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# LECTURES ON THE PROBLEM OF SPACE AND TIME IN EINSTEIN'S THEORY OF GRAVITATION

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## Preface

In 1963 these lectures gave an introduction into the concept of space and time in general relativity. The audience in Seattle consisted of mathematicians, physicists and engineers of the Boeing Scientific Research Laboratories (BSRL), who were interested in scientific experiments to add new proofs to Einstein's postulate on space and gravitation.

The lectures start with an introduction into the mathematics of 2-dimensional surfaces and illustrate the notions of geodesics and curvature. They unify many details in terms of the parallel shift of Levi-Cività.

The conceptual background of the transition to Riemannian geometry and the Minkowski continuum is assumed to be known. The computational problems are treated in the formalism of the tensor calculus.

The theory of space and gravitation is only demonstrated by the example of the Schwarzschild solution. The problem of matter and space-time geometry is not touched. Many solutions and approximations dominating nowadays practical applications were not known at the time of the lectures.

In 1963 the BSLR proposed to NASA a test of the predicted gyroscopic effect by observing the movements of a spherically symmetric satellite circling the earth.

A similar experiment with more details was suggested independently by M. Schiff and a group of physicists at Stanford. Both fell into oblivion because the necessary precision of measurement did not seem feasible.

In the meantime our knowledge and our technique have improved considerably and a new version of a test is in preparation in Stanford.

I hope, that this concentrated course may help to understand the fascination of the combination of geometry and physics.

Aachen, August 1995 Claus Müller

## I. Introduction

Recent interest in the techniques of space navigation has altered the character of the problem concerning the structure of space and time from mere speculations in theoretical mathematics to concrete practical questions. Any systematic analysis of physical space and time measurements will always be associated with the name of Albert Einstein. His great achievement was the liberation of physical statements from any dependence upon hypotheses which cannot be verified by direct measurement.

Einstein's line of thought led to a new theory of gravitation which is different from Newton's classical theory. Although predictions from both theories are essentially the same to the first order and they differ only minutely for all problems considered so far, as further explorations of outer space are contemplated, these previously negligible differences may become increasingly important. The present level of space technology has made it feasible for the first time to perform certain new experiments in the field of celestial mechanics which may serve to prove or disprove a fundamental conjecture of Einstein's theory. One such experiment which is closely related to space navigation will be suggested in the sequel.

All navigation is essentially the determination and prediction of location. Position finding within the framework of classical astronomy is based on a coordinate system which is determined by the fixed stars. A spacecraft can either refer to this system of directions or carry along an internal reference system. Such an entrained reference system can be simulated mechanically by means of a gyroscope. Although these two coordinate systems are equivalent in Newton's theory when the gyroscope is represented by a spinning spherical top, they are not equivalent in Einstein's theory of gravitation.

Although the theoretical deviations of an ideal gyroscope circling the earth are very slight, they may be quite considerable, for example, in similar movements around the sun. However, since these are systematic, they must accumulate in the course of time. The exceedingly small magnitude characteristic of the discrepancy between the Newton and Einstein theories is emphasized by the fact that heretofore the maximum measurable effect to determine deviations in favor of Einstein's theory attains a magnitude large enough to be measured only after a century has elapsed. Now, it may be possible to check the differences between the stellar and gyroscopic systems of reference by performing a new experiment in which, after only one year, the total magnitude of the effect would be large enough to be measured. Such an experiment would require astronomical observations from the earth of the rotation of a spinning object in orbit and involves two experimental difficulties. The first is attaining the necessary precision of measurement, and the second is fabricating a spinning object sufficiently close to an ideal spherical top.

It follows from the discoveries of Gauss and Riemann in the 19$^{th}$ century that the curvature of a metric continuum is characterized by the absence of a global parallelism. This means that only in a flat space can directions be shifted uniquely. Since Newton's theory of gravitation is based on a flat space, it must possess such an unambiguous transfer of directions.

Einstein formulated the relations between the geometry of the space-time continuum and matter in terms of differential equations for the components of the metric connection. In this logical system the movement of a mass point under the influence of gravitation is described in terms of the geodesics of a general metric space. Here it is of great importance that gravitation is not treated as a force, but, rather, is regarded as manifesting itself in the curvature properties of the space-time continuum. The same difficulties which make it impossible to map the curved surface of the earth on a flat plane preclude the possibility of mapping a curved space on a flat continuum of three or four dimensions.

Einstein's theory of the relation between the geometry of the space-time continuum and the physical world is based on the simulation of geometric concepts by kinematic analogues. The model best suited to the questions under consideration is the spinning mass point which is just the idealization of a small rotating rigid system having spherical symmetry. Since, according to Newton's theory, the axis of rotation of such a system would retain its orientation regardless of the gravitational forces, it provides an excellent device for the verification of Einstein's predictions.

Any evaluation of the possibilities for success of an experiment of the type suggested here must include consideration of the magnitudes involved. Mercury's perihelion undergoes a shift of 43 seconds of arc per century, and the methods of observation and calculation used in astronomy are sufficiently accurate to justify confidence in these measurements. The axis of rotation of a perfectly spherical spinning top orbiting at a distance of about 300 miles around the earth would experience a shift of roughly 4 seconds of arc per year. The path of the object may depart from an ideal circular orbit without affecting this order of magnitude essentially.

It should be mentioned that similar mathematical models were discussed by Fokker, de Sitter, and H. Weyl to explain the shift of the earth's axis of rotation. It is evident, however, that the effects which arise from the unsymmetry of the earth are much greater than the changes which can be ascribed to relativity.

The fundamental concepts of parallelism and curvature are discussed in detail. The presentation of the basic ideas relies largely on the two-dimensional case where geometric imagination is a great help. The discussion of vector fields on general surfaces is used to formulate the most important theorems in differential geometry. Many of the details of this approach seem to be new.

I should like to thank Dr. T. P. Higgins for his assistance in preparing this manuscript.

## II. Foundations of Differential Geometry

The aim of this lecture is the study of the geometrical properties of general smooth surfaces in 3-dimensional euclidean space. Although the basic concepts were first introduced by Gauss, the notations and techniques employed here make use of more recent developments.

We shall begin with the discussion of the local geometrical properties of a surface. First we introduce

Definition 1: A set $S$ of points in 3-dimensional euclidean space is called a standard surface element if there is a point $P$ of $S$, and a cartesian system of coordinates $(x^1,x^2,x^3)$ with $P$ as origin such that any point of $S$ may be represented in the form

$$x^3 = F(x^1,x^2),$$

where $F(x^1,x^2)$ is analytic in a domain $D$ of the $(x^1,x^2)$-plane, and

$$F(0,0) = 0, \quad F_{x^1}(0,0), \quad \text{and} \quad F_{x^2}(0,0) = 0.$$

Thus the $(x^1,x^2)$-plane is tangential at $P$ and the $x^3$-axis is in the direction of the normal. This system is therefore called a tangential-normal system associated with $P$. Apart from rotations around the $x^3$-axis, this system is uniquely determined. A more general representation of the surface element may be obtained by expressing the space vector

$$(1) \quad \vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = x^1\vec{e}_1 + x^2\vec{e}_2 + F(x^1,x^2)\,\vec{e}_3$$

in terms of two parameters $u^1$ and $u^2$. Then

$$(2) \qquad \vec{x} = \vec{x}(u^1,u^2).$$

This leads us to introduce

Definition 2: A set of points in 3-dimensional euclidean space will be called a surface element if there is a domain $D$ of a $u^1,u^2$-plane and a one-to-one representation of $S$ in the form

$$\vec{x} = \vec{x}(u^1,u^2) = x^1(u^1,u^2)\,\vec{e}_1 + x^2(u^1,u^2)\,\vec{e}_2 + x^3(u^1,u^2)\,\vec{e}_3$$

with $(u^1,u^2) \varepsilon D$, where the functions $x^i(u^1,u^2)$ are analytic in $D$ and satisfy

$$\left(\frac{\partial\vec{x}}{\partial u^1} \times \frac{\partial\vec{x}}{\partial u^2}\right) \neq 0.$$

Since from (1) we get

$$(3) \quad \left|\frac{\partial\vec{x}}{\partial x^1} \times \frac{\partial\vec{x}}{\partial x^2}\right| = \left|(\vec{e}_1 + \frac{\partial F}{\partial x^1}\vec{e}_3) \times (\vec{e}_2 + \frac{\partial F}{\partial x^2}\vec{e}_3)\right| = \sqrt{1 + (\frac{\partial F}{\partial x^1})^2 + (\frac{\partial F}{\partial x^2})^2},$$

it follows that the tangential-normal system always satisfies the conditions of Definition 2.

The main difficulty of Differential Geometry stems from two facts:

a) There are simple and important surfaces, such as the sphere, which are not surface elements since they cannot be represented by one system of parameters.

b) There is in general no possibility of selecting one special system of parameters which is naturally distinguished by geometrical properties.

In order to understand the importance of these facts we have to observe that all quantities which are of importance from a geometrical point of view do not depend on the system of coordinates we use to describe the surface. For example, if we want to define the curvature of a surface, we have to find a formula which leads to a value at every point of the surface element and this value must be the same for all parameter systems we may take to define the surface. This leads to

Definition 3: The expression $I$ which is formed of the coordinate functions $x^i(u^1,u^2)$ and its derivatives is called an invariant of the surface if its value is not altered by a change of the parameter. This invariant is said to be of order $n$ and degree $k$ if the maximum order of derivation is $n$ and if the highest derivatives occur in the degree $k$.

It is always possible to find invariants by specializing the system of parameters locally. Let us consider the tangential-normal system of cartesian coordinates associated with the point $P$. As we have noted, this system is uniquely determined except for rotations around the $x^3$-axis. The function $F(x^1,x^2)$ then may be expanded in a Taylor series. We use the abbreviations

$$F_1 = \frac{\partial F}{\partial x^1} \; ; \; F_2 = \frac{\partial F}{\partial x^2} \; ; \; F_{11} = \frac{\partial^2 F}{\partial x^1 \partial x^1} \; ; \; F_{12} = F_{21} = \frac{\partial^2 F}{\partial x^1 \partial x^2} \; ; \; F_{22} = \frac{\partial^2 F}{\partial x^2 \partial x^2}$$

(4)

$$\mathring{F}_1 = F_1(0,0) \; ; \; \mathring{F}_2 = F_2(0,0) \; ; \; \mathring{F}_{11} = F_{11}(0,0).$$

Then the series can be written as

$$(5) \quad F(x^1,x^2) = \frac{1}{2}\left(\mathring{F}_{11}(x^1)^2 + 2\mathring{F}_{12}x^1x^2 + \mathring{F}_{22}(x^2)^2\right) + \cdots.$$

The leading terms are a quadratic form. Substituting rotation equations of the form

$$(6)\qquad \begin{aligned} x^1 &= u^1 \cos\alpha - u^2 \sin\alpha \\ x^2 &= u^1 \sin\alpha + u^2 \cos\alpha \end{aligned}$$

we obtain

$$(7)\quad F(x^1,x^2) = G(u^1,u^2) = \frac{1}{2}\left(\mathring{G}_{11}(u^1)^2 + 2\mathring{G}_{12}u^1u^2 + \mathring{G}_{22}(u^2)^2\right) + \cdots .$$

Now, it is easy to verify by explicit calculation that

$$(8)\qquad \begin{aligned} \mathring{F}_{11} + \mathring{F}_{22} &= \mathring{G}_{11} + \mathring{G}_{22} \\ \mathring{F}_{11}\mathring{F}_{22} - \mathring{F}^2_{12} &= \mathring{G}_{11}\mathring{G}_{22} - \mathring{G}^2_{12}. \end{aligned}$$

The quantities

$$(9)\qquad \begin{aligned} 2H &= \mathring{F}_{11} + \mathring{F}_{22} \\ K &= \mathring{F}_{11}\mathring{F}_{22} - (\mathring{F}_{12})^2 \end{aligned}$$

therefore may be regarded as geometrically relevant for the point P because they are uniquely determined.

