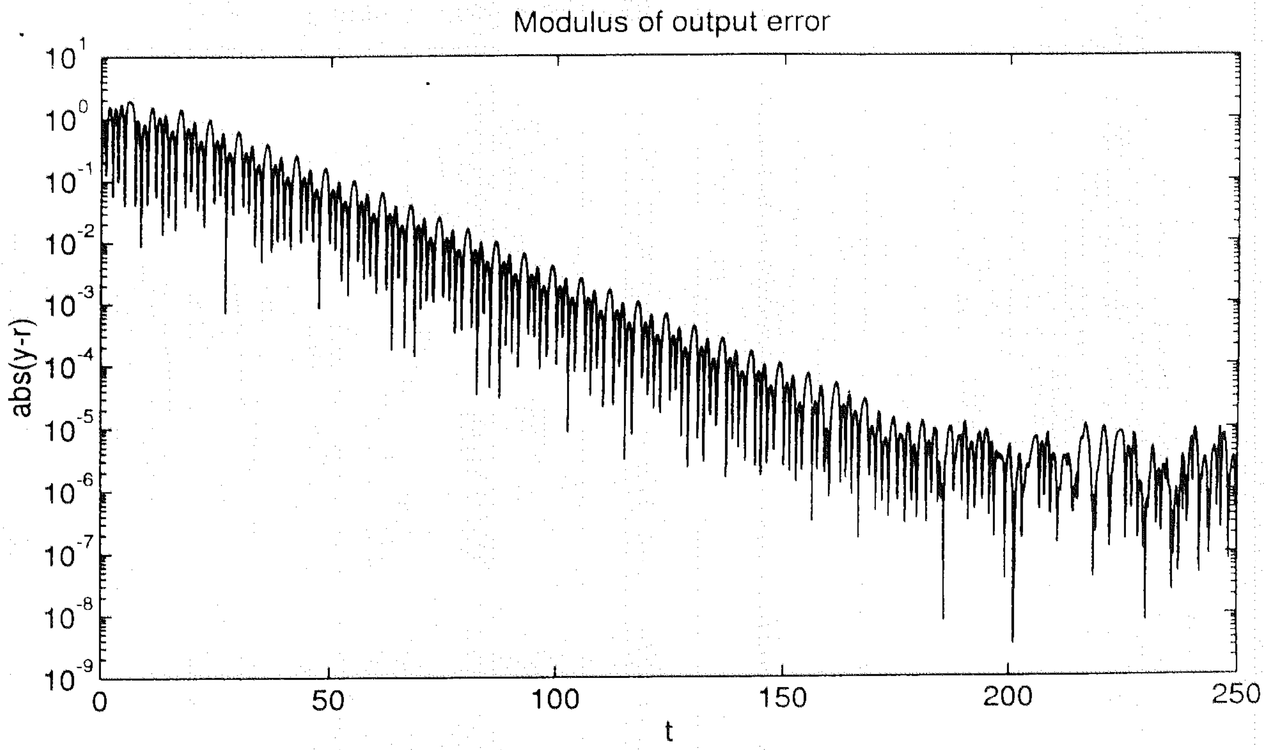


Figure 4a: Projected learning rule with $Q = I$
above: — $y(t)$, - - - $r(t)$, below: $\vartheta_i(t)$, $\vartheta^* = (4,5)^T$



