Doctoral Thesis

Mathematical modelling of German electricity prices

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Abstract

In this dissertation we apply financial mathematical modelling to electricity markets. Electricity is different from any other underlying of financial contracts: it is not storable. This means that electrical energy in one time point cannot be transferred to another. As a consequence, power contracts with disjoint delivery time spans basically have a different underlying. The main idea throughout this thesis is exactly this two-dimensionality of time: every electricity contract is not only characterized by its *trading time* but also by its *delivery time*.

The basis of this dissertation are four scientific papers corresponding to the Chapters 3 to 6, two of which have already been published in peer-reviewed journals. Throughout this thesis two model classes play a significant role: factor models and structural models. All ideas are applied to or supported by these two model classes. All empirical studies in this dissertation are conducted on electricity price data from the German market and Chapter 4 in particular studies an intraday derivative unique to the German market. Therefore, electricity market design is introduced by the example of Germany in Chapter 1. Subsequently, Chapter 2 introduces the general mathematical theory necessary for modelling electricity prices, such as Lévy processes and the Esscher transform. This chapter is the mathematical basis of the Chapters 3 to 6.

Chapter 3 studies factor models applied to the German day-ahead spot prices. We introduce a qualitative measure for seasonality functions based on three requirements. Furthermore, we introduce a relation of factor models to ARMA processes, which induces a new method to estimate the mean reversion speed.

Chapter 4 conducts a theoretical and empirical study of a pricing method for a new electricity derivative: the German intraday cap and floor futures. We introduce the general theory of derivative pricing and propose a method based on the Hull-White model of interest rate modelling, which is a one-factor model. We include week futures prices to generate a price forward curve (PFC), which is then used instead of a fixed deterministic seasonality function. The idea that we can combine all market prices, and in particular futures prices, to improve the model quality also plays the major role in Chapter 5 and Chapter 6.

In Chapter 5 we develop a Heath-Jarrow-Morton (HJM) framework that models intraday, day-ahead, and futures prices. This approach is based on two stochastic processes motivated by economic interpretations and separates the stochastic dynamics in trading and delivery time. Furthermore, this framework allows for the use of classical day-ahead spot price models such as the ones of Schwartz and Smith (2000), Lucia and Schwartz (2002) and includes many model classes such as structural models and factor models.

Chapter 6 unifies the classical theory of storage and the concept of a risk premium through the introduction of an unobservable intrinsic electricity price. Since all tradable electricity contracts are derivatives of this actual intrinsic price, their prices should all be derived as conditional expectation under the risk-neutral measure. Through the intrinsic electricity price we develop a framework, which also includes many existing modelling approaches, such as the HJM framework of Chapter 5.

Zusammenfassung

In dieser Dissertation wenden wir finanzmathematische Methoden auf Strompreismodellierung an. Anders als viele andere Anlagegüter ist Strom nicht speicherbar, weshalb Stromlieferungen zu zwei unterschiedlichen Zeitpunkten eigentlich verschiedene Güter sind. Die Hauptidee dieser Arbeit ist deshalb, dass jede Stromtransaktion nicht nur von ihrer *Handelszeit*, sondern auch von ihrer *Lieferzeit* gekennzeichnet wird. Dies führt zu einer zweidimensionalen Parametrisierung der Zeit.

Diese Dissertation ist aus vier wissenschaftlichen Artikeln entstanden, die den Kapiteln 3 bis 6 enstsprechen und von den zwei bereits in Fachzeitschriften veröffentlicht wurden. In dieser Arbeit spielen zwei Modellklassen eine besonders wichtige Rolle: Faktormodelle und strukturelle Modelle. Diese zwei Klassen werden in jedem Kapitel verwendet, um die Theorie zu veranschaulichen. Alle empirischen Auswertungen in dieser Dissertation wurden mit deutschen Strompreisten ausgeführt. Insbesondere wird in Kapitel 4 ein Intraday-Derivat behandelt, welches nur auf dem deutschen Markt existiert. Deshalb führt Kapitel 1 in das Marktdesign eines Strommarktes anhand des deutschen Marktes ein. Die mathematischen Grundlagen für Strompreismodellierung, wie Lévy Prozesse oder die Esscher Transformation, werden in Kapitel 2 vorgestellt. Dieses Kapitel stellt die mathematische Basis für die Kapitel 3 bis 6 dar.

In Kapitel 3 werden Faktormodelle für die Modellierung des deutschen Day-Ahead-Marktes verwendet. Wir führen ein Qualitätsmaß für Saisonalitätsfunktionen ein, das auf drei Kriterien basiert. Darüber hinaus wird eine Beziehung von Faktormodellen zu den ARMA Zeitreihen abgeleitet, welche zur Schätzung der Mittelwertrückkehrgeschwindigkeit benutzt werden kann.

Kapitel 4 führt eine Methode zur Bewertung neuer Intraday-Derivate, die sogenannten Intraday-Cap-Futures und Intraday-Floor-Futures, ein. Diese Methode basiert auf dem Hull-White Modell der Zinsmodellierung und ist im Wesentlichen ein 1-Faktormodell. In einer empirischen Studie benutzen wir die Preise von Wochenfutures zur Konstruktion einer Price-Forward-Curve (PFC), um eine deterministische Saisonalitätsfunktion zu ersetzen. Die Idee, dass alle Marktpreise, und insbesondere Terminpreise, benutzt werden können, um die Modellqualität zu erhöhen, wird in Kapitel 5 und 6 eingehend weiter verfolgt.

In Kapitel 5 wird ein Heath-Jarrow-Morton (HJM) Rahmen zur konsistenten Modellierung von Intraday-, Day-Ahead- und Futures-Preisen entwickelt. Die Methode basiert auf zwei stochastischen Prozessen, welche mit wirtschaftlichen Argumenten begründet werden und die Zweidimensionalität der Zeit darstellen. In diesem Rahmen können klassische Day-Ahead-Modelle verwendet werden, wie zum Beispiel Schwartz und Smith (2000), Lucia und Schwartz (2002), aber auch andere Modellklassen wie strukturelle Modelle und Faktormodelle.

Kapitel 6 vereinigt die sogenannte Theory of Storage und das Prinzip einer Risikoprämie. Dies geschieht durch Einführung eines unbeobachtbaren eigentlichen Strompreises. Alle Stromprodukte sind Derivate dieses eigentlichen Strompreises, weshalb sie als bedingte Erwartungswerte unter dem risikoneutralen Maß berechnet werden. Der neu entwickelte Rahmen umfasst viele Modellansätze, wie den HJM-Rahmen aus Kapitel 5.

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I am very grateful for the financial support I received from the Fraunhofer Institute for Industrial Mathematics ITWM.[1] The institute provided me with a professional and friendly working environment and gave me the opportunity to present my research at several conferences. In particular I want to thank my colleagues for all the fun we had in- and outside the institute.

Last but not least, I want to thank my girlfriend, Rahel, my parents, Caroline and Allard, and my sister, Renske, for their patience and support during my doctoral studies. Without them I could not have finished this thesis.

Wieger Johan Hinderks June 2019

[1]German: Fraunhofer-Institut für Techno- und Wirtschaftsmathematik ITWM.

Contents

Frequent notation

$a \wedge b := \min(a, b)$

$a \vee b := \max(a, b)$

$\stackrel{\mathrm{d}}{=}$ equality in distribution

$(\Omega, \mathcal{A}, P)$ a probability space

P the real-world measure

Q the (or a) risk-neutral measure or pricing measure

$\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = [0, \infty)$ the considered time horizon

$\mathcal{F} = \{\mathcal{F}_t; t \in \mathcal{T}\}$ a filtration

$W = \{W_t; t \in \mathcal{T}\}$ a Brownian motion

$N = \{N_t; t \in \mathcal{T}\}$ a Poisson process

$L = \{L_t; t \in \mathcal{T}\}$ a Lévy process

(γ, η^2, ℓ) generating triplet (or Lévy triplet)

$\Psi(\theta)$ characteristic exponent

$\phi_t(\theta) := \mathbb{E}e^{i\theta L_t}$ characteristic function

$\nu_t = \nu_t(\theta) := e^{\theta L_t - t\Psi(-i\theta)}$ stochastic exponential of L

$t \in \mathcal{T}$ trading time

$\tau \in \mathcal{T}$ delivery time

$I = \{I(\tau); \tau \in \mathcal{T}\}$ the intraday spot price: in this thesis the ID_3

$S = \{S(\tau); \tau \in \mathcal{T}\}$ the day-ahead spot price

$f(\tau) = \{f_t(\tau); t \in \mathcal{T}\}$ the forward price for fixed delivery time τ

$F_t(\tau_1, \tau_2)$ the futures price at time t for delivery from τ_1 to τ_2

$F_t(\mathcal{T})$ for $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_n\}$ the futures price at time t for the delivery times $\mathcal{T}$

1. Introduction

In financial mathematics we apply stochastic modelling to prices of tradable assets. The difficulty of this task lies in the fact that it is hard to capture all the characteristics of real market prices[1] in a model that can still be used to generate practical results. An important application of financial mathematical modelling lies in the field of *risk management*, where stochastic price models can be used to compute the distribution of a portfolio's value at a future time point. Consecutively, under the assumption that the model does indeed capture the characteristics correctly, this distribution can be used to compute risk indicators such as the *Value at Risk* (VaR). For an introduction to financial mathematics applied to stock markets we refer the interested reader to Korn (2014).

When constructing stochastic price models, one should be aware of a general rule of thumb: the realism of the model is inversely proportional to the simplicity of its calibration or estimation procedure to obtain the necessary model parameters. Usually, the realism of a model is proportional to its theoretical complexity. However, a too complex model might result in one that cannot be calibrated to market data or needs simplifying assumptions in its calibration procedure, which might reduce the realism of the model. Therefore, depending on the application of the model, one should always balance the theoretical complexity and the possibility to calibrate.

Another important aspect of financial mathematics is the computation of derivative prices. Derivatives are financial contracts that lead to a final payoff that is a function or functional of the price(s) of a tradable asset, which is also called the *underlying*. One of the simplest derivatives is a futures contract on a stock. This is an agreement that one party buys or sells this stock for a predetermined price at a future time point. The current value of this contract logically depends on the current price of the stock, earning its name 'derivative'. A big challenge in financial mathematics is to define models that allow for analytically tractable derivatives prices, which are usually computed by the conditional expectation under a suitable measure.

In this dissertation we apply financial mathematical modelling to electricity markets. Electricity is different from any other underlying of financial contracts: it is not storable. This means that electrical energy in one time point cannot be transferred to another. As a consequence power contracts with disjoint delivery time spans basically have a different underlying (Hinz et al., 2005). However, their prices are not necessarily uncorrelated since the price driving processes of electricity production are (auto)correlated, e.g. the demand, the weather, or other commodity prices. Because of this feature short-term electricity prices exhibit a behaviour that is not seen in other mature financial markets such as the stock or interest rate markets. For example, intensive and short-living spikes

[1] These characteristics of the market prices are also called the stylized facts or stylized features.

occur regularly in electricity markets all over the world (Higgs and Worthington, 2008; Meyer-Brandis and Tankov, 2008; Escribano et al., 2011; Janczura et al., 2013; Hagfors et al., 2016a). Chapter 3.1 gives an overview of the stylized facts and features of the German day-ahead spot electricity prices, which is a short-term power market.

Since electricity is so different, the mathematical models used for stock or interest rate prices fail to capture the stylized facts and features of power prices. Soon after the deregulation of the electricity markets, adapted and extended reduced-form modelling approaches were developed and compared in the literature: Pilipovic (1997); Clewlow and Strickland (1999); Deng (2000); Schwartz and Smith (2000); Lucia and Schwartz (2002); Vehvilainen (2002); Villaplana (2003); Hinz et al. (2005); Geman and Roncoroni (2006); Bierbrauer et al. (2007); and many others. Chapter 2.4 introduces the two most important model classes in this thesis: factor models and structural models. For an extensive literature review of reduced-form models and also energy price forecasting approaches we refer to Weron (2014).

Since all empirical studies throughout this thesis are conducted with electricity price data from the German markets, the next section briefly introduces the design of the German electricity market. This is the only prerequisite knowledge of energy finance for this thesis. Apart from a few details specific for the German markets, most liberal electricity markets have a similar market design and make use of the same terminology. Subsequently Section 1.2 outlines the structure of this thesis.

1.1. German electricity market design

German electricity is traded on two European exchanges: the Paris-based *European Power Exchange* (EPEX SPOT SE) for spot trading and the Leipzig-based *European Energy Exchange* (EEX) for derivatives trading. Figure 1.1 illustrates the market design and the different products traded at both exchanges. Classically the day-ahead spot and futures market have been of interest in the literature.

Remark 1.1 (Two-dimensionality of time). As discussed above the important characterization of traded electricity contracts is a certain two-dimensionality in time: the difference between *trading time* t and *delivery time* τ. Throughout the rest of this thesis we will denote trading time by t and delivery time by τ. The delivery time denotes the time point (or better: time period) at which the electricity is being delivered. This time basically determines the underlying of the contract, namely electricity at time point τ. However, the trading time is the time point at which a specific contract is being traded as is the case in regular stock price models. The x-axis in Figure 1.1 denotes the time to delivery, i.e. $\tau - t$. There are several products that are characterized by τ only, since the difference $\tau - t$ is fixed, e.g. the auction markets and the intraday price indices.

The most studied prices are those of the EPEX *day-ahead spot* market, see the middle block in Figure 1.1. This is an auction market, which occurs – as the name suggests – one day before delivery: at 12:00 all 24 hours of the next day are traded in an auction. The day-ahead spot price is therefore always denoted by its delivery time, but it is traded

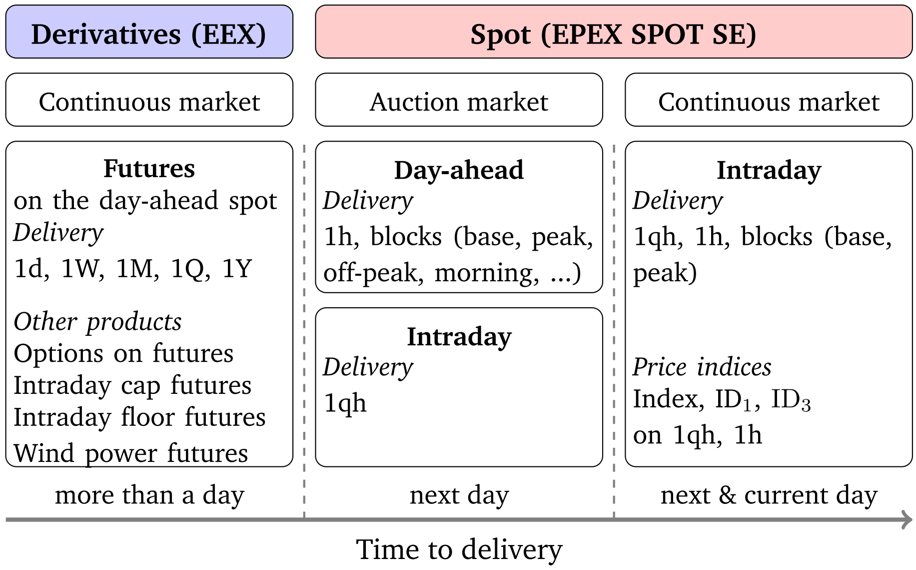


Figure 1.1.: Overview of the different German electricity markets. The delivery periods are quarter hours (1qh), hours (1h), days (1d), weeks (1W), months (1M), quarter years (1Q), and years (1Y). Electricity delivering during a certain delivery time period is first traded at the derivatives market of the EEX through futures and other derivatives. Only one day before delivery it can be traded at 12:00 at the day-ahead auction, at 15:00 at the intraday auction market, and afterwards at the continuous intraday market, which opens one day before delivery at 15:00 for 1h and at 16:00 for 1qh contracts.

(or better: made publicly known) one day before. It is most common to regard the day-ahead spot price as a time series or continuous process, cf. Benth et al. (2008a). In this thesis we will denote the day-ahead spot price by $S(\tau)$ for delivery at τ.

The second market of interest is the derivatives market of the EEX, see the left side of Figure 1.1. Here *futures* contracts on the day-ahead spot price are traded. However, these futures contracts are not just futures contracts on the single hours, but rather on blocks of hours, namely: days, weeks, months, quarters, and years are possible delivery periods. They are traded during the delivery period and their prices converge to the average of the realized day-ahead spot prices during their delivery period. There exist European call and put *options* on these futures at the EEX.

At the EEX there are other derivative products available, cf. Figure 1.1. Of particular interest for this thesis are the intraday cap and floor futures, because Chapter 4 introduces a method for pricing these derivatives. Their payoff functions are discussed in Section 4.2. Since intraday cap and floor futures are derivatives of the intraday index ID_3, this index is also discussed in more detail in this chapter, see Definition 4.1.

The day-ahead spot market has a standardised delivery with hourly granularity. However, there is also a quarter hourly auction one day before delivery: the *intraday auction*. This auction is basically the same as the day-ahead spot auction: at 15:00 all 96 quarter hours of the next day are traded. The naming stems from the fact that the intraday nominations start at 14:30, making the intraday auction officially an *intraday* event.

Finally, the *continuous intraday markets* for hourly and quarter hourly contracts of the next day open at 15:00 and 16:00, respectively. Here contracts with hourly and quarter hourly granularity are traded continuously. From all electricity markets this one is the most similar with a classical stock market, where the assets correspond to the 24 hourly and 96 quarter hourly contracts. For more, theoretical, and not German electricity contracts we refer the interested reader to Deng and Oren (2006).

1.2. Structure of this thesis

This thesis is the product of my cumulative doctoral research, meaning that each chapter was written as a scientific paper. At the time of writing this Introduction the two papers corresponding to Chapter 3 and Chapter 4 have been published in a scientific journal and the other two corresponding to Chapter 5 and Chapter 6 have been submitted. As a consequence, every one of these four chapters could be read independently.

Chapter 2 introduces the basic mathematical theory needed in the Chapters 3 to 6. We introduce the basic theory of Lévy processes and how they can be used to generalize Ornstein-Uhlenbeck processes. Furthermore, the Esscher transform is discussed as a method to generate measure changes. Finally, Section 2.4 is dedicated to the most common electricity price modelling approaches: factor models and structural models.

In Chapter 3 the German day-ahead spot prices are studied with the help of arithmetic factor models. All stylized facts and features are discussed and a new calibration method based on ARMA time series is introduced. Six different seasonality functions are compared and the new calibration technique is used to estimate a one- and a two-factor model. The distributional fit of the calibrated models is evaluated in an out-of-sample data study.

As mentioned in the previous section Chapter 4 treats the pricing of intraday cap and floor futures. These contracts are derivatives of the intraday index ID_3. A factor model in the form of the Hull-White extended Vasicek model is proposed to model the ID_3 price and from this the intraday cap and floor futures prices are derived. A new aspect of the approach is that the seasonality is based on the price forward curve (PFC), which is used to incorporate week futures into the model.

Chapter 5 continues with the idea of including the PFC into the modelling approach and proposes a structural Heath-Jarrow-Morton framework for modelling all market prices. The new aspect in this approach is that its two main components, the structural component and the market noise, are motivated by economic interpretations. The flexible set-up allows all classical day-ahead spot price models to be used within this framework. The main two explicit model choices that are discussed in this chapter are structural models and arithmetic factor models.

Finally, Chapter 6 focuses on the risk-neutral measure Q in the case of electricity. This is not a straightforward matter, as will first be discussed in Section 2.3. The chapter starts off with a discussion of the two most common approaches to connect commodity spot prices to their respective forward prices: the theory of storage and the concept of a risk premium. However, since neither is a very convenient tool to use in the case of electricity, a new approach based on the actual, intrinsic electricity price is introduced. This method unifies the theory of storage and the concept of a risk premium in one theory. In the last part of the chapter an explicit model is assumed and calibrated to real data.

This thesis is concluded in Chapter 7, in which also an outlook on open questions and successive research topics are given. Throughout this thesis all computations and data manipulations are done with the open source programming language `R`, which can be downloaded at `www.r-project.org`. In particular, we used the `tidyverse` package. This also includes the package `ggplot2`, which was used to create all figures derived from data.

2. Mathematical theory of electricity price modelling

Unless stated otherwise we assume throughout this chapter that the considered time horizon is equals $\mathcal{T} := [0, \infty)$. Furthermore, we write

$$(\Omega, \mathcal{A}, \mathcal{F} = \{\mathcal{F}_t; t \in \mathcal{T}\}, P),$$

for a complete, filtered probability space. The filtration $\mathcal{F}$ is assumed to satisfy the *usual assumptions* or *usual hypotheses* as defined in Karatzas and Shreve (1998), i.e. the filtration is right-continuous and $\mathcal{F}_0$ is assumed to contains all the P-null sets. For a discussion on why the usual conditions are a convenient assumption, we refer to the work of Protter (2005, Chapter I.5).

For completeness we recapitulate the definitions of the standard one-dimensional Brownian motion and the homogeneous Poisson process. Comparing both definitions will yield a natural generalization: Lévy processes. Section 2.1 is committed to derive characterizations of Lévy processes and to find relations with the Brownian motions and (compound) Poisson processes. The following definitions can also be found in Karatzas and Shreve (1998) and Sato (2013), where also proofs of existence are given:

Definition 2.1 (Brownian motion)**.** A real-valued stochastic process $W = \{W_t; t \in \mathcal{T}\}$ which is adapted to $\mathcal{F}$, is called a standard one-dimensional *Brownian motion* if it satisfies the following conditions:

(i) $W_0 = 0$ a.s.,
(ii) W has a.s. continuous sample paths,
(iii) W has independent increments, i.e. $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $t > s$,
(iv) W has normal increments, i.e. $W_t - W_s \sim \mathcal{N}(0, t - s)$ for all $t > s$.

Definition 2.2 (Poisson process)**.** A real-valued stochastic process $N = \{N_t; t \in \mathcal{T}\}$ which is adapted to $\mathcal{F}$, is called a homogeneous *Poisson process* with intensity $\lambda > 0$ if it satisfies the following conditions:

(i) $N_0 = 0$ a.s.,
(ii) N has a.s. càdlàg sample paths,
(iii) N has independent increments, i.e. $N_t - N_s$ is independent of $\mathcal{F}_s$ for all $t > s$,
(iv) N has Poisson increments, i.e. $N_t - N_s \sim$ Poisson$(\lambda(t - s))$ for all $t > s$.

2.1. Lévy processes

In this section we follow the introduction of Lévy processes given by Sato (2013). However, we will only consider one-dimensional processes in this section. Therefore, we will simplify the statements and notation wherever in the original work $\mathbb{R}^d$ is considered. We define a Lévy process on $(\Omega, \mathcal{A}, \mathcal{F}, P)$ as follows:

Definition 2.3 (Lévy process)**.** A real-valued stochastic process $L = \{L_t; t \in \mathcal{T}\}$ which is adapted to $\mathcal{F}$, is called a one-dimensional *Lévy process* if it satisfies the following conditions:

(i) $L_0 = 0$ a.s.,
(ii) L has a.s. càdlàg sample paths,
(iii) L has *independent increments*, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for all $t > s$,
(iv) L has *stationary increments*, i.e. $L_t - L_s \overset{\mathrm{d}}{=} L_{t-s}$ for all $t > s$.

It is clear that both the Brownian motion and the Poisson process are part of the class of Lévy processes by assuming normally and Poisson distributed increments, respectively. Therefore, Lévy processes are a natural way to generalize both. This class allows for modelling jumps and other non-Gaussian stylized facts, such as heavy-tails. However, due to the fact that the increments have to be stationary the class of Lévy processes does not allow for time-dependent behaviour like volatility clustering or seasonal jumping intensities.

2.1.1. Infinite divisibility

An important property of the distribution of Lévy processes is infinite divisibility, which is based on the self-convolution of probability measures:

Definition 2.4. The *convolution* P of two probability measures P_1 and P_2 on $(\Omega, \mathcal{A})$ is defined by

$$P(A) := \int_\Omega \int_\Omega \mathbb{1}_A(\omega_1 + \omega_2)\, dP_1(\omega_1)\, dP_2(\omega_2),$$

for any $A \in \mathcal{A}$. We write $P = P_1 * P_2$.

We denote the n-fold convolution of a measure P with itself by P^{*n} and define it by $P^{*n} := P * P * \cdots * P$, where the convolution is taken $n-1$ times.

Definition 2.5 (Infinite divisibility)**.** A probability measure P is called *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a probability measure $\tilde{P}$ such that $\tilde{P}^{*n} = P$. We call a random variable X *infinitely divisible*, if its distribution[1] is infinitely divisible.

Lemma 2.6. *A random variable X is infinitely divisible, if for all $n \in \mathbb{N}$ there exist i.i.d. random variables $X_1, \ldots, X_n$ such that*

$$X \overset{d}{=} X_1 + \cdots + X_n.$$

[1] With the distribution P_X of a random variable X we mean the probability measure on $\mathbb{R}$ which is defined as the composition $P_X := P \circ X^{-1}$.

Proof. The statement is easily confirmed: for any $B \in \mathcal{B}(\mathbb{R})$

$$P_X(B) = \mathbb{E}[\mathbb{1}_B(X)] = \mathbb{E}[\mathbb{1}_B(X_1 + \cdots + X_n)] = P_{X_1} * \cdots * P_{X_n}(B) = P_{X_1}^{*n}(B),$$

where the last two equalities follow from the i.i.d. property of the sequence. □

Corollary 2.7. *For all $t > 0$ the random variable L_t is infinitely divisible.*

Proof. We immediately see the statement holds since for any $n \in \mathbb{N}$

$$L_t = \left[L_{t/n} - L_0\right] + \left[L_{2t/n} - L_{t/n}\right] + \cdots + \left[L_t - L_{(n-1)t/n}\right] \tag{2.1}$$

by exploiting the independent and stationary increment properties. □

The following theorem illustrates the important relation between infinite divisibility and Lévy processes. For the proof we refer to Sato (2013, Lemma 7.9 and Theorem 7.10(i)).

Theorem 2.8. *Denote the distribution of a Lévy process L_t by $P_t := P_{L_t}$ for all $t \in \mathcal{T}$.*

(i) If the probability measure P_1 is infinitely divisible, then for all $t \in \mathcal{T}$ the convolution P_1^{*t} is well-defined and infinitely divisible.

(ii) The random variable L_t is infinitely divisible and we have $P_t = P_1^{*t}$ for all $t \in \mathcal{T}$.

Note that P_1 is a probability measure on $\mathbb{R}$. From the omitted proof we also find that $P_1^{*0} = \delta_0$, the Dirac measure of 0. Furthermore, it follows from Equation (2.1) that for all $n \in \mathbb{N}$ the characteristic function satisfies

$$\phi_t(\theta) = \mathbb{E}\left[e^{i\theta L_t}\right] = \mathbb{E}\left[e^{i\theta L_{t/n}}\right]^n = \phi_{t/n}(\theta)^n.$$

This relation is used in the proof of Theorem 2.8 to conclude that for infinitely divisible measures and thus for Lévy processes we have

$$\phi_t(\theta) = \phi_1(\theta)^t = e^{t \log \phi_1(\theta)}$$

for all $t \in \mathcal{T}$. Due to this equation it suffices to find $\phi_1(\theta)$ (or its logarithm) in order to find the characteristic function of a Lévy process at any time $t \in \mathcal{T}$.

We have seen that every Lévy process induces an infinitely divisible probability measure. However, the reverse is also true, as is illustrated by the next result which is shown in Sato (2013) and of which the proof relies on the Kolmogorov extension theorem.

Proposition 2.9. *If $\tilde{P}$ is an infinitely divisible probability measure on $(\mathbb{R}, \mathcal{B})$, then there exists a Lévy process $L = \{L_t; t \in \mathcal{T}\}$ such that $P_t = \tilde{P}^{*t}$ for all $t \in \mathcal{T}$.*

Proof. We combine Theorem 7.10(ii) and Theorem 11.5 from Sato (2013). The first theorem shows that there exists a stochastic process L with the wanted property and which is almost Lévy: it is only missing the property that almost all trajectories are càdlàg. The second theorem then shows that there exists an càdlàg modification of L, which yields the proof. □

Using Theorem 2.8 and Proposition 2.9 we see that Lévy processes and infinitely divisible probability measures are equivalent. This means that we can utilize infinitely divisible probability measures to learn more about Lévy processes, which we do in the following section.

2.1.2. Lévy-Khintchine representation

The following proposition is a helpful tool for finding the characteristic function of infinitely divisible probability measures on $\mathbb{R}$. The proof can be found in Sato (2013, Theorem 8.1).

Proposition 2.10 (Lévy-Khintchine representation). *If $\tilde{P}$ is an infinitely divisible probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then the logarithm of its characteristic function is given by the up to deterministic functions unique representation*

$$\log \phi_{\tilde{P}}(\theta) = i\gamma\theta - \tfrac{1}{2}\eta^2\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x\, \mathbb{1}_{|x|\leq 1}\right) d\ell(x), \tag{2.2}$$

where $\gamma \in \mathbb{R}$, $\eta^2 \in \mathbb{R}_{\geq 0}$, and ℓ is a measure on $\mathbb{R}$ with the properties that

$$\ell\left(\{0\}\right) = 0 \quad \textit{and} \quad \int_{\mathbb{R}} \left(x^2 \wedge 1\right) d\ell(x) < \infty. \tag{2.3}$$

In particular, for every triplet (γ, η^2, ℓ) satisfying the above conditions there exists an infinitely divisible probability measure $\tilde{P}$ such that the logarithm of its characteristic function is given by Equation (2.2).

We call the measure ℓ the *Lévy measure*. The triple (γ, η^2, ℓ) is also called the *generating triplet*, since it equivalently defines an infinitely divisible distribution due to *Lévy-Khintchine representation*. From Proposition 2.9 it follows that the generating triplet (γ, η^2, ℓ) equivalently defines a Lévy process with characteristic function

$$\phi_t(\theta) = \exp\left(it\gamma\theta - \tfrac{1}{2}t\eta^2\theta^2 + t\int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x\, \mathbb{1}_{|x|\leq 1}\right) d\ell(x)\right).$$

As an auxiliary function we define the characteristic exponent of a Lévy process:

Definition 2.11 (Characteristic exponent). We define the *characteristic exponent* of a Lévy process L with generating triplet (γ, η^2, ℓ) by

$$\Psi(\theta) := i\gamma\theta - \tfrac{1}{2}\eta^2\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x\, \mathbb{1}_{|x|\leq 1}\right) d\ell(x). \tag{2.4}$$

We will now compute the characteristic exponent and the generating triplet for a few well-known examples:

Example 2.12 (Poisson process). Let N be a homogeneous Poisson process with intensity λ. At $t = 1$ its characteristic function is given by

$$\phi_1(\theta) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{i\theta})^k}{k!} = e^{\lambda(e^{i\theta}-1)},$$

from which we find that the characteristic exponent equals $\Psi(\theta) = \lambda(e^{i\theta} - 1)$. We conclude by the uniqueness of the Lévy-Khintchine representation that the generating triplet is given by

$$\gamma = \lambda, \quad \eta^2 = 0, \quad \ell(x) = \lambda\, \delta_1(x),$$

where δ_1 is the Dirac measure with mass at 1.

Furthermore, we note that $\mathbb{E}[N_t] = \lambda t$. If we then consider the compensated Poisson process $\widetilde{N}_t := N_t - \lambda t$, we get

$$\phi_1(\theta) = e^{\lambda(e^{i\theta}-1-i\theta)},$$

and therefore

$$\gamma = 0, \quad \eta^2 = 0, \quad \ell(x) = \lambda\,\delta_1(x),$$

which shows us that the linear drift γ became zero.

Example 2.13 (Compound Poisson process)**.** Let N be a Poisson process given as in the previous example. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with distribution P_X. Then the sum $S_t = \sum_{k=1}^{N_t} X_k$ is a compound Poisson process, for which we compute

$$\phi_1(\theta) = e^{\lambda \int_{\mathbb{R}} (e^{i\theta x}-1)\,dP_X(x)}.$$

We conclude that $\Psi(\theta) = \lambda \int_{\mathbb{R}} (e^{i\theta x} - 1)\,dP_X(x)$. This implies that the generating triplet is given by

$$\gamma = \lambda \int_{-1}^{1} x\,dP_X(x), \quad \eta^2 = 0, \quad d\ell(x) = \lambda\,dP_X(x),$$

which shows that for $P_X = \delta_1$, i.e. jumps of size one (a.s.), we get the Poisson process back.

Example 2.14 (Brownian motion)**.** Let W be a standard Brownian motion. Using the fact that $W_1 \sim \mathcal{N}(0,1)$ we compute

$$\phi_1(\theta) = e^{-\frac{1}{2}\theta^2},$$

from which we conclude that $\Psi(\theta) = -\frac{1}{2}\theta^2$. The generating triplet is obviously given by

$$\gamma = 0 \quad \eta^2 = 1, \quad \ell \equiv 0.$$

If, on the other hand, we consider a Brownian motion with drift and different standard deviation, i.e. $W_1 \sim \mathcal{N}(\mu, \sigma^2)$, we get $\phi_1(\theta) = e^{i\mu\theta - \frac{1}{2}\sigma^2\theta^2}$ and thus

$$\gamma = \mu \quad \eta^2 = \sigma^2, \quad \ell \equiv 0.$$

In particular, we remark that the measure ℓ is zero for all Brownian motions.

A preliminary conclusion from the previous examples would be that

(i) the constant γ is related to the (linear) trend of the stochastic process (cf. Example 2.12 and 2.14),

(ii) the constant η^2 is connected to the variance of the continuous part of the process (cf. Example 2.14),

(iii) and the Lévy measure ℓ assigns a frequency to each jump size (cf. Example 2.12 and 2.13).

We will see that these observations are in some sense correct and we will formalize them with the Lévy-Itô decomposition.

2.1.3. Lévy-Itô decomposition

To make notation easier in this section we write $\mathcal{T}_0 := \mathcal{T} \setminus \{0\}$ and likewise $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Furthermore, we denote $\overline{\mathbb{Z}}_{\geq 0} = \{0, 1, 2, \dots\} \cup \{\infty\}$.

Definition 2.15 (Poisson random measure)**.** Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space.[2] A family of random variables $\{N(B) \mid B \in \mathcal{B}\}$ with $N(B) : \Omega \to \overline{\mathbb{Z}}_{\geq 0}$ is called a *Poisson random measure* with intensity measure μ, if it satisfies

(i) for all $B \in \mathcal{B}$ we have $N(B) \sim \text{Poisson}(\mu(B))$,
(ii) for disjoint $B_1, \dots, B_n \in \mathcal{B}$ with $n \in \mathbb{N}$, the random variables $N(B_1), \dots, N(B_n)$ are independent,
(iii) for all $\omega \in \Omega$ we have that $N(\cdot; \Omega)$ is a measure on X.

Definition 2.16 (Random jump measure)**.** Let $L = \{L_t; t \in \mathcal{T}\}$ be a Lévy process and let $\Omega_0 \in \mathcal{A}$ be the set with probability 1 on which L is càdlàg. We define the *random jump measure* associated to L by

$$J(B; \omega) := \begin{cases} \# \left\{ s \quad \middle| \quad (s, L(s;\omega) - L(s^-;\omega)) \in B \right\} & \text{if } \omega \in \Omega_0 \\ 0 & \text{else} \end{cases},$$

where $B \in \mathcal{B}(\mathcal{T}_0 \times \mathbb{R}_0)$, the Borel σ-algebra on $\mathcal{T}_0 \times \mathbb{R}_0$.

In particular, we denote $J(t, B; \omega) := J((0, t] \times B; \omega)$ for $t > 0$ and $B \in \mathcal{B}(\mathbb{R}_0)$. Then we find that an alternative way to represent the random jump measure is by

$$J(t, B) \overset{\text{a.s.}}{=} \sum_{0<s\leq t} \mathbb{1}_B(\Delta L_s) := \sum_{0<s\leq t} \mathbb{1}_B(L_s - L(s^-)).$$

From this representation it is clear that for fixed ω the random jump measure $J(t, B; \omega)$ is a counting measure on $\mathbb{R}_0$ that counts exactly those jumps with size in B up to time t of the ω-path of L.

Lemma 2.17. *For all $t > 0$ the random jump measure*

$$J(t, B) \overset{\text{d}}{=} t\, J(1, B),$$

where the equality holds in distribution.

Proof. This is easily seen when we consider $n, m \in \mathbb{N}$

$$\begin{aligned} J(n, B) &= \sum_{i=1}^{nm} \left(\sum_{i-1<ms\leq i} \mathbb{1}_B(\Delta L_s) \right) \\ &\overset{\text{d}}{=} nm\, J\left(\tfrac{1}{m}, B\right). \end{aligned}$$

[2] Recall that a measure space $(X, \mathcal{B}, \mu)$ is called σ-finite, if there exists a countable sequence of measurable sets with finite measure, $X_1, X_2, \dots \in \mathcal{B}$, such that $X = \cup_{i=1}^{\infty} X_i$.

By choosing first $n = 1$ and then $m = 1$ we can conclude that the statement holds for all rational $t > 0$. Noting that that the indicator function is dominated by the constant function 1 we can apply dominated convergence to see that the statement follows for all real $t > 0$ by the density of $\mathbb{Q}$ in $\mathbb{R}$. □

Using again a result without proof from the work of Sato (2013, Theorem 19.2(i)) we see that:

Theorem 2.18. *Let $L = \{L_t; t \in \mathcal{T}\}$ be a Lévy process with generating triplet (γ, η^2, ℓ). The random jump measure $\{J(B) \,|\, B \in \mathcal{B}(\mathcal{T}_0 \times \mathbb{R}_0)\}$ associated to L is a Poisson random measure on $\mathcal{T}_0 \times \mathbb{R}_0$ with intensity measure $\lambda \times \ell$, where λ is the Lebesgue measure on $\mathcal{T}_0$.*

In particular, for fixed $t \in \mathcal{T}_0$ we have that $\{J(t, B) \,|\, B \in \mathcal{B}(\mathbb{R}_0)\}$ is a Poisson random measure on $\mathbb{R}_0$ with intensity measure $t\ell(\cdot)$.

