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SOME REMARKS ON A PROBLEM OF SOUND MEASUREMENTS FROM INCOMPLETE DATA

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INTRODUCTION

In these notes we will discuss some aspects of a problem arising in car-industry.

For the sake of clarity we will set the problem into an extremely simplified scheme (see the appendix for details).

Suppose that we have a body which is emitting sound, and that the sound is measured at a finite number of points around the body. We wish to determine the intensity of the sound at an observation point which is moving.

In analytical terms: Let Ω, a bounded smooth domain in $\mathbb{R}^3$, represent the body. Let $P_1,\dots,P_M$ be points in $\mathbb{R}^3\setminus\Omega$ at which measurements are made, and let $Q = Q(t)$, $t \in (0,T)$, be a path in space-time, which represents the motion of the observation point.

The sound intensity $u = u(x,t)$, $x \in \mathbb{R}^3$ $t > 0$ is a solution of the wave equation in the exterior of the body, satisfying zero initial conditions ($t = 0$ is the time at which the sound emission starts):

$$(0.1)\qquad \begin{cases} u_{tt} - \Delta u = 0 \quad , & \text{in } (\mathbb{R}^3\setminus\Omega)\times(0,+\infty), \\ u(\cdot,0) = u_t(\cdot,0) = 0 \;, & \text{on } \mathbb{R}^3\setminus\Omega \;. \end{cases}$$

We suppose that the values of u are approximately known at the points $P_1,\dots,P_M$ for a finite number of time values $0 \le t_1 \le \dots \le t_N$

$$(0.2)\qquad u(P_i,t_j) \simeq \tilde{u}_{ij} \qquad , \; i = 1,\dots,M;\; j = 1,\dots,N.$$

We want to determine $u(Q(t),t)$, as t ranges over $(0,T)$.

This problem is ill-posed in the sense of Hadamard. In fact, for any given set of data (0.2), the solution $u(Q(t),t)$ is not uniquely determined and it is unstable. For instance we may find functions u satisfying (0,1) such that $u(P_i,t_j) = 0$ for all i's and j's and $u(Q(t),t)$ is arbitrarily large. (See section 4, Remark 4.1.)

The treatment of an ill-posed problem requires the knowledge of extra a-priori information. (See, for instance: Miller [4], Talenti [7].)

Let us give an example:
Suppose that we want to determine, on the unit disk $x^2+y^2 < 1$, a harmonic function $u(x,y)$ which takes, within a given approximation, specified values at a finite number of points (x_i,y_i) inside the disk

$$(*) \qquad \sum_{i=1}^{m} |u(x_i,y_i) - f_i|^2 \leq \varepsilon^2.$$

This is too an ill-posed problem. However, if it is known that the harmonic function we are seeking satisfies a bound like the following

$$(**) \qquad \int_{x^2+y^2<1} (u_x^2+u_y^2)\, dxdy \leq E^2$$

then it can be shown that the set of harmonic functions satisfying both (*) and (**) is bounded (for instance, with respect to the L^2-metric). Moreover an algorithm can be developed which produces a solution which may be considered the optimal one. In fact, it minimizes, up to a constant factor, the maximal distance from any other harmonic function fulfilling (*) and (**).

This algorithm consists in solving the (regularized) least-squares problem

$$(***) \qquad \sum_{i=1}^{m} |u(x_i,y_i) - f_i|^2 + \varepsilon^2/E^2 \int_{x^2+y^2<1} u_x^2+u_y^2 = \min$$

among all harmonic functions u in the disk. An interesting feature of this algorithm is that it can be reduced to a finite dimensional least-squares problem. In fact a so called reproducing kernel can be used. Namely, if we consider the Hilbert space H made of the harmonic functions u in the disk having zero average and bounded Dirichlet integral (i.e. (**) holds for some $E < +\infty$) with the scalar product

$$\langle u,v\rangle_H = \int_{x^2+y^2<1} u_x v_x + u_y v_y \ ,$$

then it turns out that, in such a space, interior point values

$$H \ni u \longrightarrow u(x,y)$$

are bounded linear functionals. Thus there exists, and it is explicitly determined, a function $K(x,y;\xi,\eta)$ (called the reproducing kernel) such that $K(x,y;\cdot,\cdot) \in H$ for every (x,y) in the disk, and, for every $u \in H$ it satisfies

$$u(x,y) = \langle K(x,y;\cdot,\cdot),u\rangle_H \quad , \quad x^2+y^2 < 1.$$

Now it is readily seen that the minimizer u^0 of (***) can be represented as follows

$$u^0(x,y) = a_0 + \sum_{i=1}^{m} a_i K(x_i,y_i;x,y)$$

and, therefore, we are led to a least-squares problem involving only the parameters $a_0,a_1,\ldots,a_m$. (See, for details, Alessandrini [1]. See also: Bergmann-Schiffer [2], for the origin of the reproducing kernels. For the application of the reproducing kernels technique to ill-posed problem see: Miller-Viano [5]. See also Secrest [6] for a prototype of this technique.)

Thus, coming back to the sound measurements problem, we need mainly two things. First an a-priori limitation which makes the set of admissible solutions bounded. Second we will need an analogue of the reproducing kernel.

Here in the sequel we will examine some types of a-priori information which may be available and are useful to our purposes. The construction of the reproducing kernel, or some suitable substitute for it, will depend on the type of a-priori information we choose. (See sections 1 and 2.)

It is convenient, at this stage, to introduce a slight change in the statement of our problem. We want to replace the input and output pointwise evaluations: $u(P_i,t_j)$ and $u(Q(t),t)$ with weighted averages on neighbourhoods of the points (P_i,t_j), $(Q(t),t)$, respectively. Let us denote

(0.4) $$\bar{\nu}_h(x,t) = \int_{\mathbb{R}^3\times(0,\infty)} \varphi_h(x-y)\psi_h(t-s)\nu(y,s)\,dyds$$

for any locally integrable function ν on $\mathbb{R}^3\times(0,\infty)$, where

(0.5) $$\varphi_h(x) = h^{-3}\varphi(|x|/h), \; \psi_h(t) = h^{-1}\psi(t/h)$$

φ, ψ being smooth, symmetric, non-negative, compactly supported, normalized functions

(i.e.: $\int_{\mathbb{R}^3} \varphi(|x|)\, dx = 1, \qquad \int_{-\infty}^{+\infty} \psi(t)\, dt = 1.$)

In most of the following discussions, the parameter h will be kept fixed, hence, when no ambiguity occurs, we are going to drop the subscript from $\bar{\nu}_h : \bar{\nu} = \bar{\nu}_h$.

