A Pyramid Scheme for Spherical Wavelets

by

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Abstract

We consider a scale discrete wavelet approach on the sphere based on spherical radial basis functions. If the generators of the wavelets have a compact support, the scale and detail spaces are finite-dimensional, so that the detail information of a function is determined by only finitely many wavelet coefficients for each scale. We describe a pyramid scheme for the recursive determination of the wavelet coefficients from level to level, starting from an initial approximation of a given function. Basic tools are integration formulas which are exact for functions up to a given polynomial degree and spherical convolutions.

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1 Introduction

In the last years there is a growing interest in wavelet methods on spherical surfaces. A number of papers from different groups have contributed, cf., e.g., [2], [1], [5], [6], [8], [9], [11], [12], [14], [16], [17]. The basic "philosophy" of our methods is the use of trial functions, which are axisymmetric, i.e., each of the trial functions depend only on the spherical distance to a certain nodal point on the sphere. This concept, of course, has close connection to the theory of radial basis functions in Euclidean spaces, for which we call the trial functions *spherical radial basis functions* (for a recent survey, see [7]). The main reasons for concentrating on this type of trial functions are: (i) they are simply structured (just given as a one–dimensional function); (ii) they are an appropriate tool for scattered data situations, which is often the case in real applications, e.g., in the geosciences, where the type and the position of measurements cannot be chosen freely; (iii) they are well–suited for the solution of boundary–value problems corresponding to spherical boundaries, since in many cases they can be easily extended to a solution of a differential equation inside or outside the sphere.

In the mean time, a series of papers have appeared, where the concept of spherical radial basis functions and wavelets have been brought together, cf. [5], [6], [8], [9], [17]. In this approach the wavelets are generated from a mother wavelet (which is a spherical radial basis function) by "moving" the function around the sphere (i.e., the corresponding nodal point is rotated according to $SO(3)$) and by a dilation operation. For the dilation there are two different approaches: in the continuous spherical wavelet theory (cf. [8], [9], [17]) one starts with a special kernel which defines a spherical singular integral and uses the free parameter of the kernel as a scale parameter. The scale discrete spherical wavelet theory (see [6]) starts from a definition of dilation which is independent of a special choice of a kernel, and can be applied to a large class of kernels. This definition is based on the existence of a *generator* of a spherical radial basis function, i.e., a function $\gamma_0 : [0, \infty) \to \mathbb{R}$ which is sampled at the integral points to define via the Legendre transform a spherical radial basis function. By the usual dilation applied to γ_0, i.e., $\gamma_j(x) = \gamma_0(2^{-j}x)$, one obtains new generators γ_j which then generate new kernels representing different frequency bands.

An important feature of this way is, that if the mother wavelet is appropriately chosen, the resulting scale and detail spaces have finite dimensions. This is the reason, why it is possible to find exact reconstruction formulas based on only finitely many wavelet coefficients at each scale. Therefore, it is, of course, enough to determine the wavelet transform at only these finitely many points at each level. This is the starting point of this publication: what is developed here is a pyramid scheme for the calculation of the wavelet transform. It turns out, that once an initial approximation of a given function is found (which is in case of a band–limited function very easy) the calculation of the wavelet coefficients can be done recursively from level to level. The major tools for this scheme are integration formulas on the sphere that are exact up to a given polynomial degree, and convolutions with spherical radial basis functions.

The outline of this paper is as follows: In Chapter 2 some preliminaries are stated, including some results on exact integration over the sphere. In order to keep the paper self-contained we present a short summary of scale discrete spherical wavelet theory as developed in [6] in the third chapter. Then the pyramid scheme for the recursive calculation of the wavelet transform is described in detail in Chapter 4. A numerical example and some concluding remarks are presented in Chapter 5.

2 Preliminaries

In this chapter we summarize some basic notations and definitions used in this paper.

2.1 Spherical Harmonics

If $x, y \in \mathbb{R}^3$, we write $x \cdot y$ for the Euclidean inner product and $|x| = \sqrt{x \cdot x}$ for the norm. We let $\Omega = \{\xi \in \mathbb{R}^3 \mid |\xi| = 1\}$ denote the unit sphere in $\mathbb{R}^3$. The standard surface measure on Ω is denoted by $d\omega$. On the space $\mathcal{L}^2(\Omega)$ we use the inner product $(F, G)_{\mathcal{L}^2(\Omega)} = \int_\Omega F(\eta)G(\eta)d\omega(\eta)$.

The *spherical harmonics* Y_n of order n are defined as the everywhere on the unit sphere Ω twice continuously differentiable eigenfunctions of the Beltrami operator Δ^* corresponding to the eigenvalues $-n(n+1), n = 0, 1, \ldots$. As it is well-known, the functions $H_n : \mathbb{R}^3 \to \mathbb{R}$ defined by $H_n(x) = r^n Y_n(\xi)$, $x = r\xi$, $r = |x|$, are polynomials in cartesian coordinates which satisfy the Laplace equation $\Delta_x H_n(x) = 0, x \in \mathbb{R}^3$. Conversely, every homogeneous harmonic polynomial of degree n when restricted to Ω is a spherical harmonic of order n. The linear space Harm_n of all spherical harmonics of order n is of dimension $2n+1$. Thus, there exist $2n+1$ linearly independent spherical harmonics $Y_{n,1}, \ldots, Y_{n,2n+1}$. Throughout the remainder of this paper we assume this system to be orthonormalized in the sense of the $\mathcal{L}^2(\Omega)$-inner product. $\mathrm{Harm}_{0,\ldots,b} = \oplus_{n=0}^{b} Harm_n$, $b \geq 0$, denotes the space of all spherical harmonics of order $\leq b$. Clearly, its dimension is $\sum_{n=0}^{b}(2n+1) = (b+1)^2$. The system $\{Y_{n,m} \mid n \in \mathbb{N}_0,\ m = 0, \ldots, 2n+1\}$ is known to be complete in $\mathcal{L}^2(\Omega)$. The *Fourier transform* of a funtion $F \in \mathcal{L}^2(\Omega)$ is denoted $F^\wedge(n, m) = (F, Y_{n,m})_{\mathcal{L}^2(\Omega)}$, $n = 0, 1, \ldots$, $m = 1, \ldots, 2n+1$.

An outstanding result of the theory of spherical harmonics is the *addition theorem*

$$\sum_{m=1}^{2n+1} Y_{n,m}(\xi)Y_{n,m}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta),\ \xi, \eta \in \Omega,$$

where P_n denotes the Legendre polynomial of degree n. The addition theorem is essential for our wavelet approach, since it relates the spherical harmonics on Ω to a univariate function, viz. the Legendre polynomial, defined on $[-1, 1]$.

The close connection between the orthogonal invariance and the addition theorem is established by the *Funk-Hecke formula* for $H \in \mathcal{L}^1[-1, 1]$,

$$\int_\Omega H(\xi \cdot \eta)P_n(\zeta \cdot \eta)d\omega(\eta) = H^\wedge(n)P_n(\xi \cdot \zeta).$$

where

$$H^\wedge(n) = 2\pi \int_{-1}^{1} H(t)P_n(t)dt,$$

$n = 0, 1, \ldots$. For more details about the theory of spherical harmonics the reader is referred, for example, to [10].

2.2 Integration Formulas on the Sphere

In this chapter we study integration formulas for the approximate integration of functions $F \in \mathcal{C}(\Omega)$. Of particular interest are those integration formulas which are exact for all $F \in \mathrm{Harm}_{0,\ldots,b}$, where $b \in \mathbb{N}_0$ is given.

Definition 2.1 *A system $\{(\eta_i, w_i)\}_{i=1,\ldots,N} \subset \Omega \times \mathbb{R}$ with pairwisely distinct $\eta_i \in \Omega$ and weights $w_i \in \mathbb{R}$ (we assume $w_i \neq 0$ for all $i = 1, \ldots, N$) defines an integration formula for the approximation of $\int_\Omega F(\eta)d\omega(\eta)$, $F \in \mathcal{C}(\Omega)$, by*

$$I(F) = \sum_{i=1}^{N} w_i F(\eta_i).$$

We call the formula exact of order b, if $\int_\Omega F(\eta)d\omega(\eta) = \sum_{i=1}^{N} w_i F(\eta_i)$ is valid for all $F \in Harm_{0,\ldots,b}$.

Obviously, the system $\{(\eta_i, w_i)\}_{i=1,\ldots,N}$ defines an exact integration formula of order b if and only if $\sum_{i=1}^{N} w_i = 4\pi$ and

$$\sum_{i=1}^{N} w_i Y_{n,m}(\eta_i) = 0 \text{ for all } n = 1, \ldots, b, \; m = 1, \ldots, 2n+1.$$

Definition 2.2 *Let $X_N = \{\eta_1, \ldots, \eta_N\} \subset \Omega$ be a system of pairwisely distinct points. We call X_N a fundamental system for $Harm_{0,\ldots,b}$, if the matrix*

$$\begin{pmatrix} Y_{0,1}(\eta_1) & \cdots & Y_{0,1}(\eta_N) \\ \vdots & \ddots & \vdots \\ Y_{b,2b+1}(\eta_1) & \cdots & Y_{b,2b+1}(\eta_N) \end{pmatrix} \tag{2.1}$$

is regular (that implies $N = (b+1)^2$). X_N is called admissible in $Harm_{0,\ldots,b}$, if it contains a subset which forms a fundamental system for $Harm_{0,\ldots,b}$.

It is not difficult to prove

Lemma 2.3 *Let $X_N = \{\eta_1, \ldots, \eta_N\} \subset \Omega$ be a system of pairwisely distinct points. Then the following three conditions are equivalent:*

(i) X_N is admissible in $Harm_{0,\ldots,b}$.