The previous result also states that $\mathbb{E}J(B) = (\lambda \times \ell)(B)$ for all $B \in \mathcal{B}(\mathcal{T}_0 \times \mathbb{R}_0)$, allowing for infinity. For B with $(s, 0) \notin \overline{B}$ for any $s \in \mathcal{T}_0$, however, we find that $J(B)$ is a.s. finite. Therefore, we define the *compensated random jump measure* by

$$\widetilde{J}(B) := J(B) - (\lambda \times \ell)(B), \quad B \in \mathcal{B}(\mathcal{T}_0 \times \mathbb{R}_0), \forall s \in \mathcal{T}_0 \,:\, (s, 0) \notin \overline{B}.$$

For fixed $t \in \mathcal{T}_0$ this simplifies to

$$\widetilde{J}(t, B) := \widetilde{J}((0, t] \times B) = J(t, B) - t\ell(B), \quad B \in \mathcal{B}(\mathbb{R}_0),\, 0 \notin \overline{B}.$$

For fixed B with $0 \notin \overline{B}$ the random jump measure $J(t, B)$ is a Poisson process with intensity $\ell(B)$. We can use the (compensated) random jump measure to characterize a Lévy process by the Lévy-Itô decomposition. We refer to Theorem 19.2 and Theorem 19.3 by Sato (2013) for the proof.

Proposition 2.19 (Lévy-Itô decomposition)**.** *Let $L = \{L_t; t \in \mathcal{T}\}$ be a Lévy process with generating triplet (γ, η^2, ℓ). Let $J(t, B)$ be the random jump measure associated to L. Then for all $t \in \mathcal{T}$ we have*

$$L_t = L_t^1 + L_t^2 + L_t^3,$$

where the L^1, L^2 and L^3 are independent Lévy processes, determined by:

(i) L_t^1 has generating triplet $(\gamma, 0, 0)$ and corresponds to a deterministic linear drift

$$L_t^1 \overset{a.s.}{=} \gamma\, t,$$

(ii) L_t^2 has generating triplet $(0, \eta^2, 0)$ and corresponds to a Brownian motion

$$L_t^2 \overset{a.s.}{=} \eta\, W_t,$$

where W_t is a standard Brownian motion, *(iii)* L_t^3 *has generating triplet* $(0, 0, \ell)$ *and is (ω-wise) defined by*

$$L_t^3 \overset{a.s.}{=} \lim_{\varepsilon \to 0^+} \int_0^t \int_{\varepsilon < |x| \le 1} x \, d\widetilde{J}(s, x) + \int_0^t \int_{|x|>1} x \, dJ(s, x), \tag{2.5}$$

and the convergence is uniform in t *on bounded intervals.*

Remark 2.20. From the proof we know that for $\omega \in \Omega$, $t \in \mathcal{T}_0$ and $\varepsilon > 0$ the mapping $x \in \mathbb{R} \mapsto J(t, \{x\} \cap (\varepsilon, \infty); \omega)$ has finite support (Sato, 2013, Lemma 20.1). However, in the first term of Equation (2.5) we have to use the compensated jump measure since the limit

$$\lim_{\varepsilon \to 0^+} \int_0^t \int_{\varepsilon < |x| \le 1} x \, dJ(s, x) \tag{2.6}$$

might not converge (a.s.). This implies that the process jumps infinitely many times with "little jumps" (with size smaller than one[3]) on the interval $(0, t]$. Furthermore, Equation (2.3) also implies that

$$\mathbb{E}J(t, \mathbb{R} \setminus [-1, 1]) = t \, \ell(\mathbb{R} \setminus [-1, 1]) < \infty,$$

i.e. the expected number of "bigger jumps" (with size bigger than one) is finite on every finite time horizon.

In order for the integral of Equation (2.6) to converge an additional condition is necessary:

Corollary 2.21 (Lévy-Itô decomposition II). *If in the setting of Proposition 2.19 it additionally holds that*

$$\int_{|x| \le 1} |x| \, d\ell(x) < \infty, \tag{2.7}$$

then we can rearrange L^1 *and* L^3 *and we find*

$$L_t^1 \overset{a.s.}{=} \widetilde{\gamma} \, t,$$

where $\widetilde{\gamma} := \gamma - \int_{|x| \le 1} x \, d\ell(x)$*, and*

$$L_t^3 \overset{a.s.}{=} \int_0^t \int_{\mathbb{R}_0} x \, dJ(s, x), \tag{2.8}$$

such that L^1 *has generating triplet* $(\widetilde{\gamma}, 0, 0)$ *and* L^3 *has* $\left(\int_{|x| \le 1} x \, d\ell(x), 0, \ell \right)$.

In the setting of Corollary 2.21 the Lévy process L^3 reduces to a compound Poisson process. This can be seen from the generating triplet, cf. Example 2.13. Moreover, the integral in Equation (2.8) can equivalently be written as

$$L_t^3 \overset{\text{a.s.}}{=} \sum_{0 < s \le t} \Delta L_s,$$

[3] Note that the boundary one is arbitrarily chosen, but generally accepted by the literature on Lévy processes. We could, however, have chosen to work with $\frac{1}{9}$ or 200 if we had preferred that.

cf. Sato (2013, Proposition 19.5) or Protter (2005, Chapter I, Theorem 36).

It becomes clear that the Lévy measure ℓ plays a big role in characterizing the Lévy process. Therefore, we note some results that follow from the Lévy-Itô decomposition and help us distinguish between different types of Lévy processes.

Theorem 2.22 (Jump times)**.** *Let L be a Lévy process with generating triplet (γ, η^2, ℓ). Then the following statements hold:*

(i) $\ell(\mathbb{R}) = 0$*, i.e. L is a combination of Brownian motion and a drift term, if and only if almost all trajectories of L are continuous,*

(ii) $0 < \ell(\mathbb{R}) < \infty$ *implies that, a.s., the jump times are infinitely many and countable in $\mathcal{T}$, but finitely many in any compact subset of $\mathcal{T}$,*

(iii) $\ell(\mathbb{R}) = \infty$ *implies that there are countably many jump times a.s. and the jump times are dense in $\mathcal{T}$.*

Proof. See Sato (2013): Theorem 21.1 for *(i)* and Theorem 21.3 for *(ii)* and *(iii)*. □

Due to this theorem the processes L^1 and L^2 from the decomposition are called the *continuous part*, whereas L^3 is the *jump part* of a Lévy process L. We remark that in case *(i)* the process L is a Brownian motion with drift and in case *(ii)* we have that the jumping part is just a compound Poisson process, since this implies Equation (2.7) and we can use the second representation in the Lévy-Itô decomposition.

We recall that for a fixed trajectory $t \mapsto L(t;\omega)$ of L, i.e. for fixed $\omega \in \Omega$, the variation up to time $t > 0$ is defined by

$$V(t;\omega) := \sup_{\Pi=\{t_0,\dots,t_n\}} \sum_{i=1}^{n} |L_{t_i}(\omega) - L_{t_{i-1}}(\omega)|,$$

where the $\Pi = \{t_0, \dots, t_n\}$, $n \in \mathbb{N}$, are partitions of the interval $(0, t]$, in other words it holds that $0 = t_0 < t_1 < \dots < t_n = t$. We conclude this section with a result on the variation of Lévy processes (Sato, 2013, Theorem 21.9).

Theorem 2.23 (Variation)**.** *Let L be a Lévy process with generating triplet (γ, η^2, ℓ). Then the following statements hold:*

(i) if $\eta^2 = 0$ *and Equation* (2.7) *holds, then* $V(t) < \infty$ *a.s. for all* $t > 0$*,*

(ii) if $\eta^2 \neq 0$ *or Equation* (2.7) *does not hold, then* $V(t) = \infty$ *a.s. for all* $t > 0$*.*

This result is somehow not surprising: the paths have finite variation if the Brownian component vanishes[4] and the jump part behaves nicely enough, cf. Remark 2.20.

[4]The Brownian motion has unbounded variation (Karatzas and Shreve, 1998, Chapter 2.9.C).

Remark 2.24 (Additive processes). The class of Lévy processes can also be generalized to the class of *additive processes* (Sato, 2013). Additive processes are also sometimes called *independent increment processes*. Their definition differs only in one point from Lévy processes: instead of the stationary increments property they are assumed to be continuous in probability.[5] This class allows for time dependent jump intensity, which is not possible for Lévy processes. However, many results of Lévy processes can be generalized to this class, including the Lévy-Itô decomposition.

2.2. Ornstein-Uhlenbeck processes

The Gaussian *Ornstein-Uhlenbeck process* (OU process) is a well-known stochastic process (Karatzas and Shreve, 1998, Chapter 5, Example 6.8.). We will discuss some results on Gaussian OU processes in Section 2.2.2. However, if we change the driving process from a Brownian motion to a Lévy process we get a more general class of OU process:

Definition 2.25 (Lévy process-driven Ornstein-Uhlenbeck process). Let $L = \{L_t; t \in \mathcal{T}\}$ be a Lévy process. Let $\lambda \geq 0$, μ, and σ be deterministic, real-valued, and continuous functions on $\mathcal{T}$. If there exists a unique strong solution X_t to the stochastic differential equation (SDE)

$$dX_t = \lambda(t)\,(\mu(t) - X_t)\,dt + \sigma(t)\,dL_t, \qquad X_0 = x_0 \in \mathbb{R}, \tag{2.9}$$

which is adapted and a.s. càdlàg, then X_t is called a *Lévy process-driven Ornstein-Uhlenbeck process*. The Lévy process L is also called the *background driving Lévy process* (BDLP).

Remark 2.26. We know that Lévy processes are semi-martingales and therefore we can integrate with respect to them (Protter, 2005, Chapter II, Corollary of Theorem 9). Due to the Lévy-Itô decomposition, Proposition 2.19, we know that every Lévy process L with generating triplet (γ, η^2, ℓ) can be decomposed into a deterministic linear drift, a Brownian motion W_t, and a pure jump process L_t^3. Assume that $\xi(t)$ is a deterministic, continuous function, then we use this decomposition to write

$$\int_0^t \xi(s)\,dL_s \overset{\text{a.s.}}{=} \gamma \int_0^t \xi(s)\,ds + \eta \int_0^t \xi(s)\,dW_s + \int_0^t \xi(s)\,dL_s^3.$$

We immediately see that we can interpret the first two terms as the Lebesgue and Itô integrals. Further inspection of the last term yields

$$\int_0^t \xi(s)\,dL_s^3 = \int_0^t \xi(s) \int_{0<|x|\leq 1} x\,d\widetilde{J}(s,x) + \int_0^t \xi(s) \int_{|x|>1} x\,dJ(s,x).$$

From Theorem 2.18 we know that the random jump measure is a Poisson random measure. Therefore, we can interpret these last two integrals ω-wise.

[5] Also called *stochastically continuous*. A process $X = \{X_t; t \in \mathcal{T}\}$ is called continuous in probability, if for all $t \in \mathcal{T}$ and for all $\varepsilon > 0$ we have that $\lim_{s\to t} P(|X_t - X_s| > \varepsilon) = 0$.

Theorem 2.27. *There exists a strong solution to the SDE* (2.9)*. This solution is unique up to null sets and given by*

$$X_t = x_0 \, e^{-\int_0^t \lambda(s)\,ds} + \int_0^t \lambda(s)\mu(s)\, e^{-\int_s^t \lambda(u)\,du}\, ds + \int_0^t \sigma(s)\, e^{-\int_s^t \lambda(u)\,du}\, dL_s. \qquad (2.10)$$

Furthermore, X_t is adapted and almost all paths are càdlàg.

Proof. First we prove the existence: define X_t as above. Clearly, X_t is an adapted and a.s. càdlàg semi-martingale. Furthermore, rewrite

$$X_t = e^{-\int_0^t \lambda(s)\,ds} \left(x_0 + \int_0^t \lambda(s)\mu(s)\, e^{\int_0^s \lambda(u)\,du}\, ds + \int_0^t \sigma(s)\, e^{\int_0^s \lambda(u)\,du}\, dL_s \right),$$

and by using integration by parts (Protter, 2005, Chapter I, Corollary of Theorem 34) we find

$$\begin{aligned} dX_t = &-\lambda(t) e^{-\int_0^t \lambda(s)\,ds} \left(x_0 + \int_0^t \lambda(s)\mu(s)\, e^{\int_0^s \lambda(u)\,du}\, ds + \int_0^t \sigma(s)\, e^{\int_0^s \lambda(u)\,du}\, dL_s \right) \\ &+ e^{-\int_0^t \lambda(s)\,ds} \left(\lambda(t)\mu(t)\, e^{\int_0^t \lambda(s)\,ds}\, dt + \sigma(t)\, e^{\int_0^t \lambda(s)\,ds}\, dL_t \right), \end{aligned}$$

which reduces to the SDE given in Equation (2.9) by plugging in Equation (2.10). Therefore X_t satisfies the SDE and is a strong solution.

Now the uniqueness: suppose that X_t and X'_t both satisfy the SDE (2.9). We give a different proof than can be found in for example Sato (2013) or Benth et al. (2008a), where they rely on a series to argue that the difference of X and X' is zero.

Let Ω_0 be the set with probability one on which $X_0 = x_0 = X'_0$ and both X and X' are càdlàg. This set exists since the intersection of finitely many sets with probability one again has probability one. Then for a fixed $\omega \in \Omega_0$ we consider the difference $f(t) := X_t(\omega) - X'_t(\omega)$. Due to our choice of Ω_0 we have that $f(0) = 0$ and that f is càdlàg. Furthermore, from Equation (2.9) we find that

$$df(t) = -\lambda(t) f(t)\, dt \implies f(t) = c \exp\left(-\int_0^t \lambda(s)\, ds \right) \quad \text{or} \quad f(t) = 0.$$

However, combined with $f(0) = 0$ this only has the trivial solution, i.e. $f \equiv 0$. This holds for all $\omega \in \Omega_0$ and therefore uniqueness up to a null set follows. □

As a direct consequence we find that OU processes allow for a certain decomposition with themselves:

Corollary 2.28. *Let X be an OU process satisfying the SDE* (2.9)*. It follows that*

$$X_{t+h} = e^{-\int_t^{t+h} \lambda(s)\,ds} X_t + \int_t^{t+h} \lambda(s)\mu(s)\, e^{-\int_s^{t+h} \lambda(u)\,du}\, ds + \int_t^{t+h} \sigma(s)\, e^{-\int_s^{t+h} \lambda(u)\,du}\, dL_s,$$

for all $t \in \mathcal{T}$ and $h \geq 0$.

This corollary is maybe the most important characteristic of OU processes. In a more general form it can be used to find a certain equivalence between stationary OU processes and Lévy self-decomposable random variables, see Proposition 2.35 in the next section. Furthermore, this property is the motivation behind the *affine decomposition* in Definition 5.15 on page 79 and it will be exploited to compute the price of options in the case of Gaussian OU processes, cf. Section 2.2.2, Chapter 4, and Chapter 6.

Lemma 2.29. *Let L be a Lévy process. Assume that $\mu \equiv 0$ and $x_0 = 0$. Then X_t is an additive process and its cumulant function is given by*

$$\log \phi_t^X(\theta) = \int_0^t \Psi\left(\sigma(s)\, e^{-\int_s^t \lambda(u)\,du}\theta\right)\, ds.$$

Proof. We see that

$$\log \mathbb{E}\left[e^{i\theta \int_0^t f(s)\,dL_s}\right] = \int_0^t \Psi(f(s)\,\theta)\,ds.$$

can be proven for piecewise constant functions f. Since we can approximate continuous functions by these simple functions, we can use monotone convergence to find that the claim holds for all continuous functions, so in particular for $f(s) = \sigma(s)\, e^{-\int_s^t \lambda(u)\,du}$. □

2.2.1. Stationary Ornstein-Uhlenbeck processes

In this section we simplify the setting to $\lambda(t) = \lambda > 0$, $\mu \equiv 0$, and $\sigma \equiv 1$. Furthermore, we assume that X_0 is a $\mathcal{F}_0$-measurable random variable independent from the Lévy process $\{L_t; t \geq 0\}$. Equation (2.10) then reduces to

$$X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)}\, dL_s. \tag{2.11}$$

From Wolfe (1982, Theorem 1) we know that this process converges as $t \to \infty$:

Lemma 2.30. *A sufficient and necessary condition for Ornstein-Uhlenbeck process of Equation* (2.11) *to converge in distribution to a random variable X_∞ as $t \to \infty$ is*

$$\mathbb{E} \log^+ |L_1| < \infty. \tag{2.12}$$

In the rest of this section we extend the time horizon to include the negative half-line, i.e. $\mathcal{T} = (-\infty, \infty)$.

Definition 2.31 (Negative time horizon)**.** The Lévy process $L = \{L_t; t \geq 0\}$ is extended to the negative half-line through independent copies, i.e. we define

$$L_t \overset{\mathrm{d}}{:=} L_{-t}$$

for $t < 0$. However, the copy on the negative half-line is modified to also be càdlàg.

If Equation (2.12) is satisfied, we can exploit the extension of Lévy processes to the negative half-line given by Definition 2.31 to rewrite

$$X_\infty \overset{\mathrm{d}}{=} \lim_{t\to\infty} \int_0^t e^{-\lambda(t-s)}\, dL_s \overset{\mathrm{d}}{=} \lim_{t\to\infty} \int_{-t}^0 e^{\lambda u}\, dL_{u+t} \overset{\mathrm{d}}{=} \int_{-\infty}^0 e^{\lambda u}\, dL_u,$$

where we substituted $u = s - t$ and used the stationary increment property.

Lemma 2.32. *If Equation* (2.12) *is satisfied and* $X_0 \overset{d}{=} X_\infty$*, then we can represent Equation* (2.11) *as*

$$X_t = \int_{-\infty}^t e^{-\lambda(t-s)}\, dL_s \tag{2.13}$$

for all $t \in \mathcal{T}$.

Proof. See the results Wolfe (1982); Jurek and Vervaat (1983); Barndorff-Nielsen and Shephard (2001), for example, and the heuristic argument above. □

Theorem 2.33 (Stationary OU process)**.** *The OU process of Equation* (2.13) *is stationary, i.e. we have that* $X_t \overset{d}{=} X_{t+h}$ *for all* $t \in \mathcal{T}$ *and* $h \geq 0$.

Proof. Rigorous proofs are given in Wolfe (1982); Barndorff-Nielsen and Shephard (2001). Heuristically, the result can be seen by substitution $u = s + h$ in the integral

$$X_t = \int_{-\infty}^t e^{-\lambda(t-s)}\, dL_s \overset{\mathrm{d}}{=} \int_{-\infty}^{t+h} e^{-\lambda(t-(u-h))}\, dL_{u-h} \overset{\mathrm{d}}{=} \int_{-\infty}^{t+h} e^{-\lambda(t+h-u)}\, dL_u \overset{\mathrm{d}}{=} X_{t+h},$$

using again the stationary increment property. □

Corollary 2.28 shows a way that a OU process decomposes with itself in distribution. This notion is formalized with the following definition:

Definition 2.34 (Self-decomposability)**.** A random variable Z is called *self-decomposable*, if for each constant $c \in (0,1)$ there exists a characteristic function ϕ_c such that the characteristic function of Z decomposes as

$$\mathbb{E}\left[e^{iuZ}\right] = \phi_c(u)\, \mathbb{E}\left[e^{icuZ}\right]$$

for all $u \in \mathbb{R}$.

The following result shows a certain equivalence between self-decomposability and OU processes of the form of Equation (2.13) (Sato, 2013, Theorem 17.5).

Proposition 2.35. *Let* Z *be a self-decomposable random variable, then there exists a Lévy process* L *satisfying Equation* (2.12) *such that the OU process induced by Equation* (2.13) *converges in distribution to* Z *as* $t \to \infty$.

Conversely, let X *be an OU process defined by Equation* (2.13) *and assume that the driving Lévy process* L *satisfies Equation* (2.12)*, then* X *converges in distribution to a random variable* X_∞*, which is self-decomposable.*

The latter statement we already knew from Lemma 2.30. However, the former statement is new and closes the bijective relation between self-decomposability and Lévy processes. For further reading on Lévy driven OU processes we refer the interested reader to Sato (2013); Maller et al. (2009); Wolfe (1982); Barndorff-Nielsen and Shephard (2001); Jurek and Vervaat (1983); Applebaum (2009); Protter (2005) and all the references therein.

2.2.2. Gaussian Ornstein-Uhlenbeck processes

In this section we consider the special case where we assume that the driving Lévy process $L = W$ is a Brownian motion. This type of OU process is called *Gaussian*.

Theorem 2.36 (Normal distribution). *For all $t \in \mathcal{T}$ and $h \geq 0$ the OU process at time $t+h$ and conditioned at time t, i.e. $X_{t+h} \,|\, X_t = x$, is normally distributed with mean*

$$\mu_{t,h} := e^{-\int_t^{t+h} \lambda(s)\, ds} x + \int_t^{t+h} \lambda(s)\mu(s)\, e^{-\int_s^{t+h} \lambda(u)\, du}\, ds$$

and variance

$$\Sigma_{t,h}^2 := \int_t^{t+h} \sigma(s)^2\, e^{-\int_s^{t+h} 2\lambda(u)\, du}\, ds.$$

Proof. From Corollary 2.28 it immediately folows that

$$(X_{t+h} \,|\, X_t = x) = \mu_{t,h} + \int_t^{t+h} \sigma(s)\, e^{-\int_s^{t+h} \lambda(u)\, du}\, dW_s.$$

Since the characteristic exponent of W equals $\Psi(\theta) = -\frac{1}{2}\theta^2$, cf. Example 2.14, it follows from Lemma 2.29 that the cumulant function of

$$Y := \int_t^{t+h} \sigma(s)\, e^{-\int_s^{t+h} \lambda(u)\, du}\, dW_s$$

is given by

$$\phi^Y(\theta) = -\frac{1}{2}\theta^2 \int_t^{t+h} \sigma(s)^2\, e^{-\int_s^{t+h} 2\lambda(u)\, du}\, ds,$$

from which the result follows by Proposition 2.10. □

As a consequence of the above theorem, we can price options on OU processes. As an auxiliary variable we define

$$\Delta_{t,h} := \frac{\mu_{t,h} - K}{\Sigma_{t,h}},$$

for $K \in \mathbb{R}$. The constant K will take the role of the strike price in the application of option pricing in Chapter 4 and Chapter 6. We omit discounting with a deterministic interest rate because this is equivalent to multiplication by a constant.

Corollary 2.37 (Conditional expectation of options' payoff)**.** *For all $t \in \mathcal{T}$ and $h \geq 0$ the following conditional expectations are given by*

$$\mathbb{E}\left[(X_{t+h} - K)^+ \mid X_t = x\right] = (\mu_{t,h} - K)\, \Phi\left(\Delta_{t,h}\right) + \frac{\Sigma_{t,h}}{\sqrt{2\pi}} e^{-\frac{1}{2}\Delta_{t,h}^2}$$

and

$$\mathbb{E}\left[(K - X_{t+h})^+ \mid X_t = x\right] = (K - \mu_{t,h})\, \Phi\left(-\Delta_{t,h}\right) + \frac{\Sigma_{t,h}}{\sqrt{2\pi}} e^{-\frac{1}{2}\Delta_{t,h}^2}$$

where $\Phi(x)$ is the cumulative distribution function of a the standard normal distribution.

Proof. Denoting ε for a standard normal distributed random variable independent of the OU process X, we directly compute with the help of Theorem 2.36

$$\begin{aligned}\mathbb{E}\left[(X_{t+h} - K)^+ \mid X_t = x\right] &= \mathbb{E}\left[(\mu_{t,h} + \Sigma_{t,h}\varepsilon - K)\, \mathbb{1}_{\mu_{t,h}+\Sigma_{t,h}\varepsilon \geq K}\right] \\ &= (\mu_{t,h} - K)\, P\left(\varepsilon \geq -\Delta_{t,h}\right) + \Sigma_{t,h}\mathbb{E}\left[\varepsilon\, \mathbb{1}_{\varepsilon \geq -\Delta_{t,h}}\right],\end{aligned}$$

from which the result for the first conditional expectation follows by direct computation of the second term. The second conditional expectation follows analogously. □

In the special case that we assume constant parameters for the OU process X, we can compute the maximum likelihood estimators (MLE) for μ, λ, and σ. First of all, we immediately recognize that Corollary 2.28 reduces to

$$X_{t+h} = e^{-\lambda h} X_t + \left(1 - e^{-\lambda h}\right)\mu + \sigma \int_t^{t+h} e^{-\lambda(t+h-s)}\, dW_s. \tag{2.14}$$

for all $t \in \mathcal{T}$ and $h \geq 0$.

Lemma 2.38 (Maximum likelihood estimation)**.** *Let $x_0, x_1, x_2, \ldots, x_n$ be equidistant observations of the Ornstein-Uhlenbeck process, i.e. we assume $x_i := X_{hi}(\omega)$ for fixed $h \geq 0$ and $i = 0, 1, 2, \ldots, n$, then the MLEs $\hat{\mu}$ and $\hat{\lambda}$ are given by solving*

$$\min_{\mu,\lambda} \sum_{i=1}^{n} \left(x_{i+1} - e^{-\lambda h} x_i - \left(1 - e^{-\lambda h}\right)\mu\right)^2.$$

The MLE of $\hat{\sigma}^2$ is given by

$$\hat{\sigma}^2(\hat{\mu}, \hat{\lambda}) := \frac{2\hat{\lambda}}{n\left(1 - e^{-2\hat{\lambda} h}\right)} \sum_{i=1}^{n} \left(x_{i+1} - e^{-\hat{\lambda} h} x_i - \left(1 - e^{-\hat{\lambda} h}\right)\hat{\mu}\right)^2.$$

Proof. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be n independent, standard normally distributed random variables. Then it is clear from Equation (2.14) that

$$x_{i+1} \overset{\mathrm{d}}{=} e^{-\lambda h} x_i + \left(1 - e^{-\lambda h}\right)\mu + \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda h}\right)\varepsilon_{i+1},$$

The log-likelihood of the ith observation is given by

$$-\frac{1}{2}\log\left(2\pi\frac{\sigma^2}{2\lambda}\left(1-e^{-2\lambda h}\right)\right)-\frac{\lambda}{\sigma^2\left(1-e^{-2\lambda h}\right)}\left(x_{i+1}-e^{-\lambda h}x_i-\left(1-e^{-\lambda h}\right)\mu\right)^2.$$

Summing over all n observations and setting the derivative with respect to σ^2 equal to zero, yields the formula for the estimator $\hat{\sigma}^2$. Plugging the estimator $\hat{\sigma}^2$ back in yields that we want to solve the minimization problem

$$\min_{\mu,\lambda}\log\left(\frac{\hat{\sigma}^2(\mu,\lambda)}{2\lambda}\left(1-e^{-2\lambda h}\right)\right).$$

Since the logarithm is an increasing function this is equivalent to solving

$$\min_{\mu,\lambda}\frac{\hat{\sigma}^2(\mu,\lambda)}{2\lambda}\left(1-e^{-2\lambda h}\right),$$

from which the result follows. □

2.3. The Esscher transform

The classical Black-Scholes setting models a complete market. It can be proven that there exist a unique equivalent martingale measure: the *risk-neutral measure*. Under this measure all risky assets have the same trend, equaling the risk-free interest rate r. Furthermore, every integrable contingent claim on these risky assets is attainable, meaning it can be replicated by a self-financing trading strategy (Harrison and Pliska, 1981). In such a setting the price of a contingent claim is clear: it should equal the price that the replicating trading strategy costs. Due to its non-storability the electricity market is in general incomplete. Since electricity contracts with fixed delivery periods cannot be traded continuously, there do not exist replication strategies. It follows that there is no unique price for contingent claims.

Many classical studies on the modelling of commodity prices take the convenience yield, cost of carriage, and storage costs into account for determining a *pricing measure* (Hull, 2000, Chapter 33). However, these concepts are redundant for electricity prices as they are non-storable (Geman and Roncoroni, 2006). Therefore the pricing methods for other storable commodities can also not be transferred to the case of electricity. Different pricing methods for electricity derivatives had to be found. Chapter 6 thoroughly discusses the relation between spot and futures prices and the role of the risk-neutral measure. It offers a new viewpoint through the introduction of a theoretical *intrinsic electricity price*.

The basis for most measure changes in the case of Lévy processes – and in particular also Brownian motions – is the Esscher transform. Although it had already been in use by actuaries to assess risk, in a non-Gaussian setting it was first introduced for valuing derivatives by Gerber and Shiu (1994). The original Esscher transform is defined as follows:

Definition 2.39 (Esscher transform)**.** Let f be a probability density. For $\theta \in \mathbb{R}$ the transformed probability density

$$f(x;\theta) := \frac{e^{\theta x} f(x)}{\int_{\mathbb{R}} e^{\theta y} f(y)\, dy}$$

is called the *Esscher transform* of f (with parameter θ).

In order to use the Esscher transform in the context of Lévy processes we need sufficient conditions that ensure that the exponential moment exists. Therefore, we recall Sato (2013, Theorem 25.17):

Theorem 2.40. *Let $L = \{L_t; t \in \mathcal{T}\}$ be a Lévy process with generating triplet (γ, η^2, ℓ). If there exists $\theta \in \mathbb{R}$ such that*

$$\int_{|x|>1} e^{\theta x}\, d\ell(x) < \infty, \tag{2.15}$$

then the moment generating function of L_t exists at θ and is given by

$$\mathbb{E}[e^{\theta L_t}] = e^{t\,\Psi(-i\theta)},$$

where Ψ is the characteristic exponent as defined in Equation (2.4).

The random variable corresponding to the probability density given by the Esscher transform can be defined in the case of Lévy processes:

Definition 2.41 (Stochastic exponential)**.** Let L be a Lévy process and let $\theta \in \mathbb{R}$ be such that Equation (2.15) is satisfied. The stochastic process $\nu(\theta) = \{\nu_t(\theta); t \in \mathcal{T}\}$, defined by

$$\nu_t(\theta) := \frac{e^{\theta L_t}}{\mathbb{E}[e^{\theta L_t}]} = e^{\theta L_t - t\,\Psi(-i\theta)},$$

is called the *stochastic exponential* or *Doléans-Dade exponential* of L.

Remark 2.42 (Notation without θ). Although the stochastic exponential $\nu_t(\theta)$ depends on θ we will often not explicitly write this dependency since it is clear from the context, cf. Chapter 6.

We will use the stochastic exponential of L to define the Radon-Nikodym derivative of a pricing measure Q_θ,

$$\frac{dQ_\theta}{dP}\bigg|_{\mathcal{F}_t} := \nu_t(\theta). \tag{2.16}$$

It is clear that this will yield a probability measure since $\mathbb{E}\nu_t(\theta) = 1$. However, in order for this measure to be well-defined, we will need that the process $\nu_t(\theta)$ is a P-martingale:

Lemma 2.43. *Assume that $\theta \in \mathbb{R}$ fulfills Equation* (2.15). *Then the stochastic exponential $\nu(\theta)$ is a P-martingale.*

Proof. We compute the conditional expectation

$$\begin{aligned}\mathbb{E}[\nu_t(\theta)\,|\,\mathcal{F}_s] &= e^{\theta L_s - t\Psi(-i\theta)}\,\mathbb{E}\left[e^{\theta(L_t-L_s)}\right]\\ &= e^{\theta L_s - t\Psi(-i\theta)}\,e^{(t-s)\Psi(-i\theta)} = \nu_s(\theta),\end{aligned}$$

where we used the independent increments property of L and to compute the expectation of the exponential Lévy process. □

Since for all $t \in \mathcal{T}$ we have that $\nu_t(\theta) > 0$ P-almost surely, we see that Q_θ is equivalent to the original measure P.

Corollary 2.44. *If $\theta \in \mathbb{R}$ fulfills Equation* (2.15)*, then the measure Q_θ defined through its Radon-Nikodym derivative by Equation* (2.16) *is a probability measure that is equivalent to P.*

Remark 2.45 (Choice of θ). Apart from the condition given in Equation (2.15) the parameter θ is free to be chosen. Usually in financial mathematics, an *equivalent martingale measure Q* is characterized by the fact that all discounted tradable assets should be martingales under Q. If one stays within the class of Esscher transform induced measures Q_θ, this yields another condition on the parameter θ: for all $0 \leq u < t$ it should hold that

$$e^{-ru}S_u \overset{!}{=} \mathbb{E}_{Q_\theta}[e^{-rt}S_t\,|\,\mathcal{F}_u] = \frac{1}{\nu_u(\theta)}\mathbb{E}_P[e^{-rt}S_t\nu_t(\theta)\,|\,\mathcal{F}_u], \tag{2.17}$$

where S_t is the stochastic vector of all risky assets in the market.

Benth et al. (2008a) argue that electricity is not a tradable asset in the classical sense, since electricity has to be consumed (immediately) after it has been delivered. Due to this non-storability it is not possible to use classical *buy and hold* strategies. If we do not see the spot price as a tradable asset, then this implies that it is not necessary for the discounted spot price to be a martingale under Q and hence we have no other restrictions on θ than Equation (2.15). This argumentation has the convenient consequence that we do not have to solve the Equation (2.17) for θ, which could be quite complicated depending on the chosen model for S or which could yield a time-dependent θ (in which case we might lose the property of the process $\nu_t(\theta)$ being a P-martingale).

Another convenient consequence of using the Esscher transform for a pricing measure Q is that Lévy processes remain Lévy processes under the measure change:

Proposition 2.46. *The Lévy process L with generating triplet (γ, η^2, ℓ) under P is also a Lévy process under Q_θ with generating triplet $(\tilde{\gamma}, \tilde{\eta}^2, \tilde{\ell})$ given by*

$$\begin{aligned}\tilde{\gamma} &= \gamma + \eta^2\theta + \int_{|x|\leq 1} x(e^{\theta x} - 1)\,d\ell(x),\\ \tilde{\eta}^2 &= \eta^2,\\ d\tilde{\ell}(x) &= e^{\theta x}d\ell(x).\end{aligned} \tag{2.18}$$

Proof. We compute the cumulant function of L,

$$\begin{aligned}\log \mathbb{E}_{Q_\theta}[e^{iwL_t}] &= \log E_P[e^{(\theta+iw)L_t}] - \log E_P[e^{\theta L_t}] \\ &= t[\Psi(-i\theta + w)) - \Psi(-i\theta)],\end{aligned}$$

where Ψ is again the characteristic exponent. Since this is linear in t, we can set $t = 1$ and compute

$$\begin{aligned}\Psi(-i\theta + w)) - \Psi(-i\theta) = iw &\left(\gamma + \eta^2\theta + \int_{|x|\le 1} x(e^{\theta x} - 1)\, d\ell(x)\right) - \tfrac{1}{2}\eta^2 w^2 \\ &+ \int_{\mathbb{R}} \left(e^{iwx} - 1 - iwx\mathbb{1}_{|x|\le 1}\right) e^{\theta x}\, d\ell(x), \qquad (2.19)\end{aligned}$$

if we assume that

$$\int_{|x|\le 1} x(e^{\theta x} - 1)\, d\ell(x) < \infty,$$

which is actually already fulfilled by just assuming that θ satisfies Equation (2.15) (Sato, 2013, Remark 33.3). From Propositon 2.9 and 2.10, we see that Equation (2.19) implies that L is also a Lévy process under Q_θ, albeit with the said generating triplet. □

Example 2.47 (Black-Scholes model)**.** In a Black-Scholes market θ is uniquely determined by Equation (2.17). It yields the well-known risk-neutral measure given by the Girsanov theorem. We remark that the approach with the Esscher transform that we discuss here, is a generalization of the Girsanov theorem for Lévy processes, cf. Equation (2.18) for the generating triplet of a standard one-dimensional Brownian motion, i.e. $(\gamma, \eta^2, \ell) \equiv (0, 1, 0)$.

As an application of the above discussed theory, we compute the Esscher transform for the stock price in a (one-dimensional) Black-Scholes model, i.e. we have one risky asset S_t given by

$$S_t = S_0\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

where W_t is a standard one-dimensional Brownian motion under P and μ, $\sigma > 0$, and $S_0 > 0$ are constant. We can now compute the Radon-Nikodym derivative

$$\nu_t(\theta) = e^{\theta W_t - \frac{1}{2}\theta^2 t}.$$

Denote the risk-free rate by r, then we can determine θ by Equation (2.17), i.e. we want to have equality of

$$S_0\, e^{(\mu - \frac{1}{2}\sigma^2 - r)u + \sigma W_u} \stackrel{!}{=} e^{-\theta W_u + \frac{1}{2}\theta^2 u} \mathbb{E}_P[S_0\, e^{(\mu - \frac{1}{2}\sigma^2 - r)t + \sigma W_t}\, e^{\theta W_t - \frac{1}{2}\theta^2 t} \mid \mathcal{F}_u].$$

Rearranging the right side (without S_0) yields

$$\begin{aligned}&e^{-\frac{1}{2}\theta^2(t-u) + (\mu - r)t + \sigma W_u - \frac{1}{2}\sigma^2 t} \cdot \mathbb{E}_P[e^{(\sigma+\theta)(W_t - W_u)} \mid \mathcal{F}_u] \\ &= e^{-\frac{1}{2}\theta^2(t-u) + (\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_u + \frac{1}{2}(\theta+\sigma)^2(t-u)},\end{aligned}$$

where we used that $W_t - W_u$ is independent of $\mathcal{F}_u$. Now using the left side of Equation (2.17) as well, we see that we require equality of the exponents,

$$(\mu - \tfrac{1}{2}\sigma^2 - r)u + \sigma W_u \stackrel{!}{=} -\tfrac{1}{2}\theta^2(t-u) + (\mu - r - \tfrac{1}{2}\sigma^2)t + \sigma W_u + \tfrac{1}{2}(\theta+\sigma)^2(t-u),$$

which yields that θ should equal

$$\theta = -\frac{\mu - r}{\sigma}.$$

Finally, we see from Equation (2.18) that the Lévy triplet of the P-Brownian motion W_t under Q_θ is given by $(\theta, 1, 0)$, i.e.

$$W_t = \tilde{W}_t - \frac{\mu - r}{\sigma}t,$$

where $\tilde{W}_t$ is a Q_θ-Brownian motion. This is exactly the statement of the Girsanov theorem.

Remark 2.48 (Time-dependence). In the case of $L = W$ one can easily generalize the Esscher transform to a time-dependent variable θ_t. We will discuss this in Section 6.2.2.

2.4. Day-ahead spot price models

As discussed in Section 1.1 in the Introduction, we denote the time-continuous spot price process by $S = \{S(\tau); \tau \in \mathcal{T}\}$. Usually in the literature on electricity price modelling continuous-time models for the day-ahead spot price are developed, e.g. the classical Schwartz and Smith (2000) and Lucia and Schwartz (2002) models discussed in Section 5.3. However, the basic two classes of models that are important throughout this thesis are the class of *factor models* and the class of *structural models*, which we will introduce in the following two sections.

The basic idea of both modelling approaches is simple. As will be discussed in Section 3.1, the day-ahead spot prices' stylized facts are not captured well by normal distributions. They are heavy-tailed and exhibit short-living spikes. Direct modelling with Gaussian OU processes is therefore not an option. The factor model approach extends the modelling to the use of Lévy processes as driving processes and thus allowing for heavy-tailed distributed prices. The structural model approach solves this problem by applying a non-linear transformation to Gaussian OU processes. This Gaussian OU process has been given the economic interpretation of the system load, which has made it very appealing and easy to calibrate. However, due to the non-linear transformation it is quite hard to compute derivatives prices, i.e. conditional expectations, in this model setting. Factor models, on the other hand, are very flexible and usually yield tractable conditional expectations, but are hard to calibrate. Therefore, these two model classes are both very useful and it depends on their intended use, which one is better suited.

