TECHNISCHE UNIVERSITÄT
KAISERSLAUTERN

FACHBEREICH MATHEMATIK

AG Funktionalanalysis und Stochastische Analysis

Operator Semigroups and Infinite Dimensional Analysis Applied to Problems from Mathematical Physics

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Tag der Disputation: 02.10.2020

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.)
genehmigte Dissertation

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Abstract

In this dissertation we treat several problems from mathematical physics via methods from functional analysis and probability theory and in particular operator semigroups. The thesis consists thematically of two parts.

In the first part we consider so-called generalized stochastic Hamiltonian systems. These are generalizations of Langevin dynamics which describe interacting particles moving in a surrounding medium. From a mathematical point of view these systems are stochastic differential equations with a degenerated diffusion coefficient. We construct weak solutions of these equations via the corresponding martingale problem. Therefore, we prove essential m-dissipativity of the degenerated and non-sectorial Itô differential operator. Further, we apply results from the analytic and probabilistic potential theory to obtain an associated Markov process. Afterwards we show our main result, the convergence in law of the positions of the particles in the overdamped regime, the so-called overdamped limit, to a distorted Brownian motion. To this end, we show convergence of the associated operator semigroups in the framework of Kuwae-Shioya. Further, we established a tightness result for the approximations which proves together with the convergence of the semigroups weak convergence of the laws.

In the second part we deal with problems from infinite dimensional Analysis. Three different issues are considered. The first one is an improvement of a characterization theorem of the so-called regular test functions and distribution of White noise analysis. As an application we analyze a stochastic transport equation in terms of regularity of its solution in the space of regular distributions. The last two problems are from the field of relativistic quantum field theory. In the first one the $(\Phi)^4_3$-model of quantum field theory is under consideration. We show that the Schwinger functions of this model have a representation as the moments of a positive Hida distribution from White noise analysis. In the last chapter we construct a non-trivial relativistic quantum field in arbitrary space-time dimension. The field is given via Schwinger functions. For these which we establish all axioms of Osterwalder and Schrader. This yields via the reconstruction theorem of Osterwalder and Schrader a unique relativistic quantum field. The Schwinger functions are given as the moments of a non-Gaussian measure on the space of tempered distributions. We obtain the measure as a superposition of Gaussian measures. In particular, this measure is itself non-Gaussian, which implies that the field under consideration is not a generalized free field.

Danksagung

An dieser Stelle möchte ich mich bei allen bedanken, die mich in den letzten drei Jahren bei meiner Promotion unterstützt haben.

Zuallererst gilt mein Dank meinem Betreuer Prof. Dr. Martin Grothaus, der mich bereits bei meiner Masterarbeit betreut und mir im Anschluss die Möglichkeit einer Promotion eröffnet hat. Danke für deine stete Unterstützung, zahlreichen Ratschläge und hilfreichen Diskussionen ohne die diese Dissertation nie zu Stande gekommen wäre.

Im selben Zug möchte ich Prof. Dr. Sergio Albeverio danken, der sich bereit erklärt hat diese Dissertation zu begutachten.

Ich möchte mich auch bei allen jetzigen sowie ehemaligen Mitgliedern aus der AG Funktionalanalysis und Stochastische Analysis und der AG Finanzmathematik für die freundliche und kollegiale Atmosphäre bedanken. Besonderer Dank gilt dabei Dr. Torben Fattler, Dr. Robert Voßhall und Dr. Wolfgang Bock für ihr Interesse an meiner Arbeit und ihre Unterstützung seit Beginn der Promotion. Unter allen Kollegen möchte ich mich ganz besonders bei meinen Freunden Dr. Sema Coşkun, Dr. Jean-Marie Lantau und Simon Wittmann bedanken. Neben euren unzähligen fachlichen Ratschlägen möchte ich mich vor allem für eure unterstützende Art, euer stets offenes Ohr und die nötige Ablenkung von der Mathematik bedanken.

Desweiteren möchte ich mich bei meinem Bruder Joachim und bei meinen Eltern bedanken ohne deren moralische Unterstützung diese Arbeit nie zu Stande gekommen wäre. Schließlich danke ich meinen Freunden für die Ermutigungen und guten Zusprüche während der Erstellung dieser Dissertation.

An letzter Stelle möchte ich mich beim Fachbereich Mathematik der TU Kaiserslautern für die finanzielle Unterstützung bedanken.

Eigene Veröffentlichungen im Zusammenhang mit dieser Arbeit

- M. Grothaus, A. Nonnenmacher. Overdamped limit of generalized stochastic Hamiltonian systems for singular interaction potentials. J. Evol. Equ. (2019).
- M. Grothaus, A. Nonnenmacher. Construction of non-trivial relativistic quantum fields in arbitrary space-time dimension via superposition of free fields. Submitted for publication to Communications in Mathematical Physics.

Contents

Chapter 0

Introduction

In this thesis we are concerned with different problems from mathematical physics. For a rigorous treatment of these problems we mainly use concepts from functional analysis and probability and measures theory. In particular, we apply the concept of operator semigroups. The problems under consideration are mutually disjoint, therefore, we describe them in the separately. In the following, we present details regarding these problems.

Overdamped Limit of Generalized Hamiltonian Systems (Part I)

In this part we consider a model for the motion of a finite number of identical particles. Let us consider a fixed number $N \in \mathbb{N}$ of particles moving in $\mathbb{R}^{\hat{d}}$, $\hat{d} \in \mathbb{N}$. We collect all positions of the particles in a vector $X \in \mathbb{R}^d$, $d = N\hat{d}$, and the velocities of the particles in a vector $V \in \mathbb{R}^d$. The motion of the particles can be modeled by Newton's Law through the differential equation $m\dot{V} = F$, where F is a force accelerating the particles and m is the mass of the particles. For the sake of simplicity, we assume that the mass m equals to one. We are interested in particles, which move in a surrounding medium. We assume that the molecules of the medium are much lighter than the particles. Therefore, the force resulting from the collisions of the particles and the molecules can be reasonably described by a friction force, proportional to the velocity of the particles and a stochastic force, see e.g. [96, Chapter 8.1]. Further the particles are exposed to a force which results from an existing potential Φ_1. The resulting forces $\nabla\Phi_1$ can be of external nature, as well as interacting forces between the particles. Thus, Newton's law becomes the Langevin equation

$$dX_t = V_t dt, \tag{0.1}$$

$$dV_t = -\nabla\Phi_1(X_t)dt - \gamma V_t dt + \sqrt{2\gamma\beta^{-1}}dB_t. \tag{0.2}$$

Here, $\nabla\Phi_1$ stands for the external and interacting forces acting on the particles arising from the potential Φ_1, $\gamma > 0$ is a constant describing the magnitude of friction, $\beta > 0$ is, up to a constant, the inverse temperature and $(B_t)_{t\geq 0}$ denotes a d-dimensional Brownian motion describing the stochastic force. In this part of the thesis, we are interested in the scaled equation, i.e.,

$$dX_t^\varepsilon = \frac{1}{\varepsilon}V_t^\varepsilon dt, \tag{0.3}$$

$$dV_t^\varepsilon = -\frac{1}{\varepsilon}\nabla\Phi_1(X_t^\varepsilon)dt - \frac{1}{\varepsilon^2}V_t^\varepsilon dt + \frac{1}{\varepsilon}\sqrt{2}dB_t, \tag{0.4}$$

compare for example the derivation of Pavliotis [83, Chapter 6.5.1] and [68, Chapter 2.2.4]. Small $\varepsilon > 0$ represents the *overdamped regime*, which physically corresponds to large friction forces and an appropriate time-scaling, see also the last mentioned reference. In the latter reference one can also find formal derivations, which indicate the convergence of $(X_t^\varepsilon)_{t\geq 0}$ to a solution of the so-called *overdamped Langevin* equation

$$dX_t^0 = -\nabla\Phi_1(X_t^0)dt + \sqrt{2}dB_t, \tag{0.5}$$

as ε tends to zero. Depending on the context a solution to (0.5) is also called a distorted Brownian motion. This convergence is known as the *overdamped limit*. More generally, we treat a scaled *generalized stochastic Hamiltonian systems* (gsHs) given by

$$dX_t^\varepsilon = \frac{1}{\varepsilon}\nabla\Phi_2(V_t^\varepsilon)dt, \tag{0.6}$$

$$dV_t^\varepsilon = -\frac{1}{\varepsilon}\nabla\Phi_1(X_t^\varepsilon)dt - \frac{1}{\varepsilon^2}\nabla\Phi_2(V_t^\varepsilon)dt + \frac{1}{\varepsilon}\sqrt{2}dB_t. \tag{0.7}$$

Here Φ_2 is a potential, generalizing the kinetic energy of the particles, i.e., the Hamiltonian is given by $\mathrm{H}_\Phi(x,v) = \Phi_1(x) + \Phi_2(v)$, $x, v \in \mathbb{R}^d$. Observe that for $\Phi_2(v) = \frac{1}{2}|v|^2$ we just recover (0.3), (0.4).

The main result of this part of the thesis can be stated as follows. We prove the existence of weak solutions $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$, $\varepsilon > 0$, of (0.6), (0.7) with initial distribution given by $h\mu_\Phi$. Here, μ_Φ is the measure which is absolutely continuous w.r.t. the Lebesgue measure $d(x,v)$ on $\mathbb{R}^{2d}$ with density $e^{-\Phi_1(x)-\Phi_2(v)}$, $(x,v) \in \mathbb{R}^{2d}$, and $h \in L^1(\mathbb{R}^{2d}, \mu_\Phi) \cap L^2(\mathbb{R}^{2d}, \mu_\Phi)$. Additionally, we construct for the overdamped Langevin equation (0.5) a weak solution $(X_t^0)_{t\geq 0}$ with initial distribution given by $h_0\mu_{\Phi_1}$. Similar as above, μ_{Φ_1} is absolutely continuous w.r.t. the Lebesgue measure dx on $\mathbb{R}^d$ with density $e^{-\Phi_1(x)}$, $x \in \mathbb{R}^d$, and $h_0 \in L^1(\mathbb{R}^d, \mu_{\Phi_1}) \cap L^2(\mathbb{R}^d, \mu_{\Phi_1})$. Then, we establish that the laws $\mathcal{L}\left((X_t^\varepsilon)_{t\geq 0}\right)$, $\varepsilon > 0$, converge towards the law $\mathcal{L}\left((X_t^0)_{t\geq 0}\right)$ as ε tends to zero. The convergence takes place in the topology of weak convergence on the space of probability measures on $C([0,\infty), \mathbb{R}^d)$. The result holds for a large class of potentials Φ_1 and Φ_2. The class of admissible potentials Φ_1 allows to consider singular interaction forces $\nabla\Phi_1$. Note that, singular interaction forces are of great physical importance, such as pair interactions with a pair potential of Lennard-Jones type. Singular interactions also prevent the solutions $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$, $\varepsilon > 0$, to pass through the points in phase space where two or more particles are simultaneously at the same position, which is physically impossible.

The mild assumptions on Φ_1 and Φ_2 do not allow us to use standard existence results from the theory of stochastic differential equations. So far, solutions for a general velocity potential are not constructed yet. For this reason, we follow the approach in [26] where weak solutions of (0.1), (0.2) are constructed via the corresponding martingale problem for singular interaction forces $\nabla\Phi_1$. We show that this construction can be adapted to a general velocity potential Φ_2. To explain this in more detail, let us consider the generator L_Φ^ε of (0.6), (0.7) obtained by Itô's lemma as follows

$$L_\Phi^\varepsilon f = \frac{1}{\varepsilon^2}\left(\Delta_v f - \nabla_v\Phi_2\cdot\nabla_v f\right) + \frac{1}{\varepsilon}\left(\nabla_v\Phi_2\cdot\nabla_x f - \nabla_x\Phi_1\cdot\nabla_v f\right) \tag{0.8}$$

for $f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$. One should observe that the generator L_Φ^ε is in general non-sectorial, because of the absence of noise in the position equation (0.6). Namely, the diffusion coefficient of (0.6), (0.7) is degenerated. By arguing similar as in [26], we show for a pair $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ that the operator $\left(L_\Phi^1, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ is essentially m-dissipative on $L^1(\mathbb{R}^{2d}, \mu_\Phi)$, where μ_Φ is an invariant measure for L_Φ^1. Then, we use the existence result in [17] which provides us with an associated Markov process. From this process, we obtain a martingale solution $\mathbb{P}^1_{h\mu_\Phi}$ for $\left(L^1_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ with initial distribution $h\mu_\Phi$. As an intermediate step in the proof of our main result, we consider a scaled pair of potentials $\Phi^\varepsilon = (\Phi_1, \Phi_2^\varepsilon)$, where $\Phi_2^\varepsilon(\cdot) = \Phi_2(\frac{\cdot}{\varepsilon}) + \ln(\varepsilon^d)$. The operators $L^1_{\Phi^\varepsilon}$ and L^ε_Φ are related by a unitary transformation U_ε. The unitary transformation U_ε is given by $U_\varepsilon f = f \circ \widetilde{U}_\varepsilon$, $f \in L^2(\mathbb{R}^{2d}, \mu_\Phi)$, where $\widetilde{U}_\varepsilon(x, v) = (x, \frac{1}{\varepsilon}v)$. Therefore, we consider the operator $L^1_{\Phi^\varepsilon}$ and the corresponding martingale solution $\mathbb{P}^1_{h_\varepsilon \mu_{\Phi^\varepsilon}}$ instead. From the martingale solution $\mathbb{P}^1_{h_\varepsilon \mu_{\Phi^\varepsilon}}$ we derive a weak solution $(X_t^{1,\Phi^\varepsilon}, V_t^{1,\Phi^\varepsilon})_{t\geq 0}$ for the equation (0.6), (0.7) for $\varepsilon = 1$ and the potentials given by the pair Φ^ε with initial distribution $h_\varepsilon \mu_{\Phi^\varepsilon}$. The major challenge is to prove the weak convergence of the position marginals $\mathbb{P}^{1,X}_{h_\varepsilon \mu_{\Phi^\varepsilon}}$ of the martingale solutions $\mathbb{P}^1_{h_\varepsilon \mu_{\Phi^\varepsilon}}$ corresponding to $L^1_{\Phi^\varepsilon}$ as $\varepsilon \to 0$. This we achieve by using analytic and probabilistic methods. In the analytic part, we show convergence of the semigroups $\left(T_t^\varepsilon\right)_{t\geq 0}$ generated by an extension of $\left(L^1_{\Phi^\varepsilon}, C_c^\infty(\{\Phi_1, \Phi_2^\varepsilon < \infty\})\right)$ on $L^2(\mathbb{R}^{2d}, \mu_{\Phi^\varepsilon})$. This convergence result for the semigroups was already shown to hold true for the case $\Phi_2(v) = |v|^2$ and for a locally Lipschitz continuous Φ_1 in [75]. We show that the results obtained in the last mentioned reference extend to our more general setting. The convergence of the operator semigroups implies the convergence of the finite dimensional distributions of the position marginals $\mathbb{P}^{1,X}_{\Phi^\varepsilon}$, $\varepsilon > 0$, as ε tends to zero. To establish the weak convergence, it suffices now to prove that the family of measures $\mathbb{P}^1_{h_\varepsilon \mu_{\Phi^\varepsilon}}$, $\varepsilon > 0$, is tight. This is the probabilistic part of the proof which is not yet shown. To this end, we use that the time-reversed laws are solutions of the martingale problem of the adjoint operator. Furthermore, we choose a different metric on the state space $\mathbb{R}^{2d}$, which is better suited to show tightness in our case. Then, we basically invert the unitary transformation U_ε. This is done by applying Itô's formula to the function $\widetilde{U}_\varepsilon$. Eventually, we establish the weak convergence of $\mathcal{L}\left((X^\varepsilon)_{t\geq 0}\right)$ towards $\mathcal{L}\left((X_t^0)_{t\geq 0}\right)$, where $(X_t^0)_{t\geq 0}$ is a weak solution of (0.5).

At this point, we aim to compare our results of this part to the existing results in the literature. Several authors already proved stronger versions of convergence, see e.g. [59, Theorem 1] for convergence of the processes in L^2–norm or [37], [55, Theorem 1] for convergence in probability. Note that, Hottovy et. al [59] and Herzog et. al [55] assume continuously differentiable coefficients and Friedlin [37] still requires Lipschitz continuity for the coefficient $\nabla\Phi_1$ of the SDE (0.1), (0.2). The reader will see, that in this work, only an integrability assumption w.r.t. the measure μ_{Φ_1} on the gradient $\nabla\Phi_1$ is made to obtain the weak convergence result. The convergence result presented here applies for a large class of interaction potentials Φ_1 as well as a general velocity potential Φ_2 which is not covered by the existing results in the literature. In particular, singular interaction forces $\nabla\Phi_1$ are admissible in our framework.

The introduction of the potential Φ_2 may seem to be arbitrary from a physical point of view. The motivation for a general potential Φ_2 comes from the study of hypocoercivity. The m-dissipativity results in Section 3.1 are of great interest to prove convergence rates for non-Gaussian generalized stochastic Hamiltonian systems, see [51].

Problems from Infinite Dimensional Analysis (Part II)

In the second part we deal with several different topics from infinite dimensional analysis, which can be separated into two subtopics.

An improved Characterization of regular generalized Functions of White Noise

The first topic is from Gaussian and, in particular, from White noise analysis. These two fields of mathematics have been intensively studied in the past decades. Driven also by applications in quantum field theory, quantum mechanics, stochastic (partial) differential equations, financial mathematics and many more, a sound mathematical theory has been developed. The necessity of a Gaussian analysis arises from the lack of a Lebesgue measure on infinite dimensional spaces, i.e., a shift invariant measure on an infinite dimensional Hilbert space is trivial. To describe the framework in more detail, let us fix a real nuclear triplet

$$\mathcal{N} \subset \mathcal{H} \subseteq \mathcal{N}',$$

where $\mathcal{N}$ is a nuclear space and continuously and densely embedded into a Hilbert space $\mathcal{H}$. Further, $\mathcal{N}'$ denotes the dual space of $\mathcal{N}$. We denote the dual pairing between elements $f \in \mathcal{N}$ and $x \in \mathcal{N}'$ by $\langle f, x\rangle := x(f)$. The starting point of Gaussian analysis is the Bochner-Minlos theorem, which gives rise to a probability measure μ defined on $\mathcal{N}'$ through its characteristic function $\hat{\mu}$, given by

$$\hat{\mu}(f) = \int_{\mathcal{N}'} \exp(i\langle f, \cdot\rangle)\, d\mu = \exp\left(-\frac{1}{2}(f,f)_{\mathcal{H}}\right), \quad f \in \mathcal{N}.$$

An important step in the development of an infinite dimensional Gaussian analysis is the Wiener-Itô-Segal isomorphism, which shows that the space $L^2(\mu) := L^2(\mathcal{N}', \mu)$ is unitarily equivalent to the symmetric Fock space over $\mathcal{H}$, i.e.,

$$L^2(\mathcal{N}', \mu) \cong \Gamma(\mathcal{H}_{\mathbb{C}}) := \oplus_{n=0}^{\infty} n! \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}. \tag{0.9}$$

By using the previous decomposition, one can set up a nuclear triplet

$$(\mathcal{N}) \subseteq L^2(\mu) \subseteq (\mathcal{N})'$$

with similar properties as the triplet given above. In particular, the space $(\mathcal{N})$ serves as a test function space and $(\mathcal{N})'$ as a space of distributions. We denote the dual pairing between elements $F \in (\mathcal{N})$ and $\Phi \in (\mathcal{N})'$ by $\langle\langle F, \Phi\rangle\rangle := \Phi(F)$.

A major part of Gaussian analysis and, in particular, White Noise analysis deals with the construction and analysis of dual pairs (X, X') of spaces, s.t. X is densely and continuously embedded into $L^2(\mu)$, i.e., we have

$$X \subseteq L^2(\mu) \subseteq X'.$$

Important examples of this are given by $(\mathcal{D}, \mathcal{D}')$, where $\mathcal{D}$ denotes the Meyer-Watanabe space together with its dual $\mathcal{D}'$, see e.g. [108],[57, Chapter 3.C] and the references therein, and also the space of Kondratiev test functions and distributions, see e.g. [65]. We also refer to [109] for more examples of pairs of spaces (X, X'). One main advantage of the pair $((\mathcal{N}), (\mathcal{N})')$ is the availability of a characterization theorem, see e.g. [63, 64, 65]. The elements of $(\mathcal{N})$ and $(\mathcal{N})'$ are characterized in terms of the so-called U-functionals. A U-functional is a map $U : \mathcal{N} \longrightarrow \mathbb{C}$, which satisfies

(U1) U is ray-analytic, i.e., for all $f_1, f_2 \in \mathcal{N}$, the function

$$\mathbb{R} \ni \lambda \mapsto U(f_1 + \lambda f_2)$$

is analytic and extends to an entire function on $\mathbb{C}$.

(U2) U is uniformly bounded of exponential order 2, i.e., there exist $A, B \geq 0$ and $p \in \mathbb{N}_0$ s.t. for all $f \in \mathcal{N}$ and $z \in \mathbb{C}$ it holds

$$|U(zf)| \leq A \exp\left(B\,|z|^2\,\|f\|_p^2\right).$$

The connection between U-functionals and the elements $\Phi \in (\mathcal{N})'$ is given by the fact that

$$S\Phi(f) := \langle\langle :\exp(\langle f, \cdot\rangle):, \Phi\rangle\rangle, \quad f \in \mathcal{N}, \tag{0.10}$$

is a U-functional, where $:\exp(\langle f, \cdot\rangle) := \exp(\langle f, \cdot\rangle - \frac{1}{2}\|f\|_{\mathcal{H}}^2) \in (\mathcal{N})$, $f \in \mathcal{N}$. The striking result regarding the above mentioned characterization theorem of $(\mathcal{N})'$ is that every U-functional U arises as the S-transform $S\Phi$ of some element $\Phi \in (\mathcal{N})'$, see [63].

In this part of the thesis, we deal with a certain dual pair $(\mathcal{G}_K, \mathcal{G}'_K)$ of spaces which satisfies

$$(\mathcal{N}) \subseteq \mathcal{G}_K \subseteq L^2(\mu) \subseteq \mathcal{G}'_K \subseteq (\mathcal{N})'. \tag{0.11}$$

This pair of spaces was introduced and characterized in [52]. Here, K stands for a self-adjoint operator on the Hilbert space $\mathcal{H}$, which leaves the space $\mathcal{N}$ invariant. The spaces $\mathcal{G}_K$ and $\mathcal{G}'_K$ can be briefly described as follows. Due to the decomposition of $L^2(\mu)$ in (0.9) we can define the second quantization $(\Gamma(K), D(\Gamma(K)))$ of the operator $(K, D(K))$ as a linear operator on $L^2(\mu)$. The random variables $\mathcal{G}_K \subset L^2(\mathcal{N}', \mu)$, we deal with, are exactly the C^∞ of $\Gamma(K)$, i.e., $\mathcal{G}_K := \cap_{s=1}^{\infty}(D(\Gamma(K)^s))$. The operator $\Gamma(K)$ induces naturally a locally convex topology on $\mathcal{G}_K$ given by the seminorms $\|\cdot\|_{K,s} := \|\Gamma(K)^s \cdot\|_{L^2(\mu)}$, $s \in \mathbb{N}$. The space of generalized random variables $\mathcal{G}'_K$ is the dual space of $\mathcal{G}_K$ w.r.t. this topology. The dual $\mathcal{G}'_K$ is given by $\mathcal{G}'_K = \cup_{s=1}^{\infty}(D(\Gamma(K)^{-s}))$, where $D(\Gamma(K)^{-s})$ is the dual space of $(D(\Gamma(K)^s), \|\cdot\|_{K,s})$, $s \in \mathbb{N}$. Important examples of spaces of random variables and their corresponding dual spaces arise in this way. For certain choices of $\mathcal{N}$ the pair $((\mathcal{N}), (\mathcal{N})')$ arises exactly in this way. Also the space $\mathcal{G}$ and its dual $\mathcal{G}'$ introduced in [87] are formed in this way for a suitable choice of the operator K.

To formulate the characterization of $(\mathcal{G}_K, \mathcal{G}'_K)$ given in [52] we introduce the following type of projections. Let $m \in \mathbb{N}$ and $(\varphi_i)_{i=1}^m \subset \mathcal{N}$ be an orthonormal system in $\mathcal{H}$. We call

$$P : \mathcal{N}'_{\mathbb{C}} \to \mathcal{N}_{\mathbb{C}},\, P\eta := \sum_{i=1}^{m} \langle \varphi_i, \eta\rangle\, \varphi_i$$

an orthogonal projection from $\mathcal{N}'_{\mathbb{C}}$ into $\mathcal{N}_{\mathbb{C}}$. We denote the set consisting of all orthogonal projections from $\mathcal{N}'_{\mathbb{C}}$ into $\mathcal{N}_{\mathbb{C}}$ by $\mathbb{P}$. The authors in [52] formulate their main result in terms of the Bargmann-Segal space over $\mathcal{H}$, which we denote by $E^2(\nu)$. This space consists of all complex-valued entire functions G defined on the complexification $\mathcal{H}_{\mathbb{C}}$, which satisfy

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |G(P\cdot)|^2 \, d\nu < \infty,$$

where $\nu = \mu_{\frac{1}{2}} \otimes \mu_{\frac{1}{2}}$, and $\hat{\mu}_{\frac{1}{2}}(f) = \exp(-(f, f)_{\mathcal{H}})$, $f \in \mathcal{N}$. In [52] the authors construct an isomorphism $\widetilde{S}$ from $L^2(\mu)$ to $E^2(\nu)$ which is closely related to the S-transform introduced above.

Hence, the space $\mathcal{G}_K$ can be characterized via its image $\widetilde{S}\mathcal{G}_K$ in $E^2(\nu)$. The characterization of $(\mathcal{G}_K, \mathcal{G}'_K)$ given in [52] can be stated as follows. An element $F \in D(\Gamma(K)^s)$, $s \in \mathbb{Z}$, satisfies $G := \widetilde{S}F \in E^2(\nu)$ and

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |G(K^s P \cdot)|^2 \, d\nu < \infty. \tag{0.12}$$

In reverse, if $G \in E^2(\nu)$ satisfies (0.12), then $\widetilde{S}^{-1}G \in D(\Gamma(K)^s)$.

The main result in this part of the thesis is an improvement of this characterization. The motivation for this new formulation was to make this characterization and, therefore, the spaces $\mathcal{G}_K$ and $\mathcal{G}'_K$ more pliable. In particular, the new characterization does not need the concept of entire functions on infinite dimensional Hilbert spaces. Instead, we use the concepts of U-functionals and the S-transform defined in (0.10). Due to (0.11) every element $\Phi \in \mathcal{G}'_K$ has a well-defined S-transform. Our improved characterization can be formulated as follows. If $\Phi \in D(\Gamma(K)^s)$, $s \in \mathbb{Z}$, then its S-transform $U := S\Phi$, which is a U-functional, satisfies

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |U(K^s P \cdot)|^2 \, d\nu < \infty. \tag{0.13}$$

In reverse, if U is a U-functional and satisfies (0.13), then $S^{-1}U \in D(\Gamma(K)^s)$, $s \in \mathbb{Z}$. The proof avoids the introduction of the Bargman-Segal space and the concept of entire functions on $\mathcal{H}_{\mathbb{C}}$ completely. Recall that the definition of a U-functional consists of a growth bound, $(U2)$, and analyticity of a function defined on $\mathbb{R}$, $(U1)$. To check real analyticity is, in general, less involved than analyticity on an infinite dimensional Hilbert space, which is an improvement to the original characterization given in [52]. Observe that in many applications elements are efficiently constructed and defined via U-functionals. In particular, elements from $(\mathcal{N})'$, which are defined via a Wick product can be efficiently treated only by means of U-functionals, see e.g. the objects constructed in [50]. Eventually, we present an application of this new result.

Axiomatic Quantum Field Theory

The second topic of this part is from axiomatic and constructive quantum field theory. Axiomatic quantum field theory (a.q.f.t.) was and still is one of the most challenging fields of mathematical physics. It is the attempt to give mathematical rigorous meaning to objects arising in the investigation of relativistic quantum phenomena. Basically, a.q.f.t. consists of two directions. The first one is to formulate systems of axioms and to show their (partially) equivalence, as well as to deduce physical properties of a field Φ satisfying these axioms. At this point, we have to mention the most important axioms as the Gårding-Whightmann axioms, the Wightmann axioms, the Osterwalder-Schrader (O.-S.) axioms, Nelsons axioms and the Axioms of Haag-Kastler, see e.g. [99, 89, 46] and the references therein. Each of theses axioms describe the field via different functional analytic, probabilistic or algebraic objects. For example, the O.-S. axioms are formulated in terms of the so-called Schwinger functions or the Euclidean Green's functions $(S_n)_{n \in \mathbb{N}_0}$. For $n \in \mathbb{N}_0$ the element S_n is a tempered distribution in $S'(\mathbb{R}^{dn})$, where d is the space-time dimension. We state the complete set of O.-S. axioms in Chapter 7.

The second direction of a.q.f.t. is to construct interesting examples of fields fulfilling theses axioms, i.e., one has to show that the formulated axioms are consistent. Since the mathematical objects of each of these sets of axioms are so involved, it is a challenging task to write down examples of such objects fulfilling all axioms under consideration. Fortunately, in any space-time dimension the so-called generalized free field models exist. This proves that the axioms are consistent. Unfortunately, these models don't incorporate any interesting physics, since they only describe non-interacting particles. A huge workload in mathematical physics was done to construct models, which include an interaction of particles. Different strategies to construct interacting models with different kinds of interactions were invented, such as the so-called Hamiltonian strategy, see e.g. [45] and [3], and the Euclidean strategy, see e.g. [36],[72] and [99] and the references therein. In this thesis, we exclusively deal with the Euclidean strategy. Therefore, we briefly describe this strategy in the following and refer the reader to the monographs [46, 99] for more details. The Euclidean strategy is related to the O.-S. axioms. The idea is that one constructs Schwinger functions $(S_n)_{n\in\mathbb{N}_0}$ as the moments of a measure μ defined on the measure space $(S'(\mathbb{R}^d),\mathcal{B})$, where d denotes the space-time dimension and $\mathcal{B}$ is the Borel σ-field of the weak topology on $S'(\mathbb{R}^d)$. One such measure is the so-called Euclidean free field measure μ_m of mass m. By the Bochner-Minlos theorem, μ_m is uniquely determined via its characteristic function $\hat{\mu}_m$ as follows

$$\int\limits_{S'(\mathbb{R}^d)} \exp(i\,\langle f,\Phi\rangle)\mu_m(d\Phi) = \exp\left(-\frac{1}{2}(f,C_m f)_{L^2(\mathbb{R}^d)}\right), \quad f\in S(\mathbb{R}^d), \tag{0.14}$$

where $C_m=(-\Delta+m^2)^{-1}$ and $m>0$. To obtain a self-interacting field one formally perturbs the measure μ_m by a density to obtain a new measure μ_V defined by

$$\mu_V(d\Phi) = \frac{1}{Z_V}\exp\left(-\int\limits_{\mathbb{R}^d} V(\Phi(x))\,dx\right)\mu_m(d\Phi), \tag{0.15}$$

where Z_V denotes a normalization constant and V a real function describing the type of interaction, e.g. V could be a real polynomial of even degree. Here, several difficulties arise, which make the measure μ_V in (0.15) not well-defined. First of all, typical configurations Φ of μ_m are distributions, which are not given by integrable functions. Therefore, $\Phi(x)$, $x\in\mathbb{R}^d$, has no meaning, in particular, powers of Φ have no meaning. Additionally, the integral $\int\limits_{\mathbb{R}^d} V(\Phi(x))\,dx$ might not converge. To make these expressions well-defined one needs to introduce cutoff parameters. Indeed, by mollifying the elements Φ with a standard approximate identity $(\chi_t)_{t>0}\subseteq S(\mathbb{R}^d)$ we can define $\Phi_t(x) := \langle \chi_t(\cdot - x),\Phi\rangle$, $x\in\mathbb{R}^d$. Additionally, one replaces powers $\Phi(x)^m$, $m\in\mathbb{N}$, by Wick powers $:\Phi_t^m(x):$. Further, let g be a smooth function with compact support on $\mathbb{R}^d$. If the interaction function V is bounded from below, then we can define a bona-fide measure $\mu_{V,t,g}$ on $(S'(\mathbb{R}^d),\mathcal{B})$, i.e.,

$$\mu_{V,t,g}(d\Phi) = \frac{1}{Z_{V,t,g}}\exp\left(-\int\limits_{\mathbb{R}^d} g(x)\,:V(\Phi_t(x)):\,dx\right)\mu_m(d\Phi). \tag{0.16}$$

Here, the difficulties of Euclidean approach start. We denote the moments $S_n^{t,g}$, $n\in\mathbb{N}$, of the measure $\mu_{V,t,g}$ by

$$S_n^{t,g}(f_1,\dots,f_n) = \frac{1}{Z_{V,t,g}} \int\limits_{S'(\mathbb{R}^d)} \prod_{i=1}^{n} \langle f_i, \cdot\rangle \exp\left(- \int\limits_{\mathbb{R}^d} g(x) :V(\Phi_t(x)): \, dx \right) \mu_m(d\Phi),$$

where $f_1,\dots,f_n \in S(\mathbb{R}^d)$. One has to show that for all $n \in \mathbb{N}$ the moment $S_n^{t,g}(f_1,\dots,f_n)$ converge in the sense of distributions as the cutoff parameter t tends to zero and the function g tends to the function, which is constantly 1 on $\mathbb{R}^d$. Additionally, one desires to obtain that the no-cutoff Schwinger functions $S_n = \lim_{g\to 1} \lim_{t\to 0} S_n^{t,g}$, $n \in \mathbb{N}$, are moments of a non-Gaussian measure. This is what we refer to as non-triviality. The difficulty of this procedure depends crucially on the space-time dimension d, which is due to the d-dependence of the singularity of the integral kernel of the operator C_m. The successful proof of such a convergence has only been achieved in a few number of cases. For example, the models with polynomial self-interaction, as the $P(\Phi)_2$ and the $(\Phi)_3^4$ model in space-time dimension $d = 2, 3$, respectively, are milestones in axiomatic quantum field theory. Interacting models in arbitrary space time dimension d are for example given by the Albeverio-Høegh-Krohn model, see [3].

In Chapter 6, we deal with the $(\Phi)_3^4$ model. The symbol $(\Phi)_3^4$ refers to an interaction V given by a polynomial of degree 4 and space-time dimension $d = 3$. We use the results from the construction of the $(\Phi)_3^4$ model in [36] to show that the Schwinger functions of this model are given as the moments of a positive Hida distribution. In particular, we show that the cutoff measures converge in the Hida distribution space. Additionally, we show how the Hida calculus can be applied to obtain, also, the convergence of the logarithmic derivatives of the approximating measures. Some additional remarks concerning the difficulties of a stochastic quantization of the $(\Phi)_3^4$ model in terms of Dirichlet forms are given.

In Chapter 7 we propose a different approach to construct non-trivial examples of relativistic quantum fields. This approach does not use any renormalization in terms the of cutoff parameters introduced above. In particular, the method applies in arbitrary space-time dimension $d \in \mathbb{N}$. The Schwinger functions are again given as moments of a probability measure μ_ϱ on the measure space $\left(S'(\mathbb{R}^d), \mathcal{B}\right)$. The measure μ_ϱ is given as the superposition of the Gaussian measures μ_m, $m > 0$, defined in (0.14). The symbol ϱ denotes a probability measure on the real positive line describing which masses m contribute to the superposition. This construction is heavily inspired by the Källen-Lehmann representation of the two point function of a relativistic quantum field, see e.g. [89, Theorem IX.34]. We also argue that the superposition μ_ϱ is not Gaussian anymore. We prove that the moments of μ_ϱ satisfy all O.-S. axioms. To the best of our knowledge, this fact has not been proven yet, in the existing literature. The advantage of the proof we give is that there are various generalizations possible. In particular, the proof basically shows that any reasonable superposition of Schwinger functions which satisfy all O.-S. axioms satisfies the O.-S. axioms, too. We state precisely in Remark 7.10 what we mean by reasonable. For example, one could superpose Schwinger functions which do not necessarily need to be moments of a Gaussian measures. We want to refer to [8] and [49] for constructions of Schwinger functions, which satisfy all O.-S. axioms except reflection positivity and do not rise as moments of a Gaussian measure. Additionally, we show certain properties of the measure μ_ϱ, which might be the starting point for a future non-Gaussian analysis as presented in [7].

Remarks on the Notation

As usual, we denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the natural, rational, real and complex numbers, respectively. For $x \in \mathbb{K}$, $\mathbb{K} \in \{\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, we denote by $|x|$ the absolute value of x. By slight abuse of notation, we also write $|x|$ to denote the Euclidean norm of an element $x \in \mathbb{K}^n$, $n \in \mathbb{N}$, i.e., $|x| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$, $x = (x_1, \ldots, x_n)$. If not further specified, we consider $\mathbb{R}^n$ and $\mathbb{C}^n$, $n \in \mathbb{N}$, to be equipped with the Euclidean topology and the corresponding Borel σ-field. Subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$ are equipped with the trace topology and trace σ-algebra, respectively.

For a topological vector space X over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $L(X)$ the space of all linear and continuous operators mapping from X into itself. The identity operator we denote by I. If $D \subseteq X$ is a subspace of X and $A : D \longrightarrow X$ is a linear map, we say that (A, D) is a linear operator on X. If the set D is unambiguously understood, we also just say A is a linear operator on X, even though A is not defined on the whole space X. For a subset Y of X, we denote by $\operatorname{span}(Y)$ the set of all finite linear combinations of elements of Y. To emphasize the role of the underlying field $\mathbb{K}$, we also write $\operatorname{span}_{\mathbb{K}}(Y)$. If X is a Banach space we denote for a closed or closable operator (A, D) on X by $\sigma(A)$ and $\varrho(A)$ the spectrum and the resolvent set of A, respectively. The scalar product $(\cdot, \cdot)$ on a complex Hilbert space X is always linear in the first component and anti-linear in the second component.

For a measure space $(\mathbb{E}, \mathcal{F}, \mu)$ and a non-negative measurable function $f : E \longrightarrow \mathbb{R}$ we denote by $f\mu$ the measure on $(E, \mathcal{F})$ with density f w.r.t. μ. Let W be some collection of functions, which are defined on E and map into some measurable space, then we denote by $\sigma(W)$ the σ-algebra generated by the elements of W. If E carries a topology, we denote by $\mathcal{B}(E)$ the Borel σ-algebra of E. For $p \in [1, \infty)$ we denote by $L^p(E, \mu)$ the space of equivalence classes of p-integrable functions w.r.t. μ, w.r.t. the equivalence relation of equality μ-a.e.. For $p = \infty$, $L^p(E, \mu)$ denotes the space of equivalence classes of μ-essentially bounded functions. To emphasize the role of the σ-algebra $\mathcal{F}$ we also write $L^p(E, \mathcal{F}, \mu)$. In the special case $E = \mathbb{R}^n$, $n \in \mathbb{N}$, we also denote by $L^p(\mathbb{R}^n)$ the space $L^p(\mathbb{R}^n, dx)$, where dx denotes the Lebesgue measure on $\mathbb{R}^n$, $p \in [1, \infty]$. Similar notations are used for the measurable subsets $\Omega \subset \mathbb{R}^n$.

For a topological space E we denote by $C(E)$ and $C_b(E)$ the space of real-valued continuous functions and real-valued continuous and bounded functions on E, respectively. For $f \in C(E)$ we denote by $\operatorname{supp}(f)$ the support of f, i.e, the set $\overline{\{x \in E \mid f(x) \neq 0\}}$.

Part I

Overdamped Limit of Generalized Stochastic Hamiltonian Systems

Chapter 1

Preliminaries from the Theory of Operator Semigroups

In this entire first chapter we denote by X a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and its dual space we denote by X'. We assume the reader is familiar with basic concepts from functional analysis such as the Hahn-Banach and the Baire category theorem as well as continuous and closed linear operators. Furthermore, it is assumed that the reader is familiar with the weak and weak-$*$ topology of Banach spaces and their duals, and the usual topologies on the space of continuous and linear operators $L(X)$ such as the uniform, strong and weak topology, see e.g. [91, 92, 27]. The content of this first chapter can be found also in any textbook on operator semigroups, such as [11, 84, 47, 31, 32]. We don't follow an approach of most generality in this presentation. We rather try to take a short route to the results needed for the applications we have in mind. Nevertheless, some proofs are presented to give several additional insights. In particular, proofs which are essential for understanding the subsequent results are presented.

In the following we use the convention that for a closable linear operator (A, D) on X we also denote the closure by $(A, D(A))$. Sometimes we also write $(\overline{A}, D(\overline{A}))$ for the closure to emphasis the difference to (A, D).

1.1 Strongly Continuous Contraction Semigroups, Generators and Resolvents

1.1.1 Definitions, Basic Properties and the Hille-Yosida Theorem

Definition 1.1. *Let $(T_t)_{t\geq 0}$ be a family in $L(X)$. Consider the following properties*

(S1) $T_{t+s} = T_t T_s$, *for all* $t, s \geq 0$,

(S2) The map $[0, \infty) \ni t \mapsto T_t \in L(X)$ *is strongly continuous*

(S3) $T_0 = I$,

(S4) $\|T_t\| \leq 1$ *for all* $t \geq 0$.

We call $(T_t)_{t\geq 0}$ a strongly continuous semigroup (s.c.s.) on X if it fulfills (S1), (S2) and (S3). If $(T_t)_{t\geq 0}$ also fulfills (S4) it is called a strongly continuous contraction semigroup (s.c.c.s.) on X.

In this thesis we sometimes also use the abbreviation s.c.c.s. for strongly continuous contraction semigroups. Depending on the context and the correct English grammar we refer to the singular or plural version of this abbreviation. Similar conventions are also used for other abbreviations. We give some generic examples of s.c.c.s. on Banach spaces.

Example 1.2. *(i) Let $A \in L(X)$. For $t \geq 0$ define the convergent series*

$$T_t := e^{(tA)} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \in L(X).$$

Then $(T_t)_{t\geq 0}$ is a strongly continuous semigroup.

(ii) Let $X = L^p((\mathbb{R}, dx)$, $p \in [1, \infty)$, where dx denotes the Lebesgue measure on $\mathbb{R}$. For $f \in L^p((\mathbb{R}, dx)$ and $t \geq 0$ define $T_t^s f(\cdot) = f(\cdot + t)$. Then $(T_t^s)_{t\geq 0}$ is called the shift semigroup

(iii) Let $X = L^p((\mathbb{R}, dx)$, $p \in [1, \infty)$. Define $T_0^h = I$ and for $f \in L^1(\mathbb{R}, dx)$ and $t > 0$ we define $T_t^h f := H_t * f$, where $H_t(\cdot) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|\cdot|^2}{4t}\right)$ and $*$ denotes the convolution. $(T_t^h)_{t\geq 0}$ is called the heat semigroup.

Lemma 1.3. *Let $(T_t)_{t\geq 0}$ be a family on X fulfilling (S1) and (S3). Furthermore, assume that $[0, \infty) \ni t \mapsto T_t \in L(X)$ is weakly continuous. Then (S2) also holds true for $(T_t)_{t\geq 0}$.*

Proof. See [112, Theorem IX.1]. □

Definition 1.4. *Let $(T_t)_{t\geq 0}$ be a family in $L(X)$. Define the linear operator $(A, D(A))$ on X via*

$$D(A) := \left\{ f \in X \mid \lim_{t \searrow 0} \frac{1}{t}(T_t f - f) \text{ exists in } X \right\},$$

$$Af := \lim_{t \searrow 0} \frac{1}{t}(T_t f - f).$$

The operator $(A, D(A))$ is called the generator of $(T_t)_{t\geq 0}$.

Example 1.5. *Recall the examples of semigroups given in Example 1.2. The corresponding generators of those semigroups are given respectively, as follows:*

(i) The generator of $(\exp(tA))_{t\geq 0}$ is given by (A, X).

(ii) Then the generator $(A, D(A))$ of $(T_t^s)_{t\geq 0}$ on $L^p(\mathbb{R}, dx)$ is given by $\left(\frac{d}{dx}, H^{1,p}(\mathbb{R})\right)$, where $H^{1,p}(\mathbb{R})$ denotes the Sobolev space of order one in $L^p(\mathbb{R}, dx)$ and $\frac{d}{dx}$ is the weak derivative, see e.g. [2].

(iii) The s.c.c.s. $(T_t^h)_{t\geq 0}$ admits the generator $\left(\frac{d^2}{dx^2}, H^{2,p}(\mathbb{R})\right)$, where $H^{2,p}(\mathbb{R})$ denotes the Sobolev space of order two in $L^p(\mathbb{R}, dx)$ and $\frac{d^2}{dx^2}$ is the weak derivative of order two.

Lemma 1.6. *Let $(T_t)_{t\geq 0}$ be a s.c.c.s. and $(A, D(A))$ be its generator. Then it holds*

(i) For $f \in X$ the Riemann integral $\int_0^t T_s f \, ds$ is an element of $D(A)$ for all $t > 0$. In particular, $D(A)$ is dense in X.

(ii) For $f \in D(A)$ it holds that $T_t f - f = \int_0^t T_s A f \, ds$ for all $t \geq 0$.

(iii) Let $(S_t)_{t\geq 0}$ be another s.c.c.s. with generator $(A, D(A))$. Then $T_t = S_t$ for all $t \geq 0$.

Proof. (i) Let $f \in X$. Then it holds by the mean value theorem

$$\left(\frac{T_r - I}{r}\right) \int_0^t T_s f \, ds = \frac{1}{r} \int_t^{t+r} T_s f \, ds - \frac{1}{r} \int_0^r T_s f \, ds \xrightarrow{r \to 0} T_t f - f.$$

Hence, by definition $\int_0^t T_s f \, ds \in D(A)$ and $A \int_0^t T_s f \, ds = T_t f - f$. In particular, we obtain $\frac{1}{t} \int_0^t T_s f \, ds \longrightarrow f$ as $t \to 0$, which proves that $D(A)$ is dense.

(ii) Observe that for $f \in D(A)$ and $s \geq 0$ it follows that $T_s f \in D(A)$ and $AT_s f = T_s Af$, i.e., the map $[0, t] \ni s \mapsto T_s f \in X$ is differentiable with derivative $T_s Af$. Hence, the assertion follows by the fundamental theorem of calculus and the Hahn-Banach theorem.

(iii) Let $(A, D(A))$ be the generator of both $(S_t)_{t \geq 0}$ and $(T_t)_{t \geq 0}$ with both being s.c.c.s.. It suffices by part (i) to show that $S_t = T_t$ on $D(A)$ for all $t \geq 0$. Therefore, let $f \in D(A)$, $t \geq 0$ and consider the map $F = F_t^f : [0, t] \longrightarrow X, s \mapsto T_{t-s} S_s f$. We show that F is differentiable with derivative zero. Let $s \in (0, t)$ and $h > 0$ s.t. $t - s - h \geq 0$. Hence, we have

$$\begin{aligned} F(s+h) - F(s) &= T_{t-s-h} \left(\frac{S_{s+h} - S_s}{h}\right) f + \left(\frac{T_{t-s-h} - T_{t-s}}{h}\right) S_s f \\ &= T_{t-s-h} \left(\frac{S_{s+h} - S_s}{h}\right) f + T_{t-s-h} \left(\frac{I - T_h}{h}\right) S_s f. \end{aligned}$$

By the proof of part (*ii*) we know that $S_s f \in D(A)$. Since $(T_t)_{t \geq 0}$ consists of contractions we can deduce by (S2)

$$\lim_{h \searrow 0} \frac{F(s+h) - F(s)}{h} = T_{t-s} S_s Af - T_{t-s} AS_s f = 0.$$

Now by the fundamental theorem of calculus we obtain

$$T_t f - S_t f = F(t) - F(0) = \int_0^t F'(s) \, ds = 0.$$

□

In the following, let $(T_t)_{t \geq 0}$ be a s.c.c.s. with generator $(A, D(A))$ and $\alpha > 0$. We denote by $\rho(A)$ the resolvent set of the operator $(A, D(A))$. Due to (S2) and (S4) we obtain that

$$[0, \infty) \ni t \mapsto e^{-\alpha t} T_t \in L(X) \tag{1.1}$$

is Bochner integrable w.r.t. the Lebesgue measure on $[0, \infty)$. Furthermore, the map (1.1) is also Riemann integrable in $L(X)$ and by using the formal notation $T_t = e^{tA}$ we obtain

$$\frac{1}{(\alpha - A)} = \int_0^\infty e^{-\alpha t} T_t \, dt =: G_\alpha. \tag{1.2}$$

The definition of G_α suggests that we call $(G_\alpha)_{\alpha>0}$ the Laplace transform of $(T_t)_{t\geq 0}$. Now we give the equation (1.2) a rigorous meaning. Therefore, we observe that $\left(e^{-\alpha t}T_t\right)_{t\geq 0}$ is again a s.c.c.s. with generator $(-\alpha + A, D(A))$. First let $f \in D(A)$. By Lemma 1.6(ii) it holds

$$e^{-\alpha t}T_t f - f = \int_0^t e^{-\alpha s}T_s(-\alpha + A)f\,ds. \tag{1.3}$$

Since $\|T_t\| \leq 1$, we can take the limit $t \longrightarrow \infty$ on both sides in (1.3) and obtain

$$G_\alpha(\alpha - A)f = f.$$

Now, let $f \in X$ be arbitrary. We need to show $G_\alpha f \in D(\alpha - A)$ and $(\alpha - A)G_\alpha f = f$. By the properties of the Bochner integral we obtain

$$\begin{aligned}\frac{1}{t}\left(e^{-\alpha t}T_t - I\right)G_\alpha f &= \frac{1}{t}\left(\int_0^\infty e^{-\alpha(s+t)}e^{(s+t)A}f\,ds - \int_0^\infty e^{-\alpha s}e^{sA}f\,ds\right)\\ &= \frac{1}{t}\int_0^t e^{-\alpha s}e^{sA}f\,ds.\end{aligned}$$

Thus, by the mean value theorem it holds $G_\alpha f \in D(\alpha - A)$ and $(\alpha - A)G_\alpha f = f$. This derivation proves the following lemma.

Lemma 1.7. *Let $(A, D(A))$ be the generator of a s.c.c.s. $(T_t)_{t\geq 0}$. Then $(0,\infty) \subseteq \rho(A)$ and*

$$(\alpha - A)^{-1} = G_\alpha = \int_0^\infty e^{-\alpha t}e^{tA}\,dt.$$

Furthermore, $\left\|\alpha(\alpha - A)^{-1}\right\| \leq 1$ for all $\alpha > 0$. In particular, $(A, D(A))$ is closed.

Definition 1.8. *In general a family $(G_\alpha)_{\alpha>0}$ in $L(X)$ fulfilling*

(R1) $\|\alpha G_\alpha\| \leq 1$ *for all* $\alpha > 0$,

(R2) $G_\alpha - G_\beta = (\beta - \alpha)G_\beta G_\alpha$ *for all* $\alpha, \beta > 0$,

(R3) $\lim\limits_{\alpha\to\infty} \alpha G_\alpha f = f$ *for all* $f \in X$.

is called a strongly continuous contraction resolvent (s.c.c.r.) on X.

Proposition 1.9. *Let $(A, D(A))$ be a densely defined linear operator on X such that $(0,\infty) \subseteq \rho(A)$ and for all $\alpha \in (0,\infty)$ it holds $\left\|\alpha(\alpha - A)^{-1}\right\| \leq 1$. Then, the family $(G_\alpha)_{\alpha>0}$ given by $G_\alpha = (\alpha - A)^{-1}$ is a s.c.c.r. on X.*

Proof. The property (R1) follows immediately. Hence, by (R1) and the density of $D(A)$ in X it suffices by an 3ε argument to prove (R3) for $f \in D(A)$. For such an f it holds

$$\|\alpha G_\alpha f - f\| = \|G_\alpha(\alpha f - (\alpha - A)f)\| \leq \frac{1}{\alpha}\|Af\| \longrightarrow 0, \text{ as } \alpha \to \infty.$$

To prove (R2) we observe that $G_\alpha f \in D(A)$ for all $f \in X$ and $\alpha > 0$. Hence, for all $f \in X$ it holds

$$\begin{aligned} G_\beta f - G_\alpha f &= G_\beta f - G_\beta(\beta - \alpha + \alpha - A)G_\alpha f \\ &= (\beta - \alpha)G_\beta G_\alpha f. \end{aligned}$$

□

Observe that for a s.c.c.r. $(G_\alpha)_{\alpha>0}$ it holds by (R2) $G_\alpha G_\beta = G_\beta G_\alpha$. We are ready to state and prove the Hille-Yosida Theorem.