(ii) The matrix (2.1) is of rank $(b+1)^2$.

(iii) For all $F \in Harm_{0,\ldots,b}$ satisfying $F(\eta_i) = 0$, $i = 1, \ldots, N$, it follows $F = 0$.

It is clear that if we have an $\mathrm{Harm}_{0,\dots,b}$-admissible system $X_N = \{\eta_1, \dots, \eta_N\} \subset \Omega$, then we can obtain an integration formula which is exact of order b, if the weights w_i solve the linear system

$$\begin{pmatrix} Y_{0,1}(\eta_1) & \cdots & Y_{0,1}(\eta_N) \\ \vdots & \ddots & \vdots \\ Y_{b,2b+1}(\eta_1) & \cdots & Y_{b,2b+1}(\eta_N) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} \int_\Omega Y_{0,1}(\eta) d\omega(\eta) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

An argument in the opposite direction is given by

Theorem 2.4 *Assume that the system $\{(\eta_i, w_i)\}_{i=1,\dots,N}$ defines an integration formula which is exact of order $2b$ for a $b \in \mathbb{N}_0$. Then:*

(i) For all $F, G \in \mathrm{Harm}_{0,\dots,b}$,

$$\int_\Omega F(\eta)G(\eta)d\omega(\eta) = \sum_{i=1}^{N} w_i F(\eta_i)G(\eta_i).$$

(ii) The system $X_N = \{\eta_1, \dots, \eta_N\}$ is admissible in $\mathrm{Harm}_{0,\dots,b}$.

Proof. Part (i) follows if we can show that the function $\eta \mapsto F(\eta)G(\eta)$ is a member of $\mathrm{Harm}_{0,\dots,2b}$. But this is a consequence of the following arguments: F and G can be seen to be restrictions of polynomials in $\mathbb{R}^3$ of degree less or equal b. Thus, the product FG can be recognized as restriction of a polynomial of degree less or equal $2b$ in $\mathbb{R}^3$. But it is known (cf., e.g., [4]) that any polynomial of degree less or equal $2b$ restricted to the unit sphere Ω is an element of $\mathrm{Harm}_{0,\dots,2b}$.

In order to prove part (ii), let $F \in \mathrm{Harm}_{0,\dots,b}$ satisfy $F(\eta_i) = 0$, $i = 1, \dots, N$. Then we have for all $n = 0, \dots, b$, $m = 1, \dots, 2n+1$,

$$F^\wedge(n,m) = \int_\Omega F(\eta) Y_{n,m}(\eta) d\omega(\eta) = \sum_{i=1}^{N} w_i F(\eta_i) Y_{n,m}(\eta_i) = 0.$$

Thus, $F = 0$, and (ii) follows from Lemma 2.3. □

2.3 Radial Basis Functions on the Sphere

Suppose that $K \in \mathcal{L}^2[-1,1]$. If $\eta \in \Omega$ is fixed, then the function

$$\Omega \ni \xi \mapsto K(\xi \cdot \eta)$$

is called a *spherical radial basis function*. These functions depend only on the spherical distance between a fixed $\eta \in \Omega$ and the argument $\xi \in \Omega$. Expanding

K in terms of spherical harmonics yields via the addition theorem and the Funk–Hecke formula

$$\begin{aligned} K(\xi\cdot\eta) &= \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\int_{\Omega} K(\zeta\cdot\eta)Y_{n,m}(\zeta)d\omega(\zeta)Y_{n,m}(\xi) \\ &= \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}K^{\wedge}(n)P_n(\xi\cdot\eta). \end{aligned}$$

The mapping $K \longmapsto \{K^{\wedge}(n)\}_{n=0,1,\ldots}$ is called the *Legendre transform* of K.

Of particular importance for this paper are *band–limited* kernels, i.e., radial basis functions K where only finitely many $K^{\wedge}(n)$ are different from zero.

Theorem 2.5 *Let*

$$K = \sum_{n=0}^{b}\frac{2n+1}{4\pi}K^{\wedge}(n)P_n$$

be a spherical radial basis functions with $K^{\wedge}(n) \neq 0$ for $n = 0,\ldots,b$ and $K^{\wedge}(n) = 0$ for $n > b$. Assume further that the system $X_N = \{\eta_1,\ldots,\eta_N\} \subset \Omega$ is admissible in $Harm_{0,\ldots,b}$. Then

$$span\{K(\eta_i) \mid i = 1,\ldots,N\} = Harm_{0,\ldots,b}.$$

Proof. The inclusion "$\subset$" is clear. In order to prove the inclusion in the opposite direction assume without loss of generality that the subset $\{\eta_1,\ldots,\eta_{(b+1)^2}\}$ forms a fundamental system in $\mathrm{Harm}_{0,\ldots,b}$. We shall show that the functions $K(\eta_1\cdot\ ),\ldots, K(\eta_{(b+1)^2}\cdot\ )$ are linearly independent. Suppose therefore that $\sum_{i=1}^{(b+1)^2} a_i K(\eta_i\cdot\ ) = 0$. By multiplying this equation with $Y_{n,m}$ and integrating over Ω we obtain

$$\begin{pmatrix} Y_{0,1}(\eta_1) & \cdots & Y_{0,1}(\eta_N) \\ \vdots & \ddots & \vdots \\ Y_{b,2b+1}(\eta_1) & \cdots & Y_{b,2b+1}(\eta_N) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But since the above matrix is regular, it follows that $a_1 = \ldots = a_N = 0$, as required. □

For later use we introduce an abbreviation for a particular band–limited kernel, namely the *Shannon kernel* SH_b for a given $b \in \mathrm{N}_0$:

$$\mathrm{SH}_b = \sum_{n=0}^{b}\frac{2n+1}{4\pi}P_n. \tag{2.2}$$

This kernel is the *reproducing kernel* of the space $\mathrm{Harm}_{0,\ldots,b}$, since

- $\mathrm{SH}_b(\eta\cdot\ ) \in \mathrm{Harm}_{0,\ldots,b}$ for all $\eta \in \Omega$.
- $F(\eta) = (F, \mathrm{SH}_b(\eta\cdot\ ))_{\mathcal{L}^2(\Omega)}$ for all $F \in \mathrm{Harm}_{0,\ldots,b}$, $\eta \in \Omega$.

If $F \in \mathcal{L}^2(\Omega)$ and $K \in \mathcal{L}^2[-1,1]$, we define the *convolution of K and F* by

$$(K * F)(\xi) = \int_\Omega K(\xi \cdot \eta) F(\eta) d\omega(\eta), \ \xi \in \Omega.$$

Note that this operation is not commutative, since K and F are defined on different sets. Obviously, we have $(K * F)^\wedge(n,m) = K^\wedge(n) F^\wedge(n,m)$ for all $n = 0, 1, \dots, m = 1, \dots, 2n+1$. If $H \in \mathcal{L}^2[-1,1]$ defines another radial basis functions, we let the *convolution of H and K* be defined as

$$(H * K)(\xi, \eta) = \int_\Omega H(\xi \cdot \zeta) K \zeta \cdot \eta) d\omega(\zeta), \ \xi, \ \eta \in \Omega.$$

It is not difficult to see that $(H * K)(\xi, \eta)$ depends again only on the inner product of ξ and η, so that we can write $(H * K)(\xi \cdot \eta) = (H * K)(\xi, \eta)$, i.e., $H * K$ is a spherical radial basis function. Obviously,

$$(H * K)^\wedge(n) = H^\wedge(n) K^\wedge(n), \ n \in \mathbb{N}_0.$$

The *iterated kernel* $K^{(2)}$ is defined by $K^{(2)} = K * K$, yielding

$$(K^{(2)})^\wedge(n) = (K^\wedge(n))^2, \ n \in \mathbb{N}_0.$$

2.4 Dilation

An important feature of Euclidean wavelet theory is the operation of contraction and dilation. There exists no obvious transition of this feature for functions defined on the unit sphere. Here we overcome this difficulty by defining dilation in the frequency space, i.e., we define the dilation of radial basis functions in terms of their Legendre transform. As a matter of fact, we have to assume that the radial basis functions under consideration are defined from a generator, i.e., a continuous version of the Legendre transform, which is evaluated (or sampled) at all integral values. These ideas were recently pointed out in [6].

Definition 2.6 *A piecewise continuous function $\gamma_0 : [0, \infty) \to \mathbb{R}$ is said to satisfy an admissibility condition, if*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\sup_{x \in [n, n+1)} |\gamma_0(x)| \right)^2 < \infty. \tag{2.3}$$

In this case, γ_0 is called an admissible generator of the function $\Gamma_0 : [-1,1] \to \mathbb{R}$ given by

$$\Gamma_0 = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \gamma_0(n) P_n, \tag{2.4}$$

i.e., $\Gamma_0^\wedge(n) = \gamma_0(n)$, $n = 0, 1, \dots$.

Remark 2.7 *(i) In many cases an admissible generator is monotonously decreasing and satisfies, in addition, the estimate $0 \le \gamma_0(x) \le 1$, $x \in [0, \infty)$. Under these assumptions, condition (2.3) simply reduces to*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} |\gamma_0(n)|^2 < \infty. \tag{2.5}$$

(ii) If γ_0 has a compact support, (2.3) is fulfilled automatically.

Based on these preliminaries it is not hard to prove (cf. [6]) that if $\gamma_0 : [0, \infty) \to \mathbb{R}$ is an admissible generator, then Γ_0 defined by (2.4) is an element of $\mathcal{L}^2[-1, 1]$, hence, $\Gamma_0(\eta \cdot)$ is an element of $\mathcal{L}^2(\Omega)$ for every $\eta \in \Omega$.

