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Quasi-isolated blocks and the Malle–Robinson conjecture

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Summary

In the representation theory of finite groups there are many longstanding open problems, one of the oldest of which is Brauer's $k(B)$ -conjecture. Let H be a finite group and let ℓ be a prime dividing $|H|$. If B is an ℓ -block of H with defect group D , then Brauer's $k(B)$ -conjecture states that

$$k(B) \leq |D|,$$

where $k(B)$ denotes the number of irreducible characters in B . Recently, Malle and Robinson proposed a modular version of this conjecture [34, Conjecture 1]. The Malle–Robinson conjecture states that

$$l(B) \leq \ell^{s(D)},$$

where $l(B)$ denotes the number of irreducible Brauer characters in B and $s(D)$ denotes the sectional ℓ -rank of D . Using the solution of the $k(GV)$ -problem [21], both of these conjectures have been proved for the blocks of p -solvable groups. However, apart from a few other relatively specific cases, not much else is known. We are concerned with the blocks of quasi-simple finite groups. Malle and Robinson proved their conjecture for the blocks of many of these groups, and all outstanding cases are blocks of finite groups of Lie type in non-defining characteristic. The overall goal of this thesis is to show that the quasi-simple finite groups of exceptional Lie type do not yield minimal counterexamples to the Malle–Robinson conjecture.

Let G be a simple, simply connected group of exceptional Lie type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. The quasi-simple finite groups of exceptional Lie type that we are interested in are the groups of the form G^F/Z where $Z \subseteq Z(G^F)$ is a central subgroup of G^F , along with the triality groups ${}^3D_4(q)$. Let H be one of these groups. Thanks to groundbreaking results of Bonnafé–Rouquier [4] and Bonnafé–Dat–Rouquier [2], the task of showing that no block of H is a minimal counterexample to the Malle–Robinson conjecture, is reduced to proving the conjecture itself for the so-called *quasi-isolated* blocks of H . The general theory splits into two cases depending on whether ℓ is *good* or *bad* for G . In both cases we have a firm grasp on the right side of the conjectured inequality. The crux of the issue is therefore to determine $l(B)$ (good primes) or at least an upper bound on $l(B)$ that is bounded from above by $\ell^{s(D)}$ (bad primes).

Recall that $\mathcal{E}(G^F, s)$ denotes the Lusztig series associated to a semisimple element $s \in G^{*F}$. Furthermore, recall that if s is an ℓ' -element, then a certain union of Lusztig series, denoted by $\mathcal{E}_\ell(G^F, s)$, is a union of ℓ -blocks of G^F (see [7]).

First, suppose that ℓ is a good prime for G . Then we have the following important result of Geck.

Theorem (Geck, [18, Theorem A]). *Suppose that ℓ is good for G . Let $s \in G^{*F}$ be a semisimple ℓ' -element. Then $\mathcal{E}(G^F, s)$ is a basic set for $\mathcal{E}_\ell(G^F, s)$*

It follows from this result that $l(B) = |\text{Irr}(B) \cap \mathcal{E}(G^F, s)|$ for every block B contained in $\mathcal{E}_\ell(G^F, s)$. To determine this cardinality, we use so-called e -cuspidal pairs. If B is an

ℓ -block parametrized by (the G^F -conjugacy class of) the e -cuspidal pair (L, λ) , we write $B = b_{G^F}(L, \lambda)$. There is a natural relation " \leq_e " on the e -cuspidal pairs. The following theorem builds on results of Cabanes–Enguehard (see [10], [9] and [8]), Enguehard (see [16]) and Malle–Kessar (see [29]), and allows us to determine $|\text{Irr}(B) \cap \mathcal{E}(G^F, s)|$. Let

$$e_\ell(q) := \text{multiplicative order of } q \text{ modulo } \begin{cases} \ell & \text{if } \ell > 2, \\ 4 & \text{if } \ell = 2. \end{cases}$$

Theorem A. *Suppose that G is a simple, simply connected group of exceptional type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or that G is simple of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Let ℓ be a prime not dividing q . Let $s \in G^{*F}$ be a quasi-isolated semisimple ℓ' -element. Let $e = e_\ell(q)$. Then*

$$\mathcal{E}(G^F, s) = \bigcup_{(L, \lambda)/G^F} \{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \leq_e (G, \chi)\},$$

where (L, λ) runs over the G^F -conjugacy classes of e -cuspidal pairs of G with $s \in L^{*F}$ and $\lambda \in \mathcal{E}(L^F, s)$. In particular, if $B = b_{G^F}(L, \lambda)$, then $\text{Irr}(B) \cap \mathcal{E}(G^F, s) = \{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \leq_e (G, \chi)\}$.

It follows that for every quasi-isolated ℓ -block $B = b_{G^F}(L, \lambda)$,

$$l(B) = |\{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \leq_e (G, \chi)\}|.$$

The cardinality $|\{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \leq_e (G, \chi)\}|$ can then be determined with the help of Chevie [36]. After that, the proof of the Malle–Robinson conjecture for the quasi-isolated blocks is straightforward.

Now, suppose that ℓ is a bad prime for G . In this case, there is no general result equivalent to Geck’s Theorem on basic sets. However, apart from four stubborn exceptions, we prove a result similar to Geck’s for the quasi-isolated blocks of quasi-simple groups of exceptional Lie type for bad primes. This gives us an upper bound $c(B)$ for $l(B)$ for every quasi-isolated block B of H . In fact we show the following even stronger inequality:

$$l(B) \leq c(B) \leq \ell^{s(D)},$$

where D is a defect group of B .

Theorem B. *Suppose that G is a simple, simply connected group of exceptional type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$, or that G is simple of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Let ℓ be a prime not dividing q and let B be a non-unipotent, quasi-isolated ℓ -block of G^F . Then the Malle–Robinson conjecture holds for B unless possibly if B is of one of the following types.*

- (i) $G^F = E_6(q)$ or ${}^2E_6(q)$ and B is the 3-block numbered 13 in Table 13, or
- (ii) $G^F = E_7(q)$ and B is either the 2-block numbered 1 or the 2-block numbered 2 in Table 15.

Finally, we come back to the minimal counterexamples to the Malle–Robinson conjecture. Theorem C shows that most blocks of the quasi-simple finite groups of exceptional Lie type are not minimal counterexamples, and if there are any at all, they only occur for the primes 2 and 3 in very specific situations.

Theorem C. *Let H be a finite quasi-simple group of exceptional Lie type. Let ℓ be a prime and let B be an ℓ -block of H . Then B is not a minimal counterexample to the Malle–Robinson conjecture for $\ell \geq 5$. More precisely, (H, B) is not a minimal counterexample, unless possibly if (H, B) is of one of the following types.*

- (i) $H = E_6(q)/Z(E_6(q))$ or ${}^2E_6(q)/Z({}^2E_6(q))$ and B is the 3-block dominated by the 3-block numbered 13 in Table 13.
- (ii) $H = E_7(q)/Z(E_7(q))$ and B is the 2-block dominated by either the 2-block numbered 1 or the 2-block numbered 2 in Table 15.
- (iii) $\ell = 3$ and $H = E_6(q)$ or ${}^2E_6(q)$ (respectively $H = E_6(q)/Z(E_6(q))$ or ${}^2E_6(q)/Z({}^2E_6(q))$) and B is a non-principal unipotent 3-block of H (respectively dominated by such a 3-block).
- (iv) $\ell = 2$, $H = E_7(q)$ (respectively $H = E_7(q)/Z(E_7(q))$) and B is a non-principal unipotent 2-block of H (respectively dominated by such a 2-block).
- (v) $\ell = 2$, $H = E_8(q)$ and B is a non-principal unipotent 2-block.

We begin with an introduction to the various objects and statements needed in this thesis in Section 1. In Section 2 we prove Theorem A. The proof of this result relies heavily on information on the relevant e -cuspidal pairs which were determined with Chevie [36] and can be found in Section 2. In Section 3 we study the quasi-isolated ℓ -blocks when ℓ is a bad prime. Considering that the assertion of [18, Theorem A] does not hold in this case, Section 3 is mainly concerned with proving analogues of this result. The final result of that section is Theorem B. In Section 4 we apply the methods developed in Section 3 to unipotent blocks. This leads to an extension of a result of Malle–Robinson (see [34, Proposition 6.10]). In Section 5 we combine all of these results to then prove Theorem C.

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1 Prerequisites

1.1 Preliminaries on the representation theory of finite groups

Let K be a field and let G be a finite group. A K -**representation** ρ of G is a group homomorphism

$$\rho : G \rightarrow \mathrm{GL}(V),$$

where V is a finite dimensional K -vector space. The dimension of V is called the **degree** of the representation ρ . Fixing an isomorphism $V \cong K^n$, where $n = \dim(V)$, we get

$$\rho : G \rightarrow \mathrm{GL}_n(K).$$

There are several equivalent ways to define representations of groups, which themselves introduce different tools. Set $KG := \{\sum_{g \in G} a_g g \mid a_g \in K \text{ for all } g \in G\}$. We define a product on KG by linearly extending the product of G . With this KG becomes a K -algebra, called the **group algebra** of G . Let ρ be a K -representation as above. Then ρ can be linearly extended to an algebra homomorphism

$$\tilde{\rho} : KG \rightarrow \mathrm{End}(V).$$

This yields a KG -module structure on V by setting $av := \tilde{\rho}(a)(v)$ for every $a \in KG$ and every $v \in V$. Conversely, every KG -module V yields a K -representation of G , because left multiplication by any element g of G , $v \mapsto gv$ gives rise to automorphisms of V .

Two K -representations ρ_1 and ρ_2 are called **similar** if there exists an invertible matrix $M \in \mathrm{GL}_n(K)$ such that $\rho_1(g) = M\rho_2(g)M^{-1}$ for every $g \in G$. In this case we write $\rho_1 \sim \rho_2$. Now, we have a bijection

$$\{K\text{-representations}\} / \sim \longleftrightarrow \{KG\text{-modules}\} / \cong .$$

The most natural way to define the important objects in representation theory is via the module theoretic approach. A nice introduction to the module theoretic approach can be found in [1]. We just need the very basics for our purposes. Let R be a K -algebra and let M be an R -module. We call $M \neq 0$ **simple** if M has no non-trivial proper R -submodules and we call it **semisimple** if it is a direct sum of simple modules. A K -representation ρ is said to be **irreducible** if the corresponding KG -module is simple. If ρ is not irreducible, we say it is **reducible**.

The representation theory of finite groups highly depends on the base field K , as shown by the following theorem.

Theorem 1.1 (Maschke, [1, I 3, Theorem 3]). *Let K be an algebraically closed field of characteristic p and G be a finite group. Then every KG -module is semisimple if and only if $p \nmid |G|$.*

Let A be a finite-dimensional K -algebra and define $\mathrm{rad}(A) = \{a \in A \mid aM = 0 \text{ for every simple } A\text{-module } M\}$; the **Jacobson radical** of A .

Definition 1.2. Let A be a finite dimensional K -algebra. We say that the algebra A is **semisimple** if $\mathrm{rad}(A) = 0$.

The following theorem illustrates why these algebras are of interest.

Theorem 1.3 ([38, (1.13) Theorem]). *Let A be a finite-dimensional K -algebra. Then A is semisimple if and only if every A -module is semisimple.*

Hence KG is a semisimple algebra if and only if $|G|$ is not divisible by $\text{char}(K)$. It follows that every \mathbb{C} -representation ρ of G is similar to a block-diagonal representation

$$\begin{pmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_s \end{pmatrix},$$

where the ρ_i 's are irreducible \mathbb{C} -representations of G . The situation is different if K is an algebraically closed field whose characteristic divides the group order. In this case the theory is much more intricate. However, because KG is a finite dimensional K -algebra (in particular, KG is artinian and noetherian), every KG -module V has a finite **composition series**

$$V = V_0 \geq V_1 \geq \cdots \geq V_s = 0$$

such that V_i/V_{i+1} is a simple KG -module. Let ρ be the representation corresponding to V . By choosing an appropriate basis of V , we see that ρ is similar to a block upper-triangular representation

$$\begin{pmatrix} \rho_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_s \end{pmatrix},$$

where the ρ_i 's correspond to the simple KG -modules V_i/V_{i+1} . By a Jordan-Hölder-type theorem for modules we know that these simple representations ρ_i are uniquely determined by ρ up to isomorphism and we call them the **irreducible constituents** of ρ .

Let $\rho : G \rightarrow \text{GL}_n(K)$ be a K -representation. The corresponding trace function

$$\chi : G \rightarrow K, \quad g \mapsto \text{tr}(\rho(g))$$

is called the **K -character** of G afforded by ρ . The number n is called the degree of χ . By properties of the trace function, it is clear that characters are class functions (i.e. constant on conjugacy classes) and that similar representations afford the same character. We say a K -character χ is **irreducible** if the representation affording it is irreducible. Due to the fact that every K -representation is similar to one in block upper-triangular form where the representations on the diagonal correspond to simple KG -modules, every K -character is a sum of irreducible K -characters. It follows that a non-zero character is irreducible if and only if it can not be written as the sum of two non-zero characters.

1.2 Ordinary characters

Let $K = \mathbb{C}$ throughout this section. The \mathbb{C} -characters of G are normally referred to as **ordinary** characters or just characters. We let $\text{Irr}(G)$ denote the set of irreducible characters of G . The character 1_G afforded by the trivial \mathbb{C} -representation ($G \rightarrow \mathbb{C}^\times, g \mapsto 1$) is called the **principal character** of G .

Theorem 1.4 ([27, (2.7) Corollary]). *Let G be a finite group. The number of irreducible characters of G is equal to the number of conjugacy classes of G .*

Clearly a lot of structural information is lost when we go from representations to their characters. But in the case where $K = \mathbb{C}$, Maschke's theorem suggests that most of the relevant information can actually be recovered from the characters.

Theorem 1.5 ([27, (2.9) Corollary]). *Let G be a finite group. Two representations are similar if and only if they afford the same character.*

Let $\text{cl}(G)$ denote the \mathbb{C} -vector space of class functions of G .

Theorem 1.6 ([27, (2.8) Theorem]). *Let G be a finite group. The set $\text{Irr}(G)$ is a basis of $\text{cl}(G)$.*

Given a character ψ , we can write

$$\psi = \sum_{\chi \in \text{Irr}(G)} a_{\chi} \chi,$$

for some non-negative integers a_{χ} . If $a_{\chi} \neq 0$ we call χ a **constituent** of ψ and a_{χ} its **multiplicity**. Let χ, θ be two characters of G and set

$$\langle \chi, \theta \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\theta(g)},$$

where the bar denotes complex conjugation. If $\chi, \theta \in \text{Irr}(G)$ then

$$\langle \chi, \theta \rangle = \delta_{\chi\theta},$$

by the First Orthogonality Relation (see [27, (2.14) Corollary]). Hence the set $\text{Irr}(G)$ is an orthonormal basis of $\text{cl}(G)$ with respect to the inner product $\langle \cdot, \cdot \rangle$.

A lot of information on the characters of G is given by the characters of subgroups of G . If φ is a class function of a subgroup $H \subseteq G$, then the restriction φ_H of φ to H is a class function of H . In particular, if φ is a character of G , then φ_H is a character of H . Conversely, given a class function θ of H we define the **induced class function** θ^G by

$$\theta^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \theta(xgx^{-1}),$$

for $g \in G$, which, as the name suggests, is itself a class function. In fact, if θ is a character, then θ^G is also a character (see [27, (5.3) Corollary]). The two concepts – restriction and induction – are closely related.

Theorem 1.7 ([27, (5.2) Lemma], Frobenius reciprocity for finite groups). *Let G be a finite group. If φ is a class function of G and θ is a class function of a subgroup $H \subseteq G$, then $\langle \varphi, \theta^G \rangle = \langle \varphi_H, \theta \rangle$.*

Restriction and induction can also be defined at the level of KG - and KH -modules (actually even as functors on the corresponding module categories) and this reciprocity phenomenon carries over to these more general settings.

1.3 Brauer characters

Let \mathbf{R} be the ring of algebraic integers in \mathbb{C} . For the rest of this section we fix a prime ℓ and choose a maximal ideal \mathfrak{m} of \mathbf{R} containing $\ell\mathbf{R}$. The quotient $F = \mathbf{R}/\mathfrak{m}$ is a field of characteristic ℓ . Let

$$* : \mathbf{R} \rightarrow F$$

denote the natural projection. Furthermore, let $S = \mathbf{R}_{\mathfrak{m}}$ be the localization of \mathbf{R} at \mathfrak{m} . We can extend $*$ to S by setting

$$\left(\frac{r}{s}\right)^* = r^*(s^*)^{-1}$$

for every $\frac{r}{s} \in S$. Now, set $\mathbf{U} = \{\xi \in \mathbb{C} \mid \exists m \in \mathbb{N} \text{ such that } \xi^m = 1 \text{ and } \ell \nmid m\}$; the multiplicative group of ℓ' -roots of unity.

Lemma 1.8 ([38, (2.1) Lemma]). *The restriction of $*$ to \mathbf{U} defines an isomorphism $\mathbf{U} \rightarrow F^\times$ of multiplicative groups. Moreover, F is the algebraic closure of its prime field $\mathbb{Z}^* \cong \mathbb{F}_\ell$.*

We say that an element $g \in G$ is ℓ -**regular** if its order is not divisible by ℓ . We denote by G° the set of ℓ -regular elements of G . Let $\rho : G \rightarrow \mathrm{GL}_n(F)$ be an F -representation of G . By Lemma 1.8, we have that for every $g \in G^\circ$ the eigenvalues of $\rho(g)$ are of the form ξ_1^*, \dots, ξ_n^* for some uniquely determined elements $\xi_1, \dots, \xi_n \in \mathbf{U}$. Then the **Brauer character** $\varphi : G^\circ \rightarrow \mathbb{C}$ of G afforded by ρ is defined by

$$\varphi(g) = \xi_1 + \dots + \xi_n.$$

We call n the **degree** of φ . We say φ is **irreducible** if ρ is irreducible and denote the set of irreducible Brauer characters of G by $\mathrm{IBr}(G)$. Similar to the ordinary case, we define the **principal** Brauer character 1_{G° to be the one afforded by the trivial F -representation. The space of complex class functions defined on G° is denoted by $\mathrm{cf}(G^\circ)$. We observe that every Brauer character φ of G is in $\mathrm{cl}(G^\circ)$: let ρ be the F -representation affording the Brauer character φ . Since $\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1}$ and similar matrices have the same eigenvalues, we have $\varphi(hgh^{-1}) = \varphi(g)$. By the block upper triangular form of F -representations, we also have the following result.

Theorem 1.9 ([38, (2.3) Theorem]). *A class function $\varphi \in \mathrm{cf}(G^\circ)$ is a Brauer character of G if and only if φ is a non-negative integral linear combination of elements of $\mathrm{IBr}(G)$.*

As one would expect, in the case that φ is a Brauer character this non-negative linear combination is uniquely determined by φ .

Theorem 1.10 ([38, (2.10) Corollary]). *The set $\mathrm{IBr}(G)$ is a basis of $\mathrm{cf}(G^\circ)$.*

If $\psi = \sum_{\varphi \in \mathrm{IBr}(G)} a_\varphi \varphi$ is a Brauer character, we call a_φ the **multiplicity** of φ in ψ .

Given an ordinary character χ of G , we denote its restriction to G° by χ° . The following theorem describes the relationship between ordinary and Brauer characters.

Theorem 1.11 ([38, (2.9) Corollary]). *If χ is an ordinary character of G , then χ° is a Brauer character of G .*

For an irreducible character $\chi \in \text{Irr}(G)$ we therefore have

$$\chi^\circ = \sum_{\varphi \in \text{IBr}(G)} d_{\chi\varphi} \varphi$$

for uniquely determined non-negative integers $d_{\chi\varphi}$. The matrix

$$D = (d_{\chi\varphi})_{\chi \in \text{Irr}(G), \varphi \in \text{IBr}(G)}$$

is called the **decomposition matrix** of G and the entries are called the **decomposition numbers** of G .

Proposition 1.12 ([38, (2.11) Corollary]). *The decomposition matrix D has rank $|\text{IBr}(G)|$.*

We call the elements of

$$\mathbb{Z} \text{Irr}(G) = \left\{ \sum_{\chi \in \text{Irr}(G)} a_\chi \chi \mid a_\chi \in \mathbb{Z} \right\}$$

generalized characters of G . Similarly, we define

$$\mathbb{Z} \text{IBr}(G) = \left\{ \sum_{\varphi \in \text{IBr}(G)} a_\varphi \varphi \mid a_\varphi \in \mathbb{Z} \right\}.$$

Theorem 1.13 ([38, (2.16) Corollary]). *Let G be a finite group. The set $\{\chi^\circ \mid \chi \in \text{Irr}(G)\}$ generates $\mathbb{Z} \text{IBr}(G)$ as a \mathbb{Z} -module.*

This generating set is in general not a \mathbb{Z} -basis. A refined version of Theorem 1.13 that we introduce in the next section plays a major role in this work.

1.4 Blocks of finite groups

Let K be any field and let A be a K -algebra. We say that A is **indecomposable** if it is nonzero and cannot be written as a direct sum of proper K -subalgebras. Similarly, we say an A -module is **indecomposable** if it is nonzero and cannot be written as a direct sum of proper A -submodules. We have the following important decomposition

Theorem 1.14. [1, IV 13, Theorem 1] *Let A be a finite dimensional K -algebra. Then A has a unique decomposition into a direct sum of subalgebras each of which is indecomposable.*

We want to decompose the set $\text{Irr}(G) \cup \text{IBr}(G)$ further into subsets which give us more information. In order to do this, we switch to a more general framework.

We call K a **splitting field** for G if $V \otimes_K L$ is a simple LG -module for every simple KG -module V and every field extension $K \subseteq L$. Let \mathcal{O} be a complete rank one valuation ring with field of fraction K of characteristic 0 and unique maximal ideal \mathfrak{m} . Then

(K, \mathcal{O}, k) is called a **splitting ℓ -modular system** for G if $k \cong \mathcal{O}/\mathfrak{m}$ has characteristic ℓ and K and k are splitting fields for G . From now on (K, \mathcal{O}, k) will be a splitting ℓ -modular system for G that is sufficiently large, that is, (K, \mathcal{O}, k) is also a splitting modular system for all subgroups of G .

Let

$$kG = B_1 + \dots + B_r$$

be the unique decomposition of kG into indecomposable subalgebras. Then we call the B_i 's the **ℓ -blocks** of G . If the prime ℓ is fixed, we will omit the ℓ and just call them **blocks**. Given a kG -module M , we say that M **lies in** the block B_i if $B_i M = M$ and $B_j M = 0$ for all $j \neq i$. If M is simple then for every i we either have $B_i M = M$ or $B_i M = 0$ and since M is nonzero, there is at least one i such that $B_i M = M$. Let B_j be another block such that $B_j M = M$. Then $B_i B_j M = B_i M = M$, but $B_i B_j \subseteq B_i \cap B_j = 0$; a contradiction. It follows that every simple kG -module lies in a unique block. With this we are able to partition the set $\text{IBr}(G)$. We say a Brauer character φ **lies in** a block B of G if any simple kG -module affording φ lies in that block. We denote the set of Brauer characters lying in a block B by $\text{IBr}(B)$.

Corresponding to the block decomposition of kG , we have a central primitive idempotent decomposition of the identity of KG (see [37, Section 8.2])

$$1 = e_1 + \dots + e_r$$

such that $B_i = e_i kG$ for $1 \leq i \leq r$. Now, a kG -module M lies in B_i if and only if $e_i M = M$ and $e_j M = 0$ for $j \neq i$. For a block B , we call the corresponding central primitive idempotent, denoted by e_B , the **block idempotent** of B . We now want to partition the set $\text{Irr}(G)$ into blocks. We can lift the central primitive idempotent decomposition of the identity of kG above to a central primitive idempotent decomposition of 1 in $\mathcal{O}G$ (see e.g [37, Theorem 4.10])

$$1 = c_1 + \dots + c_r$$

such that $c_i^* = e_i$. Let V be a simple KG -module affording an irreducible character χ . We say that χ **lies in** the block B_i if $c_i V = V$ and $c_j V = 0$ for $j \neq i$. We denote the set of irreducible characters lying in a block B by $\text{Irr}(B)$. It follows that blocks can be seen as subsets of $\text{Irr}(G) \cup \text{IBr}(G)$. The number of irreducible Brauer (respectively ordinary) characters lying in a block B will be denoted by $l(B)$ (respectively $k(B)$) and either computing or bounding $l(B)$ is the main focus of this thesis.

Definition 1.15. If U is a union of blocks (regarded as subsets of $\text{Irr}(G) \cup \text{IBr}(G)$) we set $\text{Irr}(U) = \bigcup_{B \subseteq U} \text{Irr}(B)$ and $\text{IBr}(U) = \bigcup_{B \subseteq U} \text{IBr}(B)$. Moreover, we define

$$\begin{aligned} \mathbb{Z} \text{IBr}(U) &= \left\{ \sum_{\varphi \in \text{IBr}(U)} a_\varphi \varphi \mid a_\varphi \in \mathbb{Z} \right\} \text{ and} \\ \mathbb{Q} \text{IBr}(U) &= \left\{ \sum_{\varphi \in \text{IBr}(U)} a_\varphi \varphi \mid a_\varphi \in \mathbb{Q} \right\}. \end{aligned}$$

A subset $A \subseteq \mathbb{Z}\text{IBr}(U)$ is called a **generating set** for U if it generates $\mathbb{Z}\text{IBr}(U)$ as a \mathbb{Z} -module and it is called a **basic set** for U if it is a basis of $\mathbb{Z}\text{IBr}(U)$ as a \mathbb{Z} -module. A subset $C \subseteq \text{Irr}(G)$ is called an **ordinary generating** (respectively **basic**) set for U if the set $C^\circ = \{\psi^\circ \mid \psi \in C\}$ consisting of the restrictions of the irreducible characters in C to G° is a generating (respectively basic) set for U .

Note that every basic set of a block B has cardinality $l(B)$. The cardinality of an arbitrary generating set is greater than or equal to $l(B)$. The following is a generalization of Theorem 1.13 to blocks.

Theorem 1.16 ([38, (3.16) Lemma]). *Let B be a block of a finite group G . Then $\{\chi^\circ \mid \chi \in \text{Irr}(B)\}$ is an ordinary generating set for B .*

A large share of this thesis is concerned with finding suitable, small generating sets which yield useful upper bounds on $l(B)$. This is accomplished by finding small (enough) subsets of the generating set of Theorem 1.16.

Let M be a kG -module and $H \subseteq G$ a subgroup of G . We say that M is **relatively H -free** if there is a kH -submodule X of M such that any kH -homomorphism of X to any kG -module V , extends uniquely to a kG -homomorphism of M to V . This is a generalization of the notion of a free module; M is a free module if M is relatively H -free for $H = 1$. With this we can also generalize the notion of projective modules. We say M is **relatively H -projective** if M is a direct summand of a relatively H -free module.

Theorem 1.17 ([1, III 9, Theorem 4]). *Let M be an indecomposable kG -module. Then there is an ℓ -subgroup Q of G , unique up to G -conjugacy, such that M is relatively H -projective, for a subgroup H of G , if and only if H contains a conjugate of Q .*

Every subgroup in the conjugacy class of Q is called a **vertex** of M . Let δ denote the diagonal homomorphism of G into $G \times G$, sending $g \in G$ to (g, g) .

Theorem 1.18 ([1, IV Theorem 4]). *Let B be a block of a finite group G . Then B , considered as a $k[G \times G]$ -module, has a vertex of the form δD for D an ℓ -subgroup of G which is uniquely defined up to G -conjugacy.*

The ℓ -subgroups of G conjugate to D are called **defect groups** of B . If the order of D is ℓ^d , we define the **defect** $d(B)$ of B to be d .

If $K \trianglelefteq H \subseteq G$ are two subgroups of G , we call the quotient H/K a **section** of G . The **sectional ℓ -rank** $s(G)$ of a finite group G is then defined to be the maximum of the ranks of elementary abelian ℓ -sections of G . Note that $s(H) \leq s(G)$ for every subgroup $H \subseteq G$. We are finally able to state the Malle–Robinson conjecture, which is the primary motivation for this thesis.

Conjecture 1 ([34, Conjecture 1]). *Let B be an ℓ -block of a finite group G with defect group D . Then*

$$l(B) \leq \ell^{s(D)}.$$

If strict inequality holds, we say that the conjecture holds in *strong form*. Since the defect groups of a given block B are conjugate and therefore isomorphic to each other, we often write $s(B)$ instead of $s(D)$.

This thesis is concerned with establishing the Malle–Robinson conjecture for an important class of blocks of the quasi-simple exceptional groups of Lie type. Apart from arguments stemming from the geometric nature of the character theory of finite reductive groups, we also need some block-theoretic results.

Remark 1.19. One argument we will use later on involves another description of the defect of a block (see e.g. [38, (3.15) Definition]). Let B be an ℓ -block of a finite group G with $|G|_\ell = \ell^a$. Then

$$\ell^{a-d(B)} = \min\{\chi(1)_\ell \mid \chi \in \text{Irr}(B)\}.$$

Let $N \trianglelefteq G$ be a normal subgroup of G . There is a natural action of G on kN given by

$$\left(\sum_{n \in N} a_n n\right)^g = \sum_{n \in N} a_n n^g.$$

The conjugate of an indecomposable subalgebra of kN is still indecomposable, so the action of G on kN induces an action on the blocks of N . Let $b = b_1$ be a block of N . If $\{b_1, b_2, \dots, b_t\}$ is the G -orbit of b , then the idempotent $\sum_{i=1}^t e_{b_i}$ lies in $Z(kG)$. In particular, $\sum_{i=1}^t e_{b_i}$ decomposes into a sum of primitive central idempotents of kG . Hence

$$\sum_{i=1}^t e_{b_i} = \sum_{i=1}^s e_{B_i},$$

where the B_i 's are uniquely determined blocks of G . In this case, we say that the B_i 's **cover** b . Whether or not a given block of G covers a given block of N can easily be determined by character theoretic arguments.

Theorem 1.20 ([38, (9.2) Theorem]). *Let $N \trianglelefteq G$ be a normal subgroup of a finite group G . Let b be a block of N and B be a block of G . The following statements are equivalent:*

- (a) B covers b ;
- (b) if $\chi \in \text{Irr}(B)$, then every irreducible constituent of χ_N lies in a G -conjugate of b ;
- (c) there is a $\chi \in \text{Irr}(B)$ such that χ_N has an irreducible constituent lying in b .

Set $\bar{G} = G/N$ and let $\bar{\cdot}: G \rightarrow \bar{G}$ denote the projection of G onto \bar{G} . If $\bar{\chi}$ is a character of \bar{G} , we can define a character χ of G by setting $\chi(g) = \bar{\chi}(\bar{g})$. In this way, every character of \bar{G} can be seen as a character of G with N in its kernel. Furthermore, if $\bar{\chi}$ is irreducible, then χ is irreducible as well. The same also holds for Brauer characters. This means that we can regard $\text{Irr}(\bar{G})$ (respectively $\text{IBr}(\bar{G})$) as a subset of $\text{Irr}(G)$ (respectively $\text{IBr}(G)$). The canonical projection $\bar{\cdot}: G \rightarrow \bar{G}$ induces a k -algebra homomorphism $kG \rightarrow k\bar{G}$. This homomorphism sends central elements to central elements. So, if B is a block of G , then

the image of its idempotent \bar{e}_B is either 0 or an idempotent in $Z(k\bar{G})$. In the latter case, we have

$$\bar{e}_B = \sum_{i=1}^r e_{\bar{B}_i},$$

for uniquely determined blocks $\bar{B}_1, \dots, \bar{B}_r$ of \bar{G} . We then say that B **dominates** the blocks $\bar{B}_1, \dots, \bar{B}_r$. It is easy to show that domination of blocks is the same as inclusion of blocks, where we then think of blocks of G as subsets of $\text{Irr}(G) \cup \text{IBr}(G)$. The following theorems are crucial for the proof of the Malle–Robinson conjecture for simple groups of Lie type.

Theorem 1.21 ([38, (9.9) Theorem]). *Let $N \trianglelefteq G$ be a normal subgroup of a finite group G and write $\bar{G} = G/N$.*

- (a) *Suppose that $\bar{B} \subseteq B$, where \bar{B} is a block of \bar{G} and B is a block of G . If \bar{D} is a defect group of \bar{B} , then there is a defect group D of B , such that $\bar{D} \subseteq DN/N$.*
- (b) *If N is an ℓ -group, then every block B of G contains a block \bar{B} of \bar{G} such that every defect group of \bar{B} is of the form D/N , where D is a defect group of B .*
- (c) *If N is a ℓ' -group and $\bar{B} \subseteq B$, then $\text{Irr}(B) = \text{Irr}(\bar{B})$, $\text{IBr}(B) = \text{IBr}(\bar{B})$ and every defect group of \bar{B} is of the form DN/N , where D is a defect group of B .*

Theorem 1.22 ([38, (9.10) Theorem]). *Suppose that G has a normal ℓ -subgroup P such that $G/C_G(P)$ is an ℓ -group. Let $\bar{G} = G/P$. If \bar{B} is a block of \bar{G} and B is the unique block of G containing \bar{B} , then $\text{IBr}(\bar{B}) = \text{IBr}(B)$ and every defect group \bar{D} of \bar{B} is of the form D/P for a defect group D of B .*

1.5 Connected reductive groups

From now on let $K = \bar{\mathbb{F}}_p$ be an algebraic closure of the finite field \mathbb{F}_p with p elements. Recall that the additive group $(K, +)$ and the multiplicative group $(K \setminus \{0\}, *)$ are both algebraic groups, which we denote by K_a and K_m respectively. Many arguments in this thesis use the fact that reductive algebraic groups can be classified by combinatorial data.

Let G be a connected reductive algebraic group defined over K . Let T_0 be a fixed maximal torus of G and let B_0 be a fixed Borel subgroup of G containing T_0 . Set $X(T_0) = \text{Hom}(T_0, K_m)$ and $Y(T_0) = \text{Hom}(K_m, T_0)$, where the elements are homomorphisms of algebraic groups. If $\chi \in X(T_0)$ and $\psi \in Y(T_0)$, then the composite map $\chi \circ \psi$ is an endomorphism of K_m and is therefore of the form $y \mapsto y^n$ for some $n \in \mathbb{Z}$. This yields a pairing $\langle -, - \rangle : X(T_0) \times Y(T_0) \rightarrow \mathbb{Z}$ defined by $\langle \chi, \psi \rangle = n$.

The most important object attached to a connected reductive group is its Weyl group, which we introduce now. Let U be a one dimensional connected unipotent subgroup of G that is normalized by T_0 . First of all, we have $K_a \cong U$ (see for example [25, Chapter 20]). We call an isomorphism $K_a \rightarrow U$, $t \mapsto x(t)$ a **parametrization** of U . From the structure of K_a , it follows that every other parametrization of U is of the form $t \mapsto x(ct)$, for $c \in K \setminus \{0\}$ (see for example [22, Proposition 1.3.5]).

Proposition 1.23 ([22, Proposition 1.9.2]). *If U is a one-dimensional connected unipotent subgroup of G normalized by T_0 , then there is a unique $\alpha \in X(T_0)$ such that for any parametrization $x(t)$ of U , $x(t)^s = x(\alpha(s)t)$ for $t \in K$ and $s \in T_0$.*

This α is called a **root** of G . Furthermore, it can be shown that each root α is associated to a unique one-dimensional unipotent T_0 -invariant subgroup U called a **root subgroup**, which we therefore denote by U_α (see [22, Theorem 1.9.5 (b)]). The set of roots of G will be denoted by $\Phi(T_0)$ or just Φ . All of this fits into a more general framework which we outline now.