Theorem 1.10. *Let $(A, D(A))$ be a linear operator on X which is closed and densely defined. Then $(A, D(A))$ is the generator of a s.c.c.s. $(T_t)_{t\geq 0}$ if and only if $(A, D(A))$ fulfills*

$$(0, \infty) \subseteq \rho(A) \text{ and } \left\|\alpha\,(\alpha - A)^{-1}\right\| \leq 1 \text{ for all } \alpha \in (0, \infty).$$

Proof. By Lemma 1.6 and 1.7 we already know that a generator of a s.c.c.s. satisfies the claimed conditions. Hence, let $(A, D(A))$ be closed, densely defined and fulfill $(0, \infty) \subseteq \rho(A)$ s.t. for $\alpha \in (0, \infty)$ it holds $\left\|\alpha\,(\alpha - A)^{-1}\right\| \leq 1$. By Proposition 1.9 we know that $G_\alpha = (\alpha - A)^{-1}$, $\alpha > 0$, is a s.c.c.r.. Now we define the so-called Yosida approximations

$$A_\alpha = \alpha\left(\alpha(\alpha - A)^{-1} - I\right)$$

and observe that for $f \in D(A)$ it holds $A_\alpha f = \alpha G_\alpha A f$. Furthermore, by (R3) it holds

$$A_\alpha f \longrightarrow Af \text{ for } f \in D(A) \tag{1.4}$$

Observe that

$$T_t^\alpha = e^{-\alpha t} \exp(t\alpha^2(\alpha - A)^{-1})$$

determines by Example 1.2(i) and (R1) a s.c.c.s. with generator A_α. We call the operators T_t^α, $\alpha > 0$, the Yosida approximations of T_t. The idea is now to prove that T_t^α converges as $\alpha \to \infty$ and to show that the limit forms a semigroup in t with generator $(A, D(A))$. Observe that A_β and T_t^α commute for all $t \geq 0$ and $\alpha, \beta > 0$ since the resolvents G_α and G_β commute. As in the proof of Lemma 1.6(iv) we obtain

$$T_t^\alpha f - T_t^\beta f = \int_0^t T_s^\alpha T_{t-s}^\beta (A_\alpha f - A_\beta f)\, ds$$

Hence, by the contraction property of T_t^α it follows

$$\left\|T_t^\alpha f - T_t^\beta f\right\| \leq \int_0^t \left\|A_\alpha f - A_\beta f\right\| ds = t\left\|A_\alpha f - A_\beta f\right\|. \tag{1.5}$$

Thus, for $f \in D(A)$ we obtain by (1.4) that $T_t^\alpha f$ converges locally uniformly in t to a limit $T_t f$ as α tends to infinity. Furthermore, one easily concludes that this limit is linear in $f \in D(A)$ and fulfills $\|T_t f\| \leq \|f\|$. Hence, T_t determines an element in $L(D(A), X)$ with norm less than 1. Therefore we can extend T_t to an element in $L(X)$ with norm still less than 1. Using a 3ε argument we obtain $\lim_{\alpha\to\infty} T_t^\alpha f = T_t f$ for all $f \in X$. It remains to prove that $(T_t)_{t\geq 0}$ is a s.c.c.s. with generator $(A, D(A))$. (S4) is already shown, (S1) and (S3) are inherited by the approximations. The strong continuity property follows since the convergence of T_t^α is locally uniformly in $t \in [0, \infty)$. Now let $f \in D(A)$. Hence, by taking the limit $\beta \longrightarrow \infty$ limit in (1.5) we obtain

$$\begin{aligned}\left\| \frac{(T_t f - f)}{t} - Af \right\| &\leq \left\| \frac{(T_t f - T_t^\alpha f)}{t} \right\| + \left\| \frac{(T_t^\alpha f - f)}{t} - A_\alpha f \right\| + \|A_\alpha f - Af\| \\ &\leq \left\| \frac{(T_t^\alpha f - f)}{t} - A_\alpha f \right\| + 2\,\|A_\alpha f - Af\|\,.\end{aligned}$$

Now, first choosing α large enough and then t small enough we see that $D(A)$ is indeed contained in the domain of the generator $\left(\tilde{A}, D(\tilde{A})\right)$ of $(T_t)_{t\geq 0}$ and $\tilde{A}f = Af$ on $D(A)$. By Lemma 1.7 we know that $1 \in \rho(\tilde{A})$. However, under the assumption $1 \in \rho(A)$ the operator $\left(\tilde{A}, D(\tilde{A})\right)$ can not be a proper extension. Otherwise $I - \tilde{A}$ would not be injective. □

In combination with the Hille-Yosida theorem the next theorem shows that every s.c.c.r. $(G_\alpha)_{\alpha>0}$ arises as Laplace transforms of a s.c.c.s. $(T_t)_{t\geq 0}$.

Theorem 1.11. *Let $\left(\tilde{G}_\alpha\right)_{\alpha>0}$ be a s.c.c.r. on X. Then the operator $(A, D(A))$ defined by $D(A) = \tilde{G}_\alpha X$ and $A = \alpha - \tilde{G}_\alpha^{-1}$ is densely defined, closed and satisfies $(0, \infty) \subseteq \rho(A)$.*

Proof. See [70, Proposition I.1.5] □

1.1.2 The Adjoint Semigroup and its Generator

Throughout this section let $(T_t)_{t\geq 0}$ be an s.c.c.s. on X. In the following we aim to study the dual object corresponding to $(T_t)_{t\geq 0}$, i.e., the family $\left(T_t'\right)_{t\geq 0}$ of the adjoint operators on X'. It is clear that the adjoints $\left(T_t'\right)_{t\geq 0}$ satisfy (S1),(S3) and (S4). The following examples show that strong continuity does in general not hold for the adjoint semigroups.

Example 1.12. *(i) Recall the semigroup $(T_t^s)_{t\geq 0}$ given in Example 1.2(ii). The adjoint operators $T_t^{s\prime}$ act on the dual space $L^\infty(\mathbb{R}, dx)$. For $x' = 1_{[0,1]} \in L^\infty(\mathbb{R}, dx)$, where $1_{[0,1]}$ denotes the indicator function of the interval $[0, 1]$, one easily checks $T_t^{s\prime} 1_{[0,1]} = 1_{[t,1+t]}$. Therefore, $\left\|T_t^{s\prime} 1_{[0,1]} - 1_{[0,1]}\right\|_{L^\infty(\mathbb{R},dx)} = 1$ for all $t > 0$, which shows that $(T_t^{s\prime})_{t\geq 0}$ is not strongly continuous.*

(ii) Similar as in the previous case, one can show that the adjoint of $(T_t^h)_{t\geq 0}$ given in Example 1.2(iii) is also not strongly continuous.

To overcome the lack of strong continuity we consider a weaker topology on X', namely the weak-$*$ topology.

Lemma 1.13. *The family of the adjoints $\left(T_t'\right)_{t\geq 0}$ of $(T_t)_{t\geq 0}$ fulfills (S1),(S3)and (S4) and the weak-$*$ version of (S2), i.e., $[0, \infty) \ni t \mapsto T_t' x' \in X'$ is continuous in the weak-$*$ topology on X'. Furthermore,* *the weak-∗ generator* $(A', D(A'))$ *of* $\left(T_t'\right)_{t\geq 0}$*, i.e.,*

$$D(A') := \left\{ x' \in X' \mid \lim_{t\searrow 0} \frac{1}{t}(T_t'x' - x') \text{ exists in the weak-}* \text{ topology of } X' \right\},$$
$$A'x' := \lim_{t\searrow 0} \frac{1}{t}(T_t'x' - x'), \quad x \in D(A')$$

is the adjoint of $(A, D(A))$*, where the limit in the previous line is taken in the weak-∗ topology of* X'.

Proof. The properties (S1),(S3) and (S4) for $\left(T_t'\right)_{t\geq 0}$ follow immediately from these for $(T_t)_{t\geq 0}$ as well as the weak-∗ continuity of $[0,\infty) \ni t \mapsto T_t'x' \in X'$. To see that the last assertion holds true, we only need to consider for $x' \in X'$ the map

$$D(A) \ni f \mapsto \langle Af, x' \rangle = \lim_{t\searrow 0} \left\langle f, \frac{T_t' - I}{t} x' \right\rangle.$$

□

Corollary 1.14. *If* X *is reflexive, then* $\left(T_t'\right)_{t\geq 0}$ *is a s.c.c.s. on* X'.

Proof. We only have to show that $\left(T_t'\right)_{t\geq 0}$ is strongly continuous on X'. Since X is assumed to be reflexive, the weak-∗ topology and the weak topology on X' coincide. Hence, by Lemma 1.3 the statement is proven. □

1.1.3 M-dissipativity and the Lumer-Phillips theorem

In subsection 1.1.1 we saw the theorem of Hille and Yosida characterizing the generator of an s.c.c.s.. A drawback of this result is that it is formulated in terms of the resolvent of the generator A. Often this characterization is not directly applicable since an explicit representation of the resolvent operators is not known. In general, and particularly in the cases we are interested in one merely knows a pre-domain of the a linear operator A and wants to determine whether A or rather its closure is the generator of a s.c.c.s. $(T_t)_{t\geq 0}$. Therefore, we present the theorem of Lumer-Phillips which is better suited for these cases.

For any $f \in X$ we define the set

$$\mathcal{T}_f := \left\{ x' \in X' \mid \langle f, x' \rangle = \|x'\|^2 = \|f\|^2 \right\}.$$

Note that due to the Hahn-Banach theorem the set $\mathcal{T}_f$ is non-empty for all $f \in X$.

Definition 1.15. *A linear operator* $(A, D(A))$ *on* X *is called dissipative, if for all* $f \in D(A)$ *there exists a* $x' \in \mathcal{T}_f$ *s.t.*

$$\operatorname{Re} \langle Af, x' \rangle \leq 0.$$

The next lemma and its proof is taken from [84].

Lemma 1.16. *Let* $(A, D(A))$ *be a linear operator on* X.

(i) $(A, D(A))$ *is dissipative if and only if for each* $\alpha > 0$ *and* $f \in D(A)$ *it holds* $\|(\alpha - A)f\| \geq \alpha \|f\|$.

(ii) If $(A, D(A))$ is dissipative and densely defined then it is closable and its closure $(\overline{A}, D(\overline{A}))$ is dissipative, too. Moreover, for each $\alpha > 0$ it holds $\overline{\mathcal{R}(\alpha - A)} = \mathcal{R}(\alpha - \overline{A})$.

(iii) Let $(A, D(A))$ be dissipative and $\mathcal{R}(\alpha_0 - A) = X$ for some $\alpha_0 > 0$. It follows that $(0, \infty) \subseteq \rho(A)$ and $\|G_\alpha\| = \left\|(\alpha - A)^{-1}\right\| \leq \alpha^{-1}$ for $\alpha \in (0, \infty)$. In particular, $(A, D(A))$ has no proper dissipative extension

Proof. (i) The following proof is taken from [84, Theorem 1.4.2]. Let $(A, D(A))$ be dissipative, $\alpha > 0$ and $f \in D(A)$. Choose $x' \in X'$ as in (1.15). Then

$$\|(\alpha - A)f\| \, \|f\| \geq \operatorname{Re} \langle(\alpha - A)f, x'\rangle \geq \alpha \|f\|^2 .$$

Conversely, let $\alpha > 0$, $f \in D(A)$. Wlog we may assume that $f \neq 0$. Now let $\tilde{x}'_\alpha \in \mathcal{T}_{(\alpha - A)f}$ and define $x'_\alpha := \frac{\tilde{x}'_\alpha}{\|\tilde{x}'_\alpha\|}$. Then it holds

$$\begin{aligned} \alpha \|f\| \leq \|(\alpha - A)f\| &= \left\langle(\alpha - A)f, x'_\alpha\right\rangle \\ &= \operatorname{Re}\left\langle \alpha f, x'_\alpha\right\rangle - \operatorname{Re}\left\langle Af, x'_\alpha\right\rangle \\ &\leq \alpha \|f\| - \operatorname{Re}\left\langle Af, x'_\alpha\right\rangle . \end{aligned}$$

Thus, we obtain $\operatorname{Re}\left\langle x'_\alpha, Af\right\rangle \leq 0$. By using the Banach-Alaoglu theorem we can find a subnet (x'_β) s.t. weak-$*$ $\lim_{\beta \to \infty} x'_\beta = x'$. It suffices now to show that $\|f\| \, x' \in \mathcal{T}_f$.

$$\begin{aligned} \|f\| \langle f, x'\rangle &= \|f\| \lim_{\beta \to \infty} \frac{1}{\beta} \left\langle \beta f - Af, x'_\beta\right\rangle + \lim_{\beta \to \infty} \frac{1}{\beta} \left\langle Af, x'_\beta\right\rangle \\ &= \|f\|^2 . \end{aligned}$$

(ii) To prove this part we follow the approach given in [66, Lemma 1.1]. Let $(0, g) \in \overline{\operatorname{graph}(A)}$. We need to show $g = 0$. Let $f \in D(A)$ be arbitrary and $(f_n, Af_n) \longrightarrow (0, g)$ as $n \longrightarrow \infty$ in $X \times X$. Then it holds for $\alpha > 0$ by part (i)

$$\|(\alpha - A)f - \alpha g\| = \lim_{n \to \infty} \|(\alpha - A)(\alpha f_n + f)\| \geq \lim_{n \to \infty} \alpha \|\alpha f_n + f\| = \alpha \|f\| .$$

By dividing first by $\alpha > 0$ and letting α tend to infinity we obtain $\|f - g\| \geq \|f\|$ for all f from the dense subset $D(A)$, which shows $g = 0$. The dissipativity of the closure $(\overline{A}, D(\overline{A}))$ follows immediately by part (i). To show the last assertion, we first observe that the inclusion $\mathcal{R}(\alpha - \overline{A}) \subseteq \overline{\mathcal{R}(\alpha - A)}$ holds by definition of the closure of an operator. The reverse inclusion holds, since the range $\mathcal{R}(\alpha - \overline{A})$ is closed due to part (i) and the closedness of $(\overline{A}, D(\overline{A}))$.

(iii) Let $\alpha_0 > 0$ s.t. $\mathcal{R}(\alpha_0 I - A) = X$. By part (i) it follows $\alpha_0 \in \rho(A)$ and $\left\|G_{\alpha_0}\right\| \leq \alpha_0^{-1}$. Via the Neumann series we obtain that for $\alpha_0 \in \rho(A)$ it follows $\alpha \in \rho(A)$ if $|\alpha - \alpha_0| < \frac{1}{\|G_{\alpha_0}\|}$. As this fraction is greater or equal than α_0, we have $(0, 2\alpha_0) \subseteq \rho(A)$. By proceeding inductively, we obtain $(0, \infty) \subseteq \rho(A)$ and the norm bound holds again by part (i). The last assertion also follows from part (i).

□

Before we proceed we present some examples of dissipative operators.

Example 1.17. *(i) Let $X = L^2(\mathbb{R}, dx)$. We identify $L^2(\mathbb{R}, dx)$ with its dual space via the complex conjugate Riesz isomorphism. Then for $f \in L^2(\mathbb{R}, dx)$, it holds $f \in \mathcal{T}_f$. Via integration by parts one easily obtains that the respective generators of the shift and heat semigroup on $L^2(\mathbb{R}, dx)$ are dissipative, see Example 1.5.*

(ii) This example is taken from [84, Example 1.4.7.]. Let $X = C([0,1])$ be the space of complex-valued continuous functions defined on the interval $[0,1]$ equipped with the supreme norm $\|\cdot\|_\infty$. Define the operator $(A, D(A))$ by $D(A) := \{f \in C^1([0,1]) \mid f(0) = 0\}$ and $Af := -f'$ for $f \in D(A)$. Let $\alpha \in (0,\infty)$ and $g \in C([0,1])$ be arbitrary. From the theory of ordinary differential equations and the variation of constants method, we obtain a unique $f \in D(A)$ s.t.

$$\alpha f - Af = g$$

$$f(t) = \int_0^t \exp(\alpha(s-t))g(s)\,ds, \quad t \in [0,1].$$

In particular we obtain

$$\|f\|_\infty \le \frac{1}{\alpha}(1 - \exp(-\alpha))\,\|g\|_\infty \le \frac{1}{\alpha}\,\|\alpha f - Af\|_\infty . \tag{1.6}$$

Hence, by Lemma 1.16(i) it holds that $(A, D(A))$ is dissipative.

The following lemma turns out to be useful when we consider the sum of dissipative operators.

Lemma 1.18. *Let $(A, D(A))$ be a densely defined dissipative operator on X. Then for $f \in D(A)$ it holds*

$$\mathrm{Re}\,\langle Af, x'\rangle \le 0, \text{ for all } x' \in \mathcal{T}_f.$$

Proof. See e.g. [11, A-II, Theorem 2.7.]. □

Definition 1.19. *We call a densely defined linear operator $(A, D(A))$ on X*

(i) m-dissipative, if $(A, D(A))$ is dissipative and $\mathcal{R}(\alpha - A) = X$ for one (hence all) $\alpha > 0$.

(ii) essentially m-dissipative, if the closure $(\overline{A}, D(\overline{A}))$ is m-dissipative.

The following remark, in particular the first part, is important. Since we use it later several times without mentioning it explicitly the reader should read it carefully.

Remark 1.20. *(i) The term m-dissipative originates from the expression maximal dissipative. Indeed, if $(A, D(A))$ is m-dissipative then we know by Lemma 1.16 that A is maximal w.r.t. the usual partial order on the set of linear dissipative operators, i.e. if $(B, D(B))$ is also dissipative and $(A, D(A)) \subseteq (B, D(B))$ then $(A, D(A)) = (B, D(B))$. The reverse implication is in general not true, see [69, Example p.688]. In case that X is a Hilbert space, the situation is different. Assume that $(A, D(A))$ is dissipative but not m-dissipative. Hence, there exists a $x \in \mathcal{R}(\alpha - A)^\perp \setminus \{0\}$. Assume for the sake of a contradiction that $x \in D(A)$. By the choice of x it holds*

$$(Ax, x) = \alpha\,\|x\|^2 > 0.$$

But this violates the dissipativity of $(A, D(A))$ meaning that $x \notin D(A)$. Therefore, we define the following proper linear extension

$$\begin{aligned} D(\hat{A}) &= span(D(A) \cup \{x\}), \\ \hat{A} &= A, \text{ on } D(A), \\ \hat{A}x &= -x. \end{aligned}$$

Using Lemma 1.16(i) it easily follows that $(\hat{A}, D(\hat{A}))$ is dissipative, i.e., $(A, D(A))$ is not maximal dissipative.

(ii) By Lemma 1.16(ii) a dissipative operator (A, D) is essentially m-dissipative if $(\alpha - A)D$ is dense in X for some (hence all) $\alpha > 0$.

Theorem 1.21 (Lumer-Phillips)**.** *An operator $(A, D(A))$ is the generator of an s.c.c.s. if and only if it is m-dissipative.*

Proof. Let $(A, D(A))$ generate a s.c.c.s. $(T_t)_{t\geq 0}$. Let $f \in D(A)$ and $x' \in \mathcal{T}_f$. Then, the following argument shows that A is dissipative

$$\operatorname{Re}\langle Af, x'\rangle = \lim_{t\searrow 0} \frac{1}{t} \operatorname{Re}\langle T_t f - f, x'\rangle \leq \lim_{t\searrow 0} \frac{1}{t}(\|f\|\,\|T_t f\| - \|f\|^2) \leq 0. \tag{1.7}$$

Moreover, by Theorem 1.10 it holds $(0, \infty) \subseteq \rho(A)$ which shows that $(A, D(A))$ is m-dissipative.

Now let $(A, D(A))$ be m-dissipative. Then, by Definition 1.19 and Lemma 1.16(i),(iii) it results $(0, \infty) \subseteq \rho(A)$ and $\alpha \left\|(\alpha - A)^{-1}\right\| \leq 1$ for all $\alpha \in (0, \infty)$. Hence, by Theorem 1.10 it holds that $(A, D(A))$ is the generator of an s.c.c.s. on X. □

Remark 1.22. *The calculation in (1.7) shows that the weak generator of a s.c.c.s. $(T_t)_{t\geq 0}$ defined by*

$$\begin{aligned} D(\tilde{A}) &:= \left\{ f \in X \mid \textit{weak-}\lim_{t\searrow 0} \frac{1}{t}(T_t f - f) \text{ exists in the weak-topology of } X \right\}, \\ \tilde{A}f &:= \textit{weak-}\lim_{t\searrow 0} \frac{1}{t}(T_t f - f), \end{aligned}$$

is a dissipative extension of the generator $(A, D(A))$ of $(T_t)_{t\geq 0}$. Hence by 1.20 it holds $(A, D(A)) = (\tilde{A}, D(\tilde{A}))$.

By combining Remark 1.22, Lemma 1.13 and 1.14 we obtain the following corollary.

Corollary 1.23. *Let X be reflexive and $(T_t)_{t\geq 0}$ a s.c.c.s. on X. The adjoint of the generator $(A, D(A))$ of $(T_t)_{t\geq 0}$ is the (strong) generator of the adjoint semigroup $\left(T_t'\right)_{t\geq 0}$.*

An important feature of the Lumer-Phillips theorem is that it can also be seen as a uniqueness result. This is the content of the next theorem. A proof can be found in [11, Theorem A-II.1.32].

Theorem 1.24. *Suppose there exists a s.c.c.s. $(T_t)_{t\geq 0}$ on X such that its generator $(A, D(A))$ extends the operator (A, D). Then the following assertions are equivalent.*

(i) D is a core for $(A, D(A))$.

(ii) *The closure of* (A, D) *is the generator of a s.c.c.s..*

(iii) $(T_t)_{t\geq 0}$ *is the only s.c.c.s. on* X *which has a generator that extends* (A, D)*.*

We collect some further corollaries of the Lumer-Phillips theorem which turn out to be useful every so often.

Corollary 1.25. *Let* (A, D) *be dissipative. Then the closure* $(A, D(A))$ *is m-dissipative if and only if* $\mathcal{N}(\alpha - A') = \{0\}$ *for some* $\alpha > 0$*.*

Corollary 1.26. *Let* (A, D) *be dissipative. Assume further that* $(A', D(A'))$ *is also dissipative. Then, the closure* $(A, D(A))$ *is m-dissipative.*

Theorem 1.24 indicates that for the uniqueness considerations it is important to decide whether certain spaces form a core of a generator. The following theorem gives a sufficient condition for a subset to be a core. The proof uses results about the abstract Cauchy problem related to a given generator.

Theorem 1.27. *Assume that* $(T_t)_{t\geq 0}$ *is a semigroup with generator* $(A, D(A))$*. Let* D *be a dense subset of* X *s.t.* $T_t D \subseteq D$ *for all* $t \geq 0$*. Then* D *is a core for* $(A, D(A))$*.*

Proof. See [11, Corollary A-II.1.34.]. □

The proof of the next theorem is almost trivial and therefore left out.

Theorem 1.28. *Let* Y *be another Banach space. Assume that* $U : X \longrightarrow Y$ *is an isometric isomorphism and* $(A, D(A))$ *is a densely defined operator on* X*. Then the following holds:*

(i) $(A, D(A))$ *is dissipative if and only if* $\left(UAU^{-1}, UD(A)\right)$ *is dissipative on* Y*.*

(ii) $(A, D(A))$ *is m-dissipative if and only if* $\left(UAU^{-1}, UD(A)\right)$ *is m-dissipative on* Y*.*

(iii) $(A, D(A))$ *is essentially m-dissipative if and only if* $\left(UAU^{-1}, UD(A)\right)$ *is essentially m-dissipative on* Y*. In this case* $\left(U\overline{A}U^{-1}, UD(\overline{A})\right)$ *is the unique m-dissipative extension of* $\left(UAU^{-1}, UD(A)\right)$*.*

From an analytic point of view it is sometimes advantageous to work with complex spaces. Therefore the concept of complexification is helpful to switch from the real to the complex setting. The Lemma 1.30 shows that there is no loss of generality, when it comes to the questions concerning m-dissipativity.

Definition 1.29. *Let* X *be a real vector space and define* $X_{\mathbb{C}} = X \times X$*. For* $[x_1, y_1], [x_2, y_2] \in X_{\mathbb{C}}$ *and* $a, b \in \mathbb{R}$ *we define*

$$[x_1, y_1] + [x_2, y_2] := [x_1 + x_2, y_1 + y_2], \tag{1.8}$$

$$(a + ib)[x_1, y_1] := [ax_1 - by_1, ay_1 + bx_1]. \tag{1.9}$$

If $\|\cdot\|_X$ *is a norm on* X*, we define a norm on* $X_{\mathbb{C}}$ *by*

$$\|[x_1, y_1]\|_{X_{\mathbb{C}}} := \sqrt{\|x_1\|_X^2 + \|y_1\|_X^2}. \tag{1.10}$$

In case the norm $\|\cdot\|_X$ is induced by a scalar product $(\cdot,\cdot)_X$, then the norm $\|\cdot\|_{X_\mathbb{C}}$ arises from the scalar product

$$([x_1, y_1], [x_2, y_2])_{X_\mathbb{C}} = (x_1, x_2)_X + (y_1, y_2)_X + i(y_1, x_2)_X - i(x_1, y_2)_X.$$

It is easy to check that the complexification $X_\mathbb{C}$ of a real Banach space X is a complex Banach space equipped with the addition and scalar multiplication (1.8), (1.9) and norm (1.10) .

Let $(A, D(A))$ denote a linear operator on a real Banach space X. Define its complexification $(A_\mathbb{C}, D(A_\mathbb{C}))$ by $D(A_\mathbb{C}) = D(A) \times D(A$ and $A_\mathbb{C}[x, y] = [Ax, Ay]$ for $x, y \in D(A)$, i.e., $(A_\mathbb{C}, D(A_\mathbb{C}))$ is a linear operator on the complex Banach space $X_\mathbb{C}$.

By using the characterization of dissipativity from Lemma 1.16(i) one can easily proof the first part of the following lemma. The remaining parts follow just by the definition of the complexification and therefore we omit the proof.

Lemma 1.30. *Let $(A, D(A))$ denote a linear operator on a real Banach space X. The following holds*

(i) $(A, D(A))$ is dissipative if and only if $(A_\mathbb{C}, D(A_\mathbb{C}))$ is dissipative.

(ii) $(A, D(A))$ is m-dissipative if and only if $(A_\mathbb{C}, D(A_\mathbb{C}))$ is m-dissipative.

(iii) $(A, D(A))$ is essentially m-dissipative if and only if $(A_\mathbb{C}, D(A_\mathbb{C}))$ is essentially m-dissipative.

1.1.4 The Hilbert space case and Self-adjointness

In this section we briefly compare the concepts from the previous subsections to the special case where X is a Hilbert space and the concept of self-adjointness. Hence, in the following let $(X, (\cdot,\cdot))$ be a Hilbert space. For a linear operator $(A, D(A))$ on X, we denote its Hilbert space adjoint by $(A^*, D(A^*))$. Recall that a linear operator $(A, D(A))$ is called symmetric if it is a restriction of its adjoint and $(A, D(A))$ is called self-adjoint if it coincides with its adjoint $(A^*, D(A^*))$. Further $(A, D(A))$ is called negative definite if $(Af, f) \le 0$ for all $f \in D(A)$.

Observe that for $f \in X$ the set $\mathcal{T}_f$ is single-valued since X is a Hilbert space which is strictly convex, see also [47, Chapter 1, Exercise 3.10.2.]. Hence, a negative definite operator is dissipative. Furthermore, if $(A, D(A))$ is symmetric and dissipative, then A is negative definite. The following lemma is a direct consequence of Lemma 1.13 and Corollary 1.14.

Lemma 1.31. *Let $(T_t)_{t\ge 0}$ be a s.c.c.s. with generator $(A, D(A))$ on X. Then T_t is symmetric for all $t \ge 0$ if and only if $(A, D(A))$ is self-adjoint.*

Theorem 1.32. *Let $(A, D(A))$ be a densely defined, symmetric and negative definite operator on X. Then $(A, D(A))$ is self-adjoint if and only if it is m-dissipative.*

Proof. First let $(A, D(A))$ be self-adjoint. Since A is dissipative and self-adjoint, it holds in particular that the adjoint $A^* = A$ is dissipative, too. Hence, by Corollary 1.26 $(A, D(A))$ is m-dissipative. Now, let $(A, D(A))$ be m-dissipative and denote the corresponding semigroup by $(T_t)_{t\ge 0}$. Hence, A is maximal dissipative. Observe that the symmetry of A just means that A is a restriction of its adjoint A^*. By Remark 1.22, we know that A^* generates the dual semigroup $\left(T_t^*\right)_{t\ge 0}$ and is therefore necessarily dissipative. Eventually, we conclude that $(A, D(A))$ is no proper restriction of $(A^*, D(A^*))$. □

Corollary 1.33. *Let (A, D) be a densely defined, symmetric and negative definite operator on X. Then (A, D) is essentially self-adjoint, i.e., the closure of (A, D) is self-adjoint, if and only if (A, D) is essentially m-dissipative.*

Definition 1.34. *Let $(U_t)_{t\in\mathbb{R}}$ be a family of linear an continuous operators on X. We call $(U_t)_{t\in\mathbb{R}}$ a unitary strongly continuous group (u.s.c.g.) if*

(G1) $U_0 = I$,

(G2) $U_tU_s = U_{t+s}$, *for all* $t, s \in \mathbb{R}$,

(G3) $\mathbb{R} \ni t \mapsto U_t \in L(X)$ *is strongly continuous,*

(G4) U_t *is unitary for all* $t \in \mathbb{R}$.

If $(U_t)_{t\in\mathbb{R}}$ is a u.s.c.g. Then, it holds $U_t^* = U_{-t}$ for all $t \in \mathbb{R}$. We define the generator of $(U_t)_{t\in\mathbb{R}}$ as in Definition 1.4. Recall that a linear operator $(A, D(A))$ is called skew-adjoint if $(-A, D(A)) = (A^*, D(A^*))$. We can combine our results from above to obtain Stone's theorem:

Theorem 1.35. *A linear operator $(A, D(A))$ is the genrator of a u.s.c.g. $(U_t)_{t\in\mathbb{R}}$ if and only if $(A, D(A))$ is skew-adjoint.*

Proof. Let $(U_t)_{t\in\mathbb{R}}$ be a u.s.c.g. and $(A, D(A))$ its generator. We define the s.c.c.s. $\left(U_t^-\right)_{t\geq 0}$ by $U_t^- := U_{-t}$, $t \geq 0$. It is easy to check that the generator of $\left(U_t^-\right)_{t\geq 0}$ equals $(-A, D(A))$. Since $\left(U_t^-\right)_{t\geq 0} = \left(U_t^*\right)_{t\geq 0}$ it holds by Corollary 1.14 that $(-A, D(A)) = (A^*, D(A^*))$. Now let $(A, D(A))$ be skew-adjoint. Then it holds that $(A, D(A))$ and its adjoint $(A^*, D(A^*))$ are dissipative. Hence, from Corollary 1.26 it follows that $(A, D(A))$ is m-dissipative and denote by $(U_t)_{t\geq 0}$ the s.c.c.s. generated by A. Furthermore, for $t \leq 0$ we define $U_t := U_t^*$. Then $(U_t)_{t\in\mathbb{R}}$ satisfies (G1) and (G3). Further, for $f \in D(A)$ we obtain by differentiating that $U_tU_t^*f = U_t^*U_tf$ is constant and therefore equals f. Hence, (G4) holds true and (G2) follows directly from (G4) since we have the semigroup property for $(U_t)_{t\geq 0}$ and $\left(U_t^*\right)_{t\geq 0}$. □

Remark 1.36. *Assume that X is a complex space. Then $(A, D(A))$ is skew-adjoint if and only if $(iA, D(A))$ is self-adjoint. Therefore, in this case one could also prove Stone's theorem by using the spectral theorem for self-adjoint operators, see also the proof of Theorem 7.20.*

1.2 Perturbation Theory for m-dissipative Operators

In this section we assume again that X is a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Later, we aim to establish m-dissipativity of operators $(C, D(C))$ given in the form

$$C = A + B, \quad D(C) = D(A) \subseteq D(B),$$

knowing that $(A, D(A))$ is m-dissipative. If the pertubation B is in some sense small with respect to A, then m-dissipativity is preserved unter the pertubation B. Below, we state more precisely what is meant by a small perturbation.

Definition 1.37. *Let $(A, D(A))$, $(B, D(B))$ be linear operators on X. The operator $(B, D(B))$ is called*

$A-$bounded if $D(A) \subseteq D(B)$ and there exist constants $a, b < \infty$ such that

$$\|Bf\|_X \leq a\,\|Af\|_X + b\,\|f\|_X\,, \quad f \in X. \tag{1.11}$$

The number $\inf\{a \in \mathbb{R} \mid (1.11)$ *holds for some* $b\}$ *is called the A-bound of B.*

Theorem 1.38. *Let D be a dense subspace of X and (A, D) an essentially m-dissipative operator. Let (B, D) be a dissipative operator which is A-bounded with the A-bound being strictly less than 1. Denote by $\left(\overline{A}, D(\overline{A})\right)$, $\left(\overline{B}, D(\overline{B})\right)$ the closures of (A, D), (B, D), respectively which exist due to Lemma 1.16(ii). Then, $(A + B, D)$ is essentially m-dissipative and its closure is given by $\left(\overline{A} + \overline{B}, D(\overline{A})\right)$.*

The next lemma follows immediately from the well-known inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, $a, b \in \mathbb{R}$.

Lemma 1.39. *Assume that $(X, (\cdot, \cdot))$ is a Hilbert space. Let (A, D) be an essentially m-dissipative operator and (B, D) a dissipative operator on X. Also, assume that there exist $c, d < \infty$ such that*

$$\|Bf\|^2 \leq c\,|(Af, f)| + d\,\|f\|^2$$

holds for all $f \in X$. Then B is $A-$bounded with $A-$bound equal to 0.

Definition 1.40. *A sequence $(P_n)_{n\in\mathbb{N}} \subseteq L(X)$ is called a complete othogonal family, if P_n, $n \in \mathbb{N}$, satisfies $P_n^2 = P_n$ and it holds $P_nP_m = 0$, if $n \neq m$, and $\sum_{n=1}^{\infty} P_n = I$ holds in the strong operator topology.*

Theorem 1.41. *Let (A, D) be an essentially m-dissipative operator and (B, D) a dissipative operator on X. Assume there exists a complete orthogonal family $(P_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ and $f \in D$, it holds*

$$\begin{aligned} P_nD &\subseteq D, \\ P_nAf &= AP_nf, \\ P_nBf &= BP_nf. \end{aligned}$$

Define $A_n = AP_n$ and $B_n = BP_n$ both with domain $D_n = P_nD \subseteq (P_nX) \cap D$ as operators on P_nX. Assume further that each B_n, $n \in \mathbb{N}$, is A_n-bounded with A_n-bound being less than 1. Then $(A + B, D)$ is essentially m-dissipative.

Proof. See [25, Lemma 3] where a proof is given for the case that X is a Hilbert space. Fortunately, the proof does not rely on the fact that the norm is induced by a scalar product. □

1.3 Sub-Markovian Semigroups

In this section we assume that the Banach space X under consideration is given as $X = L^p(F, \mu) := L^p(F, \mathcal{B}, \mu)$, where $p \in [1, \infty)$ and $(F, \mathcal{B}, \mu)$ is a σ-finite measure space. We assume that all elements $f \in L^p(F, \mu)$ are given as equivalence classes of real-valued functions. We deal with a certain subclass of s.c.c.s. on $L^p(F, \mu)$ which plays an important role in the connection of operator semigroup theory and probability theory. We use the usual partial orderings $\leq, \geq, <, >$ and $=$ on $L^p(F, \mathcal{B}, \mu)$, i.e., $f \leq g$ if for some respective representatives $\hat{f}, \hat{g}$ of f and g it holds $\hat{f} \leq \hat{g}$ on

F. The remaining orderings are defined in an analogous way. Furthermore, for $f \in L^p(F, \mathcal{B}, \mu)$ with representative $\hat{f}$ we denote by f^+, f^- and $|f|$ the equivalence classes of $\hat{f}^+$, $\hat{f}^-$ and $|\hat{f}|$, respectively. For a subset $S \subseteq L^p(F, \mu)$ we denote by S^+ the set $\{f^+ \mid f \in S\}$.

1.3.1 Definition and Basic Properties

Definition 1.42. *Let $(A, D(A))$ be a closed densely defined linear operator on $L^p(F, \mu)$, $p \in [1, \infty)$ and T be a linear operator on $L^{p'}(F, \mu)$, $p' \in [1, \infty]$, which is not necessarily continuous.*

(i) $(A, D(A))$ is called a Dirichlet operator if

$$\int_F Au((u-1)^+)^{p-1}\, d\mu \leq 0, \qquad \forall u \in D(A). \tag{1.12}$$

If $(A, D(A))$ is merely densely defined and fulfills (1.12) we call $(A, D(A))$ a pre-Dirichlet operator.

(ii) T is called positive or positive preserving if for all $f \in L^{p'}(F, \mu)$ with $0 \leq f$ it holds $0 \leq Tf$.

(iii) T is called sub-Markovian if for all $f \in L^{p'}(F, \mu)$ with $f \leq 1$ it holds $Tf \leq 1$.

(iv) A s.c.c.s. $(T_t)_{t \geq 0}$ (a s.c.c.r. $(G_\alpha)_{\alpha > 0}$) is called sub-Markovian or positive if T_t (αG_α) is sub-Markovian or positive for every $t \geq 0$.

Example 1.43. *The shift and heat semigroups given in Example 1.2(ii),(iii) are sub-Markovian.*

Lemma 1.44. *Let T be a linear operator on $X = L^p(F, \mu)$, $p \in [1, \infty)$ which is not necessarily continuous.*

(i) If T is positive, then T is continuous. In particular, the adjoint $T' \in L(L^q(X, \mu))$, where q is the Hölder conjugate of p, is positive, too.

(ii) If T is sub-Markovian, then T is positive.

Proof. The proof is taken from [10].

(i) Assume that T is positive. Hence, for all $f \in L^p(F, \mu)$ it holds $|Tf| \leq T|f|$ and accordingly $\|Tf\| \leq \|T|f|\|$. Since $\|f\| = \||f|\|$ we only need to show the existence of a finite positive constant C s.t. $\|Tf\| \leq C\|f\|$ for all non-negative elements $f \in L^p(F, \mu)$. Assume that such a constant does not exist. Hence, there exist non-negative $f_n \in L^p(F, \mu)$, $n \in \mathbb{N}$, s.t. $\|f_n\| \leq 2^{-n}$ and $\|Tf_n\| \geq n$. We define $f := \sum_{n=1}^{\infty} f_n \in X$. Then f is non-negative and for all $n \in \mathbb{N}$ it holds $0 \leq Tf_n \leq Tf$. Since T is positive, we have $n \leq \|Tf_n\| \leq \|Tf\|$. Knowing that Tf is an element in X and has finite norm results in a contradiction. This proves continuity. The second assertion can been seen as follows. Since $(F, \mathcal{B}, \mu)$ is σ-finite it suffices to show $\int_F T^* f 1_E\, d\mu \geq 0$ if $f \geq 0$ and $E \in \mathcal{B}$ s.t. $\mu(E) < \infty$. But this is obviously the case, since T is positive.

(ii) Let $f \leq 0$. Then, $nf \leq 1$ for all $n \in \mathbb{N}$ and further $nTf = T(nf) \leq 1$ for all $n \in \mathbb{N}$. Hence, $Tf \leq 0$ which proves positivity.

□

Remark 1.45. *In the preceding lemma we have seen that the adjoint of a positive operator T is again positive. In general, the adjoint of a sub-Markovian operator is not sub-Markovian. This can be seen by the following simple example. For certain choices of $(F, \mathcal{B}, \mu)$ it holds $L^p(F, \mu) \cong \mathbb{R}^2$. We identify a linear operator T on $\mathbb{R}^2$ with a matrix M w.r.t. the standard orthonormal basis on $\mathbb{R}^2$. Then, T is sub-Markovian if and only if the sum of the entries in every row of M is less or equal than 1. In particular, the operator corresponding to $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is therefore sub-Markovian. Its adjoint corresponds to $M^\intercal$ and is not sub-Markovian.*

Lemma 1.46. *Let $(A, D(A))$ be the generator of a s.c.c.s $(T_t)_{t\geq 0}$ on $L^p(F, \mu)$, $p \in [1, \infty)$ and $(G_\alpha)_{\alpha>0}$ be the corresponding s.c.c.r.. The following statements are equivalent:*

(i) *$(T_t)_{t\geq 0}$ is sub-Markovian.*

(ii) *$(G_\alpha)_{\alpha>0}$ is sub-Markovian, i.e., αG_α is sub-Markovian for all $\alpha > 0$.*

(iii) *$(A, D(A))$ is a Dirichlet operator.*

Proof. The proof of [70, Proposition I.4.3.] can be adapted to the general L^p setting, $p \neq 2$. □

Remark 1.47. *A similar result as in Lemma 1.46 also holds for positive preserving s.c.c.s. $(T_t)_{t\geq 0}$. Indeed, using the Laplace transform representation $G_\alpha = \int_0^\infty e^{-\alpha t} T_t \, dt$, we see that G_α is positive if T_t is positive for all $t \geq 0$. To prove the reverse implication one just uses the following approximation of T_t, i.e., it holds for all $f \in X$*

$$T_t f = \lim_{n\to\infty} \left(\frac{n}{t} G_{\frac{n}{t}} \right)^n f, \tag{1.13}$$

see e.g. [58, §2, Section 11.8]. An equivalent condition in terms of the generator is given by

$$\int_F Au(u^+)^{p-1} \, d\mu \leq 0, \qquad \forall u \in D(A).$$

For a proof see [71, Theorem 1.7.].

1.3.2 Extension and Interpolation of sub-Markovian Semigroups

The following interpolation theorem of Riesz and Thorin is essential in the upcoming considerations.

Theorem 1.48. *Let $(E, \mathcal{A}, \nu)$ be another σ-finite measure space and $p_0, p_1, q_0, q_1 \in [1, \infty]$. Suppose that $T : L^{p_0}(F, \mu) \cap L^{p_1}(F, \mu) \longrightarrow L^{q_0}(E, \nu) \cap L^{q_1}(E, \nu)$ is linear and fulfills $\|Tf\|_{L^{q_0}} \leq C_0 \|f\|_{L^{p_0}}$ and $\|Tf\|_{L^{q_1}} \leq C_1 \|f\|_{L^{p_1}}$ for all $f \in L^{p_0}(F, \mu) \cap L^{p_1}(F, \mu)$. Define for $t \in (0, 1)$ the numbers $p_t^{-1} := \frac{t}{p_0} + \frac{1-t}{p_1}$, $q_t^{-1} := \frac{t}{q_0} + \frac{1-t}{q_1}$ and $C_t := C_0^t C_1^{1-t}$. Then for each $f \in L^{p_0}(F, \mu) \cap L^{p_1}(F, \mu)$ it holds $Tf \in L^{q_t}(F, \nu)$ and $\|Tf\|_{L^{q_t}} \leq C_t \|f\|_{L^{p_t}}$.*

Proof. See e.g. [89, Theorem IX.17]. □

The following theorem and its proof are a slight modification of [24, Lemma 1.3.11]. We state some parts of the proof here, since they are important in the forthcoming argumentation.

Theorem 1.49. *Let $(T_t)_{t\geq 0}$ be a s.c.c.s. on $L^p(F,\mu)$, $p\in[1,\infty)$, with generator $(A,D(A))$.*

(i) Assume that the adjoint semigroup $\left(T_t^*\right)_{t\geq 0}$ is sub-Markovian. Then for all $t\geq 0$ it holds that

$$\tilde{T}_t : L^p(F,\mu)\cap L^1(F,\mu) \longrightarrow L^p(F,\mu)\cap L^1(F,\mu),\, f\mapsto T_t f \tag{1.14}$$

is also bounded w.r.t. $\|\cdot\|_{L^1}$. Furthermore, for all $r\in[1,p]$ there exists a sub-Markovian operator $T_{t,r}\in L(L^r(F,\mu))$ such that $T_{t,r}f=T_tf$ for all $f\in L^p(F,\mu)\cap L^1(F,\mu)$. The family $\left(T_{t,r}\right)_{t\geq 0}$ is a s.c.c.s. on $L^r(F,\mu)$.

(ii) Assume $(T_t)_{t\geq 0}$ is sub-Markovian. Then for all $t\geq 0$ it holds that

$$\tilde{T}_t : L^p(F,\mu)\cap L^\infty(F,\mu) \longrightarrow L^p(F,\mu)\cap L^\infty(F,\mu),\, f\mapsto T_t f \tag{1.15}$$

has a bounded linear extension to $T_{t,\infty}: L^\infty(F,\mu)\longrightarrow L^\infty(F,\mu)$. Furthermore, for all $r\in[p,\infty)$ there exists a sub-Markovian bounded linear operator $T_{t,r}\in L(L^r(F,\mu))$ such that $T_{t,r}f=T_tf$ for all $f\in L^p(F,\mu)\cap L^\infty(F,\mu)$. For $r\in[p,\infty]$ the family $\left(T_{t,r}\right)_{t\geq 0}$ fulfills (S1), (S3) and (S4) from Definition 1.1 and additionally (S2) if $r<\infty$. If $r=\infty$ we obtain that $\left(T_{t,\infty}\right)_{t\geq 0}$ is weak-$*$-continuous, i.e., for all $g\in L^1(F,\mu)$ and $f\in L^\infty(F,\mu)$ it holds

$$\lim_{t\to 0}\left\langle g, T_{t,\infty}f\right\rangle = \langle g,f\rangle, \tag{1.16}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $L^1(F,\mu)$ and $L^\infty(F,\mu)$.

(iii) Define for $r\in[1,\infty)$ the set $D(A)_r := \{f\in D(A)\mid f, Af\in L^r(F,\mu)\}$ as well as the set $D(A)_r^\infty := \{f\in D(A)_r\cap L^\infty(F,\mu)\}$. The generator of $\left(T_{t,r}\right)_{t\geq 0}$ is the closure of $(A,D(A)_r)$ as an operator on $L^r(F,\mu)$. In the situation of (i) (respectively (ii)) the set $D(A)_r$ is a core for the generator of $\left(T_{t,r}\right)_{t\geq 0}$, $r\in[1,p]$ ($r\in[p,\infty)$). In case $(T_t)_{t\geq 0}$ is sub-Markovian the corresponding statements with $D(A)_r^\infty$ instead of $D(A)_r$ hold true.

Proof. We only give a proof for part (i) and (ii).

(i) We need to show that $\tilde{T}_t$ is contractive w.r.t. $\|\cdot\|_{L^1}$. Let $f\in L^p(F,\mu)\cap L^1(F,\mu)$ s.t. $f\geq 0$ and $\psi_n\in L^1(f,\mu)\cap L^\infty(f,\mu)$, $n\in\mathbb{N}$, s.t. $0\leq\psi_n\nearrow 1$ as $n\to\infty$. By Lemma 1.44 we know that T_t is positive. Thus, it holds by the monotone convergence theorem

$$\int_F T_tf\,d\mu = \lim_{n\to\infty}\int_F \psi_n T_t f\,d\mu = \lim_{n\to\infty}\int_F T_t^*\psi_n f\,d\mu \leq \int_F f\,d\mu.$$

Thus, by linearity we obtain for arbitrary $f=f^+-f^-\in L^p(F,\mu)\cap L^1(F,\mu)$

$$\int_F |T_tf|\,d\mu \leq \int_F T_tf^+\,d\mu + \int_F T_tf^-\,d\mu \leq \int_F f^+\,d\mu + \int_F f^-\,d\mu = \int_F |f|\,d\mu.$$

Hence, by the Riesz-Thorin theorem and an extension we obtain for $r\in[1,p]$ that there exists a contraction $T_{t,r}\in L(L^r(F,\mu))$ such that $T_{t,r}f=T_tf$ for all $f\in L^p(F,\mu)\cap L^1(F,\mu)$. It follows immediately that $T_{t,r}$ is sub-Markovian. By the three line theorem (cf. also the proof of [89, Appendix to IX.4, Proposition 2]) it holds

$$\|\cdot\|_{L^r} \leq \|\cdot\|_{L^p}^t\,\|\cdot\|_{L^1}^{1-t} \quad\text{for some } t\in(0,1) \tag{1.17}$$

where t depends on r. From this inequality and a 3ε-argument we see that $\left(T_{t,r}\right)_{t\geq 0}$ is strongly continuous on $L^r(F,\mu)$ if $\left(T_{t,1}\right)_{t\geq 0}$ is strongly continuous on $L^1(F,\mu)$. To show this it suffices to prove that $T_{t,1}f \longrightarrow f$ in $L^1(F,\mu)$ as $t\to 0$ for $f\in L^p(F,\mu)\cap L^1(F,\mu)$. Since $T_{t,1}$ is a contraction, it suffices to prove $T_{t,1}f \longrightarrow f$ for $f\in L^p(F,\mu)\cap L^1(F,\mu)$ as $t\to 0$. To this end, let $(t_n)_{n\in\mathbb{N}}\subseteq[0,\infty)$ be a zero sequence. By the strong continuity on $L^p(F,\mu)$, we may choose a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ s.t. $T_{t_{n_k}}f\longrightarrow f$ μ-a.e. as $k\to\infty$. Thus, by the dominated convergence theorem it holds $1_{\left\{|T_{t_{n_k}}f|\leq 2|f|\right\}}T_{t_{n_k}}f \longrightarrow f$ in $L^1(F,\mu)$ as $k\to\infty$. Furthermore, we have

$$\begin{aligned}\|f\|_{L^1} &\geq \limsup_{k\to\infty}\left(\left\|1_{\left\{|T_{t_{n_k}}f|\leq 2|f|\right\}}T_{t_{n_k}}f\right\|_{L^1}+\left\|1_{\left\{|T_{t_{n_k}}f|> 2|f|\right\}}T_{t_{n_k}}f\right\|_{L^1}\right)\\ &=\|f\|_{L^1}+\limsup_{k\to\infty}\left\|1_{\left\{|T_{t_{n_k}}f|> 2|f|\right\}}T_{t_{n_k}}f\right\|_{L^1}.\end{aligned}$$

This implies that $T_{t_{n_k}}f = 1_{\left\{|T_{t_{n_k}}f|\leq 2|f|\right\}}T_{t_{n_k}}f + 1_{\left\{|T_{t_{n_k}}f|> 2|f|\right\}}T_{t_{n_k}}f \longrightarrow f$ in $L^1(F,\mu)$ as $k\to\infty$ which proves strong continuity.

(ii) Observe that the linear operator $\tilde{T}_t$ in (1.15) is well-defined since sub-Markovian operators are contractions w.r.t. $\|\cdot\|_{L^\infty}$. Now let $0\leq f\in L^\infty(F,\mu)$. Since (F,μ) is σ-finite we can choose a sequence $(f_n)_{n\in\mathbb{N}}\subseteq L^p(F,\mu)$ s.t. $0\leq f_n\nearrow f$. Hence, the limit $T_{t,\infty}f:=\lim_{n\to\infty}T_tf_n$ exists μ-a.e. in $\mathbb{R}$ since T_t is sub-Markovian. This limit is also well-defined, i.e., if $(g_n)_{n\in\mathbb{N}}$ is another sequence s.t. $0\leq g_n\nearrow f$ then $\lim_{n\to\infty}T_tf_n=\lim_{n\to\infty}T_tg_n$. It is obvious that $T_{t,\infty}$ is contraction. The existence of the operators $T_{t,r}$ for $r\in[p,\infty)$ follows immediately by the Riesz-Thorin theorem and the BLT theorem. The properties (S1), (S3), (S4) and the sub-Markovianity for $\left(T_{t,r}\right)_{t\geq 0}$ for $r\in[p,\infty]$ follow directly by those of $(T_t)_{t\geq 0}$.

To show the strong continuity in the case $r\in(p,\infty)$ one uses the estimate (1.17) and a 3ε-argument.