For a function γ_0 satisfying the admissibility condition we introduce functions $\gamma_j : [0, \infty) \to \mathbb{R}$ in the following way

$$\gamma_j(x) = D_j\gamma_0(x) = \gamma_0(2^{-j}x), \; x \in [0, \infty), \tag{2.6}$$

for $j = 0, 1, \ldots$. Then we have that also γ_j satisfies (2.3) if γ_0 does. Therefore,

$$\gamma_j = D_1\gamma_{j-1}, \; j = 1, 2, \ldots, \tag{2.7}$$

provided that γ_0 satisfies the admissibility condition (2.3). This gives rise to

Definition 2.8 *Suppose that $\gamma_0 : [0, \infty] \to \mathbb{R}$ satisfies (2.3). For the generated functions $\Gamma_j \in \mathcal{L}^2[-1, 1]$, $j = 0, 1, \ldots$, given by*

$$\Gamma_j = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \gamma_j(n) P_n$$

we let

$$\Gamma_j = D_1\Gamma_{j-1} = D_j\Gamma_0, \; j = 1, 2, \ldots.$$

D_j is called dilation operator of j-th level.

As an immediate consequence we obtain

Corollary 2.9 *If Γ_0 is generated by an admissible generator γ_0, then $\Gamma_j = D_j\Gamma_0 \in \mathcal{L}^2[-1, 1]$ for all $j \in \mathbb{N}_0$.*

3 Scale Discrete Wavelets on the Sphere

In order to keep this paper self–contained, we give a short overview on scale discrete spherical wavelets. For a more detailed discussion, see [6].

3.1 Scaling Function

In what follows, we concentrate on those admissible generators which generate scaling functions.

Definition 3.1 *Let $\varphi_0 : [0, \infty) \to \mathbb{R}$ satisfy the admissibility condition (2.3). φ_0 is called generator of a scaling function if it satisfies the following properties:*

(i) $\varphi_0(0) = 1$,

(ii) φ_0 is monotonously decreasing,

(iii) φ_0 is continuous at 0.

Under these requirements φ_0 and its dilates φ_j generate the scale discrete scaling function $\{\Phi_j\}, \Phi_j \in \mathcal{L}^2[-1,1]$, $j = 0, 1, \ldots,$ via

$$\Phi_j^\wedge(n) = \varphi_j(n), \quad n = 0, 1, \ldots .$$

A straightforeward consequence is

Lemma 3.2 *Let φ_0 and its dilates φ_j, $j \in \mathrm{N}$, generate the scale discrete scaling function $\{\Phi_j\}, \Phi_j \in \mathcal{L}^2[-1,1]$, $j \in \mathrm{N}$. Then, for $F \in \mathcal{L}^2(\Omega)$,*

$$\lim_{j\to\infty} \|F - \Phi_j * F\|_{\mathcal{L}^2(\Omega)} = 0 \tag{3.1}$$

and

$$\lim_{j\to\infty} \|F - \Phi_j^{(2)} * F\|_{\mathcal{L}^2(\Omega)} = 0. \tag{3.2}$$

It follows that the $\Phi_j^{(2)} * F$, $j = 0, 1, \ldots,$ provide us with approximations of F at different scales. In terms of filtering $\{\Phi_j^{(2)}\}$ may be interpreted as low–pass filter. The corresponding convolution operators $P_j : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$, $j \in \mathrm{N}_0$, are given by

$$P_j(F) = \Phi_j^{(2)} * F, \quad j = 0, 1, \ldots . \tag{3.3}$$

Accordingly, we understand the *scale space* V_j to be the image of $\mathcal{L}^2(\Omega)$ under the operator P_j:

$$V_j = P_j(\mathcal{L}^2(\Omega)) = \{\Phi_j^{(2)} * F | F \in \mathcal{L}^2(\Omega)\}.$$

The scale spaces V_j define a *(scale discrete) multiresolution analysis* of $\mathcal{L}^2(\Omega)$ in the following sense:

Theorem 3.3 *The scale spaces V_j satisfy the following statements:*

(i) $V_0 \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset \mathcal{L}^2(\Omega)$

(ii) $\overline{\bigcup_{j=0}^{\infty} V_j} = \mathcal{L}^2(\Omega)$

(iii) If a function $G \in \mathcal{L}^2[-1,1]$ satisfies $G(\eta\cdot) \in V_j$, then $D_{-1}G(\eta\cdot) \in V_{j-1}$, $j = 1, 2, \ldots$.

One might expect, that there is some more structure in the multiresolution, e.g., $V_0 = \{0\}$ or the fact that all scale spaces V_j are finite–dimensional with certain conditions on the dimension, etc. But these properties are, in general, not true. They depend on the special choice of the generator φ_0 of a scaling function and are discussed in more detail later on, when we restrict ourselves to compactly supported generators φ_0.

3.2 Scale Discrete Wavelet Transform

The definition of the scale discrete scaling function now allows us to introduce scale discrete wavelets on the sphere. We represent an $\mathcal{L}^2(\Omega)$-function F by a two-parameter family $(j;\eta), j \in \mathrm{N}_0$, $\eta \in \Omega$, breaking up the function F into "pieces" at different locations and different levels of resolution. An essential point is the definition of a mother wavelet and its dual wavelet starting from their generators. This definition, of course, has to be done in close relation to a given scaling function. The mother wavelet is then rotated and dilated to establish the discrete version of the wavelet transform of a function. As a matter of fact, we are able to prove a reconstruction formula.

Definition 3.4 *Let φ_0 be the generator of a scaling function (as defined by Definition 3.1). Then the piecewise continuous functions $\psi_0, \tilde{\psi}_0 : [0,\infty) \to \mathbb{R}$ are said to be generators of the mother wavelet $\Psi_0 \in \mathcal{L}^2[-1,1]$ and the dual mother wavelet $\tilde{\Psi}_0 \in \mathcal{L}^2[-1,1]$, respectively, if both of them are admissible generators and satisfy, in addition, the "refinement equation"*

$$\tilde{\psi}_0(x)\psi_0(x) = (\varphi_0(x/2))^2 - (\varphi_0(x))^2, \ x \in [0,\infty).$$

The functions $\Psi_0 \in \mathcal{L}^2[-1,1]$, $\tilde{\Psi}_0 \in \mathcal{L}^2[-1,1]$, defined via the Legendre coefficients $\Psi_0^\wedge(n), \tilde{\Psi}_0^\wedge(n)$, given by

$$\begin{aligned} \Psi_0^\wedge(n) &= \psi_0(n), \ n = 0,1,\ldots, \\ \tilde{\Psi}_0^\wedge(n) &= \tilde{\psi}_0(n), \ n = 0,1,\ldots, \end{aligned}$$

are called the mother wavelet and the dual mother wavelet, respectively.

Let us make a couple of simple observations concerning this definition.

Lemma 3.5 *The generators $\psi_0, \tilde{\psi}_0 : [0,\infty) \to \mathbb{R}$ and their dilates $\psi_j = D_j\psi_0$, $\tilde{\psi}_j = D_j\tilde{\psi}_0$ satisfy the following properties:*

(i) $\tilde{\psi}_j(0)\psi_j(0) = 0$, $j \in \mathrm{N}_0$,

(ii) $\tilde{\psi}_j(x)\psi_j(x) = (\varphi_{j+1}(x))^2 - (\varphi_j(x))^2$, $j \in \mathrm{N}_0$, $x \in [0,\infty)$,

(iii) $(\varphi_0(x))^2 + \sum_{j=0}^{J} \tilde{\psi}_j(x)\psi_j(x) = (\varphi_{J+1}(x))^2$, $J \in \mathrm{N}_0$, $x \in [0,\infty)$.

It is natural — as it was done for the scaling function — to apply the operators D_j directly to the mother wavelet and its dual. In connection with the rotation operator R_η, this will lead us to the definition of the wavelet $\Psi_{j;\eta}$ and its dual wavelet $\tilde{\Psi}_{j;\eta}$. More explicitly, we have

$$\Psi_j = D_j\Psi_0, \ \tilde{\Psi}_j = D_j\tilde{\Psi}_0, \ j \in \mathrm{N}_0, \tag{3.4}$$

and

$$\begin{aligned} (R_\eta\Psi_j)(\xi) &= \Psi_{j;\eta}(\xi) = \Psi_j(\eta\cdot\xi) \quad , \ \xi \in \Omega, \\ (R_\eta\tilde{\Psi}_j)(\xi) &= \tilde{\Psi}_{j;\eta}(\xi) = \tilde{\Psi}_j(\eta\cdot\xi) \quad , \ \xi \in \Omega. \end{aligned} \tag{3.5}$$

Putting together (3.4) and (3.5) we therefore obtain for $\xi \in \Omega$

$$\begin{aligned} \Psi_{j;\eta}(\xi) &= (R_\eta D_j \Psi_0)(\xi), \\ \tilde{\Psi}_{j;\eta}(\xi) &= (R_\eta D_j \tilde{\Psi}_0)(\xi). \end{aligned}$$

Definition 3.6 *Let Ψ_0, $\tilde{\Psi}_0 \in \mathcal{L}^2[-1,1]$ be a mother wavelet and its dual according to a scaling function $\{\Phi_j\} \in \mathcal{L}^2[-1,1]$. Then, for $F \in \mathcal{L}^2(\Omega)$, the discrete wavelet transform at scale $j \in \mathrm{N}_0$ and position $\eta \in \Omega$ is defined by*

$$\mathrm{WT}(F)(j;\eta) = (F, \Psi_{j;\eta})_{\mathcal{L}^2(\Omega)}.$$

In analogy to the definition of the operator $P_j : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ (cf. (3.3)) we consider now convolution operators (band–pass filters) $R_j : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$, $j \in \mathrm{N}_0$, defined by

$$R_j(F) = \tilde{\Psi}_j * \Psi_j * F. \tag{3.6}$$

$R_j(F)$ can be interpreted as a version of F blurred to the scale j. It describes the "detail behaviour" of F at scale j. From Lemma 3.5, (iii), we can immediately deduce that for $J \in \mathrm{N}_0$

$$\Phi_0^{(2)} + \sum_{j=0}^{J} \tilde{\Psi}_j * \Psi_j = \Phi_{J+1}^{(2)}. \tag{3.7}$$

Therefore, it follows that the operator P_{J+1} can be decomposed in the following way

$$P_{J+1} = P_0 + \sum_{j=0}^{J} R_j.$$

This gives rise to introduce the *detail space* W_j to be

$$W_j = R_j(\mathcal{L}^2(\Omega)) = \{\tilde{\Psi}_j * \Psi_j * F \mid F \in \mathcal{L}^2(\Omega)\}.$$

The space W_j contains the "detail information" needed to go from an approximation at resolution j to an approximation at resolution $j+1$. Note that

$$\begin{aligned} V_0 + \sum_{j=0}^{J} W_j &= V_{J+1}, \\ V_j &= V_{j-1} + W_{j-1}. \end{aligned} \tag{3.8}$$

It is worth mentioning that the sum decomposition in general is neither direct nor orthogonal.