The line of thought behind this approach is the following:

To a point P we first find a special tangential-normal system of coordinates. This reduces the ambiguity of the parameter representation of the surface element up to rotation of the tangential plane. We then form expressions which are not affected by these orthogonal transformations. The quantities thus obtained are well defined and will prove to represent the curvature of the surface. To this end we look at the intersection of a plane perpendicular to the plane tangential to the surface. The straight line of intersection between these two planes may be represented in the form

$$(10) \qquad x^1 = s \cos\alpha \; ; \; x^2 = s \sin\alpha$$

so that the curve of intersection may be represented by

$$(11) \qquad \varphi(s) = F(s \cos\alpha, s \sin\alpha).$$

The curvature of this curve at $s = 0$ is $\varphi''(0)$, which gives

$$(12) \qquad \varphi''(0) = \mathring{F}_{11} \cos^2\alpha + 2\mathring{F}_{12} \sin\alpha\cos\alpha + \mathring{F}_{22} \sin^2\alpha.$$

Thus the curvature of the intersection depends on the direction of the intersecting plane. The expression (12) can be written as

$$(13) \qquad \varphi''(0) = K_1 \cos^2(\alpha - \varphi) + K_2 \sin^2(\alpha - \varphi)$$

with

$$(14) \qquad \begin{aligned} K_1 + K_2 &= \mathring{F}_{11} + \mathring{F}_{22} \\ K_1 \cdot K_2 &= \mathring{F}_{11}\mathring{F}_{22} - (\mathring{F}_{12})^2. \end{aligned}$$

Thus $K_1$ and $K_2$ constitute the extremal values of the curvature of the intersecting curves. The quantities (14) therefore have an immediate geometrical interpretation,

$$(15) \qquad \begin{aligned} 2H &= K_1 + K_2 \\ K &= K_1 K_2 , \end{aligned}$$

where $H$ is called the average curvature and $K$ the Gaussian curvature.

The above procedure makes it possible to calculate the curvatures of the surface at every point, provided we know the representation of the surface in the tangential-normal system which has the point under consideration as the origin. But in order to find the curvatures it would be necessary to find a new representation of the surface for every point. This, of course, is a great complication and we shall now discuss a different way of calculating the curvatures. To this end we introduce

<u>Definition 4</u>: The quantities

$$g_{ik} = \vec{x}_i \vec{x}_k = g_{ki}$$

are called the fundamental coefficients of the first kind.

Moreover, we may introduce the normal vector by

$$(16) \qquad \vec{n} = \frac{\vec{x}_1 \times \vec{x}_2}{|\vec{x}_1 \times \vec{x}_2|}$$

and get

<u>Definition 5</u>: The quantities

$$L_{ik} = \vec{n}\,\vec{x}_{ik} = L_{ki}$$

are called the fundamental coefficients of the second kind.

From (16) we have

$$(17) \qquad \vec{n}\,\vec{x}_i = 0,$$

so that differentiation gives

(18) $$\vec{n}_k \vec{x}_i + \vec{n}\,\vec{x}_{ik} = 0$$

and then it follows from the definition that

(19) $$L_{ik} = -\vec{n}_k \vec{x}_i = -\vec{n}_i \vec{x}_k .$$

We now assume that the parameters $u^i$ depend on the parameter $w^i$ by a one-to-one analytic relation. The quantities $g_{ik}$ and $L_{ik}$ then will have different values if we express them in this new system of coordinates. Denoting the values in the w-system by an asterisk, we get from

(20) $$\frac{\partial}{\partial w^i}\vec{x} = \sum_{j=1}^{2} \frac{\partial u^j}{\partial w^i} \frac{\partial}{\partial u^j}\vec{x} = \sum_{j=1}^{2} \frac{\partial u^j}{\partial w^i}\vec{x}_j$$

the relation

(21) $$\overset{*}{g}_{ik} = \sum_{j=1}^{2} \sum_{e=1}^{2} \frac{\partial u^j}{\partial w^i} \frac{\partial u^e}{\partial w^k} g_{je} .$$

This is a law of transformation for the coefficients of the first kind which will play an important part in our theory. To simplify the notation we introduce

Einstein's convention: If in any of the following expressions the same letter is used for "upper" and "lower" indices, this denotes the summation of the terms with this letter equal to 1 and 2. This notational convention eliminates the use of the signs of summation and is very useful since all of the laws of transformation require summations of the type given in (20).

From (21) and (19) respectively we obtain

Lemma 1:
$$\overset{*}{g}_{ik} = g_{je} \frac{\partial u^j}{\partial w^i} \frac{\partial u^e}{\partial w^k}$$

$$\overset{*}{L}_{ik} = L_{je} \frac{\partial u^j}{\partial w^i} \frac{\partial u^e}{\partial w^k} .$$

We introduce

Definition 6: The quantities $g^{ik}$ are uniquely defined by the relation

$$g^{ij} g_{jk} = \delta^i_k = \begin{cases} 1 & \text{for} \quad i = k \\ 0 & \text{for} \quad i \neq k \end{cases}$$

as the elements of the inverse to the matrix with the elements $g_{ik}$.

From the relations

(22)
$$\delta^i_k = \frac{\partial u^i}{\partial u^k} = \frac{\partial u^i}{\partial w^j} \frac{\partial w^j}{\partial u^k} \; ; \; \delta^i_k = \frac{\partial w^i}{\partial w^k} = \frac{\partial w^i}{\partial u^j} \frac{\partial u^j}{\partial w^k}$$

we have

(23)
$$\overset{*}{g}{}^{ik} = g^{je} \frac{\partial w^i}{\partial u^j} \frac{\partial w^k}{\partial u^e} .$$

To prove this we verify the properties of $\overset{*}{g}{}^{ik}$ according to Definition 6. That is,

(24)
$$\overset{*}{g}{}^{ij} \overset{*}{g}_{jk} = g^{s_1 s_2} \frac{\partial w^i}{\partial u^{s_1}} \frac{\partial w^j}{\partial u^{s_2}} g_{s_3 s_4} \frac{\partial u^{s_3}}{\partial w^j} \frac{\partial u^{s_4}}{\partial w^k}$$

$$= g^{s_1 s_2} \frac{\partial w^i}{\partial u^{s_1}} \delta^{s_3}_{s_2} \frac{\partial u^{s_4}}{\partial w^k} g_{s_3 s_4} = g^{s_1 s} g_{s s_4} \frac{\partial w^i}{\partial u^{s_1}} \frac{\partial u^{s_4}}{\partial w^k}$$

$$= \delta^{s_1}_{s_4} \frac{\partial w^i}{\partial u^{s_1}} \frac{\partial u^{s_4}}{\partial w^k} = \frac{\partial w^i}{\partial u^s} \frac{\partial u^s}{\partial w^k} = \delta^i_k$$

so that $\overset{*}{g}{}^{ij}$ represents the elements of the inverse of the matrix $(\overset{*}{g}_{ik})$.

We now form

Definition 7: $L^i_k = g^{ij}L_{jk}$ ; $L^{ik} = g^{is}g^{kt}L_{st} = g^{is}L^k_s$

and obtain by calculations analogous to (24)

Lemma 2:
$$\overset{*}{L}{}^i_k = L^j_s \frac{\partial w^i}{\partial u^j} \frac{\partial u^s}{\partial w^k}$$

$$\overset{*}{L}{}^{ik} = L^{js} \frac{\partial w^i}{\partial u^j} \frac{\partial w^k}{\partial u^s} .$$

We now are able to form the first invariants by proving

Theorem 1: The quantities

$$L^i_i = 2H$$

and

$$L^i_i L^k_k - L^i_k L^k_i = 2K$$

are invariants. They are called the curvatures of the surface element. They are both second order invariants, $H$ being of first degree and $K$ of second degree.

Our theorem is proved if we show that these quantities have the same value in every system of parameters. Now we get from Lemma 2 and (22)

(25) $$\overset{*}{L}{}^i_i = L^j_s \frac{\partial w^i}{\partial u^j} \frac{\partial u^s}{\partial w^i} = L^j_s \delta^s_j = L^j_j$$

(26) $$\overset{*}{L}{}^i_k \overset{*}{L}{}^k_i = L^j_s L^r_t \frac{\partial w^i}{\partial u^j} \frac{\partial u^s}{\partial w^k} \frac{\partial w^k}{\partial u^r} \frac{\partial u^t}{\partial w^i} = L^j_s L^r_t \delta^t_j \delta^s_r = L^j_s L^s_j .$$

The terms on the left hand side of (25) and (26) therefore have the same values in all parameter systems, so that the quantities stated in Theorem 1 are invariants. Next we have to show that they are equal to the quantities defined earlier. To this end we use the tangential-normal system

(27) $$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + F(x^1,x^2)\,\vec{e}_3.$$

Let $\overset{\circ}{=}$ denote an equality for the values $x^1 = 0, x^2 = 0$. Then

(28) $$g_{ik} = (\vec{e}_i + F_i\vec{e}_3)\cdot(\vec{e}_k + F_k\vec{e}_3) = \delta_{ik} + F_iF_k \overset{\circ}{=} \delta_{ik}.$$

Observing that

(29) $$F_i \overset{\circ}{=} 0$$

we have moreover

(30) $$\frac{\partial}{\partial x^j} g_{ik} \overset{\circ}{=} 0.$$

Further

(31) $$\vec{n} = (1 + F_1^2 + F_2^2)^{-\frac{1}{2}}(\vec{e}_3 - F_1\vec{e}_1 - F_2\vec{e}_2) \overset{\circ}{=} \vec{e}_3$$

and

(32) $$\vec{x}_{ik} = F_{ik}\vec{e}_3$$

so that

(33) $$L_{ik} \overset{\circ}{=} F_{ik}.$$

With (29) we thus obtain

(34) $$L_1^1 \overset{\circ}{=} F_{11};\ L_2^2 \overset{\circ}{=} F_{22};\ L_2^1 \overset{\circ}{=} F_{12};\ L_1^2 \overset{\circ}{=} F_{21}$$

which gives us

$$L_i^i \overset{\circ}{=} F_{11} + F_{22};$$

(35)

$$L_j^i L_i^j = L_1^1 L_1^1 + L_2^1 L_1^2 + L_1^2 L_2^1 + L_2^2 L_2^2 \overset{\circ}{=} F_{11}^2 + 2F_{12}^2 + F_{22}^2 ,$$

or

$$2H \overset{\circ}{=} F_{11} + F_{22}$$

(36)

$$K \overset{\circ}{=} F_{11}F_{22} - F_{12}^2$$

so that the invariants formulated in Theorem 1 give us the curvatures we had originally defined by our tangential-normal system.

The obvious advantage of the second method for calculating the curvature is that it does not require the introduction of a new system of coordinates for every point at which we want to calculate those quantities. It should be noted, however, that the possibility of introducing a tangential-normal system for every point of our surface may still be a great help in dealing with special questions related to the neighborhood of the point.

We now want to find an interpretation of the curvature which is such that we require only a knowledge of the intrinsic properties of the surface. So far our definition of the curvature was first expressed in terms of the function $F(x^1,x^2)$ which may be regarded as a measurement of the deviation from the plane which is tangential. Obviously $H$ and $K$ vanish for $F \equiv 0$. The definitions of the curvatures which we have used so far made use of the fact that the surface was embedded in 3-dimensional space.

It was Gauss who first discovered that one of the two curvatures, i.e., $K$, can be specified solely by using the intrinsic properties of the surfaces which are based on measuring lengths and angles.

In order to prove and understand this important theorem which Gauss himself termed "theorema egregium", i.e., outstanding theorem, we first have to develop the techniques of tensor analysis by extending the concepts and methods we used to determine the curvatures.

We note that the coefficients of the fundamental form enable us to calculate the length of an arc on a surface element. That is, if we represent this arc in the parametric form $u^i = u^i(t)$ we obtain for the line element

$$(37) \qquad \left(\frac{ds}{dt}\right)^2 = \vec{x}_i \frac{du^i}{dt} \cdot \vec{x}_k \frac{du^k}{dt} = g_{ik} \frac{du^i}{dt} \frac{du^k}{dt} .$$

If we have two intersecting lines with the parameter representations $u^i_{(1)}(s)$ and $u^i_{(2)}(s)$ based on the arc lengths, we get for the angle $\alpha$ between these lines

$$(38) \qquad \cos\alpha = g_{ik} \frac{du^i_{(1)}}{ds} \frac{du^k_{(2)}}{ds} = g_{ik} \dot{u}^i_{(1)} \dot{u}^k_{(2)} .$$

The coefficients $g_{ik}$ therefore enable us to measure distances and angles on the surface. They thus determine the intrinsic metric properties.