To show the weak-$*$-continuity for $r=\infty$ we make the following observation. For $f\in L^\infty(F,\mu)$ the map $t\mapsto T_{t,\infty}f$ is bounded. Thus, it suffices to show (1.16) for $g\in L^1(F,\mu)\cap L^\infty(F,\mu)$. Now choose ψ_n, $n\in\mathbb{N}$, as in the proof of part (i) and $\tilde{p}\in(p,\infty)$, in particular, $\tilde{p}>1$. Then, by Lemma 1.14 we obtain that $\left((T_{t,\tilde{p}})^*\right)_{t\geq 0}$ is a s.c.c.s. on $L^{\frac{\tilde{p}}{\tilde{p}-1}}(F,\mu)$ with sub-Markovian adjoint semigroup. Hence, we can apply part (i), i.e., $(T_{t,\tilde{p}})^*$ is strongly continuous w.r.t. $\|\cdot\|_{L^1}$. Eventually, by the dominated convergence theorem and the construction of $T_{t,\infty}$, it holds

$$\left\langle g,T_{t,\infty}f\right\rangle=\lim_{n\to\infty}\left\langle g,T_{t,\tilde{p}}(\psi_nf)\right\rangle=\lim_{n\to\infty}\left\langle (T_{t,\tilde{p}})^*g,\psi_nf\right\rangle=\left\langle (T_{t,\tilde{p}})^*g,f\right\rangle\overset{t\to 0}{\longrightarrow}\langle g,f\rangle.$$

□

Corollary 1.50. *Let $(T_t)_{t\geq 0}$ be a sub-Markovian s.c.c.s. on $L^p(F,\mu)$, $p\in[1,\infty)$. Let $\left(T_{t,\infty}\right)_{t\geq 0}$ be the semigroup on $L^\infty(F,\mu)$ constructed in Theorem 1.49(ii). For every $t\in[0,\infty)$ the adjoint $T^*_{t,\infty}$ of $T_{t,\infty}$ leaves $L^1(F,\mu)$ invariant. In particular, $\left(T^*_{t,\infty|L^1}\right)_{t\geq 0}$ is a positive s.c.c.s. on $L^1(F,\mu)$.*

Proof. First observe that for an element $f\in L^1(F,\mu)\subseteq(L^\infty(F,\mu))'$ it holds $\|f\|_{(L^\infty)'}=\|f\|_{L^1}$. Now, let $f\in L^1(F,\mu)\cap L^{\frac{p}{p-1}}(F,\mu)$, $g\in L^\infty(F,\mu)$ and let $A_k\nearrow F$, $k\in\mathbb{N}$, with $\mu(A_k)<\infty$. Then by Hölders inequality, it holds $1_{A_k} T_t^* f \in L^1(F,\mu)$ for all $k \in \mathbb{N}$. Further, by construction of $T_{t,\infty}$ it holds

$$\left\langle g, 1_{A_k} T_t^* f \right\rangle = \left\langle T_t(1_{A_k} g), f \right\rangle \longrightarrow \left\langle T_{t,\infty} g, f \right\rangle = \left\langle g, T_{t,\infty}^* f \right\rangle,$$

for $k \to \infty$, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between L^∞ and $(L^\infty)'$. Note that $L^1(F,\mu)$ is weakly complete, see e.g. [111, Corollary III.C.14]. Thus, $T_{t,\infty}^* f \in L^1(F,\mu)$ and it holds

$$\left\| T_{t,\infty}^* f \right\|_{L^1} = \left\| T_{t,\infty}^* f \right\|_{(L^\infty)'} \le \|f\|_{(L^\infty)'} = \|f\|_{L^1}.$$

Eventually, we can extend $T_{t,\infty|L^1}^*$ to a linear contraction on $L^1(F,\mu)$. To prove the last assertion it is left to show the strong continuity of $\left(T_{t,\infty|L^1}^*\right)_{t\ge 0}$. But this is an immediate consequence of Lemma 1.3 and (1.16). □

Corollary 1.51. *Let $(T_t)_{t\ge 0}$ be a sub-Markovian s.c.c.s. on $L^1(F,\mu)$. Then there exists a sub-Markovian s.c.c.s. $\left(\hat{T}_t\right)_{t\ge 0}$ on $L^1(F,\mu)$ s.t. for all $f, g \in L^1(F,\mu) \cap L^\infty(F,\mu)$ it holds*

$$\int_F T_t f g \, d\mu = \int_F f \hat{T}_t g \, d\mu. \tag{1.18}$$

Proof. For $t \in [0,\infty)$ define $\hat{T}_t := T_{t,\infty|L^1}^*$. It remains to show that $\hat{T}_t$ is sub-Markovian. Therefore it suffices to show that $\hat{T}_t$ is contractive w.r.t. $\|\cdot\|_{L^\infty}$ on $L^1(F,\mu) \cap L^\infty(F,\mu)$. However, this is a consequence of the fact that $T_{t,\infty}^* = T_t^*$ on $L^1(F,\mu) \cap L^\infty(F,\mu)$ and the latter operator is indeed a contraction on $L^\infty(F,\mu)$. □

Convention 1.52. *In the entire thesis we will use the following convention. Let $r \in [1,\infty]$. If $(T_t)_{t\ge 0}$ is a s.c.c.s. on $L^p(F,\mu)$ fulfilling the assumptions of Theorem 1.49(i), (ii), respectively, then we denote by $\left(T_{t,r}\right)_{t\ge 0}$ the respective semigroups on $L^r(F,\mu)$ constructed in Theorem 1.49(i), (ii), respectively. Observe that in the exact same fashion as in Theorem 1.49, we can extend a s.c.c.r. $(G_\alpha)_{\alpha>0}$ on $L^p(F,\mu)$ to $L^r(F,\mu)$ for $r \in [p,\infty]$ ($r \in [1,p]$) if $(G_\alpha)_{\alpha>0}$ ($\left(G_\alpha^*\right)_{\alpha>0}$) is sub-Markovian. In this case we use the same index notation as for the semigroup case. Furthermore, if $(T_t)_{t\ge 0}$ is a sub-Markovian s.c.c.s. on $L^1(F,\mu)$, we always denote by $\left(\hat{T}_t\right)_{t\ge 0}$ the semigroup defined in Corollary 1.51. The same applies to semigroups and resolvents which we marked with sub- and superindices.*

1.3.3 Conservative Semigroups and Invariant Measures

Definition 1.53. *Let $(T_t)_{t\ge 0}$ be a sub-Markovian s.c.c.s. on $L^1(F,\mu)$.*

(i) The measure μ is called invariant w.r.t. $(T_t)_{t\ge 0}$ if $\int_F T_t f \, d\mu = \int_F f \, d\mu$ holds for all $f \in L^1(F,\mu)$ and all $t \ge 0$.

(ii) $(T_t)_{t\ge 0}$ is called conservative if $T_{t,\infty} 1 = 1$ holds for all $t \ge 0$.

Lemma 1.54. *Let $(T_t)_{t\ge 0}$ be a sub-Markovian s.c.c.s. on $L^1(F,\mu)$. Further, let $\left(\hat{T}_t\right)_{t\ge 0}$ denote the sub-Markovian s.c.c.s. on $L^1(F,\mu)$ given by Corollary 1.51. Then μ is invariant w.r.t. $(T_t)_{t\ge 0}$ if and only if $\left(\hat{T}_t\right)_{t\ge 0}$ is conservative.*

Proof. The assertion is an easy consequence of the construction of $\hat{T}_{t,\infty}$ and (1.18). □

Lemma 1.55. *Let $(T_t)_{t\geq 0}$ be a sub-Markovian s.c.c.s. on $L^1(F,\mu)$ with generator $(A, D(A))$. Then μ is invariant for $(T_t)_{t\geq 0}$ if and only if there exists a core C for $(A, D(A))$ s.t.*

$$\int_F Af\,\mu = 0 \text{ for all } f \in C. \tag{1.19}$$

Proof. Necessity follows since $\int : L^1(F,\mu) \longrightarrow \mathbb{K}, f \mapsto \int_F f\,d\mu$ is continuous. Now, let C be a core for $(A, D(A))$ and assume (1.19). Again by the continuity of the intergral and the closedness of A, it holds that $\int_F Af\,d\mu = 0$ for all $f \in D(A)$. Now, let $f \in D(A)$, then it holds by Lemma 1.6(i)

$$T_t f - f = \int_0^t T_s Af\,ds = \int_0^t AT_s f\,ds,$$

where the integral is a Riemann-integral. Further, we obtain that

$$\int_F T_t f - f\,d\mu = \int_F \int_0^t AT_s f\,ds\,d\mu = \int_0^t \int_F AT_s f\,d\mu\,ds = 0,$$

since the integrand is equal to zero. Due to continuity this extends to $f \in L^1(F,\mu)$. □

The proof of the next lemma uses similar arguments as the proof of Theorem 1.49 and is therefore omitted.

Lemma 1.56. *Let $(T_t)_{t\geq 0}$ be a sub-Markovian s.c.c.s. on $L^1(F,\mu)$ and denote by $\left(\hat{T}_t\right)_{t\geq 0}$ the sub-Markovian s.c.c.s. from Corollary 1.51.*

(i) $(T_t)_{t\geq 0}$ is conservative if and only if μ is invariant w.r.t. $\left(\hat{T}_t\right)_{t\geq 0}$.

(ii) If μ is finite, then $(T_t)_{t\geq 0}$ is conservative if and only if μ is invariant w.r.t. $(T_t)_{t\geq 0}$

1.3.4 Diffusion Operator

In the following subsection we consider a special kind of s.c.c.s. on $X = L^p(F,\mu)$. Indeed, we consider semigroups with a generator which is of diffusion type, see the next definition.

Definition 1.57. *A linear operator (A, C) on $L^p(F,\mu)$, $p \in [1,\infty)$, is called abstract diffusion operator if and only if*

(i) $\varphi(u_1, ..., u_k) \in C$ for all $k \in \mathbb{N}$, $u_1, ...u_k \in C$ and $\varphi \in C^\infty(\mathbb{R}^k)$ fulfilling $\varphi(0) = 0$ and it holds

$$A(\varphi(u_1,...,u_k)) = \sum_{i,j=1}^k \partial_i\partial_j\varphi(u_1,...,u_k)\Gamma(u_i,u_j) + \sum_{i=1}^k \partial_i\varphi(u_1,...,u_k)Lu_i,$$

where $\Gamma(f,g) = \frac{1}{2}(Lfg - fLg - gLf)$, $f, g \in C$.

(ii) $\Gamma(f,f) \geq 0, \forall f \in C$.

Example 1.58. *Let $F \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be open. Let $(a_{ij}) : F \longrightarrow \mathbb{R}^{d\times d}$ be positive semidefinite, i.e., (a_{ij}) is pointwisely a positive semidefinite matrix in $\mathbb{R}^{d\times d}$ and $\beta : F \longrightarrow \mathbb{R}^d$ such that each component of (a_{ij}) and β are measurable functions. Furthermore assume $a_{ij}, \beta_i \in L^2_{loc}(F)$ for $i,j \in \{1,\dots,d\}$ and the distributional derivatives of a_{ij} are also locally square integrable, i.e., $\partial_i a_{ij} \in L^2_{loc}(F)$ for $i,j \in \{1,\dots,d\}$. Let $f \in C_c^\infty(F)$ and define*

$$Af = \sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j f) + \sum_{i=1}^{d} \beta_i \partial_i f.$$

Then it holds that $(A, C_c^\infty(F))$ is a abstract diffusion operator on $L^2(F,dx)$.

Lemma 1.59. *Let (A,C) be an abstract diffusion operator on $L^p(F,\mu)$ and assume that μ is invariant for (A,C), i.e., for all $f \in C$ it holds that $Af \in L^1(F,\mu)$ and $\int_F Af\,d\mu = 0$. Then (A,C) is dissipative on $L^p(F,\mu)$ and fulfills (1.12) for all $f \in C$. In particular, (A,C) is closable and if the closure $(A,D(A))$ is the generator of a s.c.c.s. $(T_t)_{t\geq 0}$ then this semigroup is sub-Markovian.*

Proof. For a proof we refer to [30, Chapter 1. Appendix B, Lemma 1.8, Lemma 1.9]. □

Remark 1.60. *The subsequent application of the theory of semigroups we have in mind is roughly the following. We consider a dynamical system described by a stochastic differential equation of Itô type. Then we formally apply Itô's lemma and obtain a second order differential operator A as in Example 1.58. Formally means here that the coefficients of the stochastic differential equation are in general not regular enough to apply Itô's Lemma in its usual form. Then we try to prove that A is the generator of a corresponding sub-Markovian contraction semigroup. The previous lemma indicates that an $L^p(\mu)$ space with an invariant measure μ is a good framework for proving the existence of an associated sub-Markovian semigroup.*

1.4 Convergence of Semigroups

So far we only considered situations with one single semigroup. Recall the interpretation of a semigroup as the time evolution of a dynamical system. Often, and in particular for the applications we have in mind, one tries to describe a physical system by some corresponding approximations. Logically, one should prove that the time evolution of the approximating systems approaches in some sense the behavior of the original system. Hence, in the following section we collect several results concerning the convergence of sequences of semigroups $\left(T_t^n\right)_{t\geq 0}$, $n \in \mathbb{N}$. The semigroups $\left(T_t^n\right)_{t\geq 0}$, $n \in \mathbb{N}$, considered in this section live on different Banach spaces X_n, $n \in \mathbb{N}$.

The way this material is presented is taken from the corresponding section in [24]. We only make slight adjustments compared to [24]. Hence, we only prove a few results to provide the reader with some intuition. For the other proofs we refer instead to the last mentioned reference as well as the papers [105, 66, 67] where the material originates from.

For this section we assume that X, X_n, $n \in \mathbb{N}$ are Banach spaces over the same field $\mathbb{K}$. The respective norm on X and X_n will be denoted by $\|\cdot\|_X$ and $\|\cdot\|_{X_n}$, respectively. Further, C is assumed to be a dense subspace of X.

1.4.1 Definitions and Basic Properties

Definition 1.61. *Assume that for $n \in \mathbb{N}$ there is a linear map*

$$\Psi_n : C \longrightarrow X_n. \tag{1.20}$$

We say that the sequence $(X_n)_{n\in\mathbb{N}}$ converges towards X with respect to $(\Psi_n)_{n\in\mathbb{N}}$ if

$$\lim_{n\to\infty} \|\Psi_n(u)\|_{X_n} = \|u\|_{\mathcal{H}}, \quad \forall u \in C. \tag{1.21}$$

In this case we write $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$.

Until the end of this section we assume Equation (1.21) holds, i.e., we assume $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ for some family $(\Psi_n)_{n\in\mathbb{N}}$ given as in Equation (1.20).

Remark 1.62. *In case $C = X$ we say that X_n converges to X in the sense of Trotter, see [105]. Otherwise, we have a convergence in the sense of Kuwae-Shioya, see [67]. In case we have convergence in the sense of Trotter, we obtain by the uniform boundedness principle that Ψ_n, $n \in \mathbb{N}$, is uniformly bounded in operator norm.*

Definition 1.63. *Let $u_n \in X_n$, $n \in \mathbb{N}$ and $u \in \mathcal{H}$. The sequence $(u_n)_{n\in\mathbb{N}}$ is said to converge towards u along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ if one (hence all) sequence $(\tilde{u}_m)_{m\in\mathbb{N}} \subseteq C$ satisfies*

$$\lim_{m\to\infty} \|\tilde{u}_m - u\|_{\mathcal{H}} = 0 \tag{1.22}$$

and

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \|\Psi_n(\tilde{u}_m) - u_n\|_{X_n} = 0. \tag{1.23}$$

If (1.22) and (1.23) hold we also write $u_n \longrightarrow u$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$

The next lemma shows that the convergence of a sequence along different Banach spaces is not very different from the classical situation $X = X_n$ for all $n \in \mathbb{N}$. The proof of the lemma is elementary.

Lemma 1.64. *Let $u_n \in X_n$, $n \in \mathbb{N}$ and $u \in X$. Assume $u_n \longrightarrow u$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$.*

(i) If $u \in C$ then $u_n \longrightarrow u$ if and only if $\lim_{n\to\infty} \|u_n - \Psi_n(u)\|_{X_n} = 0$. In particular, $u_n \longrightarrow 0$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ if and only if $\lim_{n\to\infty} \|u_n\|_{X_n} = 0$.

(ii) For $\alpha, \beta \in \mathbb{K}$ it holds $\alpha u_n + \beta v_n \longrightarrow \alpha u + \beta v$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$.

(iii) Let $\hat{u} \in X$ s.t. $u_n \longrightarrow \hat{u}$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$. Then it holds $u = \hat{u}$.

(iv) Convergence along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ implies norm convergence, i.e. $\lim_{n\to\infty} \|u_n\|_{X_n} = \|u\|_X$. In

particular, if X_n, $n \in \mathbb{N}$, and X are Hilbert spaces with respective scalar products $(\cdot,\cdot)_{X_n}$ and $(\cdot,\cdot)_X$, then it holds for $v_n \longrightarrow v$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ that

$$\lim_{n\to\infty}(u_n, v_n)_{X_n} = (u,v)_X.$$

Definition 1.65. *Assume that X_n, $n \in \mathbb{N}$, and X are Hilbert spaces with respective scalar products $(\cdot,\cdot)_{X_n}$ and $(\cdot,\cdot)_X$. Let $u_n \in X_n$, $n \in \mathbb{N}$ and $u \in X$. We say that u_n converges weakly to u along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ if for every sequence $v_n \in X_n$, $n \in \mathbb{N}$, and $v \in X$ s.t. $v_n \longrightarrow v$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ it holds*

$$\lim_{n\to\infty}(u_n, v_n)_{X_n} = (u,v)_X. \tag{1.24}$$

To emphasize the difference to weak convergence we sometimes also use the term strong convergence for convergence along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$. The next two corollaries are well-known for the classical case $X = X_n$, $n \in \mathbb{N}$.

Corollary 1.66. *Assume X_n, $n \in \mathbb{N}$, and X are Hilbert spaces with respective scalar products $(\cdot,\cdot)_{X_n}$ and $(\cdot,\cdot)_X$. Let $u_n \in X_n$, $n \in \mathbb{N}$, and $u \in X$. Then it holds that $u_n \longrightarrow u$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ if and only if*

$$\lim_{n\to\infty}\|u_n\|_{X_n} = \|u\|_X \tag{1.25}$$

and

$$\lim_{n\to\infty}(u_n, \Psi_n(v))_{X_n} = (u,v)_X, \text{ for all } v \in C. \tag{1.26}$$

Proof. We only show that (1.25) and (1.26) are sufficient. Let $\tilde{u}_m \in C$, $m \in \mathbb{N}$; s.t. $\tilde{u}_m \longrightarrow u$ as $m \to \infty$. Then it holds

$$\begin{aligned}\lim_{n\to\infty}\|\Psi_n(\tilde{u}_m) - u_n\|_{X_n}^2 &= \lim_{n\to\infty}\|\Psi_n(\tilde{u}_m)\|_{X_n}^2 + \lim_{n\to\infty}\|u_n\|_{X_n}^2 - \lim_{n\to\infty} 2\operatorname{Re}(\Psi(\tilde{u}_m), u_n)_{X_n}\\ &= \|\tilde{u}_m\|_X^2 + \|u\|_X^2 - 2\operatorname{Re}(\tilde{u}_m, u)_X.\end{aligned}$$

Hence, we finally obtain for $m \longrightarrow \infty$

$$\lim_{m\to\infty}\overline{\lim_{n\to\infty}}\|\Psi_n(\tilde{u}_m) - u_n\|_{X_n} = 0.$$

□

The proof of the next corollary works similar as in the case $X_n = X$ for all $n \in \mathbb{N}$.

Corollary 1.67. *Assume X_n, $n \in \mathbb{N}$, and X are Hilbert spaces with respective scalar products $(\cdot,\cdot)_{X_n}$ and $(\cdot,\cdot)_X$. Let $u_n \in X_n$, $n \in \mathbb{N}$, and $u \in X$. Then it holds that $u_n \longrightarrow u$ weakly along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ if and only if the sequence of real numbers $\left(\|u_n\|_{X_n}\right)_{n\in\mathbb{N}}$ is bounded and*

$$\lim_{n\to\infty}(u_n, \Psi_n(v))_{X_n} = (u,v)_X, \text{ for all } v \in C. \tag{1.27}$$

In the following we clarify that in certain cases there is no difference between the notion of

Trotter and the one of Kuwae-Shioya. We elaborate this in the following.

Definition 1.68. *We say that a Banach space X has the bounded approximation property if there exists an $M \in (0, \infty)$ s.t. for any $n \in \mathbb{N}$, any $u_1, ..., u_n \in X$ and any $\varepsilon > 0$ there exists a $T \in L(X)$, s.t. $\|Tf_i - f_i\|_X < \varepsilon$, $i = 1, .., n$, and $\|T\| \leq M$.*

Example 1.69. *(i) If Y is isometric isomorphic to X and X has the bounded approximation property, then Y has this property, too.*

(ii) A separable Hilbert space has the bounded approximation property, see [111, Proposition II.F.4].

(iii) Let $(F, \mathcal{F}, \mu)$ be a σ-finite measure space. Then for $p \in [1, \infty)$ the Banach space $L^p(F, \mu)$ has the bounded approximation property, see [111, Example II.F.5(a)].

Remark 1.70. *Assume that X is separable. Then the bounded approximation property is equivalent to the existence of a sequence $T_k \in L(X)$, $k \in \mathbb{N}$, of finite rank operators converging strongly to $Id \in L(X)$.*

Lemma 1.71. *Assume that X is separable and has the bounded approximation property. Then there exist finite rank operators $\Omega_n : X \longrightarrow X_n$ s.t. $\|\Omega_n u\|_{X_n} \longrightarrow \|u\|_X$ as $n \to \infty$ for all $u \in X$. Furthermore, it holds that $u_n \longrightarrow u$ along $X_n \xrightarrow{(\Omega_n)_{n\in\mathbb{N}}} X$ if and only if $u_n \longrightarrow u$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$.*

Proof. See [24, Lemma 1.5.7.]. □

1.4.2 Convergence of Linear Operators

In the following we assume that there exists for $n \in \mathbb{N}$ a finite rank operator $\Omega_n : X \longrightarrow X_n$ s.t. the assertion in Lemma 1.71 is fulfilled. For the applications we have in mind the assumptions in Lemma 1.71 are fulfilled. Hence, this will not be an additional assumption later. In particular, we will consider Banach spaces of the type $L^p(F, \mu)$ with a σ-finite regular measure μ defined on a Borel σ-field of a second countable topological space. In that case one easily sees that $L^p(F, \mu)$ is also separable.

Definition 1.72. *Let $T_n \in L(X_n)$, $n \in \mathbb{N}$, and $T \in L(X)$. We say that the sequence of operators $(T_n)_{n\in\mathbb{N}}$ converges to T along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$, if for every sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, $n \in \mathbb{N}$, such that $u_n \longrightarrow u$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ it holds $T_n u_n \longrightarrow Tu$ along $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$.*

Definition 1.73. *For $n \in \mathbb{N}$ let $(A_n, D(A_n))$ be a linear operator on X_n. We define the set*

$$D(\operatorname*{ex\,lim}_{n\to\infty} A_n) := \left\{ u \in X \mid \exists u_n \in D(A_n), f \in X \text{ s.t. } u_n \longrightarrow u,\ A_n u_n \longrightarrow f \text{ along } X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X \right\}.$$

For $u \in D(\operatorname*{ex\,lim}_{n\to\infty} A_n)$ define

$$\operatorname*{ex\,lim}_{n\to\infty} A_n u = \left\{ f \in X \mid \exists u_n \in D(A_n) \text{ s.t. } u_n \longrightarrow u,\ A_n u_n \longrightarrow f \text{ along } X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X \right\}.$$

If $\operatorname*{ex\,lim}_{n\to\infty} A_n u$ is single-valued for every $u \in D(\operatorname*{ex\,lim}_{n\to\infty} A_n)$ one can clearly define a linear operator

$\left(\underset{n\to\infty}{ex\lim} A_n, D(\underset{n\to\infty}{ex\lim} A_n)\right)$ *on* X.

The definition of the extended limit of operators guarantees that $\left(\underset{n\to\infty}{\text{ex}\lim} A_n, D(\underset{n\to\infty}{\text{ex}\lim} A_n)\right)$ is closed as a multivalued operator, see the next lemma.

Lemma 1.74. *For $n \in \mathbb{N}$ let $(A_n, D(A_n))$ be a linear operator on X_n. Further, let $u^m \in D(\underset{n\to\infty}{ex\lim} A_n)$, $m \in \mathbb{N}$ s.t. for some $f^m \in \underset{n\to\infty}{ex\lim} A_n u^m$, $m \in \mathbb{N}$, it holds $u^m \longrightarrow u$, $f^m \longrightarrow f$ in X as $m \to \infty$. Then it holds $u \in D(\underset{n\to\infty}{ex\lim} A_n)$ and $f \in \underset{n\to\infty}{ex\lim} A_n$.*

Proof. Let u^m, f^m, $m \in \mathbb{N}$, and u, f be as above. By definition there exist for every $m \in N$ elements $u_n^m \in D(A_n)$ s.t. $u_n^m \longrightarrow u^m$ and $A_n u_n^m \longrightarrow f^m$ along $X_n \xrightarrow{(\Omega_n)_{n\in\mathbb{N}}} X$. Then for all $m \in \mathbb{N}$ there exists a $\alpha(m)$ s.t. for all $n \geq \alpha(m)$

$$\left\|\Omega_n u^m - u_n^m\right\|_{X_n} \leq \frac{1}{m}, \quad \left\|\Omega_n f^m - A_n u_n^m\right\|_{X_n} \leq \frac{1}{m}.$$

Observe that α can be chosen s.t. $\alpha(m) \leq \alpha(m+1)$ for all $m \in \mathbb{N}$. For n sufficiently large define $\beta(n) := \sup\{m \in \mathbb{N} \mid \alpha(m) \leq n\}$. Hence, by construction it holds $n \geq \alpha(\beta(n))$. Since α is strictly increasing it holds that $\beta(n) \longrightarrow \infty$ as $n \to \infty$. Recall that $(\|\Omega_n\|)_{n\in\mathbb{N}}$ is bounded. Hence we conclude that for $n \to \infty$ it holds

$$\begin{aligned} \left\|\Omega_n u - u_n^{\beta(n)}\right\|_{X_n} &\leq \left\|\Omega_n(u - u^{\beta(n)})\right\|_{X_n} + \left\|\Omega_n u^{\beta(n)} - u_n^{\beta(n)}\right\|_{X_n} \longrightarrow 0 \\ \left\|\Omega_n f - A_n u_n^{\beta(n)}\right\|_{X_n} &\leq \left\|\Omega_n(f - f^{\beta(n)})\right\|_{X_n} + \left\|\Omega_n f^{\beta(n)} - A_n u_n^{\beta(n)}\right\|_{X_n} \longrightarrow 0. \end{aligned}$$

□

This proves that $u \in D(\underset{n\to\infty}{ex\lim} A_n)$ and $f \in \underset{n\to\infty}{ex\lim} A_n u$.

The next example illustrates how multi-valued operator can arise.

Example 1.75. *Consider the space $X = L^2(\mathbb{R}, dx)$ with some orthonormal basis $(e_k)_{k\in\mathbb{N}}$ and $(A, C_c^\infty(\mathbb{R}))$ given by $Af = \sum_{k=1}^{\infty} f(k)e_k$. For $n \in \mathbb{N}$ we define $X_n = X$, $P_n = Id$ and $(A_n, D(A_n)) = (A, C_c^\infty(\mathbb{R}))$. Then it holds that $0 \in D(\underset{n\to\infty}{ex\lim} A_n)$ and $\underset{n\to\infty}{ex\lim} A_n 0 = span(e_k, k \in \mathbb{N})$. This can be seen by employing a sequence $(\varphi_l)_{l\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$, s.t. $\varphi_l(0) = 1$, $supp(\varphi_l) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $\|\varphi_l\| \longrightarrow 0$ as $l \to \infty$.*

The next lemma provides a sufficient criteria for operators of interest s.t. $\left(\underset{n\to\infty}{\text{ex}\lim} A_n, D(\underset{n\to\infty}{\text{ex}\lim} A_n)\right)$ is single-valued. We used the idea of the original proof already for proving Lemma 1.16(ii), see [66, Lemma 1.1] for the original reference.

Lemma 1.76. *Suppose $(A_n, D(A_n))$ is a dissipative operator on X_n for all $n \in \mathbb{N}$ and $D(\underset{n\to\infty}{ex\lim} A_n)$ is a dense subspace of X. Then it holds that $\underset{n\to\infty}{ex\lim} A_n u$ is single-valued for all $u \in D(\underset{n\to\infty}{ex\lim} A_n)$. Furthermore, $\left(\underset{n\to\infty}{ex\lim} A_n, D(\underset{n\to\infty}{ex\lim} A_n)\right)$ is dissipative.*

The next theorem is due to Trotter and Kurtz and is the main theorem of this section.

Theorem 1.77. *Suppose* $(A, D(A))$, $(A_n, D(A_n))$, $n \in \mathbb{N}$, *generate s.c.c.s.* $(T_t)_{t\geq 0}$, $(T_t^n)_{t\geq 0}$ *and s.c.c.r.* $(G_\alpha)_{\alpha>0}$, $(G_\alpha^n)_{\alpha>0}$ *on* X *and* X_n, *respectively. Let* $(\Omega_n)_{n\in\mathbb{N}}$ *be given as in the beginning of this subsection. Then the following statements are equivalent:*

(i) $\left(\operatorname{ex}\lim_{n\to\infty} A_n, D(\operatorname{ex}\lim_{n\to\infty} A_n)\right) = (A, D(A))$.

(ii) *For all* $t \geq 0$ *it holds* $T_t^n \longrightarrow T_t$ *along* $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$.

(iii) *For all* $t \geq 0$ *and* $u \in \mathcal{H}$ *it holds* $\lim_{n\to\infty} \sup_{0\leq s\leq t} \left\|\Omega_n T_s u - T_s^n \Omega_n u\right\|_{X_n} = 0$.

(iv) *For all* $\alpha > 0$ *it holds* $G_\alpha^n \longrightarrow G_\alpha$ *along* $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$.

Proof. See [24, Theorem 1.5.13]. □

The following corollary provides a sufficient condition for the equivalent statements of Theorem 1.77.

Corollary 1.78. *Let* $(A, D(A))$, $(A_n, D(A_n))$, $n \in \mathbb{N}$, *be as in Theorem 1.77. If there exists a core* $\hat{C} \subseteq C \cap D(A)$ *for* $(A, D(A))$ *such that* $A_n \Psi_n u \longrightarrow Au$ *along* $X_n \xrightarrow{(\Psi_n)_{n\in\mathbb{N}}} X$ *for every* $u \in \hat{C}$, *then the equivalent statements in Theorem 1.77 hold true.*

Proof. The assumption says that $\left(\operatorname{ex}\lim_{n\to\infty} A_n, D(\operatorname{ex}\lim_{n\to\infty} A_n)\right)$ is an extension of $(A, \hat{C})$ which is essentially m-dissipative. By Lemma 1.76 we obtain that $\left(\operatorname{ex}\lim_{n\to\infty} A_n, D(\operatorname{ex}\lim_{n\to\infty} A_n)\right)$ is dissipative, hence $\left(\operatorname{ex}\lim_{n\to\infty} A_n, D(\operatorname{ex}\lim_{n\to\infty} A_n)\right) = (A, D(A))$, since the latter operator is m-dissipative. □

Remark 1.79. *Corollary 1.78 is very useful in applications. Indeed, often one knows an explicit representation of an operator merely on a core. In return, in many cases the most challenging part of the analysis is to prove that a certain subspace forms a core for the limit operator or rather is a domain of essential m-dissipativity. Eventually, we conclude that to know a core of an operator is sufficient to determine which dynamics converge to the dynamic associated with the operator under consideration.*

We don't state an example of convergent semigroups here, since we prove convergence of semigroups in Chapter 3.

Chapter 2

Preliminaries from the Theory of Markov Processes and Path Space Measures

The aim of this chapter is similar as in the first chapter. We state necessary definitions and theorems to provide a solid background for the subsequent chapter. We only present results here which are needed in the further course of this thesis. More details on the respective subjects can be founded in the references mentioned in the respective Sections. Note that we don't prove any new result in this chapter. In the first Section we show some results related to probability measures $\mathbb{P}$ defined on the so-called path-space $C([0,\infty),F)$, where F is a Polish space, of continuous functions with values in F. We relate the finite dimensional distributions of such a measure $\mathbb{P}$ to sub-Markovian operator semigroups on $L^p(F,\mu)$ for a σ-finite measure μ on F. In the second section we state some definitions concerning Markov processes. Furthermore, we state an existence result for Markov processes from potential theory. This is the core result to construct weak solutions for the stochastic differential equations considered in the next chapter.

2.1 Path space measures and Operator Semigroups

In this section we collect results regarding weak convergence of probabilty measures $\mathbb{P}$ defined on the space of continuous functions $C([0,\infty),F)$, where F is a Polish space. In particular, we are concerned with measures $\mathbb{P}$ which are associated with sub-Markovian semigrous $(T_t)_{t\geq 0}$ defined on $L^p(F,\mu)$ for a σ-finite measure μ. The results stated in this section can be found in most textbooks on Markov processes and probability theory such as [32, 18, 19] and the article [110].

2.1.1 Tightness, Weak Convergence of Measures and Prohorov's Theorem

Throughout this entire subsection we consider a Hausdorff space E. We exclusively consider the Borel σ-algebra $\mathcal{F}$ on E and by $\mathcal{P}(E)$ we denote the set of probability measures on $(E,\mathcal{F})$. We denote by d a metric inducing the topology on E s.t. (E,d) is a separable, complete metric space.

Definition 2.1. *A family Y of finite measures on $(E,\mathcal{F})$ is called tight, if for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ s.t.*

$$\sup_{\mathbb{P}\in Y} \mathbb{P}(E \setminus K) < \varepsilon.$$

Example 2.2. *If E is Polish then every singleton $\{\mathbb{P}\} \subseteq \mathcal{P}(E)$ is tight, see [32, Lemma 3.2.1].*

The following lemma turns out to be useful.

Lemma 2.3. *Assume V is a second countable topological vector space equipped with the Borel σ-algebra and let I be some index set. $f_{1,i}, f_{2,i} : E \longrightarrow V$, $i \in I$, be measurable maps. Assume further that $\{\mathbb{P}_i \mid i \in I\} \subseteq \mathcal{P}(E)$ is given s.t. $\left\{\mathbb{P}_i \circ f_{j,i}^{-1} \mid i \in I\right\}$ is tight on V for $j = 1, 2$. Then $\left\{\mathbb{P}_i \circ (f_{1,i} + f_{2,i})^{-1} \mid i \in I\right\}$ is also tight on V.*

Proof. Since V is second countable, it holds that the pointwise sum $f_{1,i} + f_{2,i} : E \longrightarrow V$ is measurable. Note that for two compact sets K_1 and K_2 the product $K_1 \times K_2$ is compact in $V \times V$ equipped with the product topology. Hence, $K := K_1 + K_2 = \{v_1 + v_2 \mid v_i \in K_i, i = 1, 2\}$ is compact. Since $f_{1,i}^{-1}K_1 \cap f_{2,i}^{-1}K_2 \subseteq (f_{1,i} + f_{2,i})^{-1}K$, it holds for $i \in I$

$$\mathbb{P}_i \circ (f_{1,i} + f_{2,i})^{-1}(V \setminus K) \leq \mathbb{P} \circ f_{1,i}^{-1}(V \setminus K_1) + \mathbb{P} \circ f_{2,i}^{-1}(V \setminus K_2)$$

which proves tightness. □

In the following we restrict our considerations to the space $\mathcal{P}(E)$ and equip this space with the weak topology, i.e., the coarsest topology W s.t. all maps $\mathcal{P}(E) \ni \mathbb{P} \mapsto \int_E f\, d\mathbb{P}$, $f \in C_b(E)$, are continuous. We assume henceforth that E is a Polish space.

Theorem 2.4. *The space $\mathcal{P}(E)$ is a Polish space.*

Proof. See e.g. [18, Appendix III, Theorem 5] and [32, Theorem 3.1.7, Theorem 3.3.1]. □

The next theorem is due to Prohorov.

Theorem 2.5. *Let Y be a family in $\mathcal{P}(E)$. Then the following are equivalent.*

(i) Y is tight.

(ii) Y is pre-compact in the weak topology on $\mathcal{P}(E)$.

Proof. See e.g. [32, Theorem 3.2.2] and Theorem 2.4. □

From the proof of Theorem 2.5 one obtains:

Corollary 2.6. *Assume E is merely metrizable. Let Y be a tight family in $\mathcal{P}(E)$. Then Y is pre-compact.*

Proof. See [32, Corollary 3.2.3]. □

Remark 2.7. *To show that a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}(E)$ is weakly convergent we can use Theorem 2.5. By establishing tightness of $(\mathbb{P}_n)_{n\in\mathbb{N}}$ we obtain accumulation points of $(\mathbb{P}_n)_{n\in\mathbb{N}}$. If one can further show that all these accumulation points coincide, we obtain weak convergence by the subsubsequence criteria and Theorem 2.4. Hence we collect in the next subsection sufficient criteria for tightness for a specific type of space E, a so-called path space.*

2.1.2 Tightness for Path Space Measures

In the following we restrict the choice of E to a special case. To this end, let (F, r) be a separable complete metric space. Define E to be the space of continuous functions on $[0, \infty)$ with values in F denoted by

$$E := C([0, \infty), F) .$$

We equip $C([0, \infty), F)$ with a metric defined by

$$d(x, y) := \sum_{T=1}^{\infty} 2^{-T} \sup_{0 \le t \le T} (r(x(t), y(t)) \wedge 1), \quad x, y \in C([0, \infty), F) . \tag{2.1}$$

The topology induced by d on $C([0, \infty), F)$ is called the topology of uniform convergence on compact sets. We equip $C([0, \infty), F)$ exclusively with the Borel σ-algebra denoted by $\mathcal{B}_C$. In particular, we often do not mention the σ-algebra $\mathcal{B}_C$ explicitly. It can be shown that $C([0, \infty), F)$ is separable and complete, too. For $t \in [0, \infty)$ we denote by π_t the evaluation map

$$\pi_t : C([0, \infty), F) \longrightarrow F, x \mapsto x(t).$$

The next lemma is well-known and can be proven as in [18, p.19/20] where the case $F = \mathbb{R}$ is treated.

Lemma 2.8. *Let $D \subseteq [0, \infty)$ be dense. It holds $\mathcal{B}_C = \sigma(\pi_t, t \in D)$.*

Corollary 2.9. *Let $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(C([0, \infty), F))$. Assume that for all $n \in \mathbb{N}$ and $0 \le t_1 \le \ldots \le t_n$ it holds $\mathbb{P}_1 \circ \left(\pi_{t_1}, \ldots, \pi_{t_n}\right)^{-1} = \mathbb{P}_2 \circ \left(\pi_{t_1}, \ldots, \pi_{t_n}\right)^{-1}$. Then $\mathbb{P}_1 = \mathbb{P}_2$.*

To establish tightness for a family $Y \subseteq \mathcal{P}(C([0, \infty), F))$ we need criteria for compactness in $C([0, \infty), F)$. As a first step we reduce everything to finite time intervals and use afterwards the Arzela-Ascoli theorem. Let $T \ge 0$ and denote by $C([0, T], F)$ the space of all continuous functions from $[0, T]$ to F equipped with the topology of uniform convergence. This topology is induced by the metric $d_T(x, y) = \sup_{t \in [0,T]} r(x(t), y(t))$. As usual, we equip $C([0, T], F)$ with its Borel σ-algebra $\mathcal{B}_{C,T}$. The analog of Lemma 2.8 also holds for $\mathcal{B}_{C,T}$. We denote by R_T the time restriction operator, i.e.,

$$R_T : C([0, \infty), F) \longrightarrow C([0, T], F), x \mapsto x_{|[0,T]}.$$

Lemma 2.10. *Let $Y \subseteq \mathcal{P}(C([0, \infty), F))$. Then Y is tight if and only if $Y_T := \left\{\mathbb{P} \circ R_T^{-1} \mid \mathbb{P} \in Y\right\}$ is tight for every $T \in \mathbb{N}$.*

Proof. See e.g. [110, Corollary 5.] □

To formulate the Arzela-Ascoli version of tightness we introduce the modulus of continuity. Let $x \in C([0, \infty), F)$ ($C([0, T], F)$, $T \in \mathbb{N}$), $\delta, T > 0$.

$$w(x, \delta, T) := \sup_{\substack{s,t \in [0,T] \\ |t-s| \le \delta}} r(x(t), x(s)).$$

In fact, for every $\delta, T > 0$ the map $w(\cdot, \delta, T)$ is continuous on $C([0,\infty), F)$ $(C([0,T],F),\ T \in \mathbb{N})$, hence measurable. Observe that a set of functions $A \subseteq C([0,\infty), F)$ $(C([0,T],F),\ T \in \mathbb{N})$ is equicontinuous on $[0,T]$ if and only if $\lim_{\delta\to 0} \sup_{x\in A} w(x,\delta,T) = 0$. The next theorem is a direct consequence of the Arzela-Ascoli theorem.

Theorem 2.11. *Let $Y \subseteq \mathcal{P}(C([0,T],F))$, $T \in \mathbb{N}$, be a family of probability measures. Then Y is tight if and only if the following two conditions are fulfilled.*

(i) For every $t \geq 0$ the family $Y \circ \pi_t^{-1} := \left\{\mathbb{P} \circ \pi_t^{-1} \mid \mathbb{P} \in Y\right\}$ is tight.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$\sup_{\mathbb{P}\in Y} \mathbb{P}(w(\cdot,\delta,T) \geq \varepsilon) \leq \varepsilon. \tag{2.2}$$

Proof. See e.g. [110]. □

Remark 2.12. *Assume the special case $F = \mathbb{R}$ and the property (ii) of Theorem 2.11 holds true. Then the assumption (i) in 2.11 holds true if and only if $Y \circ \pi_0^{-1}$ is tight, i.e., the initial distributions are tight. For a proof, see [61, Proof of Theorem 2.4.9].*

The next lemma is a modification of [61, Problem 2.4.11] and can be proven in the same way.

Lemma 2.13. *A family $Y \subseteq \mathcal{P}(C([0,T],F))$ fulfills (2.2) if there exists $0 \leq \beta, C < \infty$ and $\alpha > 1$ such that for all $s,t \in [0,T]$, it holds*

$$\sup_{\mathbb{P}\in Y} \int r(\pi_t,\pi_s)^\beta \, d\mathbb{P} \leq C|t-s|^\alpha.$$

Lemma 2.14. *Let the metric r on F given by $r(x,y) = \sum_{i=1}^{l} |f_i(x) - f_i(y)|$, $l \in \mathbb{N}$, where $f_i : F \longrightarrow \mathbb{R}$. Then f_i, $i \in \{1,\ldots,l\}$, induces a Lipschitz continuous and thus a measurable map*

$$\hat{f}_i : C([0,T],F) \longrightarrow C([0,T],\mathbb{R})(x_t)_{t\in[0,T]} \mapsto (f_i(x_t))_{t\in[0,T]}\,. \tag{2.3}$$

Tightness of a family $Y \subseteq \mathcal{P}(C([0,\infty),F)$ is ensured if $Y_i := \left\{\mathbb{P} \circ \hat{f}_i^{-1} \mid \mathbb{P} \in Y\right\}$ is tight on $C([0,T],\mathbb{R})$ for every $i \in \{1,..,l\}$.

Proof. This follows directly by Theorem 2.11. □

The next remark might be obvious for readers with a strong topological background. Anyway, latter one of our main arguments rely on the next remark and therefore we want to give details at greater length.

Remark 2.15. *Observe that the tightness of a family of probability measures on a Polish space is a topological property, i.e., it does not depend on the specific metric we choose. Observe that we introduced $C([0,\infty),F)$ as a metric space via the metric d defined in (2.1) and not merely as a topological space. In particular, all results concerning tightness of probability measures on $C([0,\infty),F)$ are still true if we choose a different metric $\tilde{d}$ on $C([0,\infty),F))$ as long as it induces the same topology on $C([0,\infty),F)$. Likewise, we can choose a different metric $\tilde{r}$ on F as long as it induces the same* *topology as r on F. In particular, the topology on $C([0,\infty),F)$ depends only on the topology of F and not on the specific metric we choose for F. In the light of given arguments, we introduce a topology on $C([0,\infty),F)$ only in terms of open sets of F without using a metric. To this end we work with the equivalent concept of neighborhood filters, see e.g. [107, Chapter 1].*

Let $(T,\mathbf{P},\mathcal{U})$ be a triple consisting of $T\in\mathbb{N}$, a partition $\mathbf{P}=\{t_0,\ldots,t_n\}$ of $[0,T]$, $n\in\mathbb{N}$, and a family $\mathcal{U}=\{U_i\}_{i=1,\ldots,n}$ of open sets in F. Define the set

$$N(T,P,\mathcal{U}):=\{g\in C([0,\infty),F)\mid g(t)\in U_{i+1} \text{ if } t\in[t_i,t_{i+1}], i=0,..,n-1\}.$$

Now we define a topology on $C([0,\infty),F)$ via filters of neighborhoods $\mathcal{N}(f)$, $f\in C([0,\infty),F)$. Define for $f\in C([0,\infty),F)$ the filter $\mathcal{N}(f)$ as the filter generated by the filter base

$$B_f:=\{N(T,P,\mathcal{U})\mid T,P,\mathcal{U} \text{ as above}, f\in N(T,P,\mathcal{U})\}.$$

Now let r be a metric inducing the topology on F and define d by (2.1). Define another neighborhood filter $\widetilde{\mathcal{N}}(f)$ of $f\in C([0,\infty),F)$ generated by the filter base

$$\widetilde{B}_f:=\left\{B_{\varepsilon,d}(f)\mid\varepsilon>0\right\},$$

where $B_{\varepsilon,d}(f)$ denotes the open ball w.r.t. d around f with radius ε. We postpone the lengthy but straightforward proof of the fact $\mathcal{N}(f)=\widetilde{\mathcal{N}}(f)$ for all $f\in C([0,\infty),F)$ to the Appendix A.1.

2.1.3 Path Space Measures, associated Operator Semigroups and and the Martingale problem for L^p generators

Throughout this entire subsection we fix a separable complete metric space (F,r) equipped with the Borel σ-algebra $\mathcal{B}$. Further, we fix a σ-finite measure μ on $(F,\mathcal{B})$ space and a sub-Markovian s.c.c.s. $(T_t)_{t\geq0}$ on $L^p(F,\mu)$, $p\in[1,\infty)$. For the applications we have in mind it is necessary to extend F by an additional point, the so called cemetery $\Delta\notin E$. We adjoin Δ as an isolated point. Observe that we obtain a Polish space F_Δ. The Borel σ-algebra $\mathcal{B}_\Delta$ is given by $\mathcal{B}_\Delta=\mathcal{B}\cup\{B\cup\{\Delta\}\mid B\in\mathcal{B}\}$. In the following we use convention that every function $f:F\longrightarrow\mathbb{R}$ is extended to F_Δ via $f(\Delta)=0$. Similarly, every measure ν defined on $\mathcal{B}$ is extended to $\mathcal{B}_\Delta$ via $\nu(\{\Delta\})=0$. We consider the space $C([0,\infty),F_\Delta)$ exclusively with the topology of uniform convergence on compact sets, see the previous remark, and the corresponding Borel σ-algebra.

Convention 2.16. *For the rest of this thesis, every measure $\mathbb{P}$ on $C([0,\infty),F_\Delta)$ fulfills*

$$\mathbb{P}\left(\{x\in C([0,\infty),E_\Delta)\mid\exists s,t\in[0,\infty), s\leq t, \pi_s(x)=\Delta, \pi_t(x)\in F\}\right)=0.$$

Namely, it holds $\mathbb{P}$-a.e. that if a path x hits the cemetery it stays there forever. The map

$$\xi:C([0,\infty),F_\Delta)\longrightarrow[0,\infty], x\mapsto\inf\{t\geq0\mid\pi_t(x)=\Delta\}.$$

is called the life-time (of $(\pi_t)_{t\geq0}$). Observe that ξ is measurable.

For $q_1,q_2\in[1,\infty]$ we denote in the following by $L^{q_1}(F,\mu)+L^{q_2}(F,\mu)$ the space of all (equivalence classes of) measurable functions h which admit a decomposition into $h=h_1+h_2$ with $h_1\in L^{q_1}(F,\mu)$ and $h_2\in L^{q_2}(F,\mu)$.

Definition 2.17. *Let $\mathbb{P}$ be a measure on $C([0,\infty),F_\Delta)$ with initial distribution $P\circ\pi_0^{-1}=h\mu$, $h\in$*

$L^1(F,\mu)+L^{\frac{p}{p-1}}(F,\mu)$. $\mathbb{P}$ is called to be associated with $(T_t)_{t\geq 0}$ if for all $f_1,\dots,f_k \in L^\infty(F,\mu)\cap L^p(F,\mu)$ and $0\leq t_1<\dots<t_k$, $k\in\mathbb{N}$, it holds

$$\int_{C([0,\infty),F_\Delta)} \prod_{i=0}^{k} f_i(\pi_{t_i})\, d\mathbb{P} = \langle h, T_{t_1,\infty}(f_1 T_{t_2-t_1,\infty}(f_2 \dots T_{t_{k-1}-t_{k-2},\infty}(f_{k-1}T_{t_k-t_{k-1},\infty}f_k)\dots))\rangle, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $L^1(F,\mu)$ and $L^\infty(F,\mu)$ as well as between $L^{\frac{p}{p-1}}(F,\mu)$ and $L^p(F,\mu)$.

Remark 2.18. *(i) Sometimes we also say that a measure P defined on $C([0,T],F_\Delta)$, $T\in\mathbb{N}$, is associated with $(T_t)_{t\geq 0}$. This means that (2.4) holds for all $0\leq t_1<\dots<t_k\leq T$, $k\in\mathbb{N}$.*

(ii) In the following we also denote integration w.r.t. to a measure $\mathbb{P}$ by $\mathbb{E}_{\mathbb{P}}$ or simply by $\mathbb{E}$ if there is no ambiguity concerning $\mathbb{P}$.

Definition 2.19. *Let $\mathbb{P}$ be a measure on $C([0,\infty),F_\Delta)$ with σ-finite initial distribution μ.*

(i) $\mathbb{P}$ is called conservative if $\mathbb{P}(\{\xi<\infty\})=0$.

(ii) The measure μ is called invariant for $\mathbb{P}$, if for all $t\in[0,\infty)$ it holds $\mathbb{P}\circ\pi_t^{-1}=\mu$.

Lemma 2.20. *Let $\mathbb{P}$ be a measure on $C([0,\infty),F_\Delta)$ with σ-finite initial distribution μ which is associated with $(T_t)_{t\geq 0}$ defined on $L^1(F,\mu)$. Then it holds*

(i) $\mathbb{P}$ is conservative if and only if $(T_t)_{t\geq 0}$ is conservative.

(ii) μ is invariant for $\mathbb{P}$ if and only if μ is invariant for $(T_t)_{t\geq 0}$.

Proof. (i) Let $0<h\in L^1(F,\mu)$ be a probability density. Then the probability measure $\widetilde{\mathbb{P}}$ defined by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}=h\circ\pi_0$ has initial distribution $h\mu$ and is also associated with $(T_t)_{t\geq 0}$. Let $0\leq\psi_n\nearrow 1_F$, $n\in\mathbb{N}$, s.t. $\psi_n\in L^1(F,\mu)$. Then via the monotone convergence theorem and the construction of $T_{t,\infty}$ it holds for $t\in\mathbb{Q}_+$

$$\mathbb{E}_{\widetilde{\mathbb{P}}}[1_F\circ\pi_t]=\lim_{n\in\mathbb{N}}\mathbb{E}_{\tilde{P}}[\psi_n\circ\pi_t]=\lim_{n\in\mathbb{N}}\left\langle h,T_{t,\infty}\psi_n\right\rangle=\left\langle h,T_{t,\infty}1\right\rangle.$$

We conclude that

$$\widetilde{\mathbb{P}}(\{\pi_t\in\{\Delta\}\})=1-\mathbb{E}_{\widetilde{\mathbb{P}}}[1_F\circ\pi_t]=1-\left\langle h,T_{t,\infty}1_F\right\rangle.$$

By Convention 2.16 it holds $\widetilde{\mathbb{P}}(\{\xi<\infty\})=\widetilde{\mathbb{P}}(\cup_{t\in\mathbb{Q}_+}\{\pi_t\in\{\Delta\}\})$. Hence we see that $\widetilde{\mathbb{P}}$ is conservative if and only if $(T_t)_{t\geq 0}$ is conservative.

(ii) The statement follows immediately by Definition 2.17 and Definition 2.19.