Our problem is then rephrased as follows:
<u>To determine</u> $\bar{u}(Q(t),t)$, $t \in (0,T)$, <u>where</u> $u(x,t)$ <u>satisfies</u> (0.1) <u>and the averages</u> $\bar{u}(P_i,t_j)$ <u>are prescribed within a given approximation</u>:

$$(0.\ ) \qquad \sum_{\substack{i=1,\dots,M \\ j=1,\dots,N}} |\bar{u}(P_i,t_j) - \tilde{u}_{ij}|^2 \leq \varepsilon^2$$

It seems acceptable, from a physical point of view, to consider measurements as averages rather than point values. Moreover we have the apparent mathematical advantage that for every fixed (x,t) in space-time, the mapping

$$u \longrightarrow \bar{u}(x,t)$$

is a continuous linear functional, even if the topology of the ambient space of u is very weak.

Let us now review some different possibilities regarding the a-priori information.

We might assume that the function u satisfies the wave equation also in Ω with a source term f supported in $\Omega \times (0,T_0)$, $(T_0 \leq T)$:

$$(0.7) \qquad \begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \mathbb{R}^3 \times (0,+\infty), \\ u(\cdot,0) = u_t(\cdot,0) = 0 & \text{on } \mathbb{R}^3, \end{cases}$$

and that this source (which will be unknown) satisfies some bound. The following limitation would be extremely convenient from the mathematical point of view

$$(0.8) \qquad \int_{\Omega \times (0,T_0)} f^2(x,t)\,dxdt \leq E^2, \qquad E = \text{constant} < \infty.$$

In fact, for every (x,t), the linear functional

$$L^2(\Omega \times (0,T_0)) \ni f \;\longrightarrow\; \bar{u}(x,t) \in \mathbb{R}$$

is bounded and can be expressed, as an integral

$$\bar{u}(x,y) = \int_{\Omega\times(0,T_0)} w(x,t;y,s)f(y,s)\,dyds$$

where the kernel $w(x,t;y,s)$ can be explicitly computed. Such type of representation enables us to exploit a finite dimensional algorithm similar to the one presented above for the harmonic functions problem. We will develop this technique in section 1.

Now, it may be argued that the bound (0.8) may not be feasible concretely, and that instead, the following assumption would be more realistic

$$(0.9) \qquad \int_{\Omega\times(0,T_0)} |f| \leq E$$

where f is also allowed to be a distribution. We will show in section 3 a procedure which enables us to apply the L^2-approach developed in section 1, making use of the bound (0.9) rather than (0.8).

Finally we wish to mention a different assumption which does not require information on the sound behaviour in the interior of the body:

$$(0.10) \qquad \int_{(\mathbb{R}^3\setminus\Omega)\times(0,T)} (|D_x u(x,t)|^2 + u_t^2(x,t))\,dxdt \leq E^2.$$

This condition has an appealing physical interpretation. That is we are assuming to know the total sound-energy spent in the time-interval $(0,T)$. Theoretically speaking, also in this case, there exists a kernel $W(x,t;y,s)$ such that, for every $(x,t) \in (\mathbb{R}^3\setminus\Omega)\times(0,T)$:

$$\begin{cases} (\frac{\partial^2}{\partial s^2} - \Delta_y)W(x,t;\cdot,\cdot) = 0 & \text{in } (\mathbb{R}^3\setminus\Omega)\times(0,T) \\ W(x,t;\cdot,0) = W_s(x,t;\cdot,0) = 0 & \text{on } \mathbb{R}^3\setminus\Omega \end{cases}$$

and, for every u satisfying (0,1) and (0.8) for some $E < \infty$,

$$\bar{u}(x,t) = \int_{(\mathbb{R}^3\setminus\Omega)\times(0,T)} (D_y W(x,t;y,s)\cdot D_y u(y,s) + W_s(x,t;y,s)u_s(y,s))\,dyds.$$

This kernel might be used to reduce the problem to finite dimension as before. In section 2 we give a more detailed discussion on this matter. However, the structure of W will depend heavily on the shape of Ω, and the numerical computation of the values of W seems to be a hard task.

Let us sketch a plan of the present notes.
In section 1 we treat the problem assuming that the bound (0.8) is available. We develop an algorithm, namely, fixed $\tau \in (0,T)$ and $Q = Q(\tau)$, we are able to determine an approximation of $\overline{u}(Q,\tau)$ by solving a finite dimensional linear system. Furthermore another finite dimensional algorithm is given which yields an error estimate.

In section 2 an approach to the use of the bound (0.10) is made. Computations are made on a one-space-dimension example.

In section 3 it is shown how we may use the methods developed in section 1 when the bound (0.9) (or (0.10)) is available.

Section 4 contains a discussion on the following issue (its interest will become clear in section 1, see Remark 1.3):

Are the following M·N+1 linear functionals

$L^2(\Omega \times (0,T_0)) \ni f \to \overline{u}(P_i,t_j)$, $i = 1,\dots,M$; $j = 1,\dots,N$,

$L^2(\Omega \times (0,T_0)) \ni f \to \overline{u}(Q,\tau)$

linearly independent?

An affirmative answer is given, provided certain geometrical conditions are satisfied, and provided the support size h of the averagings is sufficiently small.

Section 5 contains an appendix in which details on the applied problem are given. A modified mathematical model is presented and briefly discussed.

1. The L^2-approach

For notation's convenience let us introduce a renumbering of the points (P_i,t_j):

$$(P_i,t_j),\ i = 1,\dots,M,\ j = 1,\dots,N \ \rightarrow (P_l,t_l),\ l = 1,\dots,L = M\cdot N.$$

We will also denote

$$(P_o,t_o) = (Q,\tau).$$

We are now concerned with the following problem:

(I) To determine $\bar{u}(P_o,t_o)$, when u is a solution of (0.7) satisfying

$$\sum_{l=1}^{L} |\bar{u}(P_l,t_l) - \tilde{u}_l|^2 \leq \varepsilon^2 \tag{1.1}$$

and the following information on the source f is given

$$\text{supp } f \subset \bar{\Omega} \times [0,T_o]$$

$$\int_{\Omega\times(0,T_o)} f^2(x,t)\, dxdt \leq E^2 . \tag{1.2}$$

Recall that $\bar{u}(x,t) = \bar{u}_h(x,t)$ denotes an average of u in a neighbourhood of (x,t) as defined in (0.4).