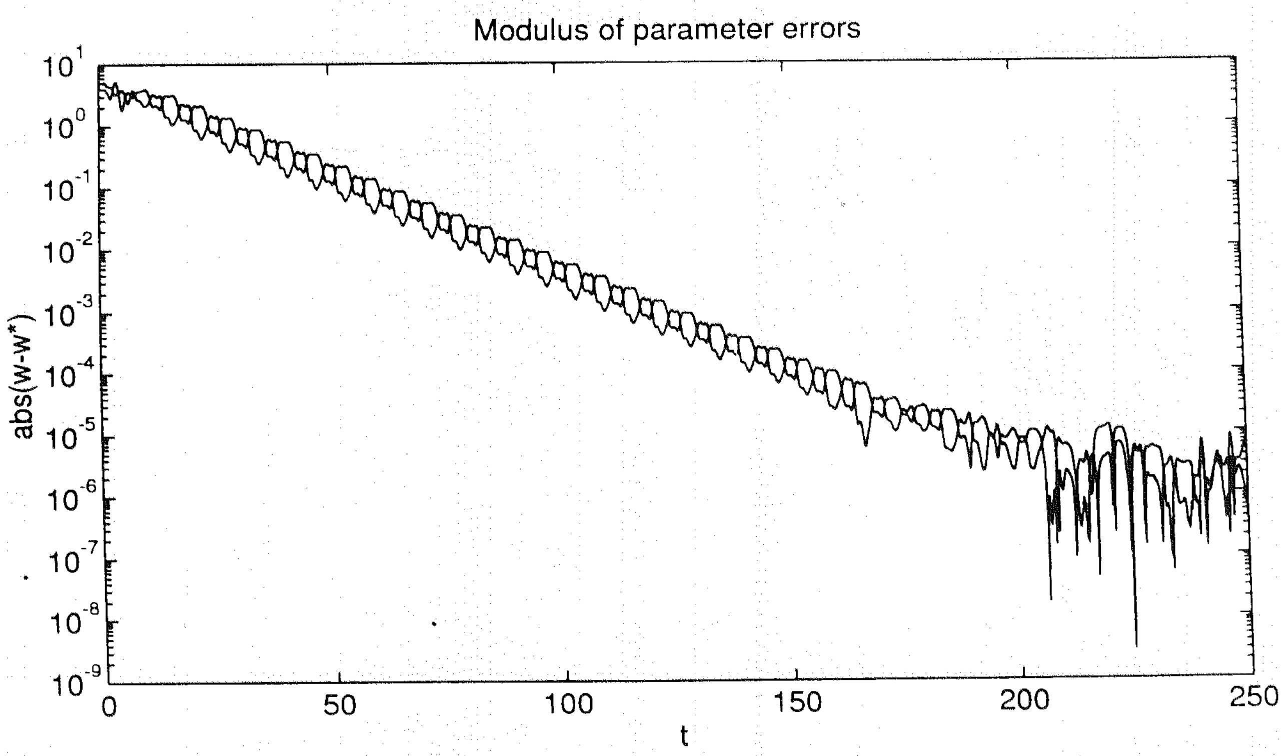


Figure 4b: Projected learning rule with $Q = I$
above: $|e_1(t)|$, below: $|e_{p_i}(t)|$