□

For $T\in\mathbb{N}$ we define time reversal operator r_T

$$r_T: C([0,T],F_\Delta)\longrightarrow C([0,T],F_\Delta),\ (x_t)_{t\in[0,T]}\mapsto(x_{T-t})_{t\in[0,T]} \tag{2.5}$$

The proof of the next lemma follows in a straightforward manner and is therefore omitted. Recall the semigroup $(\hat{T}_t)_{t\geq 0}$ defined in Corollary 1.51.

Lemma 2.21. *Let $\mathbb{P}$ be a conservative measure on $C([0,\infty), F_\Delta)$ with finite initial distribution μ. Assume that $\mathbb{P}$ is associated with a s.c.c.s. $(T_t)_{t\geq 0}$ on $L^1(F,\mu)$ and that μ is invariant for $(T_t)_{t\geq 0}$. Then for every $T \in [0,\infty)$ the law $\widehat{\mathbb{P}}^T := \mathbb{P} \circ R_T^{-1} \circ r_T^{-1}$ is associated with $(\hat{T}_t)_{t\geq 0}$.*

Lemma 2.22. *Let $\mathbb{P}$ be a measure on $C([0,\infty), F_\Delta)$ with initial distribution $h\mu$, $h \in L^1(F,\mu) \cap L^{\frac{p}{p-1}}(F,\mu)$ s.t. $\mathbb{P}$ is associated with $(T_t)_{t\geq 0}$. Let $f \in \mathcal{L}^p(F,\mu)$, i.e. f is a real valued measurable and p-integrable function w.r.t. μ.*

(i) Let $t \geq 0$. Then $f(\pi_t)$ is integrable w.r.t. $\mathbb{P}$ and the mapping

$$[0,t] \times C([0,\infty), F_\Delta) \longrightarrow \mathbb{R}, (s,x) \mapsto f \circ \pi_s(x) = f(x(s)) \tag{2.6}$$

is $\mathcal{B}([0,t]) \otimes \sigma(\pi_s \mid 0 \leq s \leq t)$-measurable and it holds

$$\mathbb{E}_{\mathbb{P}}\left[|f|(\pi_t)\right] \leq \|h\|_{L^{\frac{p}{p-1}}(F,\mu)} \|f\|_{L^p(F,\mu)},$$

$$\mathbb{E}_{\mathbb{P}}\left[\int_{[0,t]} |f|(\pi_s)\, ds\right] \leq t\, \|h\|_{L^{\frac{p}{p-1}}(E,\mu)} \|E\|_{L^p(E,\mu)}.$$

(ii) The set

$$A := \bigcup_{T\in\mathbb{N}} \left\{ \int_{[0,T]} |f|(\pi_s)\, ds = \infty \right\}$$

is measurable and $\mathbb{P}(A) = 0$. In particular, the map

$$[0,\infty) \ni t \longrightarrow \int_{[0,t]} f(x_s)\, ds \in \mathbb{R}$$

is for $\mathbb{P}$-a.e. $x \in C([0,\infty), E_\Delta)$ well-defined and continuous. Furthermore,

$$\int f(\pi_s)\, ds : C([0,\infty), F_\Delta) \longrightarrow C([0,\infty), \mathbb{R}), x \mapsto \begin{cases} \left(\int_{[0,t]} f(x_s)\, ds\right)_{t\in[0,\infty)} & , x \in A^c \\ 0 & , \text{else.} \end{cases}$$

is $\mathcal{B}(C([0,\infty), F_\Delta))/\mathcal{B}(C([0,\infty), \mathbb{R}))$ measurable.

(iii) The same statements as in (i), (ii) also hold true if we replace $[0,\infty)$ by $[0,T]$.

Proof. (i) The measurability is clear due to continuity of the process $(\pi_s)_{s\geq 0}$. The remaining assertions follow immediately from the association of $\mathbb{P}$ and $(T_t)_{t\geq 0}$.

(ii) By part (i) and Tonellis theorem we know that for $T \geq 0$ the set

$$\left\{ \int_{[0,T]} |f|(\pi_s)\, ds = \infty \right\} = \bigcap_{k\in\mathbb{N}} \left\{ \int_{[0,T]} |f|(\pi_s)\, ds \geq k \right\}$$

is measurable and negligible, hence A is measurable and negligible. The measurability of

$\int f(\pi_s)\,ds$ holds by Lemma 2.8.

□

In the following we explain the importance of the concept of association of a measure $\mathbb{P}$ on $C([0,\infty),F_\Delta)$ and a semigroup $(T_t)_{t\geq 0}$.

Theorem 2.23. *Let $0 \leq h \in L^1(F,\mu) \cap L^{\frac{p}{p-1}}(F,\mu)$, $p \in [1,\infty)$, be a probability density w.r.t. μ and $(T_t)_{t\geq 0}$ a s.c.c.s. with generator $(A, D(A))$ on $L^p(F,\mu)$. Further, let $\mathbb{P}$ be a measure on $C([0,\infty),F_\Delta)$ with initial distribution $\mathbb{P}\circ\pi_0^{-1} = h\mu$ s.t. $\mathbb{P}$ is associated with $(T_t)_{t\geq 0}$ in the sense of Definition 2.17. Then $\mathbb{P}$ solves the martingale problem for $(A, D(A))$, i.e., for $f \in D(A)$ the process $(M_t^{[f]})_{t\geq 0}$ defined by*

$$M_t^{[f]} := f(\pi_t) - f(\pi_0) - \int_{[0,t]} Af(\pi_s)\,ds, \quad t \geq 0, \tag{2.7}$$

is a martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t = \sigma(\pi_s \mid 0 \leq s \leq t)$, and $\mathbb{P}$. Additionally, if $f^2 \in D(A)$ and $Af \in L^{2p}(F,\mu)$ then the process $(N_t^{[f]})_{t\geq 0}$ defined by

$$N_t^{[f]} := \left(M_t^{[f]}\right)^2 - \int_{[0,t]} A(f^2)(\pi_s) - 2(fAf)(\pi_s)\,ds, \quad t \geq 0, \tag{2.8}$$

is also a martingale w.r.t. the measure $\mathbb{P}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Proof. See [26, Lemma 5.1.]. □

Definition 2.24. *Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a measure space with a sub-σ-field $\mathcal{G} \subset \mathcal{M}$. Denote by $\mathcal{N}^{\mathbb{P}}$ the null sets of $\mathbb{P}$, i.e., $\mathcal{N}^{\mathbb{P}} = \{B \subseteq \Omega \mid \exists A \in \mathcal{M}, B \subseteq A\}$. The σ-field $\mathcal{G}^{\mathbb{P}} := \{A \cup B \mid A \in \mathcal{G}, B \in \mathcal{N}^{\mathbb{P}}\}$ is called the $\mathbb{P}$-completion of $\mathcal{G}$ in $\mathcal{M}$. If $\mathcal{G} = \mathcal{M}$, then $\mathcal{M}^{\mathbb{P}}$ is simply called the $\mathbb{P}$-completion of $\mathcal{M}$. The restriction $\mathbb{P}_{|\mathcal{G}}$ extends uniquely to $\mathcal{G}^{\mathbb{P}}$ via $\bar{\mathbb{P}}(A \cup B) = \mathbb{P}(A)$, where $A \in \mathcal{G}$ and $B \in \mathcal{N}^{\mathbb{P}}$. We call the extension $\bar{\mathbb{P}}$ the completion of P on $\mathcal{G}^{\mathbb{P}}$. Furthermore, we define the universally measurable sets $\mathcal{M}^* = \cap_{\mathbb{P}\in\mathcal{P}(\Omega)}\mathcal{M}^{\mathbb{P}}$, where $\mathcal{P}(\Omega)$ denotes the set of all probability measure on $(\Omega, \mathcal{M})$.*

Observe that $\mathcal{G}^{\mathbb{P}}$ given in Definition 2.24 is again a σ-algebra and the extension $\bar{\mathbb{P}}$ is a well-defined measure. In the following we write $\mathbb{P}$ for various completions $\bar{\mathbb{P}}$, described in the last definition.

Remark 2.25. *(i) Observe that the random variables defined in (2.7) and (2.8) are well-defined by Lemma 2.22(i). Namely, different μ-versions of f, Af, $A(f^2)$, fAf represent the same equivalence class w.r.t $\mathbb{P}$ in (2.7) and (2.8).*

(ii) Assume we are in the set up of Theorem 2.23 and let $f \in D(A)$ have a continuous representative. Our goal is to see that $(M_t^{[f]})_{t\geq 0}$ has the martingale property w.r.t. the larger filtration $(\overline{\mathcal{F}}_{t+})_{t\geq 0}$ defined below. $(\overline{\mathcal{F}}_{t+})_{t\geq 0}$ satisfies the usual conditions, see e.g. [61, Definition 1.2.25]. W.l.o.g. we can assume that all random variables under consideration are defined on the completion $(C([0,\infty),F), \overline{\mathcal{B}}_C, \overline{\mathbb{P}})$ of $(C([0,\infty),F), \mathcal{B}_C, \mathbb{P})$. Now we consider the augmented filtration

$$\overline{\mathcal{F}}_t := \sigma(\mathcal{F}_t, \mathcal{N}^{\mathbb{P}}), t \geq 0,$$

where $\mathcal{N}^P$ is defined as in Definition 2.24. It is obvious that $(M_t^{[f]})_{t\geq 0}$ is still a martingale w.r.t. $(\overline{\mathcal{F}}_t)_{t\geq 0}$. By Lemma 2.22(ii) we know that $(M_t^{[f]})_{t\geq 0}$ has a version with continuous paths. Now define the right continuous augmentation of the filtration $(\mathcal{F}_t)_{t\geq 0}$ by

$$\overline{\mathcal{F}}_{t+} := \cap_{\varepsilon>0}\overline{\mathcal{F}}_{t+\varepsilon}.$$

Let $0 \leq s < t$. By [32, Corollary 2.2.10] there exists a sequence $s < t_n \leq t$, $n \in \mathbb{N}$, converging to s s.t.

$$\lim_{n\to\infty} M_{t_n} = \lim_{n\to\infty} E[M_t \mid \overline{\mathcal{F}}_{t_n}] = E[M_t \mid \overline{\mathcal{F}}_{s+}]\ \mathbb{P}\text{-a.s.}.$$

By the continuity it also holds $M_s = \lim_{n\to\infty} M_{t_n}$ $\mathbb{P}$-a.s., hence $M_s = E[M_t \mid \overline{\mathcal{F}}_{s+}]$ $\mathbb{P}$-a.s.. In particular, the same applies to the martingale in (2.8) if f has a continuous representative. Thus, if $A(f^2) - 2fAf \geq 0$ we obtain from (2.8) the quadratic variation process of continuous martingale $((M_t^{[f]})_{t\geq 0}, (\overline{\mathcal{F}}_{t+})_{t\geq 0})$.

2.2 Markov Processes

In the first subsection we briefly collect some definitions and remarks concerning Markov processes from [70, Chapter IV.]. We only collect material which is essentially needed for the further course of this thesis. Additional details and proofs can be found in [20, 98]. In the second subsection we state in Theorem 2.36 the main result from [17] and give a sufficient criteria to check the assumptions. This results consists of an existence result of a μ-standard process $\mathbf{M}$ associated with a s.c.c.s. $(T_t)_{t\geq 0}$ on $L^p(F,\mu)$. We use this process later on to obtain measures $\mathbb{P}$ on $C([0,\infty), F_\Delta)$ which are associated with $(T_t)_{t\geq 0}$ in the sense of Definition 2.17 and have initial distributions which are absolutely continuous w.r.t. μ.

In this section we assume that F is a topological Hausdorff space s.t. the Borel σ-algebra $\mathcal{B}$ of F is generated by the continuous real valued functions, i.e., $\mathcal{B} = \sigma(C(F))$. As above we enlarge the space F by adding a extra point Δ to F as an isolated point.

2.2.1 Definition of Markov Processes

The section is a brief summary from [70, Chapter IV.1.] and consists of the very basics to formulate the results in the next section properly.

Definition 2.26. *The quadruple $\mathbf{M} = \left(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\right)$ is called a (time-homogeneous) Markov process with state space F, life-time $\xi : \Omega \longrightarrow [0,\infty]$ and corresponding filtration $(\mathcal{M}_t)_{t\geq 0}$ on $\mathcal{M}$ if the following are fulfilled:*

(M1) $Z_t : \Omega \longrightarrow F_\Delta$ is $\mathcal{M}_t/\mathcal{B}_\Delta$-measurable for all $t \geq 0$ and $Z_t(\omega) = \Delta$ if and only if $t \geq \xi(\omega)$ for all $\omega \in \Omega$.

(M2) for each $t \geq 0$ there exists a shift operator $\theta_t : \Omega \longrightarrow \Omega$ s.t. $Z_s \circ \theta_t = Z_{t+s}$ for all $s \geq 0$.

(M3) for each $z \in F_\Delta$, $\mathbb{P}_z$ is a probability measure on $(\Omega, \mathcal{M})$ and the map $F_\Delta \ni z \mapsto \mathbb{P}_z(B) \in [0,1]$ is $\mathcal{B}_\Delta^*/\mathcal{B}([0,1])$-measurable for each $B \in \mathcal{M}$ respectively $\mathcal{B}_\Delta/\mathcal{B}([0,1])$-measurable if $B \in \sigma(Z_s, s \geq 0)$. Additionally, $\mathbb{P}_\Delta(Z_0 = \Delta) = 1$.

(M4) *for all* $A \in \mathcal{B}_\Delta$, $t, s \geq 0$ *and* $z \in F_\Delta$ *it holds*

$$\mathbb{P}_z(Z_{t+s} \in A \mid \mathcal{M}_t) = \mathbb{P}_{Z_t}(Z_s \in A), \quad \mathbb{P}_z\text{-a.s.},$$

where $\mathbb{P}_z(Z_{t+s} \in A \mid \mathcal{M}_t)$ *denotes the conditional expectation of* $1_A \circ Z_{t+s}$ *given* $\mathcal{M}_t$ *w.r.t.* $\mathbb{P}_z$.

Let $\boldsymbol{M} = \big(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\big)$ be a given as in Definition 2.26 with filtration $(\mathcal{M}_t)_{t\geq 0}$. Let τ be a $(\mathcal{M}_t)_{t\geq 0}$-stopping time, i.e., $\tau : \Omega \longrightarrow [0, \infty]$ and $\{\tau \leq t\} := \{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{M}_t$ for all $t \geq 0$. Then we define

$$\mathcal{M}_\tau := \{A \in \mathcal{M} \mid A \cap \{\tau \leq t\} \in \mathcal{M}_t \text{ for all } t \geq 0\},$$

and the stopped process $Z_\tau(\omega) = Z_{\tau(\omega)}(\omega)$, $\omega \in \Omega$ with the convention $Z_\infty(\omega) = \Delta$. For a σ-finite measure ν on $(F_\Delta, \mathcal{B}_\Delta)$ we define the measure P_ν on $(\Omega, \mathcal{M})$ via

$$\mathbb{P}_\nu(A) = \int_{F_\Delta} \mathbb{P}_z(A)\, \nu(dz), \quad A \in \mathcal{M}. \tag{2.9}$$

Due to (M3), the integrand $z \mapsto P_z(A)$ is measurable w.r.t. $\mathcal{B}^*_\Delta$, hence we integrate in (2.9) w.r.t. the completion of ν. If μ is an equivalent σ-finite measure of ν on $(F_\Delta, \mathcal{B}_\Delta)$ then $\mathbb{P}_\nu$ and $\mathbb{P}_\mu$ are equivalent, too.

Definition 2.27. *A Markov process* $\boldsymbol{M} = \big(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\big)$ *as in Definition 2.26 with state space* F*, life time* ξ *and filtration* $(\mathcal{M}_t)_{t\geq 0}$ *is called right process if additionally the following are fulfilled:*

(M5) $\mathbb{P}_z(Z_0 = z) = 1$ *for all* $z \in F_\Delta$.

(M6) $t \mapsto Z_t(\omega)$ *is right continuous on* $[0, \infty)$ *for all* $\omega \in \Omega$.

(M7) *The filtration* $(\mathcal{M}_t)_{t\geq 0}$ *is right continuous, i.e.,* $\mathcal{M}_t = \cap_{r>0}\mathcal{M}_{r+t}$ *for all* $t \geq 0$*. Furthermore, for any* $(\mathcal{M}_t)_{t\geq 0}$*-stopping time* τ *and probability measure* $\nu \in \mathcal{P}(F_\Delta)$ *the strong Markov property holds. Namely, for all* $A \in \mathcal{B}_\Delta$ *and* $s \geq 0$ *it holds*

$$\mathbb{P}_\nu(Z_{\tau+s} \in A \mid \mathcal{M}_\tau) = \mathbb{P}_{Z_\tau}(Z_s \in A), \quad \mathbb{P}_\nu\text{-a.s.}.$$

where $\mathbb{P}_\nu(Z_{t+s} \in A \mid \mathcal{M}_\tau)$ *denotes the conditional expectation of* $1_A \circ Z_{t+s}$ *given* $\mathcal{M}_\tau$ *w.r.t.* $\mathbb{P}_\nu$.

Let $\boldsymbol{M}$ be a Markov process. Define for $t \in [0, \infty]$ the following σ-algebras

$$\begin{aligned}
\mathcal{F}^0_t &= \sigma(Z_s, 0 \leq s \leq t),\\
\mathcal{F}^0_\infty &= \sigma(Z_s, 0 \leq s),\\
\mathcal{F}_t &= \cap_{\nu\in\mathcal{P}(F_\Delta)} \left(\mathcal{F}^0_t\right)^{\mathbb{P}_\nu}.
\end{aligned}$$

The filtration $(\mathcal{F}_t)_{t\geq 0}$ is called the natural filtration for $\boldsymbol{M}$. If $\boldsymbol{M} = \big(\Omega, \mathcal{M}, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\big)$ is a right process it can be shown that $\big(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\big)$ is a right process w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$, too. Hence, in the following we solely consider right processes with $\mathcal{M} = \mathcal{F}_\infty$ and with corresponding natural filtration.

Definition 2.28. *Let* $\boldsymbol{M} = \big(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\big)$ *be a right process with state space* F *and life time* ξ *and* ν *a* σ*-finite measure on* $(F_\Delta, \mathcal{B}_\Delta)$*.* $\boldsymbol{M}$ *is called* ν*-standard process if for one measure* $\gamma \in \mathcal{P}(F_\Delta)$ *which is equivalent to* ν *the following additional properties are satisfied:*

(M8) $Z_{t-} := \lim\limits_{\substack{s\uparrow t\\ s<t}}$ *exists in* F *for all* $t \in (0,\xi)$ $\mathbb{P}_\gamma$*-a.s..*

(M9) If τ, τ_n*,* $n \in \mathbb{N}$ *are* $\left(\mathcal{F}_t^{\mathbb{P}_\gamma}\right)_{t\geq 0}$*-stopping times such that* $\tau_n \uparrow \tau$ *then* $Z_{\tau_n} \longrightarrow Z_\tau$ *as* $n \to \infty$ $\mathbb{P}_\gamma$*-a.s. on* $\{\tau < \xi\}$*.*

2.2.2 Semigroup of Transition Kernels and associated Operator Semigroup

Let now $\boldsymbol{M} = \big(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\big)$ be a right process with state space F and life time ξ. Define for $t \geq 0$

$$p_t(\cdot,\cdot) : F_\Delta \times \mathcal{B}_\Delta \longrightarrow [0,1], (z, A) \mapsto \mathbb{E}_z[1_A(Z_t)],$$

where $E_z[\cdot]$ denotes the expectation w.r.t. P_z. Obviously, p_t is a kernel on $(F_\Delta, \mathcal{B}_\Delta)$, i.e., for every $z \in E_\Delta$ we obtain a measure $p_t(z,\cdot)$ on $\mathcal{B}_\Delta$ and for every $A \in \mathcal{B}_\Delta$ the function $p_t(\cdot, A)$ is $\mathcal{B}_\Delta/\mathcal{B}([0,1])$ measurable. For a non-negative $\mathcal{B}_\Delta$-measurable function f on F_Δ and $z \in F_\Delta$ we define

$$p_t f(z) := \int\limits_{F_\Delta} f(y) p_t(z, dy) = \mathbb{E}_z[f(Z_t)].$$

By linearity this definition extends to measurable f if $p_t f^+(z)$ or $p_t f^-(z)$ is finite for every $z \in F_\Delta$. The Markov property (M4) and a monotone class argument imply that the family of kernels $(p_t)_{t>0}$ satisfies the semigroup property, i.e., for a non-negative measurable function f it holds $p_{t+s} f = p_t(p_s f)$. Since the process $(Z_t)_{t\geq 0}$ is right continuous, it holds that

$$Z : [0,\infty) \times \Omega \longrightarrow F_\Delta, (t,\omega) \mapsto Z_t(\omega)$$

is $\mathcal{B}([0,\infty)) \otimes \mathcal{F}^0_\infty/\mathcal{B}_\Delta$-measurable. Hence, for every non-negative $\mathcal{B}_\Delta$-measurable function f, $z \in F_\Delta$ and $\alpha > 0$ it holds via Fubini's theorem

$$R_\alpha f(z) := \int\limits_{[0,\infty)} e^{-\alpha t} p_t f(z)\, dt = \mathbb{E}_z\left[\int\limits_{[0,\infty)} e^{-\alpha t} f(Z_t)\, dt\right]. \tag{2.10}$$

Furthermore, by Tonelli's theorem $\int\limits_{[0,\infty)} e^{-\alpha t} f(Z_t)\, dt$ is $\mathcal{F}^0_\infty/\mathcal{B}(\mathbb{R})$-measurable. Hence, by (M3) we obtain that $R_\alpha f$ is $\mathcal{B}_\Delta$-measurable. As above, this extends to general measurable f if $R_\alpha f^+(z)$ or $R_\alpha f^-(z)$ is finite for every $z \in F_\Delta$. Observe that the semigroup property of $(p_t)_{t>0}$ implies the resolvent equation is valid, i.e., for $\alpha, \beta > 0$ it holds

$$R_\alpha = R_\beta + (\beta - \alpha) R_\beta R_\alpha.$$

The two families of kernels $(p_t)_{t>0}$ and $(R_\alpha)_{\alpha>0}$ are called the associated transition semigroup and resolvent of $\boldsymbol{M}$, respectively.

Definition 2.29. *Let $\mathbf{M}$ be a right process with state space F and life time ξ with associated transition semigroup $(p_t)_{t>0}$. Let $(T_t)_{t\geq 0}$ be a sub-Markovian s.c.c.s. on $L^p(F,\mu)$, where $p \in [1,\infty)$ and μ is a σ-finite measure on $(F,\mathcal{B})$. Then $\mathbf{M}$ is called to be associated with $(T_t)_{t\geq 0}$ if for all $t \geq 0$ and all $F \in L^p(F,\mu) \cap L^\infty(F,\mu)$ with bounded μ-version $\tilde{f}$ it holds that $p_t\tilde{f}$ is a μ-version of $T_t f$.*

Equivalently, the concept described in Definition 2.29 can be expressed via the corresponding resolvents $(R_\alpha)_{\alpha>0}$ and $(G_\alpha)_{\alpha>0}$, see the next lemma.

Lemma 2.30. *Let $\mathbf{M}$ be a right process with state space F, life time ξ, associated transition semigroup $(p_t)_{t>0}$ and resolvent $(R_\alpha)_{\alpha>0}$. Further, let $(T_t)_{t\geq 0}$ be a sub-Markovian s.c.c.s. on $L^p(F,\mu)$ with corresponding s.c.c.r. $(G_\alpha)_{\alpha>0}$, where $p \in [1,\infty)$ and μ a σ-finite measure on $(F,\mathcal{B})$. Then $\mathbf{M}$ is associated with $(T_t)_{t\geq 0}$ in the sense of Definition 2.29 if and only if for all $\alpha > 0$ and all $f \in L^p(F,\mu) \cap L^\infty(F,\mu)$ with bounded μ-version $\tilde{f}$ it holds that $R_\alpha\tilde{f}$ is a μ-version of $G_\alpha f$.*

Proof. Let $f \in L^p(F,\mu) \cap L^\infty(F,\mu)$ with bounded and non-negative μ-version $\tilde{f}$. Using a monotone class argument and the assumption $\sigma(C(F)) = \mathcal{B}$ we can also assume that $\tilde{f}$ is continuous. Assume $R_\alpha\tilde{f}$ is a μ-version of $G_\alpha f$ for $\alpha > 0$. Denote by $\langle\cdot,\cdot\rangle$ the dual pairing between $L^p(F,\mu)$ and $L^q(F,\mu)$, where q is the Hölder conjugate of p. It suffices to show $\langle T_t f, g\rangle = \left\langle p_t\tilde{f}, g\right\rangle$ for all $t \geq 0$ and all $0 \leq g \in L^q(F,\mu) \cap L^1(F,\mu)$. Let $\beta > 0$ be fixed. Via Tonelli's theorem it holds for all $\alpha \geq 0$

$$\int_0^\infty e^{-\alpha t}e^{-\beta t}\left\langle p_t\tilde{f}, g\right\rangle\,dt = \left\langle R_{\alpha+\beta}\tilde{f}, g\right\rangle = \left\langle G_{\alpha+\beta}f, g\right\rangle = \int_0^\infty e^{-\alpha t}e^{-\beta t}\left\langle T_t f, g\right\rangle\,dt.$$

Hence, the finite measures $e^{-\beta t}\langle T_t f, g\rangle\,dt$ and $e^{-\beta t}\left\langle p_t\tilde{f}, g\right\rangle\,dt$ coincide on a separating algebra of continuous and bounded functions, therefore they coincide, see e.g. [32, Theorem 3.4.5]. Due to the continuity of $t \mapsto \langle T_t f, g\rangle$ and the right continuity of $t \mapsto \left\langle p_t\tilde{f}, g\right\rangle$ we obtain $\langle T_t f, g\rangle = \left\langle p_t\tilde{f}, g\right\rangle$ for every $t \geq 0$. The reverse implication works similar. □

Remark 2.31. *In Definition 2.17 we introduced what it means that a measure P on $C([0,\infty),F_\Delta)$ is associated with a sub-Markovian s.c.c.s. $(T_t)_{t\geq 0}$ on $L^p(F,\mu)$. Assume now that F is Polish. Now assume that a right process $\mathbf{M} = \left(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\right)$ is associated with $(T_t)_{t\geq 0}$ in the sense of Definition 2.17. Let $h \in L^1(F,\mu) + L^{\frac{p}{p-1}}(F,\mu)$. Furthermore, assume that the paths of $(Z_t)_{t\geq 0}$ are $P_{h\mu}$-a.s. continuous. This implies that the set*

$$C := \{\omega \in \Omega \mid (Z_t(\omega))_{t\geq 0} \in C([0,\infty),F_\Delta)\}$$

is measurable w.r.t. $\mathcal{F}_*^{P_{h\mu}}$. Consider for an arbitrary point $x_0 \in F$ the measurable map

$$\Phi : (\Omega, \mathcal{F}_*^{P_{h\mu}}) \longrightarrow (C([0,\infty),F_\Delta),\mathcal{B}_C),\ \omega \mapsto \begin{cases} (Z_t(\omega))_{t\geq 0} & \text{if } \omega \in C, \\ (x_0)_{t\geq 0} & \text{else.} \end{cases}$$

Now we claim that the image measure $P = P_{h\mu} \circ \Phi^{-1}$ is associated with $(T_t)_{t\geq 0}$ in the sense of Definition 2.17. Obviously, by (M1) the measure P satisfies Convention 2.16. Now let $f_1, f_2, .., f_n : F \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, be bounded and μ-integrable functions and $0 \leq t_1 < t_2 < .. < t_n < \infty$. Then, by *the Markov property (M4) we obtain*

$$\begin{aligned} E_\cdot\left[f_1(Z_{t_1})f_2(Z_{t_2})\right] &= E_\cdot\left[f_1(Z_{t_1})E\left[f_2(Z_{t_2-t_1+t_1}) \mid \mathcal{F}^0_{t_1}\right]\right] \\ &= E_\cdot\left[f_1(Z_{t_1})E_{Z_{t_1}}\left[f_2(Z_{t_2-t_1})\right]\right] \\ &= E_\cdot\left[\left(f_1 E_\cdot\left[f_2(Z_{t_2-t_1})\right]\right)(Z_{t_1})\right] \\ &= p_{t_1}(f_1 p_{t_2-t_1} f_2)(\cdot). \end{aligned}$$

Via induction we obtain for every $n \in \mathbb{N}$

$$E_\cdot\left[\prod_{i=1}^{n} f_i(Z_{t_i})\right] = p_{t_1}(f_1(p_{t_2-t_1}(f_2 \ldots p_{t_n-t_{n-1}} f_n)\ldots)(\cdot). \tag{2.11}$$

By assumption the right hand-side is a μ-version of $T_{t_1}(f_1(T_{t_2-t_1}(f_2 \ldots T_{t_n-t_{n-1}} f_n)\ldots)$. Now the claim follows via integrating both sides of (2.11) against the measure $h\mu$ and the transformation formula for image measures.

2.2.3 Existence Results for L^p semigroups

In this subsection we state the main result of [17]. This result consists of sufficient conditions on the generator $(A, D(A))$ of a s.c.c.s. $(T_t)_{t\geq 0}$ s.t. this semigroup is associated with a right process $\boldsymbol{M}$ in the sense of Definition 2.29. Let F be a Lusin space, i.e., $(F, \mathcal{T})$ is a topological Hausdorff space s.t. F carries a finer topology $\widetilde{\mathcal{T}}$, s.t. $(F, \widetilde{\mathcal{T}})$ is a Polish space. As a measurable space we always consider F with the Borel σ-algebra $\mathcal{B}$ induced by $\mathcal{T}$. Further, let μ be a σ-finite measure on $(F, \mathcal{B})$. Let $(T_t)_{t\geq 0}$ and $(G_\alpha)_{\alpha>0}$ be fixed corresponding sub-Markovian s.c.c.s. and sub-Markovian s.c.c.r., respectively, on $L^p(F, \mu)$, $p \in [1, \infty)$, with generator $(A, D(A))$.

Definition 2.32. *An element $u \in L^p(F, \mu)$ is called α-excessive (w.r.t. $(G_\beta)_{\beta>0}$) if $\beta G_{\beta+\alpha} u \leq u$ for all $\beta > 0$. The set of all α-excessive elements is denoted by E_α.*

In the next proposition we gather some well-known properties of α-excessive elements. For convenience we give the elementary proofs.

Proposition 2.33. *Let $u \in L^p(F, \mu)$ and $\alpha > 0$.*

(i) The element u is α-excessive if and only if $e^{-\alpha t} T_t u \leq u$ for all $t \geq 0$.

(ii) If u is α-excessive then $u \geq 0$. Furthermore, if additionally $u > 0$ then $G_\alpha u > 0$.

(iii) Let $u \in D(A)$. Then, u is α-excessive if and only if $(\alpha - A)u \geq 0$. In particular, for $u \in D(A) \cap E_\alpha$ there exists $f \geq 0$ s.t. $u = G_\alpha f$.

(iv) If u, v are α-excessive, then $u \wedge v$ is α-excessive.

(v) If $u \geq 0$, then $G_\alpha u$ is α-excessive.

(vi) Let $u_n \in E_\alpha$, $n \in \mathbb{N}$, be an increasing sequence, i.e., $u_n \leq u_{n+1}$ for all $n \in \mathbb{N}$. If $\sup_{n\in\mathbb{N}} u_n \in L^p(F;\mu)$, then $\sup_{n\in\mathbb{N}} u_n \in E_\alpha$.

Proof. (i) This part can be seen by using the Laplace transform representation of G_α, see (1.2) and the Yosida Approximations S_t^β, $\beta > 0$, for $S_t := e^{-\alpha t}T_t$, see the proof of Theorem 1.10.

(ii) If u is α-excessive we obtain $u \geq e^{-\alpha t}T_t u \xrightarrow{t\to\infty} 0$, which proves the first part. Now let $u > 0$ and denote by $q \in (1,\infty]$ the Hölder conjugate of p. Define

$$A := \{v \in L^q(F,\mu) \mid v = 0 \text{ on the set } \{G_\alpha u = 0\}\}.$$

Note that the adjoint G'_α is positive due to Lemma 1.44. Furthermore, G'_α is injective since G_α has dense range. Let $v \in A$ and assume w.l.o.g. $v \geq 0$. Then we obtain

$$0 = \langle G_\alpha u, v\rangle = \langle u, G'_\alpha v\rangle = \int_F \underbrace{u(x)}_{>0} \underbrace{G'_\alpha v(x)}_{\geq 0} \mu(dx).$$

Consequently, $G'_\alpha v = 0$ which implies $v = 0$. Hence, $\mu(\{G_\alpha u = 0\}^c) = 0$.

(iii) This property follows immediately by part (i).

(iv) Observe that T_t is positive. Thus, it holds

$$e^{-\alpha t}T_t(u \wedge v) \leq e^{-\alpha t}T_t u \wedge e^{-\alpha t}T_t v \leq u \wedge v,$$

if u and v are α-excessive.

(v) The assertion follows from the Laplace transform representation of G_α and the positivity of T_t, $t \geq 0$.

(vi) If the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and $\sup_{n\in\mathbb{N}} u_n \in L^p(F,\mu)$, then it holds $\lim_{n\to\infty} u_n = \sup_{n\in\mathbb{N}} u_n$. The assertion follows then from the continuity of T_t and part (i).

□

For an element $u \in L^p(F,\mu)$, we define the set $\mathcal{L}_u := \{v \in L^p(E,\mu) \mid v \geq u\}$.

Lemma 2.34. *Let $u \in L^p(F,\mu)$, $\alpha > 0$ and assume $\mathcal{L}_u \cap E_\alpha \neq \emptyset$. Then, there exists an element $R_\alpha u \in \mathcal{L}_u \cap E_\alpha$ s.t. $R_\alpha u \leq v$ for all $v \in \mathcal{L}_u \cap E_\alpha$. In particular, $R_\alpha u = \inf \mathcal{L}_u \cap E_\alpha$.*

Proof. See [28, Proposition 3.1.5]. □

The element $R_\alpha u$ (if it exists) given in Lemma 2.34 is called the α-reduced element of u (w.r.t. $(G_\alpha)_{\alpha>0}$).

Definition 2.35. *Let $(E_k)_{k\in\mathbb{N}}$ be an increasing sequence of closed subsets of F. We call $(E_k)_{k\in\mathbb{N}}$ a nest ($(G_\alpha)_{\alpha>0}$-nest) if $R_1(1_{E_k^c}u) \longrightarrow 0$ in $L^p(F,\mu)$ for all $u \in D(A) \cap E_1$.*

The following theorem is special formulation of the main result in [17, Theorem 1.1.], see also Remark 1.2. in the last mentioned reference.

Theorem 2.36. *Let F, μ and $(G_\alpha)_{\alpha>0}$ be given as above. We assume additionally that*

(I) There exists a nests consisting of compact sets.

(II) There exists a countable $\mathbb{Q}$-algebra $\mathcal{A} \subseteq D(A) \cap C_b(F)$ which seperates the points of F and forms a core for $(A, D(A))$.

Then there exists a μ-standard right process $\boldsymbol{M} = \left(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\right)$ with state space F equipped with the topology $\mathcal{T}_0$ generated by $\mathcal{A}$ and life-time ξ, s.t. $\boldsymbol{M}$ is associated with $(G_\alpha)_{\alpha>0}$ in the sense of Definition 2.29. The paths of $\boldsymbol{M}$ are càdlàg P_μ-a.s. w.r.t. $\mathcal{T}_0$.

Remark 2.37. *For the application of Theorem 2.36 we have in mind, the topology generated by $\mathcal{A}$ coincides with the original one, so we don't have to deal with two different topologies.*

For the rest of this section let $(T_t)_{t\geq 0}$ be a sub-Markovian s.c.c.s. on $L^1(F,\mu)$ with generator $(A, D(A))$. We recall the following notation. $(\hat{T}_t)_{t\geq 0}$ denotes the sub-Markovian semigroup on $L^1(F,\mu)$ corresponding to $(T_t)_{t\geq 0}$ via Corollary 1.51. Furthermore, $(G_\alpha)_{\alpha>0}$ and $(\hat{G}_\alpha)_{\alpha>0}$ are the corresponding sub-Markovian resolvents, respectively. The extensions of $(T_t)_{t\geq 0}$, $(\hat{T}_t)_{t\geq 0}$, $(G_\alpha)_{\alpha>0}$ and $(\hat{G}_\alpha)_{\alpha>0}$ to $L^2(F,\mu)$ given by Theorem 1.49 are denoted by the symbols $\left(T_{t,2}\right)_{t\geq 0}$, $(\hat{T}_{t,2})_{t\geq 0}$, $\left(G_{\alpha,2}\right)_{\alpha>0}$ and $(\hat{G}_\alpha)_{\alpha>0}$, respectively. The generators of $\left(T_{t,2}\right)_{t\geq 0}$ and $(\hat{T}_{t,2})_{t\geq 0}$ are denoted by $(A_2, D(A_2))$ and $(\hat{A}_2, D(\hat{A}_2))$, respectively.
The following proposition is taken from [17, Remark 2.2.] and is stated in the last mentioned reference without proof. We give a proof for the special case $p = 1$. The prove relies on the techniques from the analytic potential theory of generalized Dirichlet forms, see, e.g. [101, Chapter III.].

Proposition 2.38. *Assume $p = 1$ and let $(E_k)_{k\in\mathbb{N}}$ be an increasing sequence of closed subsets of F and let $\varphi \in L^1(F,\mu)\cap L^\infty(F,\mu)$ s.t. $\varphi > 0$. Then, $(E_k)_{k\in\mathbb{N}}$ is a $(G_\alpha)_{\alpha>0}$-nest, if $\lim_{k\to\infty} R_1(1_{E_k^c} G_1\varphi) = 0$.*

Proof. First we recall that $T^*_{t,2} = \hat{T}_{t,2}$ for all $t \geq 0$ and $G^*_{\alpha,2} = \hat{G}_{\alpha,2}$ for all $\alpha > 0$ which follows from (1.18). Denote by $\mathcal{E}$ the generalized Dirichlet form associated with $(A_2, D(A_2))$ in the sense of [101, Example I.4.9.(ii)]. Let φ and $(E_k)_{k\in\mathbb{N}}$ be given as above. Observe that an element $u \in L^1(F,\mu)\cap L^2(F,\mu)$ is 1-excessive w.r.t. $(G_\alpha)_{\alpha>0}$ if and only if it is 1-excessive w.r.t. $\left(G_{\alpha,2}\right)_{\alpha>0}$. So, we obtain for all $k\in\mathbb{N}$ that $R_1(1_{E_k^c}G_1\varphi)$ coincides with the 1-reduced function of $1_{E_k^c}G_{1,2}\varphi$ in the sense of [101, Definition III.1.8]. Moreover, we have

$$\mathrm{Cap}_\varphi(E_k^c) := \int_E R_1(1_{E_k^c}G_1\varphi)\varphi \, d\mu \longrightarrow 0 \text{ as } k\to\infty.$$

Therefore, by [101, Proposition III.2.10] we obtain that $(E_k)_{k\in\mathbb{N}}$ is a nest in the sense of [101, Definition III.1.(i)].
Now let $u \in D(A)\cap E_1$ be arbitrary. By Proposition 2.33(iii) there exists $f \in L^1(F,\mu)$ s.t. $f \geq 0$ with $u = G_1 f$. Let $n\in\mathbb{N}$ and define

$$u_n := G_1(f\wedge n) \in L^1(F,\mu)\cap L^\infty(F,\mu).$$

Then, we get

$$Au_n = u_n - f\wedge n \in L^1(F,\mu)\cap L^\infty(F,\mu).$$

We see by Proposition 2.33(v) that u_n is 1-excessive. By Proposition 2.33(v)i and the fact that G_1 is sub-Markovian, we obtain for all $k\in\mathbb{N}$ that

$$R_1(1_{E_k^c}u) = \sup_{n\in\mathbb{N}} R_1(1_{E_k^c}u_n).$$

Define $\hat{g} := \hat{G}_1\varphi$ and recall the functions $\hat{g}_{E_k^c}$ defined in [101, Section III.2]. As above, one sees that $\hat{g}_{E_k^c}$ coincides with $\hat{R}_1(1_{E_k^c}\hat{g})$, the 1-reduced of $1_{E_k^c}\hat{g}$ w.r.t. $(\hat{G}_\alpha)_{\alpha>0}$. Hence, via [101, Proposition III.2.9, III.2.10] and the monotone convergence theorem, we obtain

$$\begin{aligned}\int_F R_1(1_{E_k^c}u)\varphi\,d\mu &= \sup_{n\in\mathbb{N}}\int_F R_1(1_{E_k^c}u_n)\varphi\,d\mu\\ &= \sup_{n\in\mathbb{N}}\mathcal{E}_1(R_1(1_{E_k^c}u_n),\hat{g})\\ &= \sup_{n\in\mathbb{N}}\mathcal{E}_1(u_n,\hat{g}_{E_k^c})\\ &= \sup_{n\in\mathbb{N}}\int_F((1-A)u_n)\hat{g}_{E_k^c}\,d\mu\\ &= \sup_{n\in\mathbb{N}}\int_F(f\wedge n)\hat{g}_{E_k^c}\,d\mu\\ &= \int_F f\hat{g}_{E_k^c}\,d\mu\longrightarrow 0,\end{aligned}$$

as $k\longrightarrow\infty$. □

For subsets $B\subseteq L^1(F,\mu)$ and $G\subseteq F$ we denote by B_G the set $B_G := \{f\in B\mid f=0 \text{ on } G^c\}$. The next lemma gives a handy condition for an increasing sequence of closed sets $(E_k)_{k\in\mathbb{N}}$ to be a nest.

Lemma 2.39. *Assume $p = 1$ and let $(E_k)_{k\in\mathbb{N}}$ be an increasing sequence of closed subsets of F. Assume that $\cup_{k\in\mathbb{N}}D(A)_{E_k}$ contains some subset D which is also contained in $D(A_2)$ and is a core for $(A,D(A))$. Then, $(E_k)_{k\in\mathbb{N}}$ is a $(G_\alpha)_{\alpha>0}$-nest.*

Proof. For the proof we use again techniques from analytic potential theory of generalized Dirichlet forms, meaning that we follow the lines of [101, Remark III.2.11.]. Recall the notation used in the proof of Proposition 2.38. In particular, denote by $\mathcal{E}$ the generalized Dirichlet form associated with $(A_2,D(A_2))$ as in the previous proposition, cf. [101, Example I.4.9.(ii)].
Now, let $\varphi\in L^1(E,\mu)$ fulfill $0<\varphi\le 1$. By Proposition 2.38 and monotonicity we need to show

$$U := \lim_{k\to\infty} R_1(1_{E_k^c}G_1\varphi) = 0 \quad \mu\text{-a.e..}$$

Define $\hat{g} = \hat{G}_1\varphi$ and $h = G_1\varphi$. As in the proof of Proposition 2.38, we obtain that $R_1(1_{E_k^c}G_1\varphi)$ and $\hat{R}_1(1_{E_k^c}\hat{G}_1\varphi)$ coincide with the 1-reduced function $h_{E_k^c}$ and the 1-coreduced function $\hat{g}_{E_k^c}$ on E_k^c, respectively, cf. [101, Section III.2.]. Let $\varepsilon>0$ and $\tilde{h}\in D$ be arbitrary and choose $l\in\mathbb{N}$ with

$\tilde{h} \in D(A)_{E_l}$.

$$\begin{aligned}
\int_F U\varphi \, d\mu &= \lim_{k\to\infty} \int_F R_1(1_{E_k^c} G_1\varphi)\varphi \, d\mu \\
&= \lim_{k\to\infty} \mathcal{E}_1(R_1(1_{E_k^c} G_1\varphi), \hat{g}) \\
&= \lim_{k\to\infty} \mathcal{E}_1(G_1\varphi, \hat{g}_{E_k^c}) \\
&\leq \limsup_{k\to\infty} \int_F (1-A)(G_1\varphi - \tilde{h})\hat{g}_{E_k^c} \, d\mu + \limsup_{k\to\infty} \int_F (1-A)\tilde{h}\hat{g}_{E_k^c} \, d\mu \\
&= \limsup_{k\to\infty} \int_F (1-A)(G_1\varphi - \tilde{h})\hat{g}_{E_k^c} \, d\mu + \limsup_{k\to\infty} \mathcal{E}_1(\tilde{h}, \hat{g}_{E_k^c}) \qquad (2.12)
\end{aligned}$$

It is clear that $\hat{g}_{E_k^c} \leq \hat{g} \leq 1$, since $\hat{G}_1$ is sub-Markovian. Now, choose $\tilde{h} \in D$ s.t.

$$\left\| (1-A)(G_1\varphi - \tilde{h}) \right\|_{L^1(F,\mu)} \leq \varepsilon.$$

To estimate the second term in (2.12) we use [101, Proposition III.1.6.]. In particular, let $\hat{g}^{\alpha}_{E_k^c}$, $\alpha > 0$, be the element from [101, Proposition III.1.6.] s.t.

$$\begin{aligned}
\mathcal{E}_1(\tilde{h}, \hat{g}_{E_k^c}) &= \lim_{\alpha\to\infty} \mathcal{E}_1(\tilde{h}, \hat{g}^{\alpha}_{E_k^c}) \\
&= \lim_{\alpha\to\infty} \alpha \int_E \tilde{h} \left(\hat{g}^{\alpha}_{E_k^c} - 1_{E_k^c}\hat{g} \right)^{-} d\mu.
\end{aligned}$$

Observe that $\hat{g}^{\alpha}_{E_k^c}$, $\alpha > 0$, is 1-excessive w.r.t. $(\hat{G}_\alpha)_{\alpha>0}$, see [101, Proposition III.1.4., Proposition III.1.6.]. In particular, $\hat{g}^{\alpha}_{E_k^c}$ is non-negative. Thus, $(\hat{g}^{\alpha}_{E_k^c} - 1_{E_k^c}\hat{g})^{-} = 0$ on the set E_k. Finally, for $k \geq l$ we conclude $\int_E U\varphi \, d\mu \leq \varepsilon$ which means $U = 0$. □

The following theorem is a slight modification of [24, Lemma 2.1.10.] in consideration of Remark 2.31, see also [88, Theorem 6.3] for the original idea of the proof. It can be proven by the exact same arguments as in [24, Lemma 2.1.10.].

Theorem 2.40. *Let $\boldsymbol{M} = \left(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\right)$ be a right process associated with a sub-Markovian s.c.c.s. $(T_t)_{t\geq 0}$ on $L^1(F,\mu)$ with invariant measure μ and generator $(A, D(A))$. Furthermore, assume that there exists a set $C \subseteq D(A) \cap L^\infty(F,\mu)$ s.t. (A, C) is an abstract diffusion operator and C consists of continuous functions with the property that for every $f \in C$ it exists a $\delta > 0$ s.t. $Af \in L^{1+\delta}(F,\mu)$. Then it holds that for every $f \in C$ the paths of $(f(Z_t))_{t\geq 0}$ are P_μ-a.s. continuous.*

We obtain as a direct consequence of the last theorem a sufficient condition for the continuity of the paths of a right process.

Corollary 2.41. *Let $\boldsymbol{M} = \left(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\right)$, $(T_t)_{t\geq 0}$, μ and C be given as in Theorem 2.40. Assume that there exists a countable set $\tilde{C} \subseteq C$ which separates the points of F. Further, assume that the paths $(Z_t)_{t\geq 0}$ are P_μ-a.s. càdlàg on $[0, \xi)$. Then, the paths are P_μ-a.s. continuous on $[0, \xi)$.*

Chapter 3

Overdamped Limit of Generalized Hamiltonian Systems

In this chapter we establish the overdamped limit for generalized stochastic Hamiltonian systems. Namely, for $\varepsilon > 0$ and two potentials $\Phi_1, \Phi_2 : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$, $d \in \mathbb{N}$, we construct for the stochastic differential equation

$$dX_t^\varepsilon = \frac{1}{\varepsilon}\nabla\Phi_2(V_t^\varepsilon)dt, \tag{3.1}$$

$$dV_t^\varepsilon = -\frac{1}{\varepsilon}\nabla\Phi_1(X_t^\varepsilon)dt - \frac{1}{\varepsilon^2}\nabla\Phi_2(V_t^\varepsilon)dt + \frac{1}{\varepsilon}\sqrt{2}dB_t. \tag{3.2}$$

a weak solutions $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ with initial distribution given by $h\mu_\Phi$. Here μ_Φ is the measure which is absolutely continuous w.r.t. the Lebesgue measure $d(x,v)$ on $\mathbb{R}^{2d}$ with density $e^{-\Phi_1(x)-\Phi_2(v)}$, $x, v \in \mathbb{R}^{2d}$, and $h \in L^1(\mathbb{R}^{2d}, \mu_\Phi) \cap L^2(\mathbb{R}^{2d}, \mu_\Phi)$. Additionally, we construct for the *overdamped Langevin* equation

$$dX_t^0 = -\nabla\Phi_1(X_t^0)dt + \sqrt{2}dB_t. \tag{3.3}$$

a weak solution $\left(X_t^0\right)_{t\geq 0}$ with initial distribution given by $\tilde{h}\mu_{\Phi_1}$. Similar as above, μ_{Φ_1} is absolutely continuous w.r.t. the Lebesgue measure dx on $\mathbb{R}^d$ with density $e^{-\Phi_1(x)}$, $x \in \mathbb{R}^d$, and $\tilde{h} \in L^1(\mathbb{R}^d, \mu_{\Phi_1}) \cap L^2(\mathbb{R}^d, \mu_{\Phi_1})$. The main result of this chapter is the proof of the weak convergence of the laws $\mathcal{L}\left((X_t^\varepsilon)_{t\geq 0}\right)$, $\varepsilon > 0$, to the law $\mathcal{L}\left((X_t^0)_{t\geq 0}\right)$. The convergence takes place on the space of probability measures on the measurable space $\left(C([0,\infty), \mathbb{R}^d), \mathcal{B}\right)$, where $\mathcal{B}$ denotes the Borel σ-algebra of the topology of uniform convergence on compact sets, see Section 2.1. The specific assumptions on the potentials Φ_1 and Φ_2 are given in the course of this chapter. For the derivation and the physical framework of these equations we refer the reader to the introduction in Chapter 0.

Let us briefly give an outline of the proof of our main result. First, we recall that for the pair $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ the generator $L_{\mathbf{\Phi}}^\varepsilon$ of (3.1), (3.2) is given through Itô's formula by

$$L_{\mathbf{\Phi}}^\varepsilon f = \frac{1}{\varepsilon^2}\left(\Delta_v f - \nabla_v\Phi_2 \cdot \nabla_v f\right) + \frac{1}{\varepsilon}\left(\nabla_v\Phi_2 \cdot \nabla_x f - \nabla_x\Phi_1 \cdot \nabla_v f\right), \tag{3.4}$$

where $f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$. To keep the notation simple, we write in the following $L_{\mathbf{\Phi}}$ to denote $L_{\mathbf{\Phi}}^1$. To construct a weak solution $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ of (3.1), (3.2) and to prove the above mentioned convergence of the laws we perform several intermediate steps. These can be described as follows. In Section 1 we prove essential m-dissipativity of the operator $\left(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$, where $\mu_{\mathbf{\Phi}}$ is an invariant measure for $L_{\mathbf{\Phi}}$. In Section 2 we use the results from Section 2.2 to obtain martingale solutions $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$ for $\left(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ via an associated right process $\boldsymbol{M}$. Here, $h\mu_{\mathbf{\Phi}}$ denotes the initial distribution of $\mathbb{P}_{h\mu_{\mathbf{\Phi}}}$ and $h \in L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$. In Section

3 we introduce the generator L_{Φ_1} of (3.3) and show by the same methods as in Section 2 the existence of martingale solutions $\mathbb{P}_{h_0\mu_{\Phi_1}}$, where $h_0 \in L^1(\mathbb{R}^d, \mu_{\Phi_1})$. From theses martingale solutions we derive weak solutions $\left(X_t^0\right)_{t\geq 0}$ with corresponding initial distribution. In Section 4 we consider for ε the scaled velocity potential $\Phi_2^\varepsilon(\cdot) = \Phi_2(\frac{\cdot}{\varepsilon}) + \ln(\varepsilon^d)$ and the pair of potentials $\mathbf{\Phi}^\varepsilon = (\Phi_1, \Phi_2^\varepsilon)$. Then we prove convergence of semigroup $\left(T_t^\varepsilon\right)_{t\geq 0}$ defined on $L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}^\varepsilon})$ which are generated by an extension of $\left(L_{\mathbf{\Phi}^\varepsilon}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$. The limit semigroup $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ admits the closure of $\left(L_{\Phi_1}, C_c^\infty(\Phi_1 < \infty)\right)$ as its generator. The operators $L_{\mathbf{\Phi}^\varepsilon}$ and $L_{\mathbf{\Phi}}^\varepsilon$ are related by a unitary transformation U_ε. The unitary transformation U_ε is given by $U_\varepsilon f = f \circ \widetilde{U}_\varepsilon$, $f \in L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$, where $\widetilde{U}_\varepsilon(x,v) = (x, \frac{1}{\varepsilon}v)$. From the martingale solution $\mathbb{P}_{h_\varepsilon \mu_{\mathbf{\Phi}^\varepsilon}}$ we derive a weak solution $(X_t^{1,\mathbf{\Phi}^\varepsilon}, V_t^{1,\mathbf{\Phi}^\varepsilon})_{t\geq 0}$ to (3.1), (3.2) for $\varepsilon = 1$, with corresponding initial distribution. In Section 5 we prove tightness of the martingale solutions $(\mathbb{P}_{h_\varepsilon \mu_{\mathbf{\Phi}^\varepsilon}})_{\varepsilon>0}$. By combining the tightness result and the semigroup convergence we show that the position marginals $(\mathbb{P}^X_{h_\varepsilon \mu_{\mathbf{\Phi}^\varepsilon}})_{\varepsilon>0}$ converge weakly to the martingale solution $\mathbb{P}_{h_0\mu_{\Phi_1}}$ of L_{Φ_1} under mild assumptions on the initial densities h_ε, $\varepsilon > 0$. Eventually, in Section 6 we show how these results apply to the original problem. We basically invert the unitary transformation U_ε and apply Itô's formula to the function $\widetilde{U}_\varepsilon$ and $(X_t^{1,\mathbf{\Phi}^\varepsilon}, V_t^{1,\mathbf{\Phi}^\varepsilon})_{t\geq 0}$ to obtain a weak solution of $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ of (3.1), (3.2). The desired convergence result we deduce from the weak convergence established in Section 5. In the last section we state an example and sufficient criteria from the existing literature to check some of the assumptions we impose on the potentials.