The following lemma provides a useful representation of such averages.

Lemma 1.1

Let u be the solution of (0.7) then for every $(x,t) \in \mathbb{R}^3 \times (0,\infty)$

$$\bar{u}(x,t) = \int_{\Omega\times(0,T_o)} w(x,t;y,s) f(y,s)\, dy\, ds \tag{1.3}$$

where the kernel w is given by

$$w(x,t;y,s) = (4\pi)^{-1} \int_{|z|<t-s} |z|^{-1} \varphi_h(x-y-z)\psi_h(t-s-|z|)\, dz . \tag{1.4}$$

Proof: Combine Kirchhoff's retarded potential formula (see, e.g.: Courant-Hilbert [3])

$$(1.5) \qquad u(x,t) = (4\pi)^{-1} \int_{|z-x|<t} |z-x|^{-1} f(z,t-|z-x|)\, dz$$

and (0.4).

Remark 1.1

For every $(x,t) \in \mathbf{R}^3 \times (0,\infty)$, $w(x,t;\cdot,\cdot)$ is a bounded smooth function, in particular it belongs to $L^2(\Omega \times (0,T_0))$, thus the functional

$$L^2(\Omega \times (0,T_0)) \ni f \longrightarrow \overline{u}(x,t)$$

is bounded. Thus we get that the set of admissible values of $\overline{u}(Q_0,t_0)$ under the constraint (1.2) is bounded:

$$|\overline{u}(Q_0,t_0)| \leq \|w(Q_0,t_0;\cdot,\cdot)\|_{L^2(\Omega \times (0,T_0))} E$$

In this sense, we may say that Problem (I) is stable.

Remark 1.2

For computational purposes it may also be noteworthy that the kernel $w = w(x,t;y,s)$ depends only on $|x-y|$ and on $(t-s)$.

1.1 An Algorithm

An approximate solution to (I) can be found by an optimization procedure balancing between requirements (1.1) and (1.2), namely determining an $L^2(\Omega \times (0,T_0))$ function f which minimizes the functional

$$J : f \to \sum_{i=1}^{L} |\overline{u}(P_1,t_1) - \tilde{u}_1|^2 + \varepsilon^2/E^2 \int_{\Omega\times(0,T_0)} f^2$$

(again we refer to [4], [7] for a more precise justification). The following theorem guarantees the existence and uniqueness of such a minimizer, and gives a finite dimensional algorithm for determining it.

Let us introduce the following notation:

(1.6) $$A_{lm} = \int_{\Omega\times(0,T_o)} w(P_l,t_l;y,s)w(P_m,t_m;y,s)\,dyds;\quad l,m = 0,1,\dots,L$$

(1.7) $$A = \begin{pmatrix} A_{11} & \cdots & A_{1L} \\ \vdots & & \vdots \\ A_{1L} & & A_{LL} \end{pmatrix},\quad L\times L \text{ symmetric matrix,}$$

(1.8) $$\tilde{u} = (\tilde{u}_1,\dots,\tilde{u}_L)^T$$

Theorem 1.1

The regularized least-squares problem

(1.9) $$J(f) = \sum_{l=1}^{L} |\bar{u}(P_l,t_l) - \tilde{u}_l|^2 + \varepsilon^2/E^2 \int_{\Omega\times(0,T_o)} f^2 = \min,$$

has a unique solution f^o in $L^2(\Omega\times(0,T_o))$ which can be represented:

(1.10) $$f^o(x,t) = \sum_{l=1}^{L} b_l\, w(P_l,t_l;x,t)$$

where the vector $b = (b_1,\dots,b_L)^T$ is the solution of the linear system

(1.11) $$(A + \varepsilon^2/E^2\, I)b = \tilde{u}.$$

Note: Our goal is the evolution of $\bar{u}(P_o,t_o)$. We will obtain it as follows

$$\bar{u}(P_o,t_o) \simeq \int_{\Omega\times(0,T_o)} w(P_o,t_o;y,s)f^o(y,s)\,dyds = \sum_{l=1}^{L} A_{ol}b_l,$$

where $b = (b_1,\dots,b_L)^T$ is given by (1.11). Therefore, denoting $A_{(o)} = (A_{o1},\dots,A_{oL})^T$, we get

$$\bar{u}(P_o,t_o) \simeq A_{(o)}^T(A + \varepsilon^2/E^2\, I)^{-1}\tilde{u} = d^T\tilde{u},$$

where $d = (d_1,\dots,d_L)^T$ is determined by

(1.11)' $$(A + \varepsilon^2/E^2\, I)d = A_{(o)}$$

and this system can be solved once for all.

Proof of Theorem 1.1

Observe that $J : L^2(\Omega \times (0,T_0)) \to \mathbb{R}$ is a non-negative quadratic functional, moreover it is continuous and strictly convex, therefore it has a unique point of minimum.

Let f be the minimizer, let f^0 be its projection over the finite dimensional subspace of $L^2(\Omega \times (0,T_0))$ generated by $w(P_l,t_l;\cdot,\cdot)$ $l = 1,\dots,L$. Thus $f = f^0 + f^\perp$ where

$$f^0(x,t) = \sum_{l=1}^{m} b_l w(P_l,t_l;\cdot,\cdot)$$

and, for every $l = 1,\dots,L$,

$$\int_{\Omega\times(0,T_0)} w(P_l,t_l;\cdot,\cdot)\, f^\perp = 0.$$

Therefore we have

$$J(f) = J(f^0) + \varepsilon^2/E^2 \int_{\Omega\times(0,T_0)} (f^\perp)^2$$

in fact, by Lemma 1.1,

$$\bar{u}(P_l,t_l) = \int_{\Omega\times(0,T_0)} w(P_l,t_l;\cdot,\cdot)\, f = \int_{\Omega\times(0,T_0)} w(P_l,t_l;\cdot,\cdot)\, f^0.$$

Consequently: $f^\perp = 0$, or, as is the same, $f = f^0$.