We now introduce

$$(39) \qquad \begin{aligned} g_{ei|k} &= \frac{\partial}{\partial u^k} g_{ei} \\ \Gamma_{e,ik} &= \frac{1}{2}[g_{ie|k} + g_{ke|i} - g_{ik|e}] = \Gamma_{e,ki} . \end{aligned}$$

We define furthermore

$$(40)\qquad \Gamma^r_{jk} = g^{rs}\Gamma_{s,jk}.$$

From the definition (39) it follows

$$(41)\qquad \Gamma_{e,ik} + \Gamma_{k,ie} = \frac{1}{2}[g_{ek|i} + g_{ei|k} - g_{ik|e} + g_{ki|e} + g_{ke|i} - g_{ie|k}]$$

$$= g_{ek|i}$$

so that we get

$$(42)\qquad g_{ke|i} = g_{es}\Gamma^s_{ik} + g_{ks}\Gamma^s_{ie}.$$

Since $g^{rs}g_{st} = \delta^r_t$ , we obtain by differentiation

$$(43)\qquad g^{rs}_{|j}g_{st} + g^{rs}g_{st|j} = 0$$

and using (42) gives

$$(44)\qquad g_{st}g^{rs}_{|j} = - g^{rs}g_{sp}\Gamma^p_{jt} - g^{rs}g_{tp}\Gamma^p_{js} = - \Gamma^r_{jt} - g^{rs}g_{tp}\Gamma^p_{js}.$$

Multiplication with $g^{tn}$ now gives

$$g^{tn}g_{st}g^{rs}_{|j} = \delta^n_s g^{rs}_{|j} = g^{ru}_{|j} = - g^{tn}\Gamma^r_{jt} - g^{rs}\Gamma^n_{js}.$$

We formulate these results in

<u>Lemma 3</u>:
$$g_{ke|i} = \Gamma^s_{ik}\, g_{se} + \Gamma^s_{ie}g_{ks}$$

$$g^{ke}_{|i} = - \Gamma^k_{is}g^{se} - \Gamma^e_{is}g^{ks}.$$

Suppose now we have a curve connecting two points of our surface element which are determined by the parameter $u^i_{(1)}$ and $u^i_{(2)}$. If $u^i(t)$, $0 \leq t \leq T$, is a parameter representation of this curve, with $u^i(0) = u^i_{(1)}$; $u^i(T) = u^i_{(2)}$ the length of the arc will be

$$(46) \qquad \int_0^T \sqrt{g_{ik} \frac{du^i}{dt} \frac{du^k}{dt}}\, dt = L$$

where, of course, $g_{ik} = g_{ik}(u^1(t), u^2(t))$ depends on $t$. We now introduce the parameter $s$ of the arc as a natural parameter by the requirement

$$(47) \qquad g_{ik} \frac{du^i}{ds} \frac{du^k}{ds} = 1$$

and consider all neighboring curves with the same endpoints which may be expressed in the form

$$(48) \qquad u^i(s) + \varepsilon\eta^i(s); \; \eta^i(0) = 0; \; \eta^i(L) = 0 .$$

With the notations

$$(49) \qquad G_{ik}(\varepsilon) = g_{ik}\Big(u^1(s) + \varepsilon\eta^1(s), u^2(s) + \varepsilon\eta^2(s)\Big)$$

we have

$$(50) \qquad \frac{\partial}{\partial\varepsilon} G_{ik}(\varepsilon)\Big|_{\varepsilon=0} = g_{ik|j}\eta^j; \; G_{ik}(0) = g_{ik}$$

where the values of $g_{ik}$ and $g_{ik|j}$ are taken along the curve $u^i(s)$. The lengths of these curves are given by

$$(51) \qquad \int_0^L \sqrt{G_{ik}(\varepsilon)(\dot{u}^i + \varepsilon\dot{\eta}^i)(\dot{u}^k + \varepsilon\dot{\eta}^k)}\, ds = L(\varepsilon)$$

where $\cdot$ denotes differentiation with respect to $s$.

We introduce

Definition 8: A curve $u^i(s)$, $0 \leq s \leq L$ connecting two points $u^i(0) = u^i(1)$ and $u^i(L) = u^i(2)$ will be called geodesic if, for all functions $\eta^i(s)$ with $\eta^i(0) = \eta^i(L) = 0$,

$$\frac{d}{d\varepsilon} \int_0^L \sqrt{G_{ik}(\varepsilon)(\dot{u}^i + \varepsilon\dot{\eta}^i)(\dot{u}^k + \varepsilon\dot{\eta}^k)}\, ds\Big|_{\varepsilon=0} = L'(0) = 0.$$

Expanding in powers of $\varepsilon$ we get

$$(52)\quad \begin{aligned} G_{ik}(\varepsilon)(\dot{u}^i + \varepsilon\dot{\eta}^i)(\dot{u}^k + \varepsilon\dot{\eta}^k) &= 1 + \varepsilon(g_{ik|j}\eta^j\dot{u}^i\dot{u}^k + g_{ik}\dot{\eta}^i\dot{u}^k + g_{ik}\dot{\eta}^k\dot{u}^i) + \cdots \\ &= 1 + \varepsilon(g_{ik|j}\eta^j\dot{u}^i\dot{u}^k + 2g_{ik}\dot{\eta}^i\dot{u}^k) + \cdots. \end{aligned}$$

From (51) we therefore obtain

$$(53)\qquad L'(0) = \frac{1}{2}\int_0^L [g_{ik|j}\dot{u}^i\dot{u}^k\eta^j + 2g_{ik}\dot{\eta}^i\dot{u}^k]ds.$$

Integration by parts leads to

$$(54)\qquad \int_0^L g_{ik}\dot{\eta}^i\dot{u}^k ds = -\int_0^L \eta^i \frac{d}{ds}(g_{ik}\dot{u}^k)ds$$

where the last integrand can be expressed as

$$(55)\qquad \eta^i \frac{d}{ds} g_{ik}\dot{u}^k = (g_{ik|j}\dot{u}^j\dot{u}^k + g_{ik}\ddot{u}^k)\eta^i$$

which by changing the notation for the indices gives

$$(56)\qquad (g_{jk|i}\dot{u}^i\dot{u}^k + g_{jk}\ddot{u}^k)\eta^j = (\tfrac{1}{2}\, g_{jk|i}\dot{u}^i\dot{u}^k + \tfrac{1}{2}\, g_{ji|k}\dot{u}^i\dot{u}^k + g_{jk}\ddot{u}^k)\eta^j.$$

The integral (53) may thus be expressed as

(57)
$$-\frac{1}{2}\int_0^L \eta^j[2\ddot{u}^k g_{jk} + (g_{jk|i} + g_{ji|k} - g_{ik|j})\dot{u}^i\dot{u}^k]ds$$
$$= -\int_0^L \eta^j[\ddot{u}^r + \Gamma^r_{ik}\dot{u}^i\dot{u}^k]g_{rj}\, ds.$$

If this shall vanish for all $\eta^j(s)$, we must have

(58) $$\ddot{u}^r + \Gamma^r_{ik}\dot{u}^i\dot{u}^k = 0.$$

This gives us

Theorem 2: Let $u^i(s)$ represent a curve so that

$$g_{ik}\dot{u}^i\dot{u}^k = 1.$$

Then this curve is a geodesic if and only if

$$\ddot{u}^r + \Gamma^r_{ik}\dot{u}^i\dot{u}^k = 0$$

is satisfied for all $s$.

If our surface element is part of a plane, it is well known that the geodesics are the straight lines. The new concept may therefore be regarded as an extension of this fundamental idea to curved surfaces. So far we have only generalized the property that the straight line is the shortest connection between two points. There are, however, two more characteristic properties of these curves which we shall generalize subsequently. To this end we recall that the straight line is a curve with zero curvature, or alternately the direction of the tangent does not change, which means that the tangent vector is constant.

In elementary differential geometry the curvature is defined as a measure of the change of the direction of the tangent, so that the straight line, which does not alter its direction, obviously has zero curvature. Here the value of the curvature of the straight line is an immediate consequence of the property that the direction of the tangent is constant. This, however, requires the concept of parallelism which we have not yet formulated on a surface. Here it is easier first to generalize a formalism for the curvature and then to define parallelism.

We start by considering a curve given in the form $u^i(s)$ with $s$ as a natural parameter. Then

(59) $$g_{ik}\dot{u}^i\dot{u}^k = 1.$$

The quantities $\dot{u}^i = t^i$ may be regarded as the components of the tangent vector. Differentiation of (59) gives

(60) $$g_{ik|j}\dot{u}^i\dot{u}^k\dot{u}^j + g_{ik}\dot{u}^i\ddot{u}^k + g_{ik}\ddot{u}^i\dot{u}^k = 0.$$

This may be written as

(61) $$\dot{u}^j(2g_{jk}\ddot{u}^k + g_{ik|j}\dot{u}^i\dot{u}^k) = 0.$$

From Lemma 3 we have

(62) $$g_{ik|j} = \Gamma^s_{ji}g_{sk} + \Gamma^s_{jk}g_{is}$$

which gives

(63) $$\dot{u}^j(2g_{jk}\ddot{u}^k + \Gamma^s_{ji}g_{sk}\dot{u}^i\dot{u}^k + \Gamma^s_{jk}g_{si}\dot{u}^i\dot{u}^k) = 0.$$

All of the indices are summation indices so that we may change the notation in each of the three terms of (63). In this way we obtain

(64) $$2\dot{u}^j g_{jk}(\ddot{u}^k + \Gamma^k_{si}\dot{u}^s\dot{u}^i) = 0.$$

The vector with the components

(65) $$\ddot{u}^k + \Gamma^k_{si}\dot{u}^s\dot{u}^i$$

thus is orthogonal to the tangential vector in the sense of our metric.

Denote by $t^k = \dot{u}^k$ the components of the tangential vector. Apart from the sign, the components of the normal vector $n^k$ may then be determined by the conditions

(66) $$n^i t^k g_{ik} = n_i t^i = n^i t_i = 0$$
$$n^i n^k g_{ik} = n_i n^i = n_i n_k g^{ik} = 1$$

where we have used the abbreviations

(67) $$n_i = g_{ij}n^j; \; t_i = g_{ij}t^j.$$

Observing

(68) $$g^{11} = \frac{g_{22}}{g}; \; g^{12} = g^{21} = -\frac{g_{12}}{g} = -\frac{g_{21}}{g}; \; g^{22} = \frac{g_{11}}{g}; \; g = \det(g_{ik})$$

we see from the first condition in (66) that

(69) $$n_1 = \lambda t^2; \; n_2 = -\lambda t^1$$

so that

(70) $$n_i n_k g^{ik} = \lambda^2 t^i t^k g_{ik} \cdot \frac{1}{g} = \frac{\lambda^2}{g}$$

which gives us

(71) $$n_1 = \sqrt{g}\ t^2;\ n_2 = -\ \sqrt{g}\ t^1;\ n^i = g^{ik}n_k.$$

It follows from (64) that the vector (65) is proportional to the normal vector. The factor of proportionality may now be interpreted as the curvature of the curve. This gives

(72) $$\dot{t}^k + \Gamma^k_{si}t^s t^i = \kappa n^k$$

with

(73) $$\kappa = (\dot{t}^k + \Gamma^k_{si}t^s t^i)n_k.$$

This leads to

Definition 9: Let: $u^i(s)$ be the representation of a curve with $s$ a natural parameter;

$$g_{ik}\dot{u}^i\dot{u}^k = 1;$$

$\dot{u}^i = t^i$ components of the tangential vector; and

$$n_1 = \sqrt{g}\ t^2;\ n_2 = -\ \sqrt{g}\ t^1$$

components of the normal vector. Then

$$\kappa = (\dot{t}^k + \Gamma^k_{si}t^s t^i)n_k$$

will be called the geodesic curvature.

Comparing this definition with Theorem 2, we get

Theorem 3: The geodesic curvature of a geodesic is zero.

The geodesics may thus again be regarded as an extension of the concepts of a straight line to general surfaces.

Our approach to the concept of the curvature is a complete analogy to the usual procedure in plane euclidean geometry if we understand that the metrical properties are not those of euclidean space but are determined by the quadratic form of the first kind.

If we interpret our metric as the metric of a surface in 3-dimensional space, we may regard the curve as a curve in 3-dimensional space. This curve has a curvature in the ordinary sense, which is not like our geodesic curvature. To understand this, we go back to our representation of the surface element. We had

$$\vec{x} = \vec{x}(u^1, u^2). \tag{74}$$

The vectors $\vec{x}_1, \vec{x}_2, \vec{n}$ form a linearly independent system at every point of the surface. Therefore, we get

$$\vec{x}_{ki} = \frac{\partial^2}{\partial u^k \partial u^i} \vec{x} = \Lambda^r_{ik} \vec{x}_r + L_{ik} \vec{n}, \tag{75}$$

with the coefficients $\Lambda^r_{ik}$ still to be determined.