2.4.1. Factor models

The driving process of factor models is the sum of n different OU processes, also called factors, naming it an n-factor model. Following the general framework of Benth et al. (2008a) we will distinguish between factor models using this sum to model the day-ahead spot price itself $S(\tau)$ or its natural logarithm $\ln S(\tau)$. We call these variants *arithmetic* and *geometric*factor models, respectively.

Let $X^1, X^2, \ldots, X^n$ be OU processes with driving Lévy process $L^1, L^2, \ldots, L^n$ as defined in Section 2.2. Furthermore, let $\Lambda : \mathcal{T} \to \mathbb{R} : t \mapsto \Lambda(t)$ be a deterministic function capturing the seasonal behavior of the day-ahead spot price process $S = \{S(\tau); \tau \in \mathcal{T}\}$, then we call Λ a *seasonality function*. Section 3.2 discusses several choices of seasonality functions and assesses their performance for the German electricity market. Then we can define:

Definition 2.49 (Arithmetic factor model)**.** We define the day-ahead spot price to equal

$$S(\tau) := \Lambda(\tau) + \sum_{i=1}^{n} X^i_\tau,$$

and call this an *arithmetic factor model*.

This approach will prove to lead to more tractable models and allows for negative prices. Negative spot prices have been observed multiple times in recent data from the German spot market, as is discussed in Section 3.1.

Definition 2.50 (Geometric factor model)**.** We define the day-ahead spot price to equal

$$S(\tau) := \exp\left(\Lambda(\tau) + \sum_{i=1}^{n} X^i_\tau\right),$$

and call this a *geometric factor model*.

This approach does not allow for negative prices and has therefore been favorite over the past decades. However, as said before, the observation of negative prices in the most recent data is an argument against this type of model, but in favour of the arithmetic models. Furthermore, this type of model is less tractable when it comes to computing futures prices. For further reading we refer to Meyer-Brandis and Tankov (2008); Hambly et al. (2009); Benth et al. (2012, 2014); Gonzalez et al. (2017); Bennedsen (2017).

2.4.2. Structural models

Structural models have their roots in the work of Barlow (2002). As we said in the first part of this section, the idea behind this modelling approach is simple. To achieve the non-Gaussian behaviour we apply a non-linear transformation to one or more Gaussian Ornstein-Uhlenbeck processes.

The economic interpretation behind this approach is easy and accessible. Corresponding to a day-ahead spot price of $p \in \mathbb{R}$ let $g_\tau(p)$ and $d_\tau(p)$ be the electricity *generation* and *demand* at delivery time τ, respectively. Since supply and demand always have to be in balance, we find that

$$g_\tau(S(\tau)) = d_\tau(S(\tau))$$

for the day-ahead spot price $S(\tau)$. Barlow (2002) argues that in the case of the day-ahead spot market demand is very inelastic and therefore we assume that $D_\tau = d_\tau(p)$ independent of the price $p \in \mathbb{R}$. Furthermore, if we assume that the supply function $g(p) = g_\tau(p)$ is not random and independent of time, we find that

$$S(\tau) = g^{-1}(D_\tau),$$

which is the basis of all structural models.

Definition 2.51 (Structural model)**.** Let $D = \{D_\tau; \tau \in \mathcal{T}\}$ a real-valued stochastic process, which is adapted to the Brownian filtration, and let $M_\tau(x)$ be a (non-linear) time-dependent deterministic function. We define the day-ahead spot price to equal

$$S(\tau) := M_\tau(D_\tau),$$

and call this a *structural model*.

One simple example is to use a hyperbolic sine for the function M and model D by a Gaussian Ornstein-Uhlenbeck process. Already in this setting the day-ahead spot price is able to exhibit price spikes. However, the futures prices as discussed in the next section are harder to evaluate. For further reading we refer to Aïd et al. (2009); Lyle and Elliott (2009); Aïd et al. (2012); Wagner (2014).

2.4.3. Futures prices

A forward contract, which is entered at time t, has an effective payoff of $S(\tau) - f_t(\tau)$ at time τ. Denote the risk-free interest rate by r and assume that $S(\tau)$ has finite expectation. Assuming that we have a pricing measure Q at our disposal, the expected discounted payoff should equal

$$\mathbb{E}_Q\left[e^{-r(\tau-t)}(S(\tau) - f_t(\tau)) \,|\, \mathcal{F}_t\right] = 0,$$

since it is free to enter the forward contract. Since the value of a forward contract $f_t(\tau)$ can only be determined with the market information up to time t, it is natural to assume that it is $\mathcal{F}_t$-measurable. Therefore, we can use the previous relation to find the *forward price*

$$f_t(\tau) = \mathbb{E}_Q[S(\tau) \,|\, \mathcal{F}_t].$$

This is also an intuitive relation between the spot price and its forward: the forward price is the orthogonal projection of $S(\tau)$ on $L^2(\Omega, \mathcal{F}_t, Q)$ in $L^2(\Omega, \mathcal{A}, Q)$. This statement is formalized in the following lemma:

Lemma 2.52. *Consider $X \in L^2(\Omega, \mathcal{A}, P)$. Let $\mathcal{B}$ be a sub-σ-algebra $\mathcal{B} \subseteq \mathcal{A}$. Then for all $Y \in L^2(\Omega, \mathcal{B}, P) \subseteq L^2(\Omega, \mathcal{A}, P)$ it holds that*

$$\mathbb{E}[Y(X - \mathbb{E}[X \mid \mathcal{B}])] = 0,$$

and thus $E[X \mid \mathcal{B}]$ is an orthogonal projection of $X \in L^2(\Omega, \mathcal{A}, P)$ on $L^2(\Omega, \mathcal{B}, P)$.

Proof. The claim follows directly by direct computation

$$\begin{aligned}\mathbb{E}[Y(X - \mathbb{E}[X \mid \mathcal{B}])] &= \mathbb{E}[YX] - \mathbb{E}[Y\,\mathbb{E}[X \mid \mathcal{B}]] \\ &= \mathbb{E}[YX] - \mathbb{E}[\mathbb{E}[YX \mid \mathcal{B}]] \\ &= \mathbb{E}[YX] - \mathbb{E}[YX] = 0,\end{aligned}$$

where the second equality follows by $\mathcal{B}$-measurability of Y and the third by the law of total expectation. □

Denote $F_t(\tau_1, \tau_2)$ for the futures price at time t of a contract delivering from τ_1 to τ_2. As a technical assumption we assume that

$$\mathbb{E}_Q\left[\int_{\tau_1}^{\tau_2} |S(u)|\, du\right] < \infty. \tag{2.20}$$

If financial settlements are made continuously during the delivery period, the expected discounted payoff should equal

$$\mathbb{E}\left[\int_{\tau_1}^{\tau_2} e^{-r(u-t)}(S(u) - F_t(\tau_1, \tau_2))\, du \,\Big|\, \mathcal{F}_t\right] = 0,$$

since a futures contract is free to enter at time t. We use the same technique as we used for forward contracts to find the price of a futures contract:

$$F_t(\tau_1, \tau_2) = \frac{1}{\int_{\tau_1}^{\tau_2} e^{-ru}\, du}\mathbb{E}_Q\left[\int_{\tau_1}^{\tau_2} e^{-ru} S(u)\, du \,\Big|\, \mathcal{F}_t\right].$$

To rewrite this equation further, we need the Fubini-Tonelli theorem:

Theorem 2.53 (Fubini-Tonelli theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let $f(x, y)$ be a $(X, \mathcal{A}) \times (Y, \mathcal{B})$-measurable function. If we have that*

$$\begin{aligned}&\int_{X\times Y} |f(x,y)|\, d(\mu \times \nu)(x,y) < \infty \quad \textit{or,} \\ &\int_X \int_Y |f(x,y)|\, d\nu(y) d\mu(x) < \infty \quad \textit{or,} \\ &\int_Y \int_X |f(x,y)|\, d\mu(x) d\nu(y) < \infty,\end{aligned}$$

then $f_X(y) := \int_X f(x,y)\, d\mu(x) \in L^1(Y, \mathcal{B}, \nu)$ and $f_Y(x) := \int_Y f(x,y)\, d\nu(y) \in L^1(X, \mathcal{A}, \mu)$. Furthermore, the the following three integrals are equal

$$\int_{X\times Y} f(x,y)\, d(\mu \times \nu)(x,y) = \int_X \int_Y f(x,y)\, d\nu(y) d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x) d\nu(y).$$

Proof. See Theorem 2.37 in Folland (1999). □

Corollary 2.54 (Futures price)**.** *If Equation* (2.20) *is satisfied, then the* futures price *is also given as*

$$F_t(\tau_1, \tau_2) = \frac{1}{\int_{\tau_1}^{\tau_2} e^{-ru}\, du} \int_{\tau_1}^{\tau_2} e^{-ru} f_t(u)\, du,$$

where $f_t(\tau)$ *is the forward price.*

Proof. Direct consequence of the Fubini-Tonelli theorem with Equation (2.20). □

The above corollary is also valid in the special case that $r = 0$ assumed. When a day-ahead spot price model is chosen, the forward price $f_t(\tau)$ can be computed explicitly and, as a consequence, it can be used to compute the futures prices. In the case of factor models this is usually easier to do than in the case of structural models.

3. Stylized facts, seasonality, and calibration of day-ahead factor models[1]

In this chapter we focus on the class of *arithmetic factor models*, as already introduced in Section 2.4.1. Over the last decade research on this model class has progressed by comparing different calibration techniques and pricing new derivatives, e.g. Meyer-Brandis and Tankov (2008); Hambly et al. (2009); Benth et al. (2012, 2014); Gonzalez et al. (2017); Bennedsen (2017). However, since these studies were conducted, the German electricity market has changed significantly due to the feed-in from renewable energy sources (Hagfors et al., 2016a; Benhmad and Percebois, 2018; Paraschiv et al., 2014; Gianfreda and Bunn, 2018). Therefore, it is of interest to reevaluate how these changes have influenced the *stylized facts and features* of the energy prices and whether factor models can still model them effectively.

The goal of this chapter is to examine and to give guidelines on how to model German day-ahead spot prices by factor models such that they take the current stylized facts into account. In Section 3.1 we present all stylized facts and features of electricity spot prices and review the relevant literature on them in detail. Furthermore, we confirm them by an empirical study of the baseload EPEX Germany/Austria day-ahead spot prices from 2011 to 2016. In Section 3.2 we conduct a detailed study of six different seasonality functions and introduce three criteria to determine good seasonality functions. Section 3.3 shows how discretised factor models can be interpreted as ARMA time series. This is an essential relation that can be exploited to estimate the mean reversion speed parameters of all factors and can be used to find the number of factors that explain the stochastic behaviour optimally. Finally, in Section 3.4 we calibrate a one-factor and a two-factor model to the first five years of data and use the models to simulate the consecutive year. We introduce several methods to determine the likeliness of the realised spot price path under a given model. We apply these methods to the calibrated models and find that factor models are well-suited candidates for spot price modelling.

3.1. Stylized facts and features

Throughout the rest of this chapter we look at a data set of baseload day-ahead spot prices for Germany and Austria (Phelix) from 1 January 2011 to 31 December 2016. The data set contains 2192 daily observations. Figure 3.1 shows the daily baseload

[1]Based on published work: Hinderks and Wagner (2019a). Original title: Factor models in the German electricity market: Stylized facts, seasonality, and calibration.

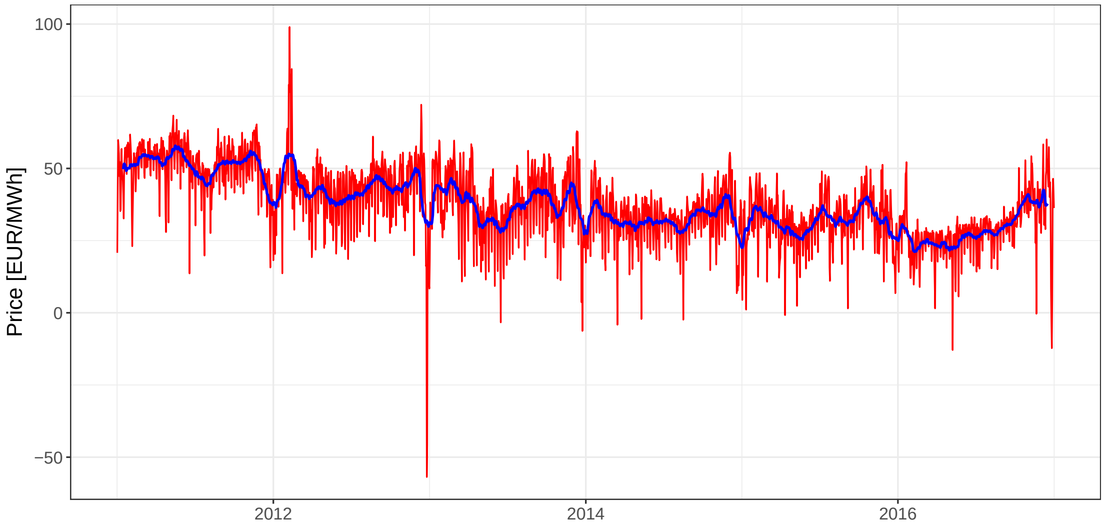


Figure 3.1.: EPEX baseload day-ahead spot price. The prices for Germany and Austria from 1 January 2011 to 31 December 2016 (red) are shown together with a 30 day centered moving average (blue). Several downward and upward spikes are visible, as well as several occurrences of negative prices.

price, i.e. the average of the 24 hourly prices quoted for each day. Table 3.2 shows several descriptive statistics of the data set grouped by year.

We recommend Meyer-Brandis and Tankov (2008) for a particular good overview of the behaviour of spot prices. They analyse the day-ahead spot prices from six different exchanges on two different continents. The data set they consider for each of these exchanges ranges from five years to well over a decade and their data set ends in November 2006. In general we follow their classification of stylized facts: spikes, negative prices, seasonality, mean reversion, stationarity, autocorrelation, and the non-Gaussian distribution.

Spikes The fundamental feature any spot price model should exhibit are spikes: occasional rapid and intense jumps away from the normal price level, almost immediately followed by a jump back to the normal price level. This feature is already incorporated in the jump-diffusion models of Deng (2000) or the regime-switching model of Geman and Roncoroni (2006) and is considered one of the most important characterizations.

Over the years there have been many studies analysing the spike behaviour in day-ahead spot prices in many different markets. Janczura et al. (2013) study techniques to identify extreme prices and show that they have a big influence on the estimation of the model. Hagfors et al. (2016a) study the impact of fundamental drivers on both positive and negative spikes in the German day-ahead spot market and conclude that both have quite different dynamics. Higgs and Worthington (2008) study the modelling of spikes in the Australian market, one of the most spiky markets in the world, by comparing a

Table 3.2.: Descriptive statistics of the data set per year and as a whole. The shown statistics are the number of observations (and between brackets the number of negative prices), the sample mean and median, the minimum and maximum, the sample standard deviation, skewness, and kurtosis (normal distribution has kurtosis 3).

Year	# (<0)	Mean	Med	Min	Max	Std	Skew	Kurt
2011	365 (0)	51.12	51.99	13.63	68.30	8.3202	-1.2891	5.7587
2012	366 (2)	42.59	43.91	-56.87	98.98	12.8210	-1.9013	19.3955
2013	365 (2)	37.78	37.13	-6.28	62.89	11.4849	-0.4040	3.4510
2014	365 (3)	32.76	33.20	-4.13	55.48	8.7367	-0.6807	4.9466
2015	365 (1)	31.62	32.14	-0.80	51.27	8.9416	-0.5952	3.9071
2016	366 (4)	28.97	28.48	-12.89	60.06	9.6530	-0.1959	5.5253
	2192 (12)	37.47	36.68	-56.87	98.98	12.6160	-0.3341	5.4258

basic stochastic model, a mean-reverting model, and a regime-switching model.

Negative prices Due to the German *Energiewende*[2] an increase in downward spikes has been observed in the most recent data from the German spot market (Hagfors et al., 2016b; Benth et al., 2014). This is caused by increased feed-in from renewable sources, which shifts the merit order curve – the so-called *merit order effect* – and induces an overall decrease of the German spot prices (Paraschiv et al., 2014; Benhmad and Percebois, 2018; Gianfreda and Bunn, 2018; Ketterer, 2014). This is also visible in the yearly decreasing mean in Table 3.2.

Due to the merit order effect and the market design the German day-ahead spot price occasionally becomes negative (Gawel et al., 2015; Paraschiv et al., 2014; Zipp, 2017). This is an extremely important new stylized fact and any realistic model should therefore be able to simulate negative spot prices. Hence, the geometric modelling of the spot price, i.e. the modelling of its logarithm as in Schwartz and Smith (2000); Bierbrauer et al. (2007); and many others, cannot be applied without modifications any more. A possible solution from interest rate models is to use a shifted model, i.e. to model $\log(S_t + c)$ instead of $\log(S_t)$. Alternatively, an arithmetic model can be used. One of the first studies to develop both geometric and arithmetic models is Lucia and Schwartz (2002).

Figure 3.1 illustrates that negative prices even occur in the daily baseload prices. In

[2]The Energiewende is the transition from using classical fossil fuel (e.g. coal) powered energy sources to exploiting modern sustainable and renewable energy sources (e.g. wind and solar power).

the whole data set there are 12 days with a negative daily baseload price. In hourly prices, negative fixings appear regularly and occurred in 1.4% of the hours in 2015 and 1.1% of the hours in 2016. Hagfors et al. (2016a) study the extreme price events in the hourly prices in more detail.

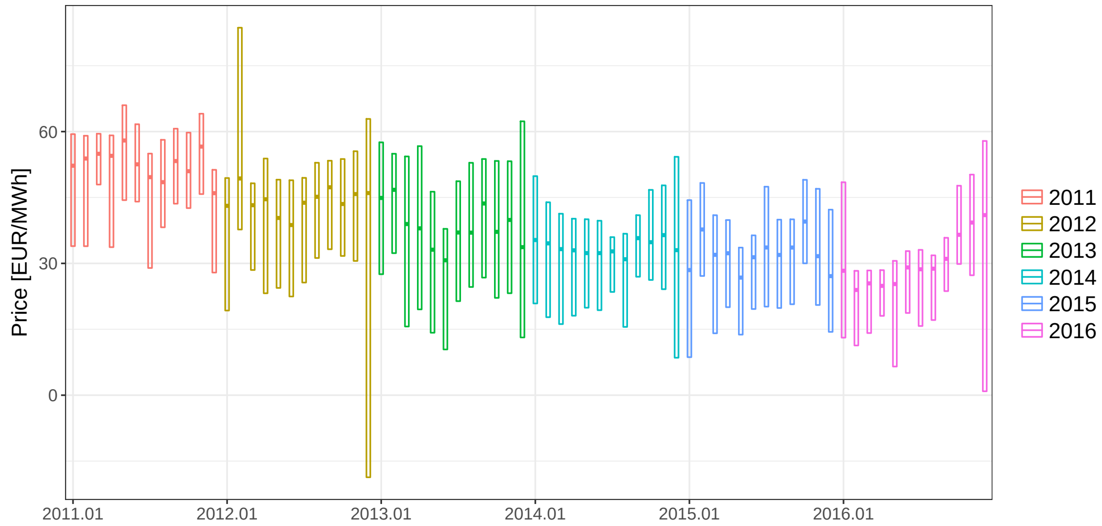
Seasonality Applications of electricity are manifold: power, light, and transportation to name a few. For heating in particular, electricity can act as a substitute for oil and gas (Geman, 2005). Therefore, consumption depends strongly on the weather, the time of the year, and the day of the week. Even the earliest academic reviews take (at least the yearly) seasonality into account. It is common practice to capture the seasonal and cyclic behaviour by a deterministic function, referred to as the *seasonality function*, cf. Section 2.4. The deseasonalised price time series is then usually assumed to follow some stochastic model.

Throughout the years two types of seasonality functions have dominated: *dummy variables* and *sinusoidal functions*. The dummy variable method is based on the use of piecewise constant functions: e.g. a constant value for every one of the twelve months, and/or every season, and/or every single day of the week, and so on. The seasonality function with sinusoidal terms combined with a linear trend and possibly dummy variables seems to be the consensus in modern literature (Lucia and Schwartz, 2002; Geman and Roncoroni, 2006; Bierbrauer et al., 2007; Benth et al., 2008a; Meyer-Brandis and Tankov, 2008; Wagner, 2014). Apart from these two methods, Weron (2014) also suggest the use of wavelets and Kiesel et al. (2018) introduce a sinusoidal spline method. In Section 3.2 we will compare several seasonality functions.

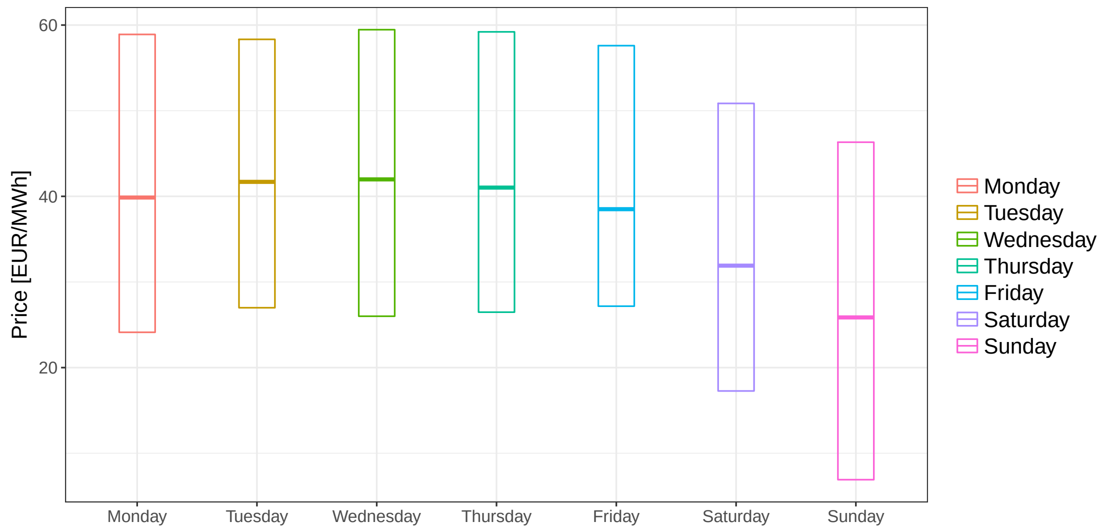
Figure 3.3a and 3.3b summarise the baseload spot prices to illustrate the seasonal behaviour. In each year the winter months are generally more expensive than the summer months – except for December, which is due to the Christmas holidays and reduced industrial production. This feature is very distinctive in the first three years of the data set. In 2016 the monthly pattern seems slightly different from the other years, with a big increase of the prices in the last months of the year. The reason for this unusual behaviour was the limited availability of French nuclear power in the winter 2016/2017 and therefore unusual high demand from France. This is visible in both Figure 3.1 and 3.3a. The weekly seasonality is observable in Figure 3.3b. In general, weekends are cheaper than the rest of the week. This effect is called the *weekend effect* by Meyer-Brandis and Tankov (2008). Furthermore, Mondays and Fridays are in general cheaper than the Tuesdays, Wednesdays, and Thursdays. Weron (2014) therefore states that it is best to use weekly dummy variables for Mondays, Fridays, Saturdays, and Sundays.

From both Figure 3.1 and 3.3a we also observe a yearly downward trend, which can be explained by the increase in renewable energy sources in the German market. We do not expect this trend to continue forever and, in fact, the last months of 2016 could be evidence that the trend is changing.

Mean reversion Like many other commodities, electricity prices show mean reversion to a seasonal mean (Pilipovic, 1997; Higgs and Worthington, 2008; Escribano et al.,



(a) Grouped per month. The plot shows the yearly seasonality and the yearly trend. In general, winters seem more expensive than summers. Noteworthy is also December 2012, when there were 2 days with negative prices, inducing a negative 5% quantile.



(b) Grouped per day of the week. Holidays are counted as Sundays, and partial holidays (i.e. holidays in some but not all German federal states) and bridge days as Saturdays. This plot shows the weekly seasonality. In general, weekends seem cheaper than the rest of the week and Mondays and Fridays seem slightly cheaper than the Tuesdays, Wednesdays, and Thursdays.

Figure 3.3.: Summary of the data set. The median, 5%, and 95% quantiles are shown.

2011). Geman and Roncoroni (2006) note that the price process shows small stochastic variations around this average seasonal trend, which represents temporary supply and demand imbalances.

Autocorrelation Meyer-Brandis and Tankov (2008) find that the autocorrelation function (ACF) of the base spot prices of several exchanges is well represented by the sum of two weighted exponentially decaying terms:

$$\rho(t) = \rho_1 \, e^{-\lambda_1 t} + \rho_2 \, e^{-\lambda_2 t}. \tag{3.1}$$

Such an autocorrelation structure can be modelled using two independent Ornstein-Uhlenbeck processes. They fit the parameters to the empirical autocorrelation function and find two mean reversion speeds: one with correlation length around 100 days, which corresponds to a slow stochastic variation around the seasonal trend, and one with correlation length of approximately four days, which corresponds to the spikes.

Equation (3.1) is often used to estimate the mean reversion speed of the different factors in factor models. In Section 3.3 we will introduce an alternative method to estimate the mean reversion speed.

Stationarity Both Lucia and Schwartz (2002) and Meyer-Brandis and Tankov (2008) conclude that the electricity prices exhibit stationary behaviour. Both conduct (augmented) Dickey-Fuller tests for unit roots[3] at 5% and 1% significance level, respectively, and both reject the hypothesis.

In Section 3.2 we also conduct our own statistical tests for unit roots on the deseasonalised prices: augmented Dickey-Fuller and Phillips-Perron tests. We conclude at 1% significance that the the residuals after deseasonalisation are stationary in the sense that they do not have a unit root. Furthermore, from descriptive statistics grouped per year, we find (approximate) moment stationarity.

Non-Gaussian distribution Lucia and Schwartz (2002); Geman and Roncoroni (2006); Bierbrauer et al. (2007); Meyer-Brandis and Tankov (2008); among others conclude from the sample skewness and kurtosis[4] that the daily spot prices and their returns of several different exchanges are not normally distributed. They find a positive skewness for both the prices itself and their returns, indicating that their distributions have a heavier right tail. Moreover, they find a sample kurtosis that is greater than three, implying that the daily spot prices and their returns are leptokurtic.[5] Due to this fact Benth et al. (2008a) state that several heavy-tailed distributions have proven to be accurate in modelling the daily spot price returns.

[3] The Dickey-Fuller test is a statistical test for autoregressive time series with null hypothesis H_0 : the time series has a unit root (i.e. it is non-stationary) and alternative H_1 : it does not have a unit root (i.e. it is stationary). The augmented Dickey-Fuller test is an extension of the normal Dickey-Fuller test such that it assumes that the time series is autoregressive with trend.

[4] The normal distribution has skewness equal to zero and kurtosis equal to three.

[5] This means that they have heavier tails than the normal distribution.

Table 3.2 shows several descriptive statistics for the daily spot prices of the data set. From the sample skewness and kurtosis we conclude that the non-Gaussian distribution seems a valid assumption, since the skewness deviates from zero and the kurtosis is greater than three. However, in contrast to the literature the sample skewness is negative, implying that we find a heavier left tail. To confirm these findings we executed a D'Agostino test of skewness (d'Agostino, 1970) and Anscombe-Glynn test of kurtosis (Anscombe and Glynn, 1983). The first test rejected the null hypothesis of having skewness equal to zero at 1% (p-value: 4.2E-10) and the latter rejected the null hypothesis of having kurtosis equal to three at 1% (p-value: 2.2E-16).

This is also in agreement with our observation of spikes. Furthermore, the skewness implies that the spikes are primarily negative. Finally, the downward trend is also clear from the decreasing mean and median and variance stationarity can be seen in the last three years of the data set.

In the next section we compare several seasonality functions and show the three remaining stylized facts on the deseasonalized data: stationarity, autocorrelation, and non-Gaussian distribution.

3.2. Seasonality functions

The seasonality function is of utmost importance to the calibration process, since a different deseasonalization results in a different calibration and therefore determines the quality of the estimation (Weron, 2014; Janczura et al., 2013). In the following we compare different seasonality functions. We define desirable properties of a seasonality function for modelling spot prices by a factor model:

- *S.1* Season: capture all seasonal and cyclic behaviour,
- *S.2* Autocorrelation: exponentially decaying empirical autocorrelation function of the residuals,
- *S.3* Noise: stationary, non-Gaussian residuals.

It is clear that a good seasonality function should satisfy S.1. Additionally, we require S.2 and S.3 in order to assess the quality of different seasonality functions that satisfy S.1. From the theory of stochastic processes we actually know that a factor model has the properties S.2 and S.3, see Section 3.3. Furthermore, for simulation purposes it is necessary that a seasonality function can be extrapolated into the future. Therefore, we do not consider wavelets in our comparison but refer the interested reader to discussion by Weron (2014).

We consider six different seasonality functions and compare them according to the criteria S.1–S.3. Five of the following six seasonality functions are of the class dummy variables and Fourier series as discussed by Kiesel et al. (2018, Section 2). Dummy variables are indicator functions that have a specific value when the time point suffices certain conditions and the Fourier series seasonality functions are a superposition of sinusoidal functions. Concerning futures prices Kiesel et al. (2018) compare one seasonality function of each of these two classes and a spline seasonality function. The spline seasonality function allows for a better fit to each separate month, which might be caused by it having more free parameters. However, we want to use as few parameters as possible to avoid overfitting and as such we do not consider the spline method here.

Unless explicitly stated otherwise we estimate the following six seasonality functions by linear least squares:

Classic Sinusoidal (CS) We fit the sinusoidal

$$\Lambda_{\text{CS}}(t) = c_1 + c_2 t + c_3 \sin(2\pi t) + c_4 \cos(2\pi t) + c_5 \sin(4\pi t) + c_6 \cos(4\pi t)\,.$$

We expect that this method is not able to capture the weekly seasonality properly.

Dummy Sinusoidal (DS) To account for the missing weekly seasonality we add dummy variables for all weekdays. We consider holidays by counting them as Sundays and partial holidays[6] and bridge days as Saturdays. This yields

$$\Lambda_{\text{DS}}(t) = \Lambda_{\text{CS}}(t) + \sum_{i=2}^{7} d_i(t),$$

where d_2 corresponds to Tuesdays and d_7 to Sundays.

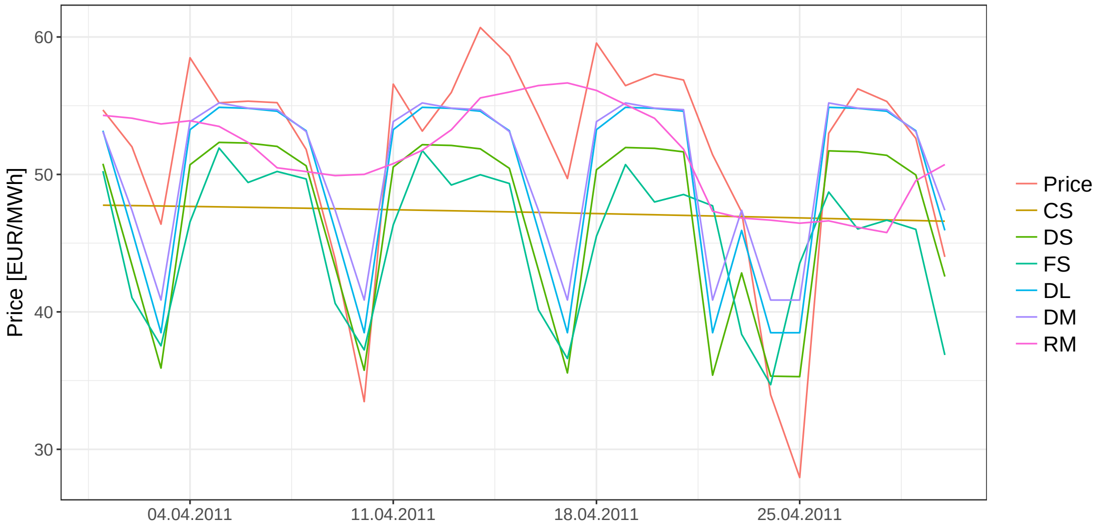
Fourier Series (FS) Inspired by Fourier series we extend the classical sinusoidal seasonality function by even more sinusoidal terms for monthly and weekly seasonality, i.e. additional sine and cosine terms with periods of a third, a fourth, a sixth, and a twelfth of a year, one week and half a week.

Dummy Linear Least Squares (DL) As an alternative to sinusoidal functions we investigate a seasonality function consisting of only dummy variables: one for each year, one for each month, and one for each day, i.e.

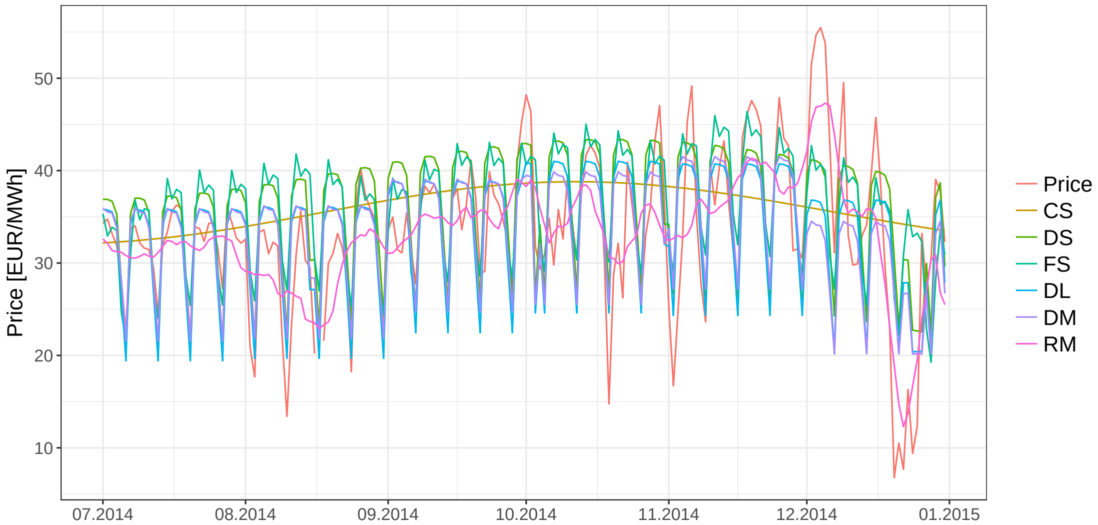
$$\Lambda_{\text{DL}}(t) = \sum_{y=2012}^{2016} a_y(t) + \sum_{m=2}^{12} c_m(t) + \sum_{i=2}^{7} d_i(t).$$

We estimate this by linear least squares.

Dummy Median (DM) This is the same as the method DL, but we successively estimate the dummy variables for each year, month, and day by the median. However, for this estimation method we need to include all dummy variables.



(a) April 2011. The effect of Easter is visible in the functions FS, DS, DL, and DM since they include a dummy variable for holidays (mapped as a Sunday).



(b) Second half of 2014. The effect of the Christmas holidays is visible in the functions DS, DL, DM, and RM.

Figure 3.4.: Baseload spot price and the estimated seasonality functions.

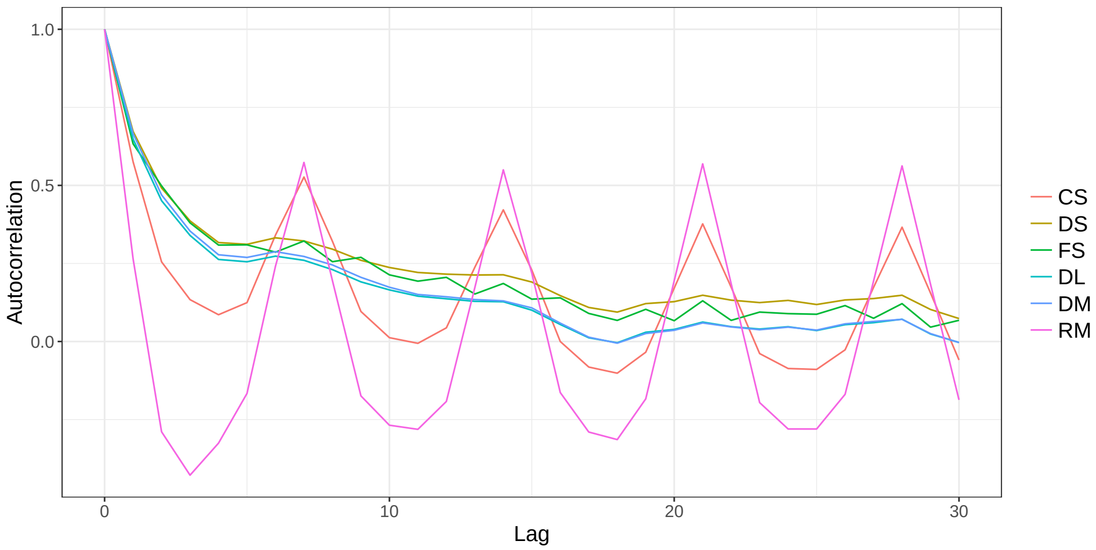


Figure 3.5.: Empirical autocorrelation function of the deseasonalised series.

Rolling Mean (RM) Inspired by Brockwell and Davis (1991, Chapter 1.4) we calculate a centered moving average with interval length of one week. For a day d we use the arithmetic mean of the spot prices from $d-3$ to $d+3$. We will use this method as a benchmark, since we can not extrapolate this seasonality function.

Remark 3.1 (Extrapolation of dummy variable methods). For the seasonality functions DL and DM it is not immediately clear how these seasonality functions can be extrapolated, since the yearly dummy variables for future years cannot be estimated. However, there are possibilities to estimate those from futures prices using price forward curve estimation techniques, cf. Caldana et al. (2017); Kiesel et al. (2018).

Remark 3.2 (Outliers removal). We estimate all seasonality functions on the whole data set. As was shown by Janczura et al. (2013), it is important for the estimation of the seasonality function to remove extreme prices. Therefore, we remove spikes by the *rule of thumb* that all values lying 1.5 times the interquartile range above or below the first or last quartile are removed. Benth et al. (2008a, Chapter 5) use the same method but with three times the interquartile range. Other techniques such as the recursive filtering procedure described by Clewlow and Strickland (2000) could also be used. With our rule of thumb we remove 21 outliers before estimating the seasonality function.