Now note that:

$$J(f) = J(f^0) = \sum_{l=1}^{L} \Big| \sum_{m=1}^{L} b_m \int_{\Omega\times(0,T_0)} w(P_l,t_l;\cdot,\cdot), w(P_m,t_m;\cdot,\cdot) - u_l \Big|^2 + \varepsilon^2/E^2 \int_{\Omega\times(0,T_0)} \Big(\sum_{l=1}^{L} b_l w(P_l,t_l;\cdot,\cdot)\Big)^2$$

and thus the coefficients $b_1,\dots,b_L$ are found by the minimization of the following L-dimensional quadratic functional (recall (1.6))

$$\sum_{l=1}^{L} \Big| \sum_{m=1}^{L} A_{lm} b_m - \tilde{u}_l \Big|^2 + \varepsilon^2/E^2 \sum_{l,m=1}^{l} A_{lm} b_l b_m .$$

The associated "Euler's equation" is, in matrix form, (recall (1.7), (1.8)):

$$A^2 b - A\tilde{u} + \varepsilon^2/E^2 Ab = 0$$

or, as is the same,

$$A((A+\varepsilon^2/E^2\ I)b-\tilde{u})=0 \tag{1.12}$$

note that $(A+\varepsilon^2/E^2\ I)$ is a positive definite $L\times L$ matrix, hence invertible. Thus $b=(A+\varepsilon^2/E^2\ I)^{-1}\tilde{u}$ yields a solution of (1.12), and this completes the proof.

Remark 1.3: Note that in the above proof we did not need the invertibility of the matrix A defined in (1.6), (1.7). The matrix A is invertible if and only if the functions $w(P_l,t_l;\cdot,\cdot)$ $l=1,\dots,L$ are linear independent elements of $L^2(\Omega\times(0,T_o))$. This matter is treated in section 4.

Here, let us just observe that it would be advantageous to know that A is invertible, in fact, in such a case the condition number of the matrix

$$A+\varepsilon^2/E^2\ I$$

does not diverge as $(\varepsilon^2/E^2)\to 0$, and thus the numerical solution of (1.11) (or (1.11)') will be manageable also for small values of ε^2/E^2.

1.2 An error estimate

Let us denote by u^1 the sound intensity produced by the (unknown) source f^1 which yields the "true" solution of problem (I). Let u^o be the solution of (0.7) when $f=f^o$ is the minimizer of the regularized least-squares problem (1.9). Note that, by (1.9), and, obviously assuming that f^1 fulfills (1.1), (1.2), we get

$$\begin{aligned}&\Big(\sum_{l=1}^{L}|\bar{u}^o(P_l,t_l)-\bar{u}^1(P_l,t_l)|^2+\varepsilon^2/E^2\int_{\Omega\times(0,T_o)}|f^o-f^1|^2\Big)^{1/2}\\&\quad\le 2\Big(\sum_{l=1}^{L}|\bar{u}^1(P_l,t_l)-\tilde{u}_l|^2+\varepsilon^2/E^2\int_{\Omega\times(0,T_o)}|f^1|^2\Big)^{1/2}\\&\quad\le 2\sqrt{2}\,\varepsilon\ .\end{aligned}$$

Note also that the following number is finite (recall Remark 1.1)

$$\mu^2=\sup_{f\in L^2(\Omega\times(0,T_o))}\frac{|\bar{u}(P_o,t_o)|^2}{\sum_{l=1}^{m}|\bar{u}(P_l,t_l)|^2+\varepsilon^2/E^2\int_{\Omega\times(0,T_o)}f^2} \tag{1.13}$$

Therefore, picking $f = f^0 - f^1$, we get

(1.14) $$|\bar{u}^0(P_0,t_0) - \bar{u}^1(P_0,t_0)| \leq 2\sqrt{2}\,\mu\,\varepsilon$$

which yields an estimate of the error made by replacing the "true" solution $u^1(P_0,t_0)$ with the one obtained via the regularized least-squares algorithm. The following theorem indicates how to compute the number μ.

Theorem 1.2

Let the point (P_0,t_0) be in the cone of influence of $\Omega \times (0,T_0)$, then there exists a function in $L^2(\Omega \times (0,T_0))$ which yields the maximum of the quotient in (1.13). This maximizing function is uniquely determined up to a constant factor, and it can be represented as a linear combination of the functions

$$w(P_l,t_0;\cdot,\cdot)\ , \qquad l = 0,1,\dots,L.$$

Proof: Note that the number μ in (1.13) is non-zero, in fact, if (P_0,t_0) is in the cone of influence of $\Omega \times (0,T_0)$ (that is: there exists $(y,s) \in \Omega \times (0,T_0)$ such that $|P_0 - y| = t_0 - s$), then we may find a source f in $L^2(\Omega \times (0,T_0))$ such that $\bar{u}(P_0,t_0) \neq 0$. Therefore the variational problem

(1.15) $$\frac{|\bar{u}(P_0,t_0)|^2}{\sum_{l=1}^{L} |\bar{u}(P_l,t_l)|^2 + \varepsilon^2/E^2 \int_{\Omega\times(0,T_0)} f^2} = \sup$$

is equivalent to

(1.16) $$\sum_{l=1}^{L} |\bar{u}(P_l,t_0)|^2 + \varepsilon^2/E^2 \int_{\Omega\times(0,T_0)} f^2 = \inf$$

under the constraint

(1.17) $$\bar{u}(P_0,t_0) = \text{constant} \neq 0.$$

Note that the norm

$$|||f||| = \Big(\sum_{l=1}^{m} |\bar{u}(P_l,t_0)|^2 + \varepsilon^2/E^2 \int_{\Omega\times(0,T_0)} f^2\Big)^{1/2}$$

is equivalent to the $L^2(\Omega \times (0,T_0))$-norm, and it gives rise to a Hilbert space structure over $L^2(\Omega \times (0,T_o))$.

Thus, solving the problem (1.16), (1.17) corresponds to find the unique projection of the origin $0 \in L^2(\Omega \times (0,T_o))$ onto the closed affine variety defined by (1.17) with respect to the $|||\ |||$-metric.

Therefore we have proved the existence of a maximizer of (1.15), and that it is uniquely determined by a constant factor.