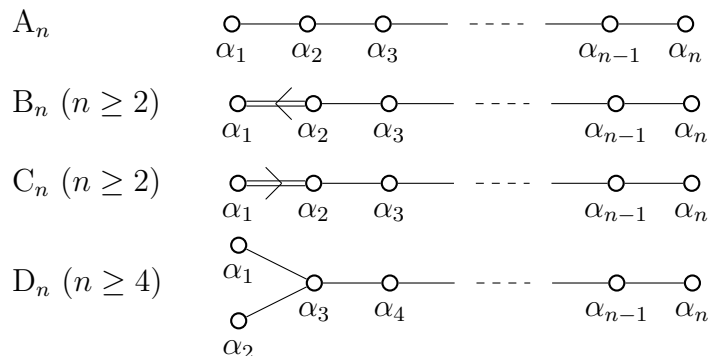
Let V be a finite dimensional euclidean vector space. Recall that a **reflection** of V is a linear map on V which sends some nonzero vector α to its negative while fixing the hyperplane H_α orthogonal to α pointwise. We denote the linear map corresponding to α by s_α . A **root system** in V consists of a subset Φ of V satisfying the following axioms:

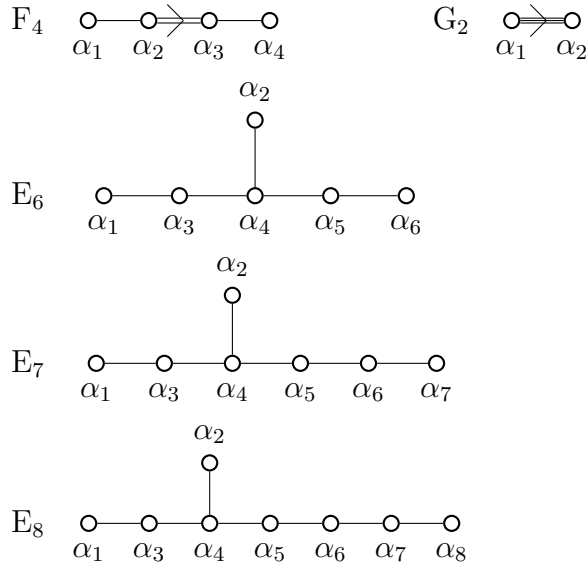
- (R1) Φ is finite, generates V and does not contain 0;
- (R2) if $\alpha \in \Phi$ and $c\alpha \in \Phi$, for a $c \in \mathbb{R}$, then $c \in \{1, -1\}$;
- (R3) if $\alpha \in \Phi$, then $s_\alpha\Phi = \Phi$;
- (R4) if $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) - \beta$ is an integral multiple of α .

We call $\dim(V)$ the **rank** of Φ . The finite group W generated by the s_α 's is called the **Weyl group** of the root system and is a group of permutations of Φ by (R3).

A subset $\Pi = \{\alpha_1, \dots, \alpha_t\}$ of Φ is called a **base** if it is a basis of V and if, for $\alpha \in \Phi$, the coefficients of α expressed as a linear combination of $\alpha_1, \dots, \alpha_t$ have the same sign. It is a nontrivial fact that bases exist (see [26, 1.3 Theorem]) and that W permutes the collection of bases simply transitively (see [26, 1.8 Theorem]). We call the elements of a given base Π the **simple roots**. The roots which are non-negative (respectively non-positive) combinations of Π comprise the set Φ^+ (respectively Φ^-) of **positive** (respectively **negative**) roots. Hence every base yields a set of positive roots. On the other hand we can equip V with a total ordering $<$ as in [26, Section 1.3] and set $\Phi^+ := \{\alpha \in \Phi \mid 0 < \alpha\}$; the corresponding set of **positive roots**. We then call an element of Φ^+ **simple** if it can not be decomposed into a sum of two or more positive roots. The set of simple roots as defined here is then a base of Φ . In particular, there is a one-to-one correspondence between bases and positive systems.

A root system is called **irreducible** if it cannot be partitioned into a union of two mutually orthogonal proper subsets. Every root system is the disjoint union of uniquely determined irreducible root systems (in suitable subspaces of V) and, up to isomorphisms, these irreducible root systems correspond to the following **Dynkin diagrams** (see [26, Chapter 2]):





Theorem 1.24 ([15, 0.31 Theorem]). *Let G be a semisimple algebraic group and let T_0 be a maximal torus of G . Then $\Phi(T_0)$ is a root system in $\mathbb{R} \otimes_{\mathbb{Z}} X(T_0)$ whose rank is $\dim(T_0)$ and whose Weyl group is isomorphic to $W(T_0) = N_G(T_0)/T_0$. Furthermore, the Borel subgroups containing T_0 are in one-to-one correspondence with the bases of Φ .*

The theorem implicitly talks about the action of $W(T_0)$ on $X(T_0)$, so let us recall that $W(T_0)$ acts on $X(T_0)$ and $Y(T_0)$ as follows:

$$\begin{aligned} (w \cdot \chi)(t) &= \chi(t^w) & \text{for all } w \in W, \chi \in X(T_0), t \in T_0; \\ (w \cdot \gamma)(c) &= \gamma(c)^{w^{-1}} & \text{for all } w \in W, \gamma \in Y(T_0), c \in K_m. \end{aligned}$$

Let Φ be a root system. A subset $\Psi \subseteq \Phi$ is said to be **closed** if

(C1) for all $\alpha, \beta \in \Psi$ we have $s_\alpha \beta \in \Psi$, and

(C2) for $\alpha, \beta \in \Psi$ with $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Psi$.

Definition 1.25. Let Φ be a root system. A prime r is called **bad** for Φ if $\mathbb{Z}\Phi/\mathbb{Z}\Psi$ has r -torsion for some closed subsystem $\Psi \subseteq \Phi$. Let G be a connected reductive group. Then a prime r is called **bad** for G , if it is bad for the root system of G and we denote the set of bad primes of G by $\gamma(G)$. Every prime that is not bad for G is called **good** for G .

There are just a few bad primes and they are given in the following table for the simple types.

Φ	bad primes
A_n	—
B_n ($n \geq 2$)	2
C_n ($n \geq 3$)	2
D_n ($n \geq 4$)	2
G_2, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

The distinction between good and bad primes is crucial in the theory of algebraic groups and in the theory of finite reductive groups. Many general results simply do not hold for bad primes. This problem arises in various different ways, as can be seen in Sections 3 and 4.

Next, we want to classify the connected reductive algebraic groups. Since the derived subgroup $[G, G]$ of a connected reductive group G is semisimple and $G = Z(G)^\circ[G, G]$ (see [25, 19.5, Lemma]), it is enough to classify the semisimple algebraic groups.

Lemma 1.26 ([35, Lemma 8.19]). *Let G be a semisimple algebraic group and let T_0 be a maximal torus of G . Let $\alpha \in \Phi = \Phi(T_0)$. Then we have the following:*

- (a) *for each root $\alpha \in \Phi$ there exists a unique $\alpha^\vee \in Y(T_0)$ such that $s_\alpha \cdot \chi = \chi - \langle \chi, \alpha^\vee \rangle \alpha$ for all $\chi \in X(T_0)$. In particular $\langle \alpha, \alpha^\vee \rangle = 2$;*
- (b) *$s_\alpha \cdot \gamma = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee$ for all $\gamma \in Y(T_0)$.*

For a root α , we call $\alpha^\vee \in Y(T_0)$ the **coroot** corresponding to α and set $\Phi^\vee = \Phi^\vee(T_0) := \{\alpha^\vee \mid \alpha \in \Phi\}$. It can be shown that Φ^\vee is a root system in $\mathbb{R} \otimes_{\mathbb{Z}} Y(T_0)$, which we call the **dual root system** of Φ .

Everything about reductive algebraic groups is encoded in the following combinatorial structure.

Definition 1.27. A quadruple (X, Φ, Y, Φ^\vee) is called a **root datum** if

- (RD1) $X \cong Y \cong \mathbb{Z}^n$ with a perfect pairing $\langle -, - \rangle : X \times Y \rightarrow \mathbb{Z}$;
- (RD2) $\Phi \subseteq X$, $\Phi^\vee \subseteq Y$ are abstract root systems in $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ respectively;
- (RD3) there exists a bijection $\Phi \rightarrow \Phi^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$; and
- (RD4) the reflections s_α of the root system Φ , respectively s_{α^\vee} of Φ^\vee are given by

$$\begin{aligned} s_\alpha \cdot \chi &= \chi - \langle \chi, \alpha^\vee \rangle \alpha && \text{for all } \chi \in X, \\ s_{\alpha^\vee} \cdot \gamma &= \gamma - \langle \alpha, \gamma \rangle \alpha^\vee && \text{for all } \gamma \in Y. \end{aligned}$$

Since the pairing $\langle -, - \rangle$ is perfect we have $X \cong \text{Hom}(Y, \mathbb{Z})$ and $Y \cong \text{Hom}(X, \mathbb{Z})$. Let $(X', \Phi', Y', \Phi'^\vee)$ be another root datum. For every group homomorphism $\phi : X' \rightarrow X$ the perfect pairing yields a dual group homomorphism $\phi^\vee : Y \rightarrow Y'$. A **homomorphism of root data** is a group homomorphism $\phi : X' \rightarrow X$ that maps Φ' bijectively to Φ and such that the dual homomorphism ϕ^\vee maps $\phi(\alpha)^\vee$ to α^\vee for every $\alpha \in \Phi'$.

If G is a semisimple algebraic group with a maximal torus T , then we observe that $(X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum. With this we are at last able to state Chevalley's Classification theorem.

Theorem 1.28 ([35, Theorem 9.13]). *Two semisimple algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum there exists a semisimple algebraic group which realizes it. This group is simple if and only if its root system is irreducible.*

The groups with root systems of type A_n, B_n, C_n, D_n , are called **groups of classical type** and the groups of type G_2, F_4, E_6, E_7, E_8 are called **groups of exceptional type**.

Let G be a semisimple algebraic group with root datum (X, Φ, Y, Φ^\vee) . By Theorem 1.24, $\mathbb{Z}\Phi$ is of finite index in X and $\mathbb{Z}\Phi^\vee$ is of finite index in Y . Furthermore, we can lift our perfect pairing $\langle -, - \rangle : X \times Y \rightarrow \mathbb{Z}$ to a non-degenerate bilinear map $\mathbb{R} \otimes_{\mathbb{Z}} X \times \mathbb{R} \otimes_{\mathbb{Z}} Y \rightarrow \mathbb{R}$, which we will still denote by $\langle -, - \rangle$. Set

$$\Omega = \{x \in \mathbb{R}\Phi \mid \langle x, \Phi^\vee \rangle \subset \mathbb{Z}\}.$$

Then Ω is a lattice with $\mathbb{Z}\Phi \subseteq X \subseteq \Omega$. We call Ω the **weight lattice**. For a given Φ we see that there are only finitely many possibilities for X . We call G **adjoint** if $X = \mathbb{Z}\Phi$ and **simply connected** if $X = \Omega$. The finite abelian group $\Delta := \Omega/\mathbb{Z}\Phi$ is called the **fundamental group** of G . Chevalley's Classification Theorem says that G is characterised by its root system and the image of X in Δ . For a given root system Φ , we denote the associated simply connected groups by G_{sc} and the adjoint groups by G_{ad} .

A surjective homomorphism $\phi : G \rightarrow H$ of algebraic groups with finite kernel is called an **isogeny**. It should be noted that this kernel lies in the center of G .

Proposition 1.29 ([35, Proposition 9.15]). *Let G be a semisimple group with root system Φ . Then there exist natural isogenies*

$$G_{sc} \rightarrow G \rightarrow G_{ad}$$

from a simply connected group G_{sc} and to an adjoint group G_{ad} , each with root system Φ .

The following well-known results are used in the proofs in later sections.

Proposition 1.30 ([35, Proposition 14.1]). *Let G be a connected reductive group, $s \in G$ semisimple, $T \leq G$ a maximal torus with corresponding root system Φ . Let $s \in T$ and $\Psi := \{\alpha \in \Phi \mid \alpha(s) = 1\}$. Then we have the following.*

- (a) $C_G^\circ(s) = \langle T, U_\alpha \mid \alpha \in \Psi \rangle$, and
- (b) $C_G(s) = \langle T, U_\alpha, \dot{w} \mid \alpha \in \Psi, w \in W \text{ with } s^w = s \rangle$, where \dot{w} denotes a representative of $w \in W$ in $N_G(T)$.

Moreover, $C_G^\circ(s)$ is reductive with root system Ψ and Weyl group $W(s) = \langle s_\alpha \mid \alpha \in \Psi \rangle$.

Theorem 1.31 (Steinberg, [35, Theorem 14.16]). *Let G be a connected reductive group such that the derived subgroup $[G, G]$ is simply connected, and let $s \in G$ be a semisimple element. Then $C_G(s)$ is connected.*

Let G be a connected reductive group and let T be a maximal torus with corresponding root system Φ and Weyl group W . Furthermore, let Π be a base of Φ . If $I \subseteq \Pi$ we write $W_I := \langle s_\alpha, \alpha \in I \rangle$ for the subgroup of W generated by the simple reflections corresponding to the simple roots in I . It turns out that this group is itself again a Weyl group corresponding to the root system $\Phi_I = \Phi \cap \sum_{\alpha \in I} \mathbb{Z}\alpha$.

Proposition 1.32 ([35, Proposition 12.2]). *Let G be connected reductive. Let Φ be the root system of G with respect to a maximal torus $T \subseteq B$ of G and let Π be the set of simple roots corresponding to B . Then we have the following.*

- (a) *Let $I \subseteq \Pi$. The group $P_I := BW_I B = \bigcup_{w \in W_I} B\dot{w}B$ is a closed, connected, self-normalizing subgroup of G which contains B .*
- (b) $P_I = \langle T, U_\alpha \mid \alpha \in \Phi^+ \cup \Phi_I \rangle$.

Moreover, all closed subgroups containing B arise in this way.

The groups P_I which arise in this way are called **standard parabolic subgroups** of G . A subgroup of G is said to be **parabolic** if it is conjugate to a standard parabolic subgroup. For $I \subseteq \Pi$ we set

$$U_I := \langle U_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_I \rangle$$

and

$$L_I := \langle T, U_\alpha \mid \alpha \in \Phi_I \rangle.$$

It can be shown that $U_I = R_u(P_I)$ and that L_I is complement to U_I in P_I , i.e. $P_I = U_I \rtimes L_I$. Moreover, L_I is a connected reductive group with root system Φ_I (see [35, Proposition 12.6]).

Definition 1.33. L_I is called the (standard) **Levi complement** of P_I . A subgroup is said to be a **Levi subgroup** of G if it is conjugate to a standard Levi complement L_I of a standard parabolic P_I for some $I \subseteq \Pi$.

It follows that every parabolic subgroup P of G has a decomposition $P = R_u(P) \rtimes L$ for some Levi subgroup L of G . As in the case of the standard parabolic subgroups, we say that L is a **Levi complement** of P .

We call a connected reductive subgroup $H \subseteq G$ a **subsystem subgroup** if its root system is a closed subsystem (see Section 1.5) of Φ , where Φ is the root system of G with respect to some maximal torus T of G . We will repeatedly make use of the following fundamental fact about bad primes.

Proposition 1.34. *If $H \subseteq G$ is a subsystem subgroup of a connected reductive group G , then $\gamma(H) \subseteq \gamma(G)$.*

Proof. We denote the root systems of H and G by Ψ and Φ respectively. By the definition of bad primes, to prove the assertion we just need to show that every closed subset Θ of Ψ is a closed subset of Φ . Since Θ is a closed subset of Ψ , (C1) is automatically satisfied. Now let $\alpha, \beta \in \Theta$ be such that $\alpha + \beta \in \Phi$. Since $\Theta \subseteq \Psi$ and Ψ is a closed subset of Φ , we have $\alpha + \beta \in \Psi$. In turn this yields $\alpha + \beta \in \Theta$ as Θ is a closed subset of Ψ . \square

Observe that the connected components of centralizers of semisimple elements in G and the Levi subgroups of G are examples of subsystem subgroups of G .

Next, we want to know what happens when you intersect Levi subgroups.

Proposition 1.35 ([15, 2.1 Proposition]). *Let P and Q be two parabolic subgroups of G with unipotent radicals U and V respectively. Let L and M be Levi complements of P and Q respectively, sharing a common maximal torus of G . Then the group $(P \cap Q).U$ is a parabolic subgroup of G with Levi complement $L \cap M$.*

In other words, the intersection of two Levi subgroup is again a Levi subgroup if they contain a common maximal torus of G .

Proposition 1.36 ([15, 1.21 Proposition, 1.22 Proposition]). *Let G be a connected reductive group. For any torus S of G , the group $C_G(S)$ is a Levi subgroup of G . Conversely, every Levi subgroup L occurs as the centraliser of a torus of G . In particular, $L = C_G(Z^\circ(L))$.*

Corollary 1.37. *Let $L \subseteq G$ be a Levi subgroup of a connected reductive group G . If $M \subseteq L$ is a Levi subgroup of L , then M is a Levi subgroup of G .*

Proof. Since M is a Levi subgroup of L , we have $M = C_L(Z^\circ(M))$ by Proposition 1.36. Similarly, we have $L = C_G(Z^\circ(L))$. Furthermore, $Z^\circ(L) \subseteq Z^\circ(M)$ because $M \subseteq L$ are both connected reductive groups and the center of a connected reductive group is the intersection of all of its maximal tori. In conclusion, we have

$$M = C_L(Z^\circ(M)) = C_{C_G(Z^\circ(L))}(Z^\circ(M)) = C_G(Z^\circ(L) \cup Z^\circ(M)) = C_G(Z^\circ(M)).$$

Hence, M is a Levi subgroup of G by Proposition 1.36. □

Proposition 1.38 (Geck–Hiss, [19, Proposition 2.1]). *Let G be a connected reductive group and let $s \in G$ be a semisimple element such that $o(s)$ is only divisible by primes which are good for G . Then $C_G^\circ(s)$ is a Levi subgroup of G .*

Proof. Let $L \subseteq G$ be the smallest Levi subgroup of G containing $C_G^\circ(s)$. Let Φ be the root system of G with respect to a maximal torus T of G . Denote the root system of L by Ψ and the root system of $C_G^\circ(s)$ by $\Phi(s)$. We have $\Phi(s) = \{\gamma \in \Phi \mid \gamma(s) = 1\}$. By the minimality of L , $\Phi(s) \subseteq \Psi$ have the same rank. In particular, $\mathbb{Z}\Psi/\mathbb{Z}\Phi(s)$ is a finite group and by definition of bad primes, its order, say n , is a product of bad primes for L and therefore of G by Proposition 1.34. We show that $\Psi \subseteq \Phi(s)$. Let $\alpha \in \Psi$. We have $n\alpha = \sum_{\gamma \in \Phi(s)} a_\gamma \gamma$. Evaluating both sides at s yields

$$\alpha(s)^n = \prod_{\gamma \in \Phi(s)} \gamma(s)^{a_\gamma} = 1.$$

It follows that $o(\alpha(s)) \mid n$ and because α is a group homomorphism, we also have $o(\alpha(s)) \mid o(s)$. Hence $o(\alpha(s)) \mid (n, o(s))$. But $(n, o(s)) = 1$, by the assumption on $o(s)$. Therefore, $\alpha(s) = 1$, i.e. $\alpha \in \Phi(s)$. □

1.6 Finite groups of Lie type

Finite reductive groups or finite groups of Lie type arise as fixed point groups of certain endomorphisms of connected reductive algebraic groups. The theory of finite groups of Lie type is therefore strongly connected to the theory of algebraic groups. To get to the finite groups of Lie type we need to discuss \mathbb{F}_q -structures (q a power of p) on algebraic groups, Frobenius endomorphisms and the relationship between them.

We denote the affine space of dimension n by \mathbb{A}_n .

Example Let $V \subseteq \mathbb{A}_n$ be an affine variety over an algebraically closed field K . Assume its vanishing ideal $I(V) \subseteq K[X_1, \dots, X_n]$ is generated by polynomials in $\mathbb{F}_q[X_1, \dots, X_n]$. In particular, $I(V)_0 := I(V) \cap \mathbb{F}_q[X_1, \dots, X_n]$ is an ideal in $\mathbb{F}_q[X_1, \dots, X_n]$ such that $I(V)_0 K[X_1, \dots, X_n] = I(V)$. Then

$$K[V] = K[X_1, \dots, X_n]/I(V)_0 K[X_1, \dots, X_n] \cong \mathbb{F}_q[X_1, \dots, X_n]/I(V)_0 \otimes_{\mathbb{F}_q} K.$$

Set $A_0 := \mathbb{F}_q[X_1, \dots, X_n]/I(V)_0$. The isomorphism above yields a natural endomorphism $F : V \rightarrow V$ given by the endomorphism $F^* : A_0 \otimes K \rightarrow A_0 \otimes K$, $a \otimes \lambda \mapsto a^q \otimes \lambda$.

With this in mind, we say that an affine variety X is **defined over** \mathbb{F}_q or is endowed with an \mathbb{F}_q -structure if its coordinate ring $K[X]$ can be written as $K[X] = A_0 \otimes K$ for a finitely generated \mathbb{F}_q -algebra A_0 . We call A_0 an \mathbb{F}_q -**structure** on X . Every \mathbb{F}_q -structure is accompanied by an endomorphism as in the example above. The **Frobenius endomorphism** $F : V \rightarrow V$ associated to a given \mathbb{F}_q -structure is the endomorphism of V defined by the endomorphism of $K[X] = A_0 \otimes_{\mathbb{F}_q} K$ that sends $a \otimes \lambda$ to $a^q \otimes \lambda$. We have the following general result.

Proposition 1.39 ([15, 3.6 Proposition]). *Let V be an affine variety over K defined over \mathbb{F}_q with corresponding Frobenius endomorphism $F : V \rightarrow V$.*

- (a) *Let φ be an automorphism of V such that $(\varphi F)^n = F^n$ for some positive integer n . Then φF is a Frobenius endomorphism associated to some \mathbb{F}_q -structure.*
- (b) *If F' is another Frobenius endomorphism of V corresponding to an \mathbb{F}_q -structure, there exists a positive integer n such that $F'^n = F^n$.*
- (c) *F^n is the Frobenius endomorphism corresponding to some \mathbb{F}_{q^n} -structure on V .*
- (d) *The set of rational points of V , $V^F := \{v \in V \mid F(v) = v\}$, is a finite set.*

Now, an algebraic group over K is said to be **defined over** \mathbb{F}_q if it has an \mathbb{F}_q -rational structure such that the corresponding Frobenius endomorphism is a group homomorphism. If G is a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$, then by Proposition 1.39 the group $G^F := \{g \in G \mid F(g) = g\}$ of F -stable points of G is a finite group. Groups that arise in this way are called **finite reductive groups** or **finite groups of Lie type**.

Theorem 1.40 ([35, Theorem 24.15 + Theorem 24.17]). *Let G be a simple, simply-connected group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Then, unless G^F is one of the groups*

$$\mathrm{SL}_2(2), \mathrm{SL}_2(3), \mathrm{SU}_3(2), \mathrm{Sp}_4(2), G_2(2), {}^2B_2(2), {}^2G_2(2), {}^2F_4(2),$$

G^F is quasi-simple.

The following theorem allows us to use geometric arguments when studying finite groups of Lie type.

Theorem 1.41 (Lang-Steinberg Theorem, [35, Theorem 21.7]). *Let G be a connected algebraic group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Then the morphism*

$$\mathcal{L} : G \rightarrow G, \quad g \mapsto F(g)g^{-1},$$

is surjective.

We collect some immediate consequences of this theorem in the following corollary.

Corollary 1.42 ([35, Section 21.2]). *Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$.*

- (a) *If H is a closed connected F -stable subgroup of G , then $(G/H)^F = G^F/H^F$.*
- (b) *There exists a pair $T \subseteq B$ consisting of an F -stable maximal torus T in an F -stable Borel subgroup B of G . All such pairs $T \subseteq B$ are G^F -conjugate.*
- (c) *Any F -stable conjugacy class of G contains an F -stable element.*
- (d) *In the abstract semidirect product $G \rtimes \langle F \rangle$, the coset $G.F$ of F consists of a single conjugacy class, that is, $G.F = F^G$. In particular, G^{g^F} and G^F are G -conjugate and hence isomorphic for any $g \in G$.*

Let G be a connected reductive algebraic group with Frobenius endomorphism F . Let T be an F -stable maximal torus contained in an F -stable Borel subgroup B of G (which exists by the previous Corollary). As T is F -stable, so is $N_G(T)$. As such F acts on the Weyl group $W = N_G(T)/T$ and we have $W^F = N_G(T)^F/T^F$ by Corollary 1.42 (a). Furthermore, F acts naturally on $X(T)$ and $Y(T)$ via

$$\begin{aligned} F(\chi)(t) &:= \chi(F(t)) \quad \text{for } \chi \in X(T), t \in T, \\ F(\gamma)(c) &:= F(\gamma(c)) \quad \text{for } \gamma \in Y(T), c \in K^\times. \end{aligned}$$

We write $\Phi \subset X(T)$ for the root system of G with respect to T and choose isomorphisms $u_\alpha : K_\alpha \rightarrow U_\alpha$ onto the root subgroups U_α . Then we have the following result.

Proposition 1.43 ([35, Proposition 22.2]).

- (a) *There exists a permutation ρ of Φ^+ and, for each $\alpha \in \Phi^+$, a positive integral power $q_\alpha > 1$ of p and $a_\alpha \in K^\times$ such that $F(\rho(\alpha)) = q_\alpha \alpha$ and $F(u_\alpha(c)) = u_{\rho(\alpha)}(a_\alpha c^{q_\alpha})$ for all $c \in K$.*
- (b) *There exists a $d \geq 1$ such that $F^d|_{X(T)} = r^d \text{id}_{X(T)}$ and $F = r\phi$ on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, for some positive integer power r of p and some $\phi \in \text{Aut}(X(T) \otimes_{\mathbb{Z}} \mathbb{R})$ of order d inducing ρ^{-1} on Φ^+ .*

If ϕ is a nontrivial automorphism of $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, we call G^F **twisted** and otherwise we call G^F **split**. Note that the original statement in [35] is slightly stronger and also includes endomorphisms of algebraic groups leading to Suzuki and Ree groups, i.e endomorphisms F such that an integer power F^d is a Frobenius endomorphism corresponding to an \mathbb{F}_{p^e} -structure for a positive integer e . Since we are not interested in Suzuki and Ree groups, we stick to Frobenius endomorphisms throughout this thesis.

Next, we briefly discuss the existence of \mathbb{F}_q -structures on algebraic groups over $K = \overline{\mathbb{F}}_p$. Since we normally assume G to be a simple algebraic group in this thesis, the following existence theorem is sufficient.

Theorem 1.44 ([35, Theorem 22.5]). *Let G be a simple, simply connected algebraic group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Then F is uniquely determined, up to inner automorphisms of G , by q and the Dynkin diagram automorphism $\rho|_{\Delta}$ in Proposition 1.43. Conversely, for every pair (q, ρ) with q an integral power of p and ρ a Dynkin diagram automorphism of Δ there exists an \mathbb{F}_q -rational structure on G yielding a Frobenius F such that the corresponding permutation of Φ^+ (see Proposition 1.43) is ρ .*

To go from algebraic groups of simply connected type to arbitrary isogeny types we need the following lifting result.

Proposition 1.45 ([35, Proposition 22.7]). *Let G be semisimple and let $\pi : G_{sc} \rightarrow G$ be the natural isogeny from a simply connected group of the same type. Then every isogeny $\sigma : G \rightarrow G$ can be lifted to a unique isogeny $\sigma_{sc} : G_{sc} \rightarrow G_{sc}$ such that $\pi \circ \sigma_{sc} = \sigma \circ \pi$.*

A general existence result can for example be found in [39]. The final result of that book is an existence theorem for structures over arbitrary fields instead of just finite ones, so the proofs are more complex than they would need to be for our purposes. For a more fitting treatment see [20, Example 1.4.15].

In light of this, we always assume the existence of Frobenius endomorphisms as in Theorem 1.44 for all connected reductive groups in the following sections.

Definition 1.46. Let G be a simple algebraic group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. If G is of type A , D or E_6 we have non-trivial Dynkin diagram automorphisms. G^F is said to be of **classical type** if $G^F \in \{A_n(q), {}^2A_n(q), B_n(q), C_n(q), D_n(q), {}^2D_n(q)\}$ (for some $n \in \mathbb{N}$) and is said to be of **exceptional type** if $G^F \in \{G_2(q), {}^3D_4(q), F_4(q), E_6(q), {}^2E_6(q), E_7(q), E_8(q)\}$, where the superscript on the left of the type indicates the order of the Dynkin diagram automorphism induced by F .

Next we introduce a generalization of root data which allows us to classify the finite groups of Lie type using combinatorial structures. Let G be a semisimple group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let T be an F -stable torus affording the root datum (X, Φ, Y, Φ^\vee) . Since $W = N_G(T)/T$ is F -stable we can define a semidirect product $W\langle F \rangle$. Every element wF in the coset $W.F$ fixes T and therefore fixes X , Φ , Y and Φ^\vee . By Corollary 1.42 (d) we have $G^{wF} \cong G^F$. Hence the isomorphism type of G^F is determined by the root datum and the coset $W\phi$, where ϕ is the automorphism associated to F by Proposition 1.43. Every finite group of Lie type therefore defines a combinatorial structure of the following form.

Definition 1.47. A tuple $\mathbb{G} = (X, \Phi, Y, \Phi^\vee, W\phi)$ is called a **complete root datum** if the tuple (X, Φ, Y, Φ^\vee) is a root datum and ϕ is an automorphism of this root datum of finite order.

Conversely, every prime power q together with a complete root datum \mathbb{G} yields a finite group of Lie type up to isomorphism. This can be seen by combining Theorem 1.24 and Theorem 1.44. We have to be careful though. It is not true that groups with different complete root data are necessarily non-isomorphic (see [35, Remark 24.9] for a list of examples).

Given a power of a prime q and a complete root datum $\mathbb{G} = (X, \Phi, Y, \Phi^\vee, W\phi)$ corresponding to an algebraic group G defined over \mathbb{F}_q with Frobenius endomorphism F , then we sometimes write $\mathbb{G}(q)$ instead of G^F in this section.

Definition 1.48. Let $\mathbb{G} = (X, \Phi, Y, \Phi, W\phi)$ be a complete root datum. Its **Ennola dual** is then defined as $\mathbb{G}^- := (X, \Phi, Y, \Phi^\vee, W(-\phi))$. If G^F corresponds to \mathbb{G} together with a prime power q , we call the group corresponding to \mathbb{G}^- together with q its Ennola dual.

Note that $\mathbb{G} = \mathbb{G}^-$ if $-\text{id} \in W$. Ennola duality is used and referred to repeatedly in order to save computing time and to half the number of tables needed in this thesis.

The fact that finite groups of Lie type are determined by combinatorial structures indicates some kind of generic behaviour. One example of this is the order formula. Its most generic form relies on a result of Shephard–Todd on finite reflection groups.

Theorem 1.49 (Shephard–Todd, [35, Theorem 24.4]). *Let $W \subseteq \text{GL}(V)$ be a finite reflection group on a real vector space V of dimension n . Then we have the following.*

- (a) *The invariants $S(V)^W$ of W in the symmetric algebra $S(V)$ of V form a polynomial algebra.*
- (b) *Let f_1, \dots, f_n denote algebraically independent generators of $S(V)^W$, homogeneous of degrees $d_i := \deg(f_i)$. Then the multiset $\{d_i \mid 1 \leq i \leq n\}$ is uniquely determined by W .*
- (c) *We have $d_1 \cdots d_n = |W|$ and $d_1 + \cdots + d_n = N + n$, where N denotes the number of reflections in W .*

Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism F . Set $V = X(T) \otimes \mathbb{R}$ where T is an F -stable maximal torus of G . Now, by Proposition 1.43, F acts on V as a scalar times an automorphism ϕ of V that normalizes W . Therefore ϕ acts on $S(V)^W$ and the generators f_i can be chosen to be eigenvectors of ϕ . The corresponding eigenvalues are denoted by ϵ_i . Given a complete root datum \mathbb{G} corresponding to G , we define the **order polynomial** of this datum to be

$$|\mathbb{G}| := X^{|\Phi^+|} \prod_{i=1}^{\text{rank}(G)} (X^{d_i} - \epsilon_i) \in \mathbb{Z}[X].$$

Proposition 1.50 ([35, Corollary 24.6]). *Let \mathbb{G} be a complete root datum. Then $|\mathbb{G}(q)| = |\mathbb{G}|(q)$ for every integral power q of p .*

It follows that the order of a finite group of Lie type is generic in q . More importantly, we see that the order of finite groups of Lie type does not depend on the isogeny type of the group. We even have a generic version of Lagrange’s theorem.

Proposition 1.51 ([35, Corollary 24.7]). *Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ and let $H \subseteq G$ be a closed connected reductive F -stable subgroup of G . Let \mathbb{G} and \mathbb{H} denote complete root data corresponding to G^F and H^F respectively. Then $|\mathbb{H}|$ divides $|\mathbb{G}|$ in $\mathbb{Z}[X]$.*

Many important objects can be defined via the order polynomial alone. Let $\mathbb{G} = (X, \Phi, Y, \Phi^\vee, W\phi)$ be a complete root datum. We observe that the eigenvalues ϵ_i appearing in the order polynomial of \mathbb{G} are roots of unity. Furthermore, their orders divide the order

of ϕ . In other words, the zeros of $|\mathbb{G}| \in \mathbb{Z}[X]$ are roots of unity. Hence we can decompose the order polynomial into a product of cyclotomic polynomials

$$|\mathbb{G}| = X^{|\Phi^+|} \prod_{i=1}^{\text{rank}(G)} (X^{d_i} - \epsilon_i) = X^{|\Phi^+|} \prod_{d \geq 1} \Phi_d(X)^{a(d)},$$

where, for every $d \geq 1$, $a(d)$ is a nonnegative integer and Φ_d denotes the d -th cyclotomic polynomial.

Definition 1.52. Let $e \geq 1$ be an integer. Suppose that S is an F -stable torus of G with complete root datum \mathbb{S} . Then S is called an e -torus if $|\mathbb{S}| = \Phi_e(X)^a$ for some nonnegative integer a .

Theorem 1.53 ([35, Proposition 25.7]). *Let S be an e -torus of G of dimension r . Then S is a direct product of $r/\varphi(e)$ F -stable tori with order polynomial $\Phi_e(X)$, where $\varphi(e) = \deg(\Phi_e)$.*

Definition 1.54. A Levi subgroup L of G is called e -split if $L = C_G(S)$ is the centralizer of an e -torus S of G .

Proposition 1.55. *If L is an e -split Levi subgroup of G , then $L = C_G(Z^\circ(L)_{\Phi_e})$, where $Z^\circ(L)_{\Phi_e}$ denotes the Φ_e -part of the torus $Z^\circ(L)$.*

Proof. Since L is e -split, there exists an e -torus S such that $L = C_G(S)$. Clearly, $S \subseteq Z^\circ(L)_{\Phi_e}$. Since $L = C_G(Z^\circ(L))$ (see Proposition 1.36), we have

$$L = C_G(S) \supseteq C_G(Z^\circ(L)_{\Phi_e}) \supseteq C_G(Z^\circ(L)) = L.$$

Hence, $L = C_G(Z^\circ(L)_{\Phi_e})$. □

The e -split Levi subgroups are strongly connected to the block theory of groups of Lie type, as was established in multiple articles by Broué–Malle–Michel [6] and Cabanes–Enguehard [8], [9], [10].

1.7 Harish-Chandra induction and restriction

Harish-Chandra induction is a generalized induction functor for groups of Lie type introduced by Harish-Chandra in [23]. This functor is strongly related to combinatorics via Howlett-Lehrer theory (see for example [12, Chapter 10]).

Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $P = U \rtimes L$ be an F -stable parabolic subgroup of G and let L be an F -stable Levi complement of P . Since P is F -stable the unipotent radical U of P is F -stable as well. As L normalizes U , there is a (G^F, L^F) -bimodule structure on $\mathbb{C}[G^F/U^F]$. We call the functor

$$R_{L \subseteq P}^G : \mathbb{C}[L^F]\text{-mod} \rightarrow \mathbb{C}[G^F]\text{-mod}, \quad V \mapsto \mathbb{C}[G^F/U^F] \otimes_{\mathbb{C}[L^F]} V$$

Harish-Chandra induction (from L to G). The adjoint functor is called **Harish-Chandra restriction** and is denoted by ${}^*R_{L \subseteq P}^G$. For our purposes it is enough to work with the induced map on the class functions. For $f \in \text{cl}(L^F)$ and $g \in G^F$ we have (see [15, 4.5 Proposition])

$$R_{L \subseteq P}^G(f)(g) = \frac{1}{|L^F|} \sum_{l \in L^F} \text{Trace}((g, l^{-1}) | \mathbb{C}[G^F/U^F]) f(l).$$

Proposition 1.56 ([15, 4.4 Proposition]). *Let P be an F -stable parabolic subgroup of G and let L be an F -stable Levi complement of P . Let Q be an F -stable parabolic subgroup of P and let M be an F -stable Levi complement of Q contained in L . Then*

$$R_{L \subseteq P}^G \circ R_{M \subseteq L \cap Q}^L = R_{M \subseteq Q}^G.$$

One key property of Harish-Chandra induction and restriction is the validity of an analogue of the classical Mackey formula. For $x \in G^F$ we define $\text{ad } x$ to be the map that sends a character to its conjugate under x .