Confirmation of S.1 Season Figures 3.4a and 3.4b illustrate the estimated seasonality functions for April 2011 and the second half of 2014. The methods CS and RM do not capture the weekly seasonality, so we conclude that we really need a method with weekly dummy variables. This is also confirmed by Figure 3.5, which shows the empirical autocorrelation function for the six deseasonalised series. For CS and RM there is a

[6] Partial holidays are holidays that are celebrated in at least one, but not all German states.

peak in the autocorrelation at lag 7 (one week). We conclude that all but CS and RM satisfy S.1: they capture all seasonal and cyclic behaviour.

Confirmation of S.2 Autocorrelation From Figure 3.5 we also recognize that deseasonalised spot prices of DS, FS, DL, and DM have exponentially decaying autocorrelation, i.e. they satisfy S.2. Furthermore, this is an argument in favour of factor models – or, equivalently, ARMA processes, see Section 3.3 in combination with Brockwell and Davis (1991).

The estimated weekly and monthly dummy variables are shown in Table 3.6. From the monthly dummy estimates we observe a yearly seasonality with a cheaper summer and a more expensive winter. The effect of the Christmas holidays is visible in the estimate for December. Furthermore, the weekly dummy estimates show that we could also group Tuesday to Thursday for dummy estimation, as also discussed by Weron (2014). We also observe an approximate difference between 6 and 7 euros between May/June and October/November.

Confirmation of S.3 Noise To check S.3 we compute several descriptive statistics of the deseasonalised spot prices for the seasonality functions DS, FS, DL, and DM and we apply stationarity tests. The statistics are shown in Table 3.7 and are computed on the data set without the 21 days determined as outliers by the rule of thumb described in Remark 3.2, so that the outliers do not have impact on the stationarity study.

All four deseasonalised series seem variance and kurtosis stationary. The skewness stationarity is broken in 2016, which is probably caused by the break of the downward trend in the middle of that year. The methods DL and DM seem favourable to achieve mean stationarity. The seasonality functions DL and DM yield approximately the same descriptive statistics for each year, which is not surprising since they are basically the same model apart from the estimation method of the dummy variables. They slightly differ in the mean and the median values for each year, which can be directly explained by the way they were estimated.

Furthermore, we conduct an augmented Dickey-Fuller test and a Phillips-Perron test for unit roots on the deseasonalised data of DL and DM. We execute both tests with the alternative hypothesis of being stationary and null hypothesis of having a unit root. For the seasonality function DL we reject both tests at 1% significance (p-value $<1\%$, ADF test statistic -9.03, PP test statistic -798.17) and for DM we also reject both tests at 1% significance (p-value $<1\%$, ADF test statistic -8.96, PP test statistic -753.71). All those findings give reason to assume the stylized fact of stationarity.

Choice of seasonality function In light of the previous analysis we will proceed with the seasonality function DM. The memthods DM and DL outperform the other four methods. The estimation of DM is more robust than that of DL since we use the median. This implies that our estimate will be less influenced by possible outliers that were not removed by the rule of thumb (Remark 3.2). With the seasonality function DM we calibrate several factor models in Section 3.4. In the next section we relate factor models to

Table 3.6.: The estimated weekly dummy parameters for the seasonality functions DS, DL, and DM. With DM* we also expressed weekly dummy variables of DM as the difference $d_i - d_1$ or $c_i - c_1$ in order to make the comparison with DS and DL easier.

	DS	DL	DM	DM*
Monday d_1			2.20	
Tuesday d_2	1.64	1.62	3.56	1.36
Wednesday d_3	1.62	1.54	3.18	0.98
Thursday d_4	1.39	1.34	3.07	0.87
Friday d_5	0.01	-0.07	1.48	-0.72
Saturday d_6	-7.34	-7.32	-4.24	-6.44
Sunday d_7	-14.81	-14.77	-10.77	-12.97
January c_1			0.16	
February c_2		1.96	1.36	1.10
March c_3		-1.26	-0.15	-0.31
April c_4		-1.28	-0.35	-0.51
May c_5		-2.63	-2.10	-2.26
June c_6		-3.22	-1.20	-1.36
July c_7		-2.12	-0.86	-1.02
August c_8		-1.85	-0.57	-0.73
September c_9		0.91	2.29	2.13
October c_{10}		3.06	3.08	2.92
November c_{11}		2.79	4.75	4.59
December c_{12}		-1.11	-2.25	-2.41

Table 3.7.: Several descriptive statistics grouped per year and per seasonality function. These statistics are computed on the data set without the 21 days determined as outliers by the rule of thumb described in Remark 3.2.

	Year	Mean	Median	Min	Max	Std	Skew	Kurt
DS	2011	2.95	3.35	-21.36	18.81	6.60	-0.65	4.14
	2012	-1.35	-0.56	-30.40	23.33	6.70	-0.64	5.46
	2013	-1.61	-2.08	-25.98	19.85	7.88	0.02	2.75
	2014	-2.35	-2.18	-23.79	14.72	5.90	-0.22	3.68
	2015	0.61	0.85	-32.35	16.27	6.64	-0.47	4.36
	2016	2.40	1.25	-16.90	28.99	7.87	0.75	3.89
FS	2011	3.00	3.62	-23.07	19.11	6.77	-0.77	4.34
	2012	-1.31	-0.79	-28.02	25.35	7.14	-0.48	4.13
	2013	-1.64	-1.93	-23.92	20.57	8.39	0.00	2.91
	2014	-2.38	-1.86	-23.46	14.80	6.28	-0.26	3.56
	2015	0.69	0.78	-26.79	16.83	6.77	-0.45	3.46
	2016	2.53	1.66	-20.83	34.08	8.02	0.73	4.48
DL	2011	-0.00	0.53	-26.11	15.60	6.64	-0.66	4.32
	2012	-0.12	0.69	-29.29	25.50	6.66	-0.61	5.39
	2013	0.16	-0.25	-24.49	22.53	7.94	0.08	2.84
	2014	0.19	0.22	-21.08	18.95	5.93	-0.06	3.90
	2015	0.16	0.35	-30.53	14.46	6.41	-0.51	4.13
	2016	0.31	-0.73	-20.77	27.18	7.98	0.68	4.02
DM	2011	-0.57	0.00	-26.38	15.51	6.36	-0.83	4.87
	2012	-1.07	-0.29	-29.02	27.22	6.86	-0.37	5.16
	2013	1.21	0.26	-22.53	26.29	8.34	0.17	2.93
	2014	0.17	0.05	-19.92	21.46	5.98	0.23	4.38
	2015	-0.04	0.19	-29.34	14.18	6.43	-0.48	4.02
	2016	1.17	-0.16	-18.41	30.65	7.90	0.99	4.72

ARMA time series and use this relationship to introduce a new way to estimate the mean reversion speed of factor models. This approach will also be investigated in Section 3.4.

3.3. Factor models

We model day-ahead power prices by the general class of *factor models*, described Section 2.4.1. Factor models use the sum of several generalized Ornstein-Uhlenbeck (OU) processes, where each OU process is called a factor. We directly model the spot price as opposed to its logarithm in order to allow for negative prices. Another advantage is that we can derive tractable futures prices analytically.

For simplicity we restrict ourselves to Lévy processes instead of the slightly more general class of additive (or independent increment) processes, which are used in Benth et al. (2008a). This implies that the spot price does not have time dependent increments and therefore simplifies the process description. Moreover, in practice, calibration of a stationary spot price process is already challenging, so a more general class of processes does not have a practical advantage.

3.3.1. Model description

For $n \geq 1$ let $L^1, \dots, L^n$, $L^i = \{L^i_t; t \geq 0\}$, be independent one-dimensional Lévy processes on the complete probability space $(\Omega, \mathcal{A}, P)$. Assume that they are adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$, which satisfies the usual conditions, meaning that the filtration is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{A}$-null sets, cf. Chapter 2.

Each of the Lévy processes induces an *Ornstein-Uhlenbeck process* (OU process) denoted by $X^i = \{X^i_t; t \geq 0\}$ with mean reversion speed $\lambda_i > 0$. The dynamics of X^i are given by

$$dX^i_t = -\lambda_i \, X^i_t \, dt + dL^i_t.$$

As discussed in Section 2.2.1, we know that this SDE has an a.s. càdlàg strong solution given by

$$X^i_t = X^i_0 \, e^{-\lambda_i t} + \int_0^t e^{-\lambda_i (t-s)} \, dL^i_s. \tag{3.2}$$

It is common knowledge that the autocorrelation function of X^i follows

$$\rho_{X^i}(t) = e^{-\lambda_i t},$$

which may be found in Benth et al. (2008a, Chapter 3.3). This corresponds well to the autocorrelation stylized fact we found in Section 3.1. We use these n OU processes to define an arithmetic factor model, see Definition 2.49 on page 27. In Section 3.4 we calibrate and compare several example models.

3.3.2. Relation to ARMA processes

Maller et al. (2009) summarize several results from Wolfe (1982); Sato (2013) on the discretization of OU processes. They state that for all $h > 0$ we have

$$\varepsilon^i := \int_0^h e^{-\lambda_i(h-s)}\, dL_s^i \overset{\mathrm{d}}{=} \int_0^h e^{-\lambda_i s}\, dL_s^i. \tag{3.3}$$

The above equality in distribution can intuitively be seen by a time-reversal argument combined with the stationary increments property of Lévy processes. For more results on properties of the integral ε^i we refer to Maller et al. (2009).

Furthermore, from Lemma 2.29 we get:

Proposition 3.3. *Let $(\gamma_i, \eta_i^2, \ell_i)$ be the generating triplet of the Lévy process L^i. Then ε^i defined by Equation (3.3) has an infinitely divisible distribution characterized by the generating triplet*

$$\begin{aligned}
\tilde{\gamma}_i &= \frac{1 - e^{-\lambda_i h}}{\lambda_i}\gamma_i + \int_0^h \int_{\mathbb{R}} e^{-\lambda_i s} x (\mathbb{1}_{e^{-\lambda_i s}|x| \le 1} - \mathbb{1}_{|x| \le 1})\, d\ell_i(x) ds, \\
\tilde{\eta}_i^2 &= \frac{1 - e^{-2\lambda_i h}}{2\lambda_i}\eta_i^2, \\
\tilde{\ell}_i(B) &= \int_0^h \int_{\mathbb{R}} \mathbb{1}_B(e^{-\lambda_i s} x)\, d\ell_i(x) ds, \quad B \in \mathcal{B}(\mathbb{R}),
\end{aligned}$$

where $\mathcal{B}(\mathbb{R})$ is the Borel-σ-algebra on $\mathbb{R}$.

It is a well-known fact that Gaussian OU processes discretise to AR processes. With the above this result can be generalized:

Lemma 3.4. *Let $h > 0$. If $\mathbb{E}[\varepsilon^i] = 0$ and $\mathrm{Var}[\varepsilon^i] < \infty$, then the discretised OU process $X_k^i := X_{kh}^i$, $k \in \mathbb{N}$, is an AR(1) process with white noise distributed as ε^i.*

Proof. For $k = 0$ the assertion holds. For $k > 0$, combining Equation (3.2), Equation (3.3), and the fact that Lévy processes have independent increments we see that

$$X_{k+1}^i = e^{-\lambda_i h} X_k^i + \int_{kh}^{(k+1)h} e^{-\lambda_i[(k+1)h - s]}\, dL_s^i \overset{\mathrm{d}}{=} e^{-\lambda_i h} X_k^i + \varepsilon_{k+1}^i,$$

which proves the result. □

Although the above result is not very surprising, we can use this fact to conclude that the discretised sum of factors is an ARMA process.

Theorem 3.5. *Let $h > 0$ and assume that $\lambda_i \neq \lambda_j$ when $i \neq j$. If for each i we have $\mathbb{E}[\varepsilon^i] = 0$ and $\mathrm{Var}[\varepsilon^i] < \infty$, then the deseasonalised spot process $\sum_{i=1}^n X_t^i$ discretised on $\{kh : k = 0, 1, 2, \ldots\}$ is an ARMA$(n, n-1)$ process.*

Furthermore, the AR coefficients are given by

$$AR_i = (-1)^{i+1} \sum_{\substack{j\in\{1,2,\dots,n\}^i \\ j_m \neq j_l \text{ if } m\neq l}} e^{\sum_{k=1}^{i} -\lambda_{j_k} h}, \tag{3.4}$$

for $i = 1, \dots, n$.

Proof. Due to Lemma 3.4 we know that each discretised factor X^i is an AR process. Using the *basic theorem* of Granger and Morris (1976) (or their comments in Section 3 of the same paper) the result follows immediately.

The statement concerning the AR coefficients has been discussed by Ku and Seneta (1998) for the case of $n = 2$. For the general case we need to prove the statement. From Lemma 3.4 we know that we can write $(1 - \theta_i B)X_k^i = \varepsilon_k^i$, where B is the backward operator[7] and $\theta_i := \exp(-\lambda_i h)$. Denote $Z_k := \sum_{i=1}^n X_k^i$, then from Equation (2.7) of Granger and Morris (1976) it follows that Z_k satisfies

$$\prod_{i=1}^{n}(1-\theta_i B)Z_k = \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}(1-\theta_j B)\,\varepsilon_k^i \sim \text{MA}(n-1).$$

To compute the AR coefficients, we compute the polynomial

$$1 - \sum_{i=1}^{n} \text{AR}_i\, B^i := \prod_{i=1}^{n}(1-\theta_i B) = 1 + \sum_{i=1}^{n}\left[\sum_{\substack{j\in\{1,2,\dots,n\}^i \\ j_m\neq j_l \text{ if } m\neq l}} \prod_{k=1}^{i}(-\theta_{j_k})\right] B^i,$$

which concludes the proof. □

Corollary 3.6. *For* $n = 2$ *Equation* (3.4) *reduces to*

$$AR_1 = e^{-\lambda_1 h} + e^{-\lambda_2 h}, \quad AR_2 = -e^{-(\lambda_1+\lambda_2)h},$$

which can be used to find the mean reversion speeds when the AR coefficients are known.

Remark 3.7 (Number of factors). In light of the above theorem, we can also determine the dimension of the model, i.e. the number of factors, by fitting ARMA time series to the deseasonalised spot prices and comparing their goodness-of-fit by information criteria such as the AIC or BIC.

3.4. Empirical analysis

In this section we calibrate two different models on the data from 2011 to 2015. Afterwards we perform an out-of-sample analysis and simulate spot prices for the year

[7] The backward operator as defined by Granger and Morris (1976) as $B^j X_k = X_{k-j}$ for all $j \geq k$.

2016 with the calibrated models. We use the seasonality function DM as described in Section 3.2, but restrict ourselves to the aforementioned calibration period for its estimation. The stochastic dynamics are then estimated from the deseasonalised price time series.

3.4.1. Estimation of the mean reversion speed

We exploit Theorem 3.5 and fit the deseasonalised spot prices to an AR(1) and an ARMA(2, 1) process in order to estimate the mean reversion speed for a one-factor and a two-factor model, respectively. The autoregressive model is fitted by solving the Yule-Walker equations. The ARMA model is fitted using the method extending these equations described by Hannan and Rissanen (1982). Furthermore, we use the empirical autocorrelation function to estimate the mean reversion parameters. The results are shown in Table 3.9.

In the one-factor case the AR estimate yields a slightly faster mean reversion, but both have a half-life of approximately two days. For the two-factor model the ARMA estimate yields a slightly slower mean reversion for both factors. The first factor has almost identical mean reversion speed with both methods. The mean reversion speed of the second factor estimated by the ARMA method is almost double the one estimated by the ACF method. Figure 3.8 illustrates the theoretical autocorrelation structure for the four estimates together with the empirical autocorrelation function. We observe that both the AR- and ARMA-estimated autocorrelation functions are closer to the empirical one than their ACF-estimated counterparts. Furthermore the ARMA method estimates the mean reversion directly from the data – whereas the ACF method first computes the empirical autocorrelation function, from which the mean reversion speed is then estimated. We also found that the ACF method is unstable in the number of lags for which the autocorrelation function is computed. Therefore we continue with the AR and ARMA estimates in the rest of this section.

In the following we define the generalized OU processes by the background driving Lévy processes or by an infinitely divisible distribution for ε_k^i. Apart from the results discussed in Section 3.3, there are also other ways to define stationary generalized OU processes, such as through the class of self-decomposable distributions. This omits the estimation of the mean reversion speed (Barndorff-Nielsen and Shephard, 2001).

Subsequently we consider the deseasonalised spot price. Using the results of Section 3.3 we assume that it has the following dynamics

$$S(kh) - \Lambda_{\mathrm{DM}}(kh) \stackrel{!}{=} \sum_{i=1}^{n} X_k^i = \sum_{i=1}^{n} \left[e^{-\lambda_i h} X_{k-1}^i + \varepsilon_k^i \right].$$

We calibrate a one-factor and a two-factor model, i.e. with $n = 1$ and $n = 2$, respectively.

3.4.2. One-factor model

For small enough h we can use the Euler-Maruyama approximation for the noise term, which is then given by $\varepsilon_k^1 \approx L^1(kh) - L^1((k-1)h) := \Delta L_k^1$. It follows that the incre-

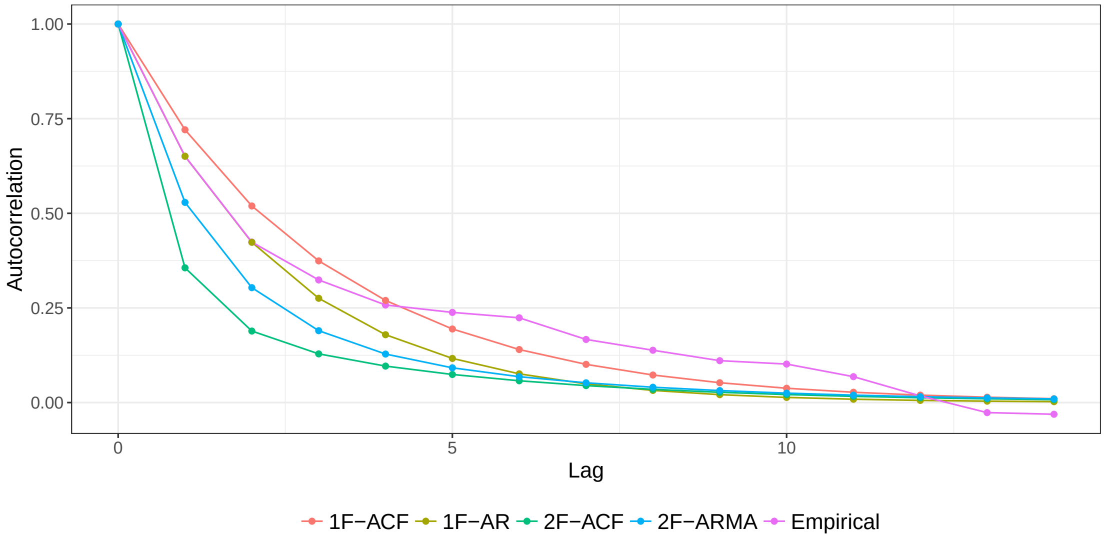


Figure 3.8.: Comparison of autocorrelation functions. The empirical autocorrelation (purple) is shown, together with the theoretical autocorrelation given by the ACF- and AR(MA)-estimated mean reversion speed.

ments ΔL_k^1 are i.i.d. random variables and we use maximum likelihood to estimate the distribution of ΔL_k^1. We need to be able to generate tails heavier than Gaussian ones, but use only one factor X^1. Therefore, we assume L^1 to be a normal inverse Gaussian process (NIG). From Barndorff-Nielsen (1997); Rydberg (1997); Barndorff-Nielsen and Shephard (2001) we know that L_1^1 has density

$$f(x) = f(x; a, b, d) = \frac{da}{\pi|x|} e^{bx} K_1(a|x|)$$

for $a \geq |b| \geq 0$, $d > 0$. K_1 is the modified Bessel function of third order and index 1. The generating triplet of L^1 is given by

$$\gamma_1 = \int_{|x|\leq 1} x f(x) dx, \quad \eta_1^2 = 0, \quad d\ell_1(x) = f(x)dx.$$

This leaves three free parameters that completely determine the model. We estimate the NIG parameters with the R-function `fit.NIGuv` from the package `ghyp`. The estimation results are shown in Table 3.10.

3.4.3. Two-factor model

For the two-factor model we assume that the stochastic spot dynamics are caused by a slower mean reverting diffusion process and a faster mean reverting jump process. Therefore, we assume that the first OU process X^1 is driven by a Brownian motion, i.e.

$$L_t^1 = \sigma_1 W_t,$$

Table 3.9.: Estimated mean reversion speed. The parameters are given per day. The value in brackets is the half-life of a deviation in days, which is given by $\ln(2)/\lambda_i$.

	ACF	AR(1)	ACF	ARMA(2,1)
λ_1	0.3275 (2.12)	0.4297 (1.61)	0.2502 (2.77)	0.2347 (2.95)
λ_2	–	–	1.5664 (0.44)	0.8264 (0.84)

Table 3.10.: Estimated model parameters. The mean reversion speeds were estimated by the AR(1) and ARMA(2,1) methods and are shown in Table 3.9.

a	0.1761
b	0.0003
d	5.5759

(a) One-factor model.

σ_1	4.7185
λ_2	0.0509
k_2	3.6701
b_2	0.2319
p_2	0.3440

(b) Two-factor model.

where W is a standard one-dimensional Brownian motion. Furthermore, we assume that the second OU process X^2 is driven by a compound Poisson process: when N_t is a homogeneous Poisson process with arrival rate $\lambda_2 > 0$, the driving proces is given by

$$L_t^2 = \sum_{i=1}^{N_t} B_i \cdot D_i,$$

where $D_1, D_2, D_3, \dots$ is a series of i.i.d. Gamma(k_2, b_2) distributed random variables and $B_1, B_2, B_3, \dots$ is a series of i.i.d. Bernoulli(p_2) distributed random variables with values in $\{-1, 1\}$.

Unfortunately, due to the non-Gaussian nature of the process X^2 the estimation of the parameters for each process is not trivial. Barlow et al. (2004) give an approach with the well-known Kalman filter, but this has the drawback that it only works for Gaussian factors. Gonzalez et al. (2017) develop a Markov Chain Monte Carlo method of estimating the different factors. Their method is very sophisticated and exploits the usage of conjugate prior distributions in Bayesian statistics. However, in this study we follow the considerations of Meyer-Brandis and Tankov (2008) and use their *hard thresholding* technique to separate a jump process from a diffusion process. The method is based on iteratively adding jumps having the maximum likelihood until some stopping criterium.

From this separation procedure we end up with two different time series corresponding to each of the factors X^1 and X^2, from which we can then estimate the parameters belonging to each process.

For the estimation of the classical diffusion OU process X^1 we refer to Proposition 3.3 in order to see that the increments ε_k^1 are i.i.d. normally distributed with variance

$$\operatorname{Var} \varepsilon_k^1 = \frac{1 - e^{-2\lambda_1 h}}{2\lambda_1} \sigma_1^2, \quad k \in \mathbb{N}$$

where σ_1 is the volatility of the driving Brownian motion.

The arrival rate λ_2 can be estimated as the number of jumps divided by the number of data points. The distribution of the jump sizes can be estimated from the separated jumps. We assume that the jump sizes are gamma times Bernoulli distributed, meaning that the sign of the jump is determined by a Bernoulli random variable and the absolute value of the jump size originates from a gamma distribution. The shape k_2, rate b_2, and Bernoulli probability p_2 of having a positive jump are estimated by maximum likelihood. The results of the estimation are again found in Table 3.10.

3.4.4. Simulation results

With the results from the previous sections we conduct a simulation study. Simulation of L^1 in the one-factor model is as described by Korn et al. (2010, Chapter 7). With the estimated parameters, we simulate from the 1st of January 2016 to 31st of December 2016. Since we want to conduct an empirical study we use the realized median as the yearly dummy variable for 2016, which was 28.48 Euro.[8]

Figure 3.12 shows the realised residuals after deseasonalisation together with the 1, 5, 25, 75, 95, and 99% quantiles per day of 10,000 simulated paths for the two-factor model described in the previous section. The difference in quantiles between the one and two-factor models is marginally, therefore we only show the figures for the two-factor model. We see that the model captures the spot prices in the summer months quite well, however in the winter months there are a few spikes beyond the shown quantiles. Remember that 2016 broke the typical seasonal structure due to high demand in France. Figure 3.13 shows the same plot, but with seasonality and without the spot price.

We now introduce a method which helps us to assess the quality of the estimated models. From the 10,000 simulated paths we compute an empirical cumulative distribution for all 366 days of 2016. We compare this with the realised deseasonalised spot prices. The method we used is as follows:

1) For each day compute the empirical cumulative distribution function from the simulated paths and compute the cumulative probability for the observed deseasonalised spot price.

[8] When this method is used to simulate future spot prices another way to determine the yearly dummy variable has to be found, see Section 3.2.

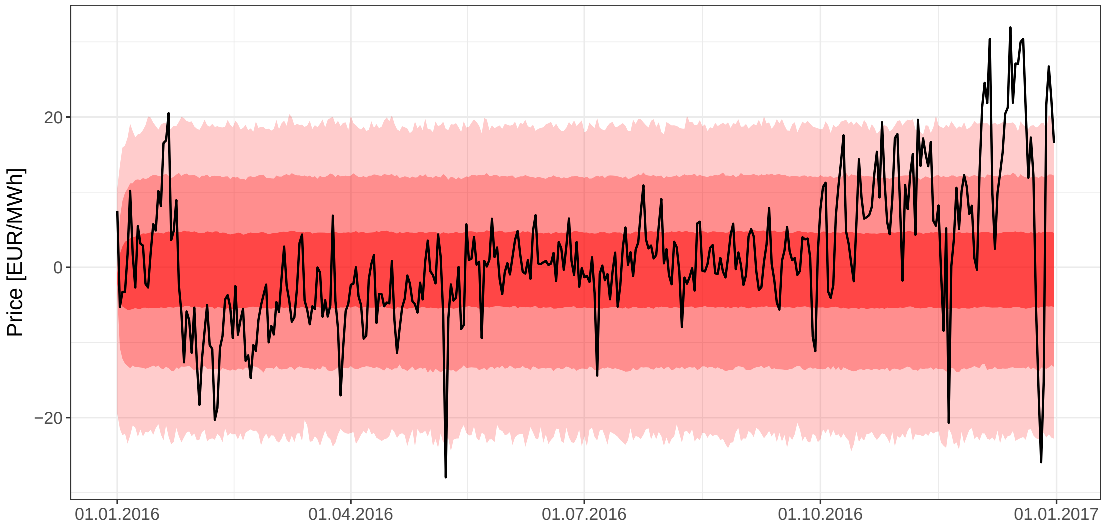


Figure 3.12.: The realised baseload spot price after deseasonalisation (black) and the 1, 5, 25, 75, 95, and 99% quantiles per day of 10,000 simulated deseasonalised paths. The simulation was done by the two-factor model.

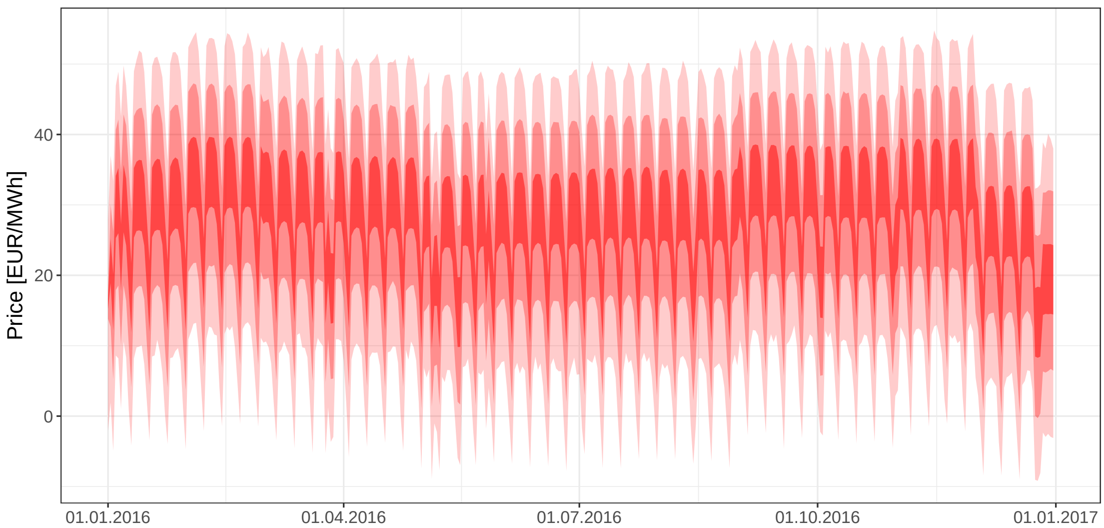


Figure 3.13.: The 1, 5, 25, 75, 95, and 99% quantiles per day of 10,000 simulated day-ahead baseload spot price paths. The simulation was done by the two-factor model.

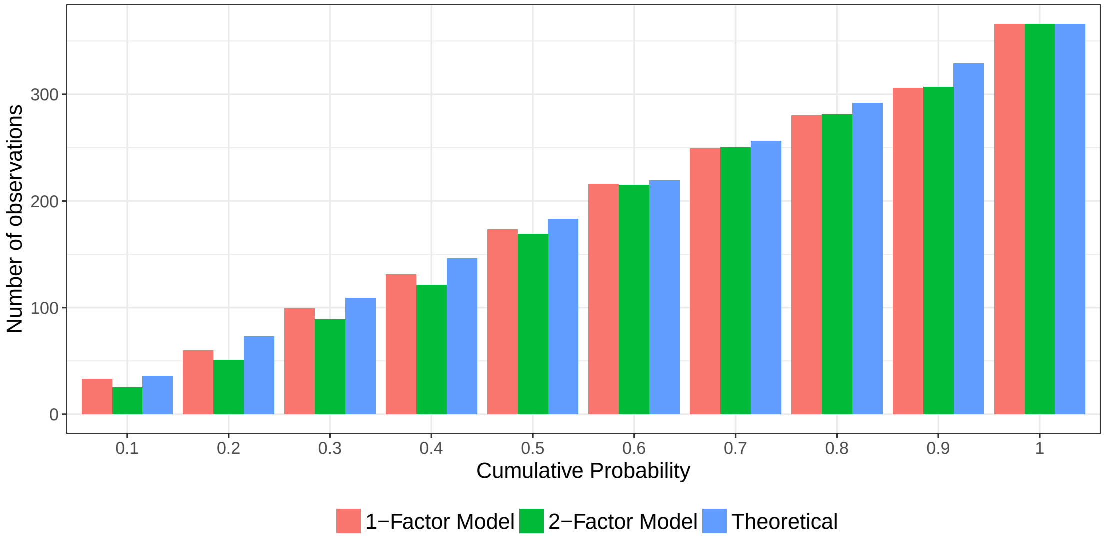


Figure 3.14.: Number of observations of the realised deseasonalised spot price falling in the α-quantile of the one-factor or two-factor model. The theoretical expectations are λN, where $N = 366$ is the total number of observations.

Table 3.15.: Descriptive statistics of the realised deseasonalised spot price and the mean and standard deviation of the descriptive statistics of all simulated paths.

		Mean	Median	Min	Max	Std	Skew	Kurt
Realised		1.38	0.45	-27.96	31.93	8.97	0.58	4.42
	Q25%	-0.56	-0.54	-28.85	21.67	6.97	-0.19	3.46
1-Factor	Mean	0.01	0.01	-25.85	25.75	7.35	-0.00	4.05
	Q75%	0.58	0.56	-21.71	28.71	7.70	0.18	4.38
	Q25%	-1.20	-1.09	-39.77	23.82	7.63	-0.47	4.03
2-Factor	Mean	-0.46	-0.33	-34.95	30.24	8.09	-0.21	5.17
	Q75%	0.29	0.42	-28.87	35.35	8.53	0.06	5.88

2) For several choices of $\alpha \in (0, 1]$ compute the number of observations that have empirical cumulative probability lower than α. In this study we used all multiples of 10%.
3) When the model captures the behaviour of the spot price well, the expected (theoretical) number of observations in the group of α would be αN, where N is the total number of observations.

We conducted this analysis for both the one-factor and two-factor model. The results are displayed in Figure 3.14. We see that the simulated quantiles correspond well to the theoretical ones, implying that the realised spot path is not unlikely under both models. The greatest deviation for the one-factor model occurs in the 90%-quantile and equals -23. For the two-factor model this happens in the 40%-quantile with -24.

However, for the extreme quantiles such as the 90%-quantile, we observe that both factor models underestimate the expectation. If one is only interested in these quantiles, one might need to use a different model. For example, Hagfors et al. (2016a) introduce a quantile-regression model using fundamental drivers to predict the extreme hourly electricity spot prices. Furthermore, one could use extreme value theory as in Paraschiv et al. (2016) to model the extreme quantiles more accurately.

Finally, we also compare descriptive statistics. We have 10,000 paths, each of which has different descriptive statistics. Therefore we compare the mean and the quartiles of all descriptive statistics in Table 3.15. We see that most of the realised descriptive statistics lie within the interquartile range. Again, we conclude that the realised spot price path is not unlikely for the given models.

As is always the case for any financial price model, it is impossible to determine with certainty whether the model captures all possible future scenarios. However, with all the analyses we conducted throughout this chapter, we are confident with the results of factor models. They are able to capture all stylized facts and features, in particular they are able to mimic spikes.

4. A factor model with futures information for pricing intraday derivatives[1]

With the introduction of the Renewable Energy Sources Act (in German: *Erneuerbare-Energien-Gesetz*, EEG) as part of the *Energiewende* the German electricity market has changed since the 2000s, which we already saw occur at the day-ahead spot market in Chapter 3. This sustainability policy sets “ambitious targets for the future share of renewable energies” (Gawel et al., 2015). Due to the *merit order effect* the increased feed-in from renewable energy sources changed the German electricity market significantly, which led to declining market prices for several successive years (Hagfors et al., 2016a; Benhmad and Percebois, 2018; Paraschiv et al., 2014; Gianfreda and Bunn, 2018). This development lead to energy price models that took renewable feed-in into account, e.g. see Wagner (2014); Ketterer (2014); Ziel and Steinert (2016).

As a consequence of the increasing need of short-term balancing due to adjusted renewable infeed forecasts, the intraday market's popularity increased in recent years. The traded volume at the German intraday market has grown by 30.3% from May 2016 to May 2018 (EPEX, 2017, 2018b). To reduce price risks for market participants in the continuous intraday market the energy exchange introduced *intraday cap futures* in 2015 and *intraday floor futures* in 2016. With the help of intraday cap futures “(..) trading participants can trade price peaks on the German intraday market, either in order to hedge against high prices (e.g. for marketers of wind power) or secure the expected revenues from price peaks (e.g. operators of highly flexible power plants)”, according to the European Energy Exchange (EEX, 2015).

To the best of our knowledge intraday cap and floor futures have not been studied in any academic work so far. In this chapter we introduce a tractable but efficient pricing method for intraday cap and floor futures. In Section 4.1 we discuss the payoff function of intraday cap and floor futures and define its underlying, the ID_3 price index. Section 4.2 introduces a general setting for pricing intraday cap and floor futures and proposes a specific model based on the Hull-White interest rate model, which is a one-factor model. We include futures prices to generate a price forward curve, which is then used in place of a fixed deterministic seasonality function. Finally, in Section 4.3 we apply the theory developed in Section 4.2 to intraday cap futures prices. We find that the proposed model fits the data very well: a nearly perfect initial fit is obtained and in a simulation study we observe that generated empirical distribution fits the realised intraday cap futures' prices well.

[1]Based on published work: Hinderks and Wagner (2019b). Original title: Pricing German Energiewende products: intraday cap/floor futures.

4.1. The German intraday market

In this chapter we focus on intraday cap and floor futures, which are traded at the derivatives exchange EEX. The underlying for these derivatives is the ID_3 price index computed from the continuous intraday market. In the following we introduce both the underlying and its derivative products in more detail.

Continuous intraday market As mentioned in the beginning of this chapter the German intraday market is a growing market. Unlike the day-ahead spot market, which is set up as an auction, the intraday market is a continuous market, meaning that market participants can place bids and asks at any time. The requests to buy or sell are stored in the order book of the exchange and the exchange determines which orders can be fulfilled (EPEX, 2018a). It is possible to trade electricity delivering for quarter hourly (qh) or hourly (h) periods every day of the week, including Sundays and holidays. The market opens at 15:00 for 1h contracts and at 16:00 for 1qh contracts delivering the next day, cf. Section 1.1.

Due to the lack of a unique continuous intraday price for each contract, EPEX created several price indices: the Index (the volume-weighted average of all trades), the ID_1, and the ID_3, for example. For an overview of all indices we refer to the market documentation by EPEX (2018c). In this chapter we focus on the price index ID_3, because this is the underlying of intraday cap and floor futures.

Definition 4.1 (ID_3 price index)**.** The price index ID_3 is the volume-weighted average of all continuous intraday trades of a certain delivery hour (1h) or quarter hour (1qh), taking only those trades into account that were fulfilled at most three hours before delivery. If no intraday trades occurred for a specific contract, the day-ahead or intraday auction results are used.

Example 4.2. The ID_3 price of the contract 21h, i.e. with delivery from 20:00 to 21:00, is computed by the volume-weighted average of all intraday trades of the 21h contract between 17:00 and 19:30, since the continuous intraday market closes half an hour before delivery. This time period before delivery, during which a contract cannot be traded anymore, is called the *leading time*.

Intraday cap futures Intraday cap futures started trading at 14 September 2015 (EEX, 2015). The payoff of an intraday cap futures "(..) corresponds to the difference between the ID3-Price of the hourly intraday products in the delivery period and the payment threshold ("cap") determined by the management of the exchange, added up for all delivery hours. If that difference is negative, it will be set to zero for the respective hour." (EEX, 2015) Equation (4.1) in Section 4.2 shows the payoff mathematically.

At the time of writing this chapter the only available delivery period is weekly, i.e. from Monday 00:00 until Sunday 24:00. Just like regular futures at the EEX intraday cap futures can only be traded on work days, but also during delivery. This means that the last trading day of an intraday cap futures is the Friday of its delivery week.