The reduction to finite dimension follows the method already used in Theorem 1.1.

Remark 1.4

Let us see how μ can be determined. Let $c = (c_o,\ldots,c_L)^T$ be such that

$$f^o(z,t) = \sum_{m=0}^{L} c_m w(P_m,t_m;x,t)$$

is the maximizer. By (1.16), (1.17) we get

$$\sum_{l=1}^{L} \sum_{m=0}^{L} A_{lk}A_{lm}c_m + \varepsilon^2/E^2 \sum_{m=0}^{L} A_{km}c_m = \lambda\, A_{ok}, \quad k = 0,1,\ldots,L;$$

here λ is a Lagrange multiplier. Multiplying both members by c_k and adding with respect to k from 0 to L, we get

$$\mu^2 = \lambda^{-1} \sum_{m=0}^{L} A_{om}c_m.$$

We may rewrite the above system as follows (note the change in the first summation)

$$\sum_{l=0}^{L} \sum_{m=0}^{L} A_{lk}A_{lm}c_m + \varepsilon^2/E^2 \sum_{m=0}^{L} A_{km}c_m = \lambda A_{ok} + \left(\sum_{m=0}^{L} A_{om}c_m\right)A_{ok}, \quad k = 0,1,\ldots,L;$$

or, as is the same,

$$(\bar{A}^2 + \varepsilon^2/E^2\bar{A})c = \lambda A_o + A_o^T c\, A_o\ ,$$

where

$$\bar{A} = \begin{pmatrix} A_{oo} & \cdots & A_{oL} \\ \vdots & & \vdots \\ A_{oL} & \cdots & A_{LL} \end{pmatrix}, \quad ((L+1)\times(L+1))$$

and

$$A_o = (A_{oo} \ \cdots \ A_{oL})^T.$$

Now: $A_o^T c = \lambda\mu^2$, and thus

$$(\bar{A}^2 + \varepsilon^2/E^2 \ \bar{A})c = \lambda(1+\mu^2)A_o \ .$$

Moreover we may normalize c in such a way that: $\lambda(1+\mu^2) = 1$, which means

$$\frac{\mu^2}{1+\mu^2} = A_o^T c \quad , \qquad \mu^2 = \frac{A_o^T c}{1 - A_o^T c}$$

where c is the solution of

$$\bar{A}(\bar{A} + \varepsilon^2/E^2 \ I)c = A_o .$$

Note that: $A_o = \bar{A}(1,0,\ldots,0)^T$, therefore c can be found as the solution of

$$(\bar{A} + \varepsilon^2/E^2 \ I)c = (1,0,\ldots,0)^T.$$

2. The bounded energy approach

2.1 The theoretical method

We start with some definitions and with two simple lemmas. Let us denote

(2.1) $$\mathcal{D} = (\mathbb{R}^3 \setminus \Omega) \times (0,T).$$

We define X as the space of all weak solutions u of (0.1) such that

(2.2) $$\langle u,u \rangle_X = \|u\|_X^2 = \int_{\mathcal{D}} (u_t^2(x,t) + |D_x u(x,t)|^2)\, dxdt < \infty .$$

In this section we will make use of a slightly different definition of average $\bar{u}$, namely, for every $(x,t) \in \mathcal{D}$

(2.3) $$\bar{u}(x,t) = \int_{\mathcal{D}} \varphi_h(x-y)\psi_h(t-s)u(y,s)\, dyds$$

where φ_h, ψ_h are the same functions defined in the introduction (see (0.4)).

Lemma 2.1

The space $(X, \langle \cdot,\cdot \rangle_X)$ is a Hilbert space.

Proof: Note that for every $u \in X$, by the initial condition $u(\cdot,0) = 0$, we have

(2.4) $$\int_{\mathcal{D}} u^2(x,t)dxdt \leq (T^2/2) \int_{\mathcal{D}} u_t^2(x,t)\, dxdt$$

and thus, on X, the H^1 Sobolev norm on $\mathcal{D}$ and the X-norm are equivalent. Now, from the weak formulation of (0.1), we have that $u \in X$ if and only if $u \in H^1(\mathcal{D})$ and

$$\int_{\mathcal{D}} (\varphi_{tt} - \Delta_x \varphi)u = 0$$

for every $\varphi \in C_0^\infty((\mathbb{R}^3 \setminus \bar{\Omega}) \times [0,T))$. It follows immediately that X is a closed linear subspace of $H^1(\mathcal{D})$, and thus the lemma is proved.

Lemma 2.2

There exists a function

$$W : \mathcal{D} \times \mathcal{D} \to \mathbb{R},$$

satisfying the following properties

(i) $W(x,t;\cdot,\cdot) \in X$, for every $(x,t) \in \mathcal{D}$,

(ii) for every $u \in X$

$$\bar{u}(x,t) = \langle W(x,t;\cdot,\cdot), u(\cdot,\cdot)\rangle_X .$$

Proof: It suffices to note that for every $(x,t) \in \mathcal{D}$ the functional

$$X \ni u \to \bar{u}(x,t)$$

is continuous, in fact

$$|\bar{u}(x,t)| \leq \|\varphi_h(x-\cdot)\psi_h(t-\cdot)\|_{L^2(\mathcal{D})} \|u\|_{L^2(\mathcal{D})}$$

and by

$$|\bar{u}(x,t)| \leq (T/\sqrt{2}) \|\varphi_h(x-\cdot)\psi_h(t-\cdot)\|_{L^2(\mathcal{D})} \|u\|_X$$

We easily infer from the above lemmas that we may mimic the procedure already used in section 1 in order to get a least-squares algorithm and an error estimate for the following problem.