Theorem 1.57 (Mackey Formula, [15, 5.1 Theorem]). *Let P and Q be two F -stable parabolic subgroups of G , and let L and M be F -stable Levi complements of P and Q respectively. Then*

$${}^*R_{L \subseteq P}^G \circ R_{M \subseteq Q}^G = \sum_x R_{L \cap {}^x M \subseteq L \cap {}^x Q}^L \circ {}^*R_{L \cap {}^x M \subseteq P \cap {}^x M} \circ \text{ad } x,$$

where x runs over a set of representatives of $L^F \backslash \mathcal{S}(L, M)^F / M^F$ with

$$\mathcal{S}(L, M) = \{x \in G \mid L \cap {}^x M \text{ contains a maximal torus of } G\}.$$

An immediate consequence of Theorem 1.57 is that Harish-Chandra induction and restriction are independent of the chosen parabolic subgroup (see [15, 6.1 Proposition]). We will therefore omit the parabolic subgroup from the subscript.

The reason Harish-Chandra induction and restriction are so important is that they yield a natural decomposition of $\text{Irr}(G^F)$. We can define a partial order on the set of pairs (L, λ) , where L is an F -stable Levi complement of an F -stable parabolic subgroup of G and $\lambda \in \text{Irr}(L^F)$, by putting $(L', \lambda') \leq (L, \lambda)$ if $L' \subseteq L$ and $\langle \lambda, R_{L'}^L \lambda' \rangle \neq 0$. If (L, λ) is minimal for this partial order, then we call (L, λ) a **cuspidal pair** of G^F .

It follows from the adjointness of induction and restriction that we can also characterize cuspidality using Harish-Chandra restriction.

Proposition 1.58 ([15, 6.3 Proposition]). *The following are equivalent.*

- (i) *The pair (L, λ) is cuspidal.*
- (ii) *For any F -stable Levi complement M of an F -stable proper parabolic subgroup of L , we have ${}^*R_M^L \lambda = 0$.*

Theorem 1.59 ([15, 6.4 Theorem]). *Let $\chi \in \text{Irr}(G^F)$. Then there exists a unique cuspidal pair (L, λ) up to G^F -conjugacy such that $(L, \lambda) \leq (G, \chi)$. In particular,*

$$\text{Irr}(G^F) = \dot{\bigcup}_{(L, \lambda)} \{\chi \in \text{Irr}(G^F) \mid \langle \chi, R_L^G \lambda \rangle \neq 0\},$$

where (L, λ) runs over a set of representatives of G^F -conjugacy classes of cuspidal pairs of G^F .

It then remains to classify the cuspidal pairs. As they are already the *minimal* objects of Harish-Chandra theory, we need something else to get information on them. This can be done by generalizing Harish-Chandra induction and restriction which are only defined for F -stable Levi complements of F -stable parabolic subgroups to F -stable Levi complements of parabolic subgroups that are not necessarily F -stable.

1.8 Lusztig induction and restriction

Let $\overline{\mathbb{Q}}_r$ be the field of r -adic numbers for a prime r different from p . From now on when we refer to ordinary representations, characters and class functions we will mean representations, characters and class function over $\overline{\mathbb{Q}}_r$. This is unproblematic as $\overline{\mathbb{Q}}_r$ and \mathbb{C} are isomorphic in a way that identifies complex conjugation in \mathbb{C} with a given involution of $\text{Aut}(\overline{\mathbb{Q}}_r)$ that sends roots of unity in $\overline{\mathbb{Q}}_r$ to their inverse.

Let P be a parabolic subgroup of G (not necessarily F -stable) and let L be an F -stable Levi complement of P . Let U denote the unipotent radical of P and let \mathcal{L} be the Lang map as defined in Theorem 1.41. We define the **Lusztig induction** $R_{L \subseteq P}^G : \mathbb{Z} \text{Irr}(L^F) \rightarrow \mathbb{Z} \text{Irr}(G^F)$ on generalized characters by

$$R_{L \subseteq P}^G(\chi)(g) = \frac{1}{|L^F|} \sum_{l \in L^F} \text{Trace}((g, l) | H_c^*(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}}_r)) \chi(l^{-1}),$$

and **Lusztig restriction** $*R_{L \subseteq P}^G : \mathbb{Z} \text{Irr}(G^F) \rightarrow \mathbb{Z} \text{Irr}(L^F)$ by

$$*R_{L \subseteq P}^G(\lambda)(g) = \frac{1}{|L^F|} \sum_{g \in G^F} \text{Trace}((g, l) | H_c^*(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}}_r)) \lambda(g^{-1}),$$

where $H_c^*(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}}_r) = \sum_i (-1)^i H_c^i(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}}_r)$ denotes the alternating sum of the groups of r -adic cohomology with compact support $H_c^i(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}}_r)$ associated to $\mathcal{L}^{-1}(U)$. For a short list of properties of these cohomology groups with the appropriate references see Chapter 10 of [15]. These groups can be given a (G^F, L^F) -bimodule structure. Therefore $H_c^*(\mathcal{L}^{-1}(U), \overline{\mathbb{Q}}_r)$ is a virtual (G^F, L^F) -bimodule. In particular, $R_{L \subseteq P}^G(\chi)$ is in general not a character even if χ is a character of L^F . It can be shown that Lusztig induction and restriction coincide with Harish-Chandra induction and restriction respectively if the parabolic subgroup P is F -stable. Thus these new induction and restriction maps properly generalize the old ones and it makes sense to use the same notation.

Even though we are working with generalized characters where cancellation can occur, Lusztig induction and restriction are still transitive.

Proposition 1.60 (Transitivity, [15, 11.5]). *Let $Q \subseteq P$ be two parabolic subgroups of G and let M and L be F -stable Levi complements of Q and P respectively. Suppose that $M \subseteq L$. Then $R_{L \subseteq P}^G \circ R_{M \subseteq L \cap Q}^L = R_{M \subseteq Q}^G$.*

The Mackey formula as proved for Harish-Chandra functors is still just conjectural for Lusztig induction and restriction. But if one of the occurring Levi subgroups is a maximal torus, the Mackey formula has been shown to hold (see [13]).

Theorem 1.61. *The Mackey formula as in Theorem 1.57 holds for Lusztig induction and restriction if either L or M is an F -stable maximal torus. In particular, Lusztig induction from an F -stable maximal torus T is independent of the Borel subgroup B that contains it.*

Using completely different (computational) arguments, Bonnafé–Michel proved an even stronger result in [3].

Theorem 1.62. *Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let P and Q be parabolic subgroups of G and let L and M be F -stable Levi complements of P and Q respectively. Then the Mackey formula holds in the following cases.*

- (i) $q > 2$, or
- (ii) G^F does not contain an F -stable quasi-simple component of type 2E_6 , E_7 or E_8 .

We say **the Mackey formula holds** for G^F , if the Mackey formula

$${}^*R_{L \subseteq P}^G \circ R_{M \subseteq Q}^G = \sum_x R_{L \cap^x M \subseteq L \cap^x Q}^L \circ {}^*R_{L \cap^x M \subseteq P \cap^x M} \circ \text{ad } x,$$

holds for every pair of parabolic subgroups P and Q of G with F -stable Levi complements L and M respectively. Since we are only interested in quasi-simple groups, the only cases where the Mackey formula does not hold are ${}^2E_6(2)$, $E_7(2)$ and $E_8(2)$ by Theorem 1.62. As for the Harish-Chandra functors, we therefore omit the parabolic subgroup in $R_{L \subseteq P}^G$ and write R_L^G throughout this work. Misunderstandings can not arise as we have to study these exceptions separately anyway; using slightly different methods.

Definition 1.63. Let T be an F -stable maximal torus of G and let $\theta \in \text{Irr}(T^F)$. Then $R_T^G(\theta)$ is called a **Deligne–Lusztig character**.

Let (T, θ) be a pair where T is an F -stable maximal torus of G and $\theta \in \text{Irr}(T^F)$. For $g \in G^F$ we write ${}^g(T, \theta)$ for $({}^gT, {}^g\theta)$. It can be shown (using either the character formula [15, 12.2 Proposition] or properties of r -adic cohomology directly) that $R_T^G(\theta) = R_{T'}^G(\theta')$, if $(T', \theta') = {}^g(T, \theta)$ for some $g \in G^F$. The converse of that statement is also true and easily follows from the Mackey formula.

Corollary 1.64 ([15, 11.15 Corollary]). *Let (T, θ) and (T', θ') be two pairs as above. Then*

$$\langle R_T^G(\theta), R_{T'}^G(\theta') \rangle_{G^F} = \frac{1}{|T^F|} |\{g \in G^F \mid {}^gT = T' \text{ and } {}^g\theta = \theta'\}|.$$

In particular, $R_T^G(\theta) = R_{T'}^G(\theta')$ if and only if $(T', \theta') = {}^g(T, \theta)$ for some $g \in G^F$.

We are interested in Deligne–Lusztig characters mainly due to the following result.

Proposition 1.65 ([15, 13.1 Proposition]). *For any $\chi \in \text{Irr}(G^F)$ there exists a pair (T, θ) such that $\langle \chi, R_T^G(\theta) \rangle_{G^F} \neq 0$.*

Many important class functions such as the trivial character and the character of the regular representation, are linear combinations of Deligne–Lusztig characters (see [15, 12. Proposition, 12.4 Corollary]).

Definition 1.66. A class function on G^F is called **uniform** if it can be written as a linear combination of Deligne–Lusztig characters.

Proposition 1.67 ([15, 12.12 Proposition]). *The orthogonal projection onto the subspace of uniform class functions is given by*

$$\pi_{uni} = \frac{1}{|G^F|} \sum_{T \in \mathcal{T}} |T^F| R_T^G \circ {}^*R_T^G,$$

where \mathcal{T} denotes the set of F -stable maximal tori of G .

By Lusztig’s work on Deligne–Lusztig characters and their decomposition into irreducible constituents in [32], we are easily able to compute the projection of any character onto the subspace of uniform class functions of every character using the formula above.

Next we want to establish a decomposition of $\text{Irr}(G^F)$ using Deligne–Lusztig characters. To be able to parametrise the irreducible characters nicely, we have to introduce dual groups.

Definition 1.68. Let G be a connected reductive group and let T be a maximal torus of G . If $(X(T), \Phi, Y(T), \Phi^\vee)$ is the root datum of G , we know that $(Y(T), \Phi^\vee, X(T), \Phi)$ is also a root datum. By Theorem 1.24, we know that there is a connected reductive group G^* together with a maximal torus T^* such that its root datum $(X(T^*), \Phi^*, Y(T^*), \Phi^{*\vee})$ is isomorphic to $(Y(T), \Phi^\vee, X(T), \Phi)$. We call G^* the **dual group** of G .

Let G and G^* be defined over \mathbb{F}_q with corresponding Frobenius endomorphisms F and F^* respectively. If T and T^* are F -stable and F^* -stable respectively and the isomorphism δ from $X(T)$ to $Y(T^*)$ given by the isomorphism of root data above is compatible with the action of F and F^* (i.e. $\delta(F(\chi)) = F^*(\delta(\chi))$ for all $\chi \in X(T)$), then we say that the pair (G^*, F^*) is **dual** to the pair (G, F) .

Remark 1.69. By the definition of duality we see that the dual of a semisimple adjoint group is simply-connected and vice versa.

Let $\nabla(G, F)$ denote the set of pairs (T, θ) where T is an F -stable maximal torus of G and $\theta \in \text{Irr}(T^F)$. Moreover, let $\nabla^*(G, F)$ denote the set of pairs (T^*, s) , where T^* is an F^* -stable maximal torus of G^* and $s \in T^{*F^*}$.

Proposition 1.70 ([15, 13.13 Proposition]). *Let (G, F) and (G^*, F^*) be dual pairs. There is a one-to-one correspondence*

$$\nabla(G, F)/G^F \leftrightarrow \nabla^*(G, F)/G^{*F^*}.$$

If (T, θ) corresponds to (T^*, s) via the bijection above, we write $R_{T^*}^G(s)$ instead of $R_T^G(\theta)$. Furthermore, we define $\nabla^*(G, F, s)$ to be the subset of $\nabla^*(G, F)$ consisting of the elements (T^*, s') such that s' and s are G^{*F^*} -conjugate.

As is common practice, we write F instead of F^* for the Frobenius endomorphism of G^* from now on.

Definition 1.71. Let $s \in G^{*F}$ be a semisimple element. We define the **(rational) Lusztig series** associated to s to be the set

$$\mathcal{E}(G^F, s) := \{\chi \in \text{Irr}(G^F) \mid \langle \chi, R_{T^*}^G(s') \rangle \neq 0 \text{ for some } (T^*, s') \in \nabla^*(G, F, s)\}.$$

Remark 1.72. There is also a notion of *geometric Lusztig series* associated to s , of which the rational Lusztig series associated to s is a subset.

Theorem 1.73 ([15, 14.41 Proposition]). *The set of irreducible characters of G^F is partitioned as follows.*

$$\text{Irr}(G^F) = \dot{\bigcup}_{[s]} \mathcal{E}(G^F, s),$$

where $[s]$ runs over all G^{*F} -conjugacy classes of semisimple elements of G^{*F} . In particular, $\mathcal{E}(G^F, s) = \mathcal{E}(G^F, s')$ if and only if s and s' are G^{*F} -conjugate

The Lusztig series corresponding to $s = 1$ is of special importance. We call the elements of $\mathcal{E}(G^F, 1)$ the **unipotent characters** of G^F . Lusztig used the unipotent characters of certain subgroups to parametrize $\text{Irr}(G^F)$. This parametrization uses the classification of the unipotent characters of connected reductive groups finished by Lusztig in [32] and another one of Lusztig's results which we introduce now.

Let \tilde{G} be a connected reductive group and let $\iota : G \rightarrow \tilde{G}$ be a homomorphism. We say ι is a **regular embedding** if \tilde{G} has connected centre, ι restricts to an isomorphism of G with a closed subgroup of \tilde{G} and $[\iota(G), \iota(G)] = [\tilde{G}, \tilde{G}]$ (see [30]). In this case we identify G with its image $\iota(G)$.

Let \tilde{G} be defined over \mathbb{F}_q with Frobenius endomorphism $\tilde{F} : \tilde{G} \rightarrow \tilde{G}$ and let $\iota : G \rightarrow \tilde{G}$ be a regular embedding compatible with \tilde{F} and F (i.e. $\iota(F(g)) = \tilde{F}(\iota(g))$ for every $g \in G$). We observe that G^F is normal in \tilde{G}^F . Hence \tilde{G}^F acts on $\text{Irr}(G^F)$ by conjugation. Since the subgroup $G^F Z(\tilde{G})^F$ acts trivial, we may consider the initial action to be an action of the quotient $\tilde{G}^F / G^F Z(\tilde{G})^F$. It can be shown that this action leaves Lusztig series invariant. Thus $\tilde{G}^F / G^F Z(\tilde{G})^F$ acts on $\mathcal{E}(G^F, s)$ for every semisimple $s \in G^F$ (see [11, Proposition 15.6]).

Let $A_{G^*}(x) := (C_{G^*}(x) / C_{G^*}^\circ(x))^F$ for $x \in G^*$. The group $A_{G^*}(s)$ acts on $\text{Irr}(C_{G^*}^\circ(s)^F)$ by conjugation. It can be shown that this action restricts to an action on $\mathcal{E}(C_{G^*}^\circ(s)^F, 1)$.

Theorem 1.74 (Jordan decomposition, [11, Corollary 15.14]). *Let $s \in G^{*F}$ be a semisimple element. There exists a bijective map between the sets of orbits*

$$\Psi : \mathcal{E}(G^F, s) / (\tilde{G}^F / G^F Z(\tilde{G})^F) \rightarrow \mathcal{E}(C_{G^*}^\circ(s)^F, 1) / A_{G^*}(s)$$

with the following properties.

- (i) If $\Omega \mapsto \Theta$, then the number of elements in the orbit Ω equals the order of the stabilizer in $A_{G^*}(s)$ of any $\lambda \in \Theta$.
- (ii) If $\rho \in \Psi^{-1}(\Theta)$ and T^* is an F -stable maximal torus of G^* containing s , then

$$\langle \rho, R_{T^*}^G(s) \rangle_{G^F} = \varepsilon_G \varepsilon_{C_{G^*}^\circ(s)} \sum_{\bar{\rho} \in \Theta} \langle \bar{\rho}, R_{T^*}^{C_{G^*}^\circ(s)}(1) \rangle_{C_{G^*}^\circ(s)^F},$$

where $\varepsilon_G = (-1)^{\mathbb{F}_q\text{-rank}(G)}$ and $\varepsilon_{C_{G^*}^\circ(s)} = (-1)^{\mathbb{F}_q\text{-rank}(C_{G^*}^\circ(s))}$ (see [15, 8.3 Definition] for a definition of \mathbb{F}_q -rank).

The map Ψ above is often just called the **Jordan decomposition** (of characters) as it parametrizes irreducible characters of G^F by pairs consisting of a *semisimple* element and an orbit of *unipotent* characters of its centraliser.

Remark 1.75. (a) Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Suppose that every simple factor of $[G, G]$ is of type A . Then the number of unipotent characters of G^F coincides with the number of G^F -conjugacy classes of maximal tori. By Corollary 1.64 we know that the set $\{R_T^G(1) \mid T \in \nabla(G, F)/G^F\}$ consists of linearly independent class functions. Hence it is a basis of $\overline{\mathbb{Q}}_r \mathcal{E}(G^F, 1)$. In short, every unipotent character of G^F is uniform.

(b) Let $s \in G^{*F}$ be semisimple with connected centraliser. Then the map Ψ in Theorem 1.74 is a bijection

$$\Psi : \mathcal{E}(G^F, s) \rightarrow \mathcal{E}(C_{G^*}(s)^F, 1).$$

In particular, Theorem 1.74 (ii) implies that Ψ induces a one-to-one correspondence between the uniform irreducible characters on both sides. In light of the remark above about groups of type A , it follows that every character of $\mathcal{E}(G^F, s)$ is uniform if the centraliser $C_{G^*}(s)$ is connected and of type A .

Definition 1.76. Let $s \in G^{*F}$ be a semisimple element (such that $C_{G^*}(s)^F/C_{G^*}^\circ(s)^F \neq 1$). An irreducible character of $C_{G^*}(s)^F$ is said to be **unipotent** if every constituent of its restriction to $C_{G^*}^\circ(s)^F$ is a unipotent character of $C_{G^*}^\circ(s)^F$. The set of unipotent characters of $C_{G^*}(s)^F$ is denoted by $\mathcal{E}(C_{G^*}(s)^F, 1)$.

By Clifford theory, the bijection of orbits in Theorem 1.74 therefore becomes a bijection

$$\Psi : \mathcal{E}(G^F, s) \rightarrow \mathcal{E}(C_{G^*}(s)^F, 1)$$

of characters.

1.9 Block theory of finite groups of Lie type

Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. From now we assume that ℓ does not divide q . This is often referred to as working in *non-defining characteristic* or *cross-characteristic*.

There is a very strong and rather surprising connection between Lusztig series on the geometric side and block theory on the algebraic side.

Theorem 1.77 ([7, 2.2 Théorème], [24, Theorem 3.1]). *Let $s \in G^{*F}$ be a semisimple ℓ' -element. Then we have the following.*

- (a) *The set $\mathcal{E}_\ell(G^F, s) := \bigcup_{t \in C_{G^*}(s)_\ell^F} \mathcal{E}(G^F, st)$ is a union of ℓ -blocks.*
- (b) *Any ℓ -block contained in $\mathcal{E}_\ell(G^F, s)$ contains a character of $\mathcal{E}(G^F, s)$.*

We can say even more about $\mathcal{E}_\ell(G^F, s)$. Let χ be an ordinary irreducible character of G^F . Recall that we denote the restriction of χ to ℓ -regular elements by χ° . We define $\hat{\mathcal{E}}(G^F, s) := \{\chi^\circ \mid \chi \in \mathcal{E}(G^F, s)\}$.

Theorem 1.78 ([18, Theorem A]). *Assume that ℓ is a good prime for G not dividing the order of $(Z(G)/Z^\circ(G))_F$ (the largest quotient of $Z(G)$ on which F acts trivially). Let $s \in G^{*F}$ be a semisimple element of order prime to ℓ . Then $\mathcal{E}(G^F, s)$ is an ordinary basic set for the union of blocks $\mathcal{E}_\ell(G^F, s)$.*

Let B be an ℓ -block contained in $\mathcal{E}_\ell(G^F, s)$ for some semisimple ℓ' -element $s \in G^{*F}$. It follows that a basic set for B is then given by $\text{Irr}(B) \cap \mathcal{E}(G^F, s)$. In particular, we have $l(B) = |\text{Irr}(B) \cap \mathcal{E}(G^F, s)|$. Fortunately, the blocks are parametrised in a way that gives us an idea of what $\text{Irr}(B) \cap \mathcal{E}(G^F, s)$ looks like. This leads us to the study of e -cuspidal pairs (see Sections 2 and 3).

2 Quasi-isolated blocks for good primes

In this section we prove Theorem A. Let G be a simple, simply connected algebraic group of exceptional Lie type defined over \mathbb{F}_q with a Frobenius endomorphism $F : G \rightarrow G$. Let ℓ be a good prime for G not dividing q .

Recall that an element s of a connected reductive group G is called **quasi-isolated** if $C_G(s)$ is not contained in any proper Levi subgroup $L \subsetneq G$. If even $C_G^\circ(s)$ is not contained in any proper Levi subgroup $L \subsetneq G$ then s is called **isolated**. For the reader's convenience we recall the classification of the quasi-isolated elements here (see [5, Proposition 4.3 and Table 3]).

Proposition 2.1 (Bonnafé). *Let G be a simple, exceptional algebraic group of adjoint type. Then the conjugacy classes of non-trivial semisimple, quasi-isolated elements, their orders, the root system of $C_G^\circ(s)$, and the group of components $A(s) := C_G(s)/C_G^\circ(s)$ are as given in Table 1.*

The order of s is denoted by $o(s)$.

Table 1: Quasi-isolated elements in exceptional groups

G	$o(s)$	$C_G^\circ(s)$	$A(s)$	isolated?
G_2	2	$A_1 \times A_1$	1	yes
	3	A_2	1	yes
F_4	2	$C_3 \times A_1, B_4$	1	yes
	3	$A_2 \times A_2$	1	yes

E_6	4	$A_3 \times A_1$	1	yes
	2	$A_5 \times A_1$	1	yes
	3	$A_2 \times A_2 \times A_2$	3	yes
	3	D_4	3	no
E_7	6	$A_1 \times A_1 \times A_1 \times A_1$	3	no
	2	$D_6 \times A_1$	1	yes
	2	A_7	2	yes
	2	E_6	2	no
E_8	3	$A_5 \times A_2$	1	yes
	4	$A_3 \times A_3 \times A_1$	2	yes
	4	$D_4 \times A_1 \times A_1$	2	no
	6	$A_2 \times A_2 \times A_2$	2	no
	2	$D_8, E_7 \times A_1$	1	yes
	3	$A_8, E_6 \times A_2$	1	yes
	4	$D_5 \times A_3, A_7 \times A_1$	1	yes
	5	$A_4 \times A_4$	1	yes
	6	$A_5 \times A_2 \times A_1$	1	yes

Definition 2.2. (a) The ℓ -blocks contained in $\mathcal{E}_\ell(G^F, s)$ for a semisimple, quasi-isolated ℓ -element $s \in G^{*F}$ are called **quasi-isolated**.

(b) Let $H = G^F/Z$, for some subgroup $Z \subseteq Z(G^F)$. A block of H is said to be **quasi-isolated** if it is dominated by a quasi-isolated block of G^F . Furthermore, it is said to be **unipotent** if the block dominating it is unipotent.

The reason we focus on these blocks are the results of Bonnafé–Rouquier [4] and more recently Bonnafé–Dat–Rouquier [2]. We use their reduction to quasi-isolated blocks to prove Theorem C later.

In order to parametrize the blocks of finite groups of Lie type we introduce the following generalization of *cuspidality*.

We say an irreducible character χ of G^F is **e -cuspidal** if $*R_{L \subseteq P}^G(\chi) = 0$ for every e -split Levi subgroup L contained in a proper parabolic subgroup $P \subsetneq G$. Let $\lambda \in \text{Irr}(L^F)$ for an e -split Levi subgroup $L \subseteq G$. Then we call (L, λ) an **e -split pair**. We define a binary relation on e -split pairs by setting $(M, \zeta) \leq_e (L, \lambda)$ if $M \subseteq L$ and $\langle *R_{M \subseteq Q}^L(\lambda), \zeta \rangle \neq 0$. Since the Lusztig restriction of a character is in general not a character, but a generalized character, the relation \leq_e might not be transitive. We denote the transitive closure of \leq_e by \ll_e . If (L, λ) is minimal for the partial order \ll_e , we call (L, λ) an **e -cuspidal pair** of G^F . Moreover, we say (L, λ) is a **proper e -cuspidal pair** if $L \subsetneq G$ is a proper F -stable Levi subgroup of G .

These e -cuspidal pairs yield a refinement of Theorem 1.77. Let $e_\ell(q)$ denote the multiplicative order of q modulo ℓ .

Theorem 2.3 ([10, Theorem 4.1]). *Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let ℓ be a good prime for G not dividing q .*

Furthermore assume that ℓ is different from 3 if G^F has a component of type ${}^3D_4(q)$. Let $s \in G^{*F}$ be a semisimple ℓ' -element. If $e = e_\ell(q)$, then we have the following.

1. There is a natural bijection

$$b_{G^F}(L, \lambda) \longleftrightarrow (L, \lambda)$$

between the ℓ -blocks of G^F contained in $\mathcal{E}_\ell(G^F, s)$ and the e -cuspidal pairs (L, λ) , up to G^F -conjugation, such that $s \in L^{*F}$ and $\lambda \in \mathcal{E}(L^F, s)$, where $b_{G^F}(L, \lambda)$ is the unique block containing the irreducible constituents of $R_L^G(\lambda)$.

2. If $B = b_{G^F}(L, \lambda)$, then $\text{Irr}(B) \cap \mathcal{E}(G^F, s) = \{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi)\}$.

Next we will introduce a notion that is meant to generalize standard Harish-Chandra theory.

For a pair (L, λ) of G^F we set $N_G^F(L, \lambda) := \{g \in N_{G^F}(L) \mid \lambda(gxg^{-1}) = \lambda(x) \text{ for all } x \in L^F\}$. The following definition can be found in [16, 2.2.1 Definition]. Let $s \in G^{*F}$ be semisimple. We say that **generalized e -Harish-Chandra theory** holds in $\mathcal{E}(G^F, s)$ if, for any $\chi \in \mathcal{E}(G^F, s)$ there exists an e -cuspidal pair (L, λ) of G^F , uniquely defined up to G^F -conjugacy, and an integer $a \neq 0$ such that

$${}^*R_{L \subseteq P}^G \chi = a \left(\sum_{g \in N_{G^F}(L)/N_{G^F}(L, \lambda)} \lambda^g \right)$$

for every parabolic subgroup $P \subseteq G$ containing L .

In this Section we will work under the following core assumption.

Assumption 2.4. *G is connected reductive, defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$, $\ell \nmid q$ is odd and good for G and $e = e_\ell(q)$. If G^F has a component of type 3D_4 then $\ell \geq 5$. Furthermore, let $s \in G^{*F}$ be a semisimple ℓ' -element.*

The following proposition shows how the notion of *generalized e -Harish-Chandra theory holding in a Lusztig series* is related to the parametrisation of blocks by the e -cuspidal pairs. Let $\mathcal{E}(G^F, (L, \lambda)) := \{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \leq_e (G, \chi)\}$ be the **e -Harish-Chandra series** associated to (L, λ) .

Note that, even though the statements about e -cuspidal pairs and e -Harish-Chandra theory in this section seem like they do not depend on ℓ , the proofs of these statements heavily rely on ℓ satisfying the conditions in Assumption 2.4 as can be seen in the proofs of the results cited in this section.

Proposition 2.5. [17, Proposition 2.2.2] *Suppose Assumption 2.4 holds. Then generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ if and only if, for any e -cuspidal pair (L, λ) of G^F with $\lambda \in \mathcal{E}(L^F, s)$, we have*

$$\mathcal{E}(G^F, (L, \lambda)) = \{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi)\}.$$

Corollary 2.6. *Suppose Assumption 2.4 holds. Then generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ if and only if*

$$\mathcal{E}(G^F, s) = \dot{\bigcup}_{(L, \lambda)/G^F} \mathcal{E}(G^F, (L, \lambda)),$$

where (L, λ) runs over the G^F -conjugacy classes of e -cuspidal pairs of G with $s \in L^{*F}$ and $\lambda \in \mathcal{E}(L^F, s)$.

Proof. By Theorem 2.3 we have

$$\begin{aligned} \mathcal{E}(G^F, s) &= \dot{\bigcup}_{(L, \lambda)/G^F} (\text{Irr}(b_{G^F}(L, \lambda)) \cap \mathcal{E}(G^F, s)) \\ &= \dot{\bigcup}_{(L, \lambda)/G^F} \{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi)\}, \end{aligned}$$

where (L, λ) runs over the G^F -conjugacy classes of e -cuspidal pairs of G with $s \in L^{*F}$ and $\lambda \in \mathcal{E}(L^F, s)$. Since $\mathcal{E}(G^F, (L, \lambda))$ is contained in $\{\chi \in \text{Irr}(G^F) \mid (L, \lambda) \ll_e (G, \chi)\}$ by definition, the assertion follows from Proposition 2.5. \square

The following result by Enguehard yields the assertion of Theorem A for the groups of type F_4 and the groups of type E_8 as long as we assume $q > 2$.

Theorem 2.7 ([17, 2.2.4 Proposition]). *Suppose that Assumption 2.4 holds. In addition suppose that the centre of G is connected and that the Mackey formula holds for every L^F where L is an F -stable Levi subgroup of G . Then generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$.*

By results of Kessar–Malle [29], the assertion of Theorem A also holds when $e = 1$ or $e = 2$ (unless $G = E_6$ or E_7 and s is semisimple, quasi-isolated of order 6). Thus, we can focus our attention on the situation where $G^F = E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(2)$ and $e \geq 3$. The proof of Theorem A follows a case-by-case approach. However, since we need to tweak our argument slightly when $q = 2$, we put that part of the proof at the end of Section 3.

For the computation of the e -cuspidal pairs we need the following results.

Theorem 2.8 ([10, Theorem 4.2.]). *Suppose that Assumption 2.4 holds. Then an element $\chi \in \mathcal{E}(G^F, s)$ is e -cuspidal if and only if it satisfies the following conditions.*

1. $Z^\circ(C_{G^*}^\circ(s))_{\Phi_e} = Z^\circ(G^*)_{\Phi_e}$ and
2. χ corresponds to a $C_{G^*}(s)^F$ -orbit of an e -cuspidal unipotent character of $C_{G^*}^\circ(s)^F$ by Jordan decomposition (see Theorem [11, Corollary 15.14]).

Using this result we can show that the assertion of Theorem A is immediate for certain numbers $e \in \mathbb{N}$.

Definition 2.9. For a semisimple element $s \in G^{*F}$ we define $\delta(G^F, s) := \{e \in \mathbb{N} \mid \exists \text{ a proper } e\text{-cuspidal pair } (L, \lambda) \text{ of } G^F \text{ with } \lambda \in \mathcal{E}(L^F, s)\}$. We say an integer e is **relevant** for a semisimple element $s \in G^{*F}$ if it occurs in $\delta(G^F, s)$.

The following easy conclusion justifies this terminology.

Proposition 2.10. *Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $s \in G^{*F}$ be semisimple. If e is not relevant for s , then an e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$.*

Proof. If e is not relevant for $\mathcal{E}(G^F, s)$, then, by definition, every character in $\mathcal{E}(G^F, s)$ is e -cuspidal. In other words, if (L, λ) is an e -cuspidal pair of G with $\lambda \in \mathcal{E}(L^F, s)$, then $L = G$. Since, clearly

$$\mathcal{E}(G^F, s) = \bigcup_{\chi \in \mathcal{E}(G^F, s)} \mathcal{E}(G^F, (G, \chi))$$

the assertion follows by Corollary 2.6. \square

Next we will show that we can determine the relevant integers for the Lusztig series associated to quasi-isolated elements by using only unipotent data.

Proposition 2.11. *Let G be a connected reductive group, $s \in G$ a semisimple element and $L \subseteq G$ a Levi subgroup of G containing s . Then $L \cap C_G^\circ(s)$ is a Levi subgroup of $C_G^\circ(s)$ and every Levi subgroup of $C_G^\circ(s)$ is of that form. In particular, $L \cap C_G^\circ(s) \subseteq C_G^\circ(s)$ is e -split if and only if $L \subseteq G$ is e -split.*

Proof. As a semisimple element, s lies in at least one maximal torus S of L , which then is a maximal torus of G . Now, $Z(L)$ lies in every maximal torus of L . In particular, $Z(L)$ lies in S . In other words, we have $Z(L) \subseteq S \subseteq C_G^\circ(s)$. As $L = C_G(Z^\circ(L))$ (see [15, 1.21 Proposition]), we have $L \cap C_G^\circ(s) = C_{C_G^\circ(s)}(Z^\circ(L))$. Since $Z^\circ(L)$ is a torus of $C_G^\circ(s)$, $C_{C_G^\circ(s)}(Z^\circ(L))$ is a Levi subgroup of $C_G^\circ(s)$, proving the first part.

Let M be a Levi subgroup of $C_G^\circ(s)$. Then $M = C_{C_G^\circ(s)}(Z^\circ(M))$. Now, $L = C_G(Z^\circ(M))$ is a Levi subgroup such that $M = L \cap C_G^\circ(s)$. The second part follows from Proposition 1.55. \square

Proposition 2.12. *Suppose that Assumption 2.4 holds with $s \in G^{*F}$ a semisimple, quasi-isolated ℓ' -element. Then $\delta(G^F, s) = \delta(C_{G^*}^\circ(s)^F, 1)$.*

Proof. Let (L, λ) be a proper e -cuspidal pair of G with $\lambda \in \mathcal{E}(L^F, s)$. Let L^* denote the dual of L in G^* . To prove the assertion we show that Jordan decomposition yields a $C_{L^*}(s)^F$ -orbit of proper unipotent e -cuspidal pairs.

By [9, Proposition 1.4], L^* is e -split. Hence $L^* = C_{G^*}(Z^\circ(L^*)_{\Phi_e})$ by Proposition 1.55. Since $s \in G^*$ is quasi-isolated, we know that $C_{G^*}^\circ(s) \not\subseteq L^*$. It follows that $C_{L^*}^\circ(s) = L^* \cap C_{G^*}^\circ(s) \subsetneq C_{G^*}^\circ(s)$ is a proper subgroup of $C_{G^*}^\circ(s)$. Furthermore, by Proposition 2.11, $C_{L^*}^\circ(s)$ is an e -split Levi subgroup of $C_{G^*}^\circ(s)$. Moreover, λ corresponds to a $C_{L^*}(s)^F$ -orbit of e -cuspidal unipotent characters of $C_{L^*}^\circ(s)^F$ by condition (ii) of Theorem 2.8. Hence, $\delta(G^F, s) \subseteq \delta(C_{G^*}^\circ(s)^F, 1)$.

Conversely, let (M, χ) be a proper e -cuspidal pair of $C_{G^*}^\circ(s)$ with $\chi \in \mathcal{E}(C_{G^*}^\circ(s), 1)$. By Proposition 2.11, there is a proper e -split Levi $L^* \subseteq G^*$ such that $L \cap C_{G^*}^\circ(s) = M$. If λ is the character in $\mathcal{E}(L^F, s)$ mapped to χ by Jordan decomposition, then (L, λ) is an e -cuspidal pair by Theorem 2.8. \square

2.1 The tables

Let G be a simple, simply connected algebraic group of exceptional type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or let G be simple, simply connected of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$.

The tables in this section are the key ingredient in the proof of Theorem A. The layout of the Tables 2, 4, 6, 8 and 10 is based on the layout of the tables in [29].