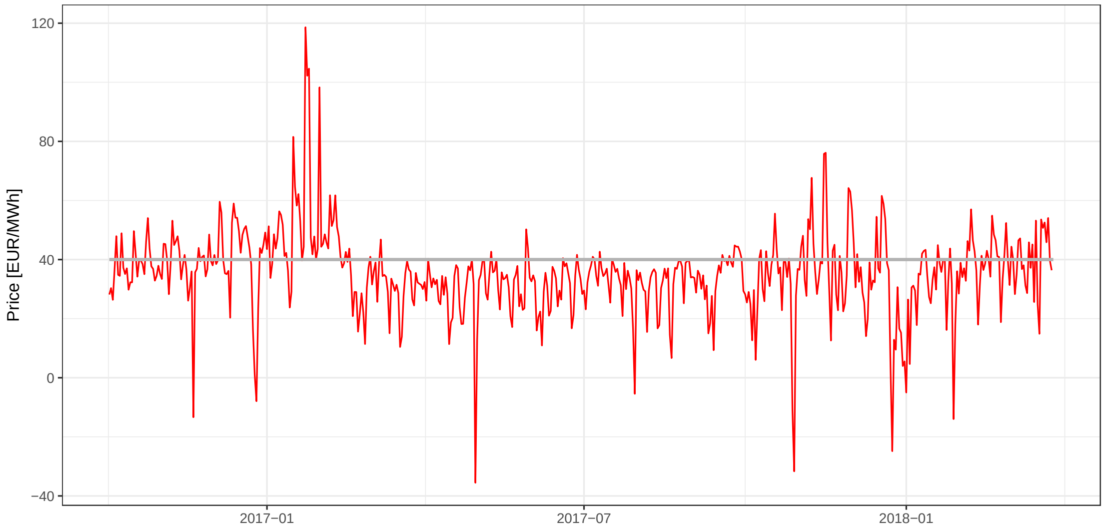


Figure 4.1.: Daily arithmetic average (baseload) of the hourly ID_3 prices (red) and cap strike price of 40 Euro/MWh (grey) from 3 October 2016 to 25 March 2018. Before 3 October 2016.

Simultaneously there are five intraday cap futures tradable: the current week and the four following weeks.

Initially the “cap”, or as we rather call it[2] *strike price*, was set to 60 Euro/MWh (EEX, 2015). However, the EEX decided to reduce the cap futures' strike price to 40 Euro/MWh starting with the intraday cap futures with delivery from 31 October 2016 to 6 November 2016 (EEX, 2016a). The first trading day for this contract was Monday, 3 October 2016. Figure 4.1 shows the daily arithmetic average of the hourly ID_3 prices (red) together with the cap strike price of 40 Euro/MWh (grey) from 3 October 2016 to 25 March 2018.

Remark 4.3 (Exclusion of strike price 60 Euro/MWh). In the following we only use the data set of the intraday cap prices with delivery after 31 October 2016, i.e. with a strike price of 40 Euro/MWh. The prices of the contracts with a strike price of 60 Euro/MWh are all approximately zero and do not show the same market price behaviour as the prices of the contracts with a strike of 40 Euro/MWh.

Intraday floor futures With the press release of the 20 September 2016 the EEX introduced the intraday floor futures (EEX, 2016b). The intraday floor futures is basically the same as the intraday cap futures apart from the fact that instead of the positive difference it pays the negative difference between the strike price and the ID_3 price. The intraday floor futures' strike price is fixed by the EEX at 10 Euro/MWh. The payoff of an intraday floor futures is formalized by Equation (4.2) in Section 4.2.

[2]For reasons that will become clear in Section 4.2.

4.2. Price of intraday cap and floor futures

In this section we consider the mathematics for pricing intraday cap and floor futures. We first discuss a general price, i.e. without any assumptions on the dynamics of the underlying ID_3 price process. In the second part of this section we introduce a Hull-White modelling approach for the ID_3 price and use this to explicitly derive the intraday cap and floor futures' prices.

4.2.1. General price

Assume that the ID_3 price index $I = \{I(\tau); \tau \geq 0\}$ is a stochastic process on a probability space $(\Omega, \mathcal{A}, P)$ equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$. We assume (the augmented version of) $\mathcal{F}_t$ to be generated by the prices observed at the EPEX and EEX markets up to trading time t. Furthermore, we assume that I is adapted to $\mathcal{F}$.

Denote the delivery times of a delivery week of an intraday cap or floor futures by

$$\mathcal{W} := \{\tau_1, \tau_2, \tau_2, \dots, \tau_n\},$$

i.e. $\mathcal{W}$ corresponds to all the hours of a week starting with the hour 00:00 until 01:00 on Monday and ending with the delivery hour 23:00 to 00:00 on Sunday. In regular weeks $|\mathcal{W}| = n = 7 \cdot 24 = 168$.[3]

Payoff functions It is clear from the description in the previous section that an intraday cap futures contract written on the week $\mathcal{W}$ has the following payoff

$$\text{Cap-payoff}(\mathcal{W}, K) = \sum_{i=1}^{n} \left(I(\tau_i) - K\right)^+ , \tag{4.1}$$

where K is the cap futures' strike price and $a^+ := \max(a, 0)$. Equivalently, the payoff of an intraday floor futures delivering during $\mathcal{W}$ equals

$$\text{Floor-payoff}(\mathcal{W}, K) = \sum_{i=1}^{n} \left(K - I(\tau_i)\right)^+ . \tag{4.2}$$

Remark 4.4 (Series of options). It is immediately visible from the payoff functions (4.1) and (4.2) that the intraday cap and floor futures are simply a series of call and put options, respectively. There is only one difference to regular options: here they act as futures contracts, meaning that the payment is made at the end of the delivery period $\mathcal{W}$. Inspired by the interest rate setting we call the call option a *caplet futures* and the put option a *floorlet futures*.

Assumption 4.5 (Pricing measure Q)**.** In the following we assume that a pricing measure Q is given or derived, which can be used to price derivatives on the ID_3 price I. The pricing measure Q measure is assumed to be equivalent to the real-world measure P.

[3] In weeks where the daylight saving time change occurs, however, n can differ by one hour.

With Assumption 4.5 we can compute the price of caplet and floorlet futures as the conditional expectation under the pricing measure Q:

Definition 4.6 (Caplet and floorlet futures)**.** For delivery time $\tau \geq 0$ let $\mathbb{E}_Q|I(\tau)| < \infty$. At trading time t the price of a caplet and a floorlet futures on $I(\tau)$ with $\tau \geq t$ and strike price K are given by

$$C_t(\tau; K) = \mathbb{E}_Q\left[(I(\tau) - K)^+ \,\middle|\, \mathcal{F}_t\right], \quad \text{and}$$
$$F_t(\tau; K) = \mathbb{E}_Q\left[(K - I(\tau))^+ \,\middle|\, \mathcal{F}_t\right],$$

respectively.

From Remark 4.4 it then follows immediately that the price of intraday cap and floor futures can be expressed in terms of caplet and floorlet futures, respectively.

Corollary 4.7 (Intraday cap futures' price)**.** *The price at time t of an intraday cap futures delivering during the period $\mathcal{W}$ and with strike price K is given by*

$$C_t(\mathcal{W}; K) = \sum_{\tau \in \mathcal{W}} C_t(\tau; K), \tag{4.3}$$

where $C_t(\tau; K)$ is the price of a caplet futures given by Definition 4.6.

Corollary 4.8 (Intraday floor futures' price)**.** *The price at time t of an intraday floor futures delivering during the period $\mathcal{W}$ and with strike price K is given by*

$$F_t(\mathcal{W}; K) = \sum_{\tau \in \mathcal{W}} F_t(\tau; K), \tag{4.4}$$

where $F_t(\tau; K)$ is the price of a floorlet futures given by Definition 4.6.

Remark 4.9 (Time-value of money). As is the case with all futures contracts, the amount of money one needs at time t to risklessly enter a long position of an intraday cap futures or an intraday floor futures equals $e^{-r(\overline{\tau}-t)}C_t(\mathcal{W}; K)$ or $e^{-r(\overline{\tau}-t)}F_t(\mathcal{W}; K)$, respectively. Here r is the risk-free interest rate of the money market account and $\overline{\tau} := \max \mathcal{W}$ is the payment date of the futures contract.

Lemma 4.10 (Cap-floor parity)**.** *The following identity holds*

$$C_t(\mathcal{W}; K) - F_t(\mathcal{W}; K) = \sum_{\tau \in \mathcal{W}} \mathbb{E}_Q[I(\tau) \,|\, \mathcal{F}_t] - |\mathcal{W}|\, K,$$

and we call it the cap-floor parity.

Proof. Follows directly from the put-call parity for European options. □

Remark 4.11 (Quoted market prices). At the EEX the market prices of intraday cap and floor futures are quoted as normalized prices per MWh. This means that the prices are the arithmetic average of the caplet or floorlet futures' prices, i.e.

$$\text{Quoted-cap-price} = \tfrac{1}{|\mathcal{W}|}\, C_t(\mathcal{W}; K), \quad \text{and}$$
$$\text{Quoted-floor-price} = \tfrac{1}{|\mathcal{W}|}\, F_t(\mathcal{W}; K),$$

where $|\mathcal{W}| = n$ is the order of the set $\mathcal{W}$.

Remark 4.12 (Strike K in practice). As discussed in Section 4.1 the tradable intraday cap and floor futures have a single fixed strike price, which is determined by the EEX. For intraday cap futures the strike price is fixed at $K = 40$ Euro/MWh and for intraday floor futures it at $K = 10$ Euro/MWh. Unfortunately, this also implies that the cap-floor parity cannot be applied in practice.

4.2.2. Hull-White modelling approach

In order to price intraday cap and floor futures we propose a *Hull-White extended Vasicek model* of interest rate theory (cf. Brigo and Mercurio, 2006, Section 3.3.2) for the ID_3 price. Then the ID_3 price has a normal distribution, allowing for analytical computation of the cap and floor prices of Equations (4.3) and (4.4), respectively. Moreover, the Hull-White model can also be seen as a one-factor model driven by a Gaussian process, cf. Section 2.4.1.

Assume that the stochastic process X_t is an Gaussian Ornstein-Uhlenbeck process with dynamics under the pricing measure Q

$$dX_t = -\lambda\, X_t\, dt + \sigma\, dW_t, \quad X_0 = 0,$$

where W is a standard one-dimensional Brownian motion under Q and $\lambda, \sigma > 0$. The above stochastic differential equation (SDE) has a strong solution given by

$$X_t = \sigma \int_0^t e^{-\lambda(t-u)} dW_u. \tag{4.5}$$

As a corollary of Theorem 2.36 on page 20, it immediately follows that X_t is normally distributed:

Corollary 4.13 (Normal distribution). *For $0 \leq t < \tau$ the conditioned Ornstein-Uhlenbeck process $(X_\tau \,|\, X_t = \tilde{x})$ is normally distributed with mean*

$$\mu_t^\tau = e^{-\lambda(\tau-t)}\, \tilde{x}$$

and variance

$$(\Sigma_t^\tau)^2 = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda(\tau-t)}\right).$$

Motivation As a motivation for our modelling approach we introduce the spread between the ID_3 and day-ahead spot prices, i.e. $X_\tau = I(\tau) - S(\tau)$. Rearranging this equation yields

$$I(\tau) = S(\tau) + X_\tau.$$

We assume that S and X are integrable with respect to Q. We denote the price forward curve (PFC) at future time t for delivery time τ by $f_t(\tau)$. We remark that $f_t(\tau)$ is an $\mathcal{F}_t$-measurable random variable. From Section 2.4.3 we know that

$$f_t(\tau) := \mathbb{E}_Q \left[S(\tau) \,|\, \mathcal{F}_t \right]. \tag{4.6}$$

In particular, we can now approximate

$$I(\tau) = (S(\tau) - \mathbb{E}_Q[S(\tau) \,|\, \mathcal{F}_t]) + f_t(\tau) + X_\tau \approx f_t(\tau) + X_\tau,$$

where we used Equation (4.6). This means that we approximate the day-ahead spot price $S(\tau)$ by its conditional expectation $\mathbb{E}_Q[S(\tau) \,|\, \mathcal{F}_t]$, i.e. by its orthogonal projection on the available information at time t, cf. Section 2.4.3. It is clear that this is exact in expectation. Economically, this approximation can be motivated by our intention to let the day-ahead spot price S determine the expectation of the ID_3 prices. The spread X_τ is assumed to be the main driver for the price peaks and thus to determine the intraday cap and futures' prices.

With the help of the Ornstein-Uhlenbeck process X_τ we now define the price of a future payment dependent on the ID_3 price as follows:

Definition 4.14. The price $p_t^g(\tau)$ at time $t \geq 0$ of a future payment g dependent on the ID_3 price $I(\tau)$ at delivery time $\tau \geq t$ is given by

$$p_t^g(\tau) = \mathbb{E}_Q \left[g(I(\tau)) \,\middle|\, \mathcal{F}_t \right] := \mathbb{E}_Q \left[g(f_t(\tau) + X_\tau) \,\middle|\, \mathcal{F}_t \right],$$

where $f_t(\tau)$ is the price forward curve (PFC) of time t for delivery time u and X_t is the Ornstein-Uhlenbeck process given by Equation (4.5).

In the following we define

$$\Delta_t^\tau := \frac{f_t(\tau) + e^{-\lambda(\tau - t)}\, X_t - K}{\Sigma_t^\tau}$$

as an auxiliary variable, where Σ_t^τ is the standard deviation of Corollary 4.13. From Corollary 2.37 on page 21 we immediately get:

Theorem 4.15 (Caplet and floorlet futures' price)**.** *At time $t > 0$ the price of a caplet futures on the ID_3 price $I(\tau)$ with $0 \leq t < \tau$ is given by*

$$C_t(\tau; K) = \left[f_t(\tau) + e^{-\lambda(\tau - t)}\, X_t - K \right] \Phi(\Delta_t^\tau) + \frac{\Sigma_t^\tau}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Delta_t^\tau)^2}$$

and the price of a floorlet futures is given by

$$F_t(\tau; K) = \left[K - f_t(\tau) - e^{-\lambda(\tau - t)}\, X_t \right] \Phi(-\Delta_t^\tau) + \frac{\Sigma_t^\tau}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Delta_t^\tau)^2},$$

where $\Phi(x)$ is the cumulative distribution function of a the standard normal distribution.

Remark 4.16 (Price forward curve). We have not specifically determined a model for the PFC $f_t(\tau)$. In Section 4.3 we will use our own method for the generation of a PFC. However, the interested reader is referred to Remark 5.4 in Chapter 5 for more literature on the construction of a price forward curve.

Remark 4.17 (Option price formula of Bachelier). The derived call option price in Theorem 4.15 is quite similar to the one derived in the original work by Bachelier (1900). His implicit assumption can be reformulated such that under the pricing measure Q the stock price S_t followed the dynamics

$$dS_t = r\, S_t\, dt + \sigma\, dW_t,$$

where $r > 0$ the positive interest rate.

4.3. Empirical analysis for intraday cap futures

In this section we apply the Hull-White modelling approach of Section 4.2.2 to market data of intraday cap futures. Section 4.3.1 discusses the available data set of ID_3 and intraday cap futures prices. We introduce a model for the PFC in Section 4.3.2 and calibrate the complete model in Section 4.3.3. Finally, Section 4.3.4 conducts an analysis of the initial fit of the calibrated model and conducts a simulation study to assess the overall goodness-of-fit of the model.

4.3.1. Data set

We study the ID_3 price data from 28 June 2015 to 25 March 2018 and the intraday cap futures' settlement prices from 3 October 2016 to 25 March 2018. As stated in Remark 4.3 we only use the intraday cap prices with strike price of 40 Euro/MWh and omit the prices before 3 October 2016. Figure 4.1 shows the daily average, or the baseload, of the ID_3 prices. A part of the intraday cap prices is illustrated in Figure 4.4a.

In Table 4.2 several descriptive statistics of the hourly ID_3 price time series are shown. From the minimum we immediately see that negative prices also occur in the continuous intraday market. Together with Figure 4.1 we conclude that spikes also occur in the ID_3 prices. The quantiles are computed because they are a rule of thumb for the choice of strike price for the intraday cap and floor futures by the EEX, cf. Remark 4.18.

In the following we use all data up until 1 October 2017 as *in-sample* data from which we can calibrate and estimate our models. All data after this date is then used for an *out-of-sample* analysis.

Remark 4.18 (Quantiles of the ID_3 price). On the EEX website[4] it is stated that the intraday cap futures' strike price is based on the 85% quantile of the ID_3 price of 2015. Likewise the intraday floor futures' strike price corresponds to the 5% quantile of the ID_3 prices of the same period. In Table 4.2 these quantiles are shown for the years 2015, 2016, and 2017. We can approximately confirm the statement of the EEX's website.

[4] `https://www.eex.com/en/products/energiewende-products/german-intraday-cap-futures` (visited at 29-06-2018).

Table 4.2.: Descriptive statistics of the hourly ID_3 prices per year. The shown statistics are the number of observations, the sample mean and median, the minimum and maximum, the sample standard deviation, and the empirical 5% and 85% quantiles. These quantiles form the basis of the intraday cap and floor futures' strike prices. The years 2015 and 2018 are marked with an asterisk, since the data for these years is incomplete. The minimum and maximum are not symmetrically around the mean since a yearly, weekly, and daily seasonality underlies the ID_3 prices.

Year	#	Mean	Med	Min	Max	Std	Q5%	Q85%
*2015	4489	32.99	32.73	-89.14	167.27	15.57	9.02	46.54
2016	8784	29.24	28.71	-184.89	134.01	13.92	9.55	40.22
2017	8760	34.31	34.03	-128.27	239.74	19.57	7.31	46.44
*2018	2015	35.89	37.50	-61.62	123.66	17.09	5.18	49.38

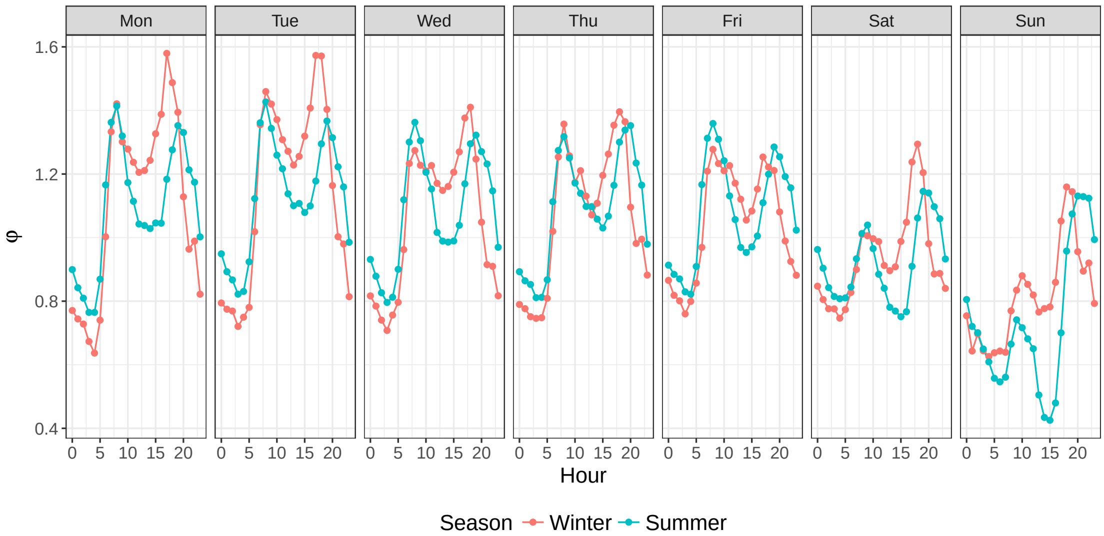
Remark 4.19 (Seasons in the data). From its behaviour in Figure 4.1 it appears that the ID_3 prices follow two regimes corresponding to summer and winter. In winter the prices are more volatile and have slightly higher mean than in summer. After a more careful analysis of the ID_3 prices we decided to classify the seven months March until September as summer and the other five months as winter. The summer months all have standard deviation less than 15, whereas the winter months have a standard deviation of around 20. This will impact our choice of PFC as will be discussed later in this section.

4.3.2. Hourly price forward curve

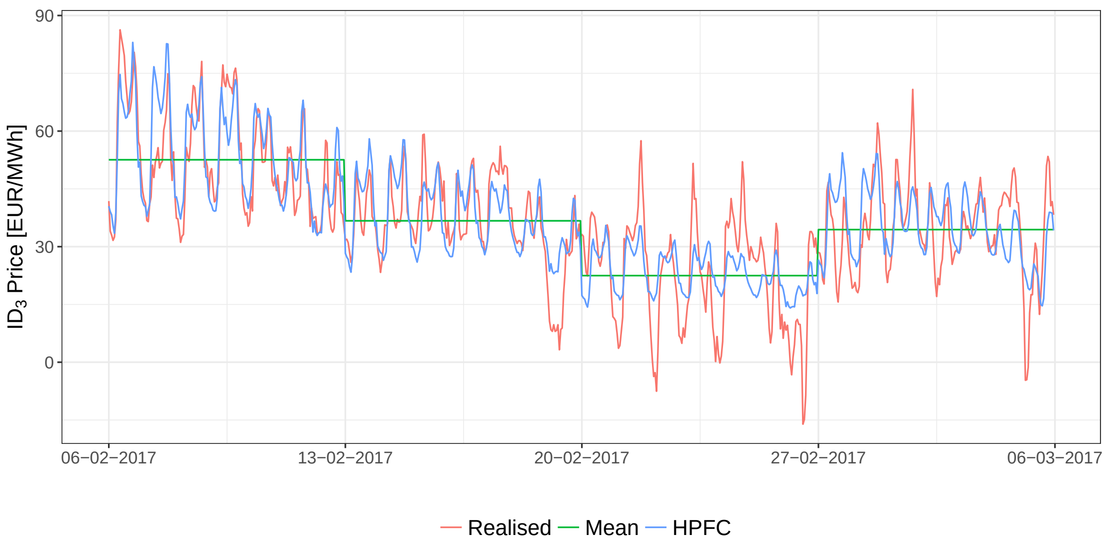
For the model of Section 4.2.2 to be completely defined we need to determine the hourly price forward curve $f_0(\tau)$. As mentioned in Remark 4.16 there exist many studies on how to adequately construct a PFC. However, since intraday cap futures are only traded four weeks in advance, we only need the PFC five weeks into the future. Therefore, we opt for a simpler method and we model the PFC by a deterministic weekly pattern φ multiplied by today's settlement price of that week's futures contract, i.e.

$$f_t(\tau) = \varphi(h_\tau, d_\tau, s_\tau)\, F_0(w_\tau)\, Z_t(s_\tau). \tag{4.7}$$

Here h_τ is the hour of delivery time τ, d_τ its day of the week, s_τ its season, and w_τ its week number. The function $\varphi(h, d, s)$ with hourly factors is shown in Figure 4.3a and $F_0(w)$ denotes today's futures price of delivery week w. As is clear from the parameter s_t we differentiate between the weekly factors for each season, the definition of which is described in Remark 4.19. For the classification of the day of the week we regarded



(a) Hourly factors for each day of the week and season. The same camel-like pattern as in the day-ahead spot prices is observable.



(b) Realised ID_3 prices (red) with the constructed hourly price forward curve (blue) and the weekly mean (green).

Figure 4.3.: Visualization of the hourly price forward curve.

bridge days and partial holidays – i.e. holidays that are a holiday in some but not all federal states of Germany – as Saturdays and holidays as Sundays.

The process Z_t is a geometric Brownian motion with season-dependent mean and standard deviation, i.e.

$$Z_t(s) = \exp\left(\left(\mu_s^Z - \tfrac{(\sigma_s^Z)^2}{2}\right) t + \sigma_s^Z\, W_t^Z\right),$$

where W^Z is a one-dimensional standard Brownian motion under P and assumed to be independent of the Q-Brownian motion W of X under Q .

4.3.3. Calibration

We calibrated the parameters λ and σ from all intraday cap futures' prices in the in-sample period using the price given in Theorem 4.15. We used the `R`-function `optim` to solve the following minimization problem

$$\min_{\lambda,\sigma^2} \frac{1}{|\mathcal{D}^{\text{in}}|} \sum_{(\mathcal{W},\mathcal{C},F)\in\mathcal{D}^{\text{in}}} \left[\mathcal{C} - C_0(\mathcal{W}, F)\right]^2,$$

where $\mathcal{D}^{\text{in}}$ denotes the in-sample data set with the realised intraday cap futures' prices denoted by $\mathcal{C}$. We denoted the modelled intraday cap futures' price by $C_0(\mathcal{W}, F)$ to emphasize the dependence on the realised week futures price F through the PFC of Equation (4.7). At each trading day t we used the week futures' settlement price F of that trading day. The results of the minimization procedure were $\lambda = 0.05487$ per hour and $\sigma = 1.9914$ Euro/MWh.

Remark 4.20 (Changing to the real-world measure P). Recall that we defined the Ornstein-Uhlenbeck X under the pricing measure Q. In order to simulate the process X we need to define a measure change. To do this we use the *stochastic exponential* as introduced in Section 2.3, i.e.

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} := e^{\theta W_t - \frac{1}{2}\theta^2 t}.$$

Under P the Q-Brownian motion is given by $W_t = \tilde{W}_t - \theta t$, where $\tilde{W}$ is a P-Brownian motion. It follows that under P the dynamics

$$dX_t = \lambda\left(-\tfrac{\theta\sigma}{\lambda} - X_t\right) dt + \sigma\, d\tilde{W}_t, \quad X_0 = 0.$$

With the help of Theorem 2.27 on page 17 we find that this SDE has the following strong solution

$$X_t = -\frac{\theta\sigma}{\lambda}\left(1 - e^{-\lambda t}\right) + \sigma \int_0^t e^{-\lambda(t-u)} d\tilde{W}_u$$

under P.

Since X is defined as the difference $X_\tau = I(\tau) - S(\tau)$, we can use these differences and Remark 4.20 to estimate θ with the help of the estimated parameters λ and σ. We estimated the mean of the differences

$$\mathbb{E}_P[X_{t+h} - e^{-\lambda h} X_t] = -\frac{\theta\sigma}{\lambda}\left(1 - e^{-\lambda h}\right)$$

from which we derived that $\theta = -0.00274$ Euro/MWh.

The hourly factors φ were estimated on the in-sample data as the median of the ID_3 prices divided by their weekly mean grouped by hour, day of the week, and season. The weekly mean was chosen to normalize the hourly observations because we expect the futures price to equal the mean of the week. The median of all observations for each group was taken to filter out the extraordinary data points (spikes) since the median is more robust than the mean. Figure 4.3a illustrates the hourly factors for both seasons. Using the weekly mean of the realised ID_3 prices instead of the week futures' price $F_0(w)$ Figure 4.3b shows the constructed PFC for four weeks in February 2017. We chose February 2017 since the mean of each week is very distinct, hence giving a good picture of the behaviour of this PFC for several very different weeks.

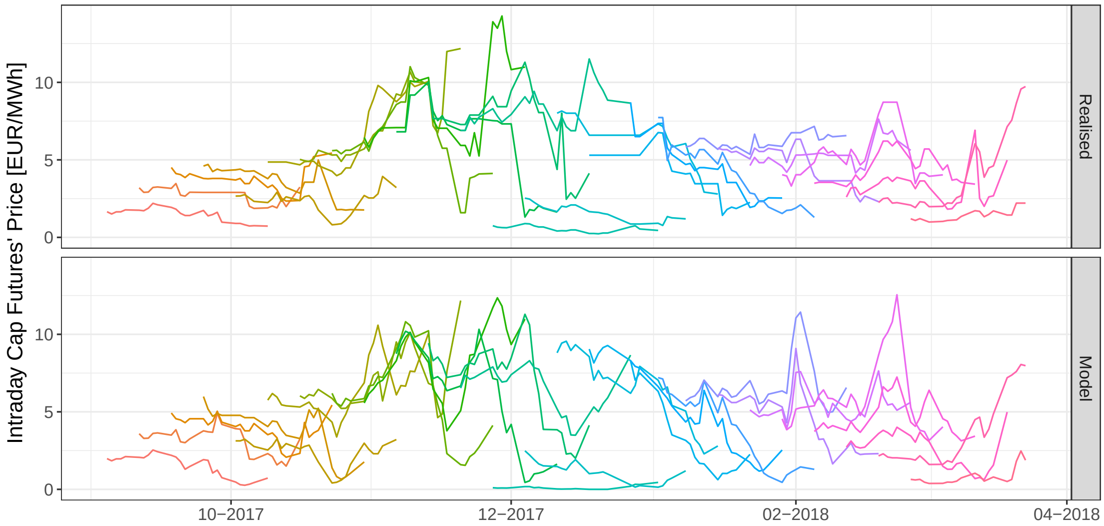
The season-dependent mean and standard deviation μ_s^Z and σ_s^Z were estimated from the weekly futures prices dynamics. The estimates for Winter were $\mu_{\text{win}}^Z = 0.000218$ per day and $\sigma_{\text{win}}^Z = 0.0350$, whereas the estimates for Summer were $\mu_{\text{sum}}^Z = 0.000343$ per day and $\sigma_{\text{sum}}^Z = 0.0214$. It is immediately clear from the values for the standard deviation that also within the futures prices the separation into two seasons is necessary.

4.3.4. Analysis

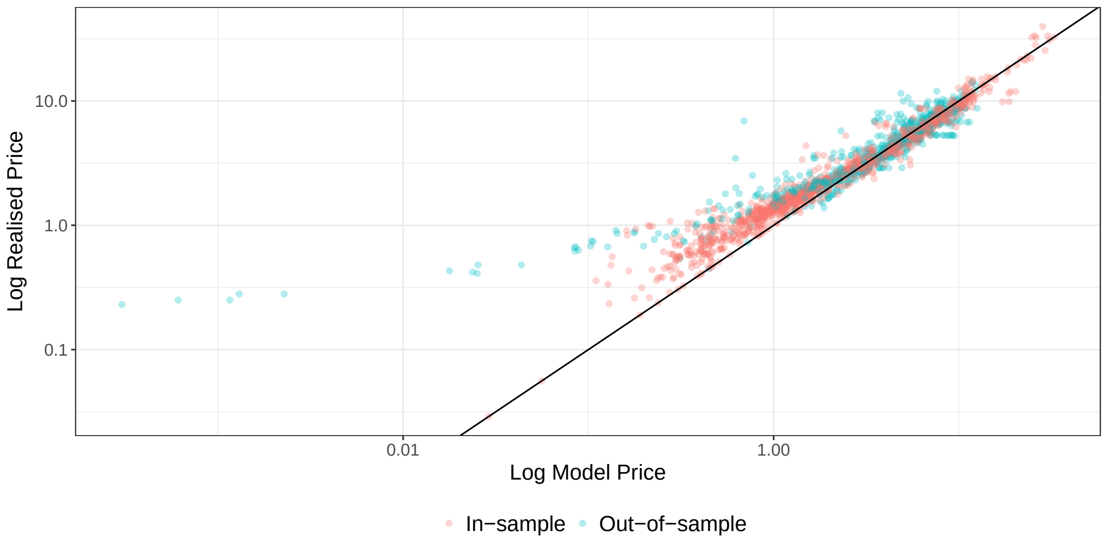
The first thing that is of interest is how well the model fits today's intraday cap futures' prices, i.e. at $t = 0$. We used the settlement prices of the week futures to compare the model with the settlement prices of the intraday cap futures at the same trading day. In this approach we assumed for each trading day that $t = 0$ and thus that $X_t = 0$ in Theorem 4.15. Figure 4.4 illustrates the performance of the model. The realised and model generated intraday cap futures' prices are shown in Figure 4.4a. In another representation this data is shown plotted against each other in Figure 4.4b. Especially from the latter it is clear that the model captures today's intraday cap futures' prices extremely well.

Furthermore, we want to see how well the model predicts the probability distribution of the intraday cap futures' prices. Figure 4.5 shows the simulated paths together with the realised intraday cap futures' price of an out-of-sample winter contract: the first full week of February 2018. The simulation started from the first trading day of the contract until its last trading day before delivery. Figure 4.5a shows five simulated paths together with the realised price. At first glance the simulated paths seem to exhibit the same behaviour as the realised path. In Figure 4.5b the evolution of the empirical distribution is shown. We observe that the realised path lies in the 25%-75% quantile range.

In order to assess the quality of the model further, it is important that we do not just simulate prices for one contract as is done in Figure 4.5. Therefore, we simulated 1,000

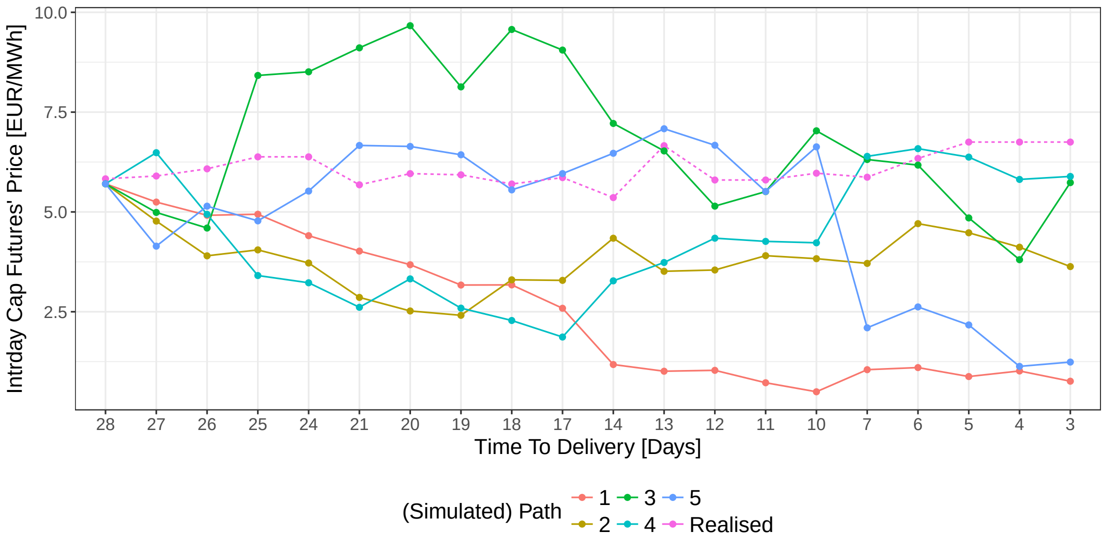


(a) Realised (top) and model (bottom) generated intraday cap futures' prices for the out-of-sample data set. Every delivery week corresponds to a single colour.

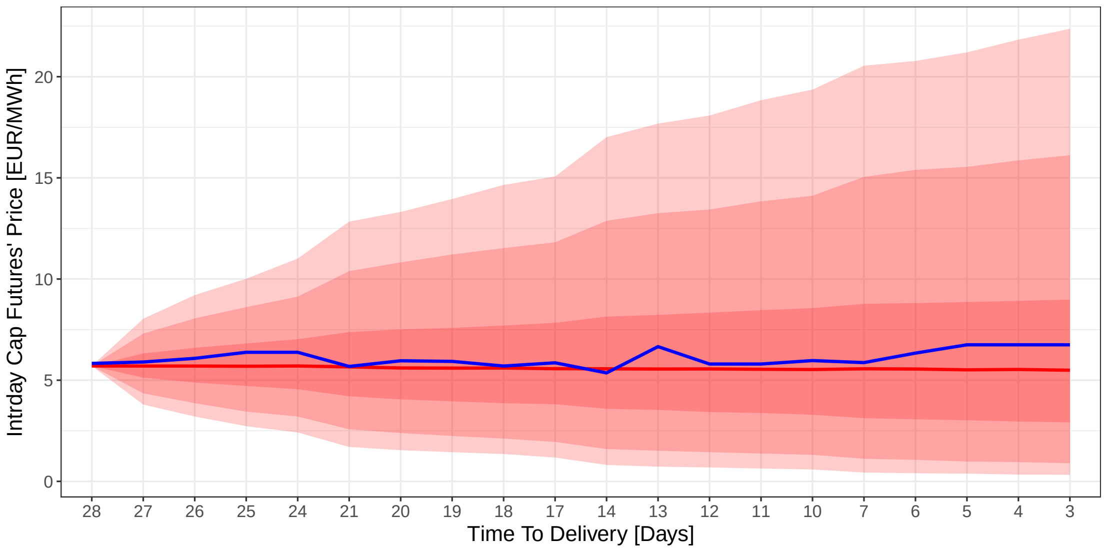


(b) Logarithm of the realised prices plotted against the logarithm of the model generated prices for the whole data set. The black line is the identity. If the model was infallible, all points would lie on the identity. The extreme deviation on the left is the contract delivering during the Christmas holidays of 2017, i.e. from the 25th to the 31st of December 2017.

Figure 4.4.: Realised and model generated intraday cap prices. We used the assumption that $t = 0$ for each trading day and used the futures price that was realised at that day for the PFC.



(a) Simulation of 5 paths of intraday cap futures' prices and its realised value. We see that the behaviour of the simulated prices is similar to the realised price.



(b) Simulation 10,000 paths of intraday cap futures' prices. The realised intraday cap futures' price (blue) is shown together with the median (red line) and the 1%, 5%, 25%, 75%, 95%, and 99% quantiles.

Figure 4.5.: Out-of-sample simulation of a winter contract with delivery starting on Monday, 5 February 2018. The initial price of the simulation is a near perfect fit to the realised price, as was discussed with Figure 4.4.

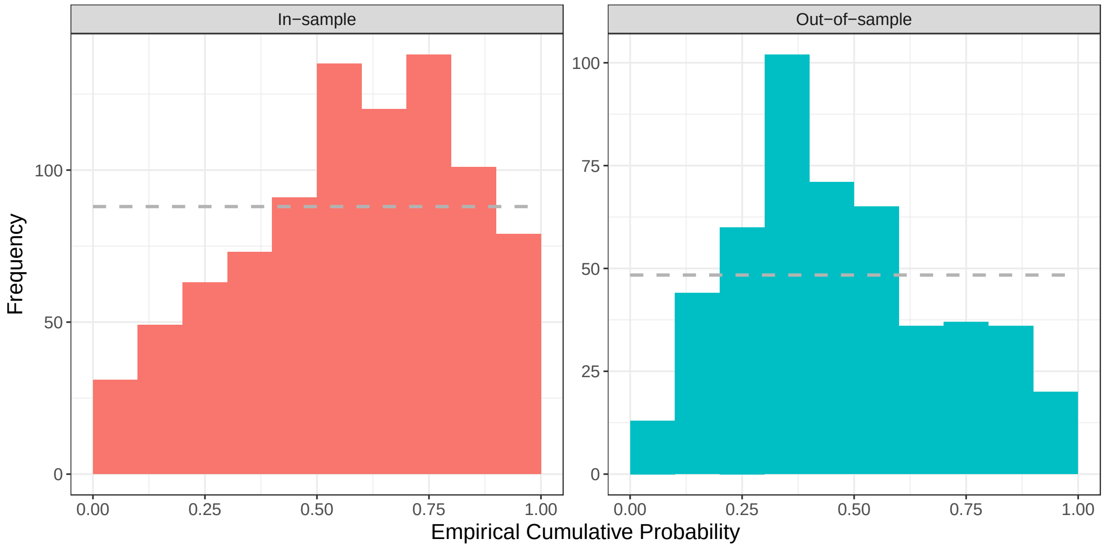


Figure 4.6.: Histogram of the empirical cumulative probabilities of the realised intraday cap futures prices of all contracts and all trading days. The empirical cumulative distribution function was constructed from 1,000 simulated paths. If the model captured the data perfectly, the grey lines show the level all bars should have theoretically.

paths for every contract starting from first trading day and ending at the last trading day before delivery, just like the simulation of Figure 4.5. This means that for every contract and every trading day we have 1,000 simulated prices and one realised price. We use the 1,000 simulated prices to construct the empirical cumulative distribution function $\hat{F}_{t,w}$ for contract w and trading day t. Using the realised price $\mathcal{C}_{t,w}$ we compute the cumulative probability

$$p_{t,w} := \hat{F}_{t,w}(\mathcal{C}_{t,w})$$

for all delivery weeks w and all trading days t. This yields 1,364 cumulative probabilities. If our model matches the distribution of the intraday cap prices perfectly, we would find that the cumulative probabilities $p_{t,w}$ would be uniformly distributed.