3.1 M-Dissipativity of the Operator $L_{\mathbf{\Phi}}$

The main goal of this section is to establish for a pair $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ of potentials essential m-dissipativity of the differential operator $\left(L_{\mathbf{\Phi}}, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ given by

$$L_{\mathbf{\Phi}} f = \Delta_v f - \nabla_v \Phi_2 \cdot \nabla_v f + \nabla_v \Phi_2 \cdot \nabla_x f - \nabla_x \Phi_1 \cdot \nabla_v f, \quad f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}) \tag{3.5}$$

on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$, $d \in \mathbb{N}$, where $\mu_{\mathbf{\Phi}}$ is absolutely continuous w.r.t. the Lebesgue measure dx on $\left(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d})\right)$. For this whole chapter, we fix $d \in \mathbb{N}$. We follow closely the argumentation in [26] and generalize the proofs therein for a general velocity potential Φ_2 fulfilling the Assumptions 3.3 below. Therefore we only prove the parts which actually differ and refer to [26] for additional details. First we prove essential m-dissipativity on $L^2(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ for locally Lipschitz continuous Φ_1. Afterwards we use this result to show the m-dissipativity of the closure of (3.5) on $L^1(\mathbb{R}^{2d}, \mu_{\mathbf{\Phi}})$ for singular Φ_1. The potentials Φ_1, Φ_2 and their derivatives are considered as functions on $\mathbb{R}^{2d}$ and $\mathbb{R}^d$ simultaneously in the following way: $\Phi_1(x,v) = \Phi_1(x)$, $\Phi_2(x,v) = \Phi_2(v)$, where $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$. For a (weakly) differentiable function f on $\mathbb{R}^{2d}$, $\nabla_x f$ denotes the d−dimensional (weak) gradient w.r.t. the first d unit vectors. Corresponding definitions hold for $\nabla_v, \Delta_x, \Delta_v, \partial_{x_i}, \partial_{v_i}$, $i = 1, \ldots, d$. Expressions like $\nabla_v \Phi_2 \cdot \nabla_v f$ from (3.5) are understood as $\nabla_v \Phi_2 \cdot \nabla_v f(x,v) = \sum_{i=1}^d \partial_{v_i} \Phi_2(x,v) \partial_{v_i} f(x,v)$. If Φ_1 and Φ_2 are considered as a function on $\mathbb{R}^d$, the gradient, the Laplacian and weak partial derivatives denoted by $\nabla, \Delta, \partial_i$, $i = 1, \ldots, d$, respectively. Additionally, we introduce the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ equipped with the topology induced by metric

$$d(x,y) = |\arctan(x) - \arctan(y)|, x, y \in \overline{\mathbb{R}}, \tag{3.6}$$

where $\arctan(\infty) = \frac{\pi}{2}$ and $\arctan(-\infty) = -\frac{\pi}{2}$. Furthermore, we endow $\overline{\mathbb{R}}$ with the Borel σ-algebra induced by the topology which is generated by the metric given in (3.6).

Notation 3.1. *For $n \in \mathbb{N}$ and a measurable function $\Psi : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ we define the measure μ_Ψ by its Radon-Nikodym derivative w.r.t. the Lebesgue measure dx on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, given by*

$$\frac{d\mu_\Psi}{dx} = e^{-\Psi},$$

where $e^{\cdot}$ denotes the continuous extension of the usual exponential function to the extended reals $\overline{\mathbb{R}}$

We state the assumptions concerning the position potential Φ_1 and the velocity potential Φ_2, as follows:

Assumption 3.2. *Let $\Phi_1 : \mathbb{R}^d \longrightarrow \overline{\mathbb{R}}$ and $q \in [2, \infty]$.*

($\Phi_1$1) *Φ_1 is Lipschitz continuous. In particular, $\Phi_1 : \mathbb{R}^d \longrightarrow \mathbb{R}$.*

($\Phi_1$2) *Φ_1 is bounded from below and $\{\Phi_1 < \infty\} := \{x \in \mathbb{R}^d \mid \Phi_1(x) < \infty\} \neq \emptyset$.*

($\Phi_1$3) *$e^{-\Phi_1}$ is continuous on $\mathbb{R}^d$.*

($\Phi_1$4)^q *Φ_1 is weakly differentiable on $\{\Phi_1 < \infty\}$ and $|\nabla\Phi_1| \in L^q_{loc}(\mathbb{R}^d, \mu_{\Phi_1})$.*

Assumption 3.3. *Let $\Phi_2 : \mathbb{R}^d \longrightarrow \overline{\mathbb{R}}$.*

($\Phi_2$1) *Φ_2 is $\mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\overline{\mathbb{R}})$ measurable and $\{\Phi_2 < \infty\} := \{x \in \mathbb{R}^d \mid \Phi_2(x) < \infty\} \neq \emptyset$ is open in $\mathbb{R}^d$.*

($\Phi_2$2) *Φ_2 is bounded from below and locally integrable on $\{\Phi_2 < \infty\}$.*

($\Phi_2$3) *For $i \in \{1, .., d\}$ the distributional derivatives satisfy $\partial_i\Phi_2 \in L^2_{loc}(\{\Phi_2 < \infty\})$ and $\partial_i^2\Phi_2 \in L^1_{loc}(\{\Phi_2 < \infty\})$.*

($\Phi_2$4) *$\big(\Delta - \nabla\Phi_2 \cdot \nabla, C_c^\infty(\{\Phi_2 < \infty\})\big)$ is essentially self-adjoint on $L^2(\mathbb{R}^d, \mu_{\Phi_2})$.*

($\Phi_2$5) *There are constants $K \in (0, \infty)$ and $\alpha \in [1, 2)$ such that it holds $|\Delta\Phi_2| \le K(1 + |\nabla\Phi_2|^\alpha)$.*

According to Notation 3.1, we denote by μ_Φ the measure $\mu_{\Phi_1+\Phi_2}$ on $\big(\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d})\big)$ and by $\mathcal{H}_\Phi$ the Hilbert space $L^2(\mathbb{R}^{2d}, \mu_\Phi)$. In the following we occasionally write ($\Phi_1$2) – ($\Phi_1$4) to denote the assumptions ($\Phi_1$2), ($\Phi_1$3), ($\Phi_1$4). We also use similar notation for Φ_2 and different subsets of other assumptions which are understood in the same way.

Remark 3.4. *(i) Let Ω be an open subset of $\mathbb{R}^d$. Then it holds $f \in H^{1,\infty}_{loc}(\Omega)$ if and only if f has a representative which is locally Lipschitz continuous in Ω (see [33, Chapter 5.8, Theorem 4]). Hence, the assumption ($\Phi_1$1) implies ($\Phi_1$2) – ($\Phi_1$4)$^\infty$ apart from the boundedness from below.*

(ii) If we assume instead of ($\Phi_2$2) the following condition:

$\widetilde{(\Phi_2 2)}$ *Φ_2 is locally bounded on $\{\Phi_2 < \infty\}$.*

Then in combination with ($\Phi_2$5) one can argue similar as in the proof of [24, Lemma A6.2.] that Φ_2 is continuously differentiable on $\{\Phi_2 < \infty\}$ and $\nabla\Phi_2$ is locally Lipschitz on $\{\Phi_2 < \infty\}$.

(iii) *Assuming* $(\Phi_1 2), (\Phi_1 4)^q, (\Phi_2 2)$ *and* $(\Phi_2 3)$ *we can consider* $\big(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\big)$ *as an operator on* $L^p(\mathbb{R}^{2d}, \mu_\Phi)$ *for every* $p \in [1, 2]$.

(iv) *Since the measure* μ_{Φ_2} *on* $\mathbb{R}^d$ *is locally finite, it holds by [23, Proposition 7.2.3] that* μ_{Φ_2} *is regular Borel measure on* $(\{\Phi_2 < \infty\}, \mathcal{B}(\{\Phi_2 < \infty\}))$. *Thus, by [23, Proposition 7.4.2] the set* $C_c^\infty(\{\Phi_2 < \infty\})$ *is dense in* $L^2(\{\Phi_2 < \infty\}, \mu_{\Phi_2}) \cong L^2(\mathbb{R}^d, \mu_{\Phi_2})$.

(v) *See Section 3.7 for explicit sufficient conditions on* Φ_2 *implying* $(\Phi_2 4)$.

Let $\Omega, \Omega' \in \mathbb{R}^n$, $n \in \mathbb{N}$. If $\overline{\Omega'}$ is compact and contained in Ω then we write $\Omega' \subset\subset \Omega$.

Proposition 3.5. *Let* $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, *be open and* $\Psi : \Omega \longrightarrow \mathbb{R}$ *be measurable and locally bounded or bounded from below and locally integrable. Assume further that the first order distributional derivatives* $\partial_i \Psi$, $i \in \{1, \dots, n\}$, *are in* $L^p_{loc}(\Omega)$, *for some* $p \in [1, \infty]$. *Then, it holds that* $e^{-\Psi} \in H^{1,p}_{loc}(\Omega)$ *and* $\partial_i \left(e^{-\Psi}\right) = -\partial_i \Psi e^{-\Psi}$.

Proof. Let $\Omega' \subset\subset \Omega$ be open. We need to show that $e^{-\Psi} \in H^{1,p}(\Omega')$. Hence, let $\varphi \in C_c^\infty(\Omega')$ be arbitrary. Since $K := \operatorname{supp}(\varphi)$ is compact, there is a non-negative $\chi \in C_c^\infty(\Omega')$ such that $\chi = 1$ on K. Obviously $e^{-\Psi} \in L^\infty(\Omega') \subseteq L^p(\Omega')$. By the compact support of χ and a regularization as in [2, Lemma 3.16] one can find a sequence $(u_k)_{k\in\mathbb{N}} \in C_c^\infty(\Omega')$ such that $u_k \longrightarrow \chi\Psi$, as $k \to \infty$, in $H^{1,1}(\Omega')$. In the case of locally bounded Ψ it holds $\|u_k\|_\infty \le \|\chi\Psi\|_\infty$, for all $k \in \mathbb{N}$. Otherwise, if $C \in \mathbb{R}$ is a lower bound of Ψ, then it holds $C \le u_k(x)$ for all $x \in \Omega'$ and all $k \in \mathbb{N}$. By switching to a subsequence which we also denote by $(u_k)_{k\in\mathbb{N}}$ we can apply the dominated convergence theorem, integration by parts and Hölders inequality to obtain

$$\int\limits_{\Omega'} e^{-\Psi} \partial_i \varphi \, dx = \lim_{k\to\infty} \int\limits_{\Omega'} e^{-u_k} \partial_i \varphi \, dx = \lim_{k\to\infty} \int\limits_{\Omega'} \partial_i u_k e^{-u_k} \varphi \, dx = \int\limits_{\Omega'} \partial_i \Psi e^{-\Psi} \varphi \, dx.$$

□

Proposition 3.6. *Let* $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, *be open,* $f \in L^\infty_{loc}(\Omega) \cap H^{1,2}_{loc}(\Omega)$ *and* $g \in L^2_{loc}(\Omega) \cap H^{1,1}_{loc}(\Omega)$. *Then it holds* $fg \in H^{1,1}_{loc}(\Omega)$.

Proof. Let $(\varphi_n)_{k\in\mathbb{N}}$ be a standard approximate identity and $\Omega' \subset\subset \Omega$. Notice that for sufficiently large k the convolution $g_k := g * \varphi_k$ are elements in $C^\infty(\Omega')$ and approximates g simultaneously in $L^2(\Omega')$ and in $H^{1,1}(\Omega')$. □

Under the assumptions $(\Phi_1 2) - (\Phi_1 4)^q$, $q \in [2, \infty]$ and $(\Phi_2 1) - (\Phi_2 3)$, we obtain the following proposition and corollary:

Proposition 3.7. *Denote by* $\{\Phi_1, \Phi_2 < \infty\}$ *the set* $\big\{(x, v) \in \mathbb{R}^{2d} \mid x \in \{\Phi_1 < \infty\}, v \in \{\Phi_2 < \infty\}\big\}$. *The operator* $\big(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\big)$ *admits a decomposition into* $L_\Phi = S + A$, *with symmetric* S *and antisymmetric* A *on* $C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$ *w.r.t. the scalar product on* $\mathcal{H}_\Phi$, *respectively.* S *and* A *are given through*

$$Sf = \Delta_v f - \nabla_v \Phi_2 \cdot \nabla_v f, \quad Af = \nabla_v \Phi_2 \cdot \nabla_x f - \nabla_x \Phi_1 \cdot \nabla_v f, \quad f \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}).$$

Proof. The proof consists of the product rule for Sobolev functions and Proposition 3.5. □

Corollary 3.8. *The measure μ_Φ is invariant for $\big(L_\Phi, C_c^\infty(\{\Phi_1,\Phi_2<\infty\})\big)$, i.e., $L_\Phi f$ is integrable w.r.t. μ_Φ for all $f\in C_c^\infty(\{\Phi_1,\Phi_2<\infty\})$ and it holds*

$$\int_{\mathbb{R}^{2d}} L_\Phi f \, d\mu_\Phi = 0. \tag{3.7}$$

In particular, $\big(L_\Phi, C_c^\infty(\{\Phi_1,\Phi_2<\infty\})\big)$ is closable and its closure $\big(L_{\Phi,p}, D(L_{\Phi,p})\big)$ is dissipative on $L^p(\mathbb{R}^{2d},\mu_\Phi)$ for every $p\in[1,2]$.

Proof. For $f\in C_c^\infty(\{\Phi_1,\Phi_2<\infty\})$ one chooses a cut off function $\eta\in C_c^\infty(\{\Phi_1,\Phi_2<\infty\})$, s.t. $\eta=1$ on $\operatorname{supp}(f)$ and uses the decomposition from Proposition 3.7. But $S\eta, A\eta$ vanish on $\operatorname{supp}(f)$, which implies (3.7). The dissipativity follows by Example 1.58 and Lemma 1.59. □

3.1.1 M-Dissipativity for Lipschitz continuous Φ_1 on $L^2(\mathbb{R}^{2d},\mu_\Phi)$

Throughout this subsection we assume that Φ_1 and Φ_2 fulfill $(\Phi_1 1)$ and $(\Phi_2 1)-(\Phi_2 5)$, respectively. In particular, it holds $\{\Phi_1<\infty\}=\mathbb{R}^d$.

Theorem 3.9. *Assume $(\Phi_1 1)$ and $(\Phi_2 1)$–$(\Phi_2 5)$ are satisfied. Then the operator $\big(L_\Phi, C_c^\infty(\{\Phi_2<\infty\})\big)$ is essentially m-dissipative on $\mathcal{H}_\Phi$. The strongly continuous contraction semigroup $\big(T_t^\Phi\big)_{t\geq 0}$ generated by the closure of $\big(L_\Phi, C_c^\infty(\{\Phi_2<\infty\})\big)$ is sub-Markovian.*

Proof. This proof follows the lines of the proof in [26, Theorem 2.1.]. All function spaces in this proof consist of complex valued functions. Observe that those spaces are isometric to the complexification of the real valued function spaces. Due to the potentials Φ_1 and Φ_2 the coefficients of the differential operator L_Φ, are real-valued. In particular, L_Φ leaves the real valued functions invariant. Hence, we show that the complexified operator is essentially m-dissipative, which proves the theorem for the real cases by Lemma 1.30.

1st part:
The basic idea is to prove the claim for a unitarily equivalent operator which comes from the unitary transformation

$$U: L^2(\mathbb{R}^{2d},\mu_\Phi)\longrightarrow L^2(\{\Phi_2<\infty\}), \quad f\mapsto \exp\Big(-\frac{\Phi_1+\Phi_2}{2}\Big)f. \tag{3.8}$$

Namely, if one assumes that $\Phi_1,\Phi_2\in C^\infty(\mathbb{R}^d)$, then one obtains that $\big(L_\Phi, C_c^\infty(\{\Phi_2<\infty\})\big)$ transforms under U into the operator

$$L = UL_\Phi U^* = \Delta_v + \frac{\Delta_v\Phi_2}{2} - \frac{|\nabla_v\Phi_2|^2}{4} + \nabla_v\Phi_2\cdot\nabla_x - \nabla_x\Phi_1\cdot\nabla_v. \tag{3.9}$$

Although our potentials Φ_1 and Φ_2 are less regular, in the following we prove the essential m-dissipativity of L on a suitably chosen domain D. Subsequently, we make the transformation in (3.9) rigorous for the class of potentials we consider. Assumption $(\Phi_2 4)$ gives us the negative definite and essentially self-adjoint operator $\big(\Delta-\nabla\Phi_2\cdot\nabla, C_c^\infty(\{\Phi_2<\infty\})\big)$ on $L^2(\mathbb{R}^d,\mu_{\Phi_2})$. Corollary 1.33 implies that $\big(\Delta-\nabla\Phi_2\cdot\nabla, C_c^\infty(\{\Phi_2<\infty\})\big)$ is essentially m-dissipative on $L^2(\mathbb{R}^d,\mu_{\Phi_2})$. Consider the unitary transformation

$$U_{\Phi_2} : L^2(\mathbb{R}^d, \mu_{\Phi_2}) \longrightarrow L^2(\{\Phi_2 < \infty\}), \quad g \mapsto \exp(-\frac{1}{2}\Phi_2)g.$$

Since unitary transformations preserve essential m-dissipativity we have that

$$L_0 = U_{\Phi_2}(\Delta - \nabla\Phi_2 \cdot \nabla)U_{\Phi_2}^* \tag{3.10}$$

defined on $U_{\Phi_2}C_c^\infty(\{\Phi_2 < \infty\})$ is an essentially m-dissipative operator on $L^2(\{\Phi_2 < \infty\})$. Let $g \in C_c^\infty(\{\Phi_2 < \infty\})$ and $f = U_{\Phi_2}g$. In the following the differential operators Δ and ∇ are understood in the weak sense. Then, due to the assumptions ($\Phi_2$2), ($\Phi_2$3) and the Propositions 3.5 and 3.6, we have

$$f \in L^\infty(\{\Phi_2 < \infty\}) \cap H^{1,2}_{loc}(\{\Phi_2 < \infty\}). \tag{3.11}$$

Furthermore, it holds

$$\partial_i^2 f \in L^1_{loc}(\{\Phi_2 < \infty\}). \tag{3.12}$$

In particular,

$$\Delta f = \Delta(U_{\Phi_2}g) = \Delta g \exp(-\frac{1}{2}\Phi_2) + 2\nabla\left(\exp(-\frac{1}{2}\Phi_2)\right) \cdot \nabla g - g\left(\frac{\Delta\Phi_2}{2} - \frac{|\nabla\Phi_2|^2}{4}\right)\exp(-\frac{1}{2}\Phi_2). \tag{3.13}$$

Now, Proposition 3.5 and (3.13) lead to

$$\begin{aligned} L^2(\{\Phi_2 < \infty\}) \ni L_0 f &= U_{\Phi_2}(\Delta - \nabla\Phi_2 \cdot \nabla)g \\ &= \Delta g \exp(-\frac{1}{2}\Phi_2) + 2\nabla\left(\exp(-\frac{1}{2}\Phi_2)\right) \cdot \nabla g \\ &= \Delta f + \left(\frac{\Delta\Phi_2}{2} - \frac{|\nabla\Phi_2|^2}{4}\right) f, \text{ for all } f \in U_{\Phi_2}C_c^\infty(\{\Phi_2 < \infty\}). \end{aligned} \tag{3.14}$$

Note: Even though the single summands $|\nabla\Phi_2|^2 f$ and $\Delta\Phi_2 f$ in (3.14) are not necessarily in $L^2(\{\Phi_2 < \infty\})$. Anyways, $L_0 f$ is an element of $L^2(\{\Phi_2 < \infty\})$. Therefore, (3.14) is a suitable representation of $L_0 f$. Furthermore, L_0 is still symmetric and negative definite because we obtained L_0 from a unitary transformation of a symmetric and negative definite operator.

So far we only worked on the velocity component. To take the position variable x into account we define a new domain $D_0 \subseteq L^2(\{\Phi_2 < \infty\})$ given by

$$\begin{aligned} D_0 &:= L_c^2(\mathbb{R}^d) \otimes U_{\Phi_2}C_c^\infty(\{\Phi_2 < \infty\}) \\ &:= \operatorname{span}_{\mathbb{C}}\left\{h \otimes g : \mathbb{R}^{2d} \ni (x,v) \mapsto h(x)g(v) \mid h \in L_c^2(\mathbb{R}^d), g \in U_{\Phi_2}C_c^\infty(\{\Phi_2 < \infty\})\right\} \end{aligned} \tag{3.15}$$

where $L_c^2(\mathbb{R}^d)$ denotes the subspace of $L^2(\mathbb{R}^d)$ with elements vanishing a.e. outside a bounded set. In the definition of $h \otimes g$ in (3.15) we consider versions of h and g which are finite everywhere, respectively, s.t. the product is well-defined. Observe that the elements in $U_{\Phi_2}C_c^\infty(\{\Phi_2 < \infty\})$ are bounded. In the following we use the tensor product $\otimes$ several times with different function spaces. All these tensor products are tacitly understood analogous as in (3.15). For $f = h \otimes g \in D_0$ we set $L_0' f := h \otimes L_0 g = \Delta_v f - \frac{|\nabla_v\Phi_2|^2}{4} f + \frac{\Delta_v\Phi_2}{2} f$. We extend L_0' linearly to D_0. In the following we denote the norm and inner product of $L^2(\{\Phi_2 < \infty\})$ by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. We make certain observations on (L_0', D_0) which follow immediately from the corresponding properties of

L_0, i.e.,

(i) (L_0', D_0) is symmetric, negative definite and densely defined.

(ii) (L_0', D_0) is essentially m-dissipative.

We perturb L_0' with the multiplication operator (B_0, D_0) given by the measurable function

$$i\nabla_v \Phi_2 \cdot x : \{\Phi_2 < \infty\} \longrightarrow \mathbb{C}, \quad (x,v) \mapsto i\nabla_v \Phi_2(x,v)\cdot x := i\sum_{l=1}^{d} \partial_l \Phi_2(v) x_l.$$

Since $\nabla_v \Phi_2 \cdot x$ is real valued it follows that B_0 is antisymmetric, i.e. $B^* = -B$, in particular, (B_0, D_0) is dissipative. We consider the complete orthogonal family of projections $(P_k)_{k\in\mathbb{N}}$ given by

$$P_k : L^2(\{\Phi_2 < \infty\}) \longrightarrow L^2(\{\Phi_2 < \infty\}), f \mapsto g_k f,$$

where $g_k(x,v) = 1_{[k-1,k]}(|x|_2)$, $k \in \mathbb{N}$. Obviously each P_k maps D_0 into itself and L_0' as well as B_0 commute with each P_k on D_0. In order to apply Theorem 1.41 we need to show that $B_0^k := P_k B_0$ is $L_k := P_k L_0'$ bounded with L_k-bound less than one. By using the Cauchy-Schwarz inequality and the definition of P_k, we have

$$|\nabla_v \Phi_2 \cdot x|^2 \, |f|^2 \le k^2 \, |\nabla_v \Phi_2|^2 \, |f|^2, \text{ for } f \in P_k D_0.$$

Hence, it suffices to show that $\| |\nabla_v \Phi_2| f\|^2 \le a(L_0' f, f) + b\,\|f\|^2$ holds for some finite constants a, b independent of $f \in P_k D_0$. Therefore, let $f \in D_0$ and observe that $-\Delta_v$ is positive definite on D_0 and $\Delta_v \Phi_2 f \in L^1(\{\Phi_2 < \infty\})$ due to assumption ($\Phi_2 3$). Thus it holds

$$\| |\nabla_v \Phi_2| f\|^2 \le 4 \int_{\mathbb{R}^{2d}} \left(\left(-\Delta_v + \frac{|\nabla_v \Phi_2|^2}{4} - \frac{\Delta_v \Phi_2}{2} \right) f \right) \overline{f} \, d(x,v) + 2 \int_{\mathbb{R}^{2d}} \Delta_v \Phi_2 \, |f|^2 \, d(x,v) \tag{3.16}$$

with both summands on the right-hand side being finite. Let $K > 0$ and $1 \le \alpha < 2$ be the constants from assumption ($\Phi_2 5$). Then we have the following estimate for the last term in (3.16)

$$\int_{\mathbb{R}^{2d}} \Delta_v \Phi_2 \, |f|^2 \, d(x,v) \le K \left(\|f\|^2 + \int_{\{\Phi_2 < \infty\}} |\nabla_v \Phi_2|^\alpha \, |f|^2 \, d(x,v) \right) \tag{3.17}$$

Hölder's and Young's inequality imply for the last integral on the right hand side of (3.17) for $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ that

$$\int_{\{\Phi_2 < \infty\}} |\nabla_v \Phi_2|^\alpha \, |f|^2 \, d(x,v) \le \frac{1}{4K} \| |\nabla_v \Phi_2| f\|^2 + \frac{(2-\alpha)(2\alpha K)^{\frac{\alpha}{2-\alpha}}}{2} \|f\|^2. \tag{3.18}$$

Consequently, for $f \in D_0$ the inequality (3.16) becomes

$$\| |\nabla_v \Phi_2| f\|^2 \le C_1 \left((1 - L_0') f, f \right), \tag{3.19}$$

where

$$C_1 = \max\left\{ 8, 4K\left(1 + \frac{(2-\alpha)(2\alpha K)^{\frac{\alpha}{2-\alpha}}}{2} \right) \right\}$$

is independent of the potential Φ_1. Since (3.19) holds we conclude that $|\nabla_v \Phi_2|\, P_k$ is L_k bounded with L_k-bound being zero and so is B_0^k for each $k \in \mathbb{N}$. Now we are able to apply Theorem 1.41 implying the essential m-dissipativity of

$$(L', D_0) := \left(L_0' + B_0, D_0\right) = \left(\Delta_v - \frac{|\nabla_v \Phi_2|^2}{4} + \frac{\Delta_v \Phi_2}{2} + i\nabla_v \Phi_2 \cdot x, D_0\right).$$

Note: The estimates (3.16),(3.17),(3.18) and (3.19) also hold for f in the larger space $L^2(\mathbb{R}^d) \otimes U_{\Phi_2} C_c^\infty(\{\Phi_2 < \infty\})$.

The set $D_1 = C_c^\infty(\mathbb{R}^d) \otimes U_{\Phi_2} C_c^\infty(\{\Phi_2 < \infty\})$ forms a core for the closure of (L', D_0), which ensures that (L', D_1) is essentially m-dissipative, too. The extension of (L', D_1) to $D_2 = S(\mathbb{R}^d) \otimes U_{\Phi_2} C_c^\infty(\{\Phi_2 < \infty\})$, where $S(\mathbb{R}^d)$ denotes the Schwartz space, is still dissipative, hence the closure of (L', D_2) is a dissipative extension of the closure of (L', D_1), meaning that their closures coincide by Remark 1.20, i.e.,

$$\left(L', S(\mathbb{R}^d) \otimes U_{\Phi_2} C_c^\infty(\{\Phi_2 < \infty\})\right) \text{ is essentially m-dissipative.}$$

Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and its inverse on $L^2(\mathbb{R}^d)$, respectively. Recall the well-known property of $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}(x^s f) = (-i)^{|s|} \partial^s (\mathcal{F}^{-1} f), \text{ for } f \in S(\mathbb{R}^d) \text{ and } s \in \mathbb{N}_0^d. \tag{3.20}$$

Let $f = f_1 \otimes f_2 \in D_2$. Define $\mathcal{F}_x f := \mathcal{F} f_1 \otimes f_2$ and extend $\mathcal{F}_x$ linearly to D_2 and afterwards to a unitary transformation on $L^2(\{\Phi_2 < \infty\})$ (similarly as one does for the construction of $\mathcal{F}$) which we also denote by $\mathcal{F}_x$. $\mathcal{F}_x$ leaves the set D_2 invariant, because $S(\mathbb{R}^d)$ is invariant under $\mathcal{F}$. By using the identity (3.20), one obtains

$$\tilde{L} f = \mathcal{F}_x^{-1} L' \mathcal{F}_x f = \left(\Delta_v + \frac{\Delta_v \Phi_2}{2} - \frac{|\nabla_v \Phi_2|^2}{4} + \nabla_v \Phi_2 \cdot \nabla_x\right) f, \quad f \in D_2.$$

Next we want to perturb $\tilde{L}$ with the antisymmetric operator (B_1, D_2) given by $B_1 f = \sum_{i=1}^{d} \partial_{x_i} \Phi_1 \partial_{v_i} f$, $f \in D_2$. Since Φ_1 is Lipschitz continuous (B_1, D_2) is well-defined on $L^2(\{\Phi_2 < \infty\}, d(x,v))$. Let $f \in D_2$. Due to (3.11), (3.12) and a similar reasoning as in the derivation of (3.19), we obtain

$$\begin{aligned}
\| |\nabla_v f| \|^2 &= \sum_{i=1}^{d} (\partial_{v_i} f, \partial_{v_i} f) = (-\Delta_v f, f) \\
&\le \left(\left(-\Delta_v + \frac{|\nabla_v \Phi_2|^2}{4} - \frac{\Delta_v \Phi_2}{2}\right) f, f\right) + \frac{1}{2}(\Delta_v \Phi_2 f, f) \\
&\le (-L_0' f, f) + \left(\underbrace{\frac{K}{2}\left(1 + \frac{(2-\alpha)(2\alpha K)^{\frac{\alpha}{2-\alpha}}}{2}\right)}_{:=K_1} \|f\|^2 + \frac{1}{8} \| |\nabla \Phi_2|\, f\|^2 \right) \\
&\le \underbrace{\left(\max\{1, K_1\} + \frac{C}{8}\right)}_{:=C_2} \left((1 - L_0') f, f\right).
\end{aligned} \tag{3.21}$$

Denote the Lipschitz constant of Φ_1 by C_{Φ_1}. Then, we obtain

$$\begin{aligned}\|B_1 f\|^2 = \|\nabla_x\Phi_1\cdot\nabla_v f\|^2 &\le C_{\Phi_1}^2\,\||\nabla_v f|\|^2\\ &\le \underbrace{C_{\Phi_1}^2 C_2}_{:=C_3}\left((1-L_0')f,f\right). \end{aligned}\tag{3.22}$$

Since (L_0', D_2) is symmetric, it holds that $(L_0' f, f) \in \mathbb{R}$, for $f \in D_2$. Let A be an arbitrary anti-symmetric linear operator on D_2. In particular, for $f \in D_2$ it holds that $(Af, f) \in i\mathbb{R}$. Hence one obtains

$$((1-L_0')f,f) \le \left|\underbrace{((1-L_0')f,f)}_{\in\mathbb{R}} + \underbrace{(Af,f)}_{\in i\mathbb{R}}\right|. \tag{3.23}$$

Applying the inequality (3.23) for the choice $A = -\nabla_v\Phi_2\cdot\nabla_x$ to (3.22), one concludes for $f \in D_2$ that

$$\|\nabla_x\Phi_1\cdot\nabla_v f\|^2 \le C_3\left|\left((1-\tilde{L})f,f\right)\right|.$$

By Lemma 1.39, we further deduce that

$$L = \tilde{L} - \nabla_x\Phi_1\cdot\nabla_v = \Delta_v - \frac{|\nabla_v\Phi_2|^2}{4} + \frac{\Delta_v\Phi_2}{2} + \nabla_v\Phi_2\cdot\nabla_x - \nabla_x\Phi_1\cdot\nabla_v$$

defined on D_2 is essentially m-dissipative on $L^2(\{\Phi_2 < \infty\})$.

We can also apply (3.23) with $A = -\nabla_v\Phi_2\cdot\nabla_x + \nabla_x\Phi_1\cdot\nabla_v$ to extend (3.19) and (3.21) for L instead of L_0'. Namely, if we let $f \in D_2$, then it holds

$$\||\nabla\Phi_2| f\|^2 \le C\,|((1-L)f,f)|, \tag{3.24}$$

$$\||\nabla_v f|\|^2 \le C\,|((1-L)f,f)|, \tag{3.25}$$

where the constant C is given by the maximum $\max\{C_1, C_2\}$ which is independent of the potential Φ_1. We restrict L to D_1 and observe that essential m-dissipativity is preserved, since $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ (w.r.t. the Schwartz space topology on $\mathcal{S}(\mathbb{R}^d)$). Now, we transform via the adjoint of unitary map from (3.8), i.e.,

$$U^* : L^2(\{\Phi_2 < \infty\}) \longrightarrow L^2(\mathbb{R}^{2d}, \mu_\Phi), f \mapsto e^{\frac{\Phi_1+\Phi_2}{2}}\tilde{f}, \tag{3.26}$$

where $\tilde{f} = 1_{\{\Phi_2<\infty\}} f$. For $f = f_1 \otimes f_2 \in D_1$ one has $U^* f = e^{\frac{\Phi_1}{2}} f_1 \otimes e^{\frac{\Phi_2}{2}} f_2$. Denote by $U_{\Phi_1}^*$ the unitary map $U_{\Phi_1}^* : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d, \mu_{\Phi_1}), f \mapsto e^{\frac{\Phi_1}{2}} f$. Due to (3.10), (3.14), the product rule for Sobolev functions and Proposition 3.5, it holds that U^* transforms L back into L_Φ.Eventually we obtain the essentially m-dissipative operator

$$(U^*LU, U^*D_1) = \left(L_\Phi, U_{\Phi_1}^* C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\})\right). \tag{3.27}$$

For $f \in U_{\Phi_1}^* C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\})$ it holds $Uf \in D_1$. hence, through (3.24) and (3.25), we obtain

$$\begin{aligned} \| |\nabla_v \Phi_2| f\|_{\mu_\Phi}^2 &= \| |\nabla_v \Phi_2| Uf\|^2 \\ &\le C\, |((1-L)Uf, Uf)| \\ &= C \left| ((1-L_\Phi)f, f)_{\mu_\Phi} \right|, \end{aligned} \tag{3.28}$$

$$\begin{aligned} \| |\nabla_v f| \|_{\mu_\Phi}^2 &= \int_{\mathbb{R}^{2d}} \left| \exp\left(-\frac{\Phi_2}{2}\right) \nabla_v f \right|^2 \exp(-\Phi_1)\, d(x,v) \\ &\le \frac{1}{2} \int_{\mathbb{R}^{2d}} \left(\left| \nabla_v \left(\exp\left(-\frac{\Phi_2}{2}\right) f \right) \right|^2 + \left| \frac{\nabla_v \Phi_2}{2} \exp\left(-\frac{\Phi_2}{2}\right) f \right|^2 \right) \exp(-\Phi_1)\, d(x,v) \\ &\le C\, |((1-L)Uf, Uf)| \\ &= C \left| ((1-L_\Phi)f, f)_{\mu_\Phi} \right| \end{aligned} \tag{3.29}$$

The Fatou lemma guarantees that (3.28) also holds for f from the domain of the closure of (3.27) denoted by $D(L_\Phi)$. By the same argument the inequality (3.29) is also preserved, where $\nabla_v f \in L^2(\mathbb{R}^{2d}, \mu_\Phi)^d$ is understood as a $L^2(\mathbb{R}^{2d}, \mu_\Phi)^d$-limit. To finish the first part we show that $C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\})$ is a domain of essential m-dissipativity for L_Φ. Since $(L_\Phi, C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\}))$ is dissipative by Corollary 3.8, it suffices thanks to the essential m-dissipativity of (3.27) to show that the closure of $(L_\Phi, C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\}))$ is an extension of (3.27). To this end, let $f = f^1 \otimes f^2 \in U_{\Phi_1}^* C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\})$. Observe that $U_{\Phi_1}^* C_c^\infty(\mathbb{R}^d)$ is by Proposition 3.5 a subset of $H^{1,2}(\mathbb{R}^d)$. Choose a sequence $(f_n^1)_{n\in\mathbb{N}}$ from $C_c^\infty(\mathbb{R}^d)$ such that $f_n^1 \longrightarrow f^1$ in $H^{1,2}(\mathbb{R}^d)$ and $\operatorname{supp}(f_n^1) \subseteq K$, $K \subseteq \mathbb{R}^d$ is compact and independent of n, which is possible, since f^1 is already compactly supported. For $f_n := f_n^1 \otimes f^2$, $n \in \mathbb{N}$, it holds by construction and the fact that the density $e^{-\Phi_1-\Phi_2}$ of μ_Φ is locally bounded which yields $f_n \longrightarrow f$, $L_\Phi f_n \longrightarrow L_\Phi f$ and $|\nabla_v \Phi_2| f_n \longrightarrow |\nabla_v \Phi_2| f$ in $\mathcal{H}_\Phi$ as $n \to \infty$. This shows that $C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\})$ is a core for the closure of (3.27).

□

Remark 3.10. *From the proof of Theorem 3.9, one sees that the condition* ($\Phi_2$5) *can also be extended to* $\alpha = 2$ *and* $0 \le K < \frac{1}{2}$.

By recalling the decomposition from Proposition 3.7 we obtain that for the adjoint $\left(\hat{L}_\Phi, D(\hat{L}_\Phi)\right)$ of $(L_\Phi, D(L_\Phi))$ it holds

$$C_c^\infty(\{\Phi_2 < \infty\}) \subseteq D(\hat{L}_\Phi), \quad \hat{L}_\Phi f = Sf - Af, \quad f \in C_c^\infty(\{\Phi_2 < \infty\}). \tag{3.30}$$

For a symmetric velocity potential Φ_2, i.e., $\Phi_2(v) = \Phi_2(-v), \forall v \in \mathbb{R}^d$, we can use the velocity reversal as in [24, p. 153]. In this case, the unitary transformation on $\mathcal{H}_\Phi$ given by

$$U : \mathcal{H}_\Phi \longrightarrow \mathcal{H}_\Phi, [f] \mapsto [(x,v) \mapsto f(x,-v)] \tag{3.31}$$

transform $(L_\Phi, C_c^\infty(\{\Phi_2 < \infty\})$ into the operator $(UL_\Phi U, UC_c^\infty(\{\Phi_2 < \infty\}) = \left(\hat{L}_\Phi, C_c^\infty(\{\Phi_2 < \infty\})\right)$. This implies that the latter is essential m-dissipative by Theorem 1.28. Since the adjoint $\left(\hat{L}_\Phi, D(\hat{L}_\Phi)\right)$

is dissipative and by (3.30) an extension of $\left(\hat{L}_\Phi, C_c^\infty(\{\Phi_2 < \infty\})\right)$, we obtain

$$\left(\hat{L}_\Phi, D(\hat{L}_\Phi)\right) = \left(\hat{L}_\Phi, \overline{C_c^\infty(\{\Phi_2 < \infty\})}\right).$$

Therefore, we make in the following the additional assumption:

Assumption 3.11.

($\Phi_2 6$) $\quad$ *Φ_2 is symmetric, i.e., $\Phi_2(v) = \Phi_2(-v)$, for all $v \in \mathbb{R}^d$.*

The next corollary recaps the previous discussion.

Corollary 3.12. *Under the assumptions of Theorem 3.9 and the additional assumption ($\Phi_2 6$), the formal adjoint $(\hat{L}_\Phi, C_c^\infty(\{\Phi_2 < \infty\}))$ is also an essentially m-dissipative Dirichlet operator. Furthermore, its closure coincides with the adjoint of $(L_\Phi, D(L_\Phi))$.*

3.1.2 M-Dissipativity for singular Φ_1 on $L^1(\mathbb{R}^{2d}, \mu_\Phi)$

In this part we only assume $(\Phi_1 2) - (\Phi_1 4)^q$, $q \in [2, \infty]$, for Φ_1. The goal is to prove essential m-dissipativity of $\left(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ on $L^1(\mathbb{R}^{2d}, \mu_\Phi)$. Observe that due to Corollary 3.8 the operator $\left(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ is closable on $L^p(\mathbb{R}^{2d}, \mu_\Phi)$ and its closure $\left(L_{\Phi,p}, D(L_{\Phi,p})\right)$ is dissipative for $p \in [1, 2]$. The next proposition is basically taken from [26, Lemma 3.7., Remark 3.8.(i)]. The proof given here is similar but requires adaption due to the general velocity potential Φ_2.

Proposition 3.13. *Let $p \in [1, 2]$. Assume that the assumptions $(\Phi_1 2) - (\Phi_1 4)^q$, $q \in [2, \infty]$, and $(\Phi_2 1) - (\Phi_2 5)$ hold true. The set $C_c^\infty(\{\Phi_2 < \infty\})$ is contained in $D(L_{\Phi,p})$ and for $f \in C_c^\infty(\{\Phi_2 < \infty\})$ it holds $L_{\Phi,p} f = L_\Phi f$.*

Proof. We complete the proof in several steps.
Step 1: Define the set

$$\mathcal{U} := \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{R} \mid f \text{ is continuous, } \operatorname{supp}(f) \subseteq \{\Phi_1 < \infty\} \text{ is compact, } f, \partial_i f \in L^2(\mathbb{R}^d) \right\},$$

where ∂_i is the distributional derivative in the direction of the i-th unit vector. Then, via convolution with a standard approximate identity and the assumptions $(\Phi_1 4)^q$, $(\Phi_2 3)$, we obtain that

$$\begin{aligned} \mathcal{U} \otimes C_c^\infty(\{\Phi_2 < \infty\}) &\subseteq D(L_{\Phi,p}), \\ L_{\Phi,p} f = L_\Phi f, \quad f &\in \mathcal{U} \otimes C_c^\infty(\{\Phi_2 < \infty\}), \end{aligned} \tag{3.32}$$

where $\otimes$ is defined analogous as in (3.15).
Step 2: Now let $f^1 \in C_c^\infty(\mathbb{R}^d)$ and $f^2 \in C_c^\infty(\{\Phi_2 < \infty\})$. For $n \in \mathbb{N}$ let $\chi_n \in C_c^\infty(\mathbb{R})$, s.t. $0 \le \chi_n \le 1$, $|\chi_n'| \le 1$, $\chi_n = 1$ on $B_n(0)$ and $\chi_n = 0$ outside of $B_{n+2}(0)$. Now define $f_n^1 := f^1 \cdot \chi_n \circ \Phi_1$. Then, it holds via the chain rule for Sobolev functions and assumption $(\Phi_1 2)$, $(\Phi_1 4)^q$ that $f_n^1 \in \mathcal{U}$. Then, by using the dominated convergence theorem and (3.32), we obtain

$$f_n^1 \otimes f^2 \longrightarrow f^1 \otimes f^2, L_\Phi(f_n^1 \otimes f^2) \longrightarrow L_\Phi(f^1 \otimes f^2),$$

where the convergence takes place in $L^2(\mathbb{R}^{2d}, \mu_\Phi)$. Thus, it holds

$$C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\}) \subseteq D(L_{\Phi,p}),$$
$$L_{\Phi,p} f = L_\Phi f, \quad f \in C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\}).$$

Step 3: To finish the proof we use the version of the Stone-Weierstrass theorem given in [76], see also [54, Theorem I.4.1.] for a reference in English. Namely, by the last reference, we obtain that

$$C_c^\infty(\mathbb{R}^d) \otimes C_c^\infty(\{\Phi_2 < \infty\}) \subseteq C_c^\infty(\mathbb{R}^d \times \{\Phi_2 < \infty\})$$

is dense w.r.t. the topology of uniform convergence on compact sets of all derivatives up to order 2. Since the measure μ_Φ is locally finite, the claim follows. □

Corollary 3.14. *Under the assumptions of Proposition 3.13, it holds that* $\left(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ *is essentially m-dissipative on* $L^p(\mathbb{R}^{2d}, \mu_\Phi)$ *if and only if its extension* $\left(L_\Phi, C_c^\infty(\{\Phi_2 < \infty\})\right)$ *is.*

The next lemma provides a sequence of smooth potentials $(\Phi_{1,n})_{n\in\mathbb{N}}$ approximating Φ_1 in a suitable sense. See [26, Lemma 3.10] for the proof.

Lemma 3.15. *Let* Φ_1 *fulfill* $(\Phi_1 2)$, $(\Phi_1 3)$, $(\Phi_1 4)^q$. *Then, there exist smooth* $\Phi_{1,n}$ *such that* $\Phi_{1,n} \le \Phi_1$ *and* $\nabla\Phi_n \overset{n\to\infty}{\longrightarrow} \nabla\Phi$ *in* $L^q_{loc}(\mathbb{R}^d, \mu_\Phi)$. *Furthermore, the family* $(\Phi_{1,n})_{n\in\mathbb{N}}$ *is uniformly bounded from below.*

In the following we state an additionally assumption on Φ_2:

Assumption 3.16. *Let* $q \in [2, \infty]$.

$(\Phi_2 7)$ μ_{Φ_2} *is a probability measure, i.e.,* $\mu_{\Phi_2}(\mathbb{R}^d) = \int_{\mathbb{R}^d} e^{-\Phi_2}\, dv < \infty$.

$(\Phi_2 8)^q$ Φ_2 *is weakly differentiable on* $\{\Phi_2 < \infty\}$ *and* $|\nabla\Phi_2| \in L^q(\mathbb{R}^d, \mu_{\Phi_1})$.

The proof of the next theorem resembles the proof in [26, Theorem 3.11]. Since the velocity component of the measure μ_Φ is not necessarily Gaussian anymore it does not hold $\int_{\mathbb{R}^d} |\nabla\Phi_2|^p\, d\mu_{\Phi_2} < \infty$ for arbitrary $p \in [1, \infty)$. Therefore we need to make some adaptions of the original proof to overcome this lack.

Theorem 3.17. *Assume* $(\Phi_1 2) - (\Phi_1 4)^q$, $q \in [2, \infty]$ *and* $(\Phi_2 1) - (\Phi_2 5)$, $(\Phi_2 7)$. *Additionally, one of the following assumptions are assumed to hold:*

(i) μ_{Φ_1} *is a probability measure on* $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$ *and* $(\Phi_2 8)^2$ *holds true.*

(ii) $q > d$ *and* $(\Phi_2 8)^{q_0}$ *holds for* $q_0 > \max\left\{d, \frac{2q}{q-2}\right\}$.

Then the operator $\left(L_{\Phi,p}, D(L_{\Phi,p})\right)$ *generates a strongly continuous contraction semigroup* $\left(T^\Phi_{t,p}\right)_{t\ge 0}$ *on* $L^p(\mathbb{R}^{2d}, \mu_\Phi)$ *for* $1 \ge p^{-1} \ge q^{-1} + \frac{1}{2}$. *Furthermore,* $\left(T^\Phi_{t,p}\right)_{t\ge 0}$ *is sub-Markovian.*

Proof. By Corollary 3.8 we know that $\left(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ is dissipative and its closure is a Dirichlet operator on $L^p(\mathbb{R}^{2d}, \mu_\Phi)$ for $p \in [1, 2]$. Due to Corollary 3.14 it suffices now to prove that for $1 \geq p^{-1} \geq q^{-1} + \frac{1}{2}$

$$(1 - L_\Phi)C_c^\infty(\{\Phi_2 < \infty\}) \subseteq L^p(\mathbb{R}^{2d}, \mu_\Phi) \text{ is dense.} \tag{3.33}$$

We split the proof into two parts.

Part 1. We first prove (3.33) for p_0 satisfying $p_0^{-1} = q^{-1} + \frac{1}{2}$ under any of the assumptions (i)-(ii). To this end let $g \in C_c^\infty(\{\Phi_2 < \infty\})$ be arbitrary. By the compactness of the support of g we can choose cut off functions $\tilde{\chi}, \tilde{\nu} \in C_c^\infty(\mathbb{R}^d)$ such that the functions defined by $\chi(x, v) = \tilde{\chi}(x)$, $\nu(x, v) = \tilde{\nu}(x)$, $x, v \in \mathbb{R}^d$ fulfill $0 \leq \chi \leq \nu \leq 1$, $\chi \equiv 1$ on $\operatorname{supp}(g)$, $\nu \equiv 1$ on $\operatorname{supp}(\chi)$. Denote by $(\Phi_{1,n})_{n\in\mathbb{N}}$ the smooth sequence constructed in Lemma 3.15. Observe that $\tilde{\nu}\Phi_{1,n} \in H^{1,\infty}(\mathbb{R}^d)$, hence, $\nu\Phi_{1,n}$ is Lipschitz continuous and define $\mathbf{\Phi}_n = (\nu\Phi_{1,n}, \Phi_2)$, $n \in \mathbb{N}$. Further, due to $(\Phi_1 2)$ and the last statement in Lemma 3.15 we can assume w.l.o.g. that $\Phi_1, \Phi_{1,n} \geq 0$.
Then, it holds for $f \in C_c^\infty(\{\Phi_2 < \infty\})$ that

$$\begin{aligned} \|(1 - L_\Phi)(\chi f) - g\|_{L^{p_0}(\mu_\Phi)} &\leq \left\|\chi\left((1 - L_{\Phi_n})f - g\right)\right\|_{L^{p_0}(\mu_{\Phi_n})} + \|f\nabla_v\Phi_2 \cdot \nabla_x\chi\|_{L^{p_0}(\mu_{\Phi_n})} \\ &\quad + \left\|\chi\nabla_v f \cdot (\nabla_x\Phi_1 - \nabla_x\Phi_{1,n})\right\|_{L^{p_0}(\mu_\Phi)} \\ &\leq \|\chi\|_{L^q(\mu_{\Phi_n})} \left\|(1 - L_{\Phi_n})f - g\right\|_{L^2(\mu_{\Phi_n})} \\ &\quad + \||\nabla_v\Phi_2| f\|_{L^2(\mu_{\Phi_n})} \||\nabla_x\chi|\|_{L^q(\mu_{\Phi_n})} \\ &\quad + \left\|\sqrt{\chi}\,|\nabla_v f|\right\|_{L^2(\mu_{\Phi_n})} \left\|\sqrt{\chi}\,\left|\nabla_x\Phi_1 - \nabla_x\Phi_{1,n}\right|\right\|_{L^q(\mu_\Phi)}. \end{aligned}$$

Since $\tilde{\nu}\Phi_{1,n}$ is Lipschitz-continuous, the results from Theorem 3.9 and its proof apply. Since $\left(L_{\Phi_n}, C_c^\infty(\{\Phi_2 < \infty\})\right)$ is dissipative and Lemma 1.16(i) holds true, we get

$$\|f\|_{L^2(\mu_{\Phi_n})} \leq \left\|(1 - L_{\Phi_n})f\right\|_{L^2(\mu_{\Phi_n})}.$$

Hence, (3.28) and (3.29) lead to

$$\begin{aligned} \||\nabla_v f|\|_{L^2(\mu_{\Phi_n})}^2 &\leq C\left\|(1 - L_{\Phi_n})f\right\|_{L^2(\mu_{\Phi_n})}^2, \\ \||\nabla_v\Phi_2| f\|_{L^2(\mu_{\Phi_n})}^2 &\leq C\left\|(1 - L_{\Phi_n})f\right\|_{L^2(\mu_{\Phi_n})}^2, \end{aligned}$$

where C is independent of n and f. Thus, we obtain

$$\begin{aligned} &\|(1 - L_\Phi)(\chi f) - g\|_{L^{p_0}(\mu_\Phi)} \\ \leq\ &\|\chi\|_{L^q(\mu_{\Phi_n})} \left\|(1 - L_{\Phi_n})f - g\right\|_{L^2(\mu_{\Phi_n})} \\ &+ C\left\|(1 - L_{\Phi_n})f\right\|_{L^2(\mu_{\Phi_n})} \left(\||\nabla\tilde{\chi}|\|_{L^q(dx)} + \left\|\sqrt{\chi}\,\left|\nabla_x\Phi_1 - \nabla_x\Phi_{1,n}\right|\right\|_{L^q(\mu_\Phi)}\right) \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Denote by $\|g\|_0$ the $L^2(\mathbb{R}^{2d}, d(x, v))$-norm of g, where $d(x, v)$ is the Lebesgue measure on $\mathbb{R}^{2d}$. Now, we specify our choice of $\tilde{\chi}$. Under any of the assumptions (i)-(ii), we can choose $\tilde{\chi}$ s.t.

$$\||\nabla\tilde{\chi}|\|_{L^q(\mathbb{R}^d, \mu_{\Phi_1})} \leq \frac{\varepsilon}{8C\,\|g\|_0}.$$

So $\tilde{\chi}, \tilde{v}$ are fixed. By Lemma 3.15, we can now choose $n \in \mathbb{N}$ s.t.

$$\left\| \sqrt{\chi} \left| \nabla_x \Phi_1 - \nabla_x \Phi_{1,n} \right| \right\|_{L^q(\mu_\Phi)} \leq \frac{\varepsilon}{8C \, \|g\|_0}.$$

And by Theorem 3.9, we can choose a $f \in C_c^\infty(\{\Phi_2 < \infty\})$ s.t.

$$\left\| (1 - L_{\Phi_n}) f - g \right\|_{L^2(\mu_{\Phi_n})} \leq \frac{\min\{\varepsilon, \|g\|_0\}}{2 \max\{\|\tilde{\chi}\|_{L^q(\mathbb{R}^d, dx)}, 1\}}.$$

Eventually, we conclude that

$$\begin{aligned}
\|(1 - L_\Phi)(\chi f) - g\|_{L^{p_0}(\mu_\Phi)} \leq& \frac{\varepsilon}{2} + C \left(\left\| (1 - L_{\Phi_n}) f - g \right\|_{L^2(\mu_{\Phi_n})} + \|g\|_0 \right) \\
&\times \left(\| |\nabla \tilde{\chi}| \|_{L^q(\mathbb{R}^d, \mu_{\Phi_1})} + \left\| \sqrt{\chi} \left| \nabla_x \Phi_1 - \nabla_x \Phi_{1,n} \right| \right\|_{L^q(\mu_\Phi)} \right) \\
\leq& \frac{\varepsilon}{2} + C 2 \, \|g\|_0 \left(\| |\nabla \tilde{\chi}| \|_{L^q(\mathbb{R}^d, dx)} + \left\| \sqrt{\chi} \left| \nabla_x \Phi_1 - \nabla_x \Phi_{1,n} \right| \right\|_{L^q(\mu_\Phi)} \right) \\
\leq& \varepsilon.
\end{aligned}$$

This proves the claim for p_0.