Any function $F \in \mathcal{L}^2(\Omega)$ can now be decomposed as follows: Starting with $P_{J+1}(F)$ for some J we have $P_{J+1}(F) = P_0(F) + \sum_{j=0}^{J} R_j(F)$. In other words, the partial reconstruction $R_{J+1}(F)$ is nothing else than the difference of two "smoothings" $P_{J+1}(F) - P_J(F)$ at two consecutive scales: $R_J(F) = P_{J+1}(F) - P_J(F)$ (cf. Lemma 3.5).

Furthermore, we have

$$\begin{aligned} (P_j(F))^\wedge(n,j) &= F^\wedge(n,j)(\Phi_j)^\wedge(n)(\Phi_j)^\wedge(n), \\ (R_j(F))^\wedge(n,j) &= F^\wedge(n,j)(\tilde{\Psi}_j)^\wedge(n)(\Psi_j)^\wedge(n). \end{aligned} \tag{3.9}$$

The fomulas (3.9), therefore, give wavelet decompositions like (3.8) an interpretation in terms of Fourier analysis by explaining how the frequency spectrum of a function $F \in V_j$ is divided up between the space V_{j-1} and W_{j-1}, which enhances our understanding of what is meant by "smoothing" and "detail".

Our definition of the discrete wavelet transform developed above enables us to prove a reconstruction formula. In other words, it is possible to reconstruct a function $F \in \mathcal{L}^2(\Omega)$ from its wavelet transform $\mathrm{WT}(F)$:

Theorem 3.7 (Reconstruction Formula) *Let Ψ_0 resp. $\tilde{\Psi}_0$ be the mother wavelet resp. the dual mother wavelet with respect to a scale discrete scaling function Φ_0. Then, for $F \in \mathcal{L}^2(\Omega)$,*

$$F = \Phi_0^{(2)} * F + \sum_{j=0}^{\infty} \int_{\Omega} \mathrm{WT}(F)(j;\eta) \tilde{\Psi}_{j;\eta}(\cdot) d\omega(\eta),$$

where the equality is understood in the $\mathcal{L}^2(\Omega)$-sense.

Up to now, the definition of the mother wavelet Ψ_0 and its dual $\tilde{\Psi}_0$ are quite general. The only condition which has to be satisfied (besides the admissibility) is the "refinement equation"

$$\tilde{\psi}_0(x)\psi_0(x) = (\varphi_1(x))^2 - (\varphi_0(x))^2, \; x \in [0, \infty), \tag{3.10}$$

where φ_0 is the generator of a scale discrete scaling function.

Two choices for ψ_0 and $\tilde{\psi}_0$ are immediately at hand. The *P-scale discrete wavelets* are defined by

$$\psi_0(x) = \tilde{\psi}_0(x) = \sqrt{[\varphi_1(x)]^2 - [\varphi_0(x)]^2}, \; x \in [0, \infty),$$

where the *M-scale discrete wavelets* can be obtained from

$$\begin{aligned} \psi_0(x) &= \varphi_1(x) - \varphi_0(x), \; x \in [0, \infty), \\ \tilde{\psi}_0(x) &= \varphi_1(x) + \varphi_0(x), \; x \in [0, \infty). \end{aligned}$$

It both cases it is not difficult to prove that the admissibility condition is satisfied by the generators.

3.3 Examples

Besides the conditions of Definition 3.1 there are no restrictions on the generator φ_0, so that many choices are at our disposal. We first mention some possibilities of globally supported generators φ_0:

(i) $\varphi_0(x) = (1+x)^{-s}$, $s > 1$

(ii) $\varphi_0(x) = (1+x^2)^{-s}$, $s > 1/2$

(iii) $\varphi_0(x) = e^{-rx}$, $r > 0$

(iv) $\varphi_0(x) = e^{-rx(x+1)}$, $r > 0$

The choice (iii) yields the Abel–Poisson kernel, whereas the generator (iv) generates the Gauß–Weierstraß kernel.

If φ_0 is chosen to be compactly supported, we get two advantages for our multiresolution analysis: the scale spaces and the detail spaces have finite dimensions, and for the reconstruction the wavelet transform $\mathrm{WT}(F)(j;\eta)$ is only required at finitely many points η.

The simplest choice for a compactly supported φ_0 is

$$\text{(v)}\quad \varphi_0(x) = \begin{cases} 1 & \text{for} \quad x \in [0,1) \\ 0 & \text{for} \quad x \in [1,\infty) \end{cases}$$

The so defined *Shannon scaling function* and *Shannon wavelets* show strong oscillations. They can be reduced, if φ_0 is smoothed out as

$$\text{(vi)}\quad \varphi_0(x) = \begin{cases} 1 & \text{for} \quad x \in [0,h) \\ (x-1)/(h-1) & \text{for} \quad x \in [h,1) \\ 0 & \text{for} \quad x \in [1,\infty) \end{cases}, \quad h \in (0,1),$$

which yields the *de la Vallé Poussin scaling function.* The perhaps best choice (from our numerical experiences) is if φ_0 is chosen to be the cubic polynomial on the interval $[0,1]$, that satisfies $\varphi_0(0) = 1$, $\varphi_0\prime(0) = 0$, $\varphi_0(1) = 0$, $\varphi_0\prime(1) = 0$, i.e.,

$$\text{(vii)}\quad \varphi_0(x) = \begin{cases} (1-x^2)(1-2x) & \text{for} \quad x \in [0,1) \\ 0 & \text{for} \quad x \in [1,\infty) \end{cases}$$

Numerical tests show, that the unpleasent oscillations, that are still existent in the de la Vallé Poussin scaling functions, are suppressed by this choice.

Of course other possibilities for φ_0 can be found. A graphical impression of the scaling functions and the corresponding P–wavelets corresponding to the compact generators are given in the pictures below.

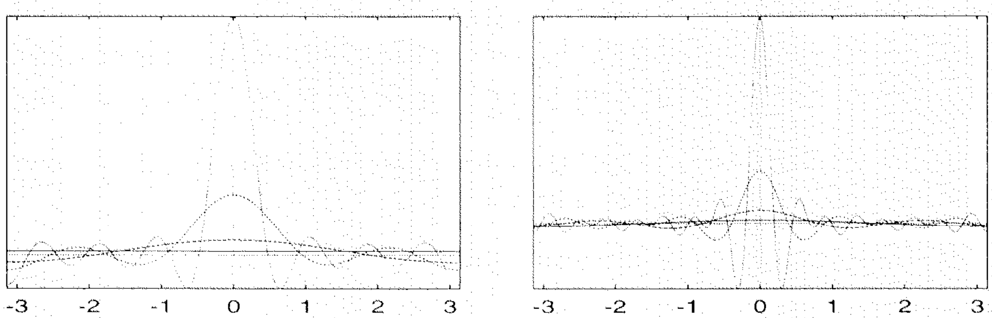

Figure 3.1: Scaling functions $\Phi_j(\cos\vartheta)$ (left) and P–wavelets $\Psi_j(\cos\vartheta)$ (right) for the generator (v).

3.4 Exact Fully Discrete Wavelet Transform

We are now interested in fully discrete wavelet approximation. For this purpose we want to show that when using band-limited wavelets, we do not need the

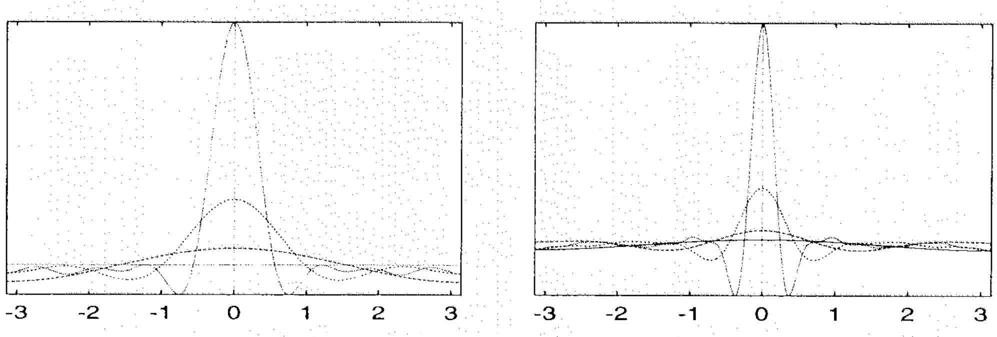

Figure 3.2: Scaling functions $\Phi_j(\cos\vartheta)$ (left) and P–wavelets $\Psi_j(\cos\vartheta)$ (right) for the generator (vi) with $h = 0.5$.