Note that we do not include tables for every relevant integer e . However, the missing tables are Ennola duals of the ones in this section and they can be obtained fairly easily. This follows from the fact that Ennola duality of finite groups of Lie type interacts nicely with Lusztig induction and restriction (see [6] and especially [6, 3.3 Theorem]). The Ennola dual cases are $e = 1 \leftrightarrow e = 2$, $e = 3 \leftrightarrow e = 6$, $e = 5 \leftrightarrow e = 10$, $e = 7 \leftrightarrow e = 14$, $e = 9 \leftrightarrow e = 18$, $e = 15 \leftrightarrow e = 30$.

In addition to the Tables 2, 4, 6, 8 and 10, Tables 3, 5, 7 and 9 contain the decomposition of $R_L^G(\lambda)$ into its irreducible constituents for every e -cuspidal pair (L, λ) for which $R_L^G(\lambda)$ is not uniform. These constituents are parametrized via Jordan decomposition (see [11, Corollary 15.14]). Since the semisimple element will always be clear from the context, we omit it from the parametrization and denote every irreducible constituent by the corresponding unipotent character. Except for the unipotent characters of classical groups (where we use the common notation using partitions and symbols), we use the notation of Chevie [36].

Remark 2.13. The e -cuspidal pairs of G^F for $e = 1 \leftrightarrow e = 2$ were already determined by Kessar and Malle in [29] and [28] except for the pairs associated to quasi-isolated elements of order 6 when $G^F = E_6(q)$, $E_7(q)$.

2.1.1 e -cuspidal pairs of F_4

Let G be simple, simply connected of type F_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. In this case, e is relevant for some quasi-isolated semisimple $1 \neq s \in G^{*F}$ if and only if $e \in \{1, 2, 3, 4, 6\}$. By Remark 2.13 and Ennola duality, it remains to determine the e -cuspidal pairs for $e = 3$ and $e = 4$.

Theorem 2.14. *Let $1 \neq s \in G^{*F}$ be semisimple and quasi-isolated. Let $e = e_\ell(q) \in \{3, 4\}$ be relevant for s . Then the e -cuspidal pairs (L, λ) of G with $\lambda \in \mathcal{E}(L^F, s)$ (up to G^F -conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 2. In particular, generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ for every quasi-isolated semisimple element $1 \neq s \in G^{*F}$.*

Table 2: Quasi-isolated blocks in $F_4(q)$

No.	$C_{G^*}(s)^F$	e	L^F	$C_{L^*}(s)^F$	λ	$ W $
1	$A_2(q)A_2(q)$	3	Φ_3^2	Φ_3^2	1	9
2	$B_4(q)$	3	$\Phi_3 \cdot \tilde{A}_2(q)$	$\Phi_1 \Phi_3 \cdot \tilde{A}_1(q)$	ϕ_{11}, ϕ_2	6
3		3	G^F	$C_{G^*}(s)^F$	13 chars.	1

4	$C_3(q)A_1(q)$	3	$\Phi_3.A_2(q)$	$\Phi_1\Phi_3.A_1(q)$	ϕ_{11}, ϕ_2	6
5		3	G^F	$C_{G^*}(s)^F$	12 chars.	1
6	$A_3(q)\tilde{A}_1(q)$	3	$\Phi_3.A_2(q)$	$\Phi_1\Phi_3.\tilde{A}_1(q)$	ϕ_{11}, ϕ_2	3
7		3	G^F	$C_{G^*}(s)^F$	4 chars.	1
1	$B_4(q)$	4	Φ_4^2	Φ_4^2	1	14
2		4	$\Phi_4.B_2(q)$	$\Phi_4.B_2(q)$	(12, 0), (01, 2)	4
3		4	G^F	$C_{G^*}(s)^F$	(13, 1), (013, 13), (014, 12)	1
4	$C_3(q)A_1(q)$	4	$\Phi_4.B_2(q)$	$\Phi_4.A_1(q)A_1(q)$	$\phi_{11} \otimes \phi_{11}, \phi_{11} \otimes \phi_2,$ $\phi_2 \otimes \phi_{11}, \phi_2 \otimes \phi_2$	4
5		4	G^F	$C_{G^*}(s)^F$	8 chars.	1
6	$A_3(q)A_1(q)$	4	$\Phi_4.B_2(q)$	$\tilde{\Phi}_2\Phi_4.A_1(q)$	ϕ_{11}, ϕ_2	4
7		4	G^F	$C_{G^*}(s)^F$	$\phi_{22} \otimes \phi_{11}, \phi_{22} \otimes \phi_2$	1
8	${}^2A_3(q)\tilde{A}_1(q)$	4	$\Phi_4.B_2(q)$	$\tilde{\Phi}_1\Phi_4.A_1(q)$	ϕ_{11}, ϕ_2	4
9		4	G^F	$C_{G^*}(s)^F$	$\phi_{22} \otimes \phi_{11}, \phi_{22} \otimes \phi_2$	1

Let π_{uni} denote the projection from the space of class functions onto the subspace of uniform functions (see [15, 12.11 Definition]). The image of a class function under π_{uni} can be explicitly computed using [15, 12.12 Proposition].

Proof. The e -cuspidal pairs can be determined with Chevie [36] using Theorem 2.8 and Proposition 2.11. To prove that generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ we use Corollary 2.6.

Let $e = 3$. By Table 2, we see that every 3-cuspidal pair (L, λ) is of the form (G, χ) , or L is a proper Levi subgroup of G of type A and λ is a uniform character. Since Lusztig induction is transitive (see Proposition 1.60) and λ is uniform, $R_L^G(\lambda)$ is uniform as well. Hence, we can determine the decomposition of $R_L^G(\lambda)$ using the formula for the uniform projection. For any semisimple, quasi-isolated element $1 \neq s \in G^{*F}$, we find that the constituents of $R_L^G(\lambda)$ for the 3-cuspidal pairs (L, λ) with $\lambda \in \mathcal{E}(L^F, s)$ given in Table 2 exhaust $\mathcal{E}(G^F, s)$. Thus, a generalized 3-Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$.

Let $e = 4$. Let (L, λ) be a 4-cuspidal pair in Table 2. Then λ is a uniform character, except for the two 4-cuspidal pairs in the line numbered 2. So the decomposition of $R_L^G(\lambda)$ can be determined using the formula for the uniform projection again, except for the two exceptions, for which we need to use a different method. For further reference we will explain this method in detail in the case $(L, \lambda) = (B_2, (12, 0))$. In this case $\pi_{uni}(R_L^G(\lambda)) = \frac{1}{4}[(1234, 012) - (123, 02) + (023, 12) - (0124, 123) + (0123, 124) - (23, 0) + (14, 0) - (02, 3) + (01, 4) + (023,) - (014,) + (0123, 2)] - \frac{3}{4}[(03, 2) - (012, 23) - (04, 1) - (01234, 12)] \in \frac{1}{4}\mathbb{Z}\mathcal{E}(G^F, s)$. Since $R_L^G(\lambda)$ is a generalized character, there exists an element $\gamma \in \mathbb{Q}\mathcal{E}(G^F, s)$ which is orthogonal to the space of uniform class functions of G^F , such that $R_L^G(\lambda) = \pi_{uni}(R_L^G(\lambda)) + \gamma \in \mathbb{Z}\mathcal{E}(G^F, s)$. A basis for the subspace of $\mathbb{Q}\mathcal{E}(G^F, s)$ orthogonal to the space of uniform class functions is given by

$$\begin{aligned}\varphi_1 &= \frac{1}{4}((1234, 012) - (0124, 123) + (0123, 124) - (01234, 12)), \\ \varphi_2 &= \frac{1}{4}((123, 02) - (023, 12) + (012, 23) - (0123, 2)),\end{aligned}$$

$$\begin{aligned}\varphi_3 &= \frac{1}{4} ((124, 01) - (014, 12) + (012, 14) - (0124, 1)), \\ \varphi_4 &= \frac{1}{4} ((23, 0) - (03, 2) + (02, 3) - (023,)), \\ \varphi_5 &= \frac{1}{4} ((14, 0) - (04, 1) + (01, 4) - (014,)). \end{aligned}$$

By the Mackey formula we know that $\|R_L^G(\lambda)\|^2 = |W_{G^F}(L, \lambda)| = 4$ and since $\|R_L^G(\lambda)\|^2 = \|\pi_{uni}(R_L^G(\lambda))\|^2 + \|\gamma\|^2$, it follows that $\gamma = -\varphi_1 + \varphi_2 + \varphi_4 - \varphi_5$. Hence, $R_L^G(\lambda) = -(03, 2) + (012, 23) + (04, 1) + (01234, 12)$. The same method yields the decomposition of $R_L^G(\lambda)$ for $(L, \lambda) = (B_2, (01, 2))$. With this we have established the decomposition for every 4-cuspidal pair in Table 3. We find that the constituents of $R_L^G(\lambda)$ for the 4-cuspidal pairs associated to a given semisimple, quasi-isolated element $s \in G^{*F}$ exhaust $\mathcal{E}(G^F, s)$. \square

Table 3: Decomposition of non-uniform $R_L^G(\lambda)$

No.	e	λ	$\pm R_L^G(\lambda)$
2	4	(12, 0)	$-(03, 2) + (012, 23) + (04, 1) + (01234, 12)$
		(01, 2)	$-(023, 12) + (0124, 123) + (23, 0) + (014, 0)$

2.1.2 e -cuspidal pairs of E_6

Let G be a simple, simply connected algebraic group of type E_6 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Then $G^F = E_{6,sc}(q)$ or ${}^2E_{6,sc}(q)$. We start with $G^F = E_{6,sc}(q)$. Here, e is relevant for some quasi-isolated $1 \neq s \in G^{*F}$ if and only if $e \in \{1, 2, 3, 4, 5, 6\}$. Since the center of G is disconnected, the situation is slightly more complicated.

In the tables, we write $C_{G^*}(s)^F = C_{G^*}^\circ(s)^F \cdot (C_{G^*}(s)^F / C_{G^*}^\circ(s)^F)$ to indicate whether or not a given centraliser is connected. A star in the first column next to the number of the line indicates that the quotient $C_{G^*}(s)^F / C_{G^*}^\circ(s)^F$ acts non-trivially on the unipotent characters of $C_{G^*}^\circ(s)^F$. To demonstrate the adjustments, we take line 4 of Table 4 for $e = 3$ as an example.

First, the star indicates that the F -stable points of the component group act non-trivially on the 14 unipotent character of $C_{G^*}^\circ(s)^F = \Phi_1^2 \cdot D_4(q)$. It can be shown that there are two orbits of order 3, and 8 trivial orbits. Thus, by Jordan decomposition, $|\mathcal{E}(G^F, s)| = 26$. Now, $C_{L^*}(s)^F / C_{L^*}^\circ(s)^F$ obviously acts trivially on the only unipotent character (which is the trivial character) of the torus $C_{L^*}^\circ(s)^F = \Phi_1^4 \Phi_3$. Hence the induction of that character to $C_{L^*}(s)^F$ yields 3 irreducible constituents. We denote them by $1^{(1)}, 1^{(2)}$ and $1^{(3)}$.

In general, if $C_{G^*}(s)^F / C_{G^*}^\circ(s)^F$ acts trivially on a given unipotent character of $C_{G^*}^\circ(s)^F$, the induction of that character always yields 3 irreducible characters of $C_{G^*}(s)^F$. In Table 4, we indicate this by adding a superscript from 1 to 3 to that unipotent character.

Theorem 2.15. *Let $1 \neq s \in G^{*F}$ be semisimple and quasi-isolated. Let $e = e_\ell(q) \in \{1, 2, 3, 4, 5, 6\}$ be relevant for s . Further, assume that $e \in \{3, 4, 5, 6\}$ if s is not of order*

6. Then the e -cuspidal pairs (L, λ) of G with $\lambda \in \mathcal{E}(L^F, s)$ (up to G^F -conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 4. In particular, generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ for every quasi-isolated semisimple element $1 \neq s \in G^{*F}$.

Table 4: Quasi-isolated blocks in $E_6(q)$

No.	$C_{G^*}(s)^F$	e	L^F	$C_{L^*}(s)^F$	λ	$ W $
1*	$\Phi_1^2.A_1(q)^4.3$	1	Φ_1^6	Φ_1^6	1	48
2	$\Phi_3.A_1(q)A_1(q^3).3$	1	$\Phi_1^2.A_2(q)^2$	$\Phi_1^2\Phi_3^2$	1	12
3	$\Phi_1\Phi_2.A_1(q)^2A_1(q^2)$	1	$\Phi_1^4.A_1(q)^2$	$\Phi_1^4\Phi_2^2$	1	8
1*	$\Phi_1^2.A_1(q)^4.3$	2	$\Phi_1^2\Phi_2^4$	$\Phi_1^2\Phi_2^4$	1	48
2	$\Phi_3.A_1(q)A_1(q^3).3$	2	$\Phi_2^2.A_2(q^2)$	$\Phi_2^2\Phi_3\Phi_6$	1	12
3	$\Phi_1\Phi_2.A_1(q)^2A_1(q^2)$	2	$\Phi_1^2\Phi_2^4$	$\Phi_1^2\Phi_2^4$	1	8
1*	$A_2(q)^3.3$	3	Φ_3^3	Φ_3^3	1	81
2	$A_2(q^3).3$	3	$\Phi_3^2.A_2(q)$	$\Phi_1^2\Phi_3^2$	1	18
3	${}^2A_2(q)A_2(q^2)$	3	$\Phi_3.{}^3D_4(q)$	$\Phi_3\Phi_6.{}^2A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	3
4*	$\Phi_1^2.D_4(q).3$	3	$\Phi_3.A_2(q)^2$	$\Phi_1^4\Phi_3.3$	$1^{(1)}, 1^{(2)}, 1^{(3)}$	6
5*		3	G^F	$C_{G^*}(s)^F$	8 chars.	1
6	$\Phi_3.{}^3D_4(q).3$	3	Φ_3^3	Φ_3^3	1	72
7		3	$\Phi_3.{}^3D_4(q)$	$\Phi_3.{}^3D_4(q)$	${}^3D_4[-1]$	3
8	$\Phi_1\Phi_2.{}^2D_4(q)$	3	$\Phi_3.A_2(q)^2$	$\Phi_1^2\Phi_2^2\Phi_3.3$	$1^{(1)}, 1^{(2)}, 1^{(3)}$	6
9		3	G^F	$C_{G^*}(s)^F$	4 chars.	1
10	$A_5(q)A_1(q)$	3	$\Phi_3^2.A_2(q)$	$\Phi_1.\Phi_3.A_1(q)$	ϕ_{11}, ϕ_2	18
11		3	G^F	$C_{G^*}(s)^F$	4 chars.	1
1	$A_5(q)A_1(q)$	4	$\Phi_1\Phi_4.{}^2A_3(q)$	$\Phi_1\Phi_2\Phi_4.A_1(q)^2$	4 chars.	4
2		4	G^F	$C_{G^*}(s)^F$	6 chars.	1
3*	$\Phi_1^2.D_4(q).3$	4	$\Phi_1^2\Phi_4^2$	$\Phi_1^2\Phi_4^2$	1	48
4*		4	G^F	$C_{G^*}(s)^F$	12 chars.	1
5	$\Phi_1\Phi_2.{}^2D_4(q)$	4	$\Phi_1\Phi_4.{}^2A_3(q)$	$\Phi_1\Phi_2\Phi_4.A_1(q)^2$	$\phi_{11} \otimes \phi_{11},$ $\phi_2 \otimes \phi_2$ $\phi_{11} \otimes \phi_2$	4 4 2
1	$A_5(q)A_1(q)$	5	$\Phi_1\Phi_5.A_1(q)$	$\Phi_1\Phi_5.A_1(q)$	ϕ_{11}, ϕ_2	5
2		5	G^F	$C_{G^*}(s)^F$	12 chars.	1
1	$A_5(q)A_1(q)$	6	$\Phi_3\Phi_6.{}^2A_2(q)$	$\Phi_2\Phi_3\Phi_6.A_1(q)$	ϕ_{11}, ϕ_2	6
2		6	G^F	$C_{G^*}(s)^F$	10 chars.	1
3*	$\Phi_1^2.D_4(q).3$	6	$\Phi_6.A_2(q^2)$	$\Phi_1^2\Phi_2^2\Phi_6.3$	$1^{(1)}, 1^{(2)}, 1^{(3)}$	6
4*		6	G^F	$C_{G^*}(s)^F$	8 chars.	1
5	$\Phi_3.{}^3D_4(q).3$	6	$\Phi_3\Phi_6^2$	$\Phi_3\Phi_6^2$	1	72
6		6	G^F	$C_{G^*}(s)^F$	$\phi_{2,1}$	1
7	$\Phi_1\Phi_2.{}^2D_4(q)$	6	$\Phi_6.A_2(q^2)$	$\Phi_1^2\Phi_2^2\Phi_6$	1	6
8		6	G^F	$C_{G^*}(s)^F$	4 chars.	1

Proof. For $q = 2$ the assertion will follow from Proposition 2.20. Suppose $q > 2$. As

for $F_4(q)$, the key step is to determine $R_L^G(\lambda)$ for the e -cuspidal pairs (L, λ) in Table 4. Except for the pairs given in Table 5, λ is uniform, so $R_L^G(\lambda)$ can be determined using the formula for the uniform projection. For the 3-cuspidal pairs $(\Phi_3.A_2(q)^2, 1^{(i)})$ ($i = 1, \dots, 3$) and the 6-cuspidal pairs $(\Phi_6.A_2(q^2), 1^{(i)})$ ($i = 1, \dots, 3$), we are not able to determine $R_L^G(\lambda)$ (see Remark 2.16). However the methods used in the proof of Theorem 2.14 give enough information to prove that an e -Harish Chandra theory holds in the Lusztig series related to the e -cuspidal pairs above. For the 3-cuspidal pair $(\Phi_3.^3D_4(q), ^3D_4[-1])$ we use a slightly different argument. Let $s \in G^{*F}$ be semisimple and quasi-isolated with $C_{G^*}(s)^F = \Phi_3.^3D_4(q).3$. By Table 4 and Theorem 2.8, $\mathcal{E}(G^F, s)$ decomposes into two blocks, namely $b_{G^F}(\Phi_3, 1)$, which contains $\mathcal{E}(G^F, (\Phi_3, 1))$ and $b_{G^F}(\Phi_3.^3D_4(q), ^3D_4[-1])$ which contains $\mathcal{E}(G^F, (\Phi_3.^3D_4(q), ^3D_4[-1]))$. Since any two different blocks, seen as subsets of $\text{Irr}(G^F) \cup \text{IBr}(G^F)$, are disjoint, we have

$$\mathcal{E}(G^F, (\Phi_3.^3D_4(q), ^3D_4[-1])) \subseteq \mathcal{E}(G^F, s) \setminus \mathcal{E}(G^F, (\Phi_3, 1))$$

and the latter is equal to $\{^3D_4[-1]^{(0)}, ^3D_4[-1]^{(1)}, ^3D_4[-1]^{(2)}\}$. Since $R_{\Phi_3.^3D_4(q)}^G(^3D_4[-1])$ has norm 3, it follows that $R_{\Phi_3.^3D_4(q)}^G(^3D_4[-1]) = ^3D_4[-1]^{(0)} + ^3D_4[-1]^{(1)} + ^3D_4[-1]^{(2)}$. Hence, an e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$. \square

Remark 2.16. The reason we are not able to determine $R_L^G(\lambda)$ in the cases numbered 4^* and 3^* in Table 5 is the following. Every constituent of $R_L^G(1^{(i)})$ in those lines is an element of an orbit of order 3. However, we are not able to determine which element of this orbit is the right constituent. We only know that it has to be one of the three. This is indicated by adding a superscript (i) to the constituents.

Table 5: Decomposition of the non-uniform $R_L^G(\lambda)$

No.	e	λ	$\pm R_L^G(\lambda)$
4^*	3	$1^{(i)}$	$(013, 123)^{(i)} + (0123, 1234)^{(i)} + (02, 13)^{(i)} + (01, 23)^{(i)} + (1, 3)^{(i)} + (0, 4)^{(i)}$
7	3	$^3D_4[-1]$	$^3D_4[-1]^{(0)} + ^3D_4[-1]^{(1)} + ^3D_4[-1]^{(2)}$
3^*	6	$1^{(i)}$	$(013, 123)^{(i)} + (0123, 1234)^{(i)} + (12, 03)^{(i)} + (1, 3)^{(i)} + (0, 4)^{(i)} + (0123,)^{(i)}$

The analogue of Table 4 for ${}^2E_6(q)$ can be obtained as follows. The $e = 3$ part of the table for ${}^2E_6(q)$ is the Ennola dual of the $e = 6$ part of Table 4 and vice-versa. The $e = 10$ part is the Ennola dual of the $e = 5$ part and the $e = 4$ part is the Ennola dual of the $e = 4$ part of Table 4. Similarly, the analogue of Table 5 for ${}^2E_6(q)$ can be obtained via Ennola duality. Thus, the assertion of Theorem 2.15 holds for ${}^2E_{6,sc}$ as well.

2.1.3 e -cuspidal pairs of E_7

Let G be a simple, simply connected group of type E_7 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. In this case, e is relevant for some quasi-isolated semisimple $1 \neq s \in G^{*F}$ if and only if $e \in \{1, 2, 3, 4, 5, 6, 7, 9, 12, 14, 18\}$. By Remark 2.13 and Ennola duality, it remains to determine the e -cuspidal pairs for $e = 3, 4, 5, 7, 9, 12$. Since the center of G is disconnected, we encounter the same issues as in Section 2.1.2.

Theorem 2.17. *Let $1 \neq s \in G^{*F}$ be semisimple and quasi-isolated. Let $e = e_\ell(q) \in \{1, 3, 4, 5, 7, 9, 12\}$ be relevant for s . Further, assume that $e \in \{3, 4, 5, 7, 9, 12\}$ if s is not of order 6. Then the e -cuspidal pairs (L, λ) of G with $\lambda \in \mathcal{E}(L^F, s)$ (up to G^F -conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 6. In particular, generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ for every quasi-isolated semisimple element $1 \neq s \in G^{*F}$.*

Table 6: Quasi-isolated blocks of $E_7(q)$

No.	$C_{G^*}(s)^F$	e	L^F	$C_{L^*}(s)^F$	λ	$ W $
1*	$\Phi_1.A_2(q)^3.2$	1	Φ_7	Φ_7	1	432
2	$\Phi_2.A_2(q)A_2(q^2).2$	1	$\Phi_1^4.A_1(q)^3$	$\Phi_1^4\Phi_2^3$	1	36
3	${}^2A_2(q)A_2(q^2).2$	1	$\Phi_1^4.A_1(q)^3$	$\Phi_1^4\Phi_2^3$	1	12
4		1	$\Phi_1^3.D_4(q)$	$\Phi_1^3\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	6
1	$A_7(q).2$	3	$\Phi_1\Phi_3^2.A_2(q)$	$\Phi_1^2\Phi_3^2.A_1(q)$	ϕ_{11}, ϕ_2	36
2		3	$\Phi_3.A_5(q)$	$\Phi_1\Phi_3.A_4(q)$	ϕ_{311}	6
3		3	G^F	$C_{G^*}(s)^F$	$\phi_{4211}^{(1,2)}$	1
4	${}^2A_7(q).2$	3	$\Phi_3.A_2(q)A_1(q^3)$	$\Phi_1\Phi_2\Phi_3\Phi_6.A_1(q).2$	$\phi_{11}^{(1,2)}, \phi_2^{(1,2)}$	6
5		3	G^F	$C_{G^*}(s)^F$	16 chars.	1
6	$\Phi_1.E_6(q).2$	3	$\Phi_1\Phi_3^3$	$\Phi_1\Phi_3^3$	1	1296
7		3	$\Phi_1\Phi_3.{}^3D_4(q)$	$\Phi_1\Phi_3.{}^3D_4(q)$	${}^3D_4[-1]$	6
8		3	G^F	$C_{G^*}(s)^F$	6 chars.	1
9	$\Phi_2.{}^2E_6(q).2$	3	$\Phi_3^2.A_1(q^3)$	$\Phi_2\Phi_3^2\Phi_6.2$	$1^{(1,2)}$	72
10		3	G^F	$C_{G^*}(s)^F$	18 chars.	1
11*	$A_3(q)^2A_1(q).2$	3	$\Phi_1\Phi_3^2.A_2(q)$	$\Phi_1^2\Phi_3^2.A_1(q)$	ϕ_{11}, ϕ_2	18
12*		3	$\Phi_1\Phi_3.A_5(q)$	$\Phi_1\Phi_3.A_1(q)A_3(q)$	4 chars.	3
13*		3	G^F	$C_{G^*}(s)^F$	10 chars.	1
14	$A_3(q^2)A_1(q).2$	3	$\Phi_1\Phi_3.{}^3D_4(q)$	$\Phi_1\Phi_2\Phi_3\Phi_6.A_1(q)$	ϕ_{11}, ϕ_2	6
15		3	G^F	$C_{G^*}(s)^F$	8 chars.	1
16*	$\Phi_1.A_2(q)^3.2$	3	$\Phi_1\Phi_3^3$	$\Phi_1\Phi_3^3$	1	54
17	$\Phi_2.A_2(q)A_2(q^2).2$	3	$\Phi_3^2.A_1(q^3)$	$\Phi_2\Phi_3^2\Phi_6$	1	18
18	$\Phi_1.{}^2A_2(q)A_2(q^2).2$	3	$\Phi_1\Phi_3.{}^3D_4(q)$	$\Phi_1\Phi_3\Phi_6.{}^2A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	6
19*	$\Phi_1.D_4(q)A_1(q)^2.2$	3	$\Phi_3.A_5(q)$	$\Phi_1^3\Phi_3.A_1(q)^2.2$	5 chars.	6
20*		3	G^F	$C_{G^*}(s)^F$	28 chars.	1
21	$\Phi_2.{}^2D_4(q)A_1(q^2).2$	3	$\Phi_3.A_5(q)$	$\Phi_1\Phi_2^2\Phi_3.A_1(q^2).2$	$\phi_{11}^{(1,2)}, \phi_2^{(1,2)}$	12
22		3	G^F	$C_{G^*}(s)^F$	8 chars.	1
23*	$\Phi_2.D_4(q)A_1(q)^2.2$	3	$\Phi_3.A_5(q)$	$\Phi_1^2\Phi_2\Phi_3A_1(q)^2.2$	5 chars.	6
24*		3	G^F	$C_{G^*}(s)^F$	28 chars.	1
25	$\Phi_1.{}^2D_4(q)A_1(q^2).2$	3	$\Phi_3.A_5(q)$	$\Phi_1^2\Phi_2\Phi_3.A_1(q^2).2$	$\phi_{11}^{(1,2)}, \phi_2^{(1,2)}$	12
26		3	G^F	$C_{G^*}(s)^F$	8 chars.	1
1	$A_7(q).2$	4	$\Phi_4^2.A_1(q)^3$	$\Phi_1\Phi_2^2\Phi_4^2.2$	$1^{(1,2)}$	32
2		4	$\Phi_4.{}^2D_4(q)A_1(q)$	$\Phi_2^2\Phi_4.A_3(q).2$	$\phi_{22}^{(1,2)}$	4
3		4	G^F	$C_{G^*}(s)^F$	4 chars.	1
4	${}^2A_7(q).2$	4	$\Phi_4^2.A_1(q)^3$	$\Phi_1^2\Phi_2\Phi_4^2.2$	$1^{(1,2)}$	32
5		4	$\Phi_4.{}^2D_4(q)A_1(q)$	$\Phi_1\Phi_2\Phi_4.{}^2A_3(q).2$	$\phi_{22}^{(1,2)}$	4
6		4	G^F	$C_{G^*}(s)^F$	4 chars.	1
7	$\Phi_1.E_6(q).2$	4	$\Phi_4^2.A_1(q)^3$	$\Phi_1^3\Phi_4^2.2$	$1^{(1,2)}$	96

8		4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_1^2\Phi_4 \cdot {}^2A_3(q) \cdot 2$	$\phi_{22}^{(1,2)}$	4
9		4	G^F	$C_{G^*}(s)^F$	20 chars.	1
10	$\Phi_2 \cdot {}^2E_6(q) \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_2^3\Phi_4^2 \cdot 2$	$1^{(1,2)}$	96
11		4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_2^2\Phi_4 \cdot A_3(q) \cdot 2$	$\phi_{22}^{(1,2)}$	4
12		4	G^F	$C_{G^*}(s)^F$	20 chars.	1
13*	$A_3(q)^2A_1(q) \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_2^2\Phi_4^2 \cdot A_1(q)$	ϕ_{11}, ϕ_2	32
14*		4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_2\Phi_4 \cdot A_3(q)A_1(q)$	$\phi_{22} \otimes \phi_{11},$ $\phi_{22} \otimes \phi_2$	4 4
15*		4	G^F	$C_{G^*}(s)^F$	4 chars.	1
16	$A_3(q^2)A_1(q) \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_1\Phi_2\Phi_4^2 \cdot A_1(q)$	ϕ_{11}, ϕ_2	16
17*	${}^2A_3(q)^2A_1(q) \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_1^2\Phi_4^2 \cdot A_1(q)$	ϕ_{11}, ϕ_2	32
18*		4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_1\Phi_4 \cdot {}^2A_3(q)A_1(q)$	$\phi_{22} \otimes \phi_{11},$ $\phi_{22} \otimes \phi_2$	4 4
19*		4	G^F	$C_{G^*}(s)^F$	4 chars.	1
20	$\Phi_2 \cdot A_2(q^2)A_2(q) \cdot 2$	4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_1\Phi_2^2\Phi_4 \cdot A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	4
21		4	G^F	$C_{G^*}(s)^F$	6 chars.	1
22	$\Phi_1 \cdot {}^2A_2(q)A_2(q^2) \cdot 2$	4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_1^2\Phi_2\Phi_4 \cdot {}^2A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	4
23		4	G^F	$C_{G^*}(s)^F$	6 chars.	1
24	$A_5(q)A_2(q)$	4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_1\Phi_2\Phi_4 \cdot A_2(q)A_1(q)$	6 chars.	4
25		4	G^F	$C_{G^*}(s)^F$	9 chars.	1
26	${}^2A_5(q)^2A_2(q)$	4	$\Phi_4 \cdot {}^2D_4(q)A_1(q)$	$\Phi_1\Phi_2\Phi_4 \cdot {}^2A_2(q)$	6 chars.	4
27		4	G^F	$C_{G^*}(s)^F$	9 chars.	1
28*	$\Phi_1 \cdot D_4(q)A_1(q)^2 \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_1\Phi_4^2 \cdot A_1(q)^2$	$\phi_{11} \otimes \phi_{11},$ $\phi_{11} \otimes \phi_2,$ $\phi_2 \otimes \phi_2$	32 32 32
29*		4	G^F	$C_{G^*}(s)^F$	18 chars.	1
30	$\Phi_2 \cdot {}^2D_4(q)A_1(q^2) \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_2\Phi_4^2 \cdot A_1(q)^2$	$\phi_{11} \otimes \phi_{11}$	16
		4			$\phi_2 \otimes \phi_2$	16
		4			$\phi_{11} \otimes \phi_2$	8
31*	$\Phi_2 \cdot D_4(q)A_1(q)^2 \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_2\Phi_4^2 \cdot A_1(q)^2$	$\phi_{11} \otimes \phi_{11},$ $\phi_{11} \otimes \phi_2,$ $\phi_2 \otimes \phi_2$	32 32 32
32*		4	G^F	$C_{G^*}(s)^F$	18 chars.	1
33	$\Phi_1 \cdot {}^2D_4(q)A_1(q^2) \cdot 2$	4	$\Phi_4^2 \cdot A_1(q)^3$	$\Phi_1\Phi_4^2 \cdot A_1(q)^2$	$\phi_{11} \otimes \phi_{11}$	16
		4			$\phi_2 \otimes \phi_2$	16
		4			$\phi_{11} \otimes \phi_2$	8
1	$A_7(q) \cdot 2$	5	$\Phi_1\Phi_5 \cdot A_2(q)$	$\Phi_1\Phi_5 \cdot A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	10
2		5	G^F	$C_{G^*}(s)^F$	14 chars.	1
3	$\Phi_1 \cdot E_6(q) \cdot 2$	5	$\Phi_1\Phi_5 \cdot A_2(q)$	$\Phi_1^2\Phi_5 \cdot A_1(q)$	ϕ_{11}, ϕ_2	10
4		5	G^F	$C_{G^*}(s)^F$	40 chars.	1
1	$A_7(q) \cdot 2$	7	$\Phi_1\Phi_7$	$\Phi_1\Phi_7$	1	14
2		7	G^F	$C_{G^*}(s)^F$	30 chars.	1
1	$\Phi_1 \cdot E_6(q) \cdot 2$	9	$\Phi_1\Phi_9$	$\Phi_1\Phi_9$	1	18
2		9	G^F	$C_{G^*}(s)^F$	42 chars.	1
1	$\Phi_1 \cdot E_6(q) \cdot 2$	12	$\Phi_{12} \cdot A_1(q^3)$	$\Phi_1\Phi_3\Phi_{12}$	1	24
2		12	G^F	$C_{G^*}(s)^F$	36 chars.	1
1	$\Phi_2 \cdot {}^2E_6(q) \cdot 2$	12	$\Phi_{12} \cdot A_1(q^3)$	$\Phi_2\Phi_6\Phi_{12}$	1	24

2		12	G^F	$C_{G^*}(s)^F$	36 chars.	1
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Proof. Similar to the proof of Theorem 2.15. \square

Table 7: Decomposition of the non-uniform $R_L^G(\lambda)$

No.	e	λ	$\pm R_L^G(\lambda)$
4	3	ϕ_{11}	$\phi_{18}^{(i)} + \phi_{2311}^{(i)} + \phi_{3221}^{(i)} + \phi_{422}^{(i)} + \phi_{62}^{(i)} + \phi_{71}^{(i)}$
	3	ϕ_2	$\phi_{216}^{(i)} + \phi_{2214}^{(i)} + \phi_{3311}^{(i)} + \phi_{431}^{(i)} + \phi_{53}^{(i)} + \phi_8^{(i)}$
7	3	${}^3D_4[-1]$	$D_4:3^{(0)} + D_4:3^{(1)} + D_4:111^{(0)} + D_4:111^{(1)} - D_4:21^{(0)} - D_4:21^{(1)}$
9	3	1	$\phi_{1,0}^{(i)} + \phi_{1,24}^{(i)} - \phi_{2,4}^{(i)} - \phi_{2,16}^{(i)} + \phi_{1,12}^{(i)} + \phi_{1,12}^{(i)} - 2\phi_{4,1}^{(i)} - 2\phi_{4,13} + 2\phi_{8,3}^{(i)} + 2\phi_{8,9}^{(i)} - \phi_{2,4}^{(i)} - \phi_{2,16}^{(i)} + \phi_{4,8}^{(i)} + 3 {}^2E_6[1]^{(i)} - 2\phi_{4,7}^{(i)} - 2\phi_{4,7}^{(i)} + 2\phi_{8,3}^{(i)} + 2\phi_{8,9}^{(i)} - 2\phi_{16,5}^{(i)} - 3 {}^2E_6[\theta]^{(i)} - 3 {}^2E_6[\theta^2]^{(i)}$
19*	3	$\phi_2 \otimes \phi_2$	$((013, 123) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((0123, 1234) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((02, 13) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((01, 23) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((1, 3) \otimes \phi_2 \otimes \phi_2)^{(i)} + ((0, 4) \otimes \phi_2 \otimes \phi_2)^{(i)}$
	3	$\phi_{11} \otimes \phi_2$	$((013, 123) \otimes \phi_{11} \otimes \phi_2)^{(i)} + ((0123, 1234) \otimes \phi_{11} \otimes \phi_2)^{(i)} + ((02, 13) \otimes \phi_{11} \otimes \phi_2)^{(i)} + ((01, 23) \otimes \phi_{11} \otimes \phi_2)^{(i)} + ((1, 3) \otimes \phi_{11} \otimes \phi_2)^{(i)} + ((0, 4) \otimes \phi_{11} \otimes \phi_2)^{(i)}$
	3	$\phi_{11} \otimes \phi_{11}$	$((013, 123) \otimes \phi_{11} \otimes \phi_{11})^{(i)} + ((0123, 1234) \otimes \phi_{11} \otimes \phi_{11})^{(i)} + ((02, 13) \otimes \phi_{11} \otimes \phi_{11})^{(i)} + ((01, 23) \otimes \phi_{11} \otimes \phi_{11})^{(i)} + ((1, 3) \otimes \phi_{11} \otimes \phi_{11})^{(i)} + ((0, 4) \otimes \phi_{11} \otimes \phi_{11})^{(i)}$
21	3	ϕ_{11}	$((123, 0) \otimes \phi_{11})^{(i)} + ((01234, 123) \otimes \phi_2)^{(i)} + ((13,) \otimes \phi_{11})^{(i)} + ((0123, 13) \otimes \phi_{11})^{(i)} + ((04,) \otimes \phi_{11})^{(i)} + ((012, 3) \otimes \phi_{11})^{(i)}$
	3	ϕ_2	$((123, 0) \otimes \phi_2)^{(i)} + ((01234, 123) \otimes \phi_2)^{(i)} + ((13,) \otimes \phi_2)^{(i)} + ((0123, 13) \otimes \phi_2)^{(i)} + ((04,) \otimes \phi_2)^{(i)} + ((012, 3) \otimes \phi_2)^{(i)}$
1	4	1	$\phi_{18}^{(i)} - \phi_{216}^{(i)} + 2\phi_{24}^{(i)} + \phi_{315}^{(i)} - 2\phi_{3221}^{(i)} + 2\phi_{332}^{(i)} - \phi_{414}^{(i)} + 2\phi_{4211}^{(i)} - 2\phi_{431}^{(i)} + 2\phi_{44}^{(i)} - \phi_{513}^{(i)} + \phi_{611}^{(i)} - \phi_{71}^{(i)} + \phi_8^{(i)}$
2	4	ϕ_{22}	$-\phi_{2214}^{(i)} + \phi_{2311}^{(i)} - \phi_{53}^{(i)} + \phi_{62}^{(i)}$
4	4	1	$\phi_{18}^{(i)} + \phi_{216}^{(i)} + 2\phi_{24}^{(i)} - \phi_{315}^{(i)} + 2\phi_{3221}^{(i)} - 2\phi_{332}^{(i)} - \phi_{414}^{(i)} - 2\phi_{4211}^{(i)} + 2\phi_{44}^{(i)} - \phi_{513}^{(i)} - \phi_{611}^{(i)} + \phi_{71}^{(i)} + \phi_8^{(i)}$
5	4	ϕ_{22}	$-\phi_{2214}^{(i)} - \phi_{2311}^{(i)} + \phi_{53}^{(i)} + \phi_{62}^{(i)}$
7	4	1	$\phi_{1,0}^{(i)} + \phi_{1,36}^{(i)} + 2\phi_{10,9}^{(i)} + 2\phi_{6,1}^{(i)} + 2\phi_{6,25}^{(i)} + \phi_{15,5}^{(i)} + \phi_{15,17}^{(i)} - 3\phi_{15,4}^{(i)} - 3\phi_{15,16}^{(i)} + 4\phi_{80,7}^{(i)} + 2\phi_{90,8}^{(i)} - 3\phi_{81,6}^{(i)} - 3\phi_{81,10}^{(i)} - 2 D_4:3 - 2 D_4:111 - 4 D_4:21$
8	4	ϕ_{22}	$\phi_{20,2}^{(i)} - \phi_{20,20}^{(i)} - \phi_{60,5}^{(i)} + \phi_{60,11}^{(i)}$
10	4	1	$\phi_{1,0}^{(i)} + \phi_{1,24}^{(i)} - 2\phi_{6,6}^{(i)} - \phi_{9,2}^{(i)} - \phi_{9,10}^{(i)} + 3\phi_{1,12}^{(i)} + 3\phi_{1,12}^{(i)} + 2\phi_{8,3}^{(i)} + 2\phi_{8,9}^{(i)} - 2\phi_{2,4}^{(i)} - 2\phi_{2,16}^{(i)} - 4 {}^2E_6[1]^{(i)} - 2\phi_{6,6}^{(i)} - 3\phi_{9,6}^{(i)} - 3\phi_{9,6}^{(i)} + 4\phi_{16,5}^{(i)}$
11	4	ϕ_{22}	$\phi_{4,1}^{(i)} - \phi_{4,13}^{(i)} - \phi_{4,7}^{(i)} + \phi_{4,7}^{(i)}$

2.1.4 e -cuspidal pairs of E_8

Let G be a simple, simply connected algebraic group of type E_8 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Here, e is relevant for some quasi-isolated $1 \neq s \in G^{*F}$ if and only if $e \in \{1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 18, 20\}$. By Remark 2.13 and Ennola duality, it remains to determine the e -cuspidal pairs for $e = 3, 4, 5, 7, 9, 12, 20$.