Part 2. Under the assumption (i) the case $p \in [1, p_0]$ follows immediately by the Hölder inequality. Indeed, let p be given s.t. $1 \leq p < p_0$. Then, $r := \frac{p_0 p}{p_0 - p}$ satisfies $p^{-1} = p_0^{-1} + r^{-1}$. Thus, we have

$$\|(1 - L_\Phi) f - g\|_{L^p(\mu_\Phi)} \leq \mu_\Phi(\mathbb{R}^{2d})^{\frac{1}{r}} \, \|(1 - L_\Phi) f - g\|_{L^{p_0}(\mu_\Phi)},$$

which can be made arbitrarily small for a suitable f since $\left(L_\Phi, C_c^\infty(\{\Phi_2 < \infty\})\right)$ is essentially m-dissipative on $L^{p_0}(\mathbb{R}^{2d}, \mu_\Phi)$.

To prove (3.33) under the assumption (ii), we proceed with a similar approach given in [26, Theorem 3.11.]. Observe that, if $q = 2$ there is nothing to show and therefore we assume $q > 2$. Let $p, p' \in [1, p_0]$, s.t. $\frac{1}{p} = \frac{1}{p'} + \frac{1}{q_0}$, $f, g \in C_c^\infty(\{\Phi_2 < \infty\})$ and χ given as in the first part. Then, we obtain the following estimate

$$\begin{aligned}
\|(1 - L_\Phi)(\chi f) - g\|_{L^p(\mu_\Phi)} \leq& \|\chi\|_{L^{q_0}(\mu_\Phi)} \, \|(1 - L_\Phi)(f) - g\|_{L^{p'}(\mu_\Phi)} \\
&+ \|f\|_{L^{p'}(\mu_\Phi)} \, \| |\nabla_x \chi| \|_{L^{q_0}(\mathbb{R}^d, \mu_{\Phi_1})} \, \| |\nabla_v \Phi_2| \|_{L^{q_0}(\mathbb{R}^d, \mu_{\Phi_2})}.
\end{aligned} \tag{3.34}$$

Hence, by the same reasoning as in the first part, we obtain that the range

$$R := (1 - L_\Phi) C_c^\infty(\{\Phi_2 < \infty\})$$

is dense in $L^p(\mathbb{R}^{2d}, \mu_\Phi)$, if R is dense in $L^{p'}(\mathbb{R}^{2d}, \mu_\Phi)$. We define the sequence $p_k \in [1, p_0]$, $k \in \{0, ..., k^*\}$, by $\frac{1}{p_k} = \frac{k}{q_0} + \frac{1}{p_0}$ and k^* is chosen s.t. $\frac{1}{p_{k^*+1}} > 1 \geq \frac{1}{p_{k^*}}$. Observe that by the assumption on q_0 it holds $k^* \geq 1$, which allows us to perform at least one inductive step. Hence, by inductively using the argument in (3.34) we obtain that the range R is dense in $L^{p_k}(\mathbb{R}^{2d}, \mu_\Phi)$ for $k \in \{1, ..., k^*\}$. Via interpolation, we obtain that R is dense in $L^p(\mathbb{R}^{2d}, \mu_\Phi)$ for every $p \in [p_{k^*}, p_0]$. Indeed, if $p \in [p_{k+1}, p_k]$, $k \in \{1, ..., k^* - 1\}$, then we can find $\theta \in [0, 1]$ s.t. $\frac{1}{p} = \frac{1-\theta}{p_k} + \frac{\theta}{p_{k-1}}$. Then it holds

$$\|(1 - L_\Phi) f - g\|_{L^p(\mu_\Phi)} \leq \|(1 - L_\Phi) f - g\|_{L^{p_k}(\mu_\Phi)}^{1-\theta} \, \|(1 - L_\Phi) f - g\|_{L^{p_{k-1}}(\mu_\Phi)}^{\theta}.$$

Hence, R is also dense in $L^p(\mathbb{R}^{2d}, \mu_\Phi)$. Now, let $p = 1$ and chose $p^* \in [1, p_0]$ s.t. $1 = \frac{1}{q_0} + \frac{1}{p^*}$. It is clear that $p^* \geq p_{k^*}$, otherwise this would contradict the choice of $k*$. Therefore, we can use the inequality (3.34) for $p = 1$ and $p' = p^*$ and obtain that R is dense in $L^1(\mathbb{R}^{2d}, \mu_\Phi)$. By using the interpolation argument again we obtain that R is dense in $L^p(\mathbb{R}^{2d,\mu_\Phi})$ for all $p \in [1, p_{k^*}]$, which completes the proof. □

Remark 3.18. *Observe that the notation $\left(T^\Phi_{t,p}\right)_{t\geq 0}$ used in the previous theorem is consistent with the notation in Theorem 1.49, see also Convention 1.52. Indeed, assume that the assumptions from Theorem 3.17 are fulfilled and let p, q be given as above. Assume additionally that $p > 1$. Hence, we obtain by Theorem 3.17 two sub-Markovian s.c.c.s. $\left(T^\Phi_{t,1}\right)_{t\geq 0}$ and $\left(T^\Phi_{t,p}\right)_{t\geq 0}$ on $L^1(\mathbb{R}^{2d}, \mu_\Phi)$ and $L^p(\mathbb{R}^{2d}, \mu_\Phi)$, respectively. Theorem 1.49(ii) provides us with extensions of the operators $T^\Phi_{t,1|L^1\cap L^\infty}$ to a semigroup on $L^p(\mathbb{R}^{2d}, \mu_\Phi)$. In addition this extension coincides with $\left(T^\Phi_{t,p}\right)_{t\geq 0}$ by Theorem 1.49(iii).*

Corollary 3.19. *Let the assumptions of Theorem 3.17 and ($\Phi_2$6) be valid. Then, the formal adjoint $\left(\hat{L}_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ is essentially m-dissipative on $L^p(\mathbb{R}^{2d}, \mu_\Phi)$ for $1 \geq p^{-1} \geq q^{-1} + \frac{1}{2}$. The s.c.c.s. $\left(\hat{T}^\Phi_{t,p}\right)_{t\geq 0}$ generated by the closure $\left(\hat{L}_{\Phi,p}, D(\hat{L}_{\Phi,p})\right)$ is sub-Markovian.*

Proof. Observe that the velocity reversal U in (3.31) is also an isometric isomorphism on the space $L^p(\mathbb{R}^{2d}, \mu_\Phi)$, $p \in [1, \infty]$. Thus, the claim follows by Theorem 1.28. Moreover, U also preserves the sub-Markovian property. □

Under the assumptions of Corollary 3.19, we obtain two sub-Markovian s.c.c.s. $\left(T^\Phi_{t,1}\right)_{t\geq 0}$ and $\left(\hat{T}^\Phi_{t,1}\right)_{t\geq 0}$ on $L^1(\mathbb{R}^{2d}, \mu_\Phi)$. Denote by $\left(T^\Phi_{t,p}\right)_{t\geq 0}$ and $\left(\hat{T}^\Phi_{t,p}\right)_{t\geq 0}$ the corresponding sub-Markovian semigroups on $L^p(\mathbb{R}^{2d}, \mu_\Phi)$, $p \in [1, \infty]$, given by Theorem 1.49. Now let $f, g \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$. Then, due to Proposition 3.7, it holds

$$\int_{\mathbb{R}^{2d}} L_\Phi f g \, d\mu_\Phi = \int_{\mathbb{R}^{2d}} f \hat{L}_\Phi g \, d\mu_\Phi. \tag{3.35}$$

Equation (3.35) indicates that the two semigroups $\left(T^\Phi_{t,1}\right)_{t\geq 0}$ and $\left(\hat{T}^\Phi_{t,1}\right)_{t\geq 0}$, $p \in [1, \infty]$, are adjoint to each other in the sense that

$$\int_{\mathbb{R}^{2d}} T^\Phi_{t,1} f g \, d\mu_\Phi = \int_{\mathbb{R}^{2d}} f \hat{T}^\Phi_{t,1} g \, d\mu_\Phi, \text{ for all } f, g \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}). \tag{3.36}$$

The next lemma contains a rigorous proof for (3.36). In particular, this implies that the notation of using the hat symbol is consistent with Corollary 1.51.

Let us introduce the following assumptions.

Assumption 3.20.

($\Phi_1$5) *Φ_1 is weakly differentiable on $\{\Phi_1 < \infty\}$ and $|\nabla\Phi_1| \in L^4_{loc}(\{\Phi_1 < \infty\}, \mu_{\Phi_1})$.*

Assumption 3.21.

($\Phi_2$9) *Φ_2 is weakly differentiable on $\{\Phi_2 < \infty\}$ and $|\nabla\Phi_2| \in L^4_{loc}(\{\Phi_2 < \infty\}, \mu_{\Phi_2})$.*

Lemma 3.22. *Assume that* $(\Phi_1 2) - (\Phi_1 4)^q$, $q \in [2, \infty]$, $(\Phi_2 1) - (\Phi_2 7)$ *and one of the assumptions (i)-(iii) of Theorem 3.17 hold true. Furthermore, let* $p \in [1, \infty)$ *and* $p^* = \frac{p}{p-1} \in (1, \infty]$. *Then it holds*

(i) *The s.c.c.s.* $\left(T^{\Phi}_{t,1}\right)_{t\geq 0}$ *and* $\left(\hat{T}^{\Phi}_{t,1}\right)_{t\geq 0}$ *satisfy (3.36). In particular, the semigroups* $\left(T^{\Phi}_{t,p^*}\right)_{t\geq 0}$ *and* $\left(\hat{T}^{\Phi}_{t,p^*}\right)_{t\geq 0}$ *are the adjoint semigroups of* $\left(\hat{T}^{\Phi}_{t,p}\right)_{t\geq 0}$ *and* $\left(T^{\Phi}_{t,p}\right)_{t\geq 0}$, *respectively.*

(ii) *The semigroups* $\left(T^{\Phi}_{t,1}\right)_{t\geq 0}$ *and* $\left(\hat{T}^{\Phi}_{t,1}\right)_{t\geq 0}$ *are conservative and admit* μ_Φ *as an invariant measure.*

Proof. (i) The same result for the case $\Phi_2(v) = \frac{1}{2}|v|^2$ is proven in [26, Lemma 3.16.] and directly extends to the general case. However, we give here an additionally proof under slightly stronger assumptions. The proof reflects the intuitive understanding of $\hat{L}_\Phi$ as the adjoint of L_Φ. We assume in the following, that additionally $(\Phi_1 5)$ and $(\Phi_2 9)$ holds true. We denote by $\langle f, g\rangle$ the integral $\int_{\mathbb{R}^{2d}} fg\, d\mu_\Phi$ where f and g are μ_Φ-classes of measurable functions s.t. fg is μ_Φ-integrable. The idea is to use the pre-dual semigroup $(S_t)_{t\geq 0} := \left(T^{\Phi\ *}_{t,\infty\,|L^1}\right)$ on $L^1(\mathbb{R}^{2d}, \mu_\Phi)$ given by Corollary 1.50.

The assumptions $(\Phi_1 5)$ and $(\Phi_2 9)$ imply that $\hat{L}_\Phi g \in L^1(\mathbb{R}^{2d}, \mu_\Phi) \cap L^4(\mathbb{R}^{2d}, \mu_\Phi)$. Furthermore, by Theorem 3.17, we know that $\left(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ is essentially m-dissipative on $L^r(\mathbb{R}^{2d}, \mu_\Phi)$, where $r = \frac{4}{3}$. Then, it holds for $t \in [0, \infty)$

$$\begin{aligned}\langle f, S_t g\rangle &= \left\langle T^{\Phi}_{t,\infty} f, g\right\rangle = \left\langle T^{\Phi}_{t,r} f, g\right\rangle \\ &= \langle f, g\rangle + \int_0^t \left\langle L_{\Phi,r} T^{\Phi}_{s,r} f, g\right\rangle\, ds \end{aligned} \tag{3.37}$$

Now, let $s \in [0, t]$ be arbitrary. Since $T^{\Phi}_{s,r} f \in D(L_{\Phi,r})$, we can find a sequence $f_n \in C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$ s.t. $f_n \longrightarrow T^{\Phi}_{s,r} f$ and $L_\Phi f_n \longrightarrow L_{\Phi,r} T^{\Phi}_{s,r} f$ in $L^r(\mathbb{R}^{2d}, \mu_\Phi)$ as $n \to \infty$. Thus, by using (3.35) and the Hölder inequality twice, we obtain

$$\left\langle L_{\Phi,r} T^{\Phi}_{s,r} f, g\right\rangle = \lim_{n\to\infty} \langle L_\Phi f_n, g\rangle = \lim_{n\to\infty} \left\langle f_n, \hat{L}_\Phi g\right\rangle = \left\langle T_{s,r} f, \hat{L}_\Phi g\right\rangle = \left\langle f, S_s \hat{L}_\Phi g\right\rangle \tag{3.38}$$

Hence, by combining (3.37) and (3.38), we obtain

$$\langle f, S_t g\rangle = \langle f, g\rangle + \left\langle f, \int_0^t S_s \hat{L}_\Phi g\, ds\right\rangle.$$

Since $S_t g$ and $g + \int_0^t S_s \hat{L}_\Phi g\, ds$ coincide as distributions on $C_c^\infty(\Phi_1, \Phi_2 < \infty)$ we obtain

$$S_t g = g + \int_0^t S_t \hat{L}_\Phi g\, ds. \tag{3.39}$$

The identity (3.39) implies that $\left(\hat{L}_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$ is a restriction of the generator of $(S_t)_{t\geq 0}$. But Corollary 3.19 states that the former operator is essentially m-dissipative on

$L^1(\mathbb{R}^{2d}, \mu_\Phi)$ and its closure generates $\left(\hat{T}^{\Phi}_{t,1}\right)_{t\geq 0}$ which yields

$$\left({T^{\Phi}_{t,\infty}}^{*}{}_{|L^1}\right)_{t\geq 0} = (S_t)_{t\geq 0} = \left(\hat{T}^{\Phi}_{t,1}\right)_{t\geq 0}.$$

Hence, it holds (3.36). The last assertion follows by the construction of the semigroups $\left(\hat{T}^{\Phi}_{t,p}\right)_{t\geq 0}$ and $\left(T^{\Phi}_{t,p}\right)_{t\geq 0}$ and (3.36).

(ii) The measure μ_Φ is invariant w.r.t. the semigroups $\left(T^{\Phi}_{t,1}\right)_{t\geq 0}$ and $\left(\hat{T}^{\Phi}_{t,1}\right)_{t\geq 0}$ by Lemma 1.55 and Corollary 3.8. Thus, by Lemma 1.54 both semigroups are conservative.

□

3.2 Existence of an associated Right Process and Martingale Solutions for $(L_{\Phi,2}, D(L_{\Phi,2}))$

In this section we use the results from Subsection 2.2.3 in particular Theorem 2.36 to obtain a Markov processes $\boldsymbol{M}$ and $\widehat{\boldsymbol{M}}$ which are associated with $(L_{\Phi,1}, D(L_{\Phi,1}))$ and $\left(\hat{L}_{\Phi,1}, D(\hat{L}_{\Phi,1})\right)$, respectively, in the sense of Definition 2.29. From these processes we derive Martingale solutions for the generators $(L_{\Phi,2}, D(L_{\Phi,2}))$ and $\left(\hat{L}_{\Phi,2}, D(\hat{L}_{\Phi,2})\right)$ of the $L^2(\mathbb{R}^{2d}, \mu_\Phi)$ semigroups $\left(T^{\Phi}_{t,2}\right)_{t\geq 0}$ and $\left(\hat{T}^{\Phi}_{t,2}\right)_{t\geq 0}$, respectively.

The first proposition seems to be a standard result. The proof relies on the Stone-Weierstrass theorem and is therefore omitted.

Proposition 3.23. *Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be open. Define on the space $C_c^\infty(\Omega)$ the following norm $\|\cdot\|_k := \sum_{|s|\leq k} \|\partial^s \cdot\|_\infty$, $k \in \mathbb{N}$, where $s \in \mathbb{N}_0^n$ and $\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$ for $f \in C_c^\infty(\Omega)$. The normed space $\left(C_c^\infty(\Omega), \|\cdot\|_k\right)$ is separable.*

Theorem 3.24. *Assume that the assumptions of Theorem 3.17 and $(\Phi_2 6)$ are satisfied. Then, there exist μ_Φ-standard processes*

$$\boldsymbol{M} = \left(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\right),$$
$$\widehat{\boldsymbol{M}} = \left(\widehat{\Omega}, \widehat{\mathcal{F}}_\infty, (\widehat{Z}_t)_{t\geq 0}, (\widehat{\mathbb{P}}_z)_{z\in F_\Delta}\right),$$

with state space $F := \{\Phi_1, \Phi_2 < \infty\}$ and life-times ξ and $\hat{\xi}$ which are associated with $(L_{\Phi,1}, D(L_{\Phi,1}))$ and $\left(\hat{L}_{\Phi,1}, D(\hat{L}_{\Phi,1})\right)$, respectively, in the sense of Definition 2.29. Moreover, the events

$$\{\xi < \infty\}, \left\{(Z_t)_{t\geq 0} \text{ is not continuous}\right\},$$
$$\left\{\hat{\xi} < \infty\right\}, \left\{\left(\widehat{Z}_t\right)_{t\geq 0} \text{ is not continuous}\right\},$$

are P_{μ_Φ}- and $\hat{P}_{\mu_\Phi}$-negligible, respectively.

Proof. We only prove the statement which refers to $(L_{\Phi,1}, D(L_{\Phi,1}))$. The remaining part is proven by using the exact same arguments and Corollary 3.19.

First, observe that the open subset $\{\Phi_1, \Phi_2 < \infty\}$ of $\mathbb{R}^{2d}$ is again a Polish space, see e.g. [97, Lemma II.2], hence a Lusin space. Under the above assumptions, we know by Theorem 3.17 that

$$\left(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right) \text{ is essentially m-dissipative on } L^1(\{\Phi_1, \Phi_2 < \infty\}, \mu_\Phi).$$

In the following we check the assumptions (I) and (II) of Theorem 2.36 to show the existence of $\boldsymbol{M}$.

Initially, we prove (II). By Proposition 3.23 there exists a dense subset $(f_k)_{k\in\mathbb{N}}$ of the space $\left(C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}), \|\cdot\|_2\right)$. Since μ_Φ is locally finite we obtain that the $\mathbb{Q}$-algebra $\mathcal{A}$ generated by $(f_k)_{k\in\mathbb{N}}$ forms a core for $\left(L_{\Phi,1}, D(L_{\Phi,1})\right)$. Furthermore, $C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})$ separates the points of $\{\Phi_1, \Phi_2 < \infty\}$ which means that the sequence $(f_k)_{k\in\mathbb{N}}$ separates points, too. Hence, $\mathcal{A}$ fulfills (II).

Now we prove (I). We choose an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of compact set, s.t. every compact subset K of the open set $\{\Phi_1, \Phi_2 < \infty\}$ is contained in some F_k, $k \in \mathbb{N}$. To verify that this choice leads to a nest, we employ Lemma 2.39. Since

$$C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}) \subseteq \cup_{k\in\mathbb{N}} D(L_{\Phi,1})_{F_k}$$

and by Theorem 1.49(iii), it holds $C_c^\infty(\{\Phi_1, \Phi_2 < \infty\}) \subseteq D(L_{\Phi,2})$ we conclude that $(F_k)_{k\in\mathbb{N}}$ is a nest of compact sets. Hence, by Theorem 2.36 there exists a μ_Φ-standard process

$$\boldsymbol{M} = \left(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in F_\Delta}\right)$$

which is associated with the semigroup $\left(T_{t,1}^\Phi\right)_{t\geq 0}$ generated by the closure $\left(L_{\Phi,1}, D(L_{\Phi,1})\right)$ of $\left(L_\Phi, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right)$. Now, let $t \geq 0$ and $h \in L^1(\{\Phi_1, \Phi_2 < \infty\}, \mu_\Phi)$. Then, it holds by Lemma 3.22(ii) that

$$\begin{aligned} P_{h\mu_\Phi}(Z_t \in \{\Delta\}) &= \int_F h((x,v))\mathbb{P}_{(x,v)}(Z_t \in \{\Delta\})\, \mu_\Phi(d(x,v)) \\ &= \int_F h((x,v))\left(1 - \mathbb{P}_{(x,v)}(Z_t \in F\right)\, \mu_\Phi(d(x,v)) \\ &= \int_F h((x,v))\left(1 - T_{t,\infty}^\Phi 1_F\right)((x,v))\, \mu_\Phi(d(x,v)) = 0. \end{aligned}$$

Thus, we get

$$P_{h\mu_\Phi}(\xi < \infty) \leq \sum_{t\in\mathbb{Q}} P_{h\mu_\Phi}(Z_t \in \{\Delta\}) = 0.$$

The statement concerning the continuity of the paths can be seen as follows. Since $\boldsymbol{M}$ is a μ_Φ-standard process we know by $(M6)$ and $(M8)$ that the paths are càdlàg $\mathbb{P}_{\mu_\Phi}$-a.s.. By assumption $(\Phi_1 4)^q$, $q \in [2, \infty]$ and $(\Phi_2 3)$ we obtain that $L_\Phi f \in L^2(\{\Phi_1, \Phi_2 < \infty\}, \mu_\Phi)$ for every f from the point separating and separable set $\mathcal{A}$. Thus, by Corollary 2.41 the claim follows. □

Note that if one only aims to prove the statements of Theorem 3.24 related to the operator $\left(L_{\Phi,1}, D(L_{\Phi,1})\right)$, then one does not have to work under the assumption $(\Phi_2 6)$. The following is basically a corollary of Remark 2.31 in combination with the previous theorem and Theorem 2.23. Since its content is of particular importance for the further course of this thesis we formulate, it as a theorem.

Theorem 3.25. *Suppose the assumptions of Theorem 3.17 holds and let $0 \leq h \in L^1(\mathbb{R}^{2d}, \mu_\Phi) \cap L^2(\mathbb{R}^{2d}, \mu_\Phi)$ be a probability density w.r.t. μ_Φ. Denote by $\langle \cdot, \cdot \rangle_{\mu_\Phi}$ the dual pairing between $L^1(\mathbb{R}^{2d}, \mu_\Phi)$ and $L^\infty(\mathbb{R}^{2d}, \mu_\Phi)$. Then there exists a probability law $\mathbb{P}_{h\mu_\Phi}$ with initial distribution $h\mu_\Phi$ on the measurable space $(C([0,\infty), \{\Phi_1, \Phi_2 < \infty\}), \mathcal{B}_C)$ which is associated with the semigroup $\left(T^\Phi_{t,1}\right)_{t\geq 0}$. Namely, for all $f_1, \ldots, f_k \in L^\infty(\mathbb{R}^{2d}, \mu_\Phi)$ and $0 \leq t_1 < \ldots < t_k$, $k \in \mathbb{N}$, it holds*

$$\mathbb{E}_{h\mu_\Phi}\left[\prod_{i=0}^{k} f_i(X_{t_i}, V_{t_i})\right] = \langle h, T^\Phi_{t_1,\infty}(f_1 T^\Phi_{t_2-t_1,\infty}(f_2 \ldots T^\Phi_{t_{k-1}-t_{k-2},\infty}(f_{k-1} T^\Phi_{t_k-t_{k-1},\infty} f_k)\ldots))\rangle_{\mu_\Phi}, \tag{3.40}$$

where $\mathbb{E}_{h\mu_\Phi}$ denotes integration w.r.t. $\mathbb{P}_{h\mu_\Phi}$. In particular, $\mathbb{P}_{h\mu_\Phi}$ solves the martingale problem for the generator $(L_{\Phi,2}, D(L_{\Phi,2}))$ of $\left(T^\Phi_{t,2}\right)_{t\geq 0}$, i.e., for $f \in D(L_{\Phi,2})$ the process $(M^{[f]}_t)_{t\geq 0}$ defined by

$$M^{[f]}_t := f(X_t, V_t) - f(X_0, V_0) - \int_{[0,t]} L_{\Phi,2} f(X_s, V_s)\, ds, \quad t \geq 0, \tag{3.41}$$

is a martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t = \sigma\left((X_s, V_s) \mid 0 \leq s \leq t\right)$, and $\mathbb{P}_{h\mu_\Phi}$. Additionally, if $f^2 \in D(L_{\Phi,2})$ and $L_{\Phi,2} f \in L^4(\mathbb{R}^{2d}, \mu_\Phi)$ then the process $(N^{[f]}_t)_{t\geq 0}$ defined by

$$N^{[f]}_t := \left(M^{[f]}_t\right)^2 - \int_{[0,t]} L_{\Phi,2}(f^2)(X_s, V_s) - 2(f L_{\Phi,2} f)(X_s, V_s)\, ds, \quad t \geq 0,$$

is also a martingale w.r.t. $\mathbb{P}_{h\mu_\Phi}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Remark 3.26.

(i) Note that the martingale property of $(M^{[f]}_t)_{t\geq 0}$ also holds for the larger filtration $\left(\overline{\mathcal{F}}_{t+}\right)_{t\geq 0}$ given in Remark 2.25 provided that f is continuous.

(ii) Under the additional assumption $(\Phi_2 6)$ the results from the previous theorem also hold for the operator $\left(\hat{L}_{\Phi,1}, D(\hat{L}_{\Phi,1})\right)$, i.e., for h given in Theorem 3.25 there exists a law $\hat{\mathbb{P}}_{h\mu_\Phi}$ on the measurable space $(C([0,\infty), \{\Phi_1, \Phi_2 < \infty\}), \mathcal{B}_C)$ with initial distribution $h\mu_\Phi$ which is associated with $\left(\hat{T}^\Phi_{t,1}\right)_{t\geq 0}$ in the sense of (3.40). We use this fact and the connection between $\hat{\mathbb{P}}_{h\mu_\Phi}$ and $\mathbb{P}_{h\mu_\Phi}$ later in the proof of Theorem 3.40.

3.3 The Limit Operator and Limit Process

This section consists of a brief summary of the functional analytic objects related to the overdamped Langevin equation (3.3) and the construction of martingale solutions for its generator. Denote by $(B_t)_{t\geq o}$ a d-dimensional Brownian motion and recall the overdamped equation (3.3)

$$dX^0_t = -\nabla\Phi_1(X^0_t)dt + \sqrt{2}dB_t. \tag{3.42}$$

The corresponding generator is obtained by the Itô lemma and given as

$$L_{\Phi_1} f = \Delta f - \nabla\Phi_1 \cdot \nabla f, \quad f \in C^\infty_c(\{\Phi_1 < \infty\}).$$

We consider the operator $\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right)$ as an operator on the Hilbert space $\mathcal{H}_{\Phi_1} := L^2(\mathbb{R}^d, \mu_{\Phi_1})$, where μ_{Φ_1} is defined according to Notation 3.1.

Proposition 3.27. *Let Φ_1 satisfy $(\Phi_1 2) - (\Phi_1 4)^2$. Then $\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right)$ is well-defined on $L^p(\mathbb{R}^d, \mu_{\Phi_1})$ for every $p \in [1,2]$. Furthermore, it is symmetric and negative definite on $\mathcal{H}_{\Phi_1}$ and for all $f \in C_c^\infty(\{\Phi_1 < \infty\})$ it holds*

$$\int_{\mathbb{R}^d} L_{\Phi_1} f \, d\mu_{\Phi_1} = 0. \tag{3.43}$$

In particular, if the closure of $\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right)$ generates a s.c.c.s. $\left(T_t^{\Phi_1}\right)_{t\geq 0}$, then this semigroup is sub-Markovian.

Proof. The proof follows from Proposition 3.5 and Lemma 1.59. □

We make additional assumptions on Φ_1.

Assumption 3.28.

$(\Phi_1 6)$ *The operator $\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right)$ is essentially self-adjoint on $\mathcal{H}_{\Phi_1}$.*

$(\Phi_1 7)$ *μ_{Φ_1} is a finite measure, i.e., $\mu_{\Phi_1}(\mathbb{R}^d) = \int_{\mathbb{R}^d} e^{-\Phi_1} dx < \infty$.*

Observe that under the assumptions of Proposition 3.27, the assumption $(\Phi_1 6)$ is equivalent to

$\widetilde{(\Phi_1 6)}$ The operator $\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right)$ is closable and its closure is the generator of a symmetric s.c.c.s. $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ on $\mathcal{H}_{\Phi_1}$.

Hence, if we assume $(\Phi_1 2)-(\Phi_1 4)^2$ and $(\Phi_1 6)$ we denote by $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ the s.c.c.s. on $\mathcal{H}_{\Phi_1}$ generated by the closure of $\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right)$.
We state a sufficient condition in Section 3.7 for the assumption $(\Phi_1 6)$.

Theorem 3.29. *Assume $(\Phi_1 2), (\Phi_1 3), (\Phi_1 4)^2, (\Phi_1 6), (\Phi_1 7)$. Then, there exists a μ_{Φ_1}-tight standard process*

$$\boldsymbol{M} = \left(\Omega, \mathcal{F}_\infty, (Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in\{\Phi_1<\infty\}_\Delta}\right)$$

which is associated with $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ in the sense of Definition 2.29. The paths are continuous and have infinite life-time $\mathbb{P}_{\Phi_1}$-a.s..

Proof. We first prove that the symmetric sub-Markovian s.c.c.s. $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ is conservative, i.e., it holds $T_{t,\infty}^{\Phi_1} 1 = 1$ for all $t \geq 0$. Since $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ is symmetric, it suffices to show that μ_{Φ_1} is invariant for $\left(T_{t,1}^{\Phi_1}\right)_{t\geq 0}$. Observe that the generator $\left(L_{\Phi_1,1}, D(L_{\Phi_1,1})\right)$ of $\left(T_{t,1}^{\Phi_1}\right)_{t\geq 0}$ is an extension of $\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right)$, see Theorem 1.49(iii). Due to (3.43) it suffices to show that $C_c^\infty(\{\Phi_1 < \infty\})$

is a core for the generator of $\left(T_{t,1}^{\Phi_1}\right)_{t\geq 0}$ which this follows directly from [30, Chapter 1.e), Lemma 1.6(ii)], see also the last mentioned reference for the definition of L^p-uniqueness, $p \in [1, \infty)$.

The remaining part, in particular, the existence of the process $\boldsymbol{M}$ follows by Theorem 2.36 and Corollary 2.41. The assumptions of Theorem 2.36 can be checked as in the proof of Theorem 3.24. □

Remark 3.30. *Another way to obtain a process $\boldsymbol{M}$ given in Theorem 3.29 is to consider the bilinear form*

$$\mathcal{E}_{\Phi_1}(f,g) := -(L_{\Phi_1} f, g)_{\mathcal{H}_{\Phi_1}}, \quad f, g \in C_c^\infty(\{\Phi_1 < \infty\}),$$

and use the theory of regular Dirichlet forms, see e.g. [39] and [70]. Proceeding this way, one obtains stronger results, e.g. concerning the continuity of paths and life-time as well as the weak solutions of 3.42 with initial distributions δ_x for all x outside a set of capacity zero, see in particular [39, Chapter 5]. In the following we only use the so-called equilibrium laws $\mathbb{P}_{h\mu_{\Phi_1}}$. In particular, we consider their image measures on $C([0,\infty), \{\Phi_1 < \infty\})$, see Remark 2.31(ii). Both approaches, i.e., the Dirichlet form approach and Theorem 3.29, lead to a measure on $C([0,\infty), \{\Phi_1 < \infty\})$ which is associated with the semigroup $\left(T_t^{\Phi_1}\right)_{t\geq 0}$. Hence, these measures coincide.

We obtain the analogous statement as in Theorem 2.23.

Corollary 3.31. *Suppose the assumptions of Theorem 3.29 hold true. Let $h \in L^1(\mathbb{R}^d, \mu_{\Phi_1}) \cap L^2(\mathbb{R}^d, \mu_{\Phi_1})$ be a probability density w.r.t. μ_{Φ_1}. Then, there exists a probability law $\mathbb{P}_{h\mu_{\Phi_1}}$ on the measurable space $C([0,\infty), \{\Phi_1 < \infty\})$ with initial distribution $h\mu_{\Phi_1}$ which is associated with the sub-Markovian s.c.c.s. $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ in the sense of Definition 2.17. In particular, the measure $\mathbb{P}_{h\mu_{\Phi_1}}$ solves the martingale problem for the generator $\left(L_{\Phi_1}, D(L_{\Phi_1})\right)$.*

3.4 Velocity Scaling and Semigroup Convergence

In this section we establish the analytic part of our main result. We show the convergence of operator semigroups in the sense of Kuwae-Shioya, introduced in Section 1.4. To this end let us introduce some notations.

Let Φ_2 be given as in Condition 3.3 and let $\varepsilon > 0$. We define a scaled velocity potential as

$$\Phi_2^\varepsilon(\cdot) = \Phi_2\left(\frac{\cdot}{\varepsilon}\right) + \ln(\varepsilon^d). \tag{3.44}$$

The constant $\ln(\varepsilon^d)$ is only a normalization constant. Before we explain how the scaled velocity potential is related to the scaling in (3.1), (3.1) we make some observations.

Similar as before we write $\boldsymbol{\Phi}^\varepsilon = \left(\Phi_1, \Phi_2^\varepsilon\right)$. We also denote by μ_ε the measure $\mu_{\boldsymbol{\Phi}^\varepsilon}$ to keep the notation simple. Observe that the assumptions $(\Phi_2 1) - (\Phi_2 7)$ hold true for Φ_2^ε if they hold true for Φ_2. Indeed, this is obvious for all assumptions except possibly for $(\Phi_2 4)$. Notice that the map

$$V_\varepsilon : L^2(\mathbb{R}^d, \mu_{\Phi_2^\varepsilon}) \longrightarrow L^2(\mathbb{R}^d, \mu_{\Phi_2}), f \mapsto f\left(\frac{\cdot}{\varepsilon}\right)$$

is a unitary transformation s.t.

$$\left(V_\varepsilon L_{\Phi_2^\varepsilon} V_\varepsilon^*, V_\varepsilon C_c^\infty(\{\Phi_2^\varepsilon < \infty\})\right) = \left(\frac{1}{\varepsilon^2} L_{\Phi_2}, C_c^\infty(\{\Phi_2 < \infty\})\right).$$

Thus, the property ($\Phi_2 4$) is also left invariant under the scaling in (3.44). Hence, under the assumptions of Theorem 3.17, we obtain that the scaled operator

$$\left(L_{\Phi^\varepsilon}, C_c^\infty(\{\Phi_1, \Phi_2^\varepsilon < \infty\})\right) \tag{3.45}$$

defined on $L^1(\mathbb{R}^{2d}, \mu_\varepsilon)$ is essentially m-dissipative. Namely, the closure $(L_{\Phi^\varepsilon}, D(L_{\Phi^\varepsilon}))$ of (3.45) is the generator of a sub-Markovian s.c.c.s. $\left(T_t^{\Phi^\varepsilon}\right)_{t\geq 0}$ on $L^1(\mathbb{R}^{2d}, \mu_\varepsilon)$ Now, we explain how the scaled velocity potential Φ_2^ε in (3.44) is related to the scaling in the original stochastic differential equation (??), (??). Let $p \in [1, \infty)$ and define via the state space transformation

$$\widetilde{U}_\varepsilon : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d}, (x, v) \mapsto \left(x, \frac{v}{\varepsilon}\right), \tag{3.46}$$

the isometric isomorphism

$$U_\varepsilon : L^p(\mathbb{R}^{2d}, \mu_\Phi) \longrightarrow L^p(\mathbb{R}^{2d}, \mu_{\Phi^\varepsilon}), f \mapsto f \circ \widetilde{U}_\varepsilon. \tag{3.47}$$

By the chain rule we easily obtain

$$\left(U_\varepsilon^* L_{\Phi^\varepsilon} U_\varepsilon, U_\varepsilon^* C_c^\infty(\{\Phi_1, \Phi_2^\varepsilon < \infty\})\right) = \left(L_\Phi^\varepsilon, C_c^\infty(\{\Phi_1, \Phi_2 < \infty\})\right),$$

as equality of operators on $L^p(\mathbb{R}^{2d}, \mu_\Phi)$, $p \in [1, 2]$. Therefore, we work in the following with the operator $\left(L_{\Phi^\varepsilon}, C_c^\infty(\{\Phi_1, \Phi_2^\varepsilon < \infty\})\right)$ and invert this transformation latter in Section 3.6.

We denote by $\left(T_{t,2}^\varepsilon\right)_{t\geq 0} = \left(T_{t,2}^{\Phi^\varepsilon}\right)_{t\geq 0}$ the semigroups on $\mathcal{H}_\varepsilon := L^2(\mathbb{R}^{2d}, \mu_\varepsilon)$ induced by $\left(T_t^{\Phi^\varepsilon}\right)_{t\geq 0}$, see Theorem 1.49. The generator $\left(L_{\Phi^\varepsilon,2}, D(L_{\Phi^\varepsilon,2})\right)$ of $\left(T_{t,2}^\varepsilon\right)_{t\geq 0}$ we abbreviated by $(L_\varepsilon, D(L_\varepsilon))$ and the norm and the scalar product on $\mathcal{H}_\varepsilon$ are denoted by $\|\cdot\|_\varepsilon$ and $(\cdot,\cdot)_\varepsilon$, respectively. Note that $(L_\varepsilon, D(L_\varepsilon))$, which is an extension of $\left(L_{\Phi^\varepsilon}, C_c^\infty(\{\Phi_1, \Phi_2^\varepsilon < \infty\})\right)$, is an operator on $\mathcal{H}_\varepsilon$. Additionally, we assume ($\Phi_1 5$) is true, i.e., we let $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ be the s.c.c.s. on $\mathcal{H}_{\Phi_1}$ from the previous section generated by the closure of

$$\left(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\})\right).$$

In the following we establish the convergence of the Hilbert spaces $\mathcal{H}_\varepsilon$ towards the Hilbert space $\mathcal{H}_{\Phi_1}$ in the sense of Kuwae-Shioya, see Section 1.4. Namely, there exists a dense subset C of $\mathcal{H}_{\Phi_1}$ and for every $\varepsilon > 0$ there exists a linear map

$$\Psi_\varepsilon : C \longrightarrow \mathcal{H}_\varepsilon,$$

such that

$$\lim_{\varepsilon\to 0} \|\Psi_\varepsilon(u)\|_{\mathcal{H}_\varepsilon} = \|u\|_{\mathcal{H}_{\Phi_1}}, \text{ for all } u \in C.$$

Furthermore, we prove the convergence of the semigroups $\left(T_t^\varepsilon\right)_{t\geq 0}$, $\varepsilon > 0$, towards the semigroup

$\left(T_t^{\Phi_1}\right)_{t\geq 0}$ along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$, i.e., for all $t \geq 0$ it holds

$$f_\varepsilon \longrightarrow f \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1} \text{ implies } T^\varepsilon_{t,2} f_\varepsilon \longrightarrow T_t^{\Phi_1} f \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}. \tag{3.48}$$

This was already done under the more restrictive assumptions of a locally Lipschitz continuous potential Φ_1 and $\Phi_2(v) = \frac{v^2}{2}$ in [75]. Fortunately, the same ideas apply in the singular case, too. For the sake of completeness we present all details here. For this purpose, we introduce the following additional assumptions for Φ_1 and Φ_2, respectively.

Assumption 3.32.

($\Phi_1$8) Φ_1 *is weakly differentiable on* $\{\Phi_1 < \infty\}$ *and* $|\nabla\Phi_1| \in L^2(\mathbb{R}^d, \mu_{\Phi_1})$.

Assumption 3.33.

($\Phi_2$10) Φ_2 *has no singularities, i.e.,* $\{\Phi_2 = \infty\} = \emptyset$.

In the following, if we assume ($\Phi_2$7), then w.l.o.g. we can assume that μ_{Φ_2} is a probability measure, i.e., $\mu_{\Phi_2}(\mathbb{R}^d) = 1$. To keep the mathematical expression readable we define the following maps $p_x, p_v, \sigma : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^d$, where $\sigma(x, v) = x + v$, $p_x(x, v) = x$, $p_v(x, v) = v$. Next, we define the maps Ψ_ε from (3.4).

Definition 3.34. *Let* $\varepsilon > 0$ *and choose a symmetric cut off function* $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, *s.t.*

(i) $\eta_\varepsilon(v) = \eta_\varepsilon(-v)$, *for all* $v \in \mathbb{R}^d$, $\eta_\varepsilon \equiv 1$ *on* $B_{\varepsilon^{-2}}(0)$ *and* $supp(\eta_\varepsilon) \subseteq B_{2\varepsilon^{-2}}(0)$,

(ii) $|\nabla\eta_\varepsilon| \leq C\varepsilon^2$ *and* $|\Delta\eta_\varepsilon| \leq C\varepsilon^4$, *for a finite constant* C *independent of* ε.

We define $C = C_c^\infty(\{\Phi_1 < \infty\})$, *where* $\{\Phi_1 < \infty\} \subseteq \mathbb{R}^d$. *Further, we define the convergence determining function* Ψ_ε *by*

$$\Psi_\varepsilon : C \longrightarrow \mathcal{H}_\varepsilon, f \mapsto (f \circ \sigma)(\eta_\varepsilon \circ p_v). \tag{3.49}$$

Observe that $\Psi_\varepsilon f$, $\varepsilon > 0$, is smooth and compactly supported for $f \in C$. In the following, we denote by ∇ and Δ the gradient and Laplacian on $\mathbb{R}^d$, respectively.

Theorem 3.35. *Assume* ($\Phi_1$2) – ($\Phi_1$4)^2, ($\Phi_1$6) – ($\Phi_1$8) *and* ($\Phi_2$1) – ($\Phi_2$10) *hold true. Then, it holds, the family of Hilbert spaces* $(\mathcal{H}_\varepsilon)_{\varepsilon>0}$ *converges along the family* $(\Psi_\varepsilon)_{\varepsilon>0}$ *defined in (3.49) towards the Hilbert space* $\mathcal{H}_{\Phi_1}$ *as* ε *tends to zero in the Kuwae-Shioya sense. Furthermore, the semigroups* $\left(T_t^\varepsilon\right)_{t\geq 0}$, $\varepsilon > 0$, *converge towards* $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ *along* $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$, *i.e., (3.48) holds true.*

Proof. We proceed as in [75, Proposition 3.21., Theorem 3.22.], where the special case $\Phi_2(v) = \frac{1}{2}|v|^2$ and stronger assumptions on Φ_1 are considered. For $f \in C$ we have to show $\|\Psi_\varepsilon f\|_{\mathcal{H}_\varepsilon} \xrightarrow{\varepsilon\to 0} \|f\|_{\mathcal{H}_{\Phi_1}}$. By using the symmetry of η_ε and Φ_2 together with the transformation $(x, v) \mapsto (x, -v)$, we rewrite the norm using the convolution $*$, i.e.,

$$\|\Psi_\varepsilon f\|_\varepsilon^2 = \int_{\mathbb{R}^d} f^2 * (\eta_\varepsilon^2 e^{-\Phi_2^\varepsilon})(x) e^{-\Phi_1}(x)\, dx. \tag{3.50}$$

For $\alpha_\varepsilon := \int_{\mathbb{R}^d} \eta_\varepsilon^2 e^{-\Phi_2^\varepsilon}(v)\,dv$ one can show $\alpha_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 1$, which assures that $\left(\alpha_\varepsilon^{-1}\eta_\varepsilon^2 e^{-\Phi_2^\varepsilon}\right)_{\varepsilon>0}$ is an approximate identity. Since $f^2 \in L^1(\mathbb{R}^d)$ and $e^{-\Phi_1} \in L^\infty(\mathbb{R}^d)$ thanks to assumption ($\Phi_1 2$), the Hölder inequality implies the desired result.

Next we prove the convergence of the semigroups generated by $(L_\varepsilon, D(L_\varepsilon))$ in $\mathcal{H}_\varepsilon$. Recall that the limit semigroup $\left(T_t^{\Phi_1}\right)_{t\geq 0}$ admits the closure of $(L_{\Phi_1}, C_c^\infty(\{\Phi_1 < \infty\}))$ as its generator. We use that the semigroup convergence is equivalent to the convergence of the generators and in particular, it suffices to have convergence of the generators on a core for the limit generator, i.e., we use Corollary 1.78. Hence, for $f \in C = C_c^\infty(\{\Phi_1 < \infty\})$, it suffices to show

$$(L_\varepsilon \Psi_\varepsilon f)_{\varepsilon>0} \longrightarrow L_0 f \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}.$$

Let $f \in C$ be given and $i \in \{1,\dots,d\}$. Observe that the function $f\circ\sigma$ fulfills $\partial_{x_i}(f\circ\sigma) = \partial_i f\circ\sigma = \partial_{v_i}(f\circ\sigma)$. We start with computing the expression $L_\varepsilon\Psi_\varepsilon f$ explicitly. According the previous observation, we obtain

$$\begin{aligned} L_\varepsilon\Psi_\varepsilon f =&(\Delta f\circ\sigma)(\eta_\varepsilon\circ p_v) + (f\circ\sigma)(\Delta\eta_\varepsilon\circ p_v) + 2(\nabla f\circ\sigma)\cdot(\nabla\eta_\varepsilon\circ p_v) \\ &- \left(\nabla_v\Phi_2^\varepsilon\cdot(\nabla\eta_\varepsilon\circ p_v)\right)(f\circ\sigma) - (\nabla_x\Phi_1\cdot(\nabla f\circ\sigma))\eta_\varepsilon\circ p_v \\ &- (\nabla_x\Phi_1\cdot(\nabla\eta_\varepsilon\circ p_v))(f\circ\sigma). \end{aligned} \tag{3.51}$$

The idea is that terms in (3.51) containing a derivative of η_ε converge to zero along and the remaining terms converge to

$$L_0 f = \Delta f - \nabla\Phi_1\cdot\nabla f,$$

along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$, respectively. More precisely, since convergence along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$ is linear by Lemma 1.64, it suffices to show convergence of the single summands in (3.51). Namely, we show

$$\left.\begin{array}{ll} 1. & (f\circ\sigma)(\Delta\eta_\varepsilon\circ p_v) \\ 2. & (\nabla f\circ\sigma)\cdot(\nabla\eta_\varepsilon\circ p_v) \\ 3. & (\nabla_x\Phi_1\cdot(\nabla\eta_\varepsilon\circ p_v))(f\circ\sigma) \\ 4. & \left(\nabla_v\Phi_2^\varepsilon\cdot(\nabla\eta_\varepsilon\circ p_v)\right)(f\circ\sigma) \end{array}\right\} \longrightarrow 0$$

$$\begin{array}{llll} 5. & (\Delta f\circ\sigma)(\eta_\varepsilon\circ p_v) & \longrightarrow & \Delta f \\ 6. & (\nabla_x\Phi_1\cdot(\nabla f\circ\sigma))(\eta_\varepsilon\circ p_v) & \longrightarrow & \nabla\Phi_1\cdot\nabla f \end{array}$$

along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$.

To prove convergence in 1.-4. we establish that the respective norms of the elements converge to zero, which implies convergence to zero by Lemma 1.64(i). This holds true due to the choice of η_ε and a convolution argument as in (3.50). The statements in 5. and 6. are obtained via Lemma 1.64(iv) by proving weak convergence and convergence of the respective norms. The underlying argument is the convolution trick which we already used in (3.50).