(II) To determine $\bar{u}(P_0,t_0)$, where u is a solution of (0.1) satisfying

$$(2.5) \qquad \sum_{l=1}^{L} |\bar{u}(P_l,t_l) - \tilde{u}_l|^2 \leq \varepsilon^2,$$

$$(2.6) \qquad \int_{\mathcal{D}} u_t^2 + |D_x u|^2 = \|u\|_X^2 \leq E^2 .$$

There is however in this case a main disadvantage: The kernel W is not explicitly defined as it was w (recall (1.4)).
It seems to us that a numerical algorithm which, given the domain $\mathcal{D}$, produces approximate values of W might be feasible. Incidentally let us remark that, for the purpose of the least-squares algorithm and of the error

estimate, it would suffice to determine the numbers

$$B_{lm} = \langle W(P_l,t_l;\cdot,\cdot),W(P_m,t_m;\cdot,\cdot)\rangle_X, \quad l,m = 0,1,\dots,L.$$

Note also that:

$$B_{lm} = \overline{W}(P_l,t_l;\cdot,\cdot)_{|(P_m,t_m)} = \int_{\mathcal{D}} \varphi_h(P_m-y)\psi_h(t_m-s)W(P_l,t_l;y,s)\ dyds.$$

In the following paragraph we show how W can be determined in a one-space-dimension setting.

2.2 A one-space-dimension example

Here we consider solutions $u = u(x,t)$ of the two dimensional wave equation

$$(2.7) \quad \begin{cases} u_{tt}(x,t)-u_{xx}(x,t) = 0, \ (x,t)\in \mathcal{D}' = [(-\infty,-1)\cup(1,+\infty)]\times(0,T) \\ u(x,0) = u_t(x,0) = 0, \quad x\in(-\infty,-1)\cup(1,+\infty). \end{cases}$$

We will look for a function $\Gamma(x,t;y,s)$ such that for every $(x,t)\in\mathcal{D}'$ $\Gamma(x,t;\cdot,\cdot)$ is a solution of (2.7) satisfying

$$(2.8) \qquad u(x,t) = \int_{\mathcal{D}} \left(\frac{\partial}{\partial s}\Gamma(x,t;y,s)\frac{\partial}{\partial s}u(y,s) + \frac{\partial}{\partial y}\Gamma(x,t;y,s)\frac{\partial}{\partial y}u(y,s)\right) dyds$$

for every u which is a solution of (2.7).

Note: The analogue W' of the kernel W will be then given by a (suitable) averaging of Γ

$$W'(x,t;y,s) = \int_{\mathcal{D}} \psi_h(x-z)\psi_h(t-\tau)\Gamma(z,\tau;y,s)\ dzd\tau$$

The domain $\mathcal{D}'$ is disconnected, and it is easily seen that $\Gamma(x,t;y,s) = 0$ if x and y have opposite sign. Furthermore, by symmetry

$$\Gamma(x,t;y,s) = \Gamma(|x|,t;|y|,s).$$

Thus we need only to determine Γ such that (2.8) holds for every u, solution of ((2.7), vanishing for $x<0$. Let u be any of such functions.
Let: $\nu(s) = u(1,s)$, and, fixed $(x,t)\in\mathcal{D}'$, $x>1$,
let: $\alpha(s) = \Gamma(x,t;1,s)$, $s\in(0,T)$.

By the initial conditions we have

$$\nu'(0+) = \nu(0+) = \alpha'(0+) = \alpha(0+) = 0$$

Moreover we note

$$u(y,s) = \begin{cases} \nu(s-(y-1)) & s \geqq (y-1) \\ 0 & s < (y-1) \end{cases}$$

$$\Gamma(x,t;y,s) = \begin{cases} \alpha(s-(y-1)) & s \geqq (y-1) \\ 0 & s < (y-1) . \end{cases}$$

Therefore we get from (2.8)

$$\nu(t-(x-1)) = \int_{\substack{1<y<1+T \\ y-1<s<t}} 2\,\nu'(s-(y-1))\alpha'(s-(y-1))\,dyds$$

$$= \int_0^T \nu'(z)\alpha'(z)\,dz \int_z^{2T-z} d\zeta = 2\int_0^T (T-z)\nu'(z)\alpha'(z)\,dz$$

$$= -2\int_0^T [(T-z)\alpha'(z)]\nu'(z)\,dz ,$$

that is, by the arbitrariness of ν, we obtain

$$\begin{cases} -2[(T-z)\alpha'(z)]' = \delta(z-t+(x-1)), z \in (0,T) \\ \alpha'(0) = \alpha(0) = 0 . \end{cases}$$

Consequently

$$\alpha(z) = \begin{cases} \frac{1}{2}\log\left[\frac{T-2}{T-(t-(x-1))}\right] & \text{if } 0 < t-(x-1) < z < T \\ 0 & \text{otherwise} . \end{cases}$$

And finally

$$\Gamma(x,t;y,s) = \begin{cases} \frac{1}{2}\log\left[\frac{T-(s-(\ y\ -1))}{T-(t-(\ x\ -1))}\right] & \text{if } xy > 0 \text{ and } 0 < t-(|x|-1) < s-(|y|-1) < T. \\ 0 & \text{otherwise} . \end{cases}$$

3. A treatment of the distribution source case

In this section we are considering the case in which u is a solution of (0.7) satisfying (1.1) and the source f is a distribution fulfilling (0.9) supported in $\bar{\Omega} \times [0,T_o]$.

We may assume with no significant loss of generality that the averaging kernels φ_h, ψ_h in (0.5) are such that

$$(3.1) \qquad \begin{cases} \varphi_h(x) = \int_{\mathbb{R}^3} \eta_h(s-y)\eta_h(y)\, dy \\ \psi_h(t) = \int_{-\infty}^{+\infty} \zeta_h(t-s)\zeta_h(s)\, ds \end{cases}$$

where η_h, ζ_h have the same properties than those stated for φ_h, ψ_h, respectively.

Let us assume, for simplicity's sake, that

$$\operatorname{supp} \eta_h = \{x \in \mathbb{R}^3 \mid |x| \leq h\} \ , \quad \operatorname{supp} \zeta_h = [-h,h].$$

Here, for any locally integrable function ν on $\mathbb{R}^3 \times (0,\infty)$, we define

$$(3.2) \qquad \check{\nu}(x,t) = \int_{\mathbb{R}^3 \times (0,\infty)} \eta_h(x-y)\zeta_h(t-s)\nu(y,s)\, dyds$$

Note that such definition of averaging can be naturally extended to distributions. Note also that

$$(\check{\nu})^{\vee}(x,t) = \bar{\nu}(x,t)$$

where $(\check{\nu})^{\vee}$ denotes the iterated averaging (3.2), and $\bar{\nu}$ is the averaging defined in (0.5).