Theorem 2.18. *Let $1 \neq s \in G^{*F}$ be semisimple and quasi-isolated. Let $e \in \{3, 4, 5, 7, 9, 12, 20\}$ be relevant for s . Then the e -cuspidal pairs (L, λ) of G with $\lambda \in \mathcal{E}(L^F, s)$ (up to G^F -conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 8. In particular, generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ for every quasi-isolated semisimple element $1 \neq s \in G^{*F}$.*

Table 8: Quasi-isolated blocks of $E_8(q)$

No.	$C_{G^*}(s)^F$	e	L^F	$C_{L^*}(s)^F$	λ	$ W $
1	$E_7(q)A_1(q)$	3	$\Phi_3^3.A_2(q)$	$\Phi_1\Phi_3^3.A_1(q)$	ϕ_1, ϕ_{22}	1296
2		3	$\Phi_3.^3D_4(q)A_2(q)$	$\Phi_1\Phi_3.^3D_4(q)A_1(q)$	$^3D_4[-1] \otimes \phi_{11}$	6
		3			$^3D_4[-1] \otimes \phi_2$	6
3		3	$\Phi_3.E_6(q)$	$\Phi_3.A_5(q)A_1(q)$	4 chars.	6
4		3	G^F	$C_{G^*}(s)^F$	20 chars.	1
5	$E_6(q)A_2(q)$	3	Φ_3^4	Φ_3^4	1	1944
6		3	$\Phi_3^2.^3D_4(q)$	$\Phi_3^2.^3D_4(q)$	$^3D_4[-1]$	9
7		3	$\Phi_3.E_6(q)$	$\Phi_3.E_6(q)$	$\phi_{81,6}, \phi_{81,10},$ $\phi_{90,8}$	3 3
8	$D_5(q)A_3(q)$	3	$\Phi_3.A_2(q)^2$	$\Phi_1^2\Phi_3^2.A_1(q)^2$	$\phi_{11} \otimes \phi_{11},$	18
		3			$\phi_2 \otimes \phi_2$	18
9		3			$\phi_{11} \otimes \phi_2$	9
10		3	$\Phi_3.E_6(q)$	$\Phi_1\Phi_3.A_3(q)A_1(q)^2$	4 chars.	6
11		3			4 chars.	3
12		3		$\Phi_1\Phi_3.D_5(q)$	5 chars.	3
13		3	G^F	$C_{G^*}(s)^F$	4 chars.	1
14	$^2D_5(q)^2A_2(q)$	3	$\Phi_3.E_6(q)$	$\Phi_1\Phi_3.^2A_3(q)A_1(q)^2$	10 chars.	6
15		3	G^F	$C_{G^*}(s)^F$	40 chars.	1
16	$A_4(q)^2$	3	$\Phi_3^2.A_2(q)^2$	$\Phi_1^2\Phi_3^2.A_1(q)^2$	4 chars.	9
17		3	$\Phi_3.E_6(q)$	$\Phi_1\Phi_3.A_4(q)A_1(q)$	4 chars.	3
18		3	G^F	$C_{G^*}(s)^F$	$\phi_{311} \otimes \phi_{311}$	1
19	$A_5(q)A_2(q)A_1(q)$	3	$\Phi_3^3.A_2(q)$	$\Phi_1\Phi_3^3.A_1(q)$	ϕ_{11}, ϕ_2	54
20		3	$\Phi_3.E_6(q)$	$\Phi_3.A_5(q)A_1(q)$	4 chars.	3
21	$A_7(q)A_1(q)$	3	$\Phi_3^2.A_2(q)^2$	$\Phi_1^2\Phi_3^2.A_1(q)^2$	4 chars.	18
22		3	$\Phi_3.E_6(q)$	$\Phi_1\Phi_3.A_4(q)A_1(q)$	$\phi_{11} \otimes \phi_{311},$	3
		3			$\phi_2 \otimes \phi_{311}$	3
23		3	G^F	$C_{G^*}(s)^F$	$\phi_{4211} \otimes \phi_{11},$ $\phi_{4211} \otimes \phi_2$	1 1
24	$A_8(q)$	3	$\Phi_3^3.A_2(q)$	$\Phi_1^2\Phi_3^3$	1	162
25		3	$\Phi_3.E_6(q)$	$\Phi_1\Phi_3.A_5(q)$	ϕ_{42}, ϕ_{2211}	3
26		3	G^F	$C_{G^*}(s)^F$	ϕ_{32211}, ϕ_{531}	1

27	$D_8(q)$	3	$\Phi_3^2.A_2(q)^2$	$\Phi_1^2\Phi_3^2.A_1(q)^2$	$\phi_{11} \otimes \phi_{11},$	72
		3			$\phi_2 \otimes \phi_2,$	72
28		3			$\phi_{11} \otimes \phi_2$	36
29		3	$\Phi_3.E_6(q)$	$\Phi_1\Phi_3.D_5(q)$	5 chars.	6
30		3	G^F	$C_{G^*}(s)^F$	18 chars.	1
1	$E_7(q)A_1(q)$	4	$\Phi_4^2.D_4(q)$	$\Phi_4^2.A_1(q)^4$	4 chars.	96
2		4			4 chars.	32
3		4	G^F	$C_{G^*}(s)^F$	32 chars.	1
4	$E_6(q)A_2(q)$	4	$\Phi_4^2.D_4(q)$	$\Phi_1^2\Phi_4^2.A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	96
5		4	$\Phi_4^2.D_6(q)$	$\Phi_1.\Phi_4^2.A_3(q)A_2(q)$	$\phi_{22} \otimes \phi_{111},$	4
		4			$\phi_{22} \otimes \phi_{21},$	4
		4			$\phi_{22} \otimes \phi_3$	4
6		4	G^F	$C_{G^*}(s)^F$	30 chars.	1
7	${}^2E_6(q)^2A_2(q)$	4	$\Phi_4^2.D_4(q)$	$\Phi_2^2\Phi_4^2.A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	96
8		4	$\Phi_4^2.D_6(q)$	$\Phi_2\Phi_4.A_3(q)^2A_2(q)$	$\phi_{22} \otimes \phi_{111},$	4
		4			$\phi_{22} \otimes \phi_{21},$	4
		4			$\phi_{22} \otimes \phi_3$	4
9		4	G^F	$C_{G^*}(s)^F$	30 chars.	1
10	$D_5(q)A_3(q)$	4	$\Phi_4^3.A_1(q^2)$	$\Phi_1\Phi_2\Phi_4^3$	1	128
11		4	$\Phi_4^2.D_4(q)$	$\Phi_1\Phi_4^2.A_3(q)$	ϕ_{22}	32
12		4	$\Phi_4^2.D_4(q)$	$\Phi_2\Phi_4^2.A_3(q)$	ϕ_{22}	16
13		4	$\Phi_4^2.D_6(q)$	$\Phi_4^2.A_3(q)A_3(q)$	$\phi_{22} \otimes \phi_{22}$	4
14		4	$\Phi_4^2.D_6(q)$	$\Phi_2\Phi_4.D_5(q)$	(12, 04),	4
		4			(123, 014)	4
15		4	G^F	$C_{G^*}(s)^F$	2 chars.	1
16	${}^2D_5(q)^2A_3(q)$	4	$\Phi_4^3.A_1(q^2)$	$\Phi_1\Phi_2\Phi_4^3$	1	128
17		4	$\Phi_4^2.D_4(q)$	$\Phi_2\Phi_4^2.A_3(q)$	ϕ_{22}	32
18		4	$\Phi_4^2.D_4(q)$	$\Phi_1\Phi_4^2.A_3(q)$	ϕ_{22}	16
19		4	$\Phi_4^2.D_6(q)$	$\Phi_4^2.A_3(q)A_3(q)$	$\phi_{22} \otimes \phi_{22}$	4
20		4	$\Phi_4^2.D_6(q)$	$\Phi_1\Phi_4^2.D_5(q)$	(014, 2),	4
		4			(0134, 12)	4
21		4	G^F	$C_{G^*}(s)^F$	2 chars.	1
22	$A_4(q)^2$	4	$\Phi_4^2.A_1(q^2)^2$	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	16
23		4	$\Phi_4^2.D_6(q)$	$\Phi_1\Phi_2\Phi_4.A_4(q)$	6 chars.	4
24		4	G^F	$C_{G^*}(s)^F$	9 chars.	1
25	${}^2A_4(q)^2$	4	$\Phi_4^2.A_1(q^2)^2$	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	16
26		4	$\Phi_4^2.D_6(q)$	$\Phi_1\Phi_2\Phi_4^2.A_4(q)$	6 chars.	4
27		4	G^F	$C_{G^*}(s)^F$	9 chars.	1
28	$A_4(q^2)$	4	Φ_4^4	Φ_4^4	1	120
29	$A_5(q)A_2(q)A_1(q)$	4	$\Phi_4^2.D_6(q)$	$\Phi_1\Phi_2\Phi_4.A_2(q)A_1(q)^2$	12 chars.	4
30		4	G^F	$C_{G^*}(s)^F$	18 chars.	1
31	${}^2A_5(q)^2A_2(q)A_1(q)$	4	$\Phi_4^2.D_6(q)$	$\Phi_1\Phi_2\Phi_4^2.A_2(q)$	12 chars.	4
32		4	G^F	$C_{G^*}(s)^F$	18 chars.	1
33	$A_7(q)A_1(q)$	4	$\Phi_4^2.D_4(q)$	$\Phi_1\Phi_2^2\Phi_4^2.A_1(q)$	ϕ_{11}, ϕ_2	32
34		4	$\Phi_4^2.D_6(q)$	$\Phi_1\Phi_2\Phi_4.A_3(q)A_1(q)$	$\phi_{11} \otimes \phi_{22},$	4
		4			$\phi_2 \otimes \phi_{22}$	4
35		4	G^F	$C_{G^*}(s)^F$	8 chars.	1

36	${}^2A_7(q)A_1(q)$	4	$\Phi_4^2.D_4(q)$	$\Phi_1^2\Phi_2\Phi_4^2.A_1(q)$	ϕ_{11}, ϕ_2	32
37		4	$\Phi_4.{}^2D_6(q)$	$\Phi_1\Phi_2\Phi_4.{}^2A_3(q)A_1(q)$	$\phi_{11} \otimes \phi_{22},$ $\phi_2 \otimes \phi_{22}$	4
38		4	G^F	$C_{G^*}(s)^F$	8 chars.	4
39	$A_8(q)$	4	$\Phi_4^2.A_1(q^2)^2$	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	32
40		4	$\Phi_4^2D_6(q)$	$\Phi_1\Phi_2\Phi_4.A_4(q)$	$\phi_{2111}, \phi_{311}, \phi_{41}$	4
41		4	G^F	$C_{G^*}(s)^F$	8 chars.	1
42	${}^2A_8(q)$	4	$\Phi_4^2.A_1(q^2)^2$	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	32
43		4	$\Phi_4.{}^2D_6(q)$	$\Phi_1\Phi_2\Phi_4.{}^2A_4(q)$	$\phi_{2111}, \phi_{311}, \phi_{41}$	4
44		4	G^F	$C_{G^*}(s)^F$	8 chars.	1
1	$E_7(q)A_1(q)$	5	$\Phi_5.A_4(q)$	$\Phi_1\Phi_5.A_2(q)A_1(q)$	6 chars.	10
2		5	G^F	$C_{G^*}(s)^F$	92 chars.	1
3	$E_6(q)A_2(q)$	5	$\Phi_5.A_4(q)$	$\Phi_1\Phi_5.A_2(q)A_1(q)$	6 chars.	5
4		5	G^F	$C_{G^*}(s)^F$	60 chars.	1
5	$D_5(q)A_3(q)$	5	$\Phi_5.A_4(q)$	$\Phi_1\Phi_5.A_3(q)$	5 chars.	5
6		5	G^F	$C_{G^*}(s)^F$	75 chars.	1
7	$A_4(q)^2$	5	Φ_5^2	Φ_5^2	1	25
8		5	$\Phi_5.A_4(q)$	$\Phi_5.A_4(q)$	4 chars.	5
9		5	G^F	$C_{G^*}(s)^F$	chars.	1
10	$A_5(q)A_2(q)A_1(q)$	5	$\Phi_5.A_4(q)$	$\Phi_1\Phi_5.A_2(q)A_1(q)$	6 chars.	5
11		5	G^F	$C_{G^*}(s)^F$	36 chars.	1
12	$A_7(q)A_1(q)$	5	$\Phi_5.A_4(q)$	$\Phi_1\Phi_5.A_2(q)A_1(q)$	6 chars.	5
13		5	G^F	$C_{G^*}(s)^F$	14 chars.	1
14	$A_8(q)$	5	$\Phi_5.A_4(q)$	$\Phi_1\Phi_5.A_3(q)$	5 chars.	5
15		5	G^F	$C_{G^*}(s)^F$	5 chars.	1
16	$D_8(q)$	5	$\Phi_5.A_4(q)$	$\Phi_1\Phi_5.A_3(q)$	5 chars.	10
17		5	G^F	$C_{G^*}(s)^F$	70 chars.	1
1	$E_7(q)A_1(q)$	7	$\Phi_1\Phi_7.A_1(q)$	$\Phi_1\Phi_7.A_1(q)$	$\phi_{11}\phi_2$	14
2		7	G^F	$C_{G^*}(s)^F$	124 chars.	1
3	$A_7(q)A_1(q)$	7	$\Phi_1\Phi_7.A_1(q)$	$\Phi_1\Phi_7.A_1(q)$	ϕ_{11}, ϕ_2	7
4		7	G^F	$C_{G^*}(s)^F$	30 chars.	1
5	$A_8(q)$	7	$\Phi_1\Phi_7.A_1(q)$	$\Phi_1\Phi_7.A_1(q)$	ϕ_{11}, ϕ_2	7
6		7	G^F	$C_{G^*}(s)^F$	16 chars.	1
7	$D_8(q)$	7	$\Phi_1\Phi_7.A_1(q)$	$\Phi_1^2\Phi_7$	1	14
8		7	G^F	$C_{G^*}(s)^F$	104 chars.	1
1	$E_7(q)A_1(q)$	8	$\Phi_8.{}^2D_4(q)$	$\Phi_8.A_1(q)^2A_1(q^2)$	8 chars.	8
2		8	G^F	$C_{G^*}(s)^F$	88 chars.	1
3	$E_6(q)A_2(q)$	8	$\Phi_8.{}^2D_4(q)$	$\Phi_1\Phi_2\Phi_8.A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	8
4		8	G^F	$C_{G^*}(s)^F$	66 chars.	1
5	${}^2E_6(q)^2A_2(q)$	8	$\Phi_8.{}^2D_4(q)$	$\Phi_1\Phi_2\Phi_8.{}^2A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	8
6		8	G^F	$C_{G^*}(s)^F$	66 chars.	1
7	$D_5(q)A_3(q)$	8	$\Phi_8.{}^2D_4(q)$	$\Phi_2\Phi_8.A_3(q)$	5 chars.	8
8		8	G^F	$C_{G^*}(s)^F$	60 chars.	1
9	${}^2D_5(q)^2A_3(q)$	8	$\Phi_8.{}^2D_4(q)$	$\Phi_1\Phi_8.{}^2A_3(q)$	5 chars.	8
10		8	G^F	$C_{G^*}(s)^F$	60 chars.	1
11	${}^2A_4(q^2)$	8	$\Phi_1\Phi_2\Phi_4\Phi_8$	$\Phi_1\Phi_2\Phi_4\Phi_8$	1	4

12		8	G^F	$C_{G^*}(s)^F$	3 chars.	1
1	$E_7(q)A_1(q)$	9	$\Phi_9.A_2(q)$	$\Phi_1\Phi_9.A_1(q)$	ϕ_{11}, ϕ_2	18
2		9	G^F	$C_{G^*}(s)^F$	116 chars.	1
3	$E_6(q)A_2(q)$	9	$\Phi_9.A_2(q)$	$\Phi_9.A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	9
4		9	G^F	$C_{G^*}(s)^F$	63 chars.	1
5	$A_8(q)$	9	$\Phi_9.A_2(q)$	$\Phi_3\Phi_9$	1	9
6		9	G^F	$C_{G^*}(s)^F$	21 chars.	1
1	$E_7(q)A_1(q)$	12	$\Phi_{12}.{}^3D_4(q)$	$\Phi_{12}.A_1(q)A_1(q^3)$	4 chars.	12
2		12	G^F	$C_{G^*}(s)^F$	124 chars.	1
3	$E_6(q)A_2(q)$	12	$\Phi_{12}.{}^3D_4(q)$	$\Phi_3\Phi_{12}.A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	12
4		12	G^F	$C_{G^*}(s)^F$	54 chars.	1
5	${}^2E_6(q)A_2(q)$	12	$\Phi_{12}.{}^3D_4(q)$	$\Phi_6\Phi_{12}.{}^2A_2(q)$	$\phi_{111}, \phi_{21}, \phi_3$	12
6		12	G^F	$C_{G^*}(s)^F$	54 chars.	1
7	${}^2A_4(q^2)$	12	$\Phi_{12}.{}^2A_2(q^2)$	$\Phi_4\Phi_{12}.A_1(q^2)$	ϕ_{11}, ϕ_2	3
8		12	G^F	$C_{G^*}(s)^F$	ϕ_{311}	1
9	$D_8(q)$	12	$\Phi_{12}.{}^2A_2(q^2)$	$\Phi_4\Phi_{12}.A_1(q^2)$	ϕ_{11}, ϕ_2	12
10		12	G^F	$C_{G^*}(s)^F$	96 chars.	1
1	${}^2A_4(q^2)$	20	Φ_{20}	Φ_{20}	1	5
2		20	G^F	$C_{G^*}(s)^F$	ϕ_{221}, ϕ_{32}	1

Proof. Similar to the proof of Theorem 2.15. \square

Table 9: Decomposition of the non-uniform $R_L^G(\lambda)$

No.	e	λ	$\pm R_L^G(\lambda)$
2	3	${}^3D_4[-1] \otimes \phi_{11}$	$D_4:.111 \otimes \phi_{11} - D_4:.21 \otimes \phi_{11} + D_4:.3 \otimes \phi_{11} + D_4:111. \otimes \phi_{11} - D_4:21. \otimes \phi_{11} + D_4:3. \otimes \phi_{11}$
	3	${}^3D_4[-1] \otimes \phi_2$	$D_4:.111 \otimes \phi_2 - D_4:.21 \otimes \phi_2 + D_4:.3 \otimes \phi_2 + D_4:111. \otimes \phi_2 - D_4:21. \otimes \phi_2 + D_4:3. \otimes \phi_2$
6	3	${}^3D_4[-1]$	$D_4:3 \otimes \phi_{111} - D_4:3 \otimes \phi_{21} + D_4:3 \otimes \phi_3 + D_4:111 \otimes \phi_{111} - D_4:111 \otimes \phi_{21} + D_4:111 \otimes \phi_3 - D_4:21 \otimes \phi_{111} + D_4:21 \otimes \phi_{21} - D_4:21 \otimes \phi_3$
12	3	(013, 124)	$(013, 124) \otimes \phi_{1111} - (013, 124) \otimes \phi_{22} + (013, 124) \otimes \phi_4$
	3	(02, 14)	$(02, 14) \otimes \phi_{1111} - (02, 14) \otimes \phi_{22} + (02, 14) \otimes \phi_4$
	3	(0124,)	$(0124,) \otimes \phi_{1111} - (0124,) \otimes \phi_{22} + (0124,) \otimes \phi_4$
	3	(01234, 1)	$(01234, 1) \otimes \phi_{1111} - (01234, 1) \otimes \phi_{22} + (01234, 1) \otimes \phi_4$
29	3	(013, 124)	$(0234, 1235) + (012346, 123457) - (124, 034) - (013, 145) + (124, 016) + (013, 127)$
	3	(02, 14)	$(1234, 0136) + (01235, 12347) - (013, 235) + (14, 05) + (02, 17)$
	3	(0124,)	$(1234,) + (0123457, 123) - (012346, 13) - (0145,) + (01235, 3) + (0127,)$
	3	(01234, 1)	$(01345, 1) + (01234567, 1234) - (01246, 1) - (012345, 23) + (01237, 1) + (01234, 4)$
14	4	(12, 04)	$-(12, 04) \otimes \phi_{1111} + (12, 04) \otimes \phi_{211} - (12, 04) \otimes \phi_{31} + (12, 04) \otimes \phi_4$

	4	(123, 014)	$-(123, 014) \otimes \phi_{1111} + (123, 014) \otimes \phi_{211} - (123, 014) \otimes \phi_{31} + (123, 014) \otimes \phi_4$
20	4	(014, 2)	$-(014, 2) \otimes \phi_{1111} - (014, 2) \otimes \phi_{211} + (014, 2) \otimes \phi_{31} + (014, 2) \otimes \phi_4$
	4	(0134, 12)	$-(0134, 12) \otimes \phi_{1111} - (0134, 12) \otimes \phi_{211} + (0134, 12) \otimes \phi_{31} + (0134, 12) \otimes \phi_4$

2.1.5 e -cuspidal pairs of $G_2(q)$ and ${}^3D_4(q)$

Let G be a simple, simply connected algebraic group of type G_2 or D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = G_2(q)$ or $G^F = {}^3D_4(q)$. Here, e is relevant for some quasi-isolated $1 \neq s \in G^{*F}$ if and only if $e \in \{1, 2, 3, 6\}$. It remains to determine the e -cuspidal pairs for $e = 3$.

Theorem 2.19. *Let $e = 3$. For any quasi-isolated semisimple element $1 \neq s \in G^{*F}$, the e -cuspidal pairs (L, λ) of G with $\lambda \in \mathcal{E}(L^F, s)$ (up to G^F -conjugacy), and the order of their relative Weyl groups $W = W_{G^F}(L, \lambda)$ are as indicated in Table 10. In particular, generalized e -Harish-Chandra theory holds in $\mathcal{E}(G^F, s)$ for every quasi-isolated semisimple element $1 \neq s \in G^{*F}$.*

Table 10: Quasi-isolated blocks of $G_2(q)$ and ${}^3D_4(q)$

No.	G^F	$C_{G^*}(s)^F$	e	L^F	$C_{L^*}(s)^F$	λ	$ W $
1	$G_2(q)$	$A_2(q)$	3	Φ_3	Φ_3	1	3
2	${}^3D_4(q)$	$A_1(q)A_1(q^3)$	3	$\Phi_1\Phi_3 \cdot A_1(q)$	$\Phi_1\Phi_3 \cdot A_1(q)$	ϕ_{11}, ϕ_2	2

Proof. Similar to the proof of Theorem 2.14. □

$${}^2E_6(2), E_7(2) \text{ AND } E_8(2)$$

Note that these groups do not have semisimple elements of even order. Furthermore, note that the Mackey Formula holds for $e = 1$ regardless of q since 1-split Levi subgroups are contained in F -stable parabolic subgroups. In this case, Lusztig induction is just ordinary Harish-Chandra induction. Consequently, the proofs of the previous section still hold for $e = 1$ for these groups.

Proposition 2.20. *The assertion of Theorems 2.15, 2.17 and 2.18 are still valid when $q = 2$.*

Proof. ${}^2E_6(2)$: In this case it remains to prove the assertion for the Lusztig series corresponding to semisimple quasi-isolated elements with centralizers of type A_2^3 and D_4 . Let (L, λ) be an e -cuspidal pair for a semisimple, quasi-isolated element with centralizer of type A_2^3 . From the tables it follows that either $L = G$ or that λ is uniform. Hence the decomposition of $R_L^G(\lambda)$ can be determined without using the Mackey formula, so the proof of Theorem 2.15 still works.

Now, let (L, λ) be an e -cuspidal pair corresponding to a quasi-isolated element $s \in G^{*F}$ with $C_{G^*}(s)^F = \Phi_2^2 \cdot D_4(2)^3 \cdot 3$. If $e = 2$ there are two 2-cuspidal pairs $(L_1, \lambda_1) = (\Phi_2^6, 1)$ and $(L_2, \lambda_2) = (\Phi_2^2 \cdot D_4(2), (02, 13))$. Since λ_1 is uniform, we can decompose $R_{L_1}^G(\lambda_1)$ without

using the Mackey formula. For the second pair we use the following argument. We observe that $\pi_{uni}(R_{L_2}^G(\lambda_2)) \in \frac{1}{4}\mathbb{Z}\mathcal{E}(G^F, s)$. Since $R_{L_2}^G(\lambda_2) \in \mathbb{Z}\mathcal{E}(G^F, s)$ is a generalized character, there exists an element $\gamma \in \mathbb{Q}\mathcal{E}(G^F, s)$ which is orthogonal to the space of uniform class functions of G^F , such that $\pi_{uni}(R_{L_2}^G(\lambda_2)) + \gamma \in \mathbb{Z}\mathcal{E}(G^F, s)$. Furthermore, we know that $R_{L_1}^G(\lambda_1)$ and $R_{L_2}^G(\lambda_2)$ do not have any irreducible constituents in common because their constituents lie in different blocks by Theorem 2.3 (a). In this particular case this already determines the constituents of γ . Without knowing the norm of $R_{L_2}^G(\lambda_2)$, we are unfortunately not able to determine the multiplicities of the individual constituents. However, it is enough for our purposes to know the constituents.

A similar argument is needed for $e = 3$ (and $e = 6$). There are four 3-cuspidal pairs (L_i, λ_i) , $i = 1, \dots, 4$ with $L := L_1 = L_2 = L_3 = \Phi_3 \cdot {}^2A_2(2)$ and $L_4 = G^F$. Again, we are able to determine the constituents of $R_{L_i}^G(\lambda_i)$ for $i = 1, 2, 3$ (the case $i = 4$ being trivial). In addition to the arguments used for $e = 2$ above, we know that $\lambda_1 + \lambda_2 + \lambda_3$ is uniform. Therefore, $R_L^G(\lambda_1 + \lambda_2 + \lambda_3)$ is also uniform by transitivity of Lusztig induction (see [15, 11.5 Transitivity]). The same arguments as for $e = 2$ yield that a generalized e -Harish Chandra theory holds.

For the quasi-isolated elements $s \in G^{*F}$ with $C_{G^*}(s)^F = \Phi_6 \cdot {}^3D_4(2).3$ we argue the same way: either λ is uniform; λ is an e -cuspidal character of G^F already; or we can determine the constituents of $R_L^G(\lambda)$ without using the Mackey formula, as for the other 3-cuspidal pairs.

$E_7(2)$: Here, we only need to consider the e -cuspidal pairs corresponding to centralizers of type $A_5 \times A_2$. Let (L, λ) be one of those e -cuspidal pairs. Checking the tables we see that either λ is uniform or $L = G$ and λ is an e -cuspidal character of G^F . Thus we can determine $R_L^G(\lambda)$ without the Mackey formula and the proof of Theorem 2.17 works.

$E_8(2)$: The only cases to consider are the ones corresponding to centralizers of type A_8 , $A_4 \times A_4$ and $E_6 \times A_2$. For every e -cuspidal pair (L, λ) corresponding to the first or second centraliser type, λ is uniform. Hence, we can determine the decomposition of $\mathcal{E}(G^F, s)$ without the Mackey-formula. For the last centraliser type we use the same arguments as for the troublesome cases of ${}^2E_6(2)$. \square

PROOF OF THEOREM A

Proof. Suppose that ℓ is either a bad prime for G or that $\ell = 3$ and $G^F = {}^3D_4(q)$. If $s = 1$, then the asserted partition follows from work of Enguehard (see [16]). If $1 \neq s$ is quasi-isolated, then the asserted partition of $\mathcal{E}(G^F, s)$ follows from [29, Theorem 1.4].

Now suppose that ℓ is good and that $\ell \neq 3$ if $G^F = {}^3D_4(q)$. If $s = 1$, the assertion follows from [6] and [8]. If $1 \neq s \in G^{*F}$ is semisimple and quasi-isolated, then the assertion follows from Theorems 2.14, 2.15, 2.17, 2.18, 2.19 and Proposition 2.20. \square

3 Quasi-isolated blocks for bad primes

Let G be a simple, simply connected algebraic group of exceptional type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or let G be simple, simply connected of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. From now on we assume that $\ell \nmid q$ is a bad prime for G . Further we assume $\ell \in \{2, 3\}$

if $G^F = {}^3D_4(q)$. In these cases the parametrization of blocks in Theorem 2.3 has to be tweaked. Moreover, except for the case where $\ell = 3$ and $G^F = {}^3D_4(q)$, the assertion of Theorem 1.78 does not hold (see section 1.2 of [19] for a counterexample). We can replace Theorem 2.3 by results of Kessar and Malle (see [29]). The crux of this section is therefore to prove a replacement for Theorem 1.78.

Let $\mathcal{E}(G^F, \ell') := \bigcup_{\ell'\text{-elements } s \in G^{*F}} \mathcal{E}(G^F, s)$ denote the union of Lusztig series corresponding to ℓ' -elements of G^{*F} .

Definition 3.1. Let $\chi \in \mathcal{E}(G^F, \ell')$. We say that χ is of **central ℓ -defect** if $|G^F|_\ell = \chi(1)_\ell |Z(G)^F|_\ell$. We say that χ is of **quasi-central ℓ -defect** if some constituent of $\chi_{[G, G]^F}$ is of central ℓ -defect.

We set

$$e = e_\ell(q) := \text{order of } q \text{ modulo } \begin{cases} \ell & \text{if } \ell > 2, \\ 4 & \text{if } \ell = 2. \end{cases}$$

Since ℓ is assumed to be a bad prime (in particular ℓ is small), the only cases that occur are $e \in \{1, 2, 4\}$.

Using characters of quasi-central ℓ -defect, we are able to parametrise the quasi-isolated blocks for bad primes. The following replaces Theorem 2.3 for quasi-isolated ℓ -blocks when ℓ is a bad prime for G .

Theorem 3.2 ([29, Theorem 1.2]). *Let G be a simple, simply connected group of exceptional Lie type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or let G be simple of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Suppose that ℓ is a prime not dividing q . If $G^F = {}^3D_4(q)$ then let $\ell \in \{2, 3\}$ otherwise let ℓ be bad for G . Suppose that $1 \neq s \in G^{*F}$ is a quasi-isolated semisimple ℓ' -element. Then we have the following.*

(a) *There is a bijection*

$$b_{G^F}(L, \lambda) \leftrightarrow (L, \lambda)$$

*between ℓ -blocks of G^F contained in $\mathcal{E}_\ell(G^F, s)$ and G^F -conjugacy classes of e -cuspidal pairs (L, λ) such that $s \in L^{*F}$ and $\lambda \in \mathcal{E}(L^F, s)$ is of quasi-central ℓ -defect.*

(b) *There is a defect group $D \leq N_G^F(L, \lambda)$ of $b_{G^F}(L, \lambda)$ with a normal series*

$$Z(L)_\ell^F \trianglelefteq P := C_D(Z(L)_\ell^F) \trianglelefteq D,$$

where D/P is isomorphic to a Sylow ℓ -subgroup of $W_{G^F}(L, \lambda)$ and $P/Z(L)_\ell^F$ is isomorphic to a Sylow ℓ -subgroup of $L^F/Z(L)_\ell^F[L, L]^F$.

(c) *If $\ell \neq 2$, then $P = Z(L)_\ell^F$ and D is a Sylow ℓ -subgroup of the extension of $Z(L)_\ell^F$ by $W_{G^F}(L, \lambda)$.*

Definition 3.3. Let (L, λ) be an e -cuspidal pair of G^F . We say that R_L^G satisfies an e -Harish-Chandra theory above (L, λ) if there exists a collection of isometries

$$I_{(L, \lambda)}^M : \mathbb{Z} \text{Irr}(W_{M^F}(L, \lambda)) \rightarrow \mathbb{Z} \mathcal{E}(M^F, (L, \lambda)),$$

where M runs over all e -split Levi subgroup of G containing L such that the following holds.

(1) For every M , we have

$$R_M^G \circ I_{(L, \lambda)}^M = I_{(L, \lambda)}^G \circ \text{Ind}_{W_{M^F}(L, \lambda)}^{W_{G^F}(L, \lambda)}.$$

(2) The collection $(I_{(L, \lambda)}^M)$ is stable under the conjugation action by $W_{G^F}(L, \lambda)$.

(3) $I_{(L, \lambda)}^L$ maps the trivial character of the trivial group $W_{L^F}(L, \lambda)$ to λ .