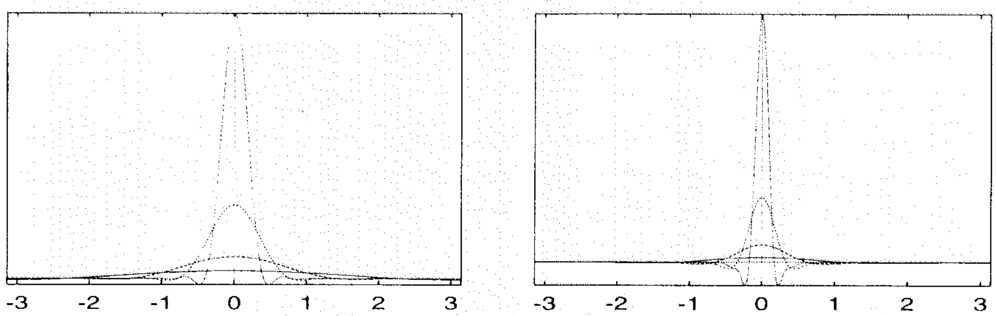

Figure 3.3: Scaling functions $\Phi_j(\cos\vartheta)$ (left) and P–wavelets $\Psi_j(\cos\vartheta)$ (right) for the generator (vii).

wavelet transform $\mathrm{WT}(F)(j;\eta)$ at all rotates $\eta \in \Omega$. It suffices to know the wavelet transform only at a finite set of rotates for each scale j. As a matter of fact, the reconstruction can be formulated in terms of simply structured sum representations. More explicitly, the reconstructed function can be expressed in each scale as a linear combination of finitely many dual wavelets $\tilde{\Psi}_j(\eta_k^j\cdot\,), \eta_k^j \in \Omega$.

Since we do not want to hide the substantial ideas behind a technical overhead, we make the following assumptions:

(i) The generator φ_0 of the scaling function satisfies $\mathrm{supp}\varphi_0 = [0,1]$ and $\varphi_0(1) = 0$.

(ii) The generators ψ_0, $\tilde{\psi}_0$ of the mother wavelet and the dual mother wavelet, respectively, satisfy $\mathrm{supp}\psi_0$, $\mathrm{supp}\tilde{\psi}_0 \subset [0,2]$ and $\psi_0(2) = \tilde{\psi}_0(2) = 0$.

Remark 3.8 *Note that all the presented generators of the band-limited scaling functions satisfy assumption (i). If ψ_0 and $\tilde{\psi}_0$ are constructed to generate the P- or M-wavelets, respectively, then requirement (ii) is also fulfilled.*

An immediate consequence of these assumptions is

$$\begin{aligned} \mathrm{supp}\varphi_j &= [0, 2^j], \\ \mathrm{supp}\psi_j,\ \mathrm{supp}\tilde{\psi}_j &\subset [0, 2^{j+1}], \\ \varphi_j(2^j) = \psi_j(2^{j+1}) = \tilde{\psi}_j(2^{j+1}) &= 0. \end{aligned}$$

Hence, we have

Lemma 3.9 *(i) Under the given assumptions the scaling functions and the wavelets satisfy for all $\eta \in \Omega$*

$$\begin{aligned} \Phi_j(\eta\cdot\ ) &\in Harm_{0,\ldots,2^j-1}, \\ \Psi_j(\eta\cdot\ ),\ \tilde{\Psi}_j(\eta\cdot\ ) &\in Harm_{0,\ldots,2^{j+1}-1}. \end{aligned}$$

(ii) The scale spaces and detail spaces fulfil

$$\begin{aligned} V_j &= Harm_{0,\ldots,2^j-1}, \\ W_j &\subset Harm_{0,\ldots,2^{j+1}-1}. \end{aligned}$$

Next we want to show, that $R_j(F)$ can be exactly determined, if the wavelet transform is known at only finitely many points. Assume therefore that the system $\{(\eta_i^j, w_i^j)\}_{i=1,\ldots,N_j} \subset \Omega \times \mathbb{R}$ defines an integration formula that is exact of order $2^{j+2}-2$. Then it follows — since $\mathrm{WT}(F)(j;\ )$ and $\tilde{\Psi}_j(\eta\cdot\ )$ are in $\mathrm{Harm}_{0,\ldots,2^{j+1}-1}$ — that for all $F \in \mathcal{L}^2(\Omega)$ and all $\xi \in \Omega$

$$\begin{aligned} R_j(F) &= \int_\Omega \mathrm{WT}(F)(j;\eta)\tilde{\Psi}_{j;\eta}(\xi)d\omega(\eta) \\ &= \sum_{i=1}^{N_j} w_i^j \mathrm{WT}(F)(j;\eta_i^j)\tilde{\Psi}_{j;\eta_i^j}(\xi). \end{aligned}$$

We summarize our results in

Theorem 3.10 *Let the generators $\varphi_0, \psi_0, \tilde{\psi}_0 : [0,\infty) \to \mathbb{R}$ satisfy the assumptions stated at the beginning of this section. Assume further that there is given a sequence of systems $\{X_j\}_{j=0,1,\ldots}$, $X_j = \{(\eta_i^j, w_i^j)\}_{i=1,\ldots,N_j}$, which defines an exact integration formula of order $2^{j+2}-2$ for each j. If $F \in \mathcal{L}^2(\Omega)$, then*

$$F(\xi) = (\Phi_0^{(2)} * F)(\xi) + \sum_{j=0}^{\infty}\sum_{i=1}^{N_j} w_i^j \mathrm{WT}(F)(j;\eta_i^j)\tilde{\Psi}_{j;\eta_i^j}(\xi),\ \xi \in \Omega.$$

Remark 3.11 *Notice that under our assumptions $\Phi_0^{(2)} = \frac{1}{4\pi}P_0$, so that*

$$(\Phi_0^{(2)} * F)(\xi) = \frac{1}{4\pi}\int_\Omega F(\eta)d\omega(\eta)$$

for all $\xi \in \Omega$. Hence, for $F \in \mathcal{L}^2(\Omega)$ with $\int_\Omega F(\eta)d\omega(\eta) = 0$,

$$F = \sum_{j=0}^{\infty}\sum_{i=1}^{N_j} w_i^j \mathrm{WT}(F)(j;\eta_i^j)\tilde{\Psi}_{j;\eta_i^j}.$$

4 Pyramid Scheme

Now we are able to describe the main results of this paper.

4.1 Introduction

We first summarize our assumptions made for the generators φ_0, ψ_0, $\tilde{\psi}_0$ and the systems X_j:

(i) $\text{supp}\varphi_0 = [0,1]$, and $\varphi_0(1) = 0$.

(ii) $\text{supp}\psi_0 = \text{supp}\tilde{\psi}_0 = [0,2]$, and $\psi_0(2) = \tilde{\psi}_0(2) = 0$.

(iii) $X_j = \{(\eta_i^j, w_i^j)\}_{i=1,\ldots,N_j}$, $j = 0, 1, \ldots$, is a sequence of systems such that the integration formula defined by X_j is exact of order $2^{j+2} - 2$. (It follows then from Theorem 2.4 that $\{\eta_i^j\}_{i=1,\ldots,N_j}$ is admissible in $\text{Harm}_{0,\ldots,2^{j+1}-1}$.)

What we are going to show in this chapter is the following. Starting from a $J \in \mathbb{N}$, there exist vectors $a^j \in \mathbb{R}^{N_j}$, $j = J, J-1, \ldots, 0$ (of course dependent on a given function $F \in \mathcal{L}^2(\Omega)$), such that

(i) $\text{WT}(F)(j;\ ) = \sum_{i=1}^{N_j} a_i^j \Psi_j(\eta_i^j \cdot\ )$, $j = J,\ J-1, \ldots, 0$.

(ii) The vector a^j is obtainable from a^{j+1}.

(iii) The vectors a^j satisfy in addition

$$\Phi_{j+1} * F = \sum_{i=1}^{N_j} a_i^j \Phi_{j+1}(\eta_i^j \cdot\ ),\ j = 0, \ldots, J.$$

Hence, we end up with the following pyramid scheme for the decomposition of an $\mathcal{L}^2(\Omega)$–signal F:

$$\begin{tikzcd}
F \arrow[r] & a^J \arrow[r] \arrow[d] & a^{J-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & a^0 \arrow[d] \\
 & \text{WT}(F)(J;\ ) & \text{WT}(F)(J-1;\ ) & & \text{WT}(F)(0;\ )
\end{tikzcd}$$

The reconstruction from the wavelet coefficients $\text{WT}(F)(j;\eta_i^j)$ can then be performed as described before:

$$R_j(F) = \tilde{\Psi}_j * \text{WT}(F)(j;\ ) = \sum_{i=1}^{N_j} w_i^j \text{WT}(F)(j;\eta_i^j)\tilde{\Psi}_j(\eta_i^j \cdot\ ). \tag{4.1}$$

That means that the reconstruction of the signal F can be written as

$$
\begin{tikzcd}
\mathrm{WT}(F)(0;\eta_i^0) \arrow[d] & & \mathrm{WT}(F)(1;\eta_i^1) \arrow[d] & & \mathrm{WT}(F)(2;\eta_i^2) \arrow[d] & & \\
R_0(F) \arrow[dr] & & R_1(F) \arrow[dr] & & R_2(F) \arrow[dr] & & \\
P_0(F) \arrow[r] & + \arrow[r] & P_1(F) \arrow[r] & + \arrow[r] & P_2(F) \arrow[r] & + \arrow[r] & \cdots
\end{tikzcd}
$$