Now we observe that $\check{u}$ has the following properties

$$(i) \qquad \sum_{l=1}^{L} |(\check{u})^{\vee}(P_l,t_l) - \tilde{u}_l|^2 \leq \varepsilon^2,$$

(ii) $$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta_x\right)\check{u} = f^{\vee}, & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \check{u}(\cdot,-h) = \check{u}_t(\cdot,-h) = 0, & \text{on } \mathbb{R}^3, \end{cases}$$

(iii) $$\text{supp } f^{\vee} \subset \overline{\Omega}^h \times [-h, T_0+h],$$

where

(3.3) $$\Omega^h = \{x \in \mathbb{R}^3 \mid d(x,\Omega) < h\},$$

and finally, by (0.9) and Young's inequality

(iv) $$\int_{\Omega^h \times (-h,T_0+h)} (f^{\vee}(x,t))^2 dxdt \leq \int_{\mathbb{R}^4} \eta_h^2(x)\zeta_h^2(t)dxdt \, E^2 \leq C\, h^{-4} E^2$$

Therefore, if we allow the following slight changes:

a) a replacement of the averaging kernels:

$$\varphi_h, \psi_h \rightarrow \eta_h, \zeta_h ,$$

b) a replacement of the a-priori bound:

$$E^2 \rightarrow C\, h^{-4} E^2 ,$$

c) a time shift:

$$t \rightarrow t+h ,$$

d) a widening of the source support:

$$\overline{\Omega} \times [0,T_0] \rightarrow \overline{\Omega}^h \times [0, T_0+2h],$$

then we see that $\check{u}$ satisfies the conditions stated in Problem (I) and the methods of section 1 can be used.

Remark 3.1

A similar reduction to the L^2-source approach can be made under the bounded energy setting of Problem (II).

Let us denote

$$2d = \min \{d(P_l,\Omega) \mid l = 0,1,\dots,L\}.$$

Consider X to be a $C_0^\infty(\mathbb{R}^3)$ function such that

$$0 \leq X \leq 1; \quad X \equiv 1 \quad \text{on} \quad \bar{\Omega} \equiv 0 \text{ outside } \Omega^d,$$

$$|D_x^k X| \leq C_k d^{-k}, \qquad k = 1,2,\dots$$

(here Ω^d is the domain defined in (3.3) with h replaced by d). Then let us define

$$\nu(x,t) = \begin{cases} X(x)u(x,t) & \text{for } (x,t) \in (\mathbb{R}^3 \setminus \Omega) \times [0,T) \\ 0 & \text{for } (x,t) \in \Omega \times [0,T). \end{cases}$$

where u satisfies (0,1) and (2.5), (2.6).
Note that

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta_x)\nu = g & \text{in } \mathbb{R}^3 \times (0,T), \\ \nu(\cdot,0) = \nu_t(\cdot,0) = 0 & \text{on } \mathbb{R}^3, \end{cases}$$

where

$$g(x,t) = \begin{cases} -(\Delta_x X(x)u(x,t) + 2D_x X(x) \cdot D_x u(x,t)), & \text{for } (x,t) \in (\Omega^d \setminus \Omega) \times (0,T), \\ 0 & \text{elsewhere.} \end{cases}$$

Now it is reasonable to assume that the supports of the averaging functions

$$\varphi_h(P_l - \cdot)\psi_h(t_l - \cdot) \quad , \qquad l = 0,1,\dots,L$$

do not intersect $\Omega^d \times (0,T)$. Therefore it turns out that

$$\bar{\nu}(P_l,t_l) = \bar{u}(P_l,t_l) \quad , \qquad l = 0,1,\dots,L,$$

and, on the other hand, we have

$$\Big(\int_{(\Omega^d \setminus \Omega)\times(0,T)} g^2\Big)^{1/2} \leq C_2 d^{-2}\Big(\int_{\mathbb{R}^3 \times(0,T)} u^2\Big)^{1/2} + C_1 d^{-1}\Big(\int_{\mathbb{R}^3 \times(0,T)} |D_x u|^2\Big)^{1/2}$$

thus, recalling (2.4), we get by (2.6)

$$\int_{(\Omega^d \setminus \Omega)\times(0,T)} g^2 \leq (C_2^2 d^{-4} T^2/2 + C_1^2 d^{-2})E^2.$$

Therefore, once again, we are dealing with a solution of the wave equation ν, with source g, which (with the due changes) satisfies the conditions of Problem (I).

4. The linear independence of the averaging functionals

For any point (P_l,t_l), $l = 0,1,\dots,L$, we consider the (characteristic) cone of dependence of (P_l,t_l):

$$C_l = \{(x,t) \in \mathbb{R}^3 \times (0,T) \mid |x-P_l| = t_l-t\}.$$

The following lemma tells under which conditions the unbounded functionals

$$L^2(\Omega \times (0,T_0)) \ni f \to u(P_l,t_l), \quad l = 0,1,\dots,L$$

are linearly independent. Here u denotes the solution of (0.7).

Lemma 4.1

Let (P_l,t_l), $l = 0,1,\dots,L$ be in the cone of influence of $\Omega \times (0,T_0)$ (that is: $C_l \cap (\Omega \times (0,T_0)) \neq \phi$, for every l).
Then there exists a number $\rho > 0$ and points $(Q_l,s_l) \in \Omega \times (0,T_0)$, $l = 0,1,\dots,L$, such that

$$(Q_l,s_l) \in C_l, \text{ for every } l = 0,1,\dots,L$$

and

$$d((Q_l,s_l),C_m) \geq \rho, \text{ for every } l,m,\ l \neq m.$$

Remark 4.1

We infer that, for every $l = 0,1,\dots,L$, we may find a source f, supported in a small neighbourhood of (P_l,t_l), such that $u(P_l,t_l)$ takes an arbitrary value, while, for every $m \neq l$,

$$u(P_m,t_m) = 0.$$

That is, the pointwise evaluation functionals

$$L^2 \ni f \longrightarrow u(P_l,t_l)$$

are linearly independent.

Proof of Lemma 4.1: Note that, if $l \neq m$, then $C_l \cap C_m$ is either empty or a 2-D manifold. On the other hand, for every l, $C_l \cap (\Omega \times (0,T_o))$ is a non-empty 3-D manifold. Thus the set

$$\Sigma_l = C_l \cap (\Omega \times (0,T_o)) \setminus (\bigcup_{m \neq l} C_m)$$

is the non-empty union of 3-D manifolds. Pick any (Q_l,s_l) in Σ_l, the number

$$\rho = \min_{l=0,\dots,L} \{\min_{m \neq l} d((Q_l,s_l),C_m)\}$$

will be positive.