1. By using Fubini's theorem and the symmetry of Φ_2, we obtain

$$
\begin{aligned}
\|(f \circ \sigma)(\Delta\eta_\varepsilon \circ p_v)\|_\varepsilon^2 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^2(x+v)\Delta_v\eta_\varepsilon(v)^2 e^{-\Phi_2^\varepsilon(v)} e^{-\Phi_1(x)}\,dv\,dx \\
&\le C\varepsilon^8 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^2(x+v) e^{-\Phi_2^\varepsilon(v)} e^{-\Phi_1(x)}\,dv\,dx \\
&= C\varepsilon^8 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^2(x-v) e^{-\Phi_2^\varepsilon(-v)} e^{-\Phi_1(x)}\,dv\,dx \\
&= C\varepsilon^8 \int_{\mathbb{R}^d} \left(f^2 * e^{-\Phi_2^\varepsilon}\right)(x) e^{-\Phi_1(x)}\,dx. \qquad (3.52)
\end{aligned}
$$

Since $f^2 \in L^1(\mathbb{R}^d)$, we obtain that $f^2 * e^{-\Phi_2^\varepsilon}$ converges in $L^1(\mathbb{R}^d)$ to f^2 as $\varepsilon \longrightarrow 0$. The assumption ($\Phi_1 2$) and the Hölder inequality imply in particular that

$$
\int_{\mathbb{R}^d} \left(f^2 * e^{-\Phi_2^\varepsilon}\right)(x) e^{-\Phi_1(x)}\,dx
$$

is bounded uniformly in ε, therefore, we conclude

$$
\lim_{\varepsilon \longrightarrow 0} \|(f \circ \sigma)(\Delta\eta_\varepsilon \circ p_v)\|_\varepsilon = 0.
$$

In the following calculations we use the convolution technique from (3.52) several times.
2. We have

$$
\begin{aligned}
\|(\nabla f \circ \sigma)\cdot(\nabla\eta_\varepsilon \circ p_v)\|_\varepsilon^2 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\nabla f(x+v)\cdot\nabla\eta_\varepsilon(v))^2 e^{-\Phi_2^\varepsilon(v)} e^{-\Phi_1(x)}\,dv\,dx \\
&\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla f(x+v)|^2\,|\nabla\eta_\varepsilon(v)|^2\,e^{-\Phi_2^\varepsilon(v)} e^{-\Phi_1(x)}\,dv\,dx \\
&\le C\varepsilon^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla f(x+v)|^2\,e^{-\Phi_2^\varepsilon(v)} e^{-\Phi_1(x)}\,dv\,dx \\
&= C\varepsilon^2 \int_{\mathbb{R}^{2d}} \left(|\nabla f|^2 * e^{-\Phi_2^\varepsilon(v)}\right)(x) e^{-\Phi_1(x)}\,dx.
\end{aligned}
$$

By applying the same argument as above leads to

$$
\lim_{\varepsilon \longrightarrow 0} \|(\nabla f \circ \sigma)\cdot(\nabla\eta_\varepsilon \circ p_v)\|_\varepsilon = 0.
$$

3. Since f^2 is bounded and $|\nabla\Phi_1|^2$ is by assumption ($\Phi_1 7$) integrable w.r.t. μ_{Φ_1}, we obtain

$$\begin{aligned}\|(\nabla_x\Phi_1\cdot(\nabla\eta_\varepsilon\circ p_v))(f\circ\sigma)\|_\varepsilon^2 &\le \left\|f^2\right\|_\infty \int\limits_{\mathbb{R}^d} |\nabla\Phi_1|^2\, e^{-\Phi_1}\,dx \int\limits_{\mathbb{R}^d} |\nabla\eta_\varepsilon|^2\, e^{-\Phi_2^\varepsilon}\,dv\\ &\le C\varepsilon^4\left\|f^2\right\|_\infty \int\limits_{\mathbb{R}^d} |\nabla\Phi_1|^2\, e^{-\Phi_1}\,dx \int\limits_{\mathbb{R}^d} e^{-\Phi_2}\,dv,\end{aligned}$$

which implies that

$$\lim_{\varepsilon\to 0}\|(\nabla_x\Phi_1\cdot(\nabla\eta_\varepsilon\circ p_v))(f\circ\sigma)\|_\varepsilon = 0.$$

4. For a function $g:\mathbb{R}^d\longrightarrow\mathbb{R}$ and $\varepsilon>0$ we denote in the following by $(g)_\varepsilon$ the function $(g)_\varepsilon:\mathbb{R}^d\longrightarrow\mathbb{R}, x\mapsto \frac{1}{\varepsilon^d}g\left(\frac{x}{\varepsilon}\right)$. Then it holds

$$\begin{aligned}\left\|\left(\nabla_v\Phi_2^\varepsilon\cdot(\nabla\eta_\varepsilon\circ p_v)\right)(f\circ\sigma)\right\|_\varepsilon^2 &\le \frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}|\nabla\eta_\varepsilon|^2(v)f^2(x+v)\left(|\nabla\Phi_2|^2\,e^{-\Phi_2}\right)_\varepsilon(v)e^{-\Phi_1}(x)\,dx\,dv\\ &\le C\varepsilon^2\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f^2(x+v)(|\nabla\Phi_2|^2\,e^{-\Phi_2})_\varepsilon(v)\,dv e^{-\Phi_1}(x)\,dx\\ &= C\varepsilon^2\int\limits_{\mathbb{R}^d}\left(f^2*\left(|\nabla\Phi_2|^2\,e^{-\Phi_2}\right)_\varepsilon\right)(x)e^{-\Phi_1}(x)\,dx.\end{aligned}$$

Due to assumption ($\Phi_2 8$), we obtain that $\int\limits_{\mathbb{R}^d}\left(f^2*\left(|\nabla\Phi_2|^2\,e^{-\Phi_2}\right)_\varepsilon\right)(x)e^{-\Phi_1}(x)\,dx$ is bounded in ε.

Hence,

$$\lim_{\varepsilon\to 0}\left\|\left(\nabla_v\Phi_2^\varepsilon\cdot(\nabla\eta_\varepsilon\circ p_v)\right)(f\circ\sigma)\right\|_\varepsilon = 0.$$

5. We begin by establishing the convergence of the norms. The same arguments as above yield

$$\|(\Delta f\circ\sigma)(\eta_\varepsilon\circ p_v)\|_\varepsilon^2 = \int\limits_{\mathbb{R}^d}(\Delta f)^2*\left(\eta_\varepsilon^2 e^{-\Phi_2^\varepsilon}\right)e^{-\Phi_1}\,dx \xrightarrow{\varepsilon\to 0}\int\limits_{\mathbb{R}^d}(\Delta f)^2 e^{-\Phi_1}\,dx.$$

Now, we show weak convergence along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$. Let $\xi\in C_c^\infty(\Phi_1<\infty)$ be arbitrary. Then, it holds

$$((\Delta f\circ\sigma)(\eta_\varepsilon\circ p_v),\Psi_\varepsilon(\xi))_\varepsilon = \int\limits_{\mathbb{R}^d}\left((\xi\Delta f)*(\eta_\varepsilon^2(e^{-\Phi_2^\varepsilon})\right)e^{-\Phi_1}\,dx \xrightarrow{\varepsilon\to 0}\int\limits_{\mathbb{R}^d}\xi\Delta f e^{-\Phi_1}\,dx.$$

Hence, by Corollary 1.66 we obtain the desired result.
6. As before, we start with the convergence of norms. Let $1\le i\le d$. Then, it holds

$$\begin{aligned}\|(\nabla_x\Phi_1\cdot(\nabla f\circ\sigma))(\eta_\varepsilon\circ p_v)\|_\varepsilon^2 = \sum_{i,j=1}^d\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}&(\partial_i f\partial_j f)(x+v)\eta_\varepsilon^2(v)e^{-\Phi_2^\varepsilon(v)}\\ &\times(\partial_i\Phi_1\partial_j\Phi_1)(x)e^{-\Phi_1(x)}\,dx\,dv\end{aligned}$$

The transformation $x \mapsto x + v$ implies for the last integral that

$$\begin{aligned}
&\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\partial_i f \partial_j f)(x+v) \eta_\varepsilon^2(v) e^{-\Phi_2^\varepsilon(v)} (\partial_i \Phi_1 \partial_j \Phi_1)(x) e^{-\Phi_1(x)} \, dx \, dv \\
&= \sum_{i,j} \int_{\mathbb{R}^d} (\partial_{x_i} f \partial_{x_j} f)(x) \int_{\mathbb{R}^d} \eta_\varepsilon^2(v) e^{-\Phi_2^\varepsilon(v)} (\partial_i \Phi_1 \partial_j \Phi_1)(x-v) e^{-\Phi_1(x-v)} \, dv \, dx \\
&= \sum_{i,j} \int_{\mathbb{R}^d} (\partial_{x_i} f \partial_{x_j} f)(x) \left(\eta_\varepsilon^2 e^{-\Phi_2^\varepsilon} * \partial_i \Phi_1 \partial_j \Phi_1 e^{-\Phi_1} \right)(x) \, dx.
\end{aligned}$$

Since $\partial_i \Phi_1 \partial_j \Phi_1 e^{-\Phi_1} \in L^1(\mathbb{R}^d)$ and $\partial_{x_i} f \partial_{x_j} f \in L^\infty(\mathbb{R}^d)$, we obtain

$$\begin{aligned}
\|(\nabla_x \Phi_1 \cdot (\nabla f \circ \sigma))(\eta_\varepsilon \circ p_v)\|_\varepsilon^2 &\xrightarrow{\varepsilon \to 0} \sum_{i,j} \int_{\mathbb{R}^d} \partial_{x_i} f \partial_{x_j} f \partial_i \Phi_1 \partial_j \Phi_1 e^{-\Phi_1} \, dx \\
&= \|\nabla_x \Phi_1 \cdot \nabla_x f\|_{\mathcal{H}_{\Phi_1}}^2
\end{aligned}$$

Now, we show weak convergence. Let $\xi \in C_c^\infty(\Phi_1 < \infty)$ be arbitrary. The same calculations as above yield

$$\begin{aligned}
(\nabla_x \Phi_1 \cdot \nabla_x f \eta_\varepsilon, \Psi_\varepsilon(\xi))_\varepsilon &= \sum_{i=1}^{d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\partial_i f \xi)(x+v) \partial_i \Phi_1(x) \eta_\varepsilon^2(v) e^{-\Phi_2^\varepsilon(v)} e^{-\Phi_1(x)} \, dx \, dv \\
&= \sum_{i=1}^{d} \int_{\mathbb{R}^d} \left((\partial_i f \xi) * \eta_\varepsilon^2 e^{-\Phi_2^\varepsilon} \right)(x) \partial_i \Phi_1(x) e^{-\Phi_1(x)} \, dx.
\end{aligned}$$

It holds $\partial_i \Phi_1 e^{-\frac{\Phi_1}{2}} \in L^2(\mathbb{R}^d)$ and $e^{-\frac{\Phi_1}{2}} \in L^\infty(\mathbb{R}^d)$ by assumption ($\Phi_1$7) and ($\Phi_1$2), respectively. Hence, we obtain by the Cauchy-Schwarz inequality

$$(\nabla_x \Phi_1 \cdot \nabla_x f \eta_\varepsilon, \Psi_\varepsilon(\xi))_\varepsilon \xrightarrow{\varepsilon \to 0} (\nabla \Phi_1 \cdot \nabla f, \xi)_{\mathcal{H}_{\Phi_1}}.$$

Taking 1.-6. together, we obtain by Lemma 1.64

$$L_\varepsilon \Psi_\varepsilon f \longrightarrow L_0 f \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_0, \quad \forall f \in C_c^\infty(\{\Phi_1 < \infty\}),$$

which proves by Corollary 1.78 the semigroup convergence (3.48). This finishes the proof. □

3.5 Weak Convergence of the Position Projections of Martingale Solutions

To formulate the results of this section let us recall some objects from previous sections and let us introduce some further notation. First of all, suppose that the assumptions of Theorem 3.35 are fulfilled. Let $h_\varepsilon \in \mathcal{H}_\varepsilon$, $\varepsilon > 0$, and $h \in \mathcal{H}_{\Phi_1}$ be probability densities w.r.t. μ_ε and μ_{Φ_1}, respectively. Furthermore, let $\mathbb{P}_{h_\varepsilon \mu_\varepsilon}$ by the martingale solution for $(L_\varepsilon, D(L_\varepsilon))$ with initial distribution $h_\varepsilon \mu_\varepsilon$ given by Theorem 3.25 and $\mathbb{P}_{h \mu_{\Phi_1}}$ be the martingale solution for $\big(L_{\Phi_1}, D(L_{\Phi_1})\big)$ from Corollary 3.31. In particular, we consider below the case $h_\varepsilon = h = 1$ for $\varepsilon > 0$. Observe that the measures

$\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ and $\mathbb{P}_{h\mu_{\Phi_1}}$ are defined on $C\left([0,\infty),\{\Phi_1<\infty\}\times\mathbb{R}^d\right)$ and $C\left([0,\infty),\{\Phi_1<\infty\}\right)$, respectively. To use the results from Section 2.1.1, we consider these measures via the continuous mappings

$$i_{2d}: C\left([0,\infty),\{\Phi_1<\infty\}\times\mathbb{R}^d\right)\longrightarrow C\left([0,\infty),\mathbb{R}^{2d}\right),(x_t,v_t)_{t\geq 0}\mapsto(x_t,v_t)_{t\geq 0},$$
$$i_d: C\left([0,\infty),\{\Phi_1<\infty\}\right)\longrightarrow C\left([0,\infty),\mathbb{R}^d\right),(x_t)_{t\geq 0}\mapsto(x_t)_{t\geq 0}.$$

as measures on the Polish space $C\left([0,\infty),\mathbb{R}^{2d}\right)$ and $C\left([0,\infty),\mathbb{R}^d\right)$, respectively. By abuse of notation we also denote by $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ and $\mathbb{P}_{h\mu_{\Phi_1}}$ the image measures $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}\circ i_{2d}^{-1}$ and $\mathbb{P}_{h\mu_{\Phi_1}}\circ i_d^{-1}$, respectively. Let us define the continuous map

$$P_X: C\left([0,\infty),\mathbb{R}^{2d}\right)\longrightarrow C\left([0,\infty),\mathbb{R}^d\right),(x_t,v_t)_{t\geq 0}\mapsto(x_t)_{t\geq 0}.$$

Now, we define the space coordinate projections of the martingale solutions $\mathbb{P}_{h\varepsilon\mu_\varepsilon}$ as $\mathbb{P}^X_{h\varepsilon\mu_\varepsilon}:=\mathbb{P}_{h\varepsilon\mu_\varepsilon}\circ P_X^{-1}$.

The main result of this section states that under suitable assumptions on the initial distributions the laws $\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}$ converge weakly to $\mathbb{P}_{h\mu_{\Phi_1}}$. The proof consists basically of two parts. In the first part, we show tightness of $\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}$, $\varepsilon>0$. Since the map P_X is continuous, we obtain tightness if the family $\left(\mathbb{P}_{h_\varepsilon\mu_\varepsilon}\right)_{\varepsilon>0}$ is tight. In the second part, we use the semigroup convergence from the previous section to obtain weak convergence of the corresponding finite dimensional distributions of $\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}$.

To establish tightness we need some prerequisites. By Remark 2.15 we can choose an appropriate metric r on the state space $\mathbb{R}^{2d}$. To this end, we choose a metric which suits better with the structure of the generators $(L_\varepsilon,D(L_\varepsilon))$. Let $i\in\{1,..,d\}$ and define the functions f_i, g_i in the following way:

$$f_i:\mathbb{R}^{2d}\longrightarrow\mathbb{R},(x,v)\mapsto x_i+v_i, \tag{3.53}$$
$$g_i:\mathbb{R}^{2d}\longrightarrow\mathbb{R},(x,v)\mapsto v_i. \tag{3.54}$$

Let the metric r on $\mathbb{R}^{2d}$ be given by

$$r((x,v),(\tilde{x},\tilde{v}))=\sum_{i=1}^{d}|f_i((x,v))-f_i((\tilde{x},\tilde{v}))|+|g_i((x,v))-g_i((\tilde{x},\tilde{v}))|\,. \tag{3.55}$$

It is obvious that r induces the Euclidean topology on $\mathbb{R}^{2d}$, since r is induced by a norm which is equivalent to the Euclidean norm. To prove the tightness of the martingale solutions we need further assumptions on the potentials Φ_1 and Φ_2, respectively.

Assumption 3.36.

$(\Phi_1 9)$ $\int\limits_{\mathbb{R}^d}|x|^{2k}\,e^{-\Phi_1}\,dx<\infty,\quad k=1,2.$

Assumption 3.37.

$(\Phi_2 11)$ $\int\limits_{\mathbb{R}^d}|v|^{2k}\,e^{-\Phi_2}\,dv<\infty,\quad k=1,2.$

Under the assumptions $(\Phi_1 6)$ and $(\Phi_2 7)$ we assume w.l.o.g. that μ_ε is a probability measure for all ε. Denote by $(\hat{L}_\varepsilon, D(\hat{L}_\varepsilon))$ the generator of the adjoint semigroup $\left(\hat{T}^\varepsilon_{t,2}\right)_{t\geq 0}$ of $\left(T^\varepsilon_{t,2}\right)_{t\geq 0}$, cf. with Lemma 3.22.

Proposition 3.38. *Assume* $(\Phi_1 2), (\Phi_1 3), (\Phi_1 5) - (\Phi_1 9)$ *and* $(\Phi_2 1) - (\Phi_2 7), (\Phi_2 9) - (\Phi_2 11)$. *For the functions* f_i, g_i, $i \in \{1, .., d\}$, *defined in (3.53) and (3.54), it holds* $f_i, f_i^2, g_i, g_i^2 \in D(L_\varepsilon) \cap D(\hat{L}_\varepsilon)$ *and*

$$
\begin{aligned}
L_\varepsilon f_i &= -\partial_{x_i}\Phi_1,\\
L_\varepsilon f_i^2 &= 2 + 2 f_i L_\varepsilon f_i,\\
L_\varepsilon g_i &= -\partial_{v_i}\Phi_2^\varepsilon - \partial_{x_i}\Phi_1,\\
L_\varepsilon g_i^2 &= 2 + 2 g_i L_\varepsilon g_i,\\
\hat{L}_\varepsilon g_i &= -\partial_{v_i}\Phi_2^\varepsilon + \partial_{x_i}\Phi_1\\
\hat{L}_\varepsilon g_i^2 &= 2 + 2 g_i \hat{L}_\varepsilon g_i,
\end{aligned}
$$

and for $i, j \in \{1, .., d\}$, $i \neq j$

$$
\begin{aligned}
L_\varepsilon(g_i g_j) &= g_i L_\varepsilon g_j + g_i L_\varepsilon g_j\\
\hat{L}_\varepsilon(g_i g_j) &= g_i \hat{L}_\varepsilon g_j + g_i \hat{L}_\varepsilon g_j.
\end{aligned}
$$

Proof. Due to Proposition 3.13 and Lemma 3.22(i), we know that $C_c^\infty(\mathbb{R}^{2d})$ is contained in $D(L_\varepsilon) \cap D(\hat{L}_\varepsilon)$. The assertions follow using suitable cut off functions. □

Remark 3.39. *In the following we fix versions of* $\nabla_x\Phi_1, \nabla_v\Phi_2$ *which we denote by the same symbol. As above, let* $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ *be the martingale solution for* $(L_\varepsilon, D(L_\varepsilon))$ *with initial distribution* $h_\varepsilon\mu_\varepsilon$ *on the measure space* $C\left([0,\infty), \mathbb{R}^{2d}\right)$. *Recall that the operator* L_ε *defined on* $C_c^\infty(\{\Phi_1 < \infty\})$ *is the generator of the stochastic differential equation*

$$
\begin{aligned}
dX_t^\varepsilon &= \nabla\Phi_2^\varepsilon(V_t^\varepsilon)dt,\\
dV_t^\varepsilon &= -\nabla\Phi_1(X_t^\varepsilon)dt - \nabla\Phi_2^\varepsilon(V_t^\varepsilon)dt + \sqrt{2}dB_t.
\end{aligned}
$$

To write this equation in a vector form let us define the following coefficients

$$
\begin{aligned}
b_\varepsilon : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d}, (x,v) \mapsto &\begin{pmatrix} \nabla_v\Phi_2^\varepsilon(v) \\ -\nabla_v\Phi_2^\varepsilon(v) - \nabla_x\Phi_1(x) \end{pmatrix}\\
\sigma = \sqrt{2}&\left(\frac{0_d}{I_d}\right) \in \mathbb{R}^{2d\times d},
\end{aligned}
$$

where 0_d *and* I_d *denote the zero* $d \times d$ *and the unit* $d \times d$ *matrix, respectively. The assumptions of the previous lemma imply that we obtain a weak solution in the sense of [61, Definition 5.3.1] with initial distributions* $h_\varepsilon\mu_\varepsilon$ *for the stochastic differential equation*

$$
dZ_t^\varepsilon = b_\varepsilon(Z_t^\varepsilon)dt + \sigma dB_t. \tag{3.56}
$$

Let $i \in \{1, \dots, d\}$. *Due to Proposition 3.38, we know that the function* g_i *is in* $D(L_\varepsilon)$ *and has obviously a continuous representative. Thus, by Remark 2.25, compare also with the notation used there, we know that the quadratic cross-variations of the continuous martingale* $\left((M_t^{[g_i]})_{t\geq 0}, \left(\overline{\mathcal{F}}_{t+}\right)_{t\geq 0}\right)$, $i \in$

$\{1, ..., d\}$, are given by

$$\left\langle M^{[g_i],\varepsilon}, M^{[g_j],\varepsilon} \right\rangle_t = 2\delta_{ij} t,$$

where δ_{ij} denotes the Kronecker delta. Using Lévy's characterization of the Brownian motion, we see that

$$(B_t)_{t\geq 0} = \left(B_t^\varepsilon\right)_{t\geq 0} := \frac{1}{\sqrt{2}} \left(M_t^{[g_i],\varepsilon}\right)_{t\geq 0}^{i=1,..,d}$$

constitutes a $d-$dimensional Brownian motion w.r.t. $\left(\overline{\mathcal{F}}_{t+}\right)_{t\geq 0}$. By computing the quadratic variation of $\left(M_t^{[f_i-g_i],\varepsilon}\right)_{t\geq 0}^{i=1,..,d}$, we obtain $\langle M^{[f_i-g_i],\varepsilon}\rangle_t = 0$ for all $t \geq 0$ which implies $M_t^{[f_i-g_i],\varepsilon} = 0$ for all $t \geq 0$. Hence, by comparing (3.56) component wise with (3.41) for $f_i - g_i$ and g_i, we see that

$$\left((X_t, V_t)_{t\geq 0}, (B_t)_{t\geq 0}\right), \quad \left(C\left([0,\infty), \mathbb{R}^{2d}\right), \overline{\mathcal{B}}_C, \mathbb{P}_{h_\varepsilon \mu_\varepsilon}\right), \quad \left(\overline{\mathcal{F}}_{t+}\right)_{t\geq 0}, \tag{3.57}$$

where $(X_t, V_t)_{t\geq 0}$ denotes the coordinate process on $C\left([0,\infty), \mathbb{R}^{2d}\right)$, constitutes a weak solution of (3.56) with initial distribution $h_\varepsilon \mu_\varepsilon$ in the sense of [61, Definition 5.3.1].

Now we are ready to prove the tightness result which is the major ingredient to prove the weak convergence of the position projections of the martingale solutions.

Theorem 3.40. *Assume $(\Phi_1 2)$, $(\Phi_1 3)$, $(\Phi_1 5) - (\Phi_1 9)$ and $(\Phi_2 1) - (\Phi_2 7)$, $(\Phi_2 9) - (\Phi_2 11)$. The family $\left(\mathbb{P}_{\mu_\varepsilon}\right)_{\varepsilon>0}$ is tight as measures on $C\left([0,\infty), \mathbb{R}^{2d}\right)$.*

Proof. In the following we always consider $\mathbb{R}^{2d}$ to be equipped with the metric r from (3.55), cf. with Remark 2.15, and let $T \in \mathbb{N}$ be arbitrary. Recall the time restriction operator R_T defined in (2.1.2). By Lemma 2.10 it suffices to show that the family of time restrictions

$$\left(\mathbb{P}_{\mu_\varepsilon}^T\right)_{\varepsilon>0} := \left(\mathbb{P}_{\mu_\varepsilon} \circ R_T^{-1}\right)_{\varepsilon>0}$$

is tight on $C([0,T], \mathbb{R}^{2d})$. Furthermore, we use Lemma 2.14. Let $i \in \{1, ..., d\}$ and denote by $\hat{f}_i, \hat{g}_i$ the measurable maps induced by f_i, g_i analogous to (2.3). Then it is enough to show separately that for $i \in \{1, ..., d\}$, the measures

I. $\left(\mathbb{P}_{\mu_\varepsilon}^T \circ \hat{f}_i^{-1}\right)_{\varepsilon>0}$,

II. $\left(\mathbb{P}_{\mu_\varepsilon}^T \circ \hat{g}_i^{-1}\right)_{\varepsilon>0}$,

are tight on $C([0,T], \mathbb{R})$. In the following, let $i \in \{1, ..., d\}$ and denote integration w.r.t. $\mathbb{P}_{\mu_\varepsilon}^T$ by $\mathbb{E}_\varepsilon^T$. We start with the family defined in I.

I. Consider on the probability space $\left(C\left([0,\infty), \mathbb{R}^{2d}\right), \mathcal{B}_C, \mathbb{P}_{\mu_\varepsilon}\right)$ the semimartingale decomposition from (3.41),

$$f_i(X_t, V_t) = M_t^{[f_i],\varepsilon} + \int_{[0,t]} L_\varepsilon f_i(X_r, V_r)\, dr + f_i(X_0, V_0), \quad t \in [0,T].$$

This implies that $\hat{f}_i$ coincides $P^T_{\mu_\varepsilon}$-a.e. with the sum of the $C([0,T],\mathbb{R})$-valued random variables $\left(M_t^{[f_i],\varepsilon}\right)_{t\in[0,T]}$, $\left(\int_{[0,t]} L_\varepsilon f_i(X_r,V_r)\,dr\right)_{t\in[0,T]}$ and $(f_i(X_0,V_0))_{t\in[0,T]}$, see also Lemma 2.22(iii). Due to Lemma 2.3, it suffices to show separately that the laws of the single summands

I.a.

$$\left(\mathbb{P}^T_{\mu_\varepsilon}\circ\left(M_t^{[f_i],\varepsilon}\right)^{-1}_{t\in[0,T]}\right)_{\varepsilon>0},$$

I.b.

$$\left(\mathbb{P}^T_{\mu_\varepsilon}\circ\left(\int_0^t L_\varepsilon f_i(X_r,V_r)\,dr\right)^{-1}_{t\in[0,T]}\right)_{\varepsilon>0},$$

I.c.

$$\left(\mathbb{P}^T_{\mu_\varepsilon}\circ(f_i(X_0,V_0))^{-1}_{t\in[0,T]}\right)_{\varepsilon>0},$$

are tight on $C([0,T],\mathbb{R})$. We proceed in the order just mentioned above.

I.a. Observe that for the initial distributions, it holds for every $\varepsilon>0$

$$\mathbb{P}^T_{\mu_\varepsilon}\circ\left(M_0^{[f_i],\varepsilon}\right)^{-1}=\delta_0,$$

where δ_0 denotes the Dirac delta measure in 0. To finish this part we use the criteria given in Lemma 2.13. Since $f_i^2\in D(L_\varepsilon)$ and $L_\varepsilon f_i\in L^4(\mathbb{R}^{2d},\mu_\varepsilon)$, (2.8) and Proposition 3.38 imply that the quadratic variation process of $\left(M_t^{[f_i],\varepsilon}\right)_{t\in[0,T]}$ is given by $2t$, $t\geq 0$. Hence, by the Burkholder-Davis-Gundy inequality, see e.g. [61, Theorem 3.3.28], it holds

$$\mathbb{E}^T_\varepsilon\left[(M_t^{[f_i],\varepsilon}-M_s^{[f_i],\varepsilon})^4\right]\leq C(t-s)^2.$$

Thus, by Lemma 2.13, we obtain the tightness of the family in I.a..

I.b. The tightness of the initial distributions follows as in I.a. We proceed as before and show that the increments fulfill an estimate as in Lemma 2.13. To this end, we recall that μ_ε is invariant for $\mathbb{P}_{\mu_\varepsilon}$. This follows by Lemma 2.20 and Lemma 3.22(ii). Eventually, we obtain by the Hölder inequality and Tonelli's theorem

$$\mathbb{E}^T_\varepsilon\left[\left(\int_s^t L_\varepsilon f_i(X_r,V_r)\,dr\right)^2\right]\leq (t-s)^2\mu_{\tilde{\Phi}_2}(\mathbb{R}^d)\int_{\mathbb{R}^d}|\partial_i\Phi_1|^2\,d\mu_{\Phi_1}.$$

By ($\Phi_1$6), ($\Phi_1$9) and ($\Phi_2$7), it holds that $\mu_{\tilde{\Phi}_2}(\mathbb{R}^d)\int_{\mathbb{R}^d}|\partial_i\Phi_1|^2\,d\mu_{\Phi_1}$ is a finite constant. Hence, we obtain the tightness of the family in I.b..

I.c. Observe that every measure of the family in I.c. is supported by functions which are constant. Hence, by Theorem 2.11 it suffices to show that the initial distributions are tight. These are given by $\left(\mu_\varepsilon\circ f_i^{-1}\right)_{\varepsilon>0}$. The tightness of this family follows directly from the tightness of $(\mu_\varepsilon)_{\varepsilon>0}$.

Eventually, by Lemma 2.3 we conclude that the family given in I. is tight.

II. Let $\hat{\mathbb{P}}_{\mu_\varepsilon}$ be the martingale solution of $\left(\hat{L}_\varepsilon, D(\hat{L}_\varepsilon)\right)$ defined on $C([0,\infty),\mathbb{R}^{2d})$. Recall the time reversal operator r_T defined in (2.5). Then we obtain by Lemma 2.21 that $\mathbb{P}^T_{\mu_\varepsilon}\circ r_T^{-1}$ is associated with the adjoint semigroup $\left(\hat{T}^\varepsilon_{t,2}\right)_{t\geq 0}$. Thus, the measures $\mathbb{P}^T_{\mu_\varepsilon}\circ r_T^{-1}$ and $\hat{\mathbb{P}}^T_{\mu_\varepsilon}$ coincide. By Proposition 3.38 it holds $g_i \in D(L_\varepsilon)\cap D(\hat{L}_\varepsilon)$. Via explicit computation we obtain the following decomposition which is motivated by [106]

$$\begin{aligned} g_i(X_t,V_t)-g_i(X_0,V_0) =& \frac{1}{2}\left(M_t^{g_i,\varepsilon}+\hat{M}_{T-t}^{g_i,\varepsilon}(r_T)-\hat{M}_T^{g_i,\varepsilon}(r_T)\right) \\ &+\frac{1}{2}\int_{[0,t]}(L_\varepsilon g_i-\hat{L}_\varepsilon g_i)(X_s,V_s)\,ds, \quad t\in[0,T]. \end{aligned} \tag{3.58}$$

Some remarks concerning the well-definedness of the decomposition (3.58) seem to be appropriate. In particular, we take a closer look at the proof of Lemma 2.22. Observe that the random variables

$$\int_{[0,t]} L_\varepsilon g_i(X_s,V_s)\,ds,\ \int_{[0,t]}\hat{L}_\varepsilon g_i(X_s,V_s)\,ds, \quad t\geq 0$$

(hence, $M_t^{g_i,\varepsilon}, \hat{M}_t^{g_i,\varepsilon}$) are first only defined up to sets of measure zero w.r.t. $\mathbb{P}_{\mu_\varepsilon}$ and $\hat{\mathbb{P}}_{\mu_\varepsilon}$, respectively. It is crucial to argue that it is possible to choose a common set which has full measure w.r.t. both measures $\mathbb{P}_{\mu_\varepsilon}$ and $\hat{\mathbb{P}}_{\mu_\varepsilon}$, otherwise the decomposition (3.58) has no meaning. Indeed, denote by g_i also its continuous version and fix μ_ε-versions $\widetilde{L_\varepsilon g_i}$ and $\widetilde{\hat{L}_\varepsilon g_i}$ of $L_\varepsilon g_i$ and $\hat{L}_\varepsilon g_i$, respectively. The random variables

$$\int_{[0,T]}\left|\widetilde{L_\varepsilon g_i}\right|(X_s,V_s)\,ds,\ \int_{[0,T]}\left|\widetilde{\hat{L}_\varepsilon g_i}\right|(X_s,V_s)\,ds$$

are both integrable w.r.t each of the measures $\mathbb{P}_{\mu_\varepsilon}$ and $\hat{\mathbb{P}}_{\mu_\varepsilon}$. Hence, for the sets

$$A := \bigcup_{T\in\mathbb{N}}\left\{\int_{[0,T]}\left|\widetilde{L_\varepsilon g_i}\right|(X_s,V_s)\,ds=\infty\right\},$$

$$\hat{A} := \bigcup_{T\in\mathbb{N}}\left\{\int_{[0,T]}\left|\widetilde{\hat{L}_\varepsilon g_i}\right|(X_s,V_s)\,ds=\infty\right\}$$

it holds that $B := A\cup\hat{A}$ is a zero set w.r.t. $\mathbb{P}_{\mu_\varepsilon}$ and $\hat{\mathbb{P}}_{\mu_\varepsilon}$ and in particular it holds for the time restricted set $B^T := R_T B$ that $B^T = r_T B^T$. Hence, the following versions of $(M_t^{g_i,\varepsilon})_{t\geq 0}$ and $(\hat{M}_t^{g_i,\varepsilon})_{t\geq 0}$ defined by

$$M_t^{g_i,\varepsilon} = \begin{cases} g_i(X_t,V_t) - g_i(X_0,V_0) - \int_{[0,t]} \widetilde{L_\varepsilon g_i}(X_s,V_s)\,ds, & \text{on } B^c \\ 0\text{ , else} \end{cases}$$

$$\hat{M}_t^{g_i,\varepsilon} = \begin{cases} g_i(X_t,V_t) - g_i(X_0,V_0) - \int_{[0,t]} \widetilde{\hat{L}_\varepsilon g_i}(X_s,V_s)\,ds, & \text{on } B^c \\ 0\text{ , else} \end{cases}$$

guarantees that the decomposition (3.58) is well-defined. Now, we can proceed as in part I. and prove that the single summands are are tight.

II.a.

$$\left(\mathbb{P}^T_{\mu_\varepsilon} \circ \left(M_t^{[g_i],\varepsilon}\right)^{-1}_{t\in[0,T]}\right)_{\varepsilon>0},$$

II.b.

$$\left(\mathbb{P}^T_{\mu_\varepsilon} \circ \left(\hat{M}_{T-t}^{g_i,\varepsilon}(r_T) - \hat{M}_T^{g_i,\varepsilon}(r_T)\right)^{-1}_{t\in[0,T]}\right)_{\varepsilon>0},$$

II.c.

$$\mathbb{P}^T_{\mu_\varepsilon} \circ \left(\left(\int_0^t (L_\varepsilon g_i - \hat{L}_\varepsilon g_i)(Z_s)\,ds\right)_{t\in[0,T]}\right)^{-1}.$$

II.a. This part can be proven by the exact same arguments as in part I.a. and the Proposition 3.38.

II.b. In the following we denote the expectation w.r.t. $\hat{\mathbb{P}}^T_{\mu_\varepsilon}$ by $\hat{\mathbb{E}}^T_\varepsilon$. We proceed similar as in part I.a. and provide a bound for the expectation of the increments. Let $s,t \in [0,T]$. Due to $\mathbb{P}^T_{\mu_\varepsilon} \circ r_T^{-1} = \hat{\mathbb{P}}^T_{\mu_\varepsilon}$, Lemma 2.21 and Proposition 3.38 we obtain by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}\mathbb{E}^T_\varepsilon\left[(\hat{M}_{T-t}^{[g_i],\varepsilon} \circ r_T - \hat{M}_{T-s}^{[g_i],\varepsilon} \circ r_T)^4\right] &= \hat{\mathbb{E}}^T_\varepsilon\left[(\hat{M}_{T-t}^{[g_i],\varepsilon} - \hat{M}_{T-s}^{[g_i],\varepsilon})^4\right] \\ &\le C(t-s)^2.\end{aligned}$$

Hence, we obtain the tightness by Lemma 2.13.

II.c. First observe that by Proposition 3.38 it holds $L_\varepsilon g_i - \hat{L}_\varepsilon g_i = -2\partial_{x_i}\Phi_1$. Now, we can proceed as in part I.b. to obtain the desired result.

Eventually, we conclude that the family of measures $\left(\mathbb{P}_{\mu_\varepsilon}\right)_{\varepsilon>0}$ is tight on $C\left([0,\infty),\mathbb{R}^{2d}\right)$ which finishes the proof.

□

Remark 3.41. *It is essential to note that the choice of the metric r defined in (3.55) was crucial. In the parts I.b. and II.c., the so-called semimartingale parts, we got rid of the ε dependence by the help of this metric. Choosing instead $f_i(x,v) = x_i$, $i = 1,\dots,d$, in (3.55) would not yield a cancellation of the ε dependent terms.*

Corollary 3.42. *For $\varepsilon > 0$, let $h_\varepsilon \in \mathcal{H}_\varepsilon$ be a probability density w.r.t. μ_ε. In addition to the assumptions of Theorem 3.40 we assume that $\|h_\varepsilon\|_{\mathcal{H}_\varepsilon}$ is uniformly bounded in $\varepsilon > 0$. Then, the measures*

$$\left(\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}\right)_{\varepsilon>0} := \left(\mathbb{P}_{h_\varepsilon\mu_\varepsilon} \circ P_X^{-1}\right)_{\varepsilon>0}$$

on $C\left([0,\infty),\mathbb{R}^d\right)$ are tight.

Proof. Since the space coordinate projection P_X is continuous it suffices to show that $\left(\mathbb{P}_{h_\varepsilon\mu_\varepsilon}\right)_{\varepsilon>0}$ is tight on $C\left([0,\infty),\mathbb{R}^{2d}\right)$. Now, let $\delta > 0$ and choose $K \subseteq C\left([0,\infty),\mathbb{R}^{2d}\right)$ to be compact s.t. $\sup_{\varepsilon>0} \mathbb{P}_{\mu_\varepsilon}(K^c) \leq \frac{\delta^2}{\sup_{\varepsilon>0}\|h_\varepsilon\|^2_{L^2(\mu_\varepsilon)}}$. Observe that the measures $\mathbb{P}_{h_\varepsilon\mu_\varepsilon}$ and $h_\varepsilon(X_0,V_0)P_{\mu_\varepsilon}$ coincide by Corollary 2.9. Denote by $\mathbb{E}_\varepsilon$ integration w.r.t. $\mathbb{P}_{\mu_\varepsilon}$, then we obtain

$$\mathbb{P}_{h_\varepsilon\mu_\varepsilon}(K^c) = \mathbb{E}_\varepsilon\left[1_{K^c} h_\varepsilon(X_0,V_0)\right] \leq \sqrt{\mathbb{P}_{\mu_\varepsilon}(K^c)}\,\|h_\varepsilon\|_{L^2(\mu_\varepsilon)} \leq \delta,$$

which completes the proof. □

Remark 3.43. *Suppose we are in the setting of Corollary 3.42. From its proof, we directly see that it is not necessary for the densities h_ε to be in $L^2(\mathbb{R}^{2d},\mu_\varepsilon)$. It suffices that their respective $L^p(\mathbb{R}^{2d},\mu_\varepsilon)$ norm is uniformly bounded for some $p > 1$.*

In the next theorem, we use the convergence of the semigroups $\left(T^\varepsilon_t\right)_{t\geq 0}$ from Theorem 3.35 and the previous corollary to prove the main result of this section. Denote in the following by δ_0 the Dirac measure in zero on $\mathbb{R}^d$.

Theorem 3.44. *For $\varepsilon > 0$ let $h_\varepsilon \in \mathcal{H}_\varepsilon$ and $h \in \mathcal{H}_{\Phi_1}$ be probability densities w.r.t. μ_ε and μ_{Φ_1}, respectively. Assume $(\Phi_1 2)-(\Phi_1 4)^2, (\Phi_1 5)-(\Phi_1 7)$ and $(\Phi_2 1)-(\Phi_2 9)$. If $h_\varepsilon\mu_\varepsilon$ converges weakly to $h\mu_{\Phi_1}\otimes\delta_0$ and $\sup_{\varepsilon>0}\|h_\varepsilon\|_{L^2(\mu_\varepsilon)}$ is finite then the measures $\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}$ converge weakly to $\mathbb{P}_{h\mu_{\Phi_1}}$ as $\varepsilon\longrightarrow 0$ as measures on $C\left([0,\infty),\mathbb{R}^d\right)$.*

Proof. After applying Corollary 3.42, we are left to show that the respective finite dimensional distributions converge weakly, see Remark 2.7 and Corollary 2.9. As mentioned above we use the semigroup convergence. The proof is essentially the same as in [75]. For the sake of consistency, we state the proof here. Let $(X_t)_{t\geq 0}$ and $(X_t,V_t)_{t\geq 0}$ be the coordinate processes on $C\left([0,\infty),\mathbb{R}^d\right)$ and $C\left([0,\infty),\mathbb{R}^{2d}\right)$, respectively. Then, it holds $X_t\circ P_X = p_x\circ(X_t,V_t)$ for all $t\geq 0$. Let $0\leq t_1 < \ldots < t_k$, $k\in\mathbb{N}$, and define

$$\begin{aligned}\mathbb{P}^{X,t_1,\ldots,t_k}_{h_\varepsilon\mu_\varepsilon} &:= \mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}\circ\left(X_{t_1},\ldots,X_{t_k}\right)^{-1}\\ \mathbb{P}^{t_1,\ldots,t_k}_{h\mu_{\Phi_1}} &:= \mathbb{P}_{h\mu_{\Phi_1}}\circ\left(X_{t_1},\ldots,X_{t_k}\right)^{-1}.\end{aligned}$$

Additionally, define for $k\in\mathbb{N}$ the class of functions

$$\mathcal{D}_k := \operatorname{span}\left\{F:\mathbb{R}^{dk}\longrightarrow\mathbb{R}\ \middle|\ \begin{aligned}&F(x_1,\ldots,x_k) = \prod_{i=1}^k f_i(x_i) \text{ for all } (x_1,\ldots,x_k)\in\mathbb{R}^{dk},\\ &f_i\in C_c^\infty(\mathbb{R}^d),\, i=1,\ldots,k\end{aligned}\right\}. \tag{3.59}$$

One easily checks that the algebra $\mathcal{D}_k$ strongly separates the points of $\mathbb{R}^{dk}$, see e.g. [32] for the precise definition of the property strongly separating points. Hence, by [32, Theorem 3.4.5] it suffices to show that

$$\int_{\mathbb{R}^{dk}} F\, d\mathbb{P}^{X,t_1,\dots,t_k}_{h_\varepsilon \mu_\varepsilon} \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^{dk}} F\, d\mathbb{P}^{t_1,\dots,t_k}_{h\mu_{\Phi_1}}.$$

Hence, let $F \in \mathcal{D}_k$ s.t. $F = \otimes_{i=1}^k f_i$ as in (3.59). Since μ_{Φ_2} is a probability measure on $\mathbb{R}^d$, we obtain $f_i \circ p_x \in L^2(\mathbb{R}^{2d}, \mu_\varepsilon)$. By the association of $\mathbb{P}_{h_\varepsilon \mu_\varepsilon}$ with $\left(T^\varepsilon_{t,2}\right)_{t\geq 0}$ and $T^\varepsilon_{t,2} = T^{\Phi^\varepsilon}_{t,\infty}$ on $L^2(\mathbb{R}^{2d}, \mu_\varepsilon) \cap L^\infty(\mathbb{R}^{2d}, \mu_\varepsilon)$ it holds

$$\int_{\mathbb{R}^{dk}} F\, d\mathbb{P}^{X,t_1,\dots,t_k}_{h_\varepsilon \mu_\varepsilon} = \int_{\mathbb{R}^{2d}} h_\varepsilon \underbrace{T^\varepsilon_{t_1,2}(f_1 \circ p_x T^\varepsilon_{t_2-t_1,2}(f_2 \circ p_x \dots T^\varepsilon_{t_k-t_{k-1},2} f_k \circ p_x))\dots)}_{F^{t_1,\dots,t_k}_\varepsilon}\, d\mu_\varepsilon.$$

By Corollary 1.67, we obtain that

$$h_\varepsilon \longrightarrow h \text{ weakly along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}. \tag{3.60}$$

Indeed, the norm boundedness is part of the assumptions. Using [32, Theorem 3.4.5] and $\mathcal{D}_2$ we obtain that $(\eta_\varepsilon \circ p_v) h_\varepsilon\, d\mu_\varepsilon$ converges by assumption weakly to $h\mu_{\Phi_1} \otimes \delta_0$. Thus, let $\varphi \in C$. Then we have

$$(h_\varepsilon, \Psi_\varepsilon(\varphi))_{\mathcal{H}_\varepsilon} = \int_{\mathbb{R}^{2d}} (\varphi \circ \sigma)(\eta_\varepsilon \circ p_v) h_\varepsilon\, d\mu_\varepsilon \longrightarrow \int_{\mathbb{R}^{2d}} (\varphi \circ \sigma) h \circ p_x\, d\mu_{\Phi_1} \otimes \delta_0 = (h, \varphi)_{\mathcal{H}_{\Phi_1}},$$

implying (3.60). Next, we argue that

$$F^{t_1,\dots,t_k}_\varepsilon \longrightarrow F^{t_1,\dots,t_k} := T^{\Phi_1}_{t_1}(f_1 T^{\Phi_1}_{t_2-t_1}(f_2 \dots T^{\Phi_1}_{t_k-t_{k-1}} f_k))\dots) \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}. \tag{3.61}$$

Corollary 1.66 implies that for $g \in C_c^\infty(\mathbb{R}^d)$ it holds

$$g \circ p_x \longrightarrow g \text{ along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}. \tag{3.62}$$

Furthermore, by the convolution argument used in the proof of Theorem 3.35 and Corollary 1.67, one immediately obtains that for every $\varphi \in C$ it holds

$$g \circ p_x \Psi_\varepsilon(\varphi) \longrightarrow g\varphi \text{ weakly along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}. \tag{3.63}$$

Now let $f_\varepsilon \longrightarrow f$ along $\mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}$. It is easy to see that $\|g \circ p_x f_\varepsilon\|_{\mathcal{H}_\varepsilon}$ is bounded since f_ε converges. For arbitrary $\varphi \in C$, it holds by (3.63) that

$$(g \circ p_x f_\varepsilon, \Psi_\varepsilon(\varphi))_{\mathcal{H}_\varepsilon} = (f_\varepsilon, g \circ p_x \Psi_\varepsilon(\varphi))_{\mathcal{H}_\varepsilon} \longrightarrow (f, g\varphi)_{\mathcal{H}_{\Phi_1}} = (gf, \varphi)_{\mathcal{H}_{\Phi_1}}.$$

Thus, by Corollary 1.67 we obtain

$$g \circ p_x f_\varepsilon \longrightarrow gf \text{ weakly along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}. \tag{3.64}$$

We conclude by (3.64) and the choice of f_ε that

$$\|g \circ p_x f_\varepsilon\|^2_{\mathcal{H}_\varepsilon} = (g^2 \circ p_x f_\varepsilon, f_\varepsilon)_{\mathcal{H}_\varepsilon} \longrightarrow (g^2 f, f)_{\mathcal{H}_{\Phi_1}} = \|gf\|^2_{\mathcal{H}_{\Phi_1}}. \tag{3.65}$$

By combining (3.64) and (3.65), we obtain through Corollary 1.66 that

$$g \circ p_x f_\varepsilon \longrightarrow gf \text{ strongly along } \mathcal{H}_\varepsilon \xrightarrow{(\Psi_\varepsilon)_{\varepsilon>0}} \mathcal{H}_{\Phi_1}. \tag{3.66}$$

Thus, we apply (3.62), (3.66) and Theorem 3.35 inductively and obtain (3.61). Eventually, through (3.60) and (3.61) we conclude that

$$\int_{\mathbb{R}^{dk}} F\, d\mathbb{P}^{X,t_1,\dots,t_k}_{h_\varepsilon\mu_\varepsilon} = \int_{\mathbb{R}^{2d}} h_\varepsilon T^\varepsilon_{t_1,2}(f_1 \circ p_x T^\varepsilon_{t_2-t_1,2}(f_2 \circ p_x \dots T^\varepsilon_{t_k-t_{k-1},2} f_k \circ p_x))\dots)\, d\mu_\varepsilon$$
$$\xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^d} h T^{\Phi_1}_{t_1}(f_1 T^{\Phi_1}_{t_2-t_1}(f_2 \dots T^{\Phi_1}_{t_k-t_{k-1}} f_k))\dots)\, d\mu_{\Phi_1} = \int_{\mathbb{R}^{dk}} F\, d\mathbb{P}^{t_1,\dots,t_k}_{h\mu_{\Phi_1}}$$

which completes the proof. □

3.6 Overdamped Limit of Generalized Stochastic Hamiltonian systems

In this last section, we use the result from Theorem 3.44 to treat the original problem. Let us recall the scaled generalized stochastic Hamiltonian system (3.1), (3.2) given by

$$dX^\varepsilon_t = \frac{1}{\varepsilon}\nabla\Phi_2(V^\varepsilon_t)dt, \tag{3.67}$$
$$dV^\varepsilon_t = -\frac{1}{\varepsilon}\nabla\Phi_1(X^\varepsilon_t)dt - \frac{1}{\varepsilon^2}\nabla\Phi_2(V^\varepsilon_t)dt + \frac{1}{\varepsilon}\sqrt{2}dB_t. \tag{3.68}$$

In the theorem below we show existence of a weak solution $(X^\varepsilon_t, V^\varepsilon_t)_{t\ge 0}$ to (3.67), (3.68) with initial distribution $h\mu_\Phi$, where $h \in L^1(\mathbb{R}^{2d}, \mu_\Phi) \cap L^2(\mathbb{R}^{2d}, \mu_\Phi)$. Furthermore we show weak convergence of the laws $\mathcal{L}\left(\left(X^\varepsilon_t\right)_{t\ge 0}\right)$ to $\mathcal{L}\left(\left(X^0_t\right)_{t\ge 0}\right)$ with $(X^0_t)_{t\ge}$ being a solution of

$$dX^0_t = -\nabla\Phi_1(X^0_t)dt + \sqrt{2}dB_t. \tag{3.69}$$

For this purpose, we use the result from Theorem 3.44 and apply Itô's formula to show how the solution (3.57) of (3.56) can be transformed into a solution of (3.67), (3.68).

Remark 3.45. *So far we only considered martingale solutions for the generator of (3.69), see Corollary 3.31. Indeed, Let $h \in L^1(\mathbb{R}^d, \mu_{\Phi_1}) \cap L^2(\mathbb{R}^d, \mu_{\Phi_1})$ be a probability density w.r.t. μ_{Φ_1}. Then, there exists a probability law $\mathbb{P}_{h\mu_{\Phi_1}}$ on $C([0,\infty), \{\Phi_1 < \infty\})$ with initial distribution $h\mu_{\Phi_1}$ which solves the martingale problem for the generator $\left(L_{\Phi_1}, D(L_{\Phi_1})\right)$. Under the assumptions of Theorem 3.46 below one can proceed as in Proposition 3.13 and show that the coordinate functions $\mathbb{R}^d \ni x \longmapsto x_i \in \mathbb{R}$ $i = 1, \dots, d$, are elements of $D(L_{\Phi_1})$. Further, the same arguments as in Remark 3.39 show that the coordinate process $(X_t)_{t\ge 0}$ on the measure space $(C([0,\infty), \{\Phi_1 < \infty\}), \mathbb{P}_{h\mu_{\Phi_1}})$ is a weak solution of (3.69) with initial distribution $h\mu_{\Phi_1}$.*

We summarize our final result in the following theorem. We want to remind the reader that we denote by μ_ε the measure on $\mathbb{R}^{2d}$ given by μ_{Φ^ε}.

Theorem 3.46. *Assume* $(\Phi_1 2) - (\Phi_1 9)$ *and* $(\Phi_2 1) - (\Phi_2 11)$. *Let* $\varepsilon > 0$ *and* $h \in L^1(\mathbb{R}^{2d}, \mu_\Phi) \cap L^2(\mathbb{R}^{2d}, \mu_\Phi)$. *Then there exists a weak solution* $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0}$ *to (3.1), (3.2) with initial distribution* $h\mu_\Phi$. *Denote by* $\tilde{h} \in L^1(\mathbb{R}^d, \mu_{\Phi_1}) \cap L^2(\mathbb{R}^d, \mu_{\Phi_1})$ *the element given by* $\tilde{h}(x) = \int_{\mathbb{R}^d} h(x, v)\, d\mu_{\Phi_2}(v)$, $x \in \mathbb{R}^d$, *and by* $\mathbb{P}_{\tilde{h}\mu_{\Phi_1}}$ *the martingale solution to the generator of (3.3) from Corollary 3.31 with initial distribution* $\tilde{h}\mu_{\Phi_1}$. *Then,* $\mathcal{L}\left((X_t^\varepsilon)_{t\geq 0}\right)$, $\varepsilon > 0$, *converge weakly to* $\mathbb{P}_{\tilde{h}\mu_{\Phi_1}}$ *as measures on* $C\left([0,\infty), \mathbb{R}^d\right)$ *as* $\varepsilon \to 0$.