We have seen that the wavelet transform $\mathrm{WT}(F)(j;\ )$ is given by the numbers $a_1^j, \ldots, a_{N_j}^j$. Therefore it is also possible to reconstruct the signal only by the use of the numbers a_i^j, rather than using the wavelet coefficients of F. Since $\{(\eta_i^j, w_i^j)\}_{i=1,\ldots,N_j}$ defines an integration formula that is exact of order $2^{j+2}-2$, we can rewrite (4.1) as

$$
\begin{aligned}
R_j(F) &= \sum_{i=1}^{N_j} w_i^j \mathrm{WT}(F)(j;\eta_i^j)\tilde{\Psi}_j(\eta_i^j\cdot\ ) \\
&= \sum_{i=1}^{N_j} w_i^j \sum_{k=1}^{N_j} a_k^j \Psi_j(\eta_k^j\cdot\eta_i^j)\tilde{\Psi}_j(\eta_i^j\cdot\ ) \\
&= \sum_{k=1}^{N_j} a_k^j \sum_{i=1}^{N_j} w_i^j \Psi_j(\eta_k^j\cdot\eta_i^j)\tilde{\Psi}_j(\eta_i^j\cdot\ ) \\
&= \sum_{k=1}^{N_j} a_k^j \int_\Omega \Psi_j(\eta_k^j\cdot\eta)\tilde{\Psi}_j(\eta\cdot\ )d\omega(\eta) \\
&= \sum_{k=1}^{N_j} a_k^j (\Psi_j * \tilde{\Psi}_j)(\eta_k^j\cdot\ ).
\end{aligned}
$$

The decomposition and reconstruction can thus be simplified to

$$
F \longrightarrow a^J \longrightarrow a^{J-1} \longrightarrow \cdots \longrightarrow a^0
$$

and

$$
\begin{tikzcd}
a^0 \arrow[d] & & a^1 \arrow[d] & & a^2 \arrow[d] & & \\
R_0(F) \arrow[dr] & & R_1(F) \arrow[dr] & & R_2(F) \arrow[dr] & & \\
P_0(F) \arrow[r] & + \arrow[r] & P_1(F) \arrow[r] & + \arrow[r] & P_2(F) \arrow[r] & + \arrow[r] & \cdots
\end{tikzcd}
$$

In this case the reconstruction of the signal is not performed with $\tilde{\Psi}_j$ but with $\Psi_j * \tilde{\Psi}_j$. It turns out later on that the vectors a^j do *not* depend on the special choice of φ_0, ψ_0 or $\tilde{\psi}_0$. This means that with the second variant one can reconstruct the signal with respect to different wavelets just by the knowledge of the vectors a^j.

4.2 The Evaluation of the First Step

The pyramid scheme starts from the vector a^J, where the a_i^J have to satisfy the equation

$$\Phi_{J+1} * F = \sum_{i=1}^{N_J} a_i^J \Phi_{J+1}(\eta_i^J \cdot\ ). \tag{4.2}$$

Theorem 4.1 *Let $F \in \mathcal{L}^2(\Omega)$, and $J \in \mathbb{N}_0$. Then there exists a vector $a^J \in \mathbb{R}^{N_J}$ such that (4.2) is fulfilled.*

Proof. From our assumptions we know that $\Phi_{J+1}^{\wedge}(n) \neq 0$ for $n = 0, \ldots, 2^{J+1} - 1$ and $\Phi_{J+1}^{\wedge}(n) = 0$ for $n \geq 2^{J+1}$. Thus, the assertion follows immediately from Theorem 2.5, since $\Phi_{J+1} * F \in \mathrm{Harm}_{0,\ldots,2^{J+1}-1}$ and $\{\eta_i^J\}_{i=1,\ldots,N_J}$ is admissible in $\mathrm{Harm}_{0,\ldots,2^{J+1}-1}$. □

Nect we turn to the question, how the a_i^J can be determined. We present two ways:

If F is band–limited such that

$$F^{\wedge}(n, m) = 0 \quad \text{for all} \quad n \geq 2^{J+1},\ m = 1, \ldots, 2n+1, \tag{4.3}$$

then it is easy to find a possible choice of a_i^J, since we have $\Phi_{J+1}(\ \cdot\xi), F \in \mathrm{Harm}_{0,\ldots,2^{J+1}-1}$, so that by Theorem 2.4

$$\Phi_{J+1} * F = \int_{\Omega} F(\eta)\Phi_{J+1}(\eta\cdot\ )d\omega(\eta) = \sum_{i=1}^{N_J} w_i^J F(\eta_i^J)\Phi_{J+1}(\eta_i^J \cdot\ ).$$

In other words, we have

$$a_i^J = w_i^J F(\eta_i^J) \quad \text{for} \quad i = 1, \ldots, N_J. \tag{4.4}$$

If F does not satisfy condition (4.3), then the numbers a_i^J can still be obtained since (4.2) is satisfied if and only if

$$(\Phi_{J+1} * F)(\eta_k^J) = \sum_{i=1}^{N_J} a_i^J \Phi_{J+1}(\eta_i^J \cdot \eta_k^J),\ k = 1, \ldots, N_J,$$

what follows from the fact that $\{\eta_i^J\}_{i=1,\ldots,N_J}$ is admissible in $\mathrm{Harm}_{0,\ldots,2^{J+1}-1}$. Hence, all possible choices of a_i^J are the solutions of the linear system

$$\begin{pmatrix} \Phi_{J+1}(\eta_1^J \cdot \eta_1^J) & \cdots & \Phi_{J+1}(\eta_{N_J}^J \cdot \eta_1^J) \\ \vdots & \ddots & \vdots \\ \Phi_{J+1}(\eta_1^J \cdot \eta_{N_J}^J) & \cdots & \Phi_{J+1}(\eta_{N_J}^J \cdot \eta_{N_J}^J) \end{pmatrix} \begin{pmatrix} a_1^J \\ \vdots \\ a_{N_J}^J \end{pmatrix} = \begin{pmatrix} (\Phi_{J+1} * F)(\eta_1^J) \\ \vdots \\ (\Phi_{J+1} * F)(\eta_{N_J}^J) \end{pmatrix}.$$

Notice that the right hand side of this equation is given by convolutions of the scaling function and F, so that there is still the problem of numerical integration. Of course, in many applications it is enough to satisfy (4.2) only approximately, so that one can often avoid the solution of a linear system of equations in the first step. A sufficiently exact integration formula (as it was in principle done for the band-limited case) is often adequate for the first step.

Next we want to show, that if the vector a^J satisfies (4.2) then the wavelet transform $\mathrm{WT}(F)(J;\ )$ is given by

$$\mathrm{WT}(F)(J;\ ) = \Psi_J * F = \sum_{i=1}^{N_J} a_i^J \Psi_J(\eta_i^J \cdot\ ). \tag{4.5}$$

But this is a consequence of

Theorem 4.2 *Let $F \in \mathcal{L}^2(\Omega)$, $K \in \mathcal{L}^2[-1,1]$ with $K^\wedge(n) \neq 0$ for $n = 0,\ldots,N$, and $K^\wedge(n) = 0$ for $n \geq N+1$. Assume further that $\eta_1,\ldots,\eta_M \in \Omega$ and that*

$$K * F = \sum_{i=1}^{M} a_i K(\eta_i \cdot\ ). \tag{4.6}$$

If $H \in \mathcal{L}^2[-1,1]$ is another kernel with $H^\wedge(n) = 0$ for all $n \geq N+1$, then

$$H * F = \sum_{i=1}^{M} a_i H(\eta_i \cdot\ ).$$

Proof. For all $n = 0,\ldots,N$, $m = 1,\ldots,2n+1$, we get from (4.6) by multiplication with $Y_{n,m}$ and integration over Ω that

$$\int_\Omega (K * F)(\eta) Y_{n,m}(\eta) d\omega(\eta) = \sum_{i=1}^{M} a_i \int_\Omega K(\eta_i \cdot \eta) Y_{n,m}(\eta) d\omega(\eta),$$

so that by the Funk–Hecke formula

$$K^\wedge(n) F^\wedge(n,m) = K^\wedge(n) \sum_{i=1}^{M} a_i Y_{n,m}(\eta_i).$$

Since $K^\wedge(n) \neq 0$ it follows that

$$F^\wedge(n,m) = \sum_{i=1}^{M} a_i Y_{n,m}(\eta_i).$$

Therefore we get for $\xi \in \Omega$

$$\begin{aligned}
(H * F)(\xi) &= \sum_{n=0}^{N} \sum_{m=1}^{2n+1} H^\wedge(n) F^\wedge(n,m) Y_{n,m}(\xi) \\
&= \sum_{n=0}^{N} \sum_{m=1}^{2n+1} H^\wedge(n) \sum_{i=1}^{M} a_i Y_{n,m}(\eta_i) Y_{n,m}(\xi) \\
&= \sum_{i=1}^{M} a_i \sum_{n=0}^{N} \frac{2n+1}{4\pi} H^\wedge(n) P_n(\eta_i \cdot \xi) \\
&= \sum_{i=1}^{M} a_i H(\eta_i \cdot \xi).
\end{aligned}$$

as required. $\square$

4.3 The Pyramid Step

Starting from $a^J \in \mathbb{R}^{N_J}$ with

$$\Phi_{J+1} * F = \sum_{i=1}^{N_J} a_i^J \Phi_{J+1}(\eta_i^J \cdot\ ),$$

we shall show now, how $a^{J-1} \in \mathbb{R}^{N_{J-1}}$ can be constructed so that

$$\Phi_J * F = \sum_{i=1}^{N_{J-1}} a_i^{J-1} \Phi_J(\eta_i^{J-1} \cdot\ ). \tag{4.7}$$