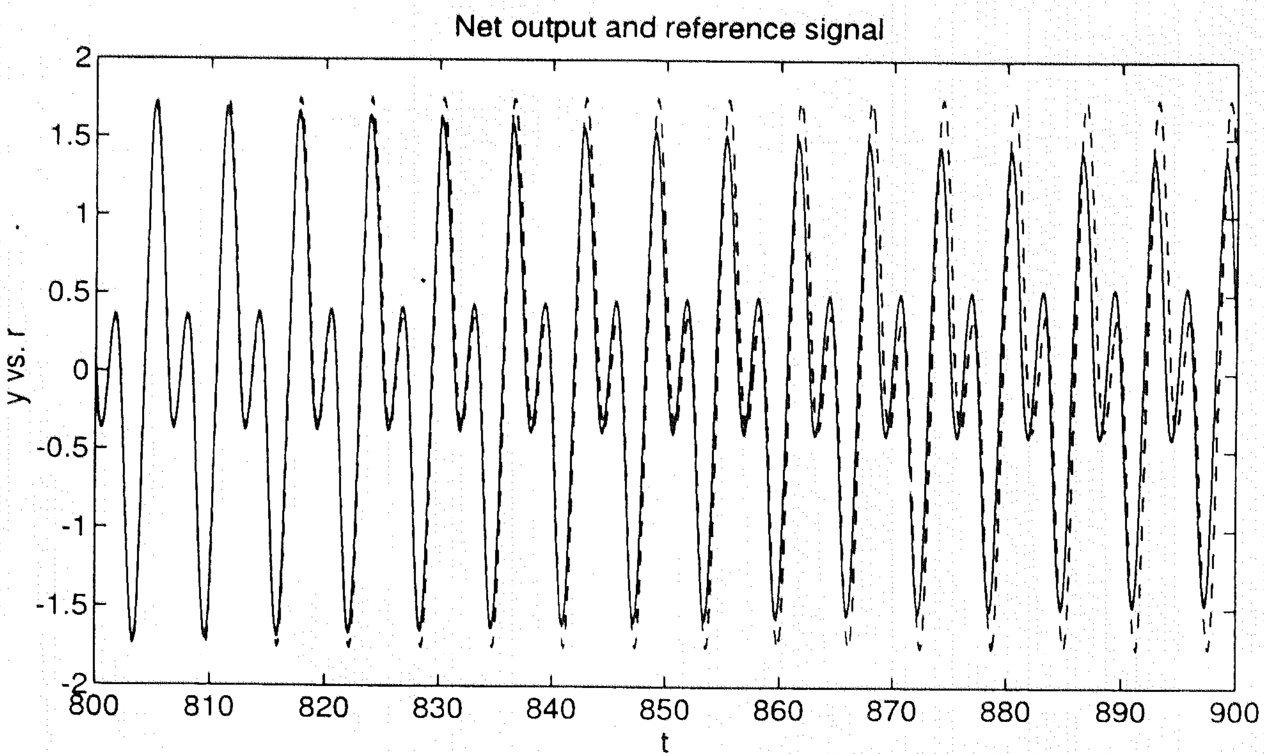
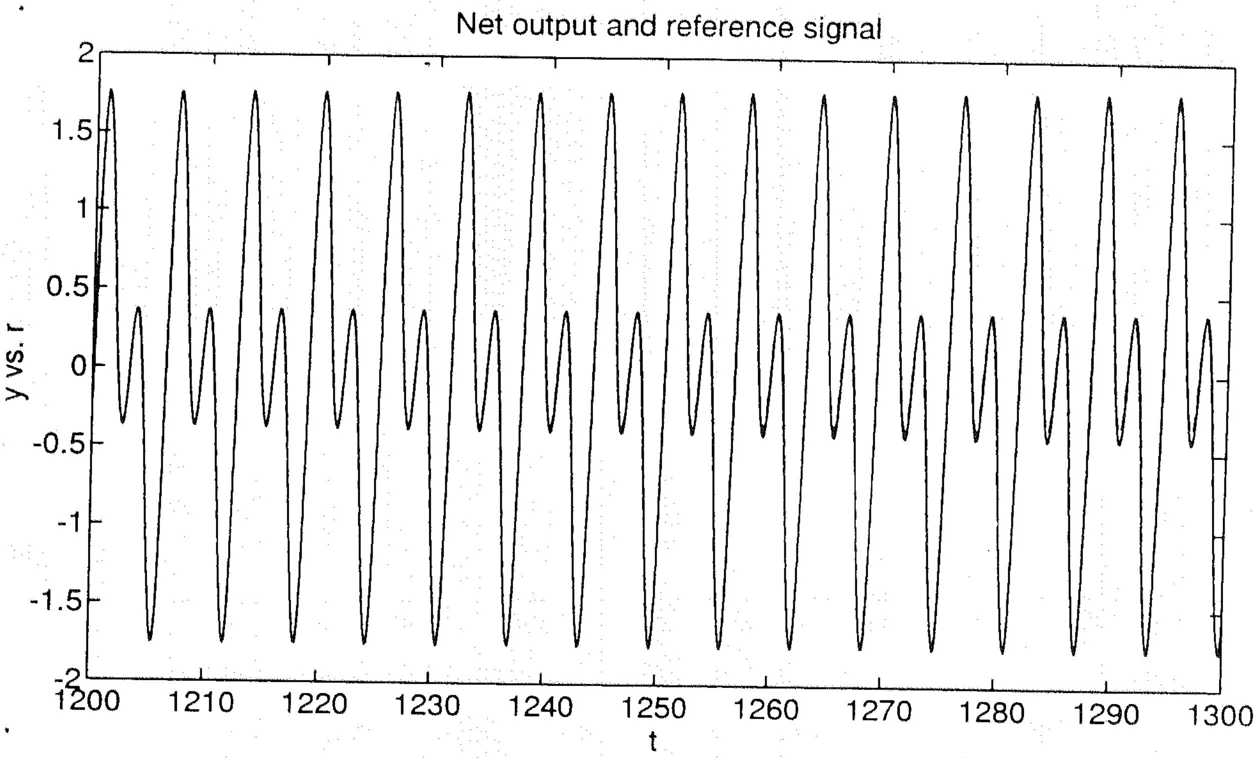