Note the difference of this definition to the notion of a generalized e -Harish-Chandra theory holding in a Lusztig series used in Section 2. However, it can be shown that Definition 3.3 implies the latter (see [29, Proposition 2.10]).

The following result was already mentioned in the proof of Theorem A. We state it here because of how important it is for some of the proofs in this section (see the proofs of Theorems 3.9, 3.22, 3.33, 3.39 and 3.42).

Theorem 3.4 ([29, Theorem 1.4]). *Let G be a simple, simply connected group of exceptional Lie type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or let G be simple of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Suppose that ℓ is a prime not dividing q . If $G^F = {}^3D_4(q)$ then let $\ell \in \{2, 3\}$ otherwise let ℓ be bad for G . Suppose that $1 \neq s \in G^{*F}$ is a quasi-isolated ℓ' -element. Then we have the following.*

(a)

$$\mathcal{E}(G^F, s) = \bigcup_{(L, \lambda)/G^F} \mathcal{E}(G^F, (L, \lambda)),$$

where (L, λ) runs over the e -cuspidal pairs of G up to G^F -conjugacy with $s \in L^{*F}$ and $\lambda \in \mathcal{E}(L^F, s)$.

(b) R_L^G satisfies an e -Harish-Chandra theory above each e -cuspidal pair (L, λ) with $s \in L^{*F}$ and $\lambda \in \mathcal{E}(L^F, s)$.

As is often the case in the representation theory of finite groups of Lie type, results are stronger if we assume the center of the underlying algebraic group to be connected. The same is true for the results in this thesis; mostly because of the following.

Lemma 3.5. *Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Suppose that $Z(G)$ is connected. Let $s \in G^{*F}$ be a quasi-isolated ℓ' -element and $t \in C_{G^*}(s)_\ell^F$ such that st is not quasi-isolated. If L^* is the minimal Levi subgroup containing $C_{G^*}(st)$, then $t \in Z(L^*)$ if one of the following conditions is satisfied:*

(a) ℓ is good for L^* , or

(b) ℓ is good for $C_{G^*}(s)$ and the order of s is not divisible by any bad primes for L^* .

Proof. Assume condition (a) to be satisfied. We have

$$C_{G^*}(st) \subseteq L^* \cap C_{G^*}(t) \subseteq C_{G^*}(t).$$

By Theorem 1.31 and our assumption on $Z(G)$, $C_{G^*}(t)$ is connected. In particular, $C_{L^*}(t) = L^* \cap C_{G^*}(t)$ is connected, as it is a Levi subgroup of $C_{G^*}(t)$ by Proposition 2.11. By Proposition 1.38, $C_{L^*}(t)$ is a Levi subgroup of L^* . By Corollary 1.37 $C_{L^*}(t)$ is therefore a Levi subgroup of G^* . Since $C_{G^*}(st) \subseteq C_{L^*}(t)$, the minimality of L^* implies $L^* = C_{L^*}(t)$; in other words $t \in Z(L^*)$.

Assume condition (b) to be satisfied. We claim that $L^* = C_{G^*}(st)$. Since s and t are commuting elements of coprime order, we have

$$C_{G^*}(st) = C_{C_{G^*}(s)}(t) = C_{C_{G^*}(t)}(s).$$

In particular, $C_{C_{G^*}(s)}(t)$ and $C_{C_{G^*}(t)}(s)$ are connected. By our assumption on the order of s , $C_{G^*}(st) = C_{C_{G^*}(s)}(t)$ is a Levi subgroup of $C_{G^*}(s)$ (see Proposition 1.38). Additionally,

$$C_{G^*}(st) \subseteq L^* \cap C_{G^*}(s) \subsetneq C_{G^*}(s),$$

where the last inclusion is proper because s is quasi-isolated and L^* is a proper Levi subgroup of G^* . By Proposition 2.11 and our assumption on the order of s , $L^* \cap C_{G^*}(s)$ is a Levi subgroup of $C_{G^*}(s)$ and the minimality of L^* yields $C_{G^*}(st) = L^* \cap C_{G^*}(s) = C_{L^*}(s)$. Applying Proposition 1.38 again, we see that $C_{L^*}(s)$ is a Levi subgroup of L^* and from Proposition 1.35 it follows that $C_{L^*}(s)$ is a Levi subgroup of G^* as well. Now, the minimality of L^* implies $L^* = C_{L^*}(s) = C_{G^*}(st)$ which proves the claim. In particular, $t \in Z(L^*)$. \square

3.1 The quasi-isolated blocks of $F_4(q)$

Let G be simple, simply connected of type F_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Recall that simple algebraic groups of type F_4 are both simply connected and adjoint. We will therefore omit any specification of the isogeny type. Let ℓ be a bad prime for G not dividing q .

Recall that the bad primes for G are just 2 and 3. Checking Table 1, we see that G^F has no quasi-isolated elements of order greater than 4. Let $1 \neq s \in G^{*F}$ be a semisimple, quasi-isolated ℓ' -element and let $1 \neq t \in C_{G^*}(s)_\ell^F$. Clearly, the order of st is greater than 4. Hence there exists a proper Levi subgroup M^* of G^* containing $C_{G^*}(st)$. Let L^* be the minimal such Levi subgroup of G^* . It is easy to see that $L^* = C_{G^*}(Z(C_{G^*}(st))^\circ)$ and since st is F -stable, L^* is also F -stable.

Proposition 3.6. *Let G be simple, simply connected of type F_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $\ell \nmid q$ be a bad prime for G . Let $1 \neq s \in G^{*F}$ be a quasi-isolated semisimple ℓ' -element and let $1 \neq t \in C_{G^*}(s)_\ell^F$. If $L^* \leq G^*$ is the minimal Levi subgroup containing $C_{G^*}(st)$, then $t \in Z(L^*)$.*

Proof. The proper Levi subgroups of G are of classical type. Hence, the only possible bad prime for L^* is 2. If $\ell = 3$, the assertion therefore follows from Lemma 3.5 (a).

If $\ell = 2$, then $o(s) = 3$ and $C_{G^*}(s)$ is of type $A_2 \times A_2$ (see Table 1). Hence, the assertion follows from Lemma 3.5 (b). \square

The following result is our aforementioned replacement for Theorem 1.78.

Theorem 3.7. *Let G be simple, simply connected of type F_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $\ell \nmid q$ be a bad prime for G . Let $1 \neq s \in G^{*F}$ be a quasi-isolated semisimple ℓ' -element. Then $\mathcal{E}(G^F, s)$ is an ordinary generating set for $\mathcal{E}_\ell(G^F, s)$. In particular, the number of irreducible Brauer characters in $\mathcal{E}_\ell(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*

The proof of this follows the proof of [19, Theorem 3.1]. Since we are working with bad primes, we have to tweak the arguments slightly. We denote the characteristic function of the set of ℓ -regular elements of G^F by $\gamma_{\ell'}$.

Proof. By Theorem 1.16 it suffices to show that $\chi^\circ \in \mathbb{Z}\hat{\mathcal{E}}(G^F, s)$ for every $\chi \in \mathcal{E}_\ell(G^F, s)$. Let $t \in C_{G^*}(s)_{\ell'}^F$ and let $\chi \in \mathcal{E}(G^F, st)$. If $t = 1$, then $\chi^\circ \in \hat{\mathcal{E}}(G^F, s)$ and we are done. Hence, assume $t \neq 1$. Let $L^* \subseteq G^*$ be the minimal Levi containing $C_{G^*}(st)$ and let L be a Levi subgroup of G dual to L^* . By [11, Theorem 9.16] there is a character $\pi \in \mathcal{E}(L^F, st)$ such that $\chi = \epsilon_{G \in L} R_L^G(\pi)$. Since $t \in Z(L^*)^F$ by Proposition 3.6, there exists a character θ_t of L^F , dual to t , such that $\pi = \theta_t \lambda$, where $\lambda \in \mathcal{E}(L^F, s)$ (see [15, 13.30 Proposition]). The order of θ_t is equal to the order of t and is therefore a power of ℓ . Thus, $\theta_t^\circ = 1_{L^F}^\circ$. We have

$$\begin{aligned} \chi \gamma_{\ell'} &= \epsilon_{G \in L} R_L^G(\theta_t \lambda) \gamma_{\ell'} \\ &= \epsilon_{G \in L} R_L^G(\theta_t \lambda \gamma_{\ell'}) \quad ([14, \text{Proposition 3.8}]) \\ &= \epsilon_{G \in L} R_L^G(\hat{\theta}_t \hat{\lambda}) \\ &= \epsilon_{G \in L} R_L^G(\hat{\lambda}) \\ &= \epsilon_{G \in L} R_L^G(\lambda) \gamma_{\ell'}. \end{aligned}$$

By [31, Corollary 6] every irreducible constituent of $R_L^G(\lambda)$ lies in $\mathcal{E}(G^F, s)$. Since $\chi^\circ = (\chi \gamma_{\ell'})_{G^F}$, it follows that $\chi^\circ \in \mathbb{Z}\hat{\mathcal{E}}(G^F, s)$. \square

If we want an ordinary generating set for a block B contained in $\mathcal{E}_\ell(G^F, s)$, we can take $\text{Irr}(B) \cap \mathcal{E}(G^F, s)$. Let $c(B) := |\text{Irr}(B) \cap \mathcal{E}(G^F, s)|$ denote the cardinality of this generating set.

To prove the Malle–Robinson conjecture for the quasi-isolated blocks B of G^F , we show that

$$l(B) \leq c(B) < \ell^{s(B)}.$$

For this, we need the classification of quasi-isolated blocks of G^F in [29].

Theorem 3.8 ([29, Proposition 3.2, Proposition 3.5]). *Let G be simple, simply connected of type F_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $\ell \nmid q$ be a bad prime for G . Let $e = e_\ell(q) = 1$. For any quasi-isolated semisimple ℓ' -element $1 \neq s \in G^{*F}$, the ℓ -block distribution of $\mathcal{E}(G^F, s)$, the decomposition of $\mathcal{E}(G^F, s)$ into e -Harish-Chandra series, and the relative Weyl groups of the e -cuspidal pairs are as indicated in Table 11.*

Table 11: Quasi-isolated blocks of $F_4(q)$

No.	$C_{G^*}(s)^F$	(ℓ, e)	L^F	λ	$W_{G^F}(L, \lambda)$
1	$A_2(q)\tilde{A}_2(q)$	(2, 1)	Φ_1^4	1	$A_2 \times A_2$
2	${}^2A_2(q)\tilde{{}^2A}_2(q)$	(2, 1)	$\Phi_1^2.A_1(q)^2$ $\Phi_1.B_3(q)$ $\Phi_1.C_3(q)$ G^F	1 ϕ_{21} ϕ_{21} $\phi_{21} \otimes \tilde{\phi}_{21}$	$A_1 \times A_1$ A_1 A_1 1
3	$B_4(q)$	(3, 1)	Φ_1^4	1	B_4
4			$\Phi_1^2.B_2(q)$	$B_2[1]$	B_2
5	$C_3(q)A_1(q)$	(3, 1)	Φ_1^4	1	$C_3 \times A_1$
6			$\Phi_1^2.B_2(q)$	$B_2[1]$	$A_1 \times A_1$
7	$A_3(q)\tilde{A}_1(q)$	(3, 1)	Φ_1^4	1	$A_3 \times A_1$
8	${}^2A_3(q)\tilde{{}^2A}_1(q)$	(3, 1)	$\Phi_1^3.\tilde{A}_1(q)$	1	$C_2 \times A_1$
2b		(2, 2)	Φ_2^4	1	$A_2 \times A_2$

Note that line 2b does not yield a new block. This line is only needed in the proof of Theorem 3.9. The table for $e = 2$ is the Ennola dual of this one. For a few remarks on how to interpret this table, see the beginning of Section 2.

Theorem 3.9. *Let G be simple, simply connected of type F_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $e = e_\ell(q) = 1$. Then Table 3 gives $c(B)$ and a lower bound for $s(B)$ for every ℓ -block B of Table 2. Moreover, the Malle–Robinson conjecture holds in strong form for these blocks.*

Table 12: $c(B)$ and lower bounds on $s(B)$ for the quasi-isolated blocks of $F_4(q)$

B	$C_{G^*}(s)^F$	ℓ	$c(B)$	$s(B) \geq$
1	$A_2(q)\tilde{A}_2(q)$	2	9	4
2	${}^2A_2(q)\tilde{{}^2A}_2(q)$	2	9	4
3	$B_4(q)$	3	20	4
4		3	5	2
5	$C_3(q)A_1(q)$	3	20	4
6		3	4	2
7	$A_3(q)\tilde{A}_1(q)$	3	10	4
8	${}^2A_3(q)\tilde{{}^2A}_1(q)$	3	10	3

Proof. Let B be a quasi-isolated block associated to a line in Table 11 and let $(L_1, \lambda_1), \dots, (L_r, \lambda_r)$ be the e -cuspidal pairs associated to that block. By Theorem 3.7 and Theorem 3.4 we conclude that

$$c(B) = \sum_{i=1}^r |\mathcal{E}(G^F, (L_i, \lambda_i))|.$$

Since R_L^G satisfies an e -Harish-Chandra theory above each (L_i, λ_i) by Theorem 3.4,

$$|\mathcal{E}(G^F, (L_i, \lambda_i))| = |\text{Irr}(W_{G^F}(L_i, \lambda_i))|,$$

and this cardinality can easily be determined using Chevie [36]. Let (L, λ) now be the unique pair parametrising $B = b_{G^F}(L, \lambda)$ by Theorem 3.2. Let D be a defect group of B . Since $Z(L)_\ell^F \leq D$, we have $s(Z(L)_\ell^F) \leq s(D)$. We prove the Malle–Robinson conjecture by establishing the stronger inequality

$$l(B) \leq c(B) < \ell^{s(Z(L)_\ell^F)} \leq \ell^{s(B)}.$$

Checking Table 11, we see that $Z(L)^F = \Phi_e^k$ is an e -torus in every case.

Let $\ell = 3$ and let B be a quasi-isolated 3-block. If $Z(L)^F = \Phi_e^m$, then $s(Z(L)_\ell^F) = m$ by Proposition 1.53. The m 's can be read off from Table 11 and we see that $c(B) < \ell^{s(Z(L)_\ell^F)}$ in every case.

Let $\ell = 2$. Let $B = b_{G^F}(L, \lambda)$ be the block corresponding to line 1 of Table 11. To prove the conjecture it is enough to take $s(Z(L)_\ell^F)$ again. Let $B = b_{G^F}(L, \lambda)$ now be the block corresponding to line 2 of Table 11. Unfortunately, $s(Z(L)_\ell^F)$ is not large enough to establish the conjecture for B in this case. We have to use line 2b of Table 11. As seen in the proof of [29, Proposition 3.5], the 1-Harish-Chandra series corresponding to line 2b actually lies in B . By [29, Proposition 2.17], $Z(M)_2^F = (\Phi_2^4)_2 \subseteq D$ where (M, ζ) is the pair of line 2b. Note that Φ_2 is always divisible by 2 unless q is a power of 2 and since we are working in cross-characteristic and assume $\ell = 2$, this can not be the case. Hence, $Z(M)_2^F = \Phi_2^4$ yields an elementary abelian 2-subgroup of D of rank 4. It follows that

$$l(B) \leq c(B) < \ell^{s(Z(L)_\ell^F)} \leq \ell^{s(B)}.$$

If $e = 2$, then the Ennola dual of line 2b gives a 1-split torus Φ_1^4 which yields an elementary abelian 2-subgroup of rank 4. The rest of the proof did not depend e . Hence, the assertion is proved. \square

3.2 The quasi-isolated blocks of $E_6(q)$ and ${}^2E_6(q)$

Let G be a simple, simply connected group of type E_6 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Then $G^F = E_{6,sc}(q)$ or ${}^2E_{6,sc}(q)$ and the dual group G^* (which is of adjoint type) contains semisimple elements whose centralisers are disconnected as $Z(G)$ is disconnected.

Remark 3.10. For the proof of Theorem 3.7 to work, our setup had to satisfy the following conditions.

- (1) To go from $\hat{\mathcal{E}}(G^F, st)$ to $\hat{\mathcal{E}}(L^F, st)$ (see [11, Theorem 9.16]), we need

$$C_{G^*}^\circ(st)C_{G^*}(st)^F \subseteq L^*,$$

where L^* is an F -stable Levi subgroup of G^* , and

- (2) to go from $\hat{\mathcal{E}}(L^F, st)$ to $\hat{\mathcal{E}}(L^F, s)$, we need

$$t \in Z(L^*)_\ell^F.$$

Both of these can fail for $G^F = E_{6,sc}(q)$ and ${}^2E_{6,sc}(q)$. The failure of the first condition is not a big deal and only leads to slightly bigger generating sets. The failure of the second condition, however, creates problems that we are not able to resolve in all cases.

Apart from conditions (1) and (2) failing in certain cases, many other things that were true for $F_4(q)$ are not necessarily true for G^F . This creates very intricate problems. For example, given a semisimple element $s \in G^*$ we do not have an explicit description of the minimal Levi subgroup L^* of G^* containing $C_{G^*}(s)$ as in Section 3.1. However, we still know that L^* is F -stable.

Lemma 3.11. *Let G be a connected reductive group defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $s \in G^{*F}$ be a semisimple element and let $L^* \subseteq G^*$ be the minimal Levi subgroup containing $C_{G^*}(s)$. Then L^* is F -stable.*

Proof. Since s is F -stable and F is an isomorphism of abstract groups we have

$$C_{G^*}(s) = C_{G^*}(F(s)) = F(C_{G^*}(s)) \subseteq F(L^*).$$

By the minimality of L^* it follows that $L \subseteq F(L^*)$ and therefore $L^* = F(L^*)$. \square

Since we are working with simple algebraic groups, we know a great deal about the Levi subgroups of G .

Lemma 3.12. *Let $L^* \subseteq G^*$ be a proper Levi subgroup of G^* . Then $[L^*, L^*]$ is simply connected unless L^* is of type A_2^2 , $A_2^2 \times A_1$ or A_5 .*

Proof. This can be checked with Chevie [36]. \square

We try to follow the idea of section 3.1 as much as possible but because of the complications addressed in the beginning of this section, this is not always possible. For example condition (1) fails if st is quasi-isolated.

Remark 3.13. Let $1 \neq s \in G^{*F}$ be quasi-isolated and let $t \in C_{G^*}(s)^F$. Then st is quasi-isolated if and only if $t = 1$ or st is of order 6 with $C_{G^*}(st) = A_1^4$ (see Table 1). It can be shown that every quasi-isolated element $z \in G^*$ of order 6 is of the form $z = st$ where s is quasi-isolated of order 3 with $C_{G^*}^\circ(s)$ of type D_4 , and t is quasi-isolated of order 2 with $C_{G^*}(t)$ of type $A_5 \times A_1$ (or vice-versa).

If $q \equiv 1 \pmod{3}$, the G^* -conjugacy classes of s and z each split into three G^{*F} -conjugacy classes (see e.g. [22, Theorem 2.1.5 (b)]) and we have the following.

- (i) If $C_{G^*}(z)^F = \Phi_1^2.A_1(q)^4.3$, then $C_{G^*}(s)^F = \Phi_1^2.D_4(q).3$.
- (ii) If $C_{G^*}(z)^F = \Phi_3.A_1(q)A_1(q^3).3$, then $C_{G^*}(s)^F = \Phi_3.{}^3D_4(q).3$.

If $q \equiv 2 \pmod{3}$, then the G^* -conjugacy classes of semisimple elements do not split and we have $C_{G^*}(z)^F = \Phi_1\Phi_2.A_1(q)^2A_1(q^2)$ and $C_{G^*}(s)^F = \Phi_1\Phi_2.{}^2D_4(q)$.

Proposition 3.14. *Let $s \in G^{*F}$ be a quasi-isolated semisimple element of order 3 with $C_{G^*}^\circ(s) = A_2^3$ and let $t \in C_{G^*}(s)^F$. If $L^* \subseteq G^*$ is the minimal Levi subgroup containing $C_{G^*}(st)$, then $t \in Z(L^*)$.*

Proof. Since $st \in G^*$ is not quasi-isolated by Remark 3.13, its centraliser in G^* is contained in a proper Levi subgroup. Note that the proper Levi subgroups of G^* are either of type D or a product of groups of type A (or maximal tori, in which case $t \in Z(L^*)$ is immediate).

Let L^* be of type A . Then 2 is not a bad prime for L^* . We have

$$C_{G^*}(st) \subseteq L^* \cap C_{G^*}(t) \subseteq C_{G^*}(t).$$

Since t is a 2-element, $C_{G^*}(t)$ is connected by [35, Proposition 14.20]. By Theorem 1.38, $C_{L^*}(t) = L^* \cap C_{G^*}(t)$ is a Levi subgroup of G^* . The minimality of L^* yields $L^* = L^* \cap C_{G^*}(t)$. In other words, $L^* \subset C_{G^*}(t)$ which implies that $t \in Z(L^*)$.

Now, let L^* be of type D . By Lemma 3.12, $[L^*, L^*]$ is simply connected. Hence, $C_{G^*}(st) = C_{L^*}(st)$ is connected by Theorem 1.31. With the same arguments as in the proof of Lemma 3.5(b), it can be shown that $L^* = C_{G^*}(st)$. Hence, $t \in Z(L^*)$. \square

Corollary 3.15. *Let $\ell = 2$ not dividing q . Let $s \in G^{*F}$ be semisimple and quasi-isolated of order 3 with $C_{G^*}^\circ(s) = A_2^3$. Then $\mathcal{E}(G^F, s)$ is an ordinary generating set for $\mathcal{E}_\ell(G^F, s)$. In particular, the number of irreducible Brauer characters in $\mathcal{E}_\ell(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*

Proof. The result follows from the fact that condition (1) and (2) of Remark 3.10 are satisfied by Proposition 3.14. \square

To be able to state the results in a concise way we focus on $G^F = E_6(q)_{sc}$. This is unproblematic as the analogous results for ${}^2E_6(q)_{sc}$ can be proved using the same type of arguments.

Theorem 3.16 ([29, Proposition 4.1, Proposition 4.3]). *Let $\ell \nmid q$ be a bad prime for G . For any quasi-isolated semisimple ℓ' -element $1 \neq s \in G^{*F}$ the ℓ -block distribution of $\mathcal{E}(G^F, s)$, the decomposition of $\mathcal{E}(G^F, s)$ into e -Harish-Chandra series, and the relative Weyl groups of the e -cuspidal pairs are as indicated in Table 13.*

See Section 2.1.2 for an explanation of how the corresponding Table for ${}^2E_6(q)_{sc}$ is obtained.

Table 13: Quasi-isolated blocks in $E_6(q)$

No.	$C_{G^*}(s)^F$	(ℓ, e)	L^F	λ	$W_{G^F}(L, \lambda)$
1	$A_2(q)^3.3$	(2, 1)	Φ_1^6	1	$A_2 \wr 3$
2	$A_2(q^3).3$	(2, 1)	$\Phi_1^2.A_2(q)^2$	1	A_2
3	$A_2(q^2).{}^2A_2(q)$	(2, 1)	$\Phi_1^3.A_1(q)^3$ $\Phi_1^2.D_4(q)$	1 ϕ_{21}	$A_2 \times A_1$ A_2
4	$\Phi_1^2.D_4(q).3$	(2, 1)	Φ_1^6 $\Phi_1^2.D_4(q)$	1 $D_4[1]$	$D_4.3$ 3
5	$\Phi_1\Phi_2.{}^2D_4(q)$	(2, 1)	$\Phi_1^4.A_1(q)^2$	1	B_3
6	$\Phi_3.{}^3D_4(q).3$	(2, 1)	$\Phi_1^2.A_2(q)^2$ G^F	1 ${}^3D_4(q)[\pm 1]$	G_2 1
7	$A_2(q)^3.3$	(2, 2)	$\Phi_1^2\Phi_2^3.A_1(q)$ $\Phi_1\Phi_2^2.A_3(q)$ $\Phi_2.A_5(q)$ G^F	1 ϕ_{21} $\phi_{21} \otimes \phi_{21}$ $\phi_{21} \otimes \phi_{21} \otimes \phi_{21}$	$A_1 \wr 3$ $A_1 \times A_1$ A_1 1

8	$A_2(q^3).3$	(2, 2)	$\Phi_2.A_2(q^2)A_1(q)$ G^F	1 ϕ, ϕ_{21}	A_1 1
9	$A_2(q^2).^2A_2(q)$	(2, 2)	$\Phi_1^2\Phi_2^4$	1	$A_2 \times A_2$
10	$\Phi_1^2.D_4(q).3$	(2, 2)	$\Phi_1^2\Phi_2^4$ G^F	1 ϕ, ϕ', ϕ''	$D_4.3$ 1
11	$\Phi_1\Phi_2.^2D_4(q)$	(2, 2)	$\Phi_1^2\Phi_2^4$	1	B_3
12	$\Phi_3.^3D_4(q).3$	(2, 2)	$\Phi_2^2.A_2(q^2)$ G^F	1 $\phi_{2,1}, \phi_{2,2}$	G_2 1
13	$A_5(q)A_1(q)$	(3, 1)	Φ_1^6	1	$A_5 \times A_1$
14	$A_5(q)A_1(q)$	(3, 2)	$\Phi_1^2\Phi_2^4$	1	$C_3 \times A_1$
15			$\Phi_2.A_5(q)$	ϕ_{321}	A_1

Remark 3.17. (a) The numbers in the first column do not count the blocks of G^F . Unlike Table 11, where every numbered line corresponds to one block, lines 2, 6, 8 and 12 of Table 13, each yield 3 blocks. We therefore either say that a quasi-isolated block B is of type k , where k is the number to which it corresponds, or we indicate this ambiguity by saying that B is numbered k, k' or k'' . If $k \notin \{2, 6, 8, 12\}$ we still say that B is numbered k .

We want to study the blocks of type 4, 5, 6, 10, 11, 12 next. Recall that every quasi-isolated element z of order 6 is of the form $z = st$, where $s \in G^*$ is quasi-isolated of order 3 with $C_{G^*}^\circ(s)$ of type D_4 , and t is quasi-isolated of order 2 with $C_{G^*}(t)$ of type $A_5 \times A_1$ (or vice-versa).

Theorem 3.18. *Let $\ell = 2$ not divide q . Let $1 \neq s \in G^{*F}$ be a quasi-isolated semisimple element of order 3. Then we have the following.*

- (a) *If $C_{G^*}(s)^F = \Phi_1^2.D_4(q).3$, then $\hat{\mathcal{E}}(G^F, s) \cup \hat{\mathcal{E}}(G^F, st)$ is a generating set of $\mathbb{Q} \text{IBr}(\mathcal{E}_2(G^F, s))$ where $t \in C_{G^*}(s)_2^F$ such that $C_{G^*}(st)^F = \Phi_1^2.A_1(q)^4.3$.*
- (b) *If $C_{G^*}(s)^F = \Phi_3.^3D_4(q).3$, then $\hat{\mathcal{E}}(G^F, s) \cup \hat{\mathcal{E}}(G^F, st)$ is a generating set of $\mathbb{Q} \text{IBr}(\mathcal{E}_2(G^F, s))$ where $t \in C_{G^*}(s)_2^F$ such that $C_{G^*}(st)^F = \Phi_3.A_1(q)A_1(q^3).3$.*

In particular, the number of irreducible Brauer characters in $\mathcal{E}_2(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)| + |\mathcal{E}(G^F, st)|$ in both cases.

- (c) *If $C_{G^*}(s)^F = \Phi_1\Phi_2.^2D_4(q)$, then $\hat{\mathcal{E}}(G^F, s)$ is a generating set of $\mathbb{Q} \text{IBr}(\mathcal{E}_2(G^F, s))$. In particular, the number of irreducible Brauer characters in $\mathcal{E}_2(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*

Note that Theorem 3.18 is about generating sets of vector spaces over \mathbb{Q} and not about generating sets as in Definition 1.15.

Proof. (a) and (b): There exist elements $t \in C_{G^*}(s)_2^F$ such that st is quasi-isolated. Clearly, condition (1) in Remark 3.10 fails in these cases, which is why the Lusztig series corresponding to quasi-isolated elements of the form st are included in the proposed generating sets.

Assume that $t \in C_{G^*}(s)_\ell^F$ such that st is not quasi-isolated. Let L^* be the minimal Levi subgroup of G with respect to $C_{G^*}(st) \subseteq L^*$. Note that proper Levi subgroups in E_6 are either of type A or type D , but not mixed.

Let L^* be of type A . We show that $t \in Z(L^*)$. Note that $C_{G^*}(t)$ is connected since t is a 2-element (see [35, Proposition 14.20]). We have

$$C_{G^*}(st) \subseteq L^* \cap C_{G^*}(t) \subseteq C_{G^*}(t).$$

Now, $L^* \cap C_{G^*}(t) = C_{L^*}(t)$ is a Levi subgroup of L^* (and therefore of G^*) by Proposition 1.38. The minimality of L^* yields $L^* = C_{L^*}(t)$. Hence, $t \in Z(L^*)$. Therefore, we can apply the methods used in the proof of Theorem 3.7 to show that every $\chi^\circ \in \hat{\mathcal{E}}(G^F, st)$ is an integral linear combination of the Brauer characters in $\hat{\mathcal{E}}(G^F, s)$.

Now, let L be of type D . We claim that $L^* = C_{G^*}^\circ(s)$. Since L^* is of type D the derived subgroup of L^* is of simply-connected type. By Theorem 1.31, $C_{L^*}(st) = C_{G^*}(st)$ is therefore connected. We have

$$C_{G^*}(st) \subseteq L^* \cap C_{G^*}^\circ(s) \subseteq C_{G^*}(s)^\circ.$$

Now, $L^* \cap C_{G^*}^\circ(s)$ is a Levi subgroup in L^* (and therefore in G^*), by Proposition 1.38 again. From the minimality of L^* it follows that $L^* = L^* \cap C_{G^*}^\circ(s) \subseteq C_{G^*}^\circ(s)$. Since $C_{G^*}^\circ(s)$ is of type D_4 , it follows that $L^* = C_{G^*}^\circ(s)$ by our assumption on L^* .

We observe that $C_{G^*}(st) = C_{L^*}(st) = C_{L^*}(t)$ because $s \in Z(L^*)$. In other words t is quasi-isolated in L^* . Using Chevie [36], we see that $C_{L^*}(t)$ is of type A_1^4 . By Lemma 3.11, L^* is F -stable. Let $L \subseteq G$ be an F -stable Levi subgroup of G dual to L^* . Then there is a natural bijection

$$\mathcal{E}(G^F, st) \rightarrow \mathcal{E}(L^F, st),$$

induced by Lusztig induction (see [15, 13.25 Theorem]). As $C_{L^*}(st) = C_{L^*}(t)$ is connected and of type A , every irreducible character in $\mathcal{E}(L^F, st)$ is uniform (see Remark 1.75). Thus, all irreducible characters in $\mathcal{E}(G^F, st)$ are uniform as well. If $\chi \in \mathcal{E}(G^F, st)$, we can therefore write

$$\chi = \sum_{T^* \subseteq C_{G^*}^\circ(st)} \alpha_{T^*} R_{T^*}^G(st),$$

where T^* runs over the F -stable maximal tori of $C_{G^*}^\circ(st)$. If we restrict χ to the ℓ -regular elements of G^F , we see that

$$\chi^\circ = \sum_{T^* \subseteq C_{G^*}^\circ(st)} \alpha_{T^*} R_{T^*}^G(s)^\circ$$

because $R_{T^*}^G(st)^\circ = R_{T^*}^G(s)^\circ$ by [24, Proposition 2.2]. Since $\alpha_{T^*} \in \mathbb{Q}$ for every F -stable maximal torus $T^* \subseteq C_{G^*}^\circ(st)$ (see Proposition 1.67), it follows that χ° is a \mathbb{Q} -linear combination of the characters in $\hat{\mathcal{E}}(G^F, s)$. This proves the assertion.

(c) The arguments from (a) and (b) also apply to this case, so we only need to study the series corresponding to the quasi-isolated elements. Let $t \in C_{G^*}(s)_2^F$ be such that st is quasi-isolated. Observe that $C_{G^*}(st)^F = C_{G^*}^\circ(st)^F$. Since $C_{G^*}(st)$ is of type A , every irreducible character in $\mathcal{E}(G^F, st)$ is uniform by Remark 1.75 (b). Let $\chi \in \mathcal{E}(G^F, st)$. With the same arguments we used for uniform characters above, we can show that χ° is an integral linear combination of characters in $\hat{\mathcal{E}}(G^F, s)$. \square

Thus far the only ℓ -blocks for which we do not have a nice generating set are the blocks numbered 13, 14 and 15, corresponding to the centralizers of type $A_5(q)A_1(q)$. The methods developed so far all fail in this case. There are $t \in C_{G^*}(s)_3^F$ such that st is not quasi-isolated and $t \notin Z(L^*)$, where L^* is the minimal Levi subgroup of G^* containing $C_{G^*}(st)$. Therefore we can not use the methods from the proof of Theorem 3.7. On top of that, not all irreducible characters in $\mathcal{E}(G^F, st)$ are uniform for these t 's, so we can not apply the methods used in the proof of Theorem 3.18 either. We were unfortunately not able to resolve these issues for the block numbered 13 so far. But for the blocks 14 and 15 we can bypass all of these issues using elementary facts on F -stable points of centralisers.

Proposition 3.19. *Let $s \in G^{*F}$ and q be the prime power corresponding to F . If*

- (a) $G^F = E_{6,sc}(q)$ and $3 \nmid (q-1)$, or
- (b) $G^F = {}^2E_{6,sc}(q)$ and $3 \nmid (q+1)$,

then $C_{G^*}^\circ(s)^F = C_{G^*}(s)^F$.

Proof. (a) By [19, Proposition 2.5], $(C_{G^*}(s)/C_{G^*}^\circ(s))^F$ is isomorphic to a subgroup of $Z(G)/(F-1)Z(G)$. It is known that, if $p \neq 3$, then $Z(G) = C_3$, where C_3 is a cyclic group of order 3, and $Z(G) = \{1\}$ when $p = 3$ (see [22, Theorem 1.12.5]). Furthermore, $|Z(G^F)| = \gcd(3, q-1)$ (see [35, Table 24.2]). Hence,

$$Z(G)/(F-1)Z(G) = \begin{cases} \{1\} & \text{if } 3 \nmid q-1 \\ C_3 & \text{if } 3 \mid q-1 \end{cases}$$

and the assertion follows. The proof of (b) is analogous. □

Theorem 3.20. *Let $\ell \nmid q$ be a bad prime for G and let $1 \neq s \in G^{*F}$ be a semisimple, quasi-isolated ℓ' -element.*

- (a) *If $3 \nmid (q-1)$, then $\mathcal{E}(G^F, s)$ is an ordinary generating set for $\mathcal{E}_\ell(G^F, s)$. In particular, the number of irreducible Brauer characters in $\mathcal{E}_\ell(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*
- (b) *If $3 \nmid (q+1)$ the analogous statement holds for ${}^2E_{6,sc}(q)$.*

Proof. (a) For blocks corresponding to centralisers of type A_2^3 the assertion follows from Corollary 3.15 (which is in fact true for all q). Now suppose that the centraliser of s is of type D_4 . By the assumption on q we have $C_{G^*}(s)^F = \Phi_1\Phi_2 \cdot {}^2D_4(q)$ (see the remark before Theorem 3.18). Thus we are done by Theorem 3.18 (c).

Now suppose that the centraliser of s is of type $A_5 \times A_1$. Let $t \in C_{G^*}(s)_3^F$ such that st is not quasi-isolated in G^* and let L^* be the minimal Levi subgroup of G^* containing $C_{G^*}^\circ(st)$. By Proposition 3.19, $C_{G^*}^\circ(st)^F = C_{G^*}(st)^F$. Therefore condition (1) of Remark 3.10 is satisfied, as $C_{G^*}^\circ(st)C_{G^*}(st)^F = C_{G^*}^\circ(st) \subseteq L^*$. If L^* is of type A , then $C_{G^*}^\circ(st) = C_{L^*}^\circ(st)$ is a Levi subgroup of G^* . It follows that $L^* = C_{G^*}^\circ(st)$ by the minimality of L^* . Hence, $t \in Z(L^*)$ and therefore condition (2) of Remark 3.10 is also satisfied. Now, assume that L^* is of type D . Since the derived subgroup of these Levis are of simply-connected type by Lemma 3.12, all centralisers of semisimple elements in L^* are connected. Hence, $C_{L^*}(t)$ is connected. As t is a 3-element, $C_{L^*}(t)$ is a Levi subgroup of L^* (and therefore of G^*). By the minimality of L^* , we have $L^* = C_{L^*}(t)$. Thus, $t \in Z(L^*)$. Since condition (1) and (2) are satisfied for every possible Levi subgroup, the assertion follows.