Since $\Phi_J * F \in \mathrm{Harm}_{0,\ldots,2^J-1}$ and $\{\eta_1^{J-1}, \ldots, \eta_{N_{J-1}}^{J-1}\} \subset \Omega$ is admissible in $\mathrm{Harm}_{0,\ldots,2^J-1}$, the existence of a_i^{J-1} is clear. Furthermore, we know from Theorem 4.2 that

$$\Phi_J * F = \sum_{i=1}^{N_J} a_i^J \Phi_J(\eta_i^J \cdot\ ),$$

so that we finally end up with the equation

$$\sum_{i=1}^{N_{J-1}} a_i^{J-1} \Phi_J(\eta_i^{J-1} \cdot\ ) = \sum_{i=1}^{N_J} a_i^J \Phi_J(\eta_i^J \cdot\ ). \tag{4.8}$$

Theorem 4.3 *A solution $a^{J-1} \in \mathbb{R}^{N_{J-1}}$ of equation (4.8) is given by*

$$a_k^{J-1} = w_k^{J-1} \sum_{i=1}^{N_J} a_i^J \mathrm{SH}_{2^J-1}(\eta_i^J \cdot \eta_k^{J-1}), \tag{4.9}$$

where the Shannon kernel SH_{2^J-1} is defined by (cf. (2.2))

$$SH_{2^J-1} = \sum_{n=0}^{2^J-1} \frac{2n+1}{4\pi} P_n.$$

If the integration weights w_i^J, w_i^{J-1} are all positive, then the solution (4.9) is characterized to minimize the discrete norm

$$\sqrt{\sum_{i=1}^{N_{J-1}} \frac{1}{w_i^{J-1}} \left(a_i^{J-1}\right)^2} \tag{4.10}$$

under all solutions of (4.8).

Proof. On $\mathbb{R}^{N_J}$ and $\mathbb{R}^{N_{J-1}}$, respectively, we introduce

$$(\alpha,\beta)_{N_j} = \sum_{i=1}^{N_j} w_i^j \alpha_i \beta_i, \quad \alpha,\ \beta \in \mathbb{R}^{N_j}, \quad j = J-1, J, \tag{4.11}$$

which defines an inner product in $\mathbb{R}^{N_j}$, if all the weights are positive. We let $x^{J-1,n,m} \in \mathbb{R}^{N_{J-1}}$ and $x^{J,n,m} \in \mathbb{R}^{N_J}$ be given by

$$x_i^{j,n,m} = Y_{n,m}(\eta_i^j),\ i = 1,\ldots,N_j,\ n = 0,\ldots,2^J-1,\ m = 1,\ldots,2n+1,$$

for $j = J-1,\ J$. Then we see that for $n\prime = 0,\ldots,2^J-1$, $m\prime = 1,\ldots,2n\prime+1$,

$$\begin{aligned}(x^{J,n,m}, x^{J,n\prime,m\prime})_{N_J} &= \sum_{i=1}^{N_J} w_i^J x_i^{J,n,m} x_i^{J,n\prime,m\prime} \\ &= \sum_{i=1}^{N_J} w_i^J Y_{n,m}(\eta_i^J) Y_{n\prime,m\prime}(\eta_i^J) \\ &= \int_\Omega Y_{n,m}(\eta) Y_{n\prime,m\prime}(\eta)\, d\omega(\eta) \\ &= \delta_{n,n\prime}\delta_{m,m\prime}.\end{aligned}$$

Similarly, it follows that

$$(x^{J-1,n,m}, x^{J-1,n\prime,m\prime})_{N_{J-1}} = \delta_{n,n\prime}\delta_{m,m\prime}.$$

The sets

$$\operatorname{span}\{x^{j,n,m} \mid n = 0,\ldots,2^J-1,\ m = 1,\ldots,2n+1\},\ j = J-1, J, \tag{4.12}$$

are subspaces of $\mathbb{R}^{N_J}$ and $\mathbb{R}^{N_{J-1}}$, respectively, which are isomorphic via the canonical basis change $x^{J-1,n,m} \mapsto x^{J,n,m}$.

Now, equation (4.8) can be transformed as follows:

$$\sum_{i=1}^{N_{J-1}} a_i^{J-1} \Phi_J(\eta_i^{J-1} \cdot\ ) = \sum_{i=1}^{N_J} a_i^J \Phi_J(\eta_i^J \cdot\ )$$

is equivalent to

$$\int_\Omega \sum_{i=1}^{N_{J-1}} a_i^{J-1} \Phi_J(\eta_i^{J-1} \cdot \eta) Y_{n,m}(\eta)\, d\omega(\eta) = \int_\Omega \sum_{i=1}^{N_J} a_i^J \Phi_J(\eta_i^J \cdot \eta) Y_{n,m}(\eta)\, d\omega(\eta)$$

for all $n = 0,\ldots,2^J-1$, $m = 1,\ldots,2n+1$. Hence, using the Funk–Hecke formula, we can write

$$\Phi_J^\wedge(n) \sum_{i=1}^{N_{J-1}} a_i^{J-1} Y_{n,m}(\eta_i^{J-1}) = \Phi_J^\wedge(n) \sum_{i=1}^{N_J} a_i^J Y_{n,m}(\eta_i^J)$$

for all $n = 0,\ldots,2^J-1$, $m = 1,\ldots,2n+1$. Since $\Phi_J^\wedge(n) \neq 0$ for all $n = 0,\ldots,2^J-1$, we may also write

$$\sum_{i=1}^{N_{J-1}} a_i^{J-1} Y_{n,m}(\eta_i^{J-1}) = \sum_{i=1}^{N_J} a_i^J Y_{n,m}(\eta_i^J),\ n = 0,\ldots,2^J-1,\ m = 1,\ldots,2n+1.$$

which shows, that there is no dependence on the scaling function anymore. If we introduce (we may without loss of generality assume that all the weights are different from zero)

$$\tilde{a}_i^{J-1} = \frac{1}{w_i^{J-1}} a_i^{J-1}, \quad \tilde{a}_i^J = \frac{1}{w_i^J} a_i^J,$$

we end up with the equations

$$(\tilde{a}^{J-1}, x^{J-1,n,m})_{N_{J-1}} = (\tilde{a}^J, x^{J,n,m})_{N_J}, \tag{4.13}$$

for $n = 0, \dots, 2^J - 1$, $m = 1, \dots, 2n+1$. A solution of (4.13) is given by

$$\tilde{a}^{J-1} = \sum_{n=0}^{2^J-1} \sum_{m=1}^{2n+1} (\tilde{a}^J, x^{J,n,m})_{N_J} x^{J-1,n,m},$$

since for $n\prime = 0, \dots, 2^J - 1$, $m\prime = 1, \dots, 2n\prime + 1$ it holds

$$\begin{aligned}
(\tilde{a}^{J-1}, x^{J-1,n\prime,m\prime})_{N_{J-1}} &= \sum_{n=0}^{2^J-1} \sum_{m=1}^{2n+1} (\tilde{a}^J, x^{J,n,m})_{N_J} (x^{J-1,n,m}, x^{J-1,n\prime,m\prime})_{N_{J-1}} \\
&= (\tilde{a}^J, x^{J,n\prime,m\prime})_{N_J}.
\end{aligned}$$

Therefore we obtain after simple manipulations

$$\begin{aligned}
a_k^{J-1} &= w_k^{J-1} \tilde{a}_k^{J-1} \\
&= w_k^{J-1} \sum_{n=0}^{2^J-1} \sum_{m=1}^{2n+1} (\tilde{a}^J, x^{J,n,m})_{N_J} x_k^{J-1,n,m} \\
&= w_k^{J-1} \sum_{n=0}^{2^J-1} \sum_{m=1}^{2n+1} \sum_{i=1}^{N_J} w_i^J \tilde{a}_i^J Y_{n,m}(\eta_i^J) Y_{n,m}(\eta_k^{J-1}) \\
&= w_k^{J-1} \sum_{i=1}^{N_J} a_i^J \sum_{n=0}^{2^J-1} \sum_{m=1}^{2n+1} Y_{n,m}(\eta_i^J) Y_{n,m}(\eta_k^{J-1}) \\
&= w_k^{J-1} \sum_{i=1}^{N_J} a_i^J \mathrm{SH}_{2^J-1}(\eta_i^J \cdot \eta_k^{J-1}),
\end{aligned}$$

which proves the first part of the theorem.

If all the weights w_i^{J-1} and w_i^J are positive, then the bilinear forms (4.11) are in fact positive definite and the $x^{j,n,m}$, $j = J-1, J$, form orthonormal bases of the subspaces (4.12), so that we know from standard arguments in linear algebra, that a^{J-1} minimizes the norm (4.10). □

This theorem shows, how a^{J-1} can be computed from a^J, so that (4.7) is fulfilled. An application of Theorem 4.2 again gives us

$$\mathrm{WT}(F)(J-1;\ ) = \sum_{i=1}^{N_{J-1}} a_i^{J-1} \Psi_{J-1}(\eta_i^{J-1} \cdot\ ).$$

Of course, this scheme (Theorem 4.3 and Theorem 4.2) can be applied recursively to get all the $a^J, \dots, a^0$ and hence $\mathrm{WT}(F)(J;\ ), \dots, \mathrm{WT}(F)(0;\ )$.

4.4 The Complete Algorithms

Now, we summarize our results and give a complete description of the algorithms for decomposition and reconstruction. We assume therefore that $F \in \mathcal{L}^2(\Omega)$ is band–limited and that J is chosen in such a way that $F^\wedge(n,m) = 0$ for all $n \geq 2^{J+1}$, $m = 1, \ldots, 2n+1$. (If F is not band–limited, the initial step can be performed as described before.) Furthermore we suppose that the requirements (i)–(iii) stated at the beginning of Section 4.1 are satisfied.