Figure 4.6 shows a histogram of the found probabilities, sorted by in-sample and out-of-sample. The grey lines show the level the bars of the histogram should have if the model captured the distribution of the intraday cap futures prices perfectly. We observe that we have a to high concentration of cumulative probabilities around 50%. This implies that the model's tails are too heavy, i.e. it gives too much probability mass to the lower and higher prices. However, mainly the lower prices have been overestimated. This can be improved by using a different stochastic driving process for Z_t. We used the simplest of all: the geometric Brownian motion. Switching to a more complex process could yield better results. We conclude that the model captures the intraday cap prices well and can easily be extended.

5. A structural Heath-Jarrow-Morton framework[1]

Since the prices of different electricity contracts exhibit different behaviour – such as spikes in the day-ahead spot but not in futures prices – it is a rising challenge in energy finance to define a single model that allows for a joint simulation of power prices at intraday, spot, and futures markets. In this chapter we suggest a Heath-Jarrow-Morton framework for modelling electricity prices. The framework is consistent with the current forward term structure (i.e. the price forward curve) and we motivate each mathematical component by an economic interpretation. Furthermore, we discuss the computation of intraday, spot, and futures prices within this framework and we show how options on futures contracts can be priced. A new approach is the use of structural models – see Section 2.4.2 – within a Heath-Jarrow-Morton framework.

5.1. Literature review on forward models

The starting point for a Heath-Jarrow-Morton (HJM[2]) approach for electricity prices is the fictitious *forward price* or *forward kernel*.[3] The forward kernel $f_t(\tau)$, $t \leq \tau$, is the price at time t of a forward contract delivering electricity instantly at time τ. As derived in Section 2.4.3 it follows that the price at t of a futures contract delivering from τ_1 to τ_2 is the averaged forward kernel during the delivery period, i.e.

$$F_t(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f_t(u)\, du, \quad t \leq \tau_1. \tag{5.1}$$

In the HJM framework for interest rates the forward rate is modelled instead of the short rate, cf. Brigo and Mercurio (2006). Therefore, modelling the forward kernel instead of the day-ahead spot price makes this an HJM approach for power prices. Furthermore, just like in the HJM framework for interest rates, the forward kernel itself is not a traded product at the market but its (integrated) derivatives are.

Several models for the forward kernel $f_t(\tau)$ have been introduced by Clewlow and Strickland (1999); Benth and Koekebakker (2008); Kiesel et al. (2009); Hinz et al. (2005); Koekebakker and Ollmar (2005). They define the forward kernel dynamics driven by Brownian motions. However, since the day-ahead spot prices show spikes,

[1] Based on published work: Hinderks et al. (2019). Original title: A structural Heath-Jarrow-Morton framework for consistent intraday, spot, and futures electricity prices.

[2] See Heath et al. (1992) for the original paper introducing this framework for interest rate modelling.

[3] Forward kernel is the name used by Caldana et al. (2017).

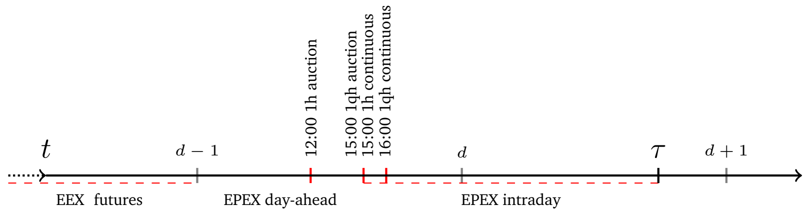


Figure 5.1.: Observation structure of $f_t(\tau)$ for the German electricity market and a fixed delivery time τ, cf. Figure 1.1 on page 3. The red marked lines and time points are the (indirect) observation moments. The notation 1h and 1qh are used for hourly and quarter hourly contracts. The lines with $d-1$ and d stand for the start of day $d-1$ and d.

these models have drawbacks. Therefore, there is a need for a forward kernel model that allows for spikes in relatively short delivery periods (day-ahead spot contracts) but smooths these out for longer delivery periods (futures contracts). The theoretical HJM framework of Benth et al. (2019) introduce forward kernel dynamics driven by Brownian motions and pure jump Lévy processes. Their framework is similar to ours but touches on the differences between the real-world and the pricing measure. A difference to our approach is that we have motivated the modelling ingredients by economic arguments and allow for an easy transfer of day-ahead structural models to a Heath-Jarrow-Morton setting, which we also show in Section 5.3.

In the literature the use of more than one probability measure has also been challenged: Lyle and Elliott (2009) and Caldana et al. (2017) assume a single probability measure, for example. This is supported by the fact that it is not clear which equivalent measure should be the pricing measure Q. Since electricity is a *non-storable* commodity and *buy and hold* strategy arguments are not valid, it is not clear what the relation between the price of electricity contracts and the money market account is (Bessembinder and Lemmon, 2002). This also implies that the market is incomplete and that there are (possibly) infinitely many equivalent martingale measures. Again, this leaves the choice of pricing measure unclear. Chapter 6 discusses this relation in more detail and offers a new approach of considering electricity prices that is consistent with the framework introduced in this chapter.

We follow the idea of Caldana et al. (2017) that the prices of day-ahead spot and futures contracts both should be computed by Equation (5.1). This actually sounds intuitively since, for example at the German markets, day-ahead spot contracts are traded at least twelve hours before delivery. In other countries such as the US the terminology is different: the day-ahead spot price is commonly referred to as the forward price (Longstaff and Wang, 2004). Even in Europe, with the increasing popularity of the intraday markets, we observe a shift in terminology: Weron (2014) remarks that the term *spot* is used more and more frequently for the real-time or intraday market. We will always explicitly state to which spot market we refer.

In this chapter we propose to extend Equation (5.1) to the intraday market. Figure 5.1

gives an example of the development of the forward kernel $f_t(\tau)$ and how it becomes observable at the German market, see also Section 1.1. We introduce a model for the forward kernel based on the economic intuition that there are two driving components behind the forward kernel. The first component is the equilibrium of supply and demand at delivery time and the second is a general noise from partially informed traders or illiquidity at trading time t. Furthermore, we show how the classical models described by Schwartz and Smith (2000) and Lucia and Schwartz (2002) fit into our framework. We also show how other more general day-ahead spot price models can be used to fit into our model. A particular new example we introduce in this chapter, is to use structural models in the context of an HJM framework. We also apply our framework to the setting of factor models.

5.2. A structural Heath-Jarrow-Morton framework

In Section 5.2.1 we will define a model for the forward kernel motivated by economic interpretations. Using this model in Sections 5.2.2 and 5.2.3 we derive the prices of futures contracts and options on futures contracts, respectively. Section 5.2.4 gives an overview of the prices for different electricity contracts for the example of the German market.

5.2.1. Forward kernel

The forward kernel $f_t(\tau)$ is the price at time t of a forward contract delivering 1 MW instantly at time τ. Like in the rest of this thesis we interpret t as the *trading time* and τ as the *delivery time*.

For $\tau \geq 0$ let $X^\tau = \{X_t^\tau; t \geq 0\}$ and $Y = \{Y_\tau; \tau \geq 0\}$ be two independent, a.s. càdlàg stochastic processes on the complete probability space $(\Omega, \mathcal{A}, P)$ taking values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Furthermore, assume that the process Y and for each $\tau \geq 0$ also the process X^τ are adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$, which satisfies the usual conditions, see Chapter 2. The filtration generated by Y and X^τ augmented by all P-null sets automatically fulfills these conditions, for example. Finally, let $g : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\{g(Y_\tau); \tau \geq 0\}$ is real-valued adapted stochastic process.

We have two strong economic interpretations for these two stochastic processes: we interpret the n-dimensional process Y_τ as the randomness or the state of the market, where each component of Y_τ represents a (random) facet of the market, e.g. demand, load, or weather predictions. The function g maps the state of the market state Y_t to its corresponding price. Combining the fact that our inspiration came from the class of structural models for day-ahead spot price modelling and the fact that it gives the basic structure to the forward kernel, we call the pair (g, Y) the *structural component*. Throughout the rest of this chapter we will often also only call Y_τ the structural component.

The process X_t^τ is called the *market noise* because it accounts for the incomplete market information of all market participants and illiquidity of the market. An example of incomplete market information is the uncertainty of weather predictions: nobody knows with complete certainty about the future weather or temperature. With these interpretations we define the forward kernel:

Definition 5.1 (Forward kernel)**.** We define the *forward kernel* at trading time t and for delivery time τ as

$$f_t(\tau) := X_t^\tau \, \mathbb{E}\left[g(Y_\tau) \,|\, \mathcal{F}_t\right],$$

where X_t^τ is the *market noise* at trading time t for the delivery time τ and Y_τ is the *structural component* at delivery time τ.

We use the notation X_t^τ to emphasize that the market noise is a stochastic process in the trading time t but can (deterministically) depend on the delivery time τ, whereas the structural component Y_τ only depends on delivery time. Economically, this makes sense since the imbalance of supply and demand at delivery time τ determines the price independent of the trading time t at which we predict this imbalance. However, the market noise is the disturbance of this prediction originating from market participants with incomplete market information, which intuitively depends on both the trading time t and the delivery time τ they are trying to predict. This also allows for seasonal volatility in the market noise, for example. Although we call X_t^τ the market noise, it can also be interpreted as a measure transformation (or Radon-Nikodym derivative, see Remark 5.7) or as a general component that introduces an additional degree of freedom in the modelling process.

Assumption 5.2 (Market noise)**.** The process $X^\tau = \{X_t^\tau; t \geq 0\}$ with its interpretation as market noise for delivery time τ is defined as multiplicative stochastic noise. Therefore, we assume that it is an a.s. positive càdlàg martingale with expectation one, i.e. $\mathbb{E}X_t^\tau = 1$ for all t and τ. In particular, we assume that the initial value $X_0^\tau = 1$ a.s. for all $\tau \geq 0$.

Assumption 5.3 (Structural component)**.** We assume that $Y = \{Y_\tau; \tau \geq 0\}$ is a $\mathbb{R}^n$-valued càdlàg stochastic process. In particular, we assume that the initial value equals $Y_0 = y_0 \in \mathbb{R}^n$ a.s. such that $g(y_0) = f_0(0)$, where $f_0(\tau)$ is the initial price forward curve (PFC) for delivery time τ, which we assume to be known (cf. Remark 5.4). Furthermore, as a technical assumption we need that $\mathbb{E}|g(Y_\tau)| < \infty$ for all $\tau \geq 0$. Finally, although we assume that $g(Y_t)$ can take all values in $\mathbb{R}$, including negative values, we assume that its expectation $\mathbb{E}g(Y_\tau) > 0$ is strictly positive. The economic interpretation behind this assumption is that we do not *expect* negative forward prices to occur.

With these assumptions the sign of the forward kernel is uniquely determined by the structural component Y and the process X^τ cannot influence it. Furthermore, the expectation $\mathbb{E}f_t(\tau)$ is fully determined by the structural component Y_τ and independent of trading time t (cf. Theorem 5.6).

Remark 5.4 (Initial price forward curve $f_0(\tau)$). In this framework the initial *price forward curve* (PFC), denoted by $f_0(\tau)$, plays an important role: it determines the expectation of the forward kernel $f_t(\tau)$. There are many studies that describe how one can construct a

PFC from market prices such as Caldana et al. (2017); Kiesel et al. (2018), for example. In practice every energy utility has an in-house PFC. Therefore, we will assume that the PFC is known in the following .

As discussed by Benth and Paraschiv (2018) another interesting possibility is to use a Musiela parametrisation for the forward kernel. This parametrisation is given by the bijective mapping $(t, \tau) \mapsto (t, u) := (t, \tau - t)$ of $\mathbb{R}^2$ on itself. In their work Benth and Paraschiv (2018) propose a spatio-temporal random field model in the context of a HJM framework under the Musiela parametrisation, where they call the time to maturity u the spatial component. They disentangled the temporal from spatial effects on the dynamics of forward prices and found that the temporal noise was non-Gaussian. In our context we could directly use the Musiela parametrisation by substituting $\tau = u + t$.

Lemma 5.5. *For fixed $\tau \geq 0$ the forward kernel process $f(\tau) := \{f_t(\tau); t \geq 0\}$ is an adapted stochastic process. Furthermore, $f(\tau)$ is a.s. càdlàg.*

Proof. By definition $f(\tau)$ is a stochastic process. Moreover, since we assumed X_t^τ to be $\mathcal{F}_t$-measurable and since the conditional expectation $Z_t^\tau := \mathbb{E}[g(Y_\tau) \mid \mathcal{F}_t]$ is always $\mathcal{F}_t$-measurable, the $\mathcal{F}_t$-measurability of $f_t(\tau)$ follows immediately. Because the filtration satisfies the usual conditions, Z_t^τ has a càdlàg modification (Karatzas and Shreve, 1998, Chapter 1, Theorem 3.13). Since the conditional expectation Z_t^τ is uniquely defined up to null sets, we can choose this modification and the result follows by the assumption that X_t^τ is càdlàg. □

Since we assume that X^τ and Y both a.s. start at a deterministic value, we assume without loss of generality that $\mathcal{F}_0$ is generated by Ω and all P-null sets. This in particular implies that $\mathbb{E}g(Y_\tau) = \mathbb{E}\left[g(Y_\tau) \mid \mathcal{F}_0\right]$, a fact we will exploit in the next theorem.

Theorem 5.6. *For fixed $\tau \geq 0$ the forward kernel process $f(\tau) := \{f_t(\tau); t \geq 0\}$ is a martingale. Furthermore, its expectation is given by*

$$\mathbb{E}f_t(\tau) = \mathbb{E}g(Y_\tau) = f_0(\tau)$$

for all $0 \leq t \leq \tau$.

Proof. The product of two independent martingales clearly is a martingale. Furthermore, it follows immediately from Assumption 5.2 and 5.3 that

$$\mathbb{E}f_t(\tau) = \mathbb{E}\left[X_t^\tau\right] \, \mathbb{E}\left[\mathbb{E}\left[g(Y_\tau) \mid \mathcal{F}_t\right]\right] = \mathbb{E}g(Y_\tau) = X_0^\tau \, \mathbb{E}\left[g(Y_\tau) \mid \mathcal{F}_0\right] = f_0(\tau)$$

by the independence of X_t^τ and Y_τ. □

Theorem 5.6 also imposes a condition for the expectation $\mathbb{E}g(Y_\tau)$ of the structural component, which can be used to calibrate the structural component Y and function g after the PFC $f_0(\tau)$ has been determined. If one wants to obtain a model that is consistent with an existing PFC $f_0(\tau)$, one needs to choose and calibrate the structural component (g, Y) such that $\mathbb{E}g(Y_\tau) = f_0(\tau)$. In Section 5.3 we give another tool to achieve a perfect initial fit by introducing the relative structural component.

Remark 5.7 (Risk-neutral measure). In the previous discussion we considered the measure space $(\Omega, \mathcal{F}, P)$ equipped with the measure P. However, as seen in Theorem 5.6 and will be seen in Lemma 5.12 the forward kernel and its induced futures prices are martingales under the measure P. This is an argument in favour of viewing P as the *risk-neutral measure* or *pricing measure* in this framework, making this framework especially suitable for pricing derivatives. However, if one wants to simulate market prices through this framework, one needs to derive the dynamics of the market prices under the real-world measure which can be done by a suitable measure change, as is discussed by Benth et al. (2019). Under the real-world measure the expectation is then no longer constant in trading time t, which is a phenomenon that is supported by plenty of empirical evidence. In particular in the case of the continuous intraday market this is studied by the findings of Kiesel and Paraschiv (2017).

Another view on this framework is achieved by defining the following measure: the τ*-forward measure* P^τ defined by its Radon-Nikodym derivative

$$\frac{dP^\tau}{dP}\Big|_{\mathcal{F}_t} = \frac{1}{X_t^\tau} \tag{5.2}$$

could be used for this purpose. Using the τ-forward measure and Bayes' theorem for conditional expectations we can rewrite Definition 5.1

$$f_t(\tau) = X_t^\tau \, \mathbb{E}_P[g(Y_\tau) \,|\, \mathcal{F}_t] = \mathbb{E}_{P^\tau}[X_\tau^\tau \, g(Y_\tau) \,|\, \mathcal{F}_t].$$

The latter term can be defined as

$$S_t := X_t^t \, g(Y_t),$$

which yields a general spot price model. This is another argument in favour of viewing P and P^τ as equivalent pricing measures (where P^τ is viewed as a forward pricing measure). As discussed in Chapter 2 electricity markets are incomplete and, therefore, it is possible that multiple equivalent pricing measures exist. In this setting the choice of the stochastic process X_t^τ can be viewed as the choice of delivery time specific pricing measure P^τ in light of Equation (5.2). If the noise $X_t := X_t^\tau$ is chosen to be independent of the delivery time τ, so is the τ-forward measure P^τ.

5.2.2. Futures prices

As discussed in Section 5.1 the forward kernel can be used to compute the price of futures contracts. In the following we assume the interest rate to equal $r = 0$ for notational convenience. On the other hand, if one assumes $r \neq 0$, discounting has to be taken into account. In Remark 5.13 we have some notes on how to change our framework to include discounting. Furthermore, we assume that all prices are normalized, meaning that we assume all prices to be in Euro/MWh as usual.

Definition 5.8 (Futures contract price). For $0 \leq t \leq \tau_1 < \tau_2$ we call

$$F_t(\tau_1, \tau_2) := \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f_t(u) \, du$$

the price of a futures contract at time t delivering 1 MW continuously from τ_1 to τ_2.

Since we denote all prices in Euro/MWh, the price that one pays at time t when one buys a futures contract delivering 1 MW from τ_1 to τ_2 is given by $(\tau_2 - \tau_1)\, F_t(\tau_1, \tau_2)$, where we assume that $\tau_2 - \tau_1$ is measured in hours.

Example 5.9 (Day-ahead spot price)**.** We compute the day-ahead spot price as a futures contract. It is auctioned at day $d-1$ at hour a:00 and delivered at day d from h:00 to $(h+1)$:00, i.e.

$$S(d,h) := F_{t^a_{d-1}}\left(t^h_d, t^{h+1}_d\right).$$

Here t^h_d denotes the time at day d and hour h.

The next result shows that the framework is consistent with cascading.[4] It also shows that there are no arbitrage opportunities in the sense that one cannot profit or lose from dividing the delivery period of a contract into multiple smaller delivery periods and buy or sell these smaller contracts separately. For example, the cost of a futures contract delivering for one year should be the same as the cost of its four quarters.

Proposition 5.10 (Consistency of cascading)**.** *Let $0 \le \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n$ be delivery times, then we have*

$$(\tau_n - \tau_0)\, F_t(\tau_0, \tau_n) = \sum_{i=1}^{n} (\tau_i - \tau_{i-1})\, F_t(\tau_{i-1}, \tau_i)$$

for all $t \ge 0$.

Proof. This statement follows directly from Definition 5.8 and the countable additivity of the Lebesgue integral. □

Lemma 5.11. *Fix $0 \le t < \tau$. If $u \mapsto f_t(u)$ is almost surely continuous on $(\tau - \epsilon, \tau]$ for some $\epsilon > 0$, then we have*

$$\lim_{s \to \tau^-} F_t(s, \tau) = f_t(\tau)$$

almost surely.

Proof. Let $\Omega_0 \subseteq \Omega$ be the set of measure one on which the continuity statement holds. For fixed $\omega \in \Omega_0$ we compute

$$\lim_{s \to \tau^-} F_t(s,\tau) = \frac{\lim_{s \to \tau^-} \int_s^\tau f_t(u)\, du}{\lim_{s \to \tau^-} \tau - s} = \frac{\lim_{s \to \tau^-} -f_t(s)}{-1} = f_t(\tau)$$

where we used L'Hôpital's rule for the second equality. □

[4] By cascading we mean the way how futures with a longer delivery period are settled. For example, a calendar year futures contract cascades (or splits up) into three monthly futures (January, February, and March) and three quarterly futures (Q2, Q3, and Q4) upon start of delivery. This way, these can be traded independently again. In the German market monthly futures do not cascade. However, the settlement price at the end of the delivery is exactly the average of the day-ahead spot prices during delivery. This could be interpreted that also monthly futures are cascading to the hourly (day-ahead) spot contracts, since their price converges to this average.

The previous lemma shows that the price of a futures contract delivering for just an instant equals the forward kernel, which supports its naming.

Lemma 5.12. *Assume that the price forward curve $\tau \mapsto f_0(\tau)$ is continuous. Then the futures price process $F(\tau_1, \tau_2) := \{F_t(\tau_1, \tau_2); t \geq 0\}$ is a martingale. Its expectation is given by*

$$\mathbb{E}F_t(\tau_1, \tau_2) = F_0(\tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f_0(u)\, du$$

for all $0 \leq t \leq \tau_1 < \tau_2$.

Proof. Since the price forward curve is continuous, it is bounded on any compact set – in particular intervals of the form $[\tau_1, \tau_2]$ – and, therefore, integrable on compacts. Direct computation with the Fubini-Tonelli theorem, Theorem 2.53, shows that for $0 \leq t < s$

$$\mathbb{E}[F_s(\tau_1, \tau_2) \,|\, \mathcal{F}_t] = \mathbb{E}\left[\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f_s(u)\, du \,|\, \mathcal{F}_t\right] = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \mathbb{E}[f_s(u) \,|\, \mathcal{F}_t]\, du,$$

where the latter exists and, therefore, all integrals exist. Combination with Theorem 5.6 now proves the result. □

Remark 5.13 (Including discounting). If we assume that $r \neq 0$, the futures price depends on the settlement date. There are two possibilities: settlement takes place either through continuous payments[5] during the delivery period or at once at the end of the delivery period. If $d_t(\tau)$ denotes the discount factor of a future payment at time τ to an earlier time t, the price of a futures contract is given by

$$F_t(\tau_1, \tau_2) = \frac{1}{\int_{\tau_1}^{\tau_2} d_t(u)\, du} \int_{\tau_1}^{\tau_2} d_t(u)\, f_t(u)\, du$$

for continuous settlement and by

$$F_t(\tau_1, \tau_2) = \frac{1}{(\tau_2 - \tau_1)\, d_t(\tau_2)} \int_{\tau_1}^{\tau_2} d_t(u)\, f_t(u)\, du$$

for settlement at the end of delivery.

5.2.3. Options on futures contracts

In this section we assume that the market noise is given by a geometric Brownian motion (GBM) without drift, i.e.

$$dX_t^\tau = X_t^\tau\, \Sigma(t, \tau)' \cdot dW_t$$

where $\Sigma(t, \tau)$ is a deterministic m-dimensional volatility vector and W_t is an m-dimensional Brownian motion. The strong solution of X_t^τ is given by

$$X_t^\tau = \exp\left(\int_0^t \Sigma(u, \tau)' \cdot dW_u - \frac{1}{2} \int_0^t \Sigma(u, \tau)' \cdot \Sigma(u, \tau)\, du\right).$$

[5] Continuous settlement of the futures contract makes it more like a swap contract on the forward kernel.

In this case, X_t^τ satisfies Assumption 5.2 if $\Sigma(u,\tau)$ is square integrable in u. But this is already a requirement for the stochastic integral to be defined.

Example 5.14 (Hull-White market noise dynamics)**.** A possible choice for Σ is a two-factor forward dynamic similar to Kiesel et al. (2009), which is also discussed in a geometric setting by Fanelli and Schmeck (2018) for pricing options on futures. This volatility structure is extended by Latini et al. (2018) in an additive setting. They discussed a two-factor volatility structure comparable to the two-factor Hull-White model for interest rate modelling (Brigo and Mercurio, 2006, Section 4.2.5). It is given by

$$\Sigma(t,\tau)' := (e^{-\lambda(\tau-t)}\sigma_1, \sigma_2(\tau)),$$

where $\sigma_1 > 0$ is the additional short-term volatility, $\lambda > 0$ is the rate of decay of the short-term volatility, and $\sigma_2(\tau) > 0$ is the long-term volatility at delivery time τ. A convenient choice for σ_2 is a piecewise constant function, being constant on delivery periods of tradable futures contracts. An advantage of this choice is that we can use the calibration methods for X_t^τ as discussed by Kiesel et al. (2009); Latini et al. (2018); Fanelli and Schmeck (2018).

Throughout the rest of this subsection we assume that the conditional expectation of the structural component decomposes into an affine structure:

Definition 5.15 (Affine decomposition)**.** We say the structural component (g, Y) allows for the *affine decomposition*, if there exist deterministic functions $(t,\tau) \mapsto A_t^\tau \in \mathbb{R}^{n\times n}$ and $(t,\tau) \mapsto B_t^\tau \in \mathbb{R}^n$ such that the following decomposition holds

$$\mathbb{E}[g(Y_\tau) \,|\, \mathcal{F}_t] = g(A_t^\tau\, Y_t + B_t^\tau) \tag{5.3}$$

a.s. for all $\tau \geq t \geq 0$.

This decomposition can be motivated by the fact that our best guess at time t for the state of market Y_τ at time τ is an affine transformation of the current state of the market Y_t. This is also the main idea behind Kalman filtering, for example. If the decomposition holds, this merely states that this best guess should hold under the transformation g, which transforms the market state into a price. Furthermore, as a mathematical motivation we offer Corollary 2.28.

It follows immediately that the forward kernel is given by

$$f_t(\tau) = X_t^\tau\, g(A_t^\tau\, Y_t + B_t^\tau), \tag{5.4}$$

when the affine decomposition assumption is satisfied. Furthermore, the futures price of Definition 5.8 can be rewritten as

$$F_t(\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} X_t^u\, g(A_t^u\, Y_t + B_t^u)\, du$$

for all $0 \leq t \leq \tau_1 < \tau_2$. As immediate consequences we obtain:

Lemma 5.16. *If the structural component (g, Y) allows for the affine decomposition, then $\mathbb{E}[g(Y_\tau) \,|\, \mathcal{F}_t] = \mathbb{E}[g(Y_\tau) \,|\, Y_t]$.*

Lemma 5.17. *Under assumption of the decomposition of Definition 5.15 the forward kernel conditioned on Y_t is lognormally distributed, i.e.*

$$(f_t(\tau) \,|\, Y_t = y) \sim LN\left(\ln[g(A_t^\tau\, y + B_t^\tau)], \int_0^t \Sigma(u,\tau)' \cdot \Sigma(u,\tau)\, du\right).$$

Proof. Using Equation (5.4) we compute

$$P\left(f_t(\tau) \le x \,|\, Y_t = y\right) = P\left(X_t^\tau\, g(A_t^\tau\, y + B_t^\tau) \le x\right),$$

which shows the result since $X_t^\tau \sim LN\left(0, \int_0^t \Sigma(u,\tau)' \cdot \Sigma(u,\tau)\, du\right)$. □

Theorem 5.18. *If the structural component (g, Y) allows for the affine decomposition, then the first two conditional moments of the futures price $F_t(\tau_1, \tau_2)$ exist and are given by*

$$\mathbb{E}[F_t(\tau_1,\tau_2) \,|\, Y_t = y] = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} g(A_t^u\, y + B_t^u)\, du$$

and

$$\mathbb{E}[F_t(\tau_1,\tau_2)^2 \,|\, Y_t = y] = \frac{1}{(\tau_2-\tau_1)^2} \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} w_t^X(u,s)\, w_t^Y(u,s,y)\, du\, ds,$$

where

$$w_t^X(u,s) := \exp\left(\int_0^t \Sigma(v,u)' \cdot \Sigma(v,s)\, dv\right) \tag{5.5}$$

and

$$w_t^Y(u,s,y) := g(A_t^u\, y + B_t^u)\, g(A_t^s\, y + B_t^s). \tag{5.6}$$

Proof. We see that the expectation follows immediately by a Fubini argument combined with the fact that $\mathbb{E}X_t^\tau = 1$ for all $\tau \ge 0$. Applying the Fubini-Tonelli theorem, Theorem 2.53, twice we find

$$\mathbb{E}[F_t(\tau_1,\tau_2)^2 \,|\, Y_t = y] = \frac{\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \mathbb{E}[X_t^u\, X_t^s]\, \mathbb{E}[Y_u\, Y_s \,|\, Y_t = y]\, du\, ds}{(\tau_2-\tau_1)^2},$$

where it is easy to verify that the expectations equal $\mathbb{E}[X_t^u\, X_t^s] = w_t^X(u,s)$ and

$$\mathbb{E}[Y_u\, Y_s \,|\, Y_t = y] = w_t^Y(u,s,y)$$

using Equation (5.3). □

Corollary 5.19. *If the structural component (g, Y) allows for the affine decomposition, then the conditional variance of the futures price $F_t(\tau_1, \tau_2)$ is given by*

$$\mathrm{Var}[F_t(\tau_1,\tau_2) \,|\, Y_t = y] = \frac{1}{(\tau_2-\tau_1)^2} \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} \left(w_t^X(u,s) - 1\right)\, w_t^Y(u,s,y)\, du\, ds,$$

where w^X and w^Y are given by Equation (5.5) *and Equation* (5.6)*, respectively.*

Proof. We directly compute

$$\mathrm{Var}[F_t(\tau_1,\tau_2)\,|\,Y_t=y] = \mathbb{E}[F_t(\tau_1,\tau_2)^2\,|\,Y_t=y] - \mathbb{E}[F_t(\tau_1,\tau_2)\,|\,Y_t=y]^2.$$

Using Theorem 5.18 the first term is immediately given and the second term can be computed using the Fubini-Tonelli theorem, Theorem 2.53

$$\begin{aligned}\mathbb{E}[F_t(\tau_1,\tau_2)\,|\,Y_t=y]^2 &= \left(\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} g(A_t^u\,y+B_t^u)\,du\right)^2\\ &= \frac{1}{(\tau_2-\tau_1)^2}\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} g(A_t^u\,y+B_t^u)\,g(A_t^s\,y+B_t^s)\,du\,ds,\end{aligned}$$

from which the result follows. □

Remark 5.20 (Lognormal approximation). Similar to the discrete approach used by Kiesel et al. (2009) we have that the futures price is an integral of lognormally distributed variables, which can be approximated by a lognormal random variable with the same mean and standard deviation. Since there is no simple expression for the convolution of lognormal distributions, this approximation of the integral (or sum) of lognormal random variables is widely used in finance, e.g. in the context of LIBOR market models by Brigo and Mercurio (2006). An analysis of this approximation, also with regard to Asian options, which may be compared to an option on a futures with delivery period, is found in Dufresne (2004), for example.

Assumption 5.21 (Lognormal approximation)**.** Assume that the first two moments of the futures price $F_t(\tau_1,\tau_2)$ exist. Justified by Remark 5.20, we then assume that

$$(F_t(\tau_1,\tau_2)\,|\,Y_t=y) \approx (\tilde{F}_t(\tau_1,\tau_2)\,|\,Y_t=y) \sim LN\left(\mu_F(y),\sigma_F^2(y)\right),$$

i.e. the futures price is approximately lognormally distributed.

As stated in Remark 5.20 we need that the first two moments of F and $\tilde{F}$ match, which is resolved by the following lemma:

Lemma 5.22. *If the structural component (g,Y) allows for the affine decomposition and Assumption 5.21 holds, then the mean and standard deviation of the lognormal distribution are given by*

$$\mu_F(y) := \ln\int_{\tau_1}^{\tau_2} g(A_t^u\,y+B_t^u)\,du - \ln(\tau_2-\tau_1) - \frac{1}{2}\sigma_F^2(y)$$

and

$$\sigma_F^2(y) := \ln\left(1+\frac{\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2}\left(w_t^X(u,s)-1\right)\,w_t^Y(u,s,y)\,du\,ds}{\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} w_t^Y(u,s,y)\,du\,ds}\right),$$

where w^X and w^Y are given by Equation (5.5) *and Equation* (5.6)*, respectively.*

Proof. For a lognormal random variable $Z \sim LN(m, s)$, the expectation and variance are given by $\mathbb{E}Z = \exp(m + s^2/2)$ and $\operatorname{Var} Z = (\mathbb{E}Z)^2(\exp(s^2) - 1)$. Using Theorem 5.18 and Corollary 5.19 the result is found by inverting these equations. □

Using this lemma we can compute the price (conditioned on Y_t) of call (and put) options on futures contracts by the Black-Scholes formula. A call option with strike price K and maturity $T < \tau_1$ has a payoff equal to

$$(\tau_2 - \tau_1)\,(F_T(\tau_1, \tau_2) - K)^+. \tag{5.7}$$

Recall that, as stated in Section 5.2.2, the price one has to pay for a futures contract at time T equals $(\tau_2 - \tau_1)\,F_T(\tau_1, \tau_2)$, since we consider normalized prices.

Proposition 5.23 (Conditional call option price)**.** *Assume that (g, Y) allows for the affine decomposition and let Assumption 5.21 hold. For simplicity we denote the futures price at maturity time T by $F := F_T(\tau_1, \tau_2)$. Let μ_F and σ_F be given by Lemma 5.22. The price of a call option at $t = 0$ with payoff given by Equation* (5.7) *conditioned on $Y_T = y$ equals*

$$C_0(T, K, \tau_1, \tau_2; y) = \Phi(d_1(y)) \int_{\tau_1}^{\tau_2} g(A_T^u\, y + B_T^u)\, du \, - (\tau_2 - \tau_1)\, K\, \Phi(d_2(y)),$$

where Φ is the cumulative distribution function of the standard normal distribution, and with $\mu_F(y)$ and $\sigma_F(y)$ as defined in Lemma 5.22 the auxiliary variables $\delta_1(y)$ and $\delta_2(y)$ are given by

$$d_2(y) := \frac{\mu_F(y) - \ln K}{\sigma_F(y)},$$

and $d_1(y) := d_2(y) + \sigma_F(y)$.

Proof. Using the discounted conditional expectation of the payoff given in Equation (5.7) yields

$$\begin{aligned} \frac{C_0(T, K, \tau_1, \tau_2; y)}{\tau_2 - \tau_1} &= \mathbb{E}\left[(F - K)^+ \,|\, Y_T = y\right] \\ &= \mathbb{E}[F\, \mathbb{1}_{\{F \geq K\}} \,|\, Y_T = y] - K\, P(F \geq K \,|\, Y_T = y), \end{aligned}$$

where noting that we have $(F \,|\, Y_T = y) \sim LN(\mu_F(y), \sigma_F^2(y))$, yields the result by direct computation. □

As an immediate consequence we have:

Corollary 5.24 (Call option price)**.** *Assume that (g, Y) allows for the affine decomposition and let Assumption 5.21 hold. Let μ_F and σ_F be given by Lemma 5.22. The price of a call option at $t = 0$ with payoff given by Equation* (5.7) *equals*

$$C_0(T, K, \tau_1, \tau_2) = \mathbb{E}C_0(T, K, \tau_1, \tau_2; Y_T), \tag{5.8}$$

where the conditional call option price $C_0(T, K, \tau_1, \tau_2; y)$ is given in Proposition 5.23.

When the distribution of Y_T is specified, the price of a call option given by Equation (5.8) might be evaluated analytically, numerically, or through simulative methods such as Monte Carlo estimation. Alternatively, with further assumptions on the distribution of Y_T this expectation could also be approximated differently.

5.2.4. Model representation of exchange traded products

In this section we give an overview of the prices of several different electricity contracts in this HJM framework. Although there is not a single unique quoted continuous electricity price we regard $F_t(\tau_1, \tau_2)$ as the true fair price for the delivery period from τ_1 to τ_2 at any trading time t.

Futures price The price of a futures contract at time t delivering 1 MW continuously from τ_1 to τ_2 is given by Definition 5.8 and denoted by $F_t(\tau_1, \tau_2)$.

Options on futures In the setting of Section 5.2.3 the price of call and put options on futures contracts can be computed by the Black-Scholes formula as given by Proposition 5.23 or Corollary 5.24.

Day-ahead spot prices The day-ahead spot price equals the futures price within this framework as discussed in Example 5.9.

ID$_1$ and ID$_3$ price The ID$_1$ and ID$_3$ price indices on the German intraday market are given as the one and three hour volume-weighted average of all intraday trades before delivery, see Definition 4.1. Therefore, we suggest the ID$_n$ price for the delivery period from τ_1 to τ_2 to equal

$$\mathrm{ID}_n(\tau_1, \tau_2) := \frac{2}{2n-1} \int_{\tau_1 - n}^{\tau_1 - 0.5} F_u(\tau_1, \tau_2)\, du,$$

where $n = 1$ or $n = 3$ and the subtraction of τ_1 is meant in hours.

5.3. Examples of the structural component

First, we show how two classical day-ahead spot price models can be used in this HJM framework. Then, we also introduce a structural model as well as a factor model approach for Y. To make choosing an explicit model easier in this framework, we introduce the relative structural component, which can be used to set the initial price forward curve (PFC) to an existing one:

Definition 5.25 (Relative structural component)**.** The additive mean-normalized version of $g(Y_\tau)$

$$I_\tau^a := g(Y_\tau) - \mathbb{E}g(Y_\tau)$$

is called the *additive relative structural component* and its multiplicative mean-normalized version

$$I_\tau^m := \frac{g(Y_\tau)}{\mathbb{E}g(Y_\tau)}$$

is called the *multiplicative relative structural component*.

We directly obtain from these definitions:

Corollary 5.26. *The relative structural components I^a and I^m are stochastic processes with constant expectation $\mathbb{E}I^a_\tau = 0$ and $\mathbb{E}I^m_\tau = 1$ for all $\tau \geq 0$.*

Corollary 5.27 (Arithmetic PFC decomposition). *Given initial price forward curve $f_0(\tau)$ the forward kernel equals*

$$f_t(\tau) = X^\tau_t \ (f_0(\tau) + \mathbb{E}[I^a_\tau \,|\, \mathcal{F}_t]) \,,$$

where I^a_τ is the arithmetic relative structural component given in Definition 5.25.