We differentiate the definition

$$g_{ik} = \vec{x}_i \cdot \vec{x}_k \tag{76}$$

to obtain

$$g_{ik|j} = \vec{x}_i \vec{x}_{kj} + \vec{x}_{ij} \vec{x}_k = \Lambda^r_{jk} g_{ri} + \Lambda^r_{ji} g_{rk} \tag{77}$$

which gives

$$g_{ik|j} - g_{ij|k} - g_{kj|i} = (\Lambda^r_{jk} - \Lambda^r_{kj}) g_{ri} + (\Lambda^r_{ji} - \Lambda^r_{ij}) g_{rk} - (\Lambda^r_{ki} + \Lambda^r_{ik}) g_{rj}. \tag{78}$$

It follows fr

(79) $\Lambda^{r}_{jk} = \Lambda^{r}_{k}$

so that (78) reduc

(80) $\Lambda^{r}_{ik} g_{rj} =$

which gives

The definition of the geodesic curvature enabled us to characterize the geodesics as the curves of zero curvature. It will now be our aim to show that the geodesics are those curves which have parallel tangents at all points. This property requires a study of the idea of parallelism which was undertaken much later than the preceding ideas. It was Levi-Civita who first introduced the following concept of parallelism.

We start by observing that at every point of one surface element we may define the space of the vectors in the plane tangential at that point. This space will be described by all pairs $v^1, v^2$ which will be called the components of the vector. The length of the vector is then given by

(85) $$g_{ik}v^i v^k$$

and the angle between two vectors $v^i, w^i$ may be defined as

(86) $$\cos\alpha = \frac{g_{ik}v^i w^k}{\sqrt{g_{ik}v^i v^k}\sqrt{g_{ik}w^i w^k}} .$$

It is easy to see that the relations

(87) $$v_i = g_{ik}v^k \quad \text{and} \quad v^k = g^{ki}v_i$$

are reciprocal, so that we may express a vector either by the components $v^i$ or the components $v_i$. The $v^i$ will be called the covariant and the $v_i$ the contravariant components of a vector.

We now want to introduce a process which makes it possible to shift a vector parallel along a curve which we assume to be given in a natural parameter representation $u^i(s)$. The process of parallel shift may then be described as a law defining the components of the vector along the curve.

Suppose we start our process at the point $s = 0$, the components of our vector being $v_0^i$ at this point. As we shift the vector along the curve the components will be

$$(88) \qquad v^i(s) = f^i(s;\ v_0^1, v_0^2)$$

where this notation expresses the fact that the components $v^i(s)$ not only depend on the parameter $s$ but also on the initial values $v_0^1, v_0^2$. We now require that the linear vector space at $s = 0$ be transformed into a linear vector space so that a linear combination $\alpha v_0^i + \beta w_0^i$ of two vectors at $s = 0$ is transformed into $\alpha v^i(s) + \beta w^i(s)$. This means that our law (88) has to satisfy

$$(89) \qquad f^i(s;\ \alpha v_0^1 + \beta w_0^1,\ \alpha v_0^2 + \beta w_0^2) = \alpha f^i(s;\ v_0^1, v_0^2) + \beta f^i(s;\ w_0^1, w_0^2)$$

for all vectors $v_0^i, w_0^i$ and all scalars $\alpha, \beta$. Considering the special values $(v_0^1, v_0^2) = (1,0)$; $(w_0^1, w_0^2) = (0,1)$, we derive from (89)

$$(90) \qquad f^i(s;\ \alpha, \beta) = \alpha f^i(s;\ 1,0) + \beta f^i(s;\ 0,1)$$

so that $f^i(s;\ v_0^1, v_0^2)$ depends linearly on the components $v_0^i$. This gives us an expression of the form

$$(91) \qquad f^i(s;\ v_0^1, v_0^2) = P_r^i(s) v_0^r = v^i(s).$$

We now require that the process may be inverted so that the matrix $P_r^i(s)$ never degenerates. If we denote the inverse by $Q_r^i(s)$, we have

(92) $$P^i_j(s)\,Q^j_k(s) = \delta^i_k$$

and

(93) $$v^i_0 = Q^i_j(s)\,v^j(s).$$

From (91) we get by differentiation

(94) $$\dot v^i(s) = \dot P^i_r(s)\,v^r_0 = \dot P^i_r(s)\,Q^r_j(s)\,v^j(s)$$

so that the parallel shift is described by a system of linear homogeneous equations of the first order. Writing as an abbreviation

(95) $$\dot v^i(s) = M^i_j(s)\,v^j(s)$$

we now require that the length of the vector be maintained. This means

(96) $$\frac{d}{ds}\, v^i v^k g_{ik} = 2\dot v^i v^k g_{ik} + v^i v^k g_{ik|j}\dot u^j = 0.$$

From Lemma 3 and (96) we get, observing that all the indices here are summation indices,

(97) $$\begin{aligned} 0 &= v^k[(v^i\Gamma^s_{kj}g_{is} + v^i\Gamma^s_{ij}g_{sk})\dot u^j + 2g_{ik}M^i_j v^j] \\ &= 2v^i v^k g_{is}(\Gamma^s_{kj}\dot u^j + M^s_k). \end{aligned}$$

This condition must be satisfied for all initial vectors and all values of $s$. It follows from (93) that $v^i(s)$ may assume any values by suitable choice of $v^i_0$. Therefore **we require**

(98) $$g_{is}(\Gamma^s_{kj}\dot u^j + M^s_k) = 0$$

which gives by multiplication with $g^{ir}$

(99) $$\delta^r_s(\Gamma^s_{kj}\dot{u}^j + M^s_k) = \Gamma^r_{kj}\dot{u}^j + M^r_k = 0.$$

We are thus led to define the process of parallel shift in the following way.

Definition 10: Suppose $u^i(s)$ is the parameter representation of a curve. Then a vector $v^i(s)$ will be said to be shifted parallel along the curve if it satisfies the differential equation

(100) $$\dot{v}^i + \Gamma^i_{rs}\dot{u}^r v^s = 0.$$

Two postulates imply this definition:

1) Parallel shift is a differentiable process of linear, isometric transformations.

2) Tangential vectors of a geodesic are parallel in the sense of the shift.

The first postulate leads to (97). Theorem 2 then shows that (98) is a consequence of the second postulate because the components $t^i$ of the tangential vector of a geodesic satisfy ($t^i = \dot{u}^i$)

(101) $$\dot{t}^i + T^i_{rs}t^r t^s = 0.$$

This gives us

Theorem 4: The tangential vectors $u^i = t^i$ of a geodesic may be regarded as shifted parallel along the curve.

In euclidean geometry we know that the process of parallel shift is independent of the path along which it is performed. We now want to show that this is not true on our surfaces, and that the curvature $K$ may be regarded as a measure of the deviation from euclidean character.

Before we can prove this we have to introduce the basic ideas of tensor analysis, which are in essence extensions and generalizations of ideas that we have already used.

We know that the only quantities that are geometrically relevant are those which do not depend on the system of paremeters we use. However many of our relations may be difficult to express in terms of these invariant quantities. It will prove to be sufficient if our quantities transform in a known way. This is more precisely formulated using the concept of the tensors.

<u>Definition 11:</u> A system of quantities $T^{i_1 \ldots i_k \ldots\ldots}_{\ldots\ldots j_1 \ldots j_e}$ will be called a tensor with $k$ covariant and $e$ contravariant indices if it transforms under a change of parameter $u^i = u^i(v^1, v^2)$; $v^i = v^i(u^1, u^2)$ according to the law

$$(102) \quad \overset{*}{T}{}^{i_1 \ldots i_k \ldots\ldots}_{\ldots\ldots j_1 \ldots j_e} = T^{s_1 \ldots s_k \ldots\ldots}_{\ldots\ldots t_1 \ldots t_e} \frac{\partial v^{i_1}}{\partial u^{s_1}} \cdots \frac{\partial v^{i_k}}{\partial u^{s_k}} \frac{\partial u^{t_1}}{\partial v^{j_1}} \cdots \frac{\partial u^{t_e}}{\partial v^{j_e}}$$

where $\overset{*}{T}$ denotes the components in the v-system and $T$ stands for the components in the u-system. The number of indices, i.e., $k + e$ is called the rank of the tensor. In particular the invariants defined earlier are tensors of rank 0 and the vectors are tensors of rank 1.

The two tensors we have introduced so far are the coefficients $g_{ik}$ and $L_{ik}$ of the fundamental form. The coefficients $g_{ik}$ and $g^{ik}$ are basic for tensor algebra of which we shall introduce here explicitly only the most important operations. They are all easily justified as tensor operations.

1. Addition and Subtraction

$$(103)\qquad T^{i_1 \cdots i_k \cdots\cdots}{}_{\cdots\cdots j_1 \cdots j_e} \pm S^{p_1 \cdots p_m \cdots\cdots}{}_{\cdots\cdots q_1 \cdots q_n} .$$

2. Multiplication

$$(104)\qquad T^{i_1 \cdots i_k \cdots\cdots}{}_{\cdots\cdots j_1 \cdots j_e} \cdot S^{p_1 \cdots p_m \cdots\cdots}{}_{\cdots\cdots q_1 \cdots q_n}$$

which give tensors of rank $(k + e) + (m + n)$ and $(k + e)(m + n)$.

3. Lowering of an upper index

$$(105)\qquad T_{i_1}{}^{\cdots i_k \cdots\cdots}{}_{\cdots j_1 \cdots j_e} = g_{i_1 s} T^{s \cdots i_k \cdots\cdots}{}_{\cdots\cdots j_1 \cdots j_e} .$$

4. Raising of a lower index

$$(106)\qquad T^{i_1 \cdots i_k j_1 \cdots\cdots}{}_{\cdots\cdots\cdots\cdots j_e} = g^{j_1 s}\, T^{i_1 \cdots i_k}{}_{\cdots\cdots s \cdots j_e} .$$

5. Reduction

$$(107)\qquad T^{i_1 . s . i_k \cdots\cdots}{}_{\cdots\cdots j_1 . s . j_k} .$$

Although the processes of raising and lowering an index do not change the rank of the tensor, the process of reduction reduces it by two. This last operation is particularly important if we want to form invariants as we did when we determined the curvatures.

From the definition of a tensor we deduce

<u>Lemma 5:</u> If two tensors have the same values in one system of parameters

at one point, they have this same property in all systems for this point. If all the components of a tensor are zero in one system, they are zero in all systems.

We shall now discuss the Christoffel symbols $\Gamma^r_{ik}$ which we found to be important for our extension of the concepts of plane euclidean geometry to curved spaces. They are not tensors. To see this we recall first that the length of an arc is an invariant as can be seen immediately from the integral

(108) $$s = \int_{t_1}^{t_1} \sqrt{g_{ik}\dot{u}^i\dot{u}^k}\, dt.$$

The differential equation of a geodesic is

(109) $$\ddot{u}^i + \Gamma^i_{rk}\dot{u}^k\dot{u}^r = 0,$$

or

(110) $$\ddot{w}^i + \overset{*}{\Gamma}{}^i_{rk}\dot{w}^r\dot{w}^k = 0$$

if we use a different system $w^i$ of parameters. We have

$$\dot{w}^i = \frac{\partial w^i}{\partial u^k}\dot{u}^k; \quad \ddot{w}^i = \frac{\partial w^i}{\partial u^k}\ddot{u}^k + \frac{\partial^2 w^i}{\partial u^i \partial u^j}\dot{u}^i\dot{u}^k; \quad \frac{\partial w^i}{\partial u^j}\frac{\partial u^j}{\partial w^k} = \delta^i_k$$

so that (110) becomes

(112) $$\frac{\partial w^i}{\partial u^j}\ddot{u}^j + \left(\overset{*}{\Gamma}{}^i_{rk}\frac{\partial w^r}{\partial u^j}\frac{\partial w^k}{\partial u^s} + \frac{\partial^2 w^i}{\partial u^j \partial u^s}\right)\dot{u}^j\dot{u}^s = 0$$

which gives, after multiplication with $\frac{\partial u^t}{\partial w^i}$,

(113) $$\ddot{u}^t + \frac{\partial u^t}{\partial w^i}\left(\overset{*}{\Gamma}{}^i_{rk}\frac{\partial w^r}{\partial u^j}\frac{\partial w^k}{\partial u^s} + \frac{\partial^2 w^i}{\partial u^j \partial u^s}\right)\dot{u}^j\dot{u}^s = 0.$$