- (b) Similar to part (a). □

Now we need to know the size of the Lusztig series appearing in the generating sets. We determine the size of the following series as an example.

Proposition 3.21. *Let $z \in G^{*F}$.*

- (a) *If $C_{G^*}(z)^F = \Phi_1^2.A_1(q)^4.3$, then $|\mathcal{E}(G^F, z)| = 16$.*
- (b) *If $C_{G^*}(z)^F = \Phi_3.A_1(q)A_1(q^3).3$, then $|\mathcal{E}(G^F, z)| = 12$.*

Proof. (a) The quotient group $N := C_{G^*}(st)^F/C_{G^*}^\circ(st)^F$ acts on $C_{G^*}(st)^F$ by permuting 3 of the $A_1(q)$ factors (can be seen using Chevie [36] for example). Since unipotent characters are trivial on the center, the unipotent character of $C_{G^*}(st)^F$ can be understood as elements in $\{1, St\}^4$ where St denotes the Steinberg character for $A_1(q)$. Let the first $A_1(q)$ factor be the one fixed by the action of N . Now we check the orbits of $\{1, St\}$ under the action of N . A system of representatives for the action of N on $C_{G^*}(st)^F$ is given by $1 \times 1 \times 1 \times 1$ (orbit of length 1), $1 \times 1 \times 1 \times St$ (orbit of length 3), $1 \times 1 \times St \times St$ (orbit of length 3), $1 \times St \times St \times St$ (orbit of length 1) and the same again where we change the first factor from 1 to St . The assertion now follows from Jordan decomposition.

(b) The connected version $\Phi_3.A_1(q)A_1(q^3)$ has only 4 unipotent characters with pairwise different degrees (can be seen with Chevie, for example). Now, $C_{G^*}(st)^F/C_{G^*}^\circ(st)^F = C_3$ acts on $\mathcal{E}(C_{G^*}^\circ(st)^F, 1)$ by conjugation which fixes the degrees. In particular, the irreducible characters in $\mathcal{E}(C_{G^*}^\circ(st)^F, 1)$ are invariant under the given operation. The assertion now follows either by Clifford theory or Theorem 1.74 again. \square

As before, the following result also holds for ${}^2E_6(q)$ and the proof is similar. We only have to keep in mind how Ennola duality acts on Table 13.

Theorem 3.22. *Let G be simple, simply connected of type E_6 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = E_6(q)$. Let $e = e_\ell(q) = 1$. Then Table 14 gives an upper bound for $c(B)$ and a lower bound for $s(B)$ for every ℓ -block B given in Table 13, except for the block numbered 13. In particular, the Malle–Robinson conjecture holds in strong form for these blocks and the corresponding blocks of $G^F/Z(G^F)$.*

Table 14: Upper bounds for $l(B)$ of quasi-isolated blocks of $E_6(q)$

B	$o(s)$	(ℓ, e)	$C_{G^*}(s)^F$	$c(B) \leq$	$\leq s(B)$	Proof for $c(B)$
1	3	(2,1)	$A_2(q)^3.3$	16	6	Prop. 3.15
2, 2', 2''	3	(2,1)	$A_2(q^3).3$	3	2	Prop. 3.15
3	3	(2,1)	${}^2A_2(q)A_2(q^2)$	9	6	Prop.3.15/Thm 3.20
4	3	(2,1)	$\Phi_1^2.D_4(q).3$	26 + 16	6	Thm 3.18
5	3	(2,1)	$\Phi_1\Phi_2.{}^2D_4(q)$	10	4	Thm 3.18/Thm 3.20
6, 6', 6''	3	(2,1)	$\Phi_3.{}^3D_4(q).3$	8 + 4	4	Thm 3.18
7	3	(2,2)	$A_2(q)^3.3$	16	6	Prop. 3.15
8, 8', 8''	3	(2,2)	$A_2(q^3).3$	3	2	Prop. 3.15
9	3	(2,2)	${}^2A_2(q)A_2(q^2)$	9	6	Prop. 3.15/Thm 3.20
10	3	(2,2)	$\Phi_1^2.D_4(q).3$	26 + 16	6	Thm 3.18
11	3	(2,2)	$\Phi_1\Phi_2.{}^2D_4(q)$	10	6	Thm 3.18/Thm 3.20
12, 12', 12''	3	(2,2)	$\Phi_3.{}^3D_4(q).3$	8 + 4	4	Thm 3.18
14	2	(3,2)	$A_5(q)A_1(q)$	20	4	Thm 3.20
15		(3,2)		2	1	

Proof. We start by proving the conjecture for G^F . In this case, the proof is similar to the proof of Theorem 3.9. The entries in the $c(B)$ -column were again obtained with the help of Chevie [36]. Except for the cases 3, 5, 7, 10, we get a sufficient lower bound on $s(B)$ by using the normal series

$$Z(L)_\ell^F \trianglelefteq P := C_P(Z(L)_\ell^F) \trianglelefteq D,$$

from Theorem 3.2. For the block B numbered 7 we use the 1-cuspidal pair (L, λ) in line 1 (see the proof of [29, Proposition 4.3]). We have $L = C_G^\circ(Z(L)_\ell^F)$ and λ is of central ℓ -defect. By combining [29, Proposition 2.13 (a)] and [29, Proposition 2.16 (3)], the pair (L, λ) satisfies the conditions of [29, Proposition 2.12]. Hence, $(Z(L)_2^F, b)$ is a B -Brauer pair where b is the block of L containing λ . In particular, $Z(L)_2^F = \Phi_1^6 \subseteq D$ where D is a defect group of B . For case 3 we use the 2-cuspidal pair from case 8. The cases 6, 6', 6'' and 12, 12', 12'' have to be proved using completely different methods (see Proposition 3.25).

Let \bar{B} now be a quasi-isolated block of $H = G^F/Z(G^F)$ with defect group \bar{D} dominated by a quasi-isolated block B of G^F .

If $\ell = 2$, we use the same line of arguments we used (in the case where $\ell = 3$) in the proof of Theorem 3.9.

If $\ell = 3$, then $l(\bar{B}) = l(B)$ and $\bar{D} = D/Z(G^F)$ for a defect group D of B by Theorem 1.22. We can not say whether or not $s(\bar{D}) = s(D)$. In the worst case, we might have $s(\bar{D}) = s(D) - 1$. In the case of block 14 we are still able to establish the conjecture, even if $s(\bar{D}) = s(D) - 1$, as $20 < 3^3$. Let B be the block numbered 15. As can be seen from the proof of [29, Proposition 4.3], the defect groups of B are cyclic. Hence the defect groups of \bar{B} are cyclic as well. In this case the conjecture for \bar{B} follows from [34, Proposition 3.1]. \square

Now we want to show the conjecture for the remaining blocks 6, 6', 6'' and 12, 12', 12''.

Let G be a connected reductive algebraic group (only for this exposition) and let G^* be a dual group. Let W and W^* be the Weyl groups of G and G^* respectively. By [12, Proposition 4.2.3] there is a natural isomorphism $W \cong W^*$. This isomorphism yields a canonical isomorphism between $N_{G^*}(L^*)/L^*$ and $N_G(L)/L$. Now, fix a semisimple ℓ' -element $s \in G^{*F}$ and let $L^* = C_{G^*}^\circ(Z^\circ(C_{G^*}^\circ(s)))$ be the minimal Levi subgroup of G^* containing $C_{G^*}^\circ(s)$. Furthermore set $N^* = C_{G^*}(s)^F.L^*$ and let L be a dual of L^* in G . Define N to be the subgroup of $N_G(L)$ containing L such that N/L corresponds to N^*/L^* via the canonical isomorphism between $N_{G^*}(L^*)/L^*$ and $N_G(L)/L$.

Let $\ell \nmid q$ be a prime. We denote the sum of the block idempotents of the ℓ -blocks contained in $\mathcal{E}_\ell(G^F, s)$ and $\mathcal{E}_\ell(L^F, s)$ by $e_s^{G^F}$ and $e_s^{L^F}$ respectively.

Theorem 3.23 (Bonnafé–Dat–Rouquier, [2, Theorem 7.7]). *Let the notations be as above. Then there exists a Morita equivalence*

$$\mathcal{O}G^F e_s^{G^F} \sim \mathcal{O}N^F e_s^{L^F}$$

together with a bijection $b \mapsto b'$ between the ℓ -blocks of both sides, preserving defect groups and such that $\mathcal{O}G^F b$ is Morita equivalent to $\mathcal{O}N^F b'$.

Let G now be a simple, simply connected algebraic group of type E_6 again. Let $s \in G^{*F}$ be a quasi-isolated semisimple element of order 3 with $C_{G^*}(s)^F = \Phi_3 \cdot {}^3D_4(q).3$ and let B be of type 6 or 12 from Table 13. We see that $L^* = C_{G^*}(s)^\circ$. Furthermore, N/L is cyclic of order 3. Hence, we are in the situation of [2, Example 7.9]. Thus, there exists a Morita equivalence $\mathcal{O}G^F e_s^{G^F} \sim \mathcal{O}N^F e_1^{L^F}$ together with a bijection as in Theorem 3.23 between the blocks on both sides that preserves defect groups and such that corresponding blocks are Morita equivalent. So every block contained in $\mathcal{E}_\ell(G^F, s)$ is Morita equivalent to a block of N^F which itself covers a unipotent block of L^F .

Remark 3.24. Let H be a finite group and $K \trianglelefteq H$. If B is a block of H covering a block b of K , then B has a defect group D such that $D \cap K$ is a defect group of b (see [38, (9.26) Theorem]). We use this fact in the case where $H = N^F$ and $K = L^F$.

Proposition 3.25. *Let $s \in G^{*F}$ be a quasi-isolated semisimple element of order 3 such that $C_{G^*}(s)^F = \Phi_3 \cdot {}^3D_4(q).3$. Let B be of type 6 or 12. Then $c(B) \leq 12$ and $4 \leq s(B)$. In particular, the Malle–Robinson conjecture holds in strong form for the blocks of type 6 and 12.*

Proof. We demonstrate the proof for case 6 as all other cases ($6', 6'', 12, 12', 12''$) are done the same way. Let B be the block numbered 6. The Lusztig series $\mathcal{E}(G^F, s)$ corresponding to s decomposes into three 2-blocks and each of those blocks contains eight out of the twenty-four irreducible characters of this Lusztig series. To get an upper bound on $c(B)$, we also need to know the decomposition of $\mathcal{E}(G^F, z)$ into 2-blocks where $z \in G^*$ with $C_{G^*}(z)^F = \Phi_3 \cdot A_1(q)A_1(q^3).3$, since the corresponding Lusztig series is part of the generating set for $\mathcal{E}_2(G^F, s)$. This series decomposes into the three 2-blocks from above; each one containing 4 out of the 12 irreducible characters of $\mathcal{E}(G^F, z)$. Hence, $c(B) \leq 12$.

For the lower bound on $s(B)$ we use Theorem 3.23 and the classification of unipotent blocks in bad characteristic obtained by Enguehard [16]. Let D be a defect group of B . We are interested in elementary abelian 2-sections of D . By Theorem 3.23 we can reduce this to the study of defect groups of the Bonnafé–Dat–Rouquier correspondent block of N^F which itself covers a unipotent block of L^F . By Remark 3.24, we are done if we can find a sufficiently large elementary abelian 2-section in the defect groups of those unipotent blocks. We can furthermore reduce this to the study of the defect groups of the unipotent blocks of the group ${}^3D_4(q) = [L, L]^F$ as can be seen as follows. Restriction of characters gives a bijection $\mathcal{E}(L^F, 1) \rightarrow \mathcal{E}([L, L]^F, 1)$ (see e.g. [15, 13.20 Proposition]). By the character-theoretic characterization of covering blocks (see Theorem 1.20), we know that the unipotent blocks of L^F cover the unipotent blocks of $[L, L]^F$. By the classification of unipotent blocks in [16], the only unipotent 2-block of ${}^3D_4(q)$ is the principal block. So it is enough to show that the Sylow 2-subgroups of ${}^3D_4(q)$ have an elementary abelian section of order 16. Checking [22, Table 4.5.1], we see that there is a subgroup C (the p' -part of the centralizer of an involution of ${}^3D_4(q)$) of type $(A_1(q) \times A_1(q^3))/S$ such that $Z(A_1(q)) = Z(A_1(q^3)) = \langle m \rangle$ and $S := \{(1, 1), (m, m)\}$. The Sylow 2-subgroups Q and Q' of the two A_1 -factors are generalized quaternion. Clearly, m is contained in both of them and is moreover also contained in their commutator subgroups. Hence, S is contained in $Q \times Q'$ and in $[Q, Q] \times [Q', Q']$. In particular $(Q \times Q')/S$ is a Sylow 2-subgroup of C and is therefore contained in a Sylow 2-subgroup of ${}^3D_4(q)$. The following yields the asserted

lower bound on $s(B)$.

$$\begin{aligned} ((Q \times Q')/S)/([Q, Q] \times [Q', Q']/S) &\cong (Q \times Q')/([Q, Q] \times [Q', Q']) \\ &\cong Q/[Q, Q] \times Q'/[Q', Q'] \\ &\cong C_2 \times C_2 \times C_2 \times C_2, \end{aligned}$$

where the last isomorphism is a general property of generalized quaternion groups. With this Theorem 3.22 is proved. \square

3.3 The quasi-isolated blocks of $E_7(q)$

Let G now be a simple, simply-connected group of type E_7 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Since the center of G is disconnected, we encounter the same intricacies we encountered for E_6 .

Let ℓ be a bad prime for G not dividing q . Let $1 \neq s \in G^{*F}$ be a semisimple, quasi-isolated ℓ' -element and let $t \in C_{G^*}(s)_\ell^F$. Checking Table 1, we see that elements of order 6 are not isolated and elements of order greater than 6 are not quasi-isolated in G^* . Hence, if $1 \neq t$, then st can not be isolated. Thus there exists a proper minimal Levi subgroup M^* of G^* containing $C_{G^*}^\circ(st)$.

Remark 3.26. Contrary to the E_6 -case, we gain nothing from this since we do not have an analogue of Proposition 3.19. To have $C_{G^*}^\circ(st)^F = C_{G^*}(st)^F$ in general, we need $2 \nmid (q-1)$. In other words, q would have to be a power of 2. Bear in mind that, since ℓ is a bad prime for G , either $\ell = 2$ and s is a semisimple 3-element, or $\ell = 3$ and s is a semisimple 2-element. The first case is in defining characteristic, for which Conjecture 1 has been proved (see [34, Theorem 3 (b)]) already and the latter case can not occur because there are no semisimple 2-elements in $E_7(2^r)$.

Lemma 3.27. *Let $L^* \subseteq G^*$ be a proper Levi subgroup of G^* . Then $[L^*, L^*]$ is simply connected unless L^* is of one of the following types: $D_6, A_5 \times A_1, A_3 \times A_2 \times A_1, D_5 \times A_1, A_5, D_4 \times A_1, A_3 \times A_1^2, A_2 \times A_1^3, A_3 \times A_1, A_1^4, A_1^3$.*

Proof. This can be checked using Chevie [36] for example. \square

Proposition 3.28. *Let $s \in G^{*F}$ be a quasi-isolated semisimple element of order 4, i.e. $C_{G^*}^\circ(s)$ is of type $A_3^2 \times A_1$ or $D_4 \times A_1^2$. Let $1 \neq t \in C_{G^*}(s)_3^F$. If L^* is the minimal proper Levi subgroup of G^* containing $C_{G^*}(st)$, then $t \in Z(L^*)$. In particular, $\mathcal{E}(G^F, s)$ is an ordinary generating set for $\mathcal{E}_3(G^F, s)$.*

Proof. First of all, the order of st is greater than 6 so st is not quasi-isolated. Hence, there exists a minimal proper Levi subgroup L^* of G^* containing $C_{G^*}(st)$. The order of t is good for $C_{G^*}^\circ(s)$ in both cases. Hence, $C_{G^*}^\circ(st)$ is a Levi subgroup of $C_{G^*}^\circ(s)$.

Assume that L^* is of type E_6 . By Lemma 3.27, $[L^*, L^*]$ is of simply connected type. Thus $C_{G^*}(st) = C_{L^*}(st)$ is connected. Since L^* is minimal, st is quasi-isolated in L^* . By [5] (or using [36]) we can determine the quasi-isolated, semisimple elements of L^* . Either st is in $Z(L^*)$, or $C_{G^*}(st) = C_{L^*}(st)$ is of type A_2^3 or $A_5 \times A_1$. As it is a Levi subgroup of $C_{G^*}^\circ(s)$, the centralizer of st can not be of those types. Hence, L^* can not be of type E_6 .

Assume L^* to be of classical type now. Then 3 is not a bad prime for L^* . Further note that centralizers of 3-elements are connected in G^* . Therefore we have

$$C_{G^*}(st) \subseteq L^* \cap C_{G^*}(t) \subseteq C_{G^*}(t).$$

Now, $C_{L^*}(t) = L^* \cap C_{G^*}(t)$ is a Levi subgroup of G^* by Proposition 1.38 and Corollary 1.37. The minimality of L^* then yields $L^* = L^* \cap C_{G^*}(t)$. Hence, $t \in Z(L^*)$.

Consequently, conditions (1) and (2) of Remark 3.10 are satisfied in every case which proves the assertion. \square

Remark 3.29. Let z be a quasi-isolated element of order 6 in G^* . It can be shown (using Chevie for example) that $z = st$ where s is quasi-isolated of order 2 with $[C_{G^*}^\circ(s), C_{G^*}^\circ(s)] = E_6$, and t is quasi-isolated of order 3 with $C_{G^*}(t) = A_5 \times A_2$ (or vice-versa).

Proposition 3.30. *Let $s \in G^{*F}$ be a semisimple, quasi-isolated 2-element such that $C_{G^*}^\circ(s)$ is either of type A_7 or of type $A_1 \times D_6$. Let $1 \neq t \in C_{G^*}(s)_3^F$. If L^* is the minimal proper Levi subgroup of G^* containing $C_{G^*}(st)$, then $t \in Z(L^*)$. Furthermore, $\mathcal{E}(G^F, s)$ is an ordinary generating set for $\mathcal{E}_3(G^F, s)$. In particular, the number of irreducible Brauer characters in $\mathcal{E}_3(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*

Proof. We can handle both cases at once. By Remark 3.29, st is not quasi-isolated. Hence, there exists a minimal proper Levi subgroup L^* of G^* containing $C_{G^*}(st)$.

If L^* is of classical type, then $t \in Z(L^*)$ by similar arguments as in the proof of Proposition 3.28.

Now suppose that L^* is non-classical, i.e L^* is of type E_6 . Since $[L^*, L^*]$ is simply connected by Lemma 3.27, $C_{G^*}(st) = C_{L^*}(st)$ is connected. Recall again that since L^* is minimal, st is quasi-isolated in L^* . Therefore $C_{G^*}(st)$ is either of type $A_1 \times A_5$ or type $A_2 \times A_2 \times A_2$. It can not be of the latter type since $C_{G^*}(st)$ is a Levi subgroup of $C_{G^*}^\circ(s)$ by Proposition 1.38. Hence, $C_{G^*}(st) = A_1 \times A_5$. Since $C_{G^*}(st)$ is connected, we know that $L^* = C_{G^*}(Z^\circ(C_{G^*}(st)))$. In particular, $Z(L^*) = Z^\circ(C_{G^*}(st))$. Checking the tables of Lübeck [33], we see that $Z(C_{G^*}(st))^F/Z^\circ(C_{G^*}(st))^F = 2$. Now, $t \in Z(C_{G^*}(st))_3^F$ which therefore implies $t \in Z^\circ(C_{G^*}(st)) = Z(L^*)$. The second part of the assertion follows since conditions (1) and (2) of Remark 3.10 are satisfied. \square

For the following statements to make sense, we cite the classification of quasi-isolated blocks for $E_7(q)$ from [29] before proceeding.

Theorem 3.31 ([29, Proposition 5.1, Proposition 5.3]). *Let $\ell \nmid q$ be a bad prime for G . Let $e = e_\ell(q) = 1$. For any quasi-isolated semisimple ℓ' -element $1 \neq s \in G^{*F}$, the ℓ -block distribution of $\mathcal{E}(G^F, s)$, the decomposition of $\mathcal{E}(G^F, s)$ into e -Harish-Chandra series, and the relative Weyl groups of the e -cuspidal pairs are as indicated in Table 15.*

Table 15: Quasi-isolated blocks of $E_7(q)$

No.	$C_{G^*}(s)^F$	(e, ℓ)	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L, \lambda)$
1	$A_5(q)A_2(q)$	(2, 1)	Φ_1^7	Φ_1^7	1	$A_5 \times A_2$
2	${}^2A_5(q)A_2(q)$	(2, 1)	$\Phi_1^4.A_1(q)^3$ $\Phi_1^3.D_4(q)$ $\Phi_1.D_6(q)$ G^F	$\Phi_1^4\Phi_2^3$ $\Phi_1^3\Phi_2^2.A_2(q)$ $\Phi_1\Phi_2^2.A_5(q)$ $C_{G^*}(s)^F$	1 ϕ_{21} ϕ_{321} $\phi_{321} \otimes \phi_{21}$	$C_3 \times A_1$ C_3 A_1 1
3	$D_6(q)A_1(q)$	(3, 1)	Φ_1^7	Φ_1^7	1	$D_6 \times A_1$
4			$\Phi_1^3.D_4(q)$	$\Phi_1^3.D_4(q)$	$D_4[1]$	$B_2 \times A_1$
5	$A_7(q).2$	(3, 1)	Φ_1^7	Φ_1^7	1	$A_7.2$
6	${}^2A_7(q).2$	(3, 1)	$\Phi_1^4.A_1(q)^3$	$\Phi_1^4\Phi_2^3.2$	1	C_4
7			$\Phi_1.D_6(q)$	$\Phi_1\Phi_2^2.A_5(q).2$	ϕ_{321}	A_1
8	$\Phi_1.E_6(q).2$	(3, 1)	Φ_1^7	Φ_1^7	1	$E_6.2$
9			$\Phi_1^3.D_4(q)$ $\Phi_1.E_6(q)$	$\Phi_1^3.D_4(q)$ $\Phi_1.E_6(q)$	$D_4[1]$ $E_6[\theta^{\pm 1}]$	$A_2.2$ 2
10	$\Phi_2^2.E_6(q).2$	(3, 1)	$\Phi_1^4.A_1(q)^3$ G^F	$\Phi_1^4\Phi_2^3.2$ $C_{G^*}(s)^F$	1 ${}^2E_6[\theta^{\pm 1}]$, ${}^2E_6[1]$	F_4 1
11			$\Phi_1.D_6(q)$	$\Phi_1\Phi_2^2.A_5(q).2$	ϕ_{321}	A_1
12	$A_3(q)^2A_1(q).2$	(3, 1)	Φ_1^7	Φ_1^7	1	$A_3 \wr 2 \times A_1$
13	${}^2A_3(q)^2A_1(q).2$	(3, 1)	$\Phi_1^5.A_1(q)^2$	$\Phi_1^5\Phi_2^2$	1	$B_2 \wr 2 \times A_1$
14	$A_3(q^2)A_1(q).2$	(3, 1)	$\Phi_1^4.A_1(q)^3$	$\Phi_1^4\Phi_2^3.2$	1	$A_3 \times A_1$
15	$\Phi_1.D_4(q)A_1(q)^2.2$	(3, 1)	Φ_1^7	Φ_1^7	1	$(D_4 \times A_1^2).2$
16			$\Phi_1^3.D_4(q)$	$\Phi_1^3.D_4(q)$	$D_4[1]$	$A_1 \wr 2$
17	$\Phi_2.D_4(q)A_1(q)^2.2$	(3, 1)	$\Phi_1^6.A_1(q)$	$\Phi_1^6\Phi_2$	1	$(D_4 \times A_1^2).2$
18			$\Phi_1^2.D_4(q)A_1(q)$	$\Phi_1^2\Phi_2.D_4(q)$	$D_4[1]$	$A_1 \wr 2$
19	$\Phi_1^2.D_4(q)A_1(q^2).2$	(3, 1)	$\Phi_1^5.A_1(q)^2$	$\Phi_1^5\Phi_2^2$	1	$(B_3 \times A_1).2$
20	$\Phi_2^2.D_4(q)A_1(q^2).2$	(3, 1)	$\Phi_1^4.A_1(q)^3$	$\Phi_1^4\Phi_2^3.2$	1	$B_3 \times A_1$
2b		(2, 2)	Φ_2^7	Φ_1^7	1	$A_5 \times A_2$

Proposition 3.32. *Let $s \in G^{*F}$ be a quasi-isolated semisimple element of order 2 with $C_{G^*}(s)^F = \Phi_1.E_6(q).2$ (respectively $C_{G^*}(s)^F = \Phi_2.{}^2E_6(q).2$). Then $\hat{\mathcal{E}}(G^F, s) \cup \hat{\mathcal{E}}(G^F, st)$ is a generating set of $\mathbb{Q} \text{IBr}(\mathcal{E}_3(G^F, s))$ where $t \in C_{G^*}(s)_3^F$ such that:*

(a) $C_{G^*}(st)^F = \Phi_1.A_2(q)^3.2$ (respectively $C_{G^*}(st)^F = \Phi_2.A_2(q)A_2(q^2).2$) if $e = 1$.

(b) $C_{G^*}(st)^F = \Phi_2.{}^2A_2(q)^3.2$ (respectively $C_{G^*}(st)^F = \Phi_1.{}^2A_2(q)A_2(q^2).2$) if $e = 2$.

In particular, the number of irreducible Brauer characters in $\mathcal{E}_3(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)| + |\mathcal{E}(G^F, st)|$.

Proof. There exist elements $t \in C_{G^*}(s)_3^F$ such that st is quasi-isolated. Clearly, condition (1) in Remark 3.10 fails in these cases, which is why the Lusztig series corresponding to quasi-isolated elements of the form st are included in the asserted generating sets. Now suppose that st is not quasi-isolated and let L^* be the minimal Levi subgroup of G^* containing $C_{G^*}(st)$.

Assume L^* to be of type E_6 . Hence, $L^* = C_{G^*}^\circ(s)$ and $C_{G^*}(st) = C_{L^*}(st) = C_{L^*}(t)$. By the minimality of L^* , t is a quasi-isolated element of L^* . The centralisers of semisimple, quasi-isolated 3-elements of L^* are of type A_2^3 (see [5]). Thus we can apply the methods of the proof of Theorem 3.18 for the case where $C_{G^*}(st)$ is connected and of type A .

Now assume L^* to be of classical type. In this case we can show that $t \in Z(L^*)$ as in the proof of Proposition 3.30. \square

With this we are able to prove the Malle–Robinson conjecture for the quasi-isolated blocks of G^F except for the ones numbered 1 and 2 in Table 15. In these cases we encounter the same issues we encountered for the block numbered 13 in Table 13.

Theorem 3.33. *Let $\ell \nmid q$ be a bad prime for G . Let $e = e_\ell(q) = 1$. Then Table 16 gives an upper bound for $c(B)$ and a lower bound for $s(B)$ for every block of Table 15, except for the blocks numbered 1 or 2. In particular, the Malle–Robinson Conjecture holds in strong form for these blocks and the corresponding blocks of $G^F/Z(G^F)$.*

As before the table for $e = 2$ is the Ennola dual of Table 16.

Table 16: Upper bounds for $l(B)$ for quasi-isolated blocks of $E_7(q)$

B	$o(s)$	(ℓ, e)	$C_{G^*}(s)^F$	$c(B) \leq$	$\leq s(B)$
3	2	(3,1)	$D_6(q)A_1(q)$	74	7
4				10	3
5	2	(3,1)	$A_7(q).2$	44	7
6, 6'	2	(3,1)	${}^2A_7(q).2$	20	4
7, 7'				2	1
8	2	(3,1)	$\Phi_1.E_6(q).2$	$50 + 27$	7
9				10	3
10, 10'	2	(3,1)	$\Phi_2.{}^2E_6(q).2$	$28 + 9$	4
11, 11'				3 ($l(B) = 2$)	1
12	4	(3,1)	$A_3(q)^2A_1(q).2$	40	7
13	4	(3,1)	${}^2A_3(q)^2A_1(q).2$	40	5
14, 14'	4	(3,1)	$A_3(q^2)A_1(q).2$	20	4
15	4	(3,1)	$\Phi_1.D_4(q)A_1(q)^2.2$	104	7
16				5	2
17	4	(3,1)	$\Phi_2.D_4(q)A_1(q)^2.2$	104	6
18				5	2
19	4	(3,1)	$\Phi_1.{}^2D_4(q)A_1(q^2).2$	40	5
20, 20'	4	(3,1)	$\Phi_2.{}^2D_4(q)A_1(q^2).2$	20	4

Proof. We start by proving the conjecture for G^F . We determine $c(B)$ as in the proof of Theorem 3.9. Except for the blocks numbered 8, 9, 10/10' and 11/11', $s(Z(L)_\ell^F)$ suffices to establish the Malle–Robinson conjecture where (L, λ) is the e -cuspidal pair associated to the given block.

To prove the conjecture for the blocks of type 8, 9, 10 or 11 in Table 15, we need to determine how the Lusztig series corresponding to the quasi-isolated elements of order 6 (see Proposition 3.32) decompose into 3-blocks. Recall that

$$|G^F| = q^{63}\Phi_1(q)^7\Phi_2(q)^7\Phi_3(q)^3\Phi_4(q)^2\Phi_5(q)\Phi_6(q)^3\Phi_7(q)\Phi_8(q)\Phi_9(q)\Phi_{10}(q)\Phi_{12}(q)\Phi_{14}(q)\Phi_{18}(q)$$

By the assumption on e , the only $\Phi_i(q)$, appearing in the expression above, which are divisible by 3 are Φ_1, Φ_3 and Φ_9 . While $\Phi_1(q)$ can be divisible by higher powers of 3

(depending on q), $\Phi_3(q)$ and $\Phi_9(q)$ are only divisible by 3 and no higher powers of 3. Hence,

$$|G^F|_3 = |\Phi_1(q)|_3^7 |\Phi_3(q)|_3^3 |\Phi_9(q)|_3 = 3^4 |\Phi_1(q)|_3^7.$$

Let $B = b_{G^F}(L, \lambda)$ be a block of type 8, 9, 10 or 11 (see Table 15) and let D be a defect group of B . By Theorem 3.2 we know that D is a Sylow 3-subgroup of the extension of $Z(L)_3^F$ by $W_G(L, \lambda)$. Hence, $|D| = |Z(L)_3^F| |W_G(L, \lambda)|_3$. By Remark 1.19,

$$|G^F|_3/|D| = \min\{\chi(1)_\ell \mid \chi \in B\}. \quad (1)$$

These cardinalities can be determined easily.

B	$ D $	$ G^F _3/ D $
8	$3^4 \Phi_1(q) _3^7$	1
9	$3 \Phi_1(q) _3^3$	$3^3 \Phi_1(q) _3^4$
10	$3^2 \Phi_1(q) _3^4$	$3^2 \Phi_1(q) _3^3$
11	$ \Phi_1(q) _3$	$3^4 \Phi_1(q) _3^6$

We start with the blocks numbered 8 and 9. By Proposition 3.32, $\hat{\mathcal{E}}(G^F, s) \cup \hat{\mathcal{E}}(G^F, st)$ is an ordinary generating set for the union of the blocks numbered 8 and 9, where $s \in G^{*F}$ with $C_{G^*}(s)^F = \Phi_1.E_6(q).2$ and $t \in C_{G^*}(s)_3^F$ such that $C_{G^*}(st)^F = \Phi_1 A_2(q)^3.2$. We claim that the series $\mathcal{E}(G^F, st)$ is contained in the block numbered 8. In this case, $s(Z(L)_3^F)$ (see lines 8 and 9 of Table 16) is enough to establish the Malle–Robinson conjecture 1 for these blocks. Let Ψ_{st} denote the Jordan decomposition associated with st (see Theorem 1.74). Let $\chi \in \mathcal{E}(G^F, st)$. By [15, 13.24 Remark] we have

$$\chi(1)_3 = \frac{|G^F|_3}{|C_{G^*}(st)^F|_3} \Psi_{st}(\chi)(1)_3.$$

The right side of this equation can easily be computed and we observe that $\chi(1)_3 < 3^3 |\Phi_1(q)|_3^4$ for every $\chi \in \mathcal{E}(G^F, st)$. Now, it follows from (1) that $\mathcal{E}(G^F, st)$ is fully contained in the block numbered 8.

We argue similarly for the blocks of type 10 and 11. It can be shown that the Lusztig series corresponding to the quasi-isolated element of order 6 (appearing in the generating set for the union of the blocks of type 10 and 11) is contained in the blocks of type 10. Unfortunately our approach does not prove the strong form of Conjecture 1 for the blocks of type 11. Let $B = b_{G^F}(L, \lambda)$ be a block of type 11. Let q be such that $|Z(L)_3^F| = \Phi_1(q) = 3$. Then we have equality:

$$3 = c(B) = 3^{s(Z(L)_3^F)} = 3.$$

Since the defect groups of the blocks of type 11 are cyclic (of order $|\Phi_1(q)|_3$) by Theorem 3.2 (c), the strong form of Conjecture 1 holds by [34, Proposition 3.1].

For the quasi-isolated blocks of $G^F/Z(G^F)$ we use the arguments of the proof of Theorem 3.22. \square

Remark 3.34. The proof of Theorem 3.33 yields another example as to why the assumption on the primes is necessary in Theorem 1.78. Let $s \in G^{*F}$ be a quasi-isolated element associated with the ℓ -blocks of type 11 in Table 15. Let B be such a block. We saw that there exist q 's such that $|\text{Irr}(B) \cap \mathcal{E}(G^F, s)| = \ell^{s(B)}$. Since the conjecture holds in strong form, we have $l(B) < \ell^{s(B)}$ for these blocks. Hence, $l(B) < |\text{Irr}(B) \cap \mathcal{E}(G^F, s)|$ for these q 's.

3.4 The quasi-isolated blocks of $E_8(q)$

Let G be simple, simply connected of type E_8 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Recall that simple algebraic groups of type E_8 are both simply connected and adjoint. We will therefore omit any specification of the isogeny type as we did in Section 3.1.

Let $\ell \nmid q$ be a bad prime for G . Let $1 \neq s \in G^{*F}$ be a semisimple, quasi-isolated ℓ' -element and let $1 \neq t \in C_{G^*}(s)_\ell^F$. Table 1 shows that there are quasi-isolated elements of order 6 in G^{*F} . As before, we know exactly what these elements look like.

Remark 3.35. Let $z \in G^{*F}$ be a quasi-isolated element of order 6. It can be shown (e.g. using Chevie [36]) that $z = st$ where s is of order 2 such that $C_{G^*}(s) = E_6 \times A_2$ and t is of order 3 such that $C_{G^*}(t) = E_7 \times A_1$ (or vice-versa).

Evidently, st is not quasi-isolated if either $C_{G^*}(s)$ or $C_{G^*}(t)$ is of classical type.