Proof. Define an extended structural component $\tilde{Y}_\tau = (Y_\tau, f_0(\tau)) \in R^{n+1}$, where $f_0(\tau)$ is the PFC, and another function $\tilde{g}(y,x) = x + g(y) - \mathbb{E}g(y)$. It is clear that $\tilde{Y}$ and $\tilde{g}$ satisfy Assumption 5.3. It follows immediately that $\tilde{g}(\tilde{Y}(\tau)) = f_0(\tau) + I^a_\tau$, which proves the result. □

Corollary 5.28 (Geometric PFC decomposition). *Given initial price forward curve $f_0(\tau)$ the forward kernel equals*

$$f_t(\tau) = f_0(\tau)\, X^\tau_t \, \mathbb{E}[I^m_\tau \,|\, \mathcal{F}_t],$$

where I^m_τ is the geometric relative structural component given in Definition 5.25.

Proof. The result can be shown analogously to the proof of Corollary 5.27. □

The interpretation of these decompositions is that today's price forward curve is the expectation of the forward kernel that is being disturbed by the market noise X^τ_t in trading time t and by the relative structural component in delivery time τ. Depending on the choice of the structural component (g, Y) this disturbance can be chosen to be multiplicatively in case of the geometric PFC decomposition or additively in case of the arithmetic PFC decomposition.

5.3.1. Classical day-ahead spot models

Given a day-ahead spot price model $S(\tau)$, we can use it in our framework by choosing the structural component so that $g(Y_\tau) := S(\tau)$. Two examples of spot price models that we explicitly compute in this section are the models by Schwartz and Smith (2000) and Lucia and Schwartz (2002).

For both examples we need the same structural component and, therefore, we assume in this subsection that it is given by $Y_\tau = (y^1_\tau, y^2_\tau) \in \mathbb{R}^2$. The first process is an Ornstein-Uhlenbeck process, i.e.

$$dy^1_\tau = -\lambda\, y^1_\tau \, d\tau + \sigma_1 \, dW^1_\tau, \quad y^1_0 = 0, \tag{5.9}$$

and the second

$$y^2_\tau = \mu_2 \tau + \sigma_2 \rho W^1_\tau + \sigma_2 \sqrt{1-\rho^2} W^2_\tau \tag{5.10}$$

is a (correlated) Brownian motion with drift. The standard one-dimensional Brownian motions W^1 and W^2 are assumed to be independent. The parameters $\lambda > 0$, $\sigma_1, \sigma_2 > 0$, $-1 \leq \rho \leq 1$, and $\mu \in \mathbb{R}$ are assumed to be real-valued.

Example 5.29 (Schwartz and Smith)**.** Schwartz and Smith (2000) define the day-ahead spot price using the function $g(y_1, y_2) = e^{y_1+y_2}$, i.e. $S(\tau) := g(Y_\tau) = \exp(y^1_\tau + y^2_\tau)$. In the HJM framework this transfers to the following forward kernel

$$f_t(\tau) = X^\tau_t \, \mathbb{E}[e^{y^1_\tau + y^2_\tau} \,|\, \mathcal{F}_t],$$

where we do not assume any extra conditions on X^τ apart from Assumption 5.2.

In this setting we can explicitly compute the conditional expectation on $g(Y_\tau)$ and we find

$$\begin{aligned}\ln \mathbb{E}[e^{y^1_\tau + y^2_\tau} \,|\, \mathcal{F}_t] &= e^{-\lambda(\tau-t)} y^1_t + y^2_t + \left(\mu_2 + \tfrac{\sigma_2^2}{2}\right)(\tau - t) \\ &\quad + \frac{\sigma_1^2}{4\lambda}\left(1 - e^{-2\lambda(\tau-t)}\right) + \frac{\rho\,\sigma_1\sigma_2}{\lambda}\left(1 - e^{-\lambda(\tau-t)}\right).\end{aligned}$$

This implies that the structural component (g, Y) satisfies the affine decomposition of Definition 5.15. The coefficient A^τ_t of the decomposition is given by

$$A^\tau_t = \begin{pmatrix} e^{-\lambda(\tau-t)} & 0 \\ 0 & 1 \end{pmatrix} \tag{5.11}$$

and B^τ_t can be chosen to be any vector in $\mathbb{R}^2$ such that

$$\ln g(B^\tau_t) = \left(\mu_2 + \frac{\sigma_2^2}{2}\right)(\tau - t) + \frac{\sigma_1^2}{4\lambda}\left(1 - e^{-2\lambda(\tau-t)}\right) + \frac{\rho\,\sigma_1\sigma_2}{\lambda}\left(1 - e^{-\lambda(\tau-t)}\right)$$

holds.

Since the function g is multiplicative in nature, the geometric PFC decomposition, i.e. Corollary 5.28, is especially suited for this model. The conditional expectation of the multiplicative relative structural component is given by

$$\begin{aligned}\ln \mathbb{E}[I^m_\tau \,|\, \mathcal{F}_t] &= \ln \frac{g(A^\tau_t\, Y_t + B^\tau_t)}{\mathbb{E} g(Y_\tau)} \\ &= e^{-\lambda(\tau-t)} y^1_t + y^2_t - \left(\mu_2 + \tfrac{\sigma_2^2}{2}\right) t \\ &\quad + \frac{\sigma_1^2 e^{-2\lambda\tau}}{4\lambda}\left(1 - e^{2\lambda t}\right) + \frac{\rho\,\sigma_1\sigma_2 e^{-\lambda\tau}}{\lambda}\left(1 - e^{\lambda t}\right),\end{aligned}$$

and the forward kernel decomposes to

$$f_t(\tau) = f_0(\tau)\, X^\tau_t\, e^{e^{-\lambda(\tau-t)} y^1_t + y^2_t - \left(\mu_2 + \frac{\sigma_2^2}{2}\right) t + \frac{\sigma_1^2 e^{-2\lambda\tau}}{4\lambda}\left(1 - e^{2\lambda t}\right) + \frac{\rho\,\sigma_1\sigma_2 e^{-\lambda\tau}}{\lambda}\left(1 - e^{\lambda t}\right)},$$

where any initial price forward curve $f_0(\tau)$ can be used.

Example 5.30 (Lucia and Schwartz)**.** Lucia and Schwartz (2002) discuss four different models. Here, we highlight their arithmetic two-factor model. This model is defined by the function $g(y_1, y_2) = y_1 + y_2$ and the forward kernel equals

$$f_t(\tau) = X^\tau_t \, \mathbb{E}[y^1_\tau + y^2_\tau \,|\, \mathcal{F}_t].$$

Again, apart from Assumption 5.2 the process X^τ can be chosen freely.

The conditional expectation can easily be computed as

$$\mathbb{E}[y_\tau^1 + y_\tau^2 \,|\, \mathcal{F}_t] = e^{-\lambda(\tau-t)}\, y_t^1 + y_t^2 + \mu_2(\tau - t)$$

and the affine decomposition of Definition 5.15 follows immediately with the coefficient A_t^τ given by Equation (5.11) and B_t^τ can be any vector in $\mathbb{R}^2$ such that it satisfies the equation $g(B_t^\tau) = \mu_2(\tau - t)$.

The additive nature of g makes the arithmetic PFC decomposition, i.e. Corollary 5.27, the best suited candidate for this model. It follows that

$$f_t(\tau) = X_t^\tau \,\left(f_0(\tau) + e^{-\lambda(\tau-t)}\, y_t^1 + y_t^2 - \mu_2 t \right)$$

for any initial price forward curve $f_0(\tau)$. We continue the study of this type of forward kernel in Section 5.3.3 with a factor model approach.

5.3.2. Structural model

We will use the HJM framework to model the structural component by a structural model, cf. Section 2.4.2. For the real-valued demand process D we use a Gaussian Ornstein-Uhlenbeck process, i.e.

$$dD_\tau = -\lambda\, D_\tau\, d\tau + \sigma\, dW_\tau, \quad D_0 = 0.$$

We choose the structural component to equal

$$Y_\tau := \begin{pmatrix} \beta(\tau) \\ D_\tau \end{pmatrix},$$

where $\beta(\tau)$ is a real-valued deterministic function. Furthermore, we define the function g as follows

$$g(y_1, y_2) = \gamma + y_1 \,\sinh(\alpha\, y_2) = \gamma + y_1 \frac{e^{\alpha\, y_2} - e^{-\alpha\, y_2}}{2}$$

for $\alpha > 0$ and $\gamma > 0$. Through the first coordinate of Y_τ, i.e. $\beta(\tau)$, we associate y_1 with the evolution of time and y_2 through the second coordinate of Y_τ, namely D_τ, with the demand. Therefore, $g(Y_\tau)$ represents the price of the delivery time τ for a load of D_τ through the *merit order curve*.

Using the auxiliary function $\nu^2(s) := \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda s})$ the affine decomposition of Definition 5.15 can be derived from the following theorem:

Theorem 5.31. *The conditional expectation of the structural component is given by*

$$\mathbb{E}[g(Y_\tau) \,|\, \mathcal{F}_t] = \gamma + \beta(\tau)\, e^{\frac{\alpha^2}{2}\nu^2(\tau-t)} \sinh(\alpha\, e^{-\lambda(\tau-t)}\, D_t)$$

for all $\tau \geq t \geq 0$.

Proof. From Theorem 2.36 we have the following decomposition

$$D_\tau \overset{\mathrm{d}}{=} e^{-\lambda(\tau-t)}\, D_t + \nu(\tau-t)\,\varepsilon, \quad \varepsilon \sim \mathcal{N}(0,1).$$

Now, exploiting the decomposition and plugging it into the definition we get

$$\begin{aligned}
\mathbb{E}[g(Y_\tau)\,|\,\mathcal{F}_t] &= \gamma + \beta(\tau)\,\mathbb{E}\left[\sinh(\alpha\, D_\tau)\,|\,\mathcal{F}_t\right] \\
&= \gamma + \beta(\tau)\sinh(\alpha\, e^{-\lambda(\tau-t)}\, D_t)\,\mathbb{E}\left[e^{\alpha\nu(\tau-t)\varepsilon}\right] \\
&= \gamma + \beta(\tau)\, e^{\frac{\alpha^2}{2}\nu^2(\tau-t)}\sinh(\alpha\, e^{-\lambda(\tau-t)}\, D_t)
\end{aligned}$$

by symmetry of the normal distribution. □

Corollary 5.32 (Affine decomposition)**.** *With coefficients given by*

$$A_t^\tau = \begin{pmatrix} \frac{\beta(\tau)}{\beta(t)} e^{\frac{\alpha^2}{2}\nu^2(\tau-t)} & 0 \\ 0 & \alpha e^{-\lambda(\tau-t)} \end{pmatrix}$$

and $B_t^\tau = 0 \in \mathbb{R}^2$ *the affine decomposition of Definition 5.15 holds.*

By Theorem 5.31 it follows immediately by taking $t = 0$ that the expectation of the structural component $\mathbb{E}g(Y_\tau) = \gamma > 0$ for all $\tau \geq 0$. We can use both the additive and geometric PFC decomposition, i.e. Corollary 5.27 and Corollary 5.28, respectively. In the additive case the forward kernel equals

$$\begin{aligned}
f_t(\tau) &= X_t^\tau\,(f_0(\tau) + g(A_t^\tau\, Y_t) - \gamma) \\
&= X_t^\tau \left(f_0(\tau) + \beta(\tau)\, e^{\frac{\alpha^2}{2}\nu^2(\tau-t)}\sinh(\alpha\, e^{-\lambda(\tau-t)}\, D_t)\right),
\end{aligned}$$

whereas in the multiplicative case it equals

$$\begin{aligned}
f_t(\tau) &= f_0(\tau)\, X_t^\tau\, \frac{g(A_t^\tau\, Y_t)}{\gamma} \\
&= f_0(\tau)\, X_t^\tau \left(1 + \frac{\beta(\tau)}{\gamma}\, e^{\frac{\alpha^2}{2}\nu^2(\tau-t)}\sinh(\alpha\, e^{-\lambda(\tau-t)}\, D_t)\right).
\end{aligned}$$

For both decompositions any initial price forward kernel can be used.

5.3.3. Arithmetic factor model

In this section we use an arithmetic factor model approach for the structural component in the HJM framework, see Section 2.4.1. More precisely, the structural component is given by an n-dimensional Lévy driven Ornstein-Uhlenbeck process

$$dY_t = -\Lambda\, Y_t\, dt + dL_t, \quad Y_0 = y_0,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^{n\times n}$ with $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ and L is an n-dimensional Lévy process.

The function g is given by the summation of all the coefficients, i.e. $g(y) = \sum_{i=1}^n y_i$. If Y_t satisfies Assumption 5.3 we can explicitly compute the conditional expectation:

Theorem 5.33. *The conditional expectation of the structural component is given by*

$$\mathbb{E}[g(Y_\tau) \,|\, \mathcal{F}_t] = g\left(e^{-\Lambda(\tau-t)}\, Y_t + \mathbb{E}\int_t^\tau e^{-\Lambda(\tau-u)}\, dL_u\right)$$

for all $\tau \geq t \geq 0$.

Proof. Using Corollary 2.28 and noting that the first term is $\mathcal{F}_t$-measurable and the second term is independent of $\mathcal{F}_t$ yields the result, as the sum g and $\mathbb{E}$ commute. □

As a direct consequence we obtain:

Corollary 5.34 (Affine decomposition)**.** *With the coefficients given by* $A_t^\tau = e^{-\Lambda(\tau-t)}$ *and* $B_t^\tau = \mathbb{E}\int_t^\tau e^{-\Lambda(\tau-u)} dL_u$ *the affine decomposition of Definition 5.15 holds.*

Due to the additive structure of g the logical PFC decomposition to choose in this setting is the arithmetic one, i.e. Corollary 5.27. From Theorem 5.33 we find that the expectation is given by

$$\mathbb{E}g(Y_\tau) = g\left(e^{-\Lambda\tau}\, y_0 + \mathbb{E}\int_0^\tau e^{-\Lambda(\tau-u)}\, dL_u\right).$$

It follows that the forward kernel is given by

$$f_t(\tau) = X_t^\tau \left(f_0(\tau) + g\left(e^{-\Lambda(\tau-t)}\, Y_t - e^{-\Lambda\tau}\, y_0 - \mathbb{E}\int_0^t e^{-\Lambda(\tau-u)}\, dL_u\right)\right),$$

where $f_0(\tau)$ can be any initial price forward curve.

6. Unifying the theory of storage and the risk premium[1]

Since electricity is such an atypical commodity, as discussed in Chapter 1, the relation between spot and forward contracts is also not obvious. In the literature several theories have been proposed to explain the relation between spot and forward prices. The two main theories are the *theory of storage* and the concept of a *risk premium*, both of which we discuss in Section 6.1. With this unclear relation between spot and forward prices also comes a lack of knowledge on what a or the risk-neutral measure Q should be for electricity markets.

This chapter uses the concept of the actual *intrinsic price* of electricity, which unifies the theory of storage and the concept of a risk premium. Chapter 4 defined a factor model under a pricing measure Q and then brought the dynamics of this model back to the real-world measure P with the help of Remark 4.20. This idea is generalized in this chapter: the intrinsic electricity price is defined under Q and only later defined in terms of the stochastic processes under P. Furthermore, in this chapter we show how this new modelling approach is related to existing methods such as the Heath-Jarrow-Morton framework described in Chapter 5. We investigate the relation between the real-world measure P and the risk-neutral measure Q and connect our theory to the theory of storage and the concept of a risk premium. In the last part of this chapter we apply this theory to real data.

6.1. Literature review on forward pricing

As discussed in Section 1.1 and Section 5.1 electricity contracts for the delivery time τ can be traded at four markets:

- the intraday spot market,
- the day-ahead spot market,
- the futures market,
- and the market for options (on futures).

This market setting is summarised in Figure 6.1. The intraday market is the last market to open and is traded in (approximately) the last 24 hours before delivery. The day-ahead market is an auction, which is held one day before delivery. On the futures market futures on the day-ahead spot price are traded up to six years before delivery and on the options market regular European call and put options on the futures contracts are

[1] Based on joint work with Prof. Dr. R. Korn and Dr. A Wagner. Original title: Unifying the theory of storage and the risk premium by an unobservable intrinsic electricity price.

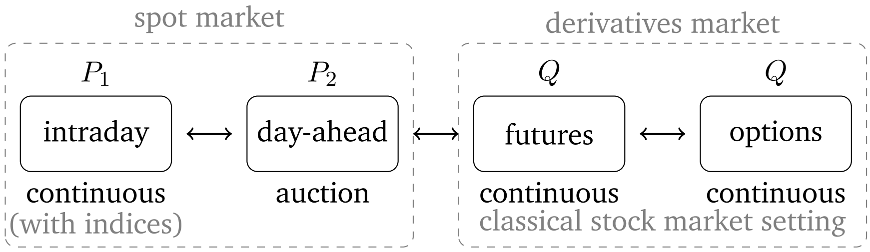


Figure 6.1.: *What is the relation between P_1, P_2, and Q?* This question cannot be answered easily. We will assume $P = P_1 = P_2$, cf. Remark 6.1.

available. For more detailed information about the different electricity markets we refer to Section 1.1.

Remark 6.1 (Relation between P_1 and P_2). In the literature the focus usually lies on either the intraday or the the day-ahead spot market. Therefore, explicit statements about the relation between P_1 and P_2 are scarce. However, since both markets trade directly in short-term power contracts and not derivatives, we assume that the spot market follows a single real-world measure, i.e. we assume $P = P_1 = P_2$.

As in the rest of this thesis we write $S(\tau)$ for the day-ahead spot price delivering 1 MW at τ and denote the price at time t of a forward on $S(\tau)$ by $f_t(\tau)$. The relation between P and Q – or in other words, the relation between the spot and futures markets – is not straightforward, since electricity is not one-dimensional in time as discussed in Remark 1.1. In the literature two main methods for pricing forward contracts can be found: the theory of storage and the concept of a risk premium, which we detail in the next two sections. Furthermore, we introduce a new idea using the notion of an unobservable intrinsic electricity price to model the relation between spot and forward markets.

6.1.1. Theory of storage

The theory of storage – as its name suggests – is based on the fact that one can buy the underlying for a forward now and sell it later (Hull, 2000; Fama and French, 1987). It follows that today's forward price should be related to today's spot price. In the theory of storage the futures price is given by

$$f_t(\tau) = e^{(r+u-y)(\tau-t)} S(t),$$

since the commodity can be bought for a price $S(t)$ at time t and sold at time τ discounted with the corresponding interest rate r and corrected by the storage costs u and the convenience yield y. In this setting it can be derived that under certain conditions there exists an equivalent measure Q such that the discounted spot price $e^{-rt} S(t)$ is a Q-martingale. This is the basis of the electricity price modelling approaches such as Vehvilainen (2002). However, as said in Chapter 1 the electricity contract delivering during t and the contract delivering during τ have different underlying commodities. Therefore, this approach cannot be taken for the modelling of electricity markets.

6.1.2. Risk premium

As discussed by Fama and French (1987) there is another line in pricing commodity forwards, which introduces the concept of the risk premium. The risk premium at time t for delivery time τ is defined as the difference

$$\pi_t(\tau) := f_t(\tau) - \mathbb{E}_P[S(\tau) \mid \mathcal{F}_t]. \tag{6.1}$$

The motivation behind this premium is that the difference between the futures price and the current spot price should equal the risk premium $\pi_t(\tau)$ plus the expected difference of the future and current spot price, i.e.

$$f_t(\tau) - S(t) = \pi_t(\tau) + \mathbb{E}_P[S(\tau) - S(t) \mid \mathcal{F}_t].$$

Rewriting this yields Equation (6.1). A common approach in electricity modelling is to assume[2] that there is an equivalent measure Q such that

$$f_t(\tau) := \mathbb{E}_Q[S(\tau) \mid \mathcal{F}_t],$$

see Benth et al. (2008b), for example. The risk premium then becomes

$$\begin{aligned} \pi_t(\tau) &= \mathbb{E}_Q[S(\tau) \mid \mathcal{F}_t] - \mathbb{E}_P[S(\tau) \mid \mathcal{F}_t] \\ &= \mathbb{E}_P\left[\left(\frac{\nu_\tau}{\nu_t} - 1\right) S(\tau) \mid \mathcal{F}_t\right], \end{aligned} \tag{6.2}$$

where $\nu_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$ is the Radon-Nikodym derivative.

Remark 6.2 (Martingale property). Usually, when we speak of *the* risk-neutral measure we mean the unique equivalent measure Q such that all discounted tradable assets are martingales, i.e.

$$e^{-rt}S(t) \overset{!}{=} \mathbb{E}_Q[e^{-r\tau}S(\tau) \mid \mathcal{F}_t],$$

cf. Remark 2.45. However, since $S(t)$ and $S(\tau)$ have different underlying commodities and $S(\tau)$ is not traded at time t, Benth et al. (2008a) argue that this relation should not hold for *a* risk-neutral measure in the electricity markets. This allows any equivalent measure to be called a pricing or risk-neutral measure.

There exist several studies investigating the risk premium for electricity contracts, e.g. Redl and Bunn (2012); Benth et al. (2008b); Lucia and Torró (2011); Viehmann (2011). However, it is hard to investigate the risk premium in the case of electricity since $S(t)$ and $S(\tau)$ have different underlying commodities. The method conducted by Fama and French (1987, Equations (6) and (7)) on a variety of other storable commodities is, therefore, not applicable in the electricity setting.

[2]Or derive an equivalent measure Q from the spot price model under P.

Redl and Bunn (2012) and Viehmann (2011) concentrate on the risk premium in the German market. They view the so-called *ex post* premium, expressed as

$$\begin{aligned} f_t(\tau) - S(\tau) &= (f_t(\tau) - \mathbb{E}_P[S(\tau) \,|\, \mathcal{F}_t]) - (S(\tau) - \mathbb{E}_P[S(\tau) \,|\, \mathcal{F}_t]) \\ &=: \pi_t(\tau) - \varepsilon_t(\tau), \end{aligned}$$

where $\varepsilon_t(\tau) \in \mathcal{F}_\tau$ is a random variable with P-mean zero. Both studies find that the risk premium is positive in mean. However,their analysis is conducted by comparing futures prices with the realized spot prices and, therefore, the error terms $\varepsilon_t(\tau)$ are assumed to be independent, which they might not be. In this case the result does not tell us anything about the risk premium, but about the average risk premium plus error term.

Benth et al. (2008b) define an arithmetic factor model for the day-ahead spot price and use a measure change from P to Q with the stochastic exponential to price futures contracts. They derive Equation (6.2) in their setting and apply their model to German market. However, they find that the majority of the contracts has a negative risk premium. This contradicts the findings of Redl and Bunn (2012) and Viehmann (2011).

In recent work a zero risk premium, i.e. $P = Q$, has been discussed for certain purposes such as constructing a PFC or forecasting prices (Caldana et al., 2017; Steinert and Ziel, 2018). Other studies do not consider a pricing measure at all and thus compute all derivatives' prices through conditional expectation under the real-world measure (Lyle and Elliott, 2009).

In light of the above discussion we find a modelling approach that just assumes that there is a pricing measure Q, which in turn induces the risk premium, not completely satisfying. Such a method cannot answer all the questions raised by the introduction of Q and it is extremely hard – if not, impossible – to verify the existence of the risk premium through empirical studies in the case of electricity prices, which is indicated by the contradictory evidence of the discussed studies.

6.1.3. An unobservable intrinsic price

In this section we introduce a new perspective: all power contracts deliver electrical energy during a certain delivery period. Surely, when looking at the system as a whole, this energy must have a true price, which is unobservable and intrinsic for that delivery period. What if we model this intrinsic electricity price instead of every market separately?

As a consequence we stop using the modelling approach displayed in Figure 6.1, i.e. a system where we model each market by its own price and try to connect two markets by the conditional expectation under some pricing measure. Instead we assume that there is an unobservable intrinsic electricity price modelled under a fixed risk-neutral Q and assume all tradable electricity contracts to be derivatives of this intrinsic electricity price. Figure 6.2 illustrates this approach.

In this approach we assume that all tradable contracts have dynamics under the real-world measure P. Therefore, it is important to define the change of measure[3] from Q to

[3]Note that this is the other way around compared to classical financial markets.

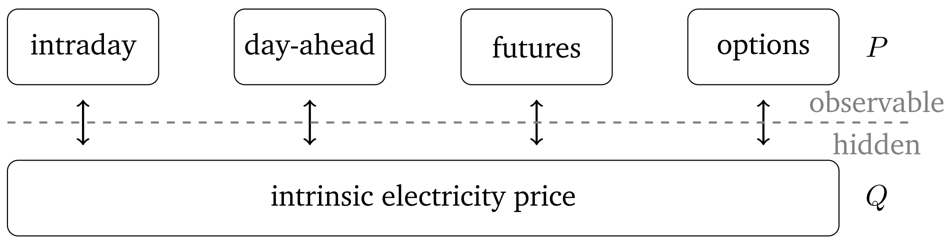


Figure 6.2.: Change of the modelling approach of Figure 6.1 to an approach with an unobservable intrinsic electricity price, which lives under the risk-neutral measure Q. All at the market tradable products have dynamics under the real-world measure P.

P, such that we can use the model we defined under Q. In the next section we pursue this idea further and develop a general theory for the intrinsic electricity price.

6.2. The intrinsic electricity price under Q

Let $(\Omega, \mathcal{A}, Q)$ be a complete probability space. On this probability space we assume $W = \{W_t; t \geq 0\}$ to be an d-dimensional Brownian motion with augmented natural filtration $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$. For technical convenience we assume that $\mathcal{F}_\infty = \mathcal{A}$. We interpret the Brownian motion W as realization of the flow of information in the electricity market. In the following we will always denote delivery time by τ and trading time by t.

Notation 6.3 (Intrinsic electricity price). We introduce the stochastic process denoted by $p = \{p(\tau); \tau \geq 0\}$ and call it the *intrinsic electricity price*.

By Notation 6.3 we mean that $p(\tau)$ is the actual price fixed for electricity being delivered from τ to $\tau + \varepsilon$. Here, $\varepsilon > 0$ is the delivery duration of our standard or smallest contract, which typically is an hour but can also be considered to be any other unit of time such as a quarter hour or a day. It is clear that $p(\tau) \notin \mathcal{F}_\tau$ since at the start of the delivery period nobody knows what the actual price of electricity is. However, after delivery it is known or can be derived and, therefore, $p(\tau) \in \mathcal{F}_{\tau+\varepsilon}$.

We assume our probability space to be equipped with a measure Q and call this *risk-neutral measure*. The following definition validates this naming:

Definition 6.4 (Tradable electricity price). At trading time t the price of electricity delivering at time τ is defined by

$$p_t(\tau) := \mathbb{E}_Q[e^{-r(\tau-t)}p(\tau) \mid \mathcal{F}_t],$$

where r is the risk-free rate. We call $p_t(\tau)$ the *tradable electricity price*.

The tradable electricity price is unobservable and, therefore, not really tradable. However, if electricity with a labeled delivery time τ would be storable like most other commodities, the tradable electricity price would be its spot price. We do not name it the spot price, since this is confusing in terms of the real day-ahead and intraday electricity spot prices. The tradable electricity price is thus an artificial price, on which we can apply the theory of storage. With this construction we artificially fit electricity in the framework of classical financial mathematics.

From the definition of the filtration $\mathcal{F}$ and the tradable electricity price $p_t(\tau)$ it is clear that $p_0(\tau) = e^{-r\tau}\mathbb{E}_Q p(\tau)$ and $p_{\tau+\varepsilon}(\tau) = e^{r\varepsilon}p(\tau)$. As mentioned in Remark 6.2 the discounted tradable assets are Q-martingales, i.e. for $t > u$ we have

$$\mathbb{E}_Q[e^{-rt}p_t(\tau)\,|\,\mathcal{F}_u] = \mathbb{E}_Q[\mathbb{E}_Q[e^{-r\tau}p(\tau)\,|\,\mathcal{F}_t]\,|\,\mathcal{F}_u] = e^{-ru}p_u(\tau).$$

This is the main reason why we define the model under the risk-neutral probability measure Q.

Definition 6.5 (Intraday price)**.** We define $I(\tau) := p_\tau(\tau)$ and call this the *intraday price*[4] for delivery time τ.

In the special, theoretical case that we assume the delivery length $\varepsilon = 0$, the intraday price equals the real electricity price $I(\tau) = p(\tau)$. Throughout the rest of this section we denote the length of one day in the chosen unit of time by δ.

Definition 6.6 (Day-ahead spot price)**.** The *day-ahead spot price* for delivery time τ is defined by $S(\tau) := p_{\tau-\delta}(\tau)$.

Note that although we write $S(\tau)$, it is $\mathcal{F}_{\tau-\delta}$-measurable. Furthermore, since the stochastic process $\{e^{-rt}p_t(\tau); t \geq 0\}$ is a Q-martingale by construction we find that $\mathbb{E}_Q[I(\tau)\,|\,\mathcal{F}_{\tau-\delta}] = e^{r\delta}S(\tau)$. This merely states that under the risk-neutral measure Q the expectation of the intraday price one day in advance, is the day-ahead spot price. Moreover, we can apply the martingale representation theorem to find:

Theorem 6.7. *For each delivery time τ there exists an a.s. unique, predictable, $\mathbb{R}^d$-valued process $\varphi(\tau) = \{\varphi_t(\tau); t \geq 0\}$ such that*

$$p_t(\tau) = e^{rt}p_0(\tau) + e^{-r(\tau-t)}\int_0^t \varphi_s(\tau)' \cdot dW_s$$

for all $t \geq 0$.

Proof. Due to the assumption that $\mathcal{F}_\infty = \mathcal{A}$ this is the exact statement of the martingale representation theorem applied to our setting (Protter, 2005). □

Definition 6.8 (Price generating process)**.** We call the a.s. unique process $\varphi(\tau)$ from Theorem 6.7 the *price generating process*.

[4] Since there is no unique intraday price, we assume that one takes an index. In this chapter we assume the German intraday index ID_3 to be 'the' intraday price, for example.

From Theorem 6.7 we can derive that the dynamics of the tradable electricity price are given by

$$dp_t(\tau) = rp_t(\tau)\,dt + e^{-r(\tau-t)}\varphi_t(\tau)' \cdot dW_t. \tag{6.3}$$

Furthermore, we immediately see that we have a recursive relation between the tradable electricity prices of a fixed delivery time τ: for $t \geq u \geq 0$ we have

$$p_t(\tau) = e^{r(t-u)}p_u(\tau) + e^{-r(\tau-t)}\int_u^t \varphi_s(\tau)' \cdot dW_s.$$

From this relation it immediately follows that:

Corollary 6.9. *An alternative representation of the intrinsic electricity price is*

$$p(\tau) = e^{r(\tau-t)}p_t(\tau) + \int_t^{\tau+\varepsilon} \varphi_s(\tau)' \cdot dW_s$$

for all $\tau + \varepsilon \geq t \geq 0$.

Proof. Follows directly by the $\mathcal{F}_{\tau+\varepsilon}$-measurability of the intrinsic electricity price: we see that $e^{r\varepsilon}p(\tau) = p_{\tau+\varepsilon}(\tau)$ □

As in the theory of storage we can now introduce the forward price of an electricity contract with delivery τ. We assume the storage costs u and convenience yield y to equal zero, since the electricity is not actually storable.

Definition 6.10 (Forward price)**.** The *forward price* is given by

$$f_t(\tau) := e^{r(\tau-t)}p_t(\tau)$$

for all $0 \leq t \leq \tau$.

It is clear that we have $f_t(\tau) = \mathbb{E}_Q[p(\tau)\,|\,\mathcal{F}_t]$ and thus that for fixed τ the forward price process $f(\tau) = \{f_t(\tau); t \geq 0\}$ is a Q-martingale. Furthermore, from Theorem 6.7 it follows that

$$f_t(\tau) = f_0(\tau) + \int_0^t \varphi_s(\tau)' \cdot dW_s$$

for all $t \geq 0$.

Idea 6.11. In light of Theorem 6.7 there are now two equivalent possibilities to assume an explicit model:

- through the intrinsic electricity price $p(\tau)$ and the computation of its conditional expectation,
- or through the initial forward price $f_0(\tau)$ (e.g. the price forward curve, see Remark 5.4) and the price generating process $\varphi(\tau)$.

We will come back to this in Section 6.3.

Remark 6.12 (Heath-Jarrow-Morton framework). Our approach is based on the actual fixed price, which can only be observed after all transactions and valuation have been made. However, as a consequence of Theorem 6.7 we derived the modelling approach of electricity prices through the price generating process φ and the initial forward curve, which usually is called a Heath-Jarrow-Morton (HJM) approach after the famous framework introduced for interest rates by Heath et al. (1992). The HJM approach has been taken by several authors, e.g. Hinz et al. (2005); Kiesel et al. (2009); Latini et al. (2018); Benth et al. (2019) or Chapter 5.

6.2.1. Futures prices

Consider a futures contract with ordered delivery times $\mathcal{T} := \{\tau_1, \tau_2, \dots, \tau_n\}$, i.e. we assume $0 \leq \tau_1 < \tau_2 < \cdots < \tau_n$, and financial fulfillment at final delivery τ_n. From the definition of the forward price it follows that the price at time t of this futures contract is given by

$$F_t(\mathcal{T}) := \tfrac{1}{n} \sum_{i=1}^{n} e^{r(\tau_n - \tau_i)} f_{t \wedge (\tau_i - \delta)}(\tau_i)$$

for all $t \geq 0$. It is clear that $F_0(\mathcal{T}) = \frac{1}{n} \sum_{i=1}^{n} e^{r(\tau_n - \tau_i)} f_0(\tau_i)$ if $\tau_1 \geq \delta$.

Theorem 6.13. *The futures price process $F(\mathcal{T}) = \{F_t(\mathcal{T}); t \geq 0\}$ is a Q-martingale.*

Proof. The statement holds since the futures price is the weighted sum of n stopped Q-martingales. □

From the definition of the tradable electricity price it is immediately clear that for all times $0 \leq t \leq \tau_1 - \delta$ the future's price is given by

$$F_t(\mathcal{T}) = \mathbb{E}_Q \left[p(\mathcal{T}) \,|\, \mathcal{F}_t \right],$$

where $p(\mathcal{T}) := \frac{1}{n} \sum_{i=1}^{n} e^{r(\tau_n - \tau_i)} p(\tau_i)$. Furthermore, with the help of Theorem 6.7 we can equivalently write for all times $0 \leq t \leq \tau_1 - \delta$

$$F_t(\mathcal{T}) = F_0(\mathcal{T}) + \int_0^t \varphi_s(\mathcal{T})' \cdot dW_s,$$

where we define $\varphi_s(\mathcal{T}) := \frac{1}{n} \sum_{i=1}^{n} e^{r(\tau_n - \tau_i)} \varphi_s(\tau_i)$. In Appendix A we discuss several settings in which options on futures can be priced.

6.2.2. Real-world measure P

Since the prices of the traded products move under the real-world measure P, see also Figure 6.2, we need a possibility to change to this measure, cf. Remark 4.20 in the setting of intraday cap and floor futures. As introduced in Section 2.3 we assume that we change from the risk-neutral measure Q to the real-world measure P by its Radon-Nikodym derivative, i.e.

$$\nu_t := \frac{dP}{dQ}\Big|_{\mathcal{F}_t}$$

for all $t \geq 0$. It is common to use the *stochastic exponential* to define the Radon-Nikodym derivative:

Definition 6.14. For an adapted $\mathbb{R}^d$-valued process $\theta = \{\theta_t; t \geq 0\}$ we define the Radon-Nikodym by

$$\nu_t := \exp\left(\int_0^t \theta_s' \cdot dW_s - \frac{1}{2}\int_0^t \theta_s' \cdot \theta_s \, ds\right),$$

i.e. by the stochastic exponential of $\int_0^t \theta_s' \cdot dW_s$.

We assume that the Novikov condition is fulfilled, i.e.

$$\mathbb{E}_Q\left[e^{-\frac{1}{2}\int_0^t \theta_s' \cdot \theta_s \, ds}\right] < \infty$$

for all $t \geq 0$. The Girsanov theorem then tells us that $\tilde{W}_t := W_t - \int_0^t \theta_s \, ds$ is a Brownian motion under P, cf. Korn and Korn (2001). Using this Brownian motion we can rewrite the tradable electricity price as

$$p_t(\tau) = e^{rt} p_0(\tau) + e^{-r(\tau-t)} \int_0^t \varphi_s(\tau)' \cdot \theta_s \, ds + e^{-r(\tau-t)} \int_0^t \varphi_s(\tau)' \cdot d\tilde{W}_s$$

under P.

Since we consider the real-world measure P and the risk-neutral measure Q to be two different measures, it follows that we can also define a risk premium in this setting as defined in Equation (6.1):

Definition 6.15 (Risk premium)**.** We call the $\mathcal{F}_t$-measurable random variable

$$\pi_t(\tau) := f_t(\tau) - \mathbb{E}_P\left[p_\tau(\tau) \,|\, \mathcal{F}_t\right]$$

the *risk premium* for delivery time τ.

Recall that $p_t(\tau)$ is the unobservable tradable electricity price and plays the same role in our theory as the spot price of storable commodities. The risk premium can alternatively be written as

$$\begin{aligned}\pi_t(\tau) &= \mathbb{E}_Q[p(\tau) \,|\, \mathcal{F}_t] - \mathbb{E}_P[\mathbb{E}_Q[p(\tau) \,|\, \mathcal{F}_\tau] \,|\, \mathcal{F}_t] \\ &= \mathbb{E}_Q\left[\left(1 - \frac{\nu_\tau}{\nu_t}\right) p(\tau) \,|\, \mathcal{F}_t\right],\end{aligned}$$

Note that here we change from P to Q instead of the other way around, which is more common in financial mathematics.

Theorem 6.16. *The risk premium is given by*

$$\pi_t(\tau) = \mathbb{E}_Q\left[\left(1 - e^{\int_t^\tau \theta_s' \cdot dW_s - \frac{1}{2}\int_t^\tau \theta_s' \cdot \theta_s \, ds}\right) \int_t^{\tau+\varepsilon} \varphi_s(\tau)' \cdot dW_s\right]$$

for all $t \leq \tau + \varepsilon$.