Comparison with (109) gives us the law of transformation for the $\Gamma^i_{rk}$

Lemma 6: Let $\overset{*}{\Gamma}{}^t_{js}$ be the Christoffel symbols in a w-system and $\Gamma^t_{sj}$ the corresponding symbols for a u-system, then

$$(114) \qquad \Gamma^t_{sj} = \frac{\partial u^t}{\partial w^i}\left(\overset{*}{\Gamma}{}^i_{rk}\,\frac{\partial w^r}{\partial u^j}\,\frac{\partial w^k}{\partial u^s} + \frac{\partial^2 w^i}{\partial u^j \partial u^s}\right).$$

This result shows that the quantities $\Gamma^t_{sj}$ do not have the properties of a tensor. However, their law of transformation enables us to derive some important relations. We want to find out if at a given point, we may introduce a system of coordinates with particularly simple properties. According to the law of tensor transformation it is always possible to introduce a system of coordinates by a linear transformation such that the point under consideration is represented by the values $u^i = 0$ and the coefficients $g_{ik}$ have the values $\delta_{ik}$ at that point. If we denote by $\overset{\circ}{=}$ a relation valid for the parameter $u^i = 0$, we have, with

$$(115) \qquad \begin{aligned} w^i &= u^i + \frac{1}{2}\overset{\circ}{\Gamma}{}^i_{js}u^j u^s \\ \Gamma^i_{js} &\overset{\circ}{=} \overset{\circ}{\Gamma}{}^i_{js}, \end{aligned}$$

a coordinate transformation which is regular in the neighborhood of the point $u^i = 0$ and which has the properties

$$(116) \qquad \frac{\partial w^i}{\partial u^j} \overset{\circ}{=} \delta^i_j ; \quad \frac{\partial u^i}{\partial w^j} \overset{\circ}{=} \delta^i_j .$$

We now get from Lemma 6

$$(117) \qquad \Gamma^t_{sj} = \frac{\partial u^t}{\partial w^i}\left(\overset{*}{\Gamma}{}^i_{rk}\,\frac{\partial w^r}{\partial u^s}\,\frac{\partial w^k}{\partial u^j} + \overset{\circ}{\Gamma}{}^i_{sj}\right)$$

so that

(118) $$\overset{*}{\Gamma}{}^t_{sj} \overset{\circ}{=} \overset{\circ}{\Gamma}{}^t_{sj} - \overset{\circ}{\Gamma}{}^t_{sj} = 0.$$

We may formulate this result as

<u>Lemma 7</u>: To any give point we may find a coordinate system such that this point is the origin, and the fundamental coefficients $g_{ik}$ and $\Gamma^i_{rk}$ satisfy

$$g_{ik} \overset{\circ}{=} \delta_{ik}; \ \Gamma^i_{kr} \overset{\circ}{=} 0.$$

We shall call this system of coordinates a Riemannian system for the point under consideration.

For the surfaces we have discussed so far, this result may be obtained by the intuitive geometrical reasoning that to every point of a surface element we may introduce a tangential-normal system with this point as origin. However, the virtue of Lemma 7 is that this result may be obtained by arguments which use only metric considerations of the surface and which do not require the concept of the 3-dimensional space in which this surface is embedded.

We see from Lemma 7 that $\Gamma^i_{rk} = 0$ implies

(119) $$g_{ik|j} \overset{\circ}{=} 0.$$

We now want to compare the derivatives of the components of a vector in two different systems. We have

(120) $$\frac{\partial \overset{*}{v}{}^i}{\partial w^j} = \frac{\partial}{\partial w^j} v^s \frac{\partial w^i}{\partial u^s} = \frac{\partial v^s}{\partial u^k} \frac{\partial w^i}{\partial u^s} \frac{\partial u^k}{\partial w^j} + v^r \frac{\partial u^k}{\partial w^j} \frac{\partial^2 w^i}{\partial u^r \partial u^k} .$$

This gives us with Lemma 6

$$\left(\frac{\partial v^s}{\partial u^k} + \Gamma^s_{kr} v^r\right)\frac{\partial w^i}{\partial u^s}\,\frac{\partial u^k}{\partial w^j} = \frac{\partial \overset{*}{v}{}^i}{\partial w^j} - v^r\,\frac{\partial u^k}{\partial w^j}\,\frac{\partial^2 w^i}{\partial u^r \partial u^k}$$

$$(121) \qquad + v^r\,\frac{\partial u^s}{\partial w^t}\left(\overset{*}{\Gamma}{}^t_{pq}\,\frac{\partial w^p}{\partial u^k}\,\frac{\partial w^q}{\partial u^r}\right)\frac{\partial w^i}{\partial u^s}\,\frac{\partial u^k}{\partial w^j}$$

$$+ v^r\,\frac{\partial u^s}{\partial w^t}\,\frac{\partial^2 w^t}{\partial u^r \partial u^k}\,\frac{\partial w^i}{\partial u^s}\,\frac{\partial u^k}{\partial w^j}\,.$$

Using the relations

$$(122) \qquad \frac{\partial u^j}{\partial w^s}\,\frac{\partial w^s}{\partial u^k} = \delta^j_k;\; \frac{\partial w^j}{\partial u^s}\,\frac{\partial u^s}{\partial w^k} = \delta^j_k;\; \overset{*}{v}{}^r = v^r\,\frac{\partial w^q}{\partial u^r}$$

we obtain for the right hand side of (121)

$$(123) \qquad \frac{\partial \overset{*}{v}{}^i}{\partial w^j} + \overset{*}{\Gamma}{}^i_{jq}\,\overset{*}{v}{}^q.$$

Our final result therefore is

$$(124) \qquad \left(\frac{\partial v^s}{\partial u^k} + \Gamma^s_{kr} v^r\right)\frac{\partial w^i}{\partial u^s}\,\frac{\partial u^k}{\partial w^j} = \frac{\partial \overset{*}{v}{}^i}{\partial w^j} + \overset{*}{\Gamma}{}^i_{jq}\,\overset{*}{v}{}^q.$$

By the same type of argument we obtain

$$(125) \qquad \left(\frac{\partial v_s}{\partial u^k} - \Gamma^r_{sk} v_r\right)\frac{\partial u^s}{\partial w^i}\,\frac{\partial u^k}{\partial w^j} = \frac{\partial \overset{*}{v}_i}{\partial w^j} - \Gamma^q_{ij}\,\overset{*}{v}_q.$$

This shows that the expressions

$$(126) \qquad \frac{\partial v^s}{\partial u^k} + \Gamma^s_{kr} v^r \quad \text{and} \quad \frac{\partial v_s}{\partial u^k} - \Gamma^r_{sk} v_r$$

transform as tensors. We use this to make

Definition 12 : The quantities

$$v^s_{\|k} = \frac{\partial v^s}{\partial u^k} + \Gamma^s_{kr} v^r$$

and

$$v_{s\|k} = \frac{\partial v_s}{\partial u^k} - \Gamma^r_{sk} v_r$$

will be called the covariant derivatives of the vector $v^s$ or $v_s$.

Comparing this definition with the concept of parallel shift we now obtain

Lemma 8 : The components $v^i$ of a vector field will be called parallel with regard to a curve $u^i(s)$ if the condition

$$v^i_{\|k} \dot{u}^k = 0$$

is satisfied.

Therefore, if $v^i_{\|k} = 0$ everywhere, we may regard $v^i$ as a parallel (or constant) vector field. It will be the main result of the next chapter to show that such vector fields do not exist in general.

Before we do this, however, we want to introduce the extension of the process of covariant differentiation to general tensors. The idea underlying these definitions is to find a process which may be regarded as the equivalent of differentiation in ordinary euclidean space because it reduces to this process in a euclidean metric, and which transforms as a tensor. The verification of the tensor properties is omitted here, as it only uses the transformation properties of the $\Gamma^i_{rk}$ and is otherwise a complete analogy to the considerations of (120) - (125).

Definition 13: The quantities

$$(T^{i_1 . s . i_k \cdots\cdots}_{\cdots\cdots j_1 \cdots j_e})_{\parallel m} = \frac{\partial}{\partial u^m} T^{i_1 \cdots i_k \cdots\cdots}_{\cdots\cdots j_1 \cdots j_e}$$

$$+ \Gamma^{i_1}_{ms} T^{s \cdots i_k \cdots\cdots}_{\cdots\cdots j_1 \cdots j_e} + \cdots + \Gamma^{i_k}_{ms} T^{i_1 \cdots s \cdots\cdots}_{\cdots\cdots j_1 \cdots j_e}$$

$$- \Gamma^{s}_{mj_1} T^{i_1 \cdots i_k \cdots\cdots}_{\cdots\cdots s \cdots j_e} - \cdots - \Gamma^{s}_{mj_e} T^{i_1 \cdots i_k \cdots\cdots}_{\cdots\cdots j \cdots s}$$

will be called the covariant derivative of the tensor.

From Lemma 8 we now obtain

Lemma 9: $g_{ik\parallel j} = 0$; $g^{ik}_{\parallel j} = 0$.

The coefficients of the metric may thus be regarded as constant with respect to this process of differentiation. Furthermore we know from Lemma 7 that to every point we may introduce a system of parameters such that the $\Gamma^i_{kr}$ vanish in that point. In this case the covariant derivatives reduce to the ordinary derivatives. This gives us, for example,

$$(v^i w_k)_{\parallel j} \stackrel{\circ}{=} \frac{\partial}{\partial u^j}(v^i w_k) \stackrel{\circ}{=} \frac{\partial v^i}{\partial u^j} w_k + v^i \frac{\partial w_k}{\partial u^j} \stackrel{\circ}{=} v^i_{\parallel j} w_k + v^i w_{k\parallel j}. \tag{127}$$

Both sides of this equation are tensors and they coincide in one system. According to Lemma 5 they therefore are identical and we have

$$(v^i w_k)_{\parallel j} = v^i_{\parallel j} w_k + v^i w_{k\parallel j}. \tag{128}$$

This example shows that the ordinary laws for differentiating sums and products may be carried over to the covariant derivatives of sums and products of tensors and vectors.

From Lemma 9 we now get

$$(129)\qquad \begin{aligned} (g_{ik}v^k)_{\|j} &= g_{ik\|j}v^k + g_{ik}v^k_{\|j} = g_{ik}v^k_{\|j} \\ (g^{ik}v_k)_{\|j} &= g^{ik}_{\|j}v_k + g^{ik}v_{k\|j} = g^{ik}v_{k\|j} \end{aligned}$$

so that the operations of lowering and raising an index commute with the covariant derivation.

We now want to calculate the tensor

$$(130)\qquad (v_{i\|k})_{\|j} - (v_{i\|j})_{\|k}.$$

In order to simplify the calculation we work with a Riemannian system of coordinates. We then obtain

$$(131)\qquad \begin{aligned} (v_{i\|k})_{\|j} - (v_{i\|j})_{\|k} &\overset{\circ}{=} \frac{\partial}{\partial u^j}(v_{i\|k}) - \frac{\partial}{\partial u^k}(v_{i\|j}) \\ &\overset{\circ}{=} \frac{\partial^2 v_i}{\partial u^j \partial u^k} - \frac{\partial}{\partial u^j}\Gamma^r_{ik}v_r - \frac{\partial^2 v_i}{\partial u^k \partial u^j} + \frac{\partial}{\partial u^k}\Gamma^r_{ij}v_r \\ &\overset{\circ}{=} v_r\left(\frac{\partial}{\partial u^k}\Gamma^r_{ij} - \frac{\partial}{\partial u^j}\Gamma^r_{ik}\right). \end{aligned}$$

To evaluate the term on the right hand side we go back to the representation of our surface in terms of the tangential-normal system. Then we get from Lemma 4

$$(132)\qquad \vec{x}_{ik} = \frac{\partial^2}{\partial x^i \partial x^k}\vec{x} = \Gamma^r_{ik}\vec{x}_r + L_{ik}\vec{n}$$

so that with

(133) $\vec{x} = x^i\vec{e}_i + F(x^1,x^2)\vec{e}_3$

we have

(134) $x_r = \vec{e}_r + F_r\vec{e}_3$; $\vec{n} = \dfrac{\vec{e}_3 - F_1\vec{e}_1 - F_2\vec{e}_2}{\sqrt{1 + F_1^2 + F_2^2}}$; $F(x^1,x^2) = \frac{1}{2}\mathring{L}_{ik}x^ix^k + \ldots$.

We now get from (132) and (133) for $r = 1,2$

(135) $\vec{e}_r \cdot \vec{x}_{ik} = \Gamma^r_{ik} + L_{ik}\vec{n}\,\vec{e}_r$.