Decomposition:

Initial Step:

for $k = 1$ to N_J do
  $a_k^J = w_k^J F(\eta_k^J)$
enddo
for $k = 1$ to N_J do
  $\mathrm{WT}(F)(J; \eta_k^J) = \sum_{i=1}^{N_J} a_i^J \Psi_J(\eta_k^J \cdot \eta_i^J)$
enddo

Pyramid Step:

for $j = J-1$ downto 0 do
  for $k = 1$ to N_j do
    $a_k^j = w_k^j \sum_{i=1}^{N_{j+1}} a_i^{j+1} \mathrm{SH}_{2^{j+1}-1}(\eta_k^j \cdot \eta_i^{j+1})$
  enddo
  for $k = 1$ to N_j do
    $\mathrm{WT}(F)(j; \eta_k^j) = \sum_{i=1}^{N_j} a_i^j \Psi_j(\eta_k^j \cdot \eta_i^j)$
  enddo
enddo

As already mentioned, there are two possibilities for the reconstruction. It is possible to reconstruct the signal from the a_i^j (in which case the calculation of $\mathrm{WT}(F)(j;\ )$ is not necessary) or from the $\mathrm{WT}(F)(j; \eta_i^j)$.

Reconstruction with the a_i^j:

$P_0(f) = \frac{1}{\sqrt{4\pi}} F^\wedge(0,1)$
for $j = 0$ to J do
  $R_j(F) = \sum_{i=1}^{N_j} a_i^j (\tilde{\Psi}_j * \Psi_j)(\eta_i^j \cdot\ )$
  $P_{j+1}(F) = P_j(F) + R_j(F)$
enddo

Or, alternatively,

Reconstruction with $\mathrm{WT}(F)(j;\eta_i^j)$**:**

$P_0(f) = \frac{1}{\sqrt{4\pi}} F^\wedge(0,1)$
for $j = 0$ **to** J **do**
  $R_j(F) = \sum_{i=1}^{N_j} w_i^j \mathrm{WT}(F)(j;\eta_i^j)\tilde{\Psi}_j(\eta_i^j\cdot\ )$
  $P_{j+1}(F) = P_j(F) + R_j(F)$
enddo

5 Numercal Example and Conclusions

As a numerical example we present the wavelet decomposition and reconstruction of the gravitational potential of the earth. Our calculations are based on the OSU–model (cf. [13]) which gives a series expansion of the earth's gravitational potential in terms of spherical harmonics. We have used the data up to a polynomial degree of 180. The systems X_j are chosen as described in [3]. The applied P–wavelets are based on the generator $\varphi_0(x) = (1-x^2)(1-2x)$ (example (vii) of Section 3.3). The pictures below show the reconstruction of the signal F for the levels 7, 5, and 3. The degrees of longitude range from 170° west up to 190° east.

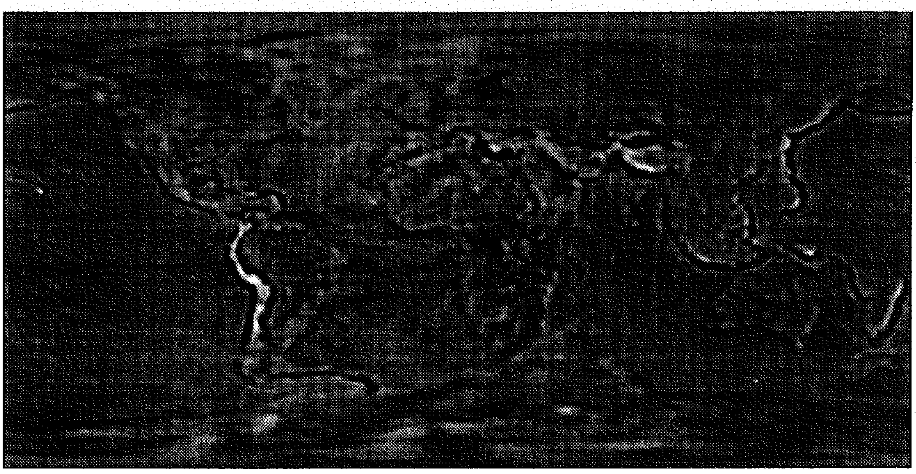

Figure 5.1: $R_7(F)$.

Let us finally make some concluding remarks.

- First of all it is obvious that all what was done before can also be done in the higher dimensional case.

- For the question what type of wavelet one should use, a first answer is, that the oscillations of the Shannon and the de la Vallé Poussin wavelet cause oscillations in the reconstructions $R_j(F)$. If the wavelets are built

Figure 5.2: $R_5(F)$.

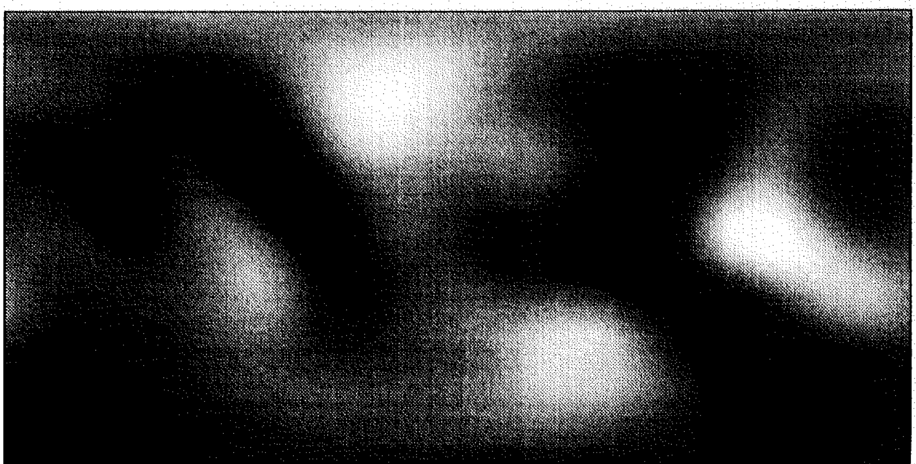

Figure 5.3: $R_3(F)$.

according to $\varphi_0(x) = (1 - x^2)(1 - 2x)$ (example (vii) of Section 3.3), oscillations were suppressed.

- The numerical effort of one pyramid step $j + 1 \to j$ is of order $N_j N_{j+1}$. This effort can be drastically reduced by two methods. The first possibility is to use gridded pointsystems and then to apply FFT–methods. To make this statement more concrete we introduce spherical coordinates $[0, \pi] \times [0, 2\pi) \to \mathbb{R}^3$ by (ϑ: co–latitude, λ: longitude)

$$(\vartheta, \lambda) \mapsto \begin{pmatrix} \sin\vartheta\cos\lambda \\ \sin\vartheta\sin\lambda \\ \cos\vartheta \end{pmatrix}.$$

We assume that for each j the system $X_j = \{(\eta^j_{i,k}, w^j_{i,k})\}$ is given by $\eta^j_{i,k} = (\vartheta^j_i, \lambda^j_k)$, $i = 1, \ldots, T_j$, $k = 1, \ldots, L_j$, where $0 \leq \vartheta^j_1 < \ldots < \vartheta^j_{T_j} \leq$

π, $\lambda_k^j = 2\pi(k-1)/L_j$, and L_j is a power of 2. We suppose further that the corresponding integration weights $w_{i,k}^j$ depend only on the co–latitude, i.e. $w_{i,k}^j = w_i^j$. The pyramid step can then be written as

$$a_{i,k}^j = w_i^j \sum_{i'=1}^{T_{j+1}} \sum_{k'=1}^{L_{j+1}} a_{i',k'}^{j+1} \mathrm{SH}_{2^{j+1}-1}(\eta_{i,k}^j \cdot \eta_{i',k'}^{j+1}),$$

where $i = 1, \ldots, T_j$, $k = 1, \ldots, L_j$. If we introduce the vectors

$$\begin{aligned} a_i^j &= (a_{i,1}^j, \ldots, a_{i,L_j}^j), \\ S_{i,i'}^{j+1\to j} &= (\mathrm{SH}_{2^{j+1}-1}(\eta_{i,1}^j \cdot \eta_{i',1}^{j+1}), \ldots, \mathrm{SH}_{2^{j+1}-1}(\eta_{i,1}^j \cdot \eta_{i',L_{j+1}}^{j+1})), \end{aligned}$$

it is obvious that the pyramid step can be written as

$$a_i^j = w_i^j \sum_{i'=1}^{T_{j+1}} a_{i'}^{j+1} * S_{i,i'}^{j+1\to j},$$

where $*$ means the cyclic discrete convolution and $i = 1, \ldots, T_j$. Thus applying a FFT method for the discrete convolution the numerical effort can be reduced.

In [3] there is described a pointsystem with corresponding integration weights with an equiangular distribution of the ϑ_i. Using a Gauß quadrature rule in north–south direction, the numbers T_j can be reduced, cf., e.g., [15].

A complete different idea for making the pyramid step more efficient is the use of a panel–clustering method. One takes advantage of the localizing structure of the kernel $\mathrm{SH}_{2^{j+1}-1}$. The kernel is splitted into a near field and a far field component. The far field component is then approximated by Legendre polynomials up to a given low degree. For the evaluation one uses for points near at the evaluation position the exact near field of the corresponding kernel. For the remaining points, the approximated far fields are glued together, what is via the addition theorem no problem. A numerical realization of this idea is under development.

- Two possible applications of the described wavelet decompostion and reconstruction should be mentioned: we are interested in data compression, particularly for representations of the gravitational field of the earth. Another application is, that for the evaluation of the signal at a certain point, only those wavelet coefficients that are near to the point under consideration have to be taken into account, which gives the possibility for the combination of global and local models of the earth's gravitational field.

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