Figure 5: Reproduction phase of the basic learning rule, $t_s = 800$(s)
— $y(t)$, - - - $r(t)$



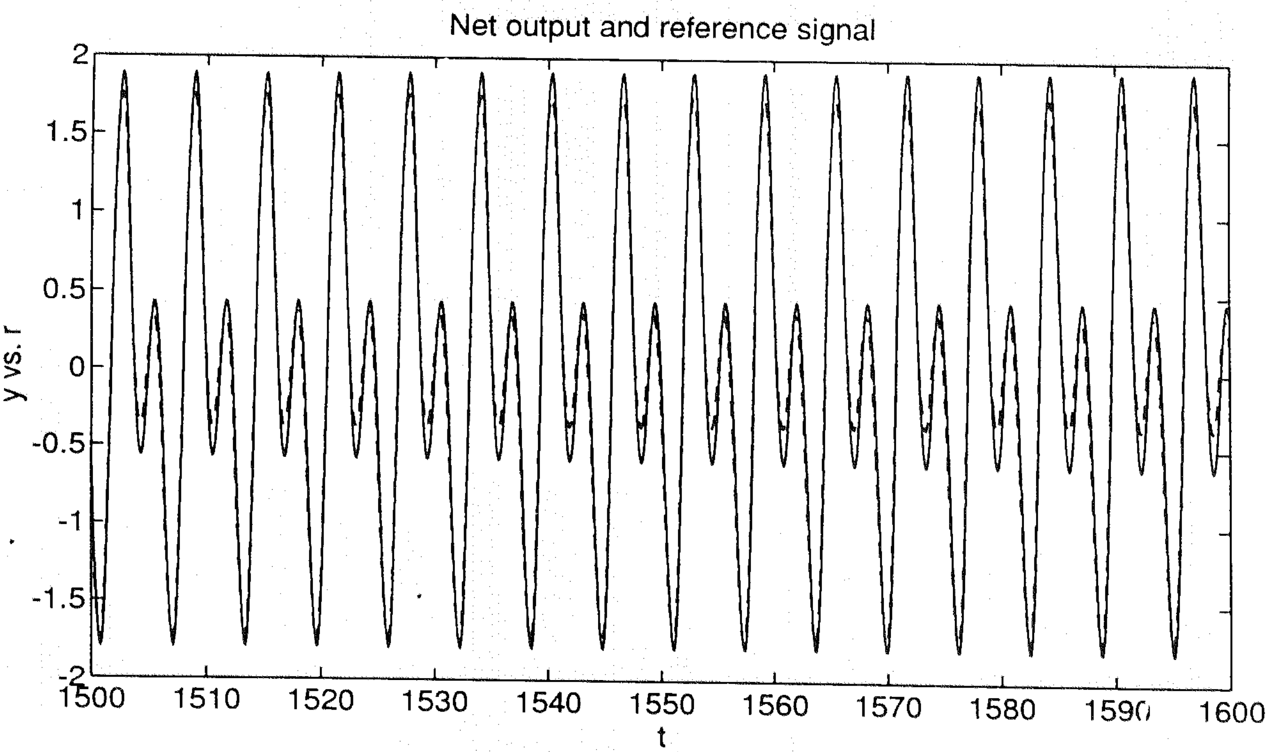


Figure 6: Reproduction phase of the basic learning rule, $t_s = 1200$(s)
— $y(t)$, - - - $r(t)$