The left hand side vanishes as $\vec{x}_{ik}$ is proportional to $\vec{e}_3$ so that

(136) $0 = \Gamma^r_{ik} - \mathring{L}_{ik}\mathring{L}_{sj}x^j\delta^{sr} + \ldots$

which gives us

(137) $\Gamma^r_{ik} = \mathring{L}_{ik}\mathring{L}_{sj}x^j\delta^{sr} + \ldots$

where $\mathring{L}_{ik}$ denotes the values of these quantities for $x^i = 0$. This gives us

(138) $\dfrac{\partial}{\partial x^j}\Gamma^r_{ik} \overset{\circ}{=} \mathring{L}_{ik}\mathring{L}_{sj}\delta^{sr}$

and we obtain

(139) $\dfrac{\partial}{\partial x^k}\Gamma^r_{ij} - \dfrac{\partial}{\partial x^j}\Gamma^r_{ik} \overset{\circ}{=} (\mathring{L}_{ij}\mathring{L}_{sk} - \mathring{L}_{ik}\mathring{L}_{sj})\delta^{sr}$.

Now the expression

(140) $\mathring{L}_{ij}\mathring{L}_{sk} - \mathring{L}_{ik}\mathring{L}_{sj}$

changes sign when $i$ and $s$ are interchanged. It suffices therefore to study

(141) $$\mathring{L}_{1j}\mathring{L}_{2k} - \mathring{L}_{1k}\mathring{L}_{2j}$$

which vanishes except when $j = 1, k = 2$. It then assumes the value

(142) $$\mathring{L}_{11}\mathring{L}_{22} - \mathring{L}_{12}\mathring{L}_{21} = \mathring{F}_{11}\mathring{F}_{22} - (\mathring{F}_{12})^2 = K$$

so that

(143) $$L_{ij}L_{sk} - L_{sk}L_{sj} \mathrel{\mathring{=}} K(g_{ij}g_{sk} - g_{ik}g_{sj}).$$

From Theorem 1 we now obtain

(144) $$L_{ij}L_{sk} - L_{sk}L_{sj} = K(g_{ij}g_{sk} - g_{ik}g_{sj})$$

which gives us

(145) $$\begin{aligned} v_{i\|k\|j} - v_{i\|j\|k} &\mathrel{\mathring{=}} K(g_{ij}g_{sk} - g_{ik}g_{sj})g^{sr}v_r \\ &\mathrel{\mathring{=}} K(g_{ij}g_{sk} - g_{ik}g_{sj})v^s. \end{aligned}$$

This leads us to

<u>Theorem 5</u>: For any vector field $v_i$

$$v_{i\|k\|j} - v_{i\|j\|k} = K(g_{ij}v_k - g_{ik}v_j)$$

and

$$v^i_{\|k\|j} - v^i_{\|j\|k} = K(\delta^i_j v_k - \delta^i_k v_j).$$

The last equation was obtained by raising the index $i$ since this process commutes with the covariant differentiation.

An immediate application of this result may be obtained in the following way. Suppose $p^i$ are the components of a vector of unit length. We then have

$$(146)\qquad p_i p^i = 1;\; p_{i\|j}p^i + p_i p^i_{\|j} = 2p_i p^i_{\|j} = 0$$

where the last equation is obtained by differentiating the first one. This last equation shows that the determinant

$$(147)\qquad p^1_{\|1}p^2_{\|2} - p^1_{\|2}p^2_{\|1} = 0 .$$

We therefore obtain

$$(148)\qquad \begin{aligned}(p^i p^j_{\|j} - p^j p^i_{\|j})_{\|i} &= p^i_{\|i}p^j_{\|j} - p^j_{\|i}p^i_{\|j} + p^i p^j_{\|j\|i} - p^j p^i_{\|j\|i}\\ &= p^1_{\|1}p^2_{\|2} - p^1_{\|2}p^2_{\|1} + p^i(p^j_{\|j\|i} - p^j_{\|i\|j})\\ &= p^i K(\delta^j_i p_j - \delta^j_j p_i) = K(p^i p_i - 2p^i p_i) = -K\end{aligned}$$

which gives us

<u>Lemma 10</u>: Suppose $p^i$ is a vector field of unit length, then

$$(p^i p^j_{\|j} - p^j p^i_{\|j})_{\|i} = -K.$$

Since the vector field $p^i$ may be regarded as an arbitrary field of directions this reveals a surprising fact first discovered by Gauss.

<u>Theorem 6 (Theorem egregium)</u>: The curvature $K$ may be calculated in terms of only the coefficients $g_{ik}$.

It is therefore connected with the intrinsic metric properties of the surface, and further it must be possible to determine this quantity simply by measuring lengths and angles. Since the determination and description of the geometrical properties of the space we live in is done using solely the measurements of distances and angles, the problem of the nature of space was thus opened.

In order to find direct geometric interpretations of this result we have to introduce the concept of the measure of the area. This we do by means of the orientated differential element

$$dS = \gamma(du^1, du^2)$$

with

$$\gamma^2 = g = g_{11}g_{22} - g_{12}^2 = \det|g_{ik}| .$$

It is easy to verify by direct calculation

$$\overset{*}{g} = g\left|\frac{\partial(u^1,u^2)}{\partial(v^1,v^2)}\right|^2 \quad \text{and} \quad \sqrt{g}(du^1,du^2) = \pm\sqrt{\overset{*}{g}}(dv^1,dv^2) .$$

The ambiguity of the sign in the second equation reflects the fact that the orientation of the area may have been changed.

In order to conserve the orientation, we have to regard $\gamma$ as an independent quantity which satisfies $|\gamma| = \sqrt{g}$ and transforms according to

$$\overset{*}{\gamma} = \gamma \frac{\partial(v^1,v^2)}{\partial(u^1,u^2)} . \tag{149}$$

Suppose now that $S_{ik} = -S_{ki}$ is a skew symmetric tensor. Then

$$\overset{*}{S}_{12} = S_{ik}\frac{\partial u^i}{\partial v^1}\frac{\partial u^k}{\partial v^2} = S_{12}\left(\frac{\partial u^1}{\partial v^1}\frac{\partial u^2}{\partial v^2} - \frac{\partial u^2}{\partial v^1}\frac{\partial u^1}{\partial v^2}\right) \tag{150}$$

so that

$$(151) \qquad \frac{\overset{*}{S}_{12}}{\overset{*}{\Upsilon}} = \frac{S_{12}}{\Upsilon}$$

is an invariant.

Before we proceed to discuss the situation in a 3-dimensional space or in the 4-dimensional space-time continuum, let us discuss the importance of this discovery in 2-dimensional space where the geometric intuition may be a great help.

## III Riemannian Geometry

We shall now discuss the basic problems of a general metric geometry, as they were first formulated by B. Riemann. Beginning with the two-dimensional case, we first express those properties of the curvature $K$ which relate it to the generalized concept of parallelism. Starting with

(1) $$q^j_{\|i\|k} - q^j_{\|k\|i} = K(\delta^j_k q_i - \delta^j_i q_k)$$

we obtain for any two vector fields $p^i, q^i$

(2) $$p_j(q^j_{\|i\|k} - q^j_{\|k\|i}) = K(p_k q_i - p_i q_k).$$

Suppose now that these vector fields satisfy

(3) $$p_j p^j = 1; \; q_j q^j = 1; \; p_j q^j = q_j p^j = 0,$$

and introduce

(4) $$v_i = p_j q^j_{\|i}.$$

Then

(5) $$v_{i\|k} - v_{k\|i} = v_{i|k} - v_{k|i} = p_{j\|k} q^j_{\|i} - p_{j\|i} q^j_{\|k} + K(p_k q_i - p_i q_k).$$

We want to show that

(6) $$p_{j\|k} q^j_{\|i} - p_{j\|i} q^j_{\|k} = 0.$$

To this end we assume that at a given point we have a system of coordinates with the following properties

(7)
$$g_{ik} \overset{\circ}{=} \delta_{ik}; \; \Gamma^{r}_{ik} \overset{\circ}{=} 0$$
$$p^1 \overset{\circ}{=} p_1 \overset{\circ}{=} 1; \; p^2 \overset{\circ}{=} p_2 \overset{\circ}{=} 0$$
$$q^1 \overset{\circ}{=} q_1 \overset{\circ}{=} 0; \; q^2 \overset{\circ}{=} q_2 \overset{\circ}{=} 1.$$

Differentiating (3) we then obtain

(8)
$$p_j p^j_{\,||k} = 0 \overset{\circ}{=} p^1_{\,||k} \overset{\circ}{=} p_{1\,||k}$$
$$q_j q^j_{\,||k} = 0 \overset{\circ}{=} q^2_{\,||k} \overset{\circ}{=} q_{2\,||k}$$

so that

(9) $$p_{j\,||i}\, q^j_{\,||k} \overset{\circ}{=} p_{2\,||i}\, q^2_{\,||k} \overset{\circ}{=} 0,$$

which gives us

<u>Lemma 11</u>: Suppose $p^i, q^i$ represent two vector fields with $p_i p^i = 1$ and $q_i q^i = 1$, $p^i q_i = p_i q^i = 0$. Then with

$$v_i = p_j q^j_{\,||i}$$

we have

$$v_{i\,||k} - v_{k\,||i} = K(p_k q_i - p_i q_k).$$

If we now use the fact that the two vector fields are perpendicular, we obtain

(10) $$p_1 q_2 - p_2 q_1 \overset{\circ}{=} 1 \overset{\circ}{=} \sqrt{g}\ .$$

Since for any skew symmetric tensor $S_{ik}$, the expression

(11) $$\gamma^{-1}(S_{12} - S_{21})$$

is an invariant in the sense of (151), we get

$$v_{1||2} - v_{2||1} = K \cdot \gamma$$

which may be formulated as

Theorem 7: Suppose $p^i, q^i$ are vector fields of unit lengths, normalized such that

$$p_1 q_2 - p_2 q_1 = \gamma.$$

Then

$$\frac{\partial}{\partial u^2}(p_j q^j_{\ ||1}) - \frac{\partial}{\partial u^1}(p_j q^j_{\ ||2}) = K\gamma.$$

Suppose now that $C$ is the boundary of a simply connected domain $G$ measured in such a way that the parameters $s = 0$ and $s = L$ correspond to the same point $P$. We then obtain

(12) $$\iint_G K\gamma\,(du^1, du^2) = -\oint_C [p_j q^j_{\ ||i}\, du^i] = -\int_0^L p_j q^j_{\ ||i}\, \dot{u}^i ds.$$

We now introduce two vectors $r^i, t^i$ by parallel shift around $C$. Assuming that they are orthonormalized at $P$ they will retain this property along $C$. However, in general we cannot assume that they return to the same values after their shift around the curve and we must expect that they will have been rotated by a certain angle. The determination of this angle is the purpose of the following considerations.