Proof. We directly compute

$$\pi_t(\tau) = \mathbb{E}_Q\left[\left(1 - e^{\int_t^\tau \theta_s' \cdot dW_s - \frac{1}{2}\int_t^\tau \theta_s' \cdot \theta_s \, ds}\right) p(\tau) \,|\, \mathcal{F}_t\right]$$

where the result follows by noting that in Corollary 6.9 the first term is $\mathcal{F}_t$-measurable and the second term is independent of $\mathcal{F}_t$. □

The interpretation of the above theorem is clear: the risk premium is the expected uncertainty left in the intrinsic price, i.e. the integral over the price generating process from t to $\tau + \varepsilon$, weighted with the change induced through the measure change.

6.3. Explicit model choice and empirical results

In this section we assume an explicit model for the intrinsic electricity price $p(\tau)$ by using a structural model approach. Section 6.3.1 proposes the explicit model and Section 6.3.2 discusses its empirical results.

6.3.1. Structural model

Structural models have their roots in the work of Barlow (2002), see Section 2.4.2. We assume that the ex post[5] *system load* or *system generation*[6] G_τ is defined by

$$G_\tau := g(\tau) + X_\tau,$$

where $g(\tau)$ is a deterministic seasonality function capturing all cyclic and seasonal behaviour and X_τ is a Gaussian Ornstein-Uhlenbeck (OU) process, see Section 2.2. The mean-reverting process X_τ is the solution of the following stochastic differential equation under Q:

$$dX_\tau = -\lambda X_\tau \, d\tau + \sigma \, dW_\tau, \quad X_0 = x_0 \in \mathbb{R}$$

where W is a one-dimensional Brownian motion and $\lambda > 0$, $\sigma > 0$, and μ are real-valued model parameters. Its strong solution is given by

$$X_\tau = e^{-\lambda\tau} x_0 + \int_0^\tau \sigma e^{-\lambda(\tau - s)} \, dW_s.$$

Recall that $\varepsilon > 0$ is the duration of the delivery period, which is fixed. As an auxiliary time variable we define ex post delivery time $\tau_e := \tau + \varepsilon$. We define the intrinsic electricity price as

$$p(\tau) := e^{\alpha_1 (G_{\tau_e} - \beta_1)} - e^{\alpha_2 (G_{\tau_e} - \beta_2)} + \gamma_3(\tau), \tag{6.4}$$

[5]With ex post we mean that the system load $G_{\tau+\varepsilon}$ is the system load for the delivery period from τ to $\tau + \varepsilon$.

[6]In the context of a structural model we use the words (system) load, demand, and generation interchangeably.

where $\alpha_1 > 0$, $\alpha_2 < 0$, β_1, and β_2 are real-valued parameters, and $\gamma_3(\tau)$ is a deterministic function. With the help of the auxiliary process

$$\gamma_i(t;\tau) := \exp\left\{\alpha_i\left(g(\tau_e) + e^{-\lambda(\tau_e - t)}X_t + \frac{\alpha_i\sigma^2}{4\lambda}\left(1 - e^{-2\lambda(\tau_e - t)}\right) - \beta_i\right)\right\}$$

for $i = 1, 2$, we derive an analytical result for the tradable electricity price:

Lemma 6.17 (Tradable electricity price)**.** *The tradable electricity price is given by*

$$p_t(\tau) = e^{-r(\tau - t)}\left(\gamma_1(t;\tau) - \gamma_2(t;\tau) + \gamma_3(\tau)\right)$$

for all $t \leq \tau_e$.

Proof. Using Corollary 2.28, i.e.

$$X_{\tau_e} = e^{-\lambda(\tau_e - t)}X_t + \int_t^{\tau_e} \sigma e^{-\lambda(\tau_e - s)}\, dW_s,$$

we see that

$$\mathbb{E}_Q\left[e^{\alpha_i X_{\tau_e}} \,|\, \mathcal{F}_t\right] = e^{\alpha_i e^{-\lambda(\tau_e - t)}X_t}\mathbb{E}_Q\left[e^{\alpha_i \int_t^{\tau_e} \sigma e^{-\lambda(\tau_e - s)}\, dW_s}\right]$$

for $i = 1, 2$. From this the result follows by explicit computation of the expectation of the lognormal distribution. ☐

It follows directly that

$$p_t(\tau) = e^{rt}p_0(\tau) + e^{-r(\tau - t)}\left\{[\gamma_1(t;\tau) - \gamma_1(0;\tau)] - [\gamma_2(t;\tau) - \gamma_2(0;\tau)]\right\}.$$

and in particular

$$f_t(\tau) = \gamma_1(t;\tau) - \gamma_2(t;\tau) + \gamma_3(\tau)$$

for all $t \leq \tau_e$. From the above equation we can derive the price generating process with the help of Theorem 6.7:

Proposition 6.18 (Price generating process)**.** *The price generating process process is given by*

$$\varphi_t(\tau) = \begin{cases} \sigma e^{-\lambda(\tau_e - t)}\left[\alpha_1\gamma_1(t;\tau) - \alpha_2\gamma_2(t;\tau)\right], & \text{if } t \leq \tau_e, \\ 0, & \text{else}, \end{cases}$$

for all $\tau \geq 0$.

Proof. From Theorem 6.7 we know that we should find $\varphi_t(\tau)$ such that

$$\int_0^t \varphi_s(\tau)\, dW_s = [\gamma_1(t;\tau) - \gamma_1(0;\tau)] - [\gamma_2(t;\tau) - \gamma_2(0;\tau)].$$

We introduce an auxiliary processs

$$dM_t = \sigma e^{-\lambda(\tau_e - t)}\, dW_t, \quad M_0 = 0,$$

and rewrite

$$\gamma_i(t;\tau) = e^{\alpha_i\left(g(\tau_e)+e^{-\lambda\tau_e}x_0+M_t+\frac{\alpha_i\sigma^2}{4\lambda}\left(1-e^{-2\lambda(\tau_e-t)}\right)-\beta_i\right)}$$

We apply Itô's lemma on γ_i and M_t to find that

$$d\gamma_i = \left(\frac{\partial}{\partial t}\gamma_i + \frac{\sigma^2}{2}e^{-2\lambda(\tau_e-t)}\,\frac{\partial^2}{\partial x^2}\gamma_i\right)dt + \sigma e^{-\lambda(\tau_e-t)}\,\frac{\partial}{\partial x}\gamma_i\,dW_t.$$

Recalling that $\alpha_i^{-2}\frac{\partial^2}{\partial x^2}\gamma_i = \alpha_i^{-1}\frac{\partial}{\partial x}\gamma_i = \gamma_i$ and computing the derivative with respect to time

$$\frac{\partial}{\partial t}\gamma_i = -\frac{\alpha_i^2\sigma^2}{2}e^{-2\lambda(\tau_e-t)}\,\gamma_i,$$

then yields

$$d\gamma_i = \alpha_i\sigma e^{-\lambda(\tau_e-t)}\gamma_i\,dW_t,$$

which shows the result. □

6.3.2. Empirical results

In this section we calibrate the model to real data, see Remark 6.19 for a description of the data set. Throughout the rest of this section we assume that we measure time in hours. Therefore, we assume $\varepsilon = 1$ and $\delta = 24$. We will evaluate contracts with delivery times of the form $\tau = k\varepsilon$ for $k \in \mathbb{N}$. For the annual risk-free interest rate we choose $r = 0.001$, i.e. 0.1% which is a reasonable assumption in Germany at the moment.

Remark 6.19 (Data set). We have the following data from the German/Austrian market:

- the hourly system load G_{τ_e} from 1 January 2014 to 15 April 2018,
- the hourly day-ahead spot prices $S^M(\tau)$ and the hourly ID_3 prices $I^M(\tau)$ from 28 June 2015 to 15 April 2018.

We use the whole data set for the estimation and analysis.

Remark 6.20 (Dynamics under P). Assuming that the Girsanov parameter as introduced in Section 6.2.2 is constant $\theta_t \equiv \lambda\theta \in \mathbb{R}$, we find that the Ornstein-Uhlenbeck process X_{τ_e} can be rewritten under P as

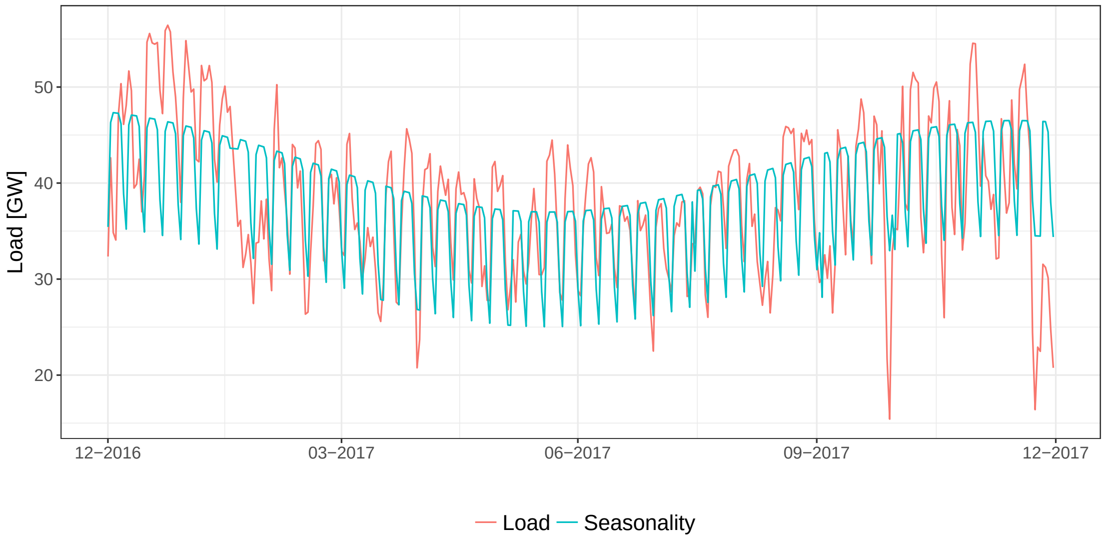
$$X_\tau = e^{-\lambda\tau}x_0 + \left(1 - e^{-\lambda\tau}\right)\sigma\theta + \int_0^\tau \sigma e^{-\lambda(\tau-s)}\,d\tilde{W}_s,$$

where $\tilde{W}_t$ is a P-Brownian motion. It follows that we can split $G_\tau = \tilde{g}(\tau) + \tilde{X}_\tau$ under P, if $\tilde{X}$ is a P-Gaussian Ornstein-Uhlenbeck process defined by

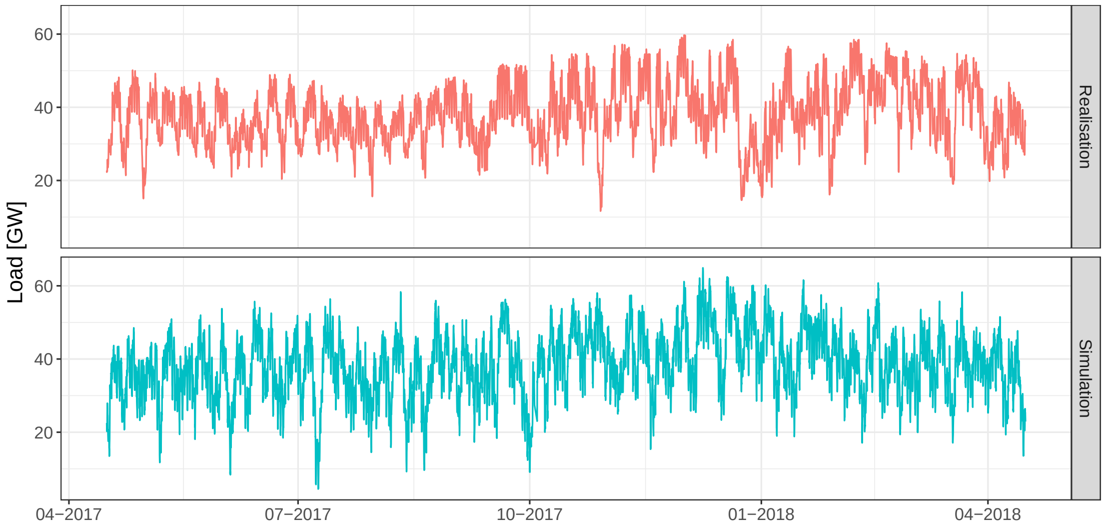
$$d\tilde{X}_\tau = -\lambda\tilde{X}_\tau\,d\tau + \sigma\,d\tilde{W}_\tau, \quad \tilde{X}_0 = x_0$$

and

$$\tilde{g}(\tau) := g(\tau) + \left(1 - e^{-\lambda\tau}\right)\sigma\theta.$$



(a) Actual baseload system load G_τ for the year 2017 together with the estimated seasonality function $\tilde{g}(\tau)$.



(b) Simulation of a path of the system generation G_τ with hourly granularity for the last year of the data set, i.e. from 16 April 2017 to 15 April 2018.

Figure 6.3.: The realised system load compared with the model.

Assuming the mean reversion speed λ is small we can use the first order approximation $1 - e^{-\lambda\tau} \approx \lambda\tau$ to find

$$\tilde{g}(\tau) \approx g(\tau) + \lambda\sigma\theta\tau,$$

which we will use to deseasonalize the system load G_τ under P. Furthermore, in the approximated setting we have the following relation $X_\tau = \tilde{X}_\tau + \lambda\sigma\theta\tau$ between the two Ornstein-Uhlenbeck processes.

As discussed in Remark 6.20 the system load G_τ moves under P. We define the P-load seasonality function

$$\tilde{g}(\tau) := z_0 + z_1\tau + z_2 \sin\left(\tfrac{2\pi}{365\cdot 24}\tau\right) + z_3 \cos\left(\tfrac{2\pi}{365\cdot 24}\tau\right) + \mathrm{DoW}_\tau + \mathrm{HoD}_\tau, \tag{6.5}$$

where DoW_τ and HoD_τ are *dummy variables*[7] for the day of the week[8] and hour of the day. We directly estimate $\tilde{g}$ by linear least squares from the load data. Figure 6.3a shows the estimated seasonality together with the system load for the year 2017. The estimate $\tilde{g}$ can be used to deseasonalize the data $\tilde{X}_\tau = G_\tau - \tilde{g}(\tau)$, after which λ and σ can be estimated by maximum likelihood, cf. Lemma 2.38 on page 21. The estimates of λ and σ are shown in Table 6.4. Figure 6.3b illustrates a sample path of the system load G_τ modelled with the estimated parameters.

In order to proceed with the estimation from market prices we need an estimate of the seasonality function γ_3. We estimate the same type of formula as for $\tilde{g}$, cf. Equation (6.5). We estimated γ_3 with linear least squares to a mixture of the day-ahead and intraday spot prices $\frac{I^M + S^M}{1+e^{-r\delta}}$. This corresponds approximately to the seasonality of the intrinsic price.

We can combine the above Remark 6.20 to calibrate the supply function parameters α_1, α_2, β_1, and β_2 together with θ. We use the `R` function `optim` with method the `BFGS` to minimize the mean squared error of the realized and theoretical day-ahead and intraday prices. The theoretical prices are given by Lemma 6.17. This means that we minimize

$$\min_{\alpha_1,\alpha_2,\beta_1,\beta_2,\theta} \frac{1}{2N}\sqrt{\sum_{k=1}^{N}(I^M(k\varepsilon) - I(k\varepsilon))^2 + \sum_{k=1}^{N}(S^M(k\varepsilon) - S(k\varepsilon))^2}, \tag{6.6}$$

where the superscript M stands for the market price. As initial parameters we used the ones obtained from fitting the intraday prices directly to the formula for the intrinsic electricity price of Equation (6.4). The results of the estimation procedure are given in Table 6.4. Figure 6.5 shows the estimated intrinsic electricity price compared to the realised intraday and day-ahead spot prices.

Analogously to the proof of Lemma 6.17 we can derive an explicit formula for the risk premium:

$$\pi_t(\tau) = [\gamma_1(t;\tau) - \gamma_2(t;\tau)] - [\tilde{\gamma}_1(t;\tau) - \tilde{\gamma}_2(t;\tau)]$$

[7] This means they take a different constant value for a different *day of the week* (DoW) and *hour of the day* (HoD). Mathematically, they are just the sum of weighted indicator functions, cf. Section 3.2.

[8] We define four classes of weekdays: Mondays and Fridays; Tuesdays, Wednesdays, and Thursdays; Saturdays, bridge days (i.e. a days between a holiday and a weekend), and partial holidays (i.e. holidays in some but not all German federal states); Sundays and holidays.

Table 6.4.: Estimated parameters of the structural model.

Parameter	Value
λ	0.0298
σ	1.4988
x_0	-12.5776
α_1	0.1949
α_2	-0.1796
β_1	43.8799
β_2	37.4548
θ	-0.0036

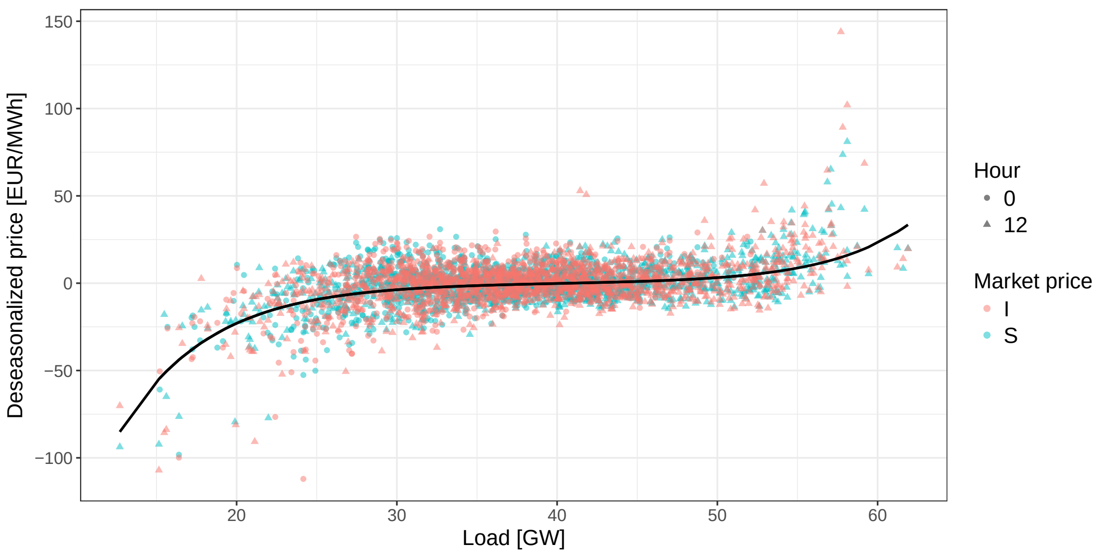
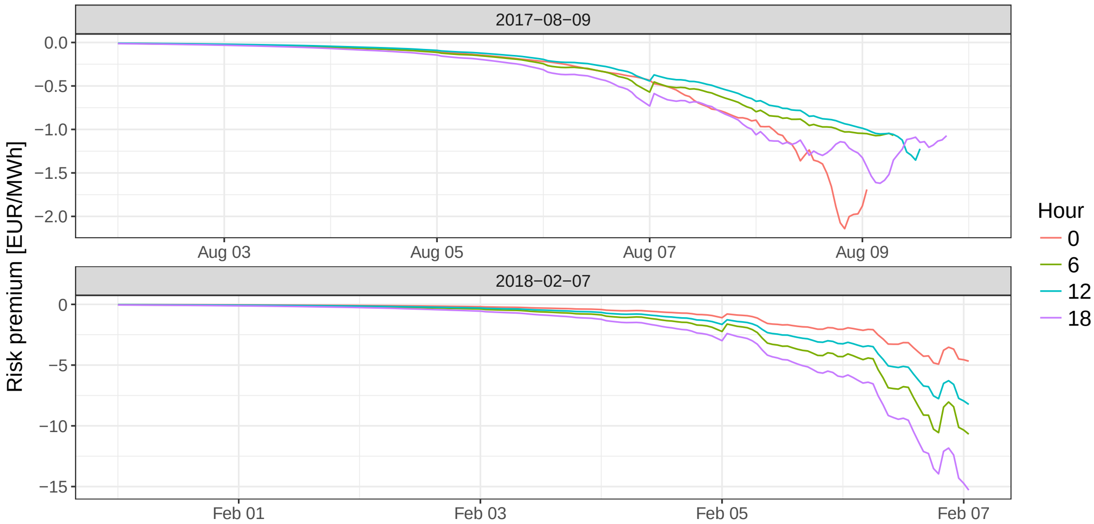
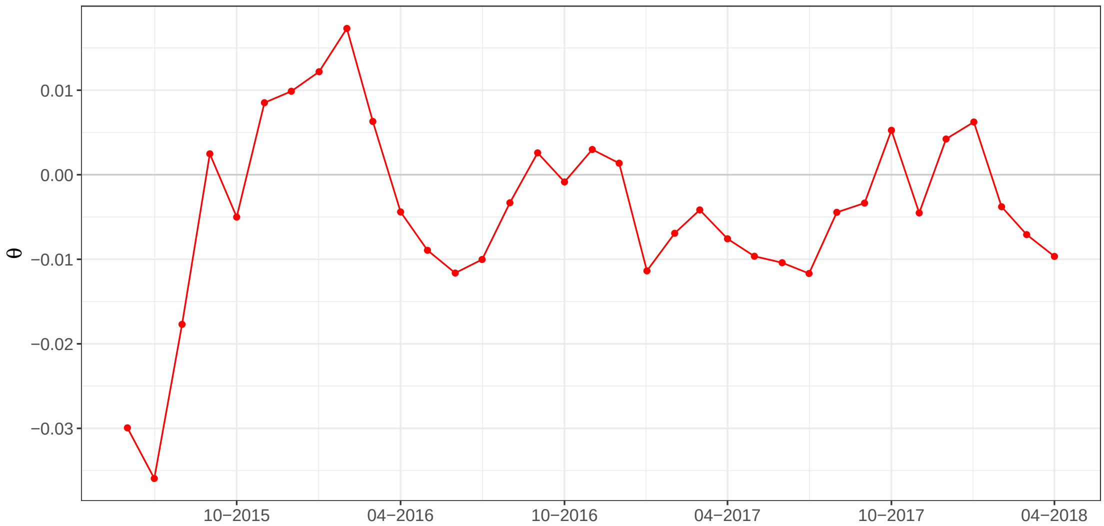


Figure 6.5.: Deseasonalized market intraday $I^M - \gamma_3$ and day-ahead $S^M - \gamma_3$ spot prices for the delivery hours 0–1 and 12–13 together with the intrinsic price curve $p - \gamma_3$ (black).



(a) Risk premia through time for two different delivery dates: the second Wednesday of August 2017 (Summer) and of February 2018 (Winter).



(b) Monthly implied Girsanov parameter θ.

Figure 6.6.: Difference between the real-world measure P and the risk-neutral measure Q.

for all $t \leq \tau_e$, if we define

$$\tilde{\gamma}_i(t;\tau) := e^{\alpha_i \left(g(\tau_e) + e^{-\lambda\varepsilon}\left(1-e^{-\lambda\tau}\right)\sigma\theta + e^{-\lambda(\tau_e - t)}\tilde{X}_t + \frac{\alpha_i \sigma^2}{4\lambda}\left(1-e^{-2\lambda(\tau_e-t)}\right) - \beta_i \right)},$$

where $\tilde{X}$ is given in Remark 6.20. Figure 6.6a illustrates the evolution of the risk premium through time. We see that we find an overall negative risk premium for all the plotted contracts, indicating that the *"producers' desire to hedge their positions outweights that of the consumers"* (Benth et al., 2008b). In that sense our findings support to the results of Benth et al. (2008b), and not those of Redl and Bunn (2012); Viehmann (2011).

In Figure 6.6b the implied Girsanov parameter θ per month is shown. These were computed by solving Equation (6.6) for each month with the parameters α_1, α_2, β_1, and β_2 fixed at the values we estimated before. The first thing we notice is the change in level from August 2015 to September 2015, where the value jumps from around -0.03 to approximately zero. We see that the implied θ changes sign at least twice a year but is negative for most months (in 23 of the 35 months). We see that the positive values all occur during the months September to March. Furthermore, the implied θ shows that the assumption of a constant value might have been an oversimplification, which should be investigated in future work.

Appendix A Options on futures

Keeping in mind that the price generating process φ can also be used as one of the modelling ingredients, we can formulate results for the price of European call and put options in for two special cases of the price generating process, which yield normally or lognormally distributed prices.

Normal distribution For deterministic price generating processes we can find:

Proposition 6.21 (Normal distribution). *If $\varphi(\tau)$ is deterministic process for all τ, then the conditional futures price $F_t(\mathcal{T}) \,|\, \mathcal{F}_u$ is normally distributed under Q with mean*

$$\mu_u := F_u(\mathcal{T}) = F_0(\mathcal{T}) + \int_0^u \varphi_s(\mathcal{T})' \cdot dW_s$$

and variance

$$\sigma^2_{u,t} := \int_u^t \varphi_s(\mathcal{T})' \cdot \varphi_s(\mathcal{T})\, ds$$

for all $u \leq t \leq \tau_1 - \delta$.

Proof. For deterministic φ we know from the proof of Lemma 2.29 that the integral $\int_0^t \varphi_s(\mathcal{T})' \cdot dW_s$ is normally distributed with mean 0 and variance $e^{-2rt}\sigma^2_{0,t}$. This is easily extended to any u. □

With the help of this proposition and the following auxilary variable

$$\Delta_{u,t} := \frac{F_u(\mathcal{T}) - K}{\sigma_{u,t}}$$

we can compute the price of European put and call options on the futures price $F_t(\mathcal{T})$. As a direct consequence of Corollary 2.37 we find:

Lemma 6.22 (Call and put options). *If $\varphi(\tau)$ is deterministic process for all τ, then for all $u \leq t \leq \tau_1 - \delta$ the price at time u of a European option with strike K on the futures contract $F_t(\mathcal{T})$ is given by*

$$C_u(F_t(\mathcal{T}); K) = e^{-r(t-u)} \left(F_u(\mathcal{T}) - K\right) \Phi\left(\Delta_{u,t}\right) + \frac{e^{-r(t-u)}\sigma_{u,t}}{\sqrt{2\pi}} e^{-\frac{1}{2}\Delta_{u,t}^2}$$

for a call and by

$$P_u(F_t(\mathcal{T}); K) = e^{-r(t-u)} \left(K - F_u(\mathcal{T})\right) \Phi\left(-\Delta_{u,t}\right) + \frac{e^{-r(t-u)}\sigma_{u,t}}{\sqrt{2\pi}} e^{-\frac{1}{2}\Delta_{u,t}^2}$$

for a put option. Here Φ is the cumulative distribution function of the standard normal distribution.

Lognormal distribution In contrast to Proposition 6.21 we can derive a lognormal distribution in the following case:

Proposition 6.23 (Lognormal distribution). *If the price generating process is of the form*

$$\varphi_t(\tau) = \sigma_t\, f_t(\tau) \tag{6.7}$$

for an $\mathbb{R}^d$-valued, deterministic, quadratic integrable process σ_t independent of the delivery time τ, then forward price is given by

$$f_t(\tau) = f_0(\tau)\, e^{-\frac{1}{2}\int_0^t \sigma_s' \cdot \sigma_s\, ds + \int_0^t \sigma_s' \cdot dW_s},$$

and, in particular, $f_t(\tau)$ has a lognormal distribution.

Proof. Follows directly from the SDE in Equation (6.3) and Karatzas and Shreve (1998, Chapter 5.6C). □

From the definition of the futures contract it follows immediately that:

Corollary 6.24. *If the price generating process is of the form of Equation* (6.7)*, then the futures price is given by*

$$F_t(\mathcal{T}) = F_0(\mathcal{T})\, e^{-\frac{1}{2}\int_0^t \sigma_s' \cdot \sigma_s\, ds + \int_0^t \sigma_s' \cdot dW_s}$$

for all $t \leq \tau_1 - \delta$ and has a lognormal distribution.

As for any lognormally distributed asset we can apply the Black-76 formula to derive the price of European call and put options (Black, 1976). Therefore, let us define the common auxiliary variables

$$d_{\pm}^{u,t} := \frac{\ln F_u(\mathcal{T}) - \ln K \pm \int_u^t \sigma_s' \cdot \sigma_s \, ds}{\sqrt{\int_u^t \sigma_s' \cdot \sigma_s \, ds}}$$

for any $u \leq t$.

Lemma 6.25 (Call and put options). *If the price generating process is of the form of Equation* (6.7)*, then for all $u \leq t \leq \tau_1 - \delta$ the price at time u of a European option with strike K on the futures contract $F_t(\mathcal{T})$ is given by*

$$C_u(F_t(\mathcal{T}); K) = e^{-r(t-u)} \left[F_u(\mathcal{T}) \, \Phi\left(d_+^{u,t}\right) - K \, \Phi\left(d_-^{u,t}\right) \right]$$

for call and by

$$P_u(F_t(\mathcal{T}); K) = e^{-r(t-u)} \left[K \, \Phi\left(-d_-^{u,t}\right) - F_u(\mathcal{T}) \, \Phi\left(-d_+^{u,t}\right) \right]$$

for put options. Here Φ is the cumulative distribution function of the standard normal distribution.

7. Conclusion and outlook

The basis of this thesis were the four (peer-reviewed) papers presented in Chapters 3 to 6. In this dissertation we applied financial mathematical modelling to the German electricity market. One of the main ideas throughout this thesis was the fact that every electricity contract is characterized by its trading time and its delivery time, giving electricity a certain two-dimensionality of time as presented in Remark 1.1.

Throughout this thesis the two model classes that played a significant role were factor models and structural models, introduced in Section 2.4.1 and Section 2.4.2. All chapters used one or both of these two model classes to give an explicit model choice or to conduct a data analysis. Chapter 3 and Chapter 4 solely used factor models, whereas Chapter 6 only discussed structural models. Chapter 5 applied its theory on both model classes.

Another main line in this thesis was the idea to include all market information in the modelling procedure. This is first introduced in Chapter 4, where the price forward curve (PFC) is used to model the mean of the intraday price index ID_3. In the proposed model the PFC was then constructed from futures prices. In Chapter 5 this idea was developed further by introducing a Heath-Jarrow-Morton framework (HJM) to model intraday, day-ahead, and futures prices. Finally, in chapter 6 we introduced a framework, which is consistent with Chapter 5 and with which we could investigate the relation between the real-world measure P and the risk-neutral measure Q in more detail.

In the following we give a more detailed, separate conclusion and outlook for each of the Chapters 3 to 6. We highlight the contributions of each of these chapters and discuss what possible extensions or future research topics could be.

Chapter 3 We studied the German day-ahead spot prices and surveyed the literature on the classical stylized facts of power prices. We discussed six different seasonality functions, each of which we judged by three requirements introduced in Section 3.2. The best seasonality function for the German day-ahead spot prices was found to be a seasonality function consisting of only dummy variables that were estimated by the median. To model the spot dynamics we used the class of arithmetic factor models. This class is especially suited because it allows negative prices, which we characterized as an important stylized fact of the German spot market. This class of models yields tractable derivative prices, e.g. futures prices.

Usually the mean reversion speed of the factors is estimated by fitting exponentially decaying functions to the empirical autocorrelation. However, we introduced a relation to ARMA processes which also can be used to estimate the mean reversion speed. We applied both these techniques to estimate the mean reversion speed parameters for a one-factor and two-factor model and concluded that the mean reversion estimates of the ARMA method fit the empirical autocorrelation structure better. We calibrated a one-factor and a two-factor model to the deseasonalised day-ahead spot prices and we performed an out-of-sample analysis.

In future work it would be interesting to study seasonal volatility. This would depart from the stationarity assumption, but could leave parts of the year stationary. This might yield even more realistic market prices. In a purely diffusive setting this has already been studied, for example, cf. Latini et al. (2018); Fanelli and Schmeck (2018).

Chapter 4 This chapter conducts a theoretical and empirical study of a pricing method for a relatively new electricity derivative: the German intraday cap and floor futures. We introduced the general theory of pricing these derivative and proposed a model based on the Hull-White model from interest rate modelling. In this modelling approach we derived the intraday cap and floor prices analytically. In a further empirical analysis of intraday cap futures' settlement prices we found that we can find a near perfect fit of today's prices without using intraday cap futures' price data. Moreover, we showed several simulated paths to see the stylized facts of the prices and we discussed the evolution of the empirical distribution in a simulation study. We conclude that the model captures the intraday cap prices well and can easily be extended.

As possible extensions we propose that the stochastic process Z_t, which we used to model the PFC through time, could be modelled by more realistic stochastic processes. In general, one could try to improve the initial PFC, e.g. by use of the techniques of Caldana et al. (2017); Kiesel et al. (2018). Furthermore, the spread process X_t could be improved. To do so, it might also be beneficial to include non-Gaussian factors and then use option pricing techniques, for example, as discussed by Kleinert and Korbel (2016).

Chapter 5 In this chapter we developed a unifying Heath-Jarrow-Morton (HJM) framework that models intraday, day-ahead, and futures prices. This approach is based on two stochastic processes motivated by economic interpretations and separates the stochastic dynamics in trading and delivery time. Within this framework it is possible to price options on futures by means of the Black-Scholes formula. Furthermore this framework allows for the use of classical day-ahead spot price models such as Schwartz and Smith (2000); Lucia and Schwartz (2002) and includes many model classes such as structural models and factor models. To further the development of this framework empirical studies are needed: statistical evaluations but also calibration methods need to be discussed. The theoretical applications of Section 5.3 need to be specified and calibrated to real data from intraday, day-ahead, futures, and option prices.

Chapter 6 This chapter introduced a new theory for modelling electricity prices. We have discussed how this theory unifies the classical theory of storage and the concept of a risk premium through the introduction of an unobservable intrinsic electricity price $p(\tau)$. Since all tradable electricity contracts are derivatives of the actual intrinsic price, their prices should all be derived under the risk-neutral measure Q. Based on this assumption we derived the prices for all common contracts such as the intraday spot price, the day-ahead spot price, and futures prices. Furthermore, we have shown how this framework relates to existing modelling approaches such as the HJM modelling approach, e.g. see Hinz et al. (2005); Kiesel et al. (2009); Latini et al. (2018); Benth et al. (2019) or Chapter 5.

In the last part of Chapter 6 we estimated a structural model from the difference between the intraday and day-ahead spot prices. By construction of this framework we could directly estimate the measure change between real-world measure P and the risk-neutral measure Q. With this result we derived and computed the risk premium for several delivery times. We found that the risk premium is negative, indicating that the *"producers' desire to hedge their positions outweights that of the consumers"* (Benth et al., 2008b).

For further research it is of interest to investigate the many possibilities for modelling the intrinsic electricity price and develop calibration methods that use all market data, i.e. from intraday, day-ahead spot, and futures markets, in the spirit of Caldana et al. (2017). Existing models could be fitted to this framework and the results on the measure change could be investigated. In particular, the Girsanov parameter θ could be made time-dependent. Finally, the framework as it is presented in this chapter is based on a probability space with the natural Brownian filtration. This setting could possibly be extended to a more general setting, in which also jump processes are allowed.

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Scientific career

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2011 – 2014	B.SC. MATHEMATICS Universiteit van Amsterdam, Amsterdam, the Netherlands Cum laude, Cum honore B.SC. PHYSICS AND ASTRONOMY Universiteit van Amsterdam, Amsterdam, the Netherlands Cum laude
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Publications

Chapter 3 W. J. Hinderks and A. Wagner. Factor models in the German electricity market: Stylized facts, seasonality, and calibration. *Energy Economics*, Apr. 2019. doi: 10.1016/j.eneco.2019.03.024.

Chapter 4 W. J. Hinderks and A. Wagner. Pricing German Energiewende products: intraday cap/floor futures. *Energy Economics*, 81:287–296, June 2019. doi: 10.1016/j.eneco.2019.04.005.

Chapter 5 W. J. Hinderks, R. Korn and A. Wagner. A structural Heath-Jarrow-Morton framework for consistent intraday, spot, and futures electricity prices. *Quantitative Finance*, pages 1–11, Dec. 2019. doi: 10.1080/14697688.2019.1687927.

Chapter 6 W. J. Hinderks, R. Korn and A. Wagner. Unifying the theory of storage and the risk premium by an unobservable intrinsic electricity price. *Manuscript submitted for publication*.

Presentations

2019 | SEMINAR WITH PROF. DR. RENÉ AÏD
Kaiserslautern, Germany
Unifying the theory of storage and the risk premium

THE MATHEMATICS OF ENERGY SYSTEMS
Cambridge, United Kingdom
Pricing German Energiewende products: intraday cap/floor futures

2018 | PHD SEMINAR FINANCIAL MATHEMATICS
Kaiserslautern, Germany
Pricing German Energiewende products: intraday cap futures

INTERNATIONAL RUHR ENERGY CONFERENCE 2018
Essen, Germany
Pricing German Energiewende products: intraday cap futures

COMMODITY AND ENERGY MARKETS ASSOCIATION ANNUAL MEETING 2018
Rome, Italy
A structural Heath-Jarrow-Morton framework for consistent intraday, spot, and futures electricity prices

PHD SEMINAR FINANCIAL MATHEMATICS
Kaiserslautern, Germany
Electricity exchanges as financial markets

2017 | INTERNATIONAL RUHR ENERGY CONFERENCE 2017
Essen, Germany
Factor models in the German electricity market: Stylized facts, seasonality, and calibration

2016 | ENERGY FINANCE ITALIA II
Padua, Italy
An extension of arithmetic factor models for spot and futures price modelling in an industrial project (poster presentation)

Wissenschaftlicher Werdegang

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Veröffentlichungen

Kapitel 3 W. J. Hinderks and A. Wagner. Factor models in the German electricity market: Stylized facts, seasonality, and calibration. *Energy Economics*, Apr. 2019. doi: 10.1016/j.eneco.2019.03.024.

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Kapitel 6 W. J. Hinderks, R. Korn and A. Wagner. Unifying the theory of storage and the risk premium by an unobservable intrinsic electricity price. *Zur Veröffentlichung eingereicht*.

Vorträge

2019 | Seminar mit Prof. Dr. René Aïd
Kaiserslautern
Unifying the theory of storage and the risk premium

The mathematics of energy systems
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Pricing German Energiewende products: intraday cap/floor futures

2018 | Doktorandenseminar der AG Finanzmathematik
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Pricing German Energiewende products: intraday cap futures

International Ruhr Energy Conference 2018
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Pricing German Energiewende products: intraday cap futures

Commodity and Energy Markets Association Annual Meeting 2018
Rom, Italien
A structural Heath-Jarrow-Morton framework for consistent intraday, spot, and futures electricity prices

Doktorandenseminar der AG Finanzmathematik
Kaiserslautern
Electricity exchanges as financial markets

2017 | International Ruhr Energy Conference 2017
Essen
Factor models in the German electricity market: Stylized facts, seasonality, and calibration

2016 | Energy Finance Italia II
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An extension of arithmetic factor models for spot and futures price modelling in an industrial project (Posterpräsentation)