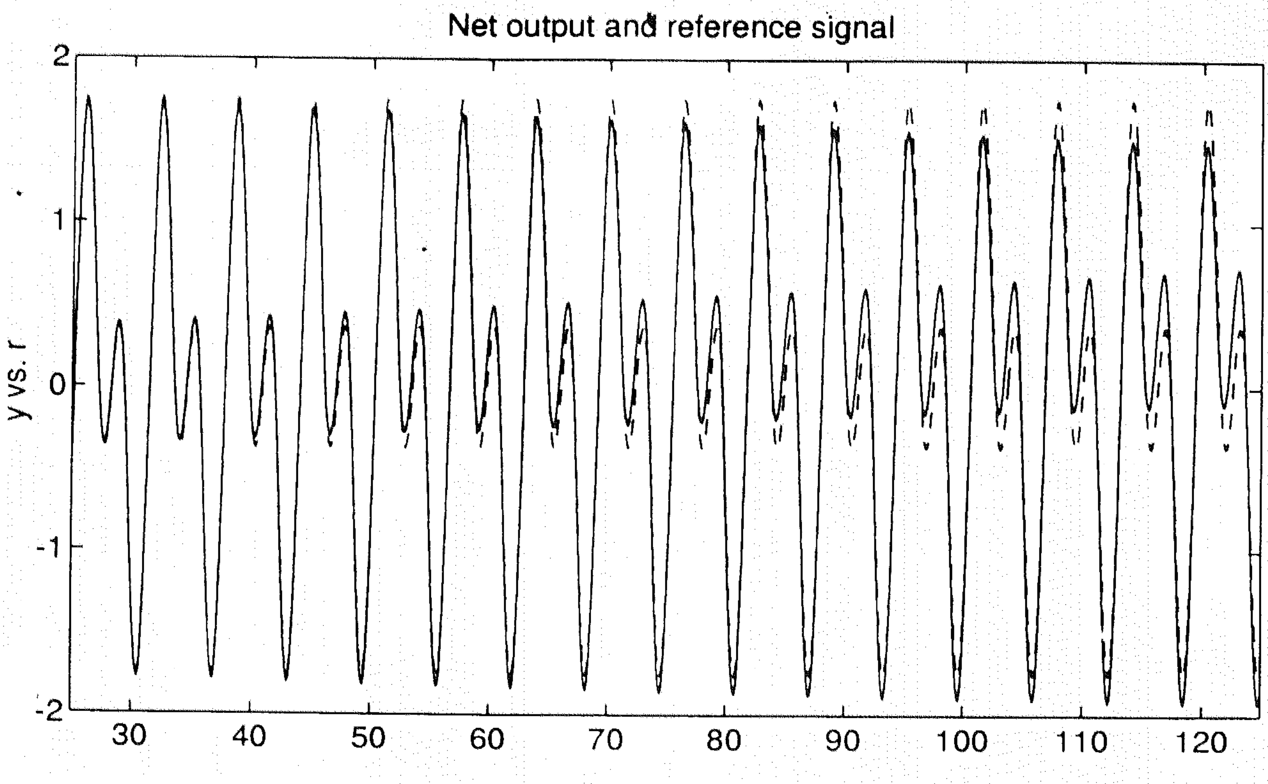


Figure 7: Reproduction phase of the projected algorithm, $t_s = 25$(s)
— $y(t)$, - - - $r(t)$