Along $C$ we have

$$p_i = r_i \cos\alpha + t_i \sin\alpha ;\ p^i = r^i \cos\alpha + t^i \sin\alpha$$

(13) $$q_i = - r_i \sin\alpha + t_i \cos\alpha ;\ q^i = - r^i \sin\alpha + t^i \cos\alpha$$

$$\dot{r}^i + \Gamma^i_{ks} r^k \dot{u}^s = 0;\ \dot{t}^j + \Gamma^i_{ks} t^k \dot{u}^s = 0.$$

It is easily verified with the above equations that

(14) $$\dot{q}^j = - \dot{\alpha} p^j - \Gamma^j_{ks} q^k \dot{u}^s .$$

Furthermore,

(15) $$p_j q^j_{\,||i}\, \dot{u}^i = p_j \left(\frac{\partial q^j}{\partial u^i} \frac{du^i}{ds} + \Gamma^j_{ik} q^k \dot{u}^i\right) = p_j (\dot{q}^j + \Gamma^j_{ik} q^k \dot{u}^i).$$

Combining these two equations we get

(16) $$p_j q^j_{\,||i}\, \dot{u}^i = - \dot{\alpha} p_j p^j = - \dot{\alpha}.$$

Thus we get from (12)

$$\int\!\!\int_G K \;\; \gamma(du^1, du^2) = \int\!\!\int_G K dS = \alpha(L) - \alpha(0) = \delta.$$

Let us now choose $r^i$ and $t^i$ so that

(17) $$r^i(0) = p^i(0);\ t^i(0) = q^i(0).$$

Then $\alpha(0) = 0$ and $\alpha(L) = \delta$. Because $p^i(L) = p^i(0)$ and $q^i(L) = q^i(0)$, we obtain from (13)

(18) $$r^i(L) = r^i(0) \cos\delta - t^i(0) \sin\delta$$
$$t^i(L) = r^i(0) \sin\delta + t^i(0) \cos\delta .$$

The vectors $r^i$ and $t^i$ therefore have been turned by the angle $\delta$ due to the shift around C. This gives us

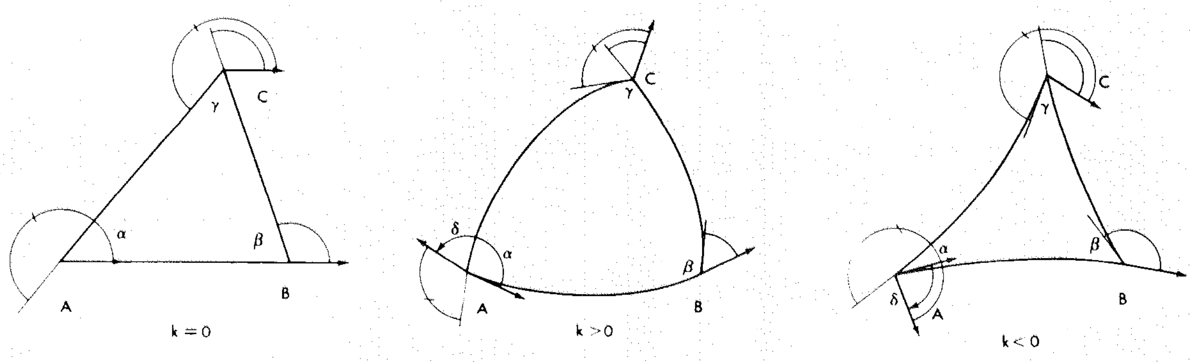
<u>Theorem 8</u>: Suppose C is the piecewise continuously differentiable boundary of a simply connected domain G. Then every vector shifted parallel around C in a positive sense will be turned through the angle $\delta$, where

$$\delta = \iint_G K dS$$

and dS is the surface differential of the element. This shows that for surfaces with $K \neq 0$, parallel shift is not integrable.

Consider now a figure consisting of three geodesics which we may regard as a generalized triangle. A vector shifted parallel around the sides of this figure does not, in general, return to the same position at the initial point and this difference is expressed by Theorem 8.

Referring to Figure 1 we now derive the law of the sum




a b c

Figure 1

of the angles in a triangle. Starting at the point A the vector may be in the same direction as the tangent to the geodesic. At B the angle between the tangent to the next side and the vector will be $\pi - \beta$. At C the angle between the tangent to the third geodesic and the vector will be $\pi - \beta + \pi - \gamma = 2\pi - (\beta + \gamma)$. Coming back to A then brings the vector in a position which forms the angle $3\pi - (\alpha + \beta + \gamma)$ with the initial direction. Thus we see that the angle $\delta$ defined earlier is equal to

$$(19) \qquad \delta = 2\pi + (\alpha + \beta + \gamma) - 3\pi = (\alpha + \beta + \gamma) - \pi$$

and we obtain

Lemma 12: Suppose $\alpha, \beta, \gamma$ are the angles at the corners of a figure formed by three geodesics. Then

$$\alpha + \beta + \gamma - \pi = \iint K dS$$

where the integral is to be taken over the interior.

These last results were obtained without using the fact that the surface is embedded in a 3-dimensional space. The basis of the concepts was the fundamental metric tensor $g_{ik}$.

It is not difficult now to generalize these techniques to geometries of more than two dimensions. We start by assuming that a n-dimensional manifold may be used as a parameter representation. The metric properties of this manifold are then described by the law

$$(20) \qquad ds^2 = g_{ik} dx^i dx^k$$

for the line element. The tensor definition, the Christoffel symbols and the covariant differentiations may then be generalized by immediate formal considerations.

The curvatures, however, require a different treatment. Although there is no analogue to the tensor $L_{ik}$, the curvature $K$ may be recognized as a special case of a more general tensor. To understand this we assume the covariant derivatives to be introduced and consider

$$(21)\qquad v^i_{\|k\|j} - v^i_{\|j\|k}.$$

This obviously is a tensor which depends linearly on the components $v^i$. We express this by introducing Riemann's tensor of curvature by the definition

$$(22)\qquad v^i_{\|k\|j} - v^i_{\|j\|k} = R^i_{.sjk} v^s .$$

In a two-dimensional space this tensor reduces to

$$(23)\qquad R^i_{.sjk} = K(\delta^i_j g_{sk} - \delta^i_k g_{sj}) .$$

In Einstein's theory of gravitation the reduced tensors

$$(24)\qquad R_{ik} = R^s_{.isk};\ g^{ik} R_{ik} = R$$

are of great importance.

We shall go no further into the details of the n-dimensional Riemannian geometry. Instead, we study the application of these concepts in the theory of time and space.

## IV The Space-Time Continuum

In Einstein's theory of gravitation, the basis of all physical measurements is a four-dimensional metric continuum which can be constructed of space-time elements which are defined as follows:

Definition 14: A four-dimensional continuum is called a space-time element if it is an isomorphic image of the four-sphere **(ball) with the** metric fundamental form

$$ds^2 = g_{ik}dx^i dx^k$$

which is such that to every point $P$ of the set, a system of coordinates satisfying

$$g_{11} = -1;\ g_{22} = -1, g_{33} = -1;\ g_{44} = c^2;$$

$$g_{12} = g_{13} = g_{23} = g_{14} = g_{24} = g_{34} = 0;\ \Gamma^i_{kr} = 0$$

can be found. This system of coordinates will be called geodesic at $P$.

A vector $\xi^i$ will be called timelike, if

$$g_{ik}\xi^i\xi^k > 0$$

and spacelike if

$$g_{ik}\xi^i\xi^k < 0.$$

The vectors $\xi^i$ with

$$g_{ik}\xi^i\xi^k = 0$$

are called nullvectors.

It should be noted here that the situation is very similar to that for two-dimensional surfaces with the exception that the metric fundamental form is no longer positive definite. The exact requirements are specified by the above definition. These requirements mean that for every world point a system of coordinates can be found which has the properties of the reference system used in special relativity.

Special relativity differs from general relativity in that for special relativity a system of coordinates can be found which has the properties required in our definition, not only at one point but at all points. As this situation is very similar to the relation between a four-dimensional euclidean space and a general metric space of that dimension, we introduce

Definition 15: A space-time element will be called flat if it is possible to introduce a system of coordinates $x^1, x^2, x^3, x^4$ such that

$$ds^2 = c^2(dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

We now prove

Theorem 9: A space-time element with integrable parallelism is flat.

In order to prove the theorem we choose one point $P$ and consider parallel shifts starting from this point. These are described by the differential equations

$$\dot{v}^i + \Gamma^i_{rk} v^r \dot{x}^k = 0 \qquad (1)$$

and we may cnoose the vector at $P$ arbitrarily. Since the process of parallel shift is assumed to be integrable, the vector parallel to $v^i$ at any other point can be uniquely determined. The vector field thus defined is differentiable, and we have for every curve represented in the form $x^i(s)$

$$(2) \qquad \frac{dv^i}{ds} = \frac{\partial v^i}{\partial x^k}\frac{dx^k}{ds} = \frac{\partial v^i}{\partial x^k}\dot{x}^k = -\Gamma^i_{rk}v^r\dot{x}^k.$$

This last equation may be written

$$(3) \qquad \left(\frac{\partial v^i}{\partial x^k} + \Gamma^i_{kr}v^r\right)\dot{x}^k = 0.$$

Since the values $\dot{x}^k$ can be chosen arbitrarily, we obtain

$$(4) \qquad v^i_{\parallel k} = 0$$

which gives us

Lemma 13: A space-time continuum with integrable parallelism possesses 4 vector fields $p^i, q^i, r^i, s^i$ of which three are spacelike and one is timelike, which are mutually orthogonal and of constant length. Moreover, they satisfy

$$p^i_{\parallel k} = q^i_{\parallel k} = r^i_{\parallel k} = s^i_{\parallel k} = 0.$$

This is obtained by observing that four orthonormalized vectors can be defined at P which may then be shifted parallel. These vector fields may be regarded as constant vector fields which shows that the existence of a constant vector field and the integrability of parallelism are closely connected.

From Definition 12 we get

$$(5)\qquad \begin{aligned} p_{i\|k} - p_{k\|i} &= \frac{\partial p_i}{\partial x^k} - \frac{\partial p_k}{\partial x^i} = 0 \\ q_{i\|k} - q_{k\|i} &= \frac{\partial q_i}{\partial x^k} - \frac{\partial q_k}{\partial x^i} = 0 \end{aligned}$$

so that the integrals

$$(6)\qquad \int_P^Q p_i dx^i = \varphi(Q) : \int_P^Q q_i dx^i = \psi(Q)$$

are independent of the path. We thus get

$$(7)\qquad p_i = \frac{\partial \varphi}{\partial x^i};\ q_i = \frac{\partial \psi}{\partial x^i}.$$

Similarly $r^i$ and $s^i$ may be derived from potentials. These potentials can now be introduced as coordinates. If $\varphi$ and $\psi$ are denoted by the indices '1' and '2', the corresponding components of the metric tensor may be calculated. They are

$$(8)\qquad \begin{aligned} \overset{*}{g}{}^{11} &= g^{ik} \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^k} = g^{ik} p_i p_k = -1 \\ \overset{*}{g}{}^{22} &= g^{ik} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^k} = g^{ik} q_i q_k = -1 \\ \overset{*}{g}{}^{12} = \overset{*}{g}{}^{21} &= g^{ik} \frac{\partial \varphi}{\partial x^i} \frac{\partial \psi}{\partial x^k} = g^{ik} p_i q_k = 0. \end{aligned}$$

The other components may be determined in a similar manner to complete the proof. Einstein's basic assumption on gravitation states that, due to gravitational masses, the space-time continuum is not flat. In this way fundamental physical and geometric concepts are connected. Einstein formulates the relation between the components $g_{ik}$ of the metric tensor and the components $T_{ik}$ of the energy-stress tensor in terms of the equation

$$(9) \qquad R_{ik} - \frac{1}{2} g_{ik} R = T_{ik}$$

which is regarded as a system of partial differential equations of the second order for the $g_{ik}$. As it is not our purpose here to discuss the validity of this equation or the general solution of it, we refer to the literature on these points.

We want to discuss the main conjecture on the curvature of the space-time continuum. To this end we have to simulate the basic geometric concepts by kinematic models. Just as we employ the test body to determine the forces in Newtonian mechanics, we introduce here the test system of the spinning mass point. The concept of the mass point was introduced by Einstein. However, this concept does not fully simulate the finer structure of the vector space connected with the point. This additional structure may be simulated by the model of a rotating mechanical system.

For the purpose of demonstrating the physical meaning we consider a mechanical system of a finite number of mass points under rigid motion. The coordinates will be determined in the form $x^i(s) + \xi^i(s)$ where the $x^i(s)$ represent the coordinates of the center, and the $\xi^i(s)$ denote the parameters with respect to the center. If we regard the $\xi^i$ as differentials, they transform like vectors, and we may use them to impose a finer structure on the mass point.

In order to avoid confusion we shall not distinguish in our notations between masses, the coordinates, and the velocities of the individual points with regard to the center. We first suppose

(10) $\Sigma m \xi^i = 0$

so that $x^i$ represents the center of gravity. Denoting the velocities with

(11) $u^i = \frac{d\xi^i}{ds}$,

we get from (10)

(12) $\Sigma m u^i = 0.$

We now want to formulate the laws for the movement of this system. Here we proceed as is done in Newtonian mechanics, by separating the description into a law for the center of gravity and a law for the movement of the spinning system. This means we must first describe the movement as a whole by a law for the center of gravity. Then we can describe the movement around the center in analogy to the theory of the spinning top.

The movement of the system is described by the path of the center which is a curve in our continuum. We shall call this curve the world line of our system. The tangential vector,

(13) $v^i(s) = \frac{dx^i}{ds}$; $v^i v^k g_{ik} = 1$,

is the vector of velocity.

We now return to the idea of the geodesic coordinate system. This coordinate system is not yet uniquely determined since a linear transformation of the form of the Lorentz transform can still be applied without changing the geodesic properties of the coordinate system. We specify that this undetermined linear transformation be chosen in such a way that the velocity vector $v^i$ be transformed into the unit vector in the direction of the $x^4$-axis, which, recall, is the time-axis. This means that we introduce a system of reference, relative to which the center of gravity is at rest.