Theorem 3.36 ([29, Proposition 6.1, Proposition 6.4]). *Let $e = e_\ell(q) = 1$. For any quasi-isolated 3- or 5-element (respectively 2- or 5-element) of G^{*F} , the 2-block (respectively 3-block) distribution of $\mathcal{E}(G^F, s)$, the decomposition of $\mathcal{E}(G^F, s)$ into e -Harish-Chandra series and the relative Weyl groups of the e -cuspidal pairs are as indicated in Table 17 (respectively Table 18).*

Table 17: Quasi-isolated 2-blocks of $E_8(q)$

No.	$C_{G^*}(s)^F$	e	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L, \lambda)$
1	$A_8(q)$	1	Φ_1^8	Φ_1^8	1	A_8
2	${}^2A_8(q)$	1	$\Phi_1^4.A_1(q)^4$	$\Phi_1^4\Phi_2^4$	1	B_4
			$\Phi_1^3D_4(q)A_1(q)$	$\Phi_1^3\Phi_2^3.{}^2A_2(q)$	ϕ_{21}	B_3
3	$E_6(q).A_2(q)$	1	Φ_1^8	Φ_1^8	1	$E_6 \times A_2$
			$\Phi_1^4.D_4(q)$	$\Phi_1^4.D_4(q)$	$D_4[1]$	$G_2 \times A_2$
4			$\Phi_1^2.E_6(q)$	$\Phi_1^2.E_6(q)$	$E_6[\theta^{\pm 1}]$	A_2
5	${}^2E_6(q).{}^2A_2(q)$	1	$\Phi_1^5.A_1(q)^3$	$\Phi_1^5\Phi_2^3$	1	$F_4 \times A_1$
			$\Phi_1^4.D_4(q)$	$\Phi_1^4\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	F_4
			$\Phi_1^2.D_6(q)$	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	$A_1 \times A_1$
			$\Phi_1.E_7(q)$	$\Phi_1.{}^2A_5(q).{}^2A_2(q)$	$\phi_{321} \otimes \phi_{21}$	A_1
			$\Phi_1.E_7(q)$	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[1]$	A_1
6			G^F	$C_{G^*}(s)^F$	${}^2E_6[1] \otimes \phi_{21}$	1
			$\Phi_1.E_7(q)$	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[\theta^{\pm 1}]$	A_1
			G^F	$C_{G^*}(s)^F$	${}^2E_6[\theta^{\pm 1}] \otimes \phi_{21}$	1

7	$A_4(q)^2$	1	Φ_1^8	Φ_1^8	1	A_4^2
8	${}^2A_4(q^2)$	1	$\Phi_1^2.A_3(q)^2$ $\Phi_1.D_7(q)$	$\Phi_1^2\Phi_2^2\Phi_4^2$ $\Phi_1\Phi_2\Phi_4.{}^2A_2(q^2)$	1 ϕ_{21}	B_2 A_1
9	${}^2A_4(q)^2$	1	$\Phi_1^4.A_1(q)^4$ $\Phi_1^3.D_4(q)4A_1(q)$ $\Phi_1^2.D_6(q)$	$\Phi_1^4\Phi_2^4$ $\Phi_1^3\Phi_2^3.{}^2A_2(q)$ $\Phi_1^2\Phi_2^2.{}^2A_2(q)^2$	1 ϕ_{21} (2 x) $\phi_{21} \otimes \phi_{21}$	B_2^2 $B_2 \times A_1$ A_1^2
5b	${}^2E_6(q).{}^2A_2(q)$	2	Φ_2^8	Φ_2^8	1	$E_6 \times A_2$

Table 18: Quasi-isolated 3-blocks of $E_8(q)$, $q \equiv 1 \pmod 3$

No.	$C_{G^*}(s)^F$	L^F	$C_{L^*}(s)^F$	λ	$W_{G^F}(L, \lambda)$
1	$D_8(q)$	Φ_1^8	Φ_1^8	1	D_8
2		$\Phi_1^4.D_4(q)$	$\Phi_1^4.D_4(q)$	$D_4[1]$	B_4
3	$E_7(q)A_1(q)$	Φ_1^8	Φ_1^8	1	$E_7 \times A_1$
4		$\Phi_1^4.D_4(q)$	$\Phi_1^4.D_4(q)$	$D_4[1]$	$C_3 \times A_1$
		$\Phi_1^2.E_6(q)$	$\Phi_1^2.E_6(q)$	$E_6[\theta^{\pm 1}]$	$A_1 \times A_1$
5		$\Phi_1.E_7(q)$	$\Phi_1.E_7(q)$	$E_7[\pm \xi]$	A_1
6	$D_5(q)A_3(q)$	Φ_1^8	Φ_1^8	1	$D_5 \times A_3$
7		$\Phi_1^4.D_4(q)$	$\Phi_1^4.D_4(q)$	$D_4[1]$	$A_3 \times A_1$
8	${}^2D_5(q).{}^2A_3(q)$	$\Phi_1^6.A_1(q)^2$	$\Phi_1^6\Phi_2^2$	1	$B_4 \times C_2$
9	$A_7(q)A_1(q)$	Φ_1^8	Φ_1^8	1	$A_7 \times A_1$
10	${}^2A_7(q)A_1(q)$	$\Phi_1^5.A_1(q)^3$	$\Phi_1^5\Phi_2^3$	1	$C_4 \times A_1$
11		$\Phi_1^2.D_6(q)$	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	A_1^2
12	$A_4(q)^2$	Φ_1^8	Φ_1^8	1	A_4^2
13	${}^2A_4(q^2)$	$\Phi_1^2.A_3(q)^2$	$\Phi_1^2\Phi_2^2\Phi_4^2$	1	B_2
14		$\Phi_1.D_7(q)$	$\Phi_1\Phi_2\Phi_4.{}^2A_2(q^2)$	ϕ_{21}	A_1
15	${}^2A_4(q)^2$	$\Phi_1^4.A_1(q)^4$	$\Phi_1^4\Phi_2^4$	1	B_2^2
16		$\Phi_1^3.D_4(q)A_1(q)$	$\Phi_1^3\Phi_2^3.{}^2A_2(q)$	ϕ_{21} (2x)	$B_2 \times A_1$
17		$\Phi_1^2.D_6(q)$	$\Phi_1^2\Phi_2^2.{}^2A_2(q)^2$	$\phi_{21} \otimes \phi_{21}$	A_1^2

Since no real intricacies occur for the 5-blocks of G^F , we refer to the classification of the quasi-isolated 5-blocks in [29, Tables 7 and 8] without recalling the tables here. Note that these tables do not include the quasi-isolated 5-blocks corresponding to the quasi-isolated elements of order 6. For these blocks we refer to [28].

Proposition 3.37. *Let $\ell \nmid q$ be a bad prime of G . Let $1 \neq s \in G^{*F}$ be a quasi-isolated semisimple ℓ -element and let $1 \neq t \in C_{G^*}(s)_\ell^F$. If*

- (i) $\ell = 5$, or
- (ii) $C_{G^*}(s)$ is classical and $\ell \in \{2, 3\}$,

then $t \in Z(L^)$ where L^* is the minimal proper Levi subgroup of G^* containing $C_{G^*}(st)$. In particular, $\mathcal{E}(G^F, s)$ is an ordinary generating set for $\mathcal{E}_\ell(G^F, s)$. Therefore, the number of irreducible Brauer characters in $\mathcal{E}_\ell(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*

Proof. (i) Since 5 is a good prime for every proper Levi subgroup L^* of G^* , the first part follows from Lemma 3.5 (a). The second part follows since conditions (1) and (2) of Remark 3.10 are satisfied.

(ii) Note that $s, t \in Z(C_{G^*}(st))^F$. Set $n = |Z(C_{G^*}(st))^F / Z^\circ(C_{G^*}(st))^F|$. Now, n can be read off from [33] for every centralizer (of semisimple elements). It can be seen that $n = 2, 3, 4$ or 5 in every case except for the centralizer of the quasi-isolated element of order 6 (where $n = 6$). By the assumption on $C_{G^*}(s)$ and Remark 3.35, it follows that st is not quasi-isolated. Thus, either $(o(t), n) = 1$ or $(o(s), n) = 1$. In the first case, $t \in Z^\circ(C_{G^*}(st))$. Hence, $t \in Z(L^*)$ and we are done. If $(o(s), n) = 1$, then $s \in Z(L^*)$. In particular, $L^* \subseteq C_{G^*}(s)$ and it follows that L^* is of classical type. In every case $C_{G^*}(st) = C_{L^*}(t)$ is a Levi subgroup of G^* by Proposition 1.38 and Corollary 1.37. By the minimality of L^* we therefore have $C_{G^*}(st) = L^*$. Thus, $t \in Z(L^*)$. The second part follows since conditions (1) and (2) of Remark 3.10 are satisfied in every case. \square

Proposition 3.38. *Let $\ell \nmid q$ be a bad prime for G . Let $s \in G^{*F}$ be a quasi-isolated semisimple ℓ' -element with $C_{G^*}(s) = E_6 \times A_2$ or $E_7 \times A_1$. Then $\hat{\mathcal{E}}(G^F, s)$ is a generating set of $\mathbb{Q}\text{IBr}(\mathcal{E}_\ell(G^F, s))$. In particular, the number of irreducible Brauer characters in $\mathcal{E}_\ell(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*

Proof. In both cases there exist elements $1 \neq t \in C_{G^*}(s)_\ell^F$ of order 3 or 2 respectively such that st is quasi-isolated of order 6. Since the centralizer of such an element is connected of type $A_5 \times A_2 \times A_1$, we can proceed as in the proof of Theorem 3.18 and show that $\chi^\circ \in \mathbb{Q}\hat{\mathcal{E}}(G^F, s)$ for every $\chi \in \mathcal{E}(G^F, st)$ in that case.

Now assume that st is not quasi-isolated. Let $L^* = C_{G^*}(Z^\circ(C_{G^*}(st)))$ be the minimal Levi subgroup of G^* containing $C_{G^*}(st)$. Recall that $Z(L^*) = Z^\circ(C_{G^*}(st))$. As in the proof of Proposition 3.37, we conclude that either s or t lie in $Z^\circ(C_{G^*}(st))$. If $t \in Z^\circ(C_{G^*}(st))$ we are done since conditions (1) and (2) are satisfied for these t . Thus, suppose $s \in Z^\circ(C_{G^*}(st))$. In other words $L^* \subseteq C_{G^*}(s)$ and $C_{G^*}(st) = C_{L^*}(st) = C_{L^*}(t)$.

First, suppose $C_{G^*}(s) = E_6 \times A_2$ (i.e. $o(s) = 3$ and t is a 2-element). If L^* is of type A , it follows from Theorem 3.5 that $t \in Z(L^*)$. For the Levi subgroups $E_6 \times A_1$, E_6 , $D_5 \times A_1$, $D_4 \times A_2$, $D_4 \times A_1$ or D_4 we can determine the quasi-isolated 2-elements (e.g. using Chevie) and observe that their centralisers are all of type A . Thus, we can apply the methods used in the proof of Theorem 3.18 for connected centralisers of type A again.

Now suppose that $C_{G^*}(s) = E_7 \times A_1$ (i.e. $o(s) = 2$ and t is a 3-element). If L is of classical type, it follows from Theorem 3.5 that $t \in Z(L^*)$. As above, it can be shown (using Chevie) that the centralisers of the semisimple, quasi-isolated 3-elements of the non-classical Levi subgroups of type E_7 , $E_6 \times A_1$ or E_6 are of type A . Hence, we can apply the methods used in the proof of Theorem 3.18 as before. In conclusion, the assertion is proved. \square

Theorem 3.39. *Let G be simple, simply connected of type E_8 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $e = e_\ell(q) = 1$. Then the Tables 19, 20 and 21 give $c(B)$ and a lower bound for $s(B)$ for every non-unipotent quasi-isolated ℓ -block B of G^F . In particular, the Malle–Robinson conjecture holds for these blocks. Moreover, for the non-unipotent quasi-isolated 2- and 3-blocks of G^F the strong form of the conjecture holds.*

The numbering in the Tables 19 and 20 below is taken from the Tables 5 and 6 in [29]. Similarly, the numbering of the first 44 entries of Table 21 agrees with the numbering of the Tables 7 and 8 in [29]. However, the entries 45 to 51 correspond to blocks that were not included in [29] but will be included in [28].

Table 19: Upper bounds for $l(B)$ for quasi-isolated 2-blocks of $E_8(q)$

B	$o(s)$	$C_{G^*}(s)^F$	$c(B)$	$\leq s(B)$
1	3	$A_8(q)$	30	8
2	3	${}^2A_8(q)$	30	8
3	3	$E_6(q)A_2(q)$	84	8
4	3		3	2
5	3	${}^2E_6(q)A_2(q)$	84	8
6	3		3	2
7	5	$A_4(q)^2$	49	8
8	5	${}^2A_4(q^2)$	7	4
9	5	${}^2A_4(q)^2$	49	8

Table 20: Upper bounds for $l(B)$ for quasi-isolated 3-blocks of $E_8(q)$

B	$o(s)$	$C_{G^*}(s)^F$	$c(B)$	$\leq s(B)$
1	2	$D_8(q)$	100	8
2	2		40	4
3	2	$E_7(q)A_1(q)$	120	8
4	2		28	4
5	2		2	1
6	4	$D_5(q)A_3(q)$	90	8
7	4		10	4
8	4	${}^2D_5(q).{}^2A_3(q)$	100	6
9	4	$A_7(q)A_1(q)$	44	8
10	4	${}^2A_7(q)A_1(q)$	40	5
11	4		4	2
12	5	$A_4(q)^2$	49	8
13	5	${}^2A_4(q^2)$	5	2
14	5		2	1
15	5	${}^2A_4(q)^2$	25	4
16	5		10	3
17	5		4	2

Table 21: Upper bounds for $l(B)$ of quasi-isolated 5-blocks of $E_8(q)$

B	$o(s)$	e	$C_{G^*}(s)$	$c(B)$	$\leq s(B)$
1	2	1	$D_8(q)$	100	8
2				20	4
3	2	1	$E_7(q)A_1(q)$	120	8
4				20	4
5				4	2
6				2	1
7	4	1	$D_5(q)A_3(q)$	90	8
8				10	4

9	4	1	${}^2D_5(q).{}^2A_3(q)$	100	6
10	4	1	$A_7(q)A_1(q)$	44	8
11	4	1	${}^2A_7(q)A_1(q)$	40	5
12				4	2
13	3	1	$A_8(q)$	30	8
14	3	1	${}^2A_8(q)$	20	4
15				10	3
16	3	1	$E_6(q)A_2(q)$	75	8
17				18	4
18				3	2
19	3	1	${}^2E_6(q).{}^2A_2(q)$	50	5
20				25	4
21				4	2
22				2	1
23				2	1
24				1	0
25	2	4	$D_8(q)$	60	4
26				14	2
27				1	0
28	2	4	$E_7(q)A_1(q)$	16	2
29				14	2
30				1	0
31	4	4	$D_5(q)A_3(q)$	56	3
32				14	2
33				16	2
34				4	1
35				1	0
36	4	4	$A_7(q)A_1(q)$	14	2
37				4	1
38				1	0
39	3	4	$A_8(q)$	14	2
40				4	1
41				1	0
42	3	4	$E_6(q)A_2(q)$	16	2
43				4	1
44				1	0
45	6	1	$A_5(q)A_2(q)A_1(q)$	66	8
46	6	1	${}^2A_5(q){}^2A_2(q)A_1(q)$	40	5
47				20	4
48				4	2
49				2	1
50	6	4	$A_5(q)A_2(q)A_1(q)$	4	1
51				1	0

Proof. First, assume that $\ell = 2$. Let $B = b_{G^F}(L, \lambda)$ be a quasi-isolated 2-block of G^F . Except for the blocks of type 2, 8 and 9 in Table 19, $s(Z(L)_2^F)$ suffices to establish the Malle–Robinson conjecture. For the blocks 2, 8 and 9, $s(Z(L)_2^F)$ is not enough (e.g. $c(B_2) = 30$ but $2^{s(Z(L)_2^F)} = 2^4 = 16$ in this case). Let $B = b_{G^F}(L, \lambda)$ be one of those blocks. Recall that we have a normal series

$$Z(L)_i^F \trianglelefteq P := C_P(Z(L)_i^F) \trianglelefteq D,$$

where D is a defect group of B . Furthermore, by [29, Proposition 2.1 + Proposition 2.7], P is a defect group of the block of L^F containing λ . Now, in all cases (2, 8 and 9) $C_{L^*}(s)$ is a maximal torus of L^* . Let $M \subseteq G$ be an F -stable torus dual to $C_{L^*}(s)$. There is a Morita equivalence

$$\mathcal{O}L^F e_s^{L^F} \sim \mathcal{O}M^F e_1^{M^F},$$

(see Theorem 3.23) with a bijection between the blocks on both sides preserving defect groups. In particular, $s(D) = s(D')$ where D' is a defect group corresponding to D by this bijection. Since M is a torus, there is only one block on the right side of the equivalence, namely the principal block of M^F . Every defect group of that block is a Sylow 2-subgroup of M^F . The structure of M^F can be read off from Table 17. Hence we can determine $s(D)$ and find that $c(B) \leq 2^{s(D)}$.

Now suppose that $\ell = 3$ or 5 . Let $B = b_{G^F}(L, \lambda)$ be a quasi-isolated ℓ block of G^F . In this case $s(Z(L)_\ell^F)$ suffices to establish the conjecture. \square

Note that $1 = l(B) = \ell^{s(D)} = 1$ for blocks of defect zero. Hence, the conjecture can not hold in strong form for the quasi-isolated 5-blocks of G^F .

3.5 The quasi-isolated blocks of $G_2(q)$ and ${}^3D_4(q)$

Both of these groups have been studied in great detail. As a result the character tables are known. Furthermore, the Malle–Robinson conjecture is known to hold for all blocks of these groups by [34, Proposition 6.5]. However, in order for this thesis to stay somewhat self-contained, we prove the conjecture for the quasi-isolated blocks of these groups using the methods that were established so far.

It might seem strange that we have to check the conjecture for the 3-blocks of ${}^3D_4(q)$ even though 3 is not a bad prime for the underlying algebraic group. Note that throughout Section 2 we assumed ℓ to be different from 3 if G^F has a component of type ${}^3D_4(q)$. The reason for this can be found in the proof of [8, Theorem 1.5].

Let G be simple, simply-connected of type G_2 or D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = G_2(q)$ or $G^F = {}^3D_4(q)$ respectively. Note that there are no semisimple, quasi-isolated elements of order 6 in both cases (for D_4 check [5, Table 2]). Note that D_4 has no quasi-isolated semisimple element of order bigger than 2.

Theorem 3.40 ([29, Lemma 6.13]). *Let $\ell \in \{2, 3\}$ not dividing q . Let $e = e_\ell(q) = 1$. For every quasi-isolated semisimple ℓ' -element $1 \neq s \in G^{*F}$, the ℓ -block distribution of $\mathcal{E}(G^F, s)$, the decomposition of $\mathcal{E}(G^F, s)$ into e -Harish-Chandra series, and the relative Weyl groups of the e -cuspidal pairs are as indicated in Table 22.*

Table 22: Quasi-isolated blocks of $G_2(q)$ and ${}^3D_4(q)$

G^F	No.	$C_{G^*}(s)^F$	ℓ	L^F	λ	$W_{G^F}(L, \lambda)$
G_2	1	$A_2(q)$	2	Φ_1^2	1	A_2
	2	${}^2A_2(q)$	2	$\Phi_1\Phi_2$	1	A_1
				G^F	ϕ_{21}	1
	3	$A_1(q)A_1(q)$	3	Φ_1^2	1	$A_1 \times A_1$
${}^3D_4(q)$	1	$A_1(q)A_1(q^3)$	3	$\Phi_1^2\Phi_3$	1	$A_1 \times A_1$

Proposition 3.41. *Let $\ell \in \{2, 3\}$ not dividing q . Let $1 \neq s \in G^{*F}$ be a quasi-isolated semisimple ℓ -element and let $1 \neq t \in C_{G^*}(s)_\ell^F$. If $L^* \subseteq G^*$ is the minimal Levi subgroup containing $C_{G^*}(st)$, then $t \in Z(L^*)$. In particular, $\mathcal{E}(G^F, s)$ is an ordinary generating set for $\mathcal{E}_\ell(G^F, s)$. Therefore, the number of irreducible Brauer characters in $\mathcal{E}_\ell(G^F, s)$ is less than or equal to $|\mathcal{E}(G^F, s)|$.*

Proof. Since the simple groups of type G_2 or D_4 do not have quasi-isolated elements of order greater than 3, st is not quasi-isolated. Thus, there exists a minimal proper Levi subgroup L^* of G . The proper Levi subgroups of simple groups of type G_2 or D_4 are of type A . By Theorem 3.5, $t \in Z(L^*)$. Therefore, the assertion follows since conditions (1) and (2) of Remark 3.10 are satisfied. \square

Theorem 3.42. *Let $e = e_\ell(q) = 1$. Then Table 23 gives an upper bound for $c(B)$ and a lower bound for $s(B)$ for every quasi-isolated ℓ -block B of G^F as in Table 22. Moreover, the Malle–Robinson conjecture holds in strong form for these blocks.*

 Table 23: Upper bounds for $l(B)$ for quasi-isolated blocks of $G_2(q)$ and ${}^3D_4(q)$

G^F	B	$C_{G^*}(s)^F$	ℓ	$c(B) \leq$	$\leq s(B)$
G_2	1	$A_2(q)$	2	3	2
	2	${}^2A_2(q)$	2	2	2
				3	4
	3	$A_1(q)A_1(q)$	3	4	2
${}^3D_4(q)$	1	$A_1(q)A_1(q^3)$	3	4	2

Proof. For every quasi-isolated block $B = b_{G^F}(L, \lambda)$ as in Table 23, $s(Z(L)_\ell^F)$ is enough to establish the strong form of the conjecture. \square

The proof of Theorem B

Theorem B. *Let G be a simple, simply connected group of exceptional type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or let G be simple of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Let ℓ be a prime not dividing q . Let B be a non-unipotent, quasi-isolated ℓ -block of G^F . Then the Malle–Robinson conjecture holds for B unless possibly if B is of one of the following types.*

- (i) $G^F = E_6(q)$ or ${}^2E_6(q)$ and B is the 3-block numbered 13 in Table 13, or
- (ii) $G^F = E_7(q)$ and B is either the 2-block numbered 1 or the 2-block numbered 2 in Table 15.

Proof. Suppose that ℓ is either a good prime for G and further assume that $\ell \neq 3$ if $G^F = {}^3D_4(q)$. The Malle–Robinson conjecture holds for the unipotent blocks of G^F and $G^F/Z(G^F)$ by [34, Proposition 6.10]. Suppose $B = b_{G^F}(L, \lambda)$ is a non-unipotent, quasi-isolated block of G^F and let $q > 2$ when G^F is of type ${}^2E_6, E_7$ or E_8 . By the results in Section 2 and Theorem 1.78, we conclude that $l(B) = |\mathcal{E}(G^F, (L, \lambda))|$. These cardinalities can be read off from the tables in Section 2 (see the Tables 2, 4, 6, 8 and 10). In every case $s(Z(L)_\ell^F)$ is enough to establish the conjecture and its structure can be read off from the tables in Section 2. We find that $l(B) \leq \ell^{s(Z(L)_\ell^F)}$ in every case, which proves the conjecture for these cases. When $q = 2$ and G^F is of type ${}^2E_6, E_7$ or E_8 then the assertion follows from Proposition 2.20 with similar arguments as the ones used above.

Suppose that \bar{B} is a non-unipotent, quasi-isolated block of $H = G^F/Z(G^F)$ with defect group \bar{D} . Let B be the quasi-isolated block of G^F that dominates \bar{B} . The order of $Z(G^F)$ is either 1 or a bad prime for G . Thus, $Z(G^F)$ is an ℓ' -subgroup by our assumption on ℓ . By [38, (9.9) Theorem], $l(\bar{B}) = l(B)$ and \bar{D} is of the form $DZ(G^F)/Z(G^F)$, for a defect group D of B . Note that $DZ(G^F)$ is a direct product since the two groups commute and $D \cap Z(G^F) = \{1\}$. It follows that $DZ(G^F)/Z(G^F) \cong D$, i.e. $s(\bar{D}) = s(D)$. Thus, the conjecture holds for \bar{B} since it holds for B .

Lastly, if ℓ is either a bad prime for G or $\ell = 3$ and $G^F = {}^3D_4(q)$ then the assertion follows from Theorems 3.9, 3.22, 3.33, 3.39 and 3.42. \square

4 Some notes on unipotent blocks

In [34], Malle and Robinson already proved their conjecture for many unipotent blocks for quasi-simple groups of Lie type. The only cases missing are the unipotent (non-principal) ℓ -blocks of quasi-simple groups of Lie type of exceptional type for bad primes.

In this section we apply the methods used for the quasi-isolated blocks, to study the unipotent blocks for bad primes and see how far these methods take us. As before, we first need a parametrisation of these blocks that allows us to extract the information needed to establish the Malle–Robinson conjecture. The right parametrisation was found by Enguehard in [16].

Theorem 4.1 ([16, Théorème A]). *Let G be a simple, simply connected group of exceptional Lie type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or let G be simple, simply connected of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Suppose that ℓ is a prime not dividing q . If $G^F = {}^3D_4(q)$ then let $\ell \in \{2, 3\}$ otherwise let ℓ be bad for G . Let $e = e_\ell(q)$. Then we have the following.*

(a) *There is a bijection*

$$b_{G^F}(L, \lambda) \longleftrightarrow (L, \lambda),$$

between the unipotent ℓ -blocks of G^F and the e -cuspidal pairs (L, λ) of G^F , up to G^F -conjugation, such that $\lambda \in \mathcal{E}(L^F, 1)$ is of central ℓ -defect.

(b) We have the following partition of $\mathcal{E}(G^F, s)$.

$$\mathcal{E}(G^F, 1) = \bigcup_{(L, \lambda)/G^F} \mathcal{E}(G^F, (L, \lambda))$$

where (L, λ) runs over the e -cuspidal pairs of G , up to G^F -conjugacy, with $\lambda \in \mathcal{E}(L^F, 1)$.

Enguehard also describes the defect groups in a way that gives enough information to get sufficient lower bounds on their ℓ -sectional ranks.

Theorem 4.2. *Let G be a simple, simply connected group of exceptional type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ or let G be simple, simply connected of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Suppose that ℓ is a prime not dividing q . If $G^F = {}^3D_4(q)$ then let $\ell \in \{2, 3\}$ otherwise let ℓ be bad for G . Then $\hat{\mathcal{E}}(G^F, 1)$ is a generating set of $\mathbb{Q}\text{IBr}(\mathcal{E}_\ell(G^F, 1))$ if one of the following holds.*

- (i) $G^F = G_2(q), {}^3D_4(q)$,
- (ii) $\ell = 3$ and $G^F = F_4(q)$,
- (iii) $\ell = 2$ and $G^F = E_6(q)$ or ${}^2E_6(q)$,
- (iv) $\ell = 3$ and $G^F = E_7(q)$, or
- (v) $\ell = 5$ and $G^F = E_8(q)$.

In particular, the number of irreducible Brauer characters in $\mathcal{E}_\ell(G^F, 1)$ is less than or equal to $|\mathcal{E}(G^F, 1)|$.

Proof. Recall that it suffices to prove that $\chi^\circ \in \mathbb{Q}\hat{\mathcal{E}}(G^F, 1)$ for every $\chi \in \mathcal{E}(G^F, 1)_\ell$. Let $t \in G^{*F}$ be a semisimple ℓ -element and let $\chi \in \mathcal{E}(G^F, t)$. We claim that $\chi^\circ \in \mathbb{Q}\hat{\mathcal{E}}(G^F, 1)$. The claim is clearly true for $t = 1$, which is why we suppose that $1 \neq t$ from now on.

(i) If t is quasi-isolated, checking Table 22, we see that $C_{G^*}(t)$ is of type A . Hence, the claim follows by the argument of the proof of Theorem 3.18. If t is not quasi-isolated, there exists a proper Levi subgroup L^* containing $C_{G^*}(t)$. Since the proper Levi subgroups of G_2 and D_4 are of type A , $C_{G^*}(t)$ is of type A as well and the claim follows by the proof of Theorem 3.18 again.

(ii) If t is quasi-isolated, checking Table 11, we see again that $C_{G^*}(t)$ is of type A . Let t be semisimple and not quasi-isolated and let L^* be the minimal Levi subgroup of G^* containing $C_{G^*}(t)$. Since t is a 3-element by assumption, and the proper Levi subgroups of F_4 are of classical type, it follows from Proposition 1.38 and the minimality of L^* that $C_{G^*}(t) = L^*$. In particular, $t \in Z(L^*)$. The claim follows since conditions (1) and (2) of Remark 3.10 are satisfied.

(iii) Checking Table 13, we observe that $C_{G^*}(t)$ is a connected reductive group of type A (the connectedness follows since t is a 2-element) if t is quasi-isolated. Suppose t is not quasi-isolated and let L^* be the proper minimal Levi subgroup containing $C_{G^*}(t)$. Thus, t is quasi-isolated in L^* . It can be shown (using Chevie [36]) that every centralizer of a quasi-isolated 2-element of L^* is of type A . Hence, the claim follows.

For (iv) we argue as in (iii) and for (v) we argue as in (ii). □

Proposition 4.3. *Let G be a simple, simply connected group of type F_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $\ell = 2$. Then $\hat{\mathcal{E}}(G^F, 1) \cup \hat{\mathcal{E}}(G^F, t_1) \cup \hat{\mathcal{E}}(G^F, t_2)$ is a generating set of $\mathbb{Q}\text{IBr}(\mathcal{E}_2(G^F, 1))$, where $t_1, t_2 \in G^{*F}$ are quasi-isolated semisimple 2-elements with $C_{G^*}(t_1)^F = B_4(q)$ and $C_{G^*}(t_2)^F = C_3(q)A_1(q)$.*

Proof. Let $1 \neq t \in G_2^{*F}$ be semisimple. If t is quasi-isolated, we see that $C_{G^*}(t)$ is either of type A and we are done or $C_{G^*}(t) = B_4(q)$ or $C_3(q)A_1(q)$ in which case we can not use any of our tools. Therefore, the series corresponding to these types are part of our generating set. Now suppose that t is not quasi-isolated. Let L^* be the minimal Levi subgroup of G^* containing $C_{G^*}(t)$. Using Chevie again, we observe that the centralisers of quasi-isolated elements of L^* are of type A . Hence the assertion follows from the argument of the proof of Theorem 3.18 again. \square

Proposition 4.4. *Let G be a simple, simply connected group of type E_8 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$. Let $\ell = 3$. Then $\hat{\mathcal{E}}(G^F, 1) \cup \hat{\mathcal{E}}(G^F, t_1)$ is a generating set of $\mathbb{Q}\text{IBr}(\mathcal{E}_3(G^F, 1))$, where $t_1 \in G^{*F}$ is a quasi-isolated semisimple 3-element with*

- (i) $C_{G^*}(t_1)^F = E_6(q)A_2(q)$ if $q \equiv 1 \pmod{3}$, or
- (ii) $C_{G^*}(t_1)^F = {}^2E_6(q)^2A_2(q)$ if $q \equiv 2 \pmod{3}$.

The reason for the distinction in Proposition 4.4 can be found in [22, Table 4.7.3A].

Proof. The proof is essentially the same as the proof of Proposition 4.3. Let $1 \neq t \in G_3^{*F}$ be semisimple. It can be shown that $C_{G^*}(t)$ is either of type A , type $E_6 \times A_2$, or a proper Levi subgroup of G^* . Hence, the assertion follows. \square

Corollary 4.5. *Let (G^F, ℓ) be a pair as in Theorem 4.2, Proposition 4.3, or Proposition 4.4. Then the Malle–Robinson conjecture holds in strong form for the unipotent ℓ -blocks of G^F and $G^F/Z(G^F)$.*

Proof. For the generating sets in Propositions 4.3 and 4.4 we can show that the series corresponding to the non-trivial quasi-isolated elements are contained in the principal block by applying the argument of the proof of Theorem 3.33. Let $B = b_{G^F}(L, \lambda)$ for some pair (L, λ) as in Theorem 4.1. Except for the principal 2-block of $F_4(q)$, the conjecture can be established using $s(Z(L)_\ell^F)$. Recall that the defect groups of principal ℓ -blocks of a finite group are the Sylow ℓ -subgroups of that group. (For the structure of Sylow subgroups of groups of Lie type in cross characteristic see [22, Section 4.10]). By Table [22, Table 4.10.6], $F_4(q)$ contains a central product of 4 commuting A_1 -type subgroups. An argument similar to the one in the proof of Proposition 3.25 yields an elementary abelian 2-subgroup of rank 8 which is enough to prove the strict form of the Malle–Robinson conjecture for the principal 2-block of $F_4(q)$. For the second part of the assertion see the proof of Theorem 3.22. \square

5 On minimal counterexamples

In this section we prove Theorem C on minimal counterexamples to the Malle–Robinson conjecture. A pair (G, B) is called a **minimal counterexample** to the conjecture if

1. the conjecture does not hold for B , and
2. the conjecture holds for all ℓ -blocks B' of groups H with $|H/Z(H)|$ strictly smaller than $|G/Z(G)|$ having defect groups isomorphic to those of B .

We also say that B is a minimal counterexample if the group G is understood.

Theorem 5.1. *Let G be a simple, simply connected group of exceptional Lie type defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$, or let G be simple, simply-connected of type D_4 defined over \mathbb{F}_q with Frobenius endomorphism $F : G \rightarrow G$ such that $G^F = {}^3D_4(q)$. Let ℓ be a prime and let $1 \neq s \in G^{*F}$ be a semisimple ℓ' -element. Let B be an ℓ -block of G^F contained in $\mathcal{E}_\ell(G^F, s)$. Then (G^F, B) is not a minimal counterexample to the Malle–Robinson Conjecture.*

Proof. Suppose that B is a minimal counterexample to the Malle–Robinson conjecture. By [34, Proposition 6.1], ℓ does not divide q . Hence, by Theorem 3.23, B is Morita equivalent to an ℓ -block b of a subgroup N of G^F and their defect groups are isomorphic. In particular, $l(B) = l(b)$ and $s(B) = s(b)$. If s is not quasi-isolated, then N is a proper subgroup. By the minimality of B , B is therefore a quasi-isolated block of G^F . Now, by the results in Section 3 (see Theorem 3.9, Theorem 3.22, Theorem 3.33, Theorem 3.39 and Theorem 3.42), either $G^F = E_6(q)$ or ${}^2E_6(q)$ and B is the 3-block numbered 13 in Table 13 or $G^F = E_7(q)$ and B is the block numbered 1 or 2 in Table 15. However, then B is not minimal by [34, Lemma 3.2]. In conclusion, the assertion is proved. \square

Theorem C. *Let H be a finite quasi-simple group of exceptional type. Let ℓ be a prime and let B be an ℓ -block of H . Then B is not a minimal counterexample to the Malle–Robinson conjecture for $\ell \geq 5$. More precisely, (H, B) is not a minimal counterexample, unless possibly if (H, B) is of one of the following types.*

- (i) $H = E_6(q)/Z(E_6(q))$ or ${}^2E_6(q)/Z({}^2E_6(q))$ and B is the 3-block dominated by the 3-block numbered 13 in Table 13.
- (ii) $H = E_7(q)/Z(E_7(q))$ and B is the 2-block dominated by either the 2-block numbered 1 or the 2-block numbered 2 in Table 15.
- (iii) $\ell = 3$ and $H = E_6(q)$ or ${}^2E_6(q)$ (respectively $H = E_6(q)/Z(E_6(q))$ or ${}^2E_6(q)/Z({}^2E_6(q))$) and B is a non-principal unipotent 3-block of H (respectively dominated by such a 3-block)
- (iv) $\ell = 2$ and $H = E_7(q)$ (respectively $H = E_7(q)/Z(E_7(q))$) and B is a non-principal unipotent 2-block of H (respectively dominated by such a 2-block)
- (v) $\ell = 2$ and $H = E_8(q)$ is a non-principal unipotent 2-block.

Proof. Suppose that (H, B) is a minimal counterexample to the Malle–Robinson conjecture. Let D be a defect group of B . By [34, Proposition 6.4], H is not an exceptional covering group of a finite group of exceptional Lie type. By [34, Proposition 6.5], H is not of Lie type 2B_2 , 2G_2 , G_2 , 3D_4 or 2F_4 . Hence, $H = G^F/Z$, where G is a simple, simply connected group of exceptional type (F_4, E_6, E_7 or E_8), $F : G \rightarrow F$ is a Frobenius endomorphism and $Z \subseteq Z(G^F)$ is a central subgroup. By [34, Proposition 6.1], ℓ does not divide q . Let B' be the unique block of G^F that dominates B and let D' be a defect

group of B' . In particular, $l(B) = l(B')$ and $s(D) = s(D')$. By [2, Theorem 7.7], B' is Morita equivalent to an ℓ -block b of a subgroup N of G^F and their defect groups are isomorphic. In particular, $l(B') = l(b)$ and $s(B') = s(b)$. If s is not quasi-isolated, then N is a proper subgroup. By the minimality of (H, B) , B is therefore a quasi-isolated block of H . Moreover, $\ell < 5$ by Theorem 3.39 and Corollary 4.5. Suppose that B is a unipotent block of H . Then (H, B) can only be of the types (iii), (iv) and (v) by Corollary 4.5. If we suppose that B is non-unipotent and quasi-isolated, then (H, B) can only be of types (i) and (ii) by the results in Section 3 (see Theorem 3.9, Theorem 3.22, Theorem 3.33, Theorem 3.39). \square

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