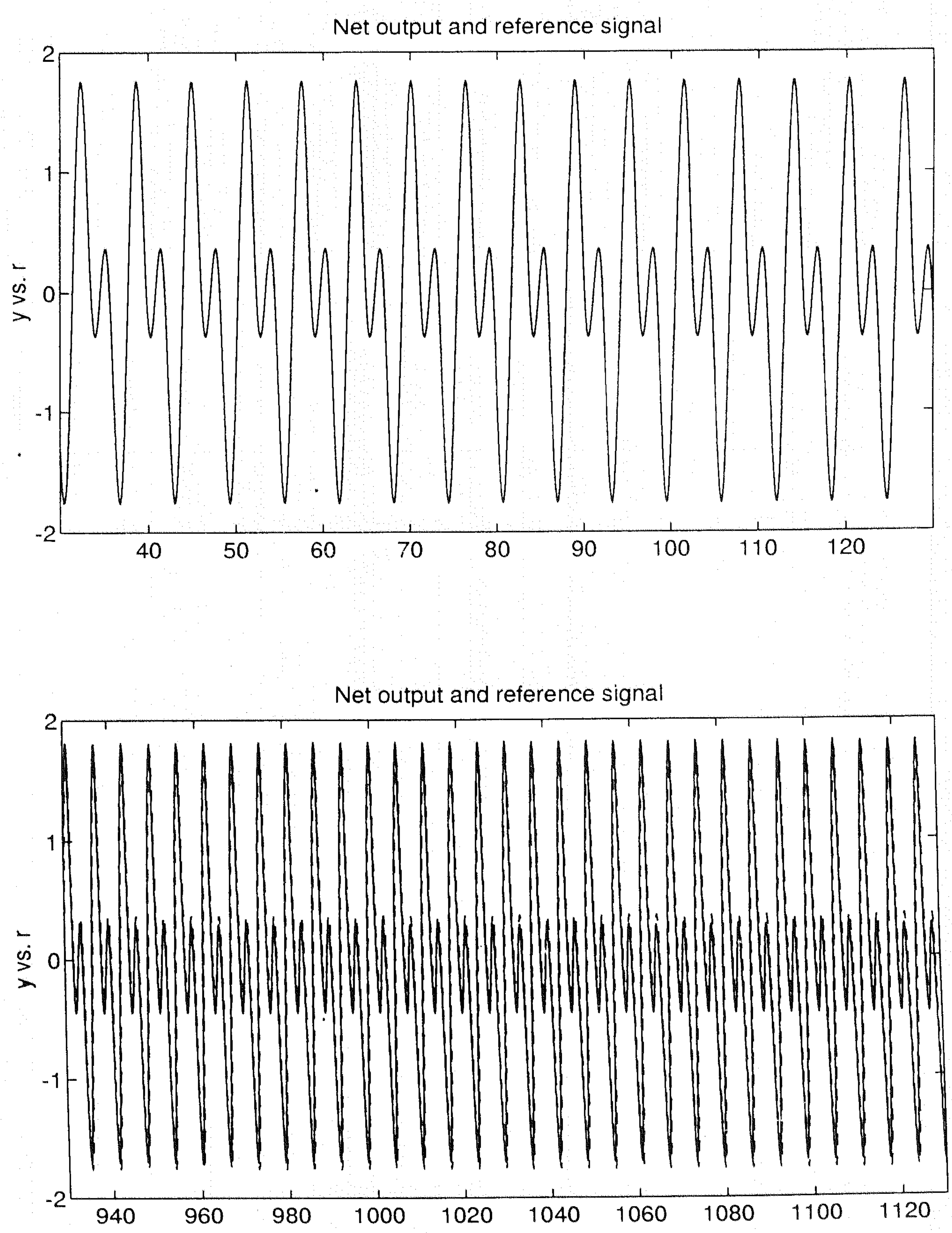


Figure 8: Reproduction phase of the projected algorithm, $t_s = 30$(s)
— $y(t)$, - - - $r(t)$