It is Einstein's hypothesis that to a point moving under the influence of gravitation, a system of reference exists at every point of the world line such that the point seems to be moving uniformly in the system of reference. This is often expressed by saying that a system of reference can be found in such a way that the forces of gravitation disappear. For our purpose this means that

$$(14) \qquad \ddot{x}^i \overset{\circ}{=} 0$$

where $\overset{\circ}{=} 0$ expresses the equality at the point $P(x^i(s))$ which is the center of our geodesic system. In a general system of reference this requires that

$$(15) \qquad \ddot{x}^i + \Gamma^i_{rk}\dot{x}^r\dot{x}^k = 0.$$

This is the differential equation of a geodesic. The condition (14) is therefore satisfied at every point of the world line if this line is a geodesic.

We now have to clarify the meaning of rigid motion of our small system. Using the notation (11), we now require for the coordinates $\xi^i(x)$ and the velocities $u^i(x)$ of all the mass points of the system that in the coordinate system in which the center of gravity is at rest ,

$$u^4(x) = d; \; \xi^4(x) = \xi_4(x) = 0 \tag{16}$$

$$\xi^1(x)u_1(x) + \xi^2(x)u_2(x) + \xi^3(x)u_3(x) = 0.$$

Introducing the angular momentum

$$L_{ik} = \Sigma m(\xi_i u_k - \xi_k u_i) \tag{17}$$

we find

$$L_{ik} = - L_{ki} \tag{18}$$

and we get from (10) and (16)

$$L_{4k} = \Sigma m(\xi_4 u_k - \xi_k u_4) = - d\Sigma m\xi_k = 0. \tag{19}$$

The second of the conditions (16) is satisfied if we introduce the skew-symmetric tensor $\alpha_{ik}$ of the angular velocity by means of the substitution

$$u_i = \alpha_{ik}\xi^k; \; \alpha_{4i} = 0; \; \alpha_{ik} = - \alpha_{ki}. \tag{20}$$

From (17) we now obtain

$$L_{ik} = \Sigma m(\xi_i \alpha_{kr}\xi^r - \xi_k \alpha_{ir}\xi^r). \tag{21}$$

Using the tensor of the inertia momentum

$$T_i^r = \Sigma m\xi_i\xi^r; \; T_4^r = T_r^4 = 0 , \tag{22}$$

(21) becomes

(23) $$L_{ik} = \alpha_{kr}T^r_i - \alpha_{ir}T^r_k.$$

A small kinematic system will be called spherical if

(24) $$T^r_i = T\delta^r_i.$$

Then

(25) $$L_{ik} = T\alpha_{ki} - T\alpha_{ik} = 2T\alpha_{ki}.$$

In the absence of external forces the angular momentum is constant. Thus in our system of reference, we get

(26) $$\dot{L}_{ik} = 0.$$

The extension of this relation to our general formulation is the parallel shift of the tensor which is formulated as

(27) $$\dot{L}_{ik} - \Gamma^r_{is}L_{rk}\dot{x}^s - L^r_{ks}L_{ir}\dot{x}^s = 0.$$

In a general system of reference (19) and (20) imply that

(28) $$L_{ik}v^k = \alpha_{ik}v^k = 0$$

where $v^k = \frac{dx^k}{ds}$ is the tangent vector of the world line.

Because $L_{ik}$ and $\alpha_{ik}$ are skew-symmetric, the property (28) implies the existence of two vectors $p^i$ and $q^i$ such that

(29) $$L_{ik}p^k = 0;\ \alpha_{ik}q^k = 0;\ p^k v_k = 0;\ q^k v_k = 0.$$

If we assume the system to be spherical, the two vectors $p^k$ and $q^k$ are proportional. The vector $q^k$ determines the direction of the axis of rotation.

The conservation of the angular momentum can be expressed as the parallel shift of the tensor $L_{ik}$. Since parallel shift conserves the scalar product, the vectors $p^k(s)$ may be obtained by parallel shift as well. A small kinematic system of spherical symmetry, moving as a rigid body will now be regarded as being represented by the model of the spinning mass point. We then obtain

Theorem 10: The parallel shift of a spacelike vector along a geodesic may be simulated kinematically by the shift of the axis of rotation of a spinning mass point.

If parallelism is not integrable, the axis of rotation will come to a position which depends on the particular world line along which the point moves. Two spinning mass points starting **movements** at the same point simultaneously and having initially parallel axes of rotation will not, in general, be spinning around the same axis if they meet again after some time. This effect, which H. Weyl called geodesic precession, may now provide an experiment by which Einstein's main hypothesis can be tested.

In order to get an estimate for the order of magnitude of this effect, we study Schwarzschild's exact solution of Einstein's equations which would represent the space-time continuum around a gravitational center.

This metric is given by

$$ds^2 = f^2(dx^4)^2 - \sum_{i=1}^{3}\sum_{k=1}^{3}(\delta_{ik} + \ell(r)x^i x^k)dx^i dx^k;$$

(30) $$f^2(r) = 1 - \frac{2m}{r};\quad \ell = \frac{2m}{r^3 f^2} = \frac{2m}{r^3 - 2mr^2};$$

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2.$$

For the Christoffel symbols we obtain by calculation, ($\ell' = \frac{d\ell}{dr}$; $f' = \frac{df}{dr}$)

$$\Gamma^e_{ik} = \frac{1}{2}x^e f^2(\frac{\ell'}{r}x^i x^k + 2\ell\delta^{ik});$$

(31) $$\Gamma^4_{4k} = \frac{f'}{f}\frac{x^k}{r};\quad \Gamma^k_{44} = f^3 f'\frac{x^k}{r} = f^2\frac{m}{r^3}x^k;$$

$$\Gamma^e_{4k} = 0;\quad \Gamma^4_{44} = 0;\quad \Gamma^4_{ek} = 0$$

where latin indices denote the numbers 1,2,3.

We assume now that a geodesic may be obtained in the form

(32) $$x^4 = \alpha s;\quad x^1 = R\cos\omega s;\quad x^2 = R\sin\omega s;\quad x^3 = 0.$$

From (30) we get

(33) $$f^2\alpha^2 - R^2\omega^2 = 1$$

so that

(34) $$\alpha^2 = \frac{1 + R^2\omega^2}{1 - 2m/R} = \frac{1 + R^2\omega^2}{f^2}.$$

For $x^3 = 0$ all $\Gamma^3_{..}$ are zero and the differential equation of the geodesic corresponding to this upper index is satisfied by $x^3 = 0$. Observing that

(35) $\dot{x}^1 = -\omega x^2;\ \dot{x}^2 = \omega x^1;\ \dot{x}^4 = \alpha$

we obtain

$$\ddot{x}^1 + x^1(f^2\ell R^2 w^2 + \frac{f^3 f'}{R}\alpha^2) = 0$$

(36) $$\ddot{x}^2 + x^2(f^2\ell R^2 w^2 + \frac{f^3 f'}{R}\alpha^2) = 0$$

$$\ddot{x}^4 = 0.$$

This gives

(37) $$-\omega^2 + [\frac{2m}{R}\omega^2 + (1 + R^2\omega^2)\frac{m}{R^3}] = -\omega^2 + [\frac{3m}{R}\omega^2 + \frac{m}{R^3}] = 0$$

from which we get [compare (34)]

(38) $$\omega^2 = \frac{m}{R^3}(1 - \frac{3m}{R})^{-1} \quad \text{and} \quad \alpha^2 = (1 - \frac{3m}{R})^{-1}.$$

This relation between m,R and $\omega$ therefore leads to a geodesic which may be regarded as a circular movement in space.

The components $u^1(s) \cdots u^4(s)$ of the tangent to the world line are

(39) $u^1(s) = -\omega x^2;\ u^2(s) = \omega x^1;\ u^3(s) = 0;\ u^4(s) = \alpha.$

The parallel shift of a vector with the components $\xi^1,\ldots,\xi^4$ is now described by the differential equations

$$\dot{\xi}^1 + x^1[A(\xi^2 x^1 - \xi^1 x^2) + B\xi^4] = 0$$

(40)
$$\dot{\xi}^2 + x^2[A(\xi^2 x^1 - \xi^1 x^2) + B\xi^4] = 0$$

$$\dot{\xi}^3 = 0$$

$$\dot{\xi}^4 + c(x^1\xi^1 + x^2\xi^2) = 0$$

where

$$A = f^2 \ell \omega = \frac{2m}{R^3}\,\omega = 2\omega^3\left(1 - \frac{3m}{R}\right)$$

(41)
$$B = \sigma f^2 \frac{m}{R^3} = \sigma f^2 \omega^2\left(1 - \frac{3m}{R}\right)$$

$$C = \frac{f'\sigma}{fR} = \sigma f^{(-2)}\omega^2\left(1 - \frac{3m}{R}\right).$$

We now introduce the new variables $\zeta^1, \zeta^2, \zeta^3$ by setting

(42)
$$\zeta^1 = \frac{1}{R}(x^1\xi^1 + x^2\xi^2);\ \zeta^2 = \frac{1}{R}(x^1\xi^2 - x^2\xi^1);\ \zeta^3 = \xi^4$$

which gives

$$\dot{\xi}^1 = -\, x^1(RA\zeta^2 + B\zeta^3)$$

(43)
$$\dot{\xi}^2 = -\, x^2(RA\zeta^2 + B\zeta^3)$$

$$\dot{\xi}^3 = -\, CR\zeta^1.$$

From this we see that

$$x^1\dot{\xi}^1 + x^2\dot{\xi}^2 = -\, R^2(RA\zeta^2 + B\zeta^3)$$

(44)
$$x^1\dot{\xi}^2 - x^2\dot{\xi}^1 = 0$$

and we obtain by differentiating (42)

$$\dot{\zeta}^1 = (w - R^2A)\zeta^2 - BR\,\zeta^3$$
(45)
$$\dot{\zeta}^2 = -w\zeta^1$$
$$\dot{\zeta}^3 = -CR\zeta^1.$$

For the point $s = 0$, the components of the vector tangential to our geodesic are

(46) $$u^1(0) = 0;\ u^2(0) = \omega;\ u^3(0) = 0;\ u^4(0) = \alpha.$$

Setting

(47) $$\xi^1(0) = 1;\ \xi^2(0) = 0;\ \xi^3(0) = 0;\ \xi^4(0) = 0$$

we obviously get a vector which is orthogonal to the tangent vector. The corresponding values for the $\zeta$-variables are

(48) $$\zeta^1(0) = 1;\ \zeta^2(0) = 0;\ \zeta^3(0) = 0.$$

It now follows from (45) that

(49) $$CR\zeta^2 - \omega\zeta^3 = \text{const.}$$

Since the value of the constant is zero for $s = 0$, we get

(50) $$\zeta^3 = \frac{CR}{\omega}\zeta^2.$$

This reduces the three equations (45) to

$$\dot{\zeta}^1 = (\omega - R^2A - \frac{BCR^2}{w})\zeta^2$$
(51)
$$\dot{\zeta}^2 = -\omega\zeta^1.$$

Repeated differentiation shows that both functions satisfy

(52) $$\ddot{y} + \overset{*}{\omega}{}^2 y = 0$$

with

(53) $$\overset{*}{\omega}{}^2 = \omega^2\left(1 - \frac{R^2 A}{\omega} - \frac{BCR^2}{\omega^2}\right).$$

From (41) and (38) we obtain

(54) $$1 - \frac{R^2A}{\omega} - \frac{BCR^2}{\omega^2} = 1 - \frac{2m}{R} - \alpha^2\omega^2R^2\left(1 - \frac{3m}{R}\right)^2 = 1 - \frac{3m}{R}.$$

This gives us

(55) $$\overset{*}{\omega} = \omega\sqrt{1 - 3m/R} \approx \omega\left(1 - \frac{3}{2}\frac{m}{R}\right).$$

The difference between $\omega$ and $\overset{*}{\omega}$ means that the vector $\xi^1, \xi^2$, when shifted parallel around the circle, has not returned to its original position after one period. The angle between the initial position and the position after one period is

(56) $$2\pi\left(1 - \sqrt{1 - 3m/R}\right) \approx \frac{3\pi m}{R}.$$

For an orbit around the earth with a radius of 7000 km, this amount is approximately

$$6.10^{-9} \text{ radians.}$$

After a year the satellite has made between 3000 and 4000 rounds. The shift then would be in the order of $2.10^{-5}$ radians or approximately 4 seconds of arc.

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