Theorem 4.1

Let $(P_l,t_l) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times (0,T)$ be as in Lemma 4.1. Then there exists a number $h_o > 0$, such that for every $h \leq h_o$ the linear functionals

$$L^2(\Omega \times (0,T_o)) \ni f \longrightarrow \overline{u}_h(P_l,t_l), \quad l = 0,1,\dots,L$$

are linearly independent.

Proof: The above statement is equivalent to say that the functions $w(P_l,t_l;\cdot,\cdot)$, $l = 0,1,\dots,L$ given by (1.4) are linearly independent elements of $L^2(\Omega \times (0,T_o))$ when $h \leq h_o$.

Moreover, since such functions are continuous, we may replace $L^2(\Omega \times (0,T_o))$ with $C(\Omega \times (0,T_o))$. Thus the theorem will remain proved once we have shown that the matrix

$$M = \{w(P_l,t_l;Q_m,s_m)\}, \quad l,m = 0,1,\dots,L$$

is invertible. Here the points (Q_m,s_m) are those found in Lemma 4.1.

Making use of (0,5), (1.4) and of Lemma 4.1 the following estimates can be obtained by rather lengthy, but straightforward, computations:

$$w(P_l,t_l;Q_l,s_l) \geq C_1 h^{-1}, \text{ for every } l,$$

and, if $h \leq h_1 = \min_l d(P_l,\Omega)/2$, then

$$0 \leq w(P_l,t_l;Q_m,s_m) \leq C_2\, h^{-5}(\max\{0,h-\rho\})^3, \text{ for every } l,m,\ l \neq m,$$

here C_1, C_2 are constants independent of h.

Therefore, if $h \leqq \min\{h_1,\rho\}$ then the matrix M is diagonal and positive definite. Note also that, by continuity, if $\rho < h_1$, then we may find $\delta > 0$ such that for $h \leqq \rho+\delta \leqq h_1$ M is still invertible. Thus the theorem is proved when

$$h_o = \begin{cases} h_1 & \text{if } h_1 \leqq \rho \\ \rho+\delta & \text{if } h_1 > \rho. \end{cases}$$

5. Appendix

Our problem originates from the following car-industry issue.

Law regulations prescribe an upper noise-level for cars. Standard measurements are made as follows: As the car goes at prescribed speeds on a straight road, the sound is measured at a point on the side of the road.
The distance from the road and the height on the ground of such a point are also prescribed.
It would be desirable to forecast such car-noise from laboratory experiments. Simulations are made with a car kept steady in an acoustic chamber. Sound measurements are taken at points around the car body.

We assume that noise-sources in the car are the same in the laboratory and on the road. The only difference comes then from the motion.
Let $y = x+x^0(t)$, $t \in (0,T)$, be the translation in space which represents the motion of the car on the road (actually, since the road is straight we may assume: $x^0(t) = \lambda(t)x^0$, where λ is a function and x^0 is a constant vector). Let u be the sound intensity in the laboratory and let ν be the sound-intensity on the road. Let $f = f(x,t)$ be the sound-source inside the car, that is: $\operatorname{supp} f \subset \bar{\Omega} \times [0,T_0]$. We obtain:

$$(5.1)\qquad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta_x\right)u(x,t) = f(x,t), & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = u_t(x,0) = 0 & \text{, on } \mathbb{R}^3, \end{cases}$$

$$(5.2)\qquad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta_x\right)\nu(x,t) = f(x-x_0(t),t), & \text{in } \mathbb{R}^3 \times (0,T), \\ \nu(x,0) = \nu_t(x,0) = 0 & \text{, on } \mathbb{R}^3. \end{cases}$$

The information on laboratory measurements is schematized by

$$(5.3)\qquad \sum_{l=1}^{L} |\bar{u}(P_l,t_l)-\tilde{u}_l|^2 \leq \varepsilon^2.$$

Moreover we assume the following bound on the source (see Introduction and Section 3):

$$(5.4)\qquad \int_{\Omega\times(0,T_0)} f^2(x,t)\, dxdt \leq E^2.$$

Note: The presence of the ground should be taken into account. That is equations (5.1), (5.2) should hold only in the half space $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathbb{R}^3 , x_3 > 0\}$. Moreover a boundary condition at $x_3 = 0$ should be added. However, for the sake of clarity we will continue to consider u, ν as solutions in all of space. Note also that, if homogeneous Dirichlet or Neumann conditions at $x_3 = 0$ hold, then we may introduce suitable symmetrizations which extend the solutions to all of space.

Our goal will be the determination of

(5.5) $\bar{\nu}(Q,t)$, q fixed, $t \in (0,T)$.

Now we see that

(5.6) $$\bar{\nu}(Q,t) = \int_{\mathbb{R}^3 \times (0,T)} w(Q,t;x,s)f(x-x^0(s),s)\,dxds$$

where w is the kernel defined in (1.4) (recall Lemma 1.1).

Therefore, since by (5.1), (5.2), (5.3) we have on the source f the same information as in Problem (I), we may adapt to the present case the algorithms developed in section 1. And precisely we see the following:

(i) The least-squares algorithm given in Theorem 1.1 may be applied as it is.

For every fixed $\tau \in (0,T)$ an approximate value of $\bar{\nu}(Q,\tau)$, can be obtained by

$$\bar{\nu}(Q,\tau) \simeq \sum_{l=1}^{L} b_l \int_{\substack{t \in (0,T_0) \\ x-x^0(t) \in \Omega}} w(Q,\tau;x,t)w(P_l,t_l;x-x^0(t),t)\,dxdt$$

where $b = (b_1,\ldots,b_L)^T$ is given by (1.11).

(ii) An error estimate can be developed as the one given in paragraph 1.2. We will just have to replace at all steps the functional

$$L^2(\Omega \times (0,T_0)) \ni f \longrightarrow \bar{u}(P_0,t_0)$$

with the functional

$$L^2(\Omega \times (0,T_0)) \ni f \longrightarrow \overline{v}(Q,\tau)$$

given by (5.6), which is too linear and bounded.

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