Proof. Let $\varepsilon > 0$ and h be given as above. Recall the state space transformation $\widetilde{U}_\varepsilon$ defined by $\widetilde{U}_\varepsilon(x, v) = (x, \frac{1}{\varepsilon}v)$, $(x, v) \in \mathbb{R}^{2d}$. It is clear that the densities $h_\varepsilon := h \circ \widetilde{U}_\varepsilon$, $\varepsilon > 0$, satisfy $\int_{\mathbb{R}^{2d}} h_\varepsilon^2\, d\mu_\varepsilon = \int_{\mathbb{R}^{2d}} h^2\, d\mu_\Phi$ and the measures $h_\varepsilon\mu_\varepsilon$ converge weakly to $\tilde{h}\mu_{\Phi_1} \otimes \delta_0$ as $\varepsilon \to 0$. Now, let

$$\left((X_t, V_t)_{t\geq 0}, (B_t)_{t\geq 0}\right), \quad \left(C\left([0,\infty), \mathbb{R}^{2d}\right), \overline{\mathcal{B}}_C, \mathbb{P}_{h_\varepsilon\mu_\varepsilon}\right), \quad \left(\overline{\mathcal{F}}_{t+}\right)_{t\geq 0},$$

be the weak solution of (3.56) with initial distribution $h_\varepsilon\mu_\varepsilon$ given in Remark 3.39. Recall that $(X_t, V_t)_{t\geq 0}$ denotes the coordinate process on $C\left([0,\infty), \mathbb{R}^{2d}\right)$. Then, by applying Itô's formula, see e.g. [61, Theorem 3.3.6] to the function $f_\varepsilon(x, v) = (x, \frac{1}{\varepsilon}v)$ we obtain that the process $(X_t^\varepsilon, V_t^\varepsilon)_{t\geq 0} := (f_\varepsilon(X_t, V_t))_{t\geq 0}$ is a weak solution to (3.67), (3.68) with initial distribution $h\mu_\Phi$. Furthermore, the laws $\mathcal{L}\left((X_t^\varepsilon)_{t\geq 0}\right)$ and $\mathcal{L}\left((X_t)_{t\geq 0}\right)$ coincide and are equal to $\mathbb{P}^X_{h_\varepsilon\mu_\varepsilon}$. Hence, the last assertion follows by Theorem 3.44. □

3.7 Examples and Sufficient Conditions

So far we did not present any example for Φ_1 and Φ_2 which satisfy the assumptions $(\Phi_1 1) - (\Phi_1 9)$ and $(\Phi_2 1) - (\Phi_2 11)$, respectively. To make this chapter consistent, in the sense that there are physical relevant examples of Φ_1 and Φ_2, we present some examples taken from [24, Section 6.6.2 and 6.6.3] and [51]. Further, we present a lemma which can be used to check the assumptions $(\Phi_1 6)$ and $(\Phi_2 4)$. These are the only assumptions which are not explicit in terms Φ_1 and Φ_2 and their respective weak derivatives. This Lemma is also taken from [24, Lemma 6.6.7.].

3.7.1 A sufficient condition for $(\Phi_1 6)$ and $(\Phi_2 4)$

Recall that $(\Phi_1 6)$ and $(\Phi_2 4)$ are the same condition. Therefore, we only work in the following with one potential $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$. We introduce two conditions on Φ:

(ΦA) $\Phi = \Phi_s + \Phi_r$, where Φ_s and Φ_r are given as follows. Φ_s satisfying $(\Phi_1 2)$, $(\Phi_1 3)$ and $(\Phi_1 8)$, is continuously differentiable and has weak second derivatives on $\{\Phi < \infty\}$. There exists a $K \in [0,\infty)$ and $\alpha \in [1,2)$ s.t.

$$\left|\partial_i\partial_j\Phi_s(x)\right| \leq K\left(1 + \sum_{i=1}^d |\partial_i\Phi_s(x)|^\alpha\right), \tag{3.70}$$

holds for all $i, j \in \{1, \ldots, d\}$ and μ_Φ-a.e. $x \in \{\Phi < \infty\}$. Φ_r is bounded and Lipschitz continuous on $\mathbb{R}^d$. The measure μ_Φ satisfies $\mu_\Phi(\mathbb{R}^d) = 1$.

(ΦB) Φ satisfies $(\Phi_1 4)^4_{loc}$ and $\nabla\Phi e^{-\frac{\Phi}{2}}$ is μ_Φ-essentially bounded.

If Φ satisfies (ΦA) then we denote as in Section 3.3 by L_Φ the differential operator

$$L_\Phi f = \Delta f - \nabla\Phi \cdot \nabla f, \quad f \in C_c^\infty(\{\Phi < \infty\}).$$

Further we denote by $\mathcal{H}_\Phi$ the Hilbert space $\mathcal{H}_\Phi = L^2(\mathbb{R}^d, \mu_\Phi)$. The following lemma is taken from [24, Lemma 6.6.7.].

Lemma 3.47. *Let Φ satisfy (ΦA).*

(i) If Φ satisfies additionally $(\Phi_1 4)^4_{loc}$, then $\left(L_\Phi, C_c^\infty(\{\Phi < \infty\})\right)$ is essentially self-adjoint if and only if $\left(L_\Phi, C_c^\infty(\mathbb{R}^d)\right)$ is essentially self-adjoint on $\mathcal{H}_\Phi$.

(ii) If Φ satisfies additionally (ΦB), then $\left(L_\Phi, C_c^\infty(\{\Phi < \infty\})\right)$ is essentially self-adjoint on $\mathcal{H}_\Phi$.

3.7.2 Examples of Φ_1 and Φ_2

In the following examples we show some pairs of potentials s.t. the assumptions in Theorem 3.46 are fulfilled.

Example 3.48. *The following examples of $\Phi_2 : \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfy the assumptions $(\Phi_2 1)$)-$(\Phi_2 11)$ up to an additive constant, i.e., μ_{Φ_2} is only a finite measure.*

(i) Let $\Phi_2(v) := |v|^2$, $v \in \mathbb{R}^d$.

(ii) Let $K \in (0,\infty)$ and $\lambda > \frac{d}{2}$ and define $\Phi_2(v) := K\ln(1 + |v|^{2\lambda})$, $v \in \mathbb{R}^d$.

(iii) Let $K \in (0,\infty)$ and $\lambda \geq 1$ and define $\Phi_2(v) := K(1 + |v|^2)^\lambda$.

All but the assumption $(\Phi_2 6)$ can be checked by simple calculation for each of the potentials in (i)-(iii). In particular, the gradient $\nabla\Phi_2$ satisfies $|\nabla\Phi_2| \in \cap_{p\in[1,\infty)} L^p(\mathbb{R}^d, \mu_{\Phi_2})$. The assumption $(\Phi_2 6)$ can be checked with Lemma 3.47 by choosing $\Phi_{2,r} = 0$ in all three cases. In the first case, where $\Phi_2(v) = |v|^2$ one can also use the Hermite polynomials to establish self-adjointness. By using cutoff functions one shows that the Hermite polynomials are contained in the domain of the closure of $\left(L_{\Phi_2}, C_c^\infty(\{\Phi_2 < \infty\})\right)$. Further they are eigenvectors of L_{Φ_2} with real eigenvalues and form an orthonormal basis of $\mathcal{H}_{\Phi_2}$, see Section (B.9).

Before we state some examples for Φ_1 let us recall that the Langevin equation (0.1), (0.2) describes the motion of interacting particles. The interaction forces as well as an external forces on the particles are described through the gradient $\nabla\Phi_1$ of the potential Φ_1. To reflect this meaning of Φ_1 and to ease the notation we replace d by $N\tilde{d}$, with $N, \tilde{d} \in \mathbb{N}$. Here, N represents the number of particles under consideration and $\tilde{d}$ denotes the space dimension. Namely, we consider N particles moving in $\mathbb{R}^{\tilde{d}}$. We also write $(x_1, ..., x_N) = x \in \mathbb{R}^{Nd}$, where $x_i \in \mathbb{R}^{\tilde{d}}$, $i = 1, ..., N$. Many physical interesting interactions between particles are so-called pair interactions. In that case the interaction part Φ_i of Φ is described through one single pair potential $\varphi : \mathbb{R}^{\tilde{d}} \longrightarrow \mathbb{R} \cup \{\infty\}$ via

$$\Phi_i(x_1, ..., x_N) = \frac{1}{2}\sum_{i\neq j} \varphi(x_i - x_j). \tag{3.71}$$

Interactions with external fields can be described via a field potential $\psi : \mathbb{R}^d :\longrightarrow \mathbb{R} \cup \{\infty\}$ through

$$\Phi_e(x_1, \dots, x_N) = \sum_{i=1}^{N} \psi(x_i). \tag{3.72}$$

In the following we present sufficient conditions for φ and ψ s.t. $\Phi = \Phi_i + \Phi_e$ satisfies the assumptions $(\Phi_1 2) - (\Phi_1 4)^2, (\Phi_1 5) - (\Phi_1 9)$. We denote for $x \in \mathbb{R}^d \setminus \{0\}$ by $\hat{x}$ the normalized vector $\hat{x} := \frac{x}{|x|}$.

Assumption 3.49. *Assume that* $\varphi, \psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ *admit a decomposition* $\varphi = \varphi_1 + \varphi_2$ *and* $\psi = \psi_1 + \psi_2$ *with* $\varphi_1, \varphi_2, \psi_1, \psi_2 : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$.

(H1) φ_1 *is a radially symmetric and decreasing function, i.e., there exists a decreasing function* $\theta : [0, \infty) \longrightarrow \mathbb{R} \cup \{\infty\}$ *s.t.* $\varphi_1(x) = \theta(|x|)$. *The support of* φ_1 *is contained in the open ball* $B_r(0)$ *for some* $r > 0$.

(H2) φ_1 *is continuously differentiable in* $\mathbb{R}^d \setminus \{0\}$ *and* $\varphi_1(x) \longrightarrow \infty$ *and* $|\nabla \varphi_1(x)| \longrightarrow \infty$ *as* $x \to 0$. *Moreover,* φ_1 *has second weak derivatives in* $\mathbb{R}^d \setminus \{0\}$ *which satisfy (3.70).*

(H3) $|\nabla \varphi_1| \in L^4(\mathbb{R}^d, \mu_{\varphi_1})$ *and* $\nabla \varphi_1 e^{-\frac{\varphi_1}{2}}$ *is bounded on* $\mathbb{R}^d \setminus \{0\}$.

(H4) φ_2 *is Lipschitz continuous and bounded.*

(H5) ψ_1 *is continuously differentiable. There exists* $\kappa > 0$ *and* $R > 0$ *s.t.* $\hat{x}\nabla\psi_1(x) \geq \kappa |\nabla \psi_2(x)|$ *for a.e.* $x \in \mathbb{R}^d \setminus B_R(0)$. *Moreover* $|\nabla \psi_1(x)| \longrightarrow \infty$ *as* $|x| \to \infty$ *and* ψ_1 *has second weak derivatives which satisfy (3.70). In particular, there exist* $a > 0$ *and* $b \in \mathbb{R}$ *s.t.* $\psi_1(x) \geq a|x| - b$ *for all* $x \in \mathbb{R}^d$.

(H6) $|\nabla \psi_1| \in L^4(\mathbb{R}^d, \mu_{\psi_1})$ *and* $\nabla \psi_1 e^{-\frac{\psi_1}{2}}$ *is bounded on* $\mathbb{R}^d \setminus \{0\}$.

(H7) ψ_2 *is Lipschitz continuous and bounded.*

Lemma 3.50. *Assume* $d \geq 2$ *and let* $\varphi_1, \varphi_2, \psi_1, \psi_2 : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$ *satisfy (H1)-(H7) from Assumption 3.49. Let* $\varphi = \varphi_1 + \varphi_2$ *and* $\psi = \psi_1 + \psi_2$ *and* $\Phi_i, ,\Phi_e$ *be given by (3.71) and (3.72), respectively. Then* $\Phi_1 := \Phi_i + \Phi_e$ *satisfies* (ΦA) *and* (ΦB). *In particular,* Φ_1 *satisfies* $(\Phi_1 2) - (\Phi_1 4)^2$, $(\Phi_1 5) - (\Phi_1 9)$.

Proof. See [24, Theorem 6.6.11.]. □

Remark 3.51. *If* $\Phi_1 : \mathbb{R}^{Nd} \longrightarrow \mathbb{R} \cup \{\infty\}$ *is given as in the previous lemma, then it holds*

$$\{\Phi_1 < \infty\} = \{(x_1, \dots, x_N) \in \mathbb{R}^{Nd} \mid x_i \neq x_j \text{ if } i \neq j\}. \tag{3.73}$$

The complement of the set on the right-hand side of (3.73) is called the set of coinciding points. In particular, the martingale solutions $\mathbb{P}_{h\mu_\Phi}$ *constructed in Theorem 3.25 are supported by the trajectories which don't pass through coinciding points. This is favorable from a physical point of view, since two different particles are not allowed to be at the same point at the same time.*

The next example shows that pair interactions of Lennard-Jones type are possible in this framework.

Example 3.52. *Let A, B be two positive constants. Define $V_{LJ}(r) := V_{LJ}^{A,B}(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$, $r > 0$. Further define*

$$\varphi : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}, x \mapsto \begin{cases} \infty, & x=0, \\ V_{LJ}(|x|), & \text{else}, \end{cases}$$
$$\psi : \mathbb{R}^d \longrightarrow \mathbb{R}, x \mapsto |x|^2 .$$

Choose $\psi_1 := \psi$, $\psi_2 = 0$. To decompose φ into φ_1 and φ_2 according to the assumptions in (H1)-(H4) we decompose V_{LJ} accordingly. We observe that V_{LJ} attains its minimum at $r_0 = \left(\frac{2A}{B}\right)^{\frac{1}{6}}$ and its value at r_0 is denoted by $V_0 := V_{LJ}(r_0)$. Now define for $r > 0$

$$V_1(r) := 1_{(0,r_0]}(r) \left(V_{LJ}(r) - V_0\right),$$
$$V_2(r) := V_0 + 1_{(r_0,\infty)}(r) \left(V_{LJ}(r) - V_0\right),$$

where $1_{(0,r_0]}$ and $1_{(r_0,\infty)}$ denote the indicator functions of the intervals $(0, r_0]$ and (r_0, ∞), respectively. See Figure 3.1 for a graphical representation of V_{LJ}, V_1, V_2.

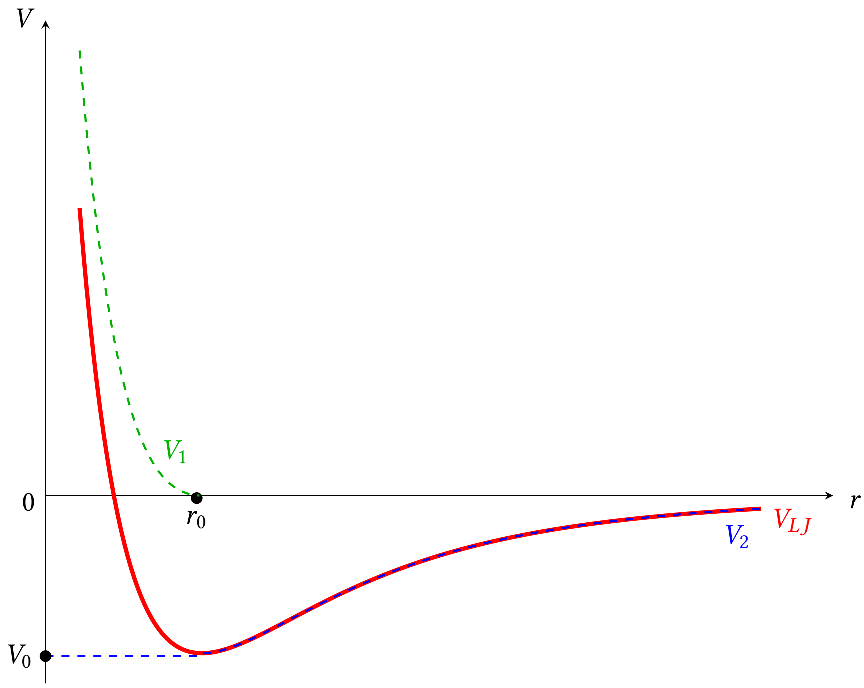

Figure 3.1: Decomposition of V_{LJ}

Now we define φ_1 and φ_2 as

$$\varphi_1 : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}, x \mapsto \begin{cases} \infty, & x=0, \\ V_1(|x|), & \text{else}, \end{cases}$$
$$\varphi_2 : \mathbb{R}^d \longrightarrow \mathbb{R}, x \mapsto V_2(|x|).$$

Then $\varphi_1, \varphi_2, \psi_1, \psi_2$ satisfy (H1)-(H7).

Part II

Problems from Infinite Dimensional Analysis

Chapter 4

Preliminaries from the theory of Gaussian and White Noise analysis

In this chapter we present basic facts from Gaussian analysis and White noise analysis. Similar as in Chapter 1 and Chapter 2 we try to present only the results and their proofs which are necessary for the applications we have in mind. In particular, we do not aim for the most possible generality in this chapter. The content of this chapter can be also found in the standard literature on Gaussian analysis and White noise analysis, such as [16, 65, 57, 42, 79].

We give a short overview on nuclear spaces in the first section. Apart from the fact that Gaussian analysis takes place on the dual space of a nuclear space, many important test function spaces in Gaussian analysis and White noise analysis are nuclear spaces. In the second section we present basics from Gaussian analysis, such as the Bochner-Minlos theorem and the Wiener-Itô-Segal isomorphsim. Thereafter, we present the Hida test function and distribution space together with the most important properties, like the characterization theorem for Hida distributions. Further, we introduce differential operators in the framework of infinite dimensional Gaussian analysis.

4.1 Nuclear countably Hilbert spaces

The aim of this section is to present the basic definitions and properties of nuclear countable Hilbert spaces. We only present content we need in the further course of this thesis. The presentation chosen here is of course not the most general one. For more details, generalizations and proofs we refer to [16, 41, 43, 42, 93, 107, 13, 14].

4.1.1 Definition and Basic Properties

Throughout this entire chapter we fix a real vector space $\mathcal{N}$ and assume there exists a family of real scalar products $\left\{(\cdot,\cdot)_p \mid p \in \mathbb{N}\right\}$ on $\mathcal{N}$. The induced norms we denote by $\|\cdot\|_p$, $p \in \mathbb{N}$, respectively. We can w.l.o.g. choose the system of scalar products in such a way that the norms $\|\cdot\|_p$, $p \in \mathbb{N}$, are increasing, i.e., $\|\cdot\|_p \leq \|\cdot\|_{p+1}$ for all $p \in \mathbb{N}$. We define a locally convex vector space topology $\mathcal{T}$ on $\mathcal{N}$ via defining a neighborhood base of zero $\left(U_{p,\varepsilon}\right)_{p\in\mathbb{N},\varepsilon>0}$, across the family of norms, where for $p \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$U_{p,\varepsilon} := \left\{f \in \mathcal{N} \mid \|f\|_p < \varepsilon\right\}. \tag{4.1}$$

Remark 4.1. *Of course, several different families of scalar products or even just norms which are not derived from a scalar product can induce the same topology in the fashion just described. We take advantage of this fact later. Depending on the situation one can choose a different family which* *is more appropriate and simplifies calculations.*

The abstract completion of $\mathcal{N}$ w.r.t. the norm $\|\cdot\|_p$, $p \in \mathbb{N}$, is denoted by $\mathcal{H}_p$ and the respective extensions of $(\cdot,\cdot)_p$ and $\|\cdot\|_p$ to $\mathcal{H}_p$ are represented by the same symbols. The natural embedding of $\mathcal{N}$ into $\mathcal{H}_p$ is denoted by I_p. Since the norms are increasing, we obtain for $p, q \in \mathbb{N}$ with $p \le q$ that the identity map on $\mathcal{N}$ extends to a continuous linear operator

$$I_{q,p} : \mathcal{H}_q \longrightarrow \mathcal{H}_p.$$

The family of norms $(\|\cdot\|_p)_{p\in\mathbb{N}}$ is called compatible, if for all $p, q \in \mathbb{N}$ and every sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathcal{N}$ which is a zero sequence w.r.t. the norm $\|\cdot\|_p$ and a Cauchy sequence w.r.t. $\|\cdot\|_q$ the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to zero w.r.t. $\|\cdot\|_q$. If compatibility holds true, we obtain that the linear operator $I_{q,p}$ is also injective and we obtain the chain of continuous embeddings

$$\mathcal{N} \subseteq \mathcal{H}_q \subseteq \mathcal{H}_p \subseteq \mathcal{H}_1, \quad 1 \le p \le q. \tag{4.2}$$

In the following we assume additionally that as sets it holds

$$\mathcal{N} = \cap_{p\in\mathbb{N}} \mathcal{H}_p. \tag{4.3}$$

The condition (4.3) is equivalent to the completeness of $\mathcal{N}$, i.e., if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{N}$ which is Cauchy w.r.t. $\|\cdot\|_p$ for every $p \in \mathbb{N}$, then there exists a $f \in \mathcal{N}$, s.t. $\|f - f_n\|_p \xrightarrow{n\to\infty} 0$ for every $p \in \mathbb{N}$.

Definition 4.2. *Assume the scalar products $\{(\cdot,\cdot)_p \mid p \in \mathbb{N}\}$ on $\mathcal{N}$ determine an increasing family of norms which is compatible and it holds (4.3). Then, we call $\mathcal{N}$ a countably Hilbert space. Sometimes the tuples $(\mathcal{N}, \mathcal{T})$ and $(\mathcal{N}, \{(\cdot,\cdot)_p \mid p \in \mathbb{N}\})$ are also called countably Hilbert space.*

In the following let $\mathcal{N}$ be a countably Hilbert space. Assume for the moment that $\mathcal{N}$ is separable. Then it immediately follows that the Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, are separable, too, since $\mathcal{N}$ is densely embedded into $\mathcal{H}_p$. In reverse, if each $\mathcal{H}_p$, $p \in \mathbb{N}$, is separable then we can choose a countable dense set D_p in $\mathcal{N}$ s.t. $I_p D_p$ is dense in $\mathcal{H}_p$. Hence, the countable set $\cup_{p\in\mathbb{N}} D_p$ is dense in $\mathcal{N}$ w.r.t. $\mathcal{T}$.

Definition 4.3. *Assume that the countably Hilbert space $\mathcal{N}$ is separable and for every $p \in \mathbb{N}$ we can find $q \ge p$ s.t. $I_{q,p}$ is of Hilbert-Schmidt type. Then, we call $\mathcal{N}$ $\left((\mathcal{N},\mathcal{T}), (\mathcal{N}, \{(\cdot,\cdot)_p \mid p \in \mathbb{N}\})\right)$ a nuclear countably Hilbert space or simply a nuclear space.*

Example 4.4. *For $d \in \mathbb{N}$ denote by $S(\mathbb{R}^d)$ the space of Schwartz functions over $\mathbb{R}^d$, i.e.,*

$$S(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \mid \text{for all } \alpha, \beta \in \mathbb{N}_0^d \text{ it holds } \sup_{x\in\mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| < \infty \right\}.$$

Denote by H the differential operator defined on $S(\mathbb{R}^d)$ given as follows

$$Hf = \prod_{i=1}^{d} \left(-\frac{\partial^2}{\partial x_i^2} + x_i^2 + 1 \right) f, \quad f \in S(\mathbb{R}^d).$$

Define the scalar products $(f,g)_p := (H^p f, H^p g)_2 := \int_{\mathbb{R}^d} H^p f(x) H^p g(x)\, dx$, $p \in \mathbb{N}$, $f, g \in S(\mathbb{R}^d)$.

Then $\left(S(\mathbb{R}^d), \left\{(\cdot,\cdot)_p \mid p \in \mathbb{N}\right\}\right)$ *defines a nuclear countably Hilbert space, see also [91, Section V.3] for details.*

Remark 4.5. *It is not difficult to check that nuclearity is independent of the choice of the system of scalar products. Namely, if* $(\widetilde{(\cdot,\cdot)}_q)_{q\in\mathbb{N}}$ *is another system of scalar products on* $\mathcal{N}$ *which induces also the topology* $\mathcal{T}$*, then it holds that* $\left\{(\cdot,\cdot)_p \mid p \in \mathbb{N}\right\})$ *is nuclear if and only if* $\left\{\widetilde{(\cdot,\cdot)}_q \mid q \in \mathbb{N}\right\})$ *is nuclear, see also [42, Section I.3.3] for a purely topological formulation of nuclearity of a space* $\mathcal{N}$*.*

We denote the dual space of $\mathcal{N}$ by $\mathcal{N}'$. Further, we denote by $\mathcal{H}_{-p}$ the dual Hilbert space $\mathcal{H}_p'$ of $\mathcal{H}_p$, $p \in \mathbb{N}$. From the definition of the neighborhoods $U_{p,\varepsilon}$ in (4.1) one immediately obtains that as sets it holds

$$\mathcal{N}' = \cup_{p\in\mathbb{N}}\mathcal{H}_{-p}, \tag{4.4}$$

i.e., every element $F \in \mathcal{N}'$ extends continuously to some $\mathcal{H}_p$, $p \in \mathbb{N}$, and the elements from $\cup_{p\in\mathbb{N}}\mathcal{H}_{-p}$ restricted to $\mathcal{N}$ are continuous in the topology of $\mathcal{N}$. This can be realized by the operators I_{-p}, $p \in \mathbb{N}$,

$$I_{-p} : \mathcal{H}_{-p} \longrightarrow \mathcal{N}', \Phi \mapsto \Phi \circ I_p.$$

The dual pairing between elements $f \in \mathcal{N}$ and $F \in \mathcal{N}'$ is denoted by $\langle f, F\rangle := F(f)$.

Typical topologies on the dual space $\mathcal{N}'$ are the weak, strong and inductive limit topology. To define these topologies we need to define boundedness in $\mathcal{N}$. A set $B \subseteq \mathcal{N}$ is called bounded, if it is bounded w.r.t. $\|\cdot\|_p$ for every $p \in \mathbb{N}$. For an arbitrary subset $A \subset \mathcal{N}$ we define the semi-norm $\|\cdot\|_A$ on $\mathcal{N}'$ by

$$\|\cdot\|_A : \mathcal{N}' \longrightarrow \mathbb{R}, \Phi \mapsto \sup_{f\in A} |\langle f, \Phi\rangle| .$$

Now we can define the weak topology β_w as the topology given by the local base of neighborhoods of zero $\{U_{A,\varepsilon} \mid A \subset \mathcal{N}' \text{ is finite}, \varepsilon > 0\}$, where $U_{A,\varepsilon} := \|\cdot\|_A^{-1}[0,\varepsilon)$, $A \subseteq \mathcal{N}'$, $\varepsilon > 0$. Analogue, we define the strong topology β_s as the topology given by the local base of neighborhoods of zero $\{U_{A,\varepsilon} \mid A \subset \mathcal{N}' \text{ is bounded}, \varepsilon > 0\}$. The inductive limit topology β_i is defined as the finest locally convex topology s.t. the maps I_{-p}, $p \in \mathbb{N}$, are continuous. A neighborhood base of zero in β_i is given by the sets balanced[1] convex hulls[2] of sets of the form $\bigcup_{p\in\mathbb{N}} I_{-p}B_{\varepsilon_p}^{-p}(0)$, where $B_{\varepsilon_p}^{-p}(0)$ denotes the ball with radius $\varepsilon_p > 0$ in $\mathcal{H}_{-p}$ with center 0, see e.g. [13].

Lemma 4.6. *For a countably Hilbert space* $\mathcal{N}$ *the topologies* β_s *and* β_i *on* $\mathcal{N}'$ *coincide.*

Proof. See [13, Theorem 4.16.]. □

If we identify the Hilbert space $\mathcal{H}_1$ via the Riesz isomorphism with itself and equip $\mathcal{N}'$ with one of the topologies β_w, β_s or β_i, then we can extend the chain from (4.2) and obtain the continuous embeddings

$$\mathcal{N} \subseteq \mathcal{H}_q \subseteq \mathcal{H}_p \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_{-p} \subseteq \mathcal{H}_{-q} \subseteq \mathcal{N}', \quad 1 \le p \le q. \tag{4.5}$$

[1] A subset $W \subseteq \mathcal{N}'$ is called balanced if for all $F \in W$ and $\lambda \in \mathbb{R}$ with $|\lambda| \le 1$ it holds $\lambda F \in W$.

[2] The balanced convex hull of a subset $W \subseteq \mathcal{N}'$ is the smallest balanced and convex set containing W.

Lemma 4.7. *Let $\mathcal{N}$ be a nuclear space and $(F_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{N}'$ which converges to $F \in \mathcal{N}'$ w.r.t. β_w. Then the convergence takes place in β_s, too.*

Proof. See e.g. [43, 42]. □

4.1.2 Tensor Powers of Countably Hilbert spaces and the Kernel Theorem

Throughout this subsection we fix a countably Hilbert space $(\mathcal{N}, \{(\cdot,\cdot)_p \mid p \in \mathbb{N}\})$ with corresponding separable Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$. Furthermore, we assume that the Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$ are seperable. We denote the complexifications, see Definition 1.29, of $\mathcal{N}$ and $\mathcal{H}_p$, $p \in \mathbb{N}$, by $\mathcal{N}_{\mathbb{C}}$ and $\mathcal{H}_{p,\mathbb{C}}$, $p \in \mathbb{N}$, respectively. For a number $n \in \mathbb{N}_0$ we denote by $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{p,\mathbb{C}}^{\otimes n}$ the usual real and complex $n-$fold tensor product of $\mathcal{H}_p$ and $\mathcal{H}_{p,\mathbb{C}}$, $p \in \mathbb{N}$, respectively. We occasionally also denote the scalar product and norm on $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{p,\mathbb{C}}^{\otimes n}$ simply by $(\cdot,\cdot)_p$ and $\|\cdot\|_p$. Further, denote by $\mathrm{sym}_p^{(n)}$ the symmetrization operator on $\mathcal{H}_p^{\otimes n}$, i.e.,

$$\mathrm{sym}_p^{(n)} \otimes_{i=1}^n f_i := \widehat{\otimes}_{i=1}^n f_i := \frac{1}{n!} \sum_{\pi \in \Sigma_n} \otimes_{i=1}^n f_{\pi(i)}, \quad f_i \in \mathcal{H}_p, i \in \{1, \dots, n\},$$

where Σ_n denotes the permutation group of n elements. We extend $\mathrm{sym}_p^{(n)}$ linearly to the span

$$D^{(n)} := \mathrm{span}_{\mathbb{R}} \left\{ \otimes_{i=1}^n f_i \mid f_i \in \mathcal{H}_p, i \in \{1, \dots, n\} \right\},$$

which is well-defined. By continuity, $\mathrm{sym}_p^{(n)}$ extends to a contraction on $\mathcal{H}_p^{\otimes n}$ which is also denote by $\mathrm{sym}_p^{(n)}$. In the same way we introduce the analog operator $\mathrm{sym}_{p,\mathbb{C}}^{(n)}$ on $\mathcal{H}_{p,\mathbb{C}}^{\otimes n}$. Both operators $\mathrm{sym}_p^{(n)}$ and $\mathrm{sym}_{p,\mathbb{C}}^{(n)}$ are projections on $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{p,\mathbb{C}}^{\otimes n}$, respectively. We define the subspaces of symmetric tensors by

$$\mathcal{H}_p^{\widehat{\otimes} n} = \mathrm{sym}_p^{(n)} \mathcal{H}_p^{\otimes n},$$
$$\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n} = \mathrm{sym}_p^{(n)} \mathcal{H}_{p,\mathbb{C}}^{\otimes n}.$$

Recall the continuous operators $I_{q,p}$, $q, p \in \mathbb{N}$, from the previous subsection. Their tensor powers are contractions

$$I_{q,p}^{\otimes n} : \mathcal{H}_q^{\otimes n} \longrightarrow \mathcal{H}_p^{\otimes n},$$
$$I_{q,p,\mathbb{C}}^{\otimes n} : \mathcal{H}_{q,\mathbb{C}}^{\otimes n} \longrightarrow \mathcal{H}_{p,\mathbb{C}}^{\otimes n},$$

for every $n \in \mathbb{N}$ and and commute with $\mathrm{sym}_p^{(n)}$, i.e.,

$$I_{q,p}^{\otimes n} \mathcal{H}_q^{\widehat{\otimes} n} \subseteq \mathcal{H}_p^{\widehat{\otimes} n},$$
$$I_{q,p,\mathbb{C}}^{\otimes n} \mathcal{H}_{q,\mathbb{C}}^{\widehat{\otimes} n} \subseteq \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n},$$

see also [91, Section VIII.10]. The families of maps $\left(I_{q,p}^{\otimes n}\right)_{p,q\in\mathbb{N}}$ and $\left(I_{q,p,\mathbb{C}}^{\otimes n}\right)_{p,q\in\mathbb{N}}$ are consistent, i.e.,

$$\begin{aligned} I_{q,q}^{\otimes n} &= Id, \quad q \in \mathbb{N}, \\ I_{q,r}^{\otimes n} &= I_{p,r}^{\otimes n} I_{q,p}^{\otimes n}, \quad r \le p \le q, \end{aligned}$$

and the analog statements holds for $\left(I_{q,p,\mathbb{C}}^{\otimes n}\right)_{p,q\in\mathbb{N}}$, too. Hence, we can define the projective limits of the respective Hilbert spaces, see e.g. [93, Section 2.5.2],

$$\mathcal{N}^{\widehat{\otimes} n} = \lim_{\longleftarrow p\in\mathbb{N}} \mathcal{H}_p^{\widehat{\otimes} n}, \quad \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n} = \lim_{\longleftarrow p\in\mathbb{N}} \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}.$$

From general duality theory we obtain that the respective dual space are given as the inductive limits of the dual Hilbert spaces, see [93, Section 2.6.3],

$$\mathcal{N}'^{\widehat{\otimes} n} := \left(\mathcal{N}^{\widehat{\otimes} n}\right)' = \lim_{\longrightarrow p\in\mathbb{N}} \mathcal{H}_p^{\widehat{\otimes} n}, \quad \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} n} := \left(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}\right)' = \lim_{\longrightarrow p\in\mathbb{N}} \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}.$$

We call a map $M : \times_{i=1}^n \mathcal{N} \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$, symmetric, if for all $f_i \in \mathcal{N}$, $i = 1, ..n$, and all $\pi \in \Sigma_n$, it holds $M(f_1, ..., f_n) = M(f_{\pi(1)}, ..., f_{\pi(n)})$. Next, we state the polarization formula which simplifies many computations in Gaussian analysis. A proof can be found in [104].

Lemma 4.8. *Let W, V be $\mathbb{K}$-vector spaces, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $n \in \mathbb{N}$ and $F : V^n \longrightarrow W$ be $\mathbb{K}$-multilinear and symmetric. Define $\Delta_n : V \longrightarrow V^n, v \mapsto \Delta_n(v) = (v)_{i=1}^n$. Then it holds*

$$F(v_1, ..., v_n) = \frac{1}{2^n n!} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{\pm 1\}} \varepsilon_1 \dots \varepsilon_n F \circ \Delta_n \left(\sum_{i=1}^n \varepsilon_i v_i \right).$$

Theorem 4.9 (Kernel Theorem)**.** *Let $\mathcal{N}$ be a nuclear countably Hilbert space and $F : \mathcal{N}_{\mathbb{C}}^n \longrightarrow \mathbb{C}$, $n \in \mathbb{N}$, be symmetric, $\mathbb{C}$-multilinear and separately continuous, i.e., there exists a positive constant C and $p \in \mathbb{N}$ s.t. for all $f_i \in \mathcal{N}_{\mathbb{C}}$, $i = 1, ..n$, it holds*

$$|F(f_1, ..., f_n)| \le C \prod_{i=1}^n \|f_i\|_p .$$

Let $q \in \mathbb{N}$ s.t. $I_{q,p}$ is of Hilbert-Schmidt type. Then there exists a unique $\Phi_{-q}^{(n)} \in \mathcal{H}_{-q,\mathbb{C}}^{\hat{\otimes} n}$ s.t. for all $f_i \in \mathcal{N}_{\mathbb{C}}$, $i = 1, ..n$, it holds

$$F(f_1, ..., f_n) = \left\langle \widehat{\otimes}_{i=1}^n f_i, \Phi_{-q}^{(n)} \right\rangle$$

Proof. See e.g. [16]. □

Remark 4.10. *The proof of Theorem 4.9 also implies that a real version of the Kernel Theorem holds true.*

4.2 Gaussian Analysis and White Noise Analysis

In this section we present basic facts from Gaussian Analysis and White Noise Analysis and present the theory of Hida test functions and Hida distributions. All the content in this section can be found in the monographs [16, 65, 57, 42, 79]. For further reading, we also want to mention the well-written paper [64] for a very general approach to infinite dimensional analysis. As usual in Gaussian analysis we start our presentation with the Bochner-Minlos theorem. Afterwards, we give a detailed presentation of the famous Wiener-Itô-Segal isomorphism. In the last subsection we introduce the Hida test functions $(\mathcal{N})$ and Hida distributions $(\mathcal{N})'$. Important aspects of this infinite dimensional distribution theory, such as the characterization theorem and the differential calculus, are presented.

The usage of Hermite polynomials in Gaussian analysis is omnipresent. For this purpose, we state a collection of formulas related to Hermite polynomials in the Appendix B.1.

4.2.1 Bochner-Minlos Theorem

The starting point of Gaussian analysis and White noise analysis is a real nuclear countably Hilbert space $\mathcal{N}$ which we fix throughout this entire section. As usual, its dual space is denoted by $\mathcal{N}'$. We introduced in section 4.1.1 several topologies on $\mathcal{N}'$. Our aim is to construct a measure on the space $\mathcal{N}'$. To this end we have to define a σ-field $\mathcal{B}$ on $\mathcal{N}'$ which contains enough events of interest. On the other hand, to construct a measure it is preferable to choose the σ-field as small as possible. From this point of view, we choose $\mathcal{B}$ as the Borel σ-field of the weak topology β_w. It turns out that there is no difference if we choose the strong topology instead, as the next lemma shows.

Lemma 4.11. *The Borel σ-fields of the weak, strong and inductive limit topology on $\mathcal{N}'$ coincide, i.e.,*

$$\sigma(\beta_w) = \sigma(\beta_s) = \sigma(\beta_i).$$

Proof. See e.g. [13]. □

Remark 4.12. *Legitimated by the previous lemma we just call $\mathcal{B} = \sigma(\beta_w)$ the Borel σ-field on $\mathcal{N}'$.*

In finite dimensional spaces Bochner's theorem provides a handy tool to define and analyze finite measures, see e.g. [89, Theorem IX.9]. Its generalization to dual spaces of nuclear countably Hilbert spaces is the celebrated Bochner-Minlos theorem:

Theorem 4.13. *Let $\rho : \mathcal{N} \longrightarrow \mathbb{C}$ be a function satisfying*

(i) *ρ is positive definite, i.e., for all $n \in \mathbb{N}$ and $f_i \in \mathcal{N}$, $\alpha_i \in \mathbb{C}$, $i = 1, \dots, n$, it holds*

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \rho(f_i - f_j) \geq 0,$$

(ii) $\rho(0) = 1$,

(iii) ρ *is continuous.*

Then there exists a unique probability measure μ_ρ *on* $(\mathcal{N}', \mathcal{B})$ *s.t.*

$$\rho(f) = \int_{\mathcal{N}'} \exp(i \langle f, \cdot \rangle) \, d\mu_\rho, \quad f \in \mathcal{N}. \tag{4.6}$$

Moreover, if ρ *is continuous in zero w.r.t. the norm* $\|\cdot\|_p$ *and for* $q \in \mathbb{N}$ *the operator* $I_{q,p}$ *is of Hilbert-Schmidt type, then* $\mu_\rho(\mathcal{H}_{-q}) = 1$. *On the contrary, if* μ *is a probability measure on* $(\mathcal{N}', \mathcal{B})$, *then the function* $\rho : \mathcal{N} \longrightarrow \mathbb{C}$ *defined by (4.6) satisfies the conditions (i)-(iii) defined above.*

Proof. See e.g. [56] or [42]. □

If μ is a probability measure on $(\mathcal{N}', \mathcal{B})$ then the function $\rho : \mathcal{N} \longrightarrow \mathbb{C}$ given as in (4.6) is called the characteristic function or Fourier transform of μ.

Example 4.14. *Let* $b : \mathcal{N} \times \mathcal{N} \longrightarrow \mathbb{C}$ *be a positive definite, continuous and symmetric bilinear form. Then the map*

$$\hat{\mu}_b : \mathcal{N} \longrightarrow \mathbb{C}, f \mapsto \exp(-\frac{1}{2} b(f, f)) \tag{4.7}$$

satisfies (i)-(iii) from Theorem 4.13. Property (i) follows from the fact that the Hadamard product of two positive definite matrices is again positive definite. Hence there exists a corresponding measure μ_b.

Proposition 4.15. *Let* b, μ_b *be given as in Example 4.14(i) and let* $f_1, \ldots, f_n \in \mathcal{N}$, $n \in \mathbb{N}$. *The image measure of* μ_b *under the map* $T_{f_1,\ldots,f_n} : \mathcal{N}' \longrightarrow \mathbb{R}^n, F \mapsto (\langle f_i, F \rangle)_{i=1,\ldots,n}$ *is the Gaussian measure with mean zero and covariance matrix* $\Lambda = \big(b(f_i, f_j)\big)_{1 \le i,j \le n}$ *on* $\mathbb{R}^n$, *i.e.,*

$$\mu_b \circ T^{-1}_{f_1,\ldots,f_n} = N(0, \Lambda).$$

The previous proposition justifies the next definition.

Definition 4.16. *Let* b *and* μ_b *be given as in Example 4.14. We call the measure* μ_b *the Gaussian measure with mean zero and covariance* b *on* $\mathcal{N}'$. *Occasionally, we just call* μ_b *the Gaussian measure with covariance* b.

We denote by τ_x the translation given by $x \in \mathcal{N}'$, i.e.,

$$\tau_x : \mathcal{N}' \longrightarrow \mathcal{N}', y \mapsto y + x.$$

Lemma 4.17. *Let* b *be given as in Example 4.14(i). Define for* $f \in \mathcal{N}$ *the element* $F_{b,f}(\cdot) := b(\cdot, f) \in \mathcal{N}'$. *The Gaussian measure* μ_b *is quasi-shift invariant w.r.t. the shift* $\tau_{F_{b,f}}$, *i.e., it holds that* $\mu_b \circ \tau^{-1}_{F_{b,f}}$ *is absolutely continuous w.r.t.* μ_b *and the Radon-Nikodym derivative is given by*

$$\frac{d\mu_b \circ \tau^{-1}_{F_{b,f}}}{d\mu_b} = \exp\left(\langle f, \cdot \rangle - \frac{1}{2} b(f, f)\right).$$

Proof. By the uniqueness part of Bochner-Minlos theorem it suffices to check that the Fourier transforms of $\mu_b \circ \tau_{F_{b,f}}^{-1}$ and μ_b coincide. But this can be done via Proposition 4.15 and a straightforward calculation for finite dimensional Gaussian integrals. □

Sometimes we simply write f instead of $F_{b,f}$ if there is no room for confusion.

We sometime call the property quasi-translation invariant of a measure also quasi-shift invariant. From the quasi-translation invariance we obtain a nice result concerning the topological support of the measure μ_b.

Corollary 4.18. *Let $b : \mathcal{N} \times \mathcal{N} \longrightarrow \mathbb{C}$ be a positive definite, continuous, symmetric and non-degenerated bilinear form, i.e., $b(f,f) > 0$ if $f \in \mathcal{N} \setminus \{0\}$. The Gaussian measure μ_b has full topological support, i.e., for all $O \in \beta_s$ it holds $\mu_b(O) > 0$.*

Proof. See e.g. [79, Proof of Proposition 3.2.2]. □

4.2.2 Wiener-Ito-Segal Isomorphism

Throughout this subsection we fix a positive definite, continuous, symmetric and non-degenerated bilinear form $b : \mathcal{N} \times \mathcal{N} \longrightarrow \mathbb{C}$. We denote the associated Gaussian measure by μ_b. To ease the notation we also write μ instead of μ_b in this section. In the following we give a complete description of the space

$$L^2(\mu) = L^2(\mu_b) := L^2(\mathcal{N}', \mathcal{B}, \mu_b; \mathbb{C})$$

of complex valued (classes of) square μ-integrable functions. We define the Hilbert space $\mathcal{H}$ as the abstract completion of $\mathcal{N}$ w.r.t. norm $\|f\|_{\mathcal{H}} := b(f,f)^{\frac{1}{2}}$.

Definition 4.19. *Let $n \in \mathbb{N}_0$ and define the complex space of polynomials of maximal degree n over $\mathcal{N}'$ by*

$$\mathcal{P}_n(\mathcal{N}') := \{p(\langle f_1, \cdot\rangle, \ldots, \langle f_m, \cdot\rangle) : \mathcal{N}' \longrightarrow \mathbb{C} \mid m \in \mathbb{N}, f_1, \ldots, f_m \in \mathcal{N}, p \in \mathbb{C}[x_1, \ldots, x_m], deg(p) \leq n\}$$

and the space of all polynomials by

$$\mathcal{P}(\mathcal{N}') := \cup_{n\in\mathbb{N}} \mathcal{P}_n(\mathcal{N}').$$

Remark 4.20. *(i) Elements of $\mathcal{P}(\mathcal{N}')$ are p-integrable w.r.t. μ for all $p \in [1, \infty)$. This can be seen by Proposition 4.15 and the fact that Gaussian measures on $\mathbb{R}^n$ have moments of all orders.*

(ii) Observe that the elements $P \in \mathcal{P}(\mathcal{N}')$ are continuous functions on $\mathcal{N}'$ w.r.t. the weak topology β_w on $\mathcal{N}'$. Due to Corollary 4.18 there is no need to distinguish between the function P on $\mathcal{N}'$ and the μ-class of P since μ-equivalence class contain at most one element from $\mathcal{P}(\mathcal{N}')$.

(iii) Observe that an element $P \in \mathcal{P}_n(\mathcal{N}')$ can be represented via the polarization identity as

$$P(x) = \sum_{l=0}^{n} \left\langle x^{\otimes l}, f^{(l)} \right\rangle, \quad x \in \mathcal{N}',$$

where the elements $f^{(l)}$ are given as

$$f^{(l)} = \sum_{s \in \mathcal{I}_l} \alpha_s f_s^{\otimes n}$$

with a finite index set $\mathcal{I}_l$ and complex coefficients $\alpha_s \in \mathbb{C}$ and $f_s \in \mathcal{N}$ for all $s \in \mathcal{I}_l$.

As a first step we show that the subspace $\mathcal{P}(\mathcal{N}')$ of $L^2(\mu)$ is dense.

Proposition 4.21. *$\mathcal{P}(\mathcal{N}')$ is dense in $L^2(\mu)$.*

Proof. It suffices to show that $\mathcal{P}_{\mathbb{R}}(\mathcal{N}')$ is dense in $L^2_{\mathbb{R}}(\mu)$. Let $F \in L^2_{\mathbb{R}}(\mu)$ s.t. $F \in \mathcal{P}_{\mathbb{R}}(\mathcal{N}')^{\perp}$. We need to show that $F = 0$. We decompose F into $F = F_+ - F_-$, where $F_+, F_- \geq 0$ μ-a.e.. It suffices to show that the measures $\nu_+ = F_+\mu$ and $\nu_- = F_-\mu$ coincide. Indeed, if $\nu_+ = \nu_-$ then it holds

$$\int_{\mathcal{N}'} (F_+ - F_-)^2 \, d\mu = \underbrace{\int_{\mathcal{N}'} F_+ \, d\nu_+ - \int_{\mathcal{N}'} F_+ \, d\nu_-}_{=0} + \underbrace{\int_{\mathcal{N}'} F_- \, d\nu_- - \int_{\mathcal{N}'} F_- \, d\nu_+}_{=0} = 0.$$

Since F is orthogonal to 1 we obtain that ν_+ and ν_- have the same mass. Hence, we can assume that both are probability measures. Due to Theorem 4.13 it suffices to show that their one dimensional distributions coincide, i.e., that for every $f \in \mathcal{N}$ it holds $\nu_+ \circ \langle f, \cdot \rangle^{-1} = \nu_- \circ \langle f, \cdot \rangle^{-1}$. Since F is orthogonal to $\mathcal{P}(\mathcal{N}')$ we obtain that the measures $\nu_+ \circ \langle f, \cdot \rangle^{-1}$ and $\nu_- \circ \langle f, \cdot \rangle^{-1}$ have the same moments. Now we check that both measures satisfy Cramér's condition, which implies uniqueness of the associated Hamburger moment problem, see e.g. [40, Theorem 1]. Therefore, let $t \in \mathbb{R}$ be arbitrary,

$$\int_{\mathbb{R}} e^{tx} \, \nu_+ \circ \langle f, \cdot \rangle^{-1} (dx) \leq \|F_+\|_{L^2(\mu)} \left\| e^{t\langle f, \cdot \rangle} \right\|_{L^2(\mu)} < \infty$$

since μ is Gaussian. The same holds for $\nu_- \circ \langle f, \cdot \rangle^{-1}$ which finishes the proof. □

To start with the characterization of $L^2(\mu)$ we make the following observation based on Proposition 4.15 . For $f = f_1 + i f_2 \in \mathcal{N}_{\mathbb{C}}$ with $f_1, f_2 \in \mathcal{N}$ it holds

$$\int_{\mathcal{N}'} |\langle f, \cdot \rangle|^2 \, d\mu = \int_{\mathcal{N}'} \langle f_1, \cdot \rangle^2 + \langle f_2, \cdot \rangle^2 \, d\mu = b(f_1, f_1) + b(f_2, f_2) = \|f\|^2_{\mathcal{H}_{\mathbb{C}}}.$$

Since $L^2(\mu)$ is complete, we obtain for every sequence $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{N}_{\mathbb{C}}$ which is Cauchy w.r.t. $\|\cdot\|_{\mathcal{H}_{\mathbb{C}}}$, an element in $L^2(\mu)$, which we denote formally by $\langle f, \cdot \rangle$. Furthermore, this construction can be regarded as an isometric linear map from the complexification $\mathcal{H}_{\mathbb{C}}$ of $\mathcal{H}$, see Definition 1.29, to $L^2(\mu)$, i.e.,

$$I_1 : \mathcal{H}_{\mathbb{C}} \longrightarrow L^2(\mu), f = [(f_n)_{n \in \mathbb{N}}] \mapsto \lim_{n \to \infty} \langle f_n, \cdot \rangle = \langle f, \cdot \rangle. \tag{4.8}$$

Now, we define the following subspaces of $L^2(\mu)$

$$\mathcal{W}_0(\mathcal{N}') := \mathrm{span}_{\mathbb{C}}\{1\}$$
$$\mathcal{W}_n(\mathcal{N}') := \overline{\mathcal{P}_n(\mathcal{N}')} \cap \mathcal{P}_{n-1}(\mathcal{N}')^{\perp}, n \in \mathbb{N}. \tag{4.9}$$

From Proposition 4.21 we obtain that the space $L^2(\mu)$ decomposes into the orthogonal sum of subspaces $\mathcal{W}_n(\mathcal{N}')$, $n \in \mathbb{N}$, i.e.,

$$L^2(\mu) = \perp_{n\in\mathbb{N}_0} \mathcal{W}_n(\mathcal{N}').$$

In the following we give a unitary representation of the orthogonal subspaces $\mathcal{W}_n(\mathcal{N}')$, $n \in \mathbb{N}$. Observe that we already established above

$$\mathcal{W}_1(\mathcal{N}') = I_1\mathcal{H}_{\mathbb{C}}.$$

Now let $f, g \in \mathcal{N}$. Denote by $\left(H_{n,\sigma^2}\right)_{n\in\mathbb{N}}$ the family of Hermite polynomials with parameter σ^2, see B.1. By using the relations (B.3), (B.6) and Proposition 4.15 we obtain

$$\int_{\mathcal{N}'} H_{n,b(f,f)}(\langle f,\cdot\rangle)H_{m,b(g,g)}(\langle g,\cdot\rangle)\, d\mu = \delta_{n,m}n!b(f,g)^n = \delta_{n,m}n!\left(f^{\otimes n}, g^{\otimes n}\right)_{\mathcal{H}^{\widehat{\otimes} n}}. \tag{4.10}$$

The polarization identity, the formula (B.5) (4.10) imply that $H_{n,b(f,f)}(\langle f,\cdot\rangle) \in \mathcal{W}_n(\mathcal{N}')$. Now, we define $\mathrm{tr}_b \in \mathcal{N}'^{\widehat{\otimes} 2}$ via the Theorem 4.9 through

$$\mathrm{tr}_b(f\widehat{\otimes} g) = b(f,g), \quad f, g \in \mathcal{N}.$$

Then, we can write, see (B.4),

$$H_{n,b(f,f)}(\langle f,x\rangle) = \left\langle f^{\otimes n}, :x^{\otimes n}:_b\right\rangle,$$

where $:x^{\otimes n}:_b$ is called the n-th Wick power of x (w.r.t. b) and is given by the kernel theorem as an element of $\mathcal{N}'^{\widehat{\otimes} n}$ defined by

$$:x^{\otimes n}:_b = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \left(-\frac{1}{2}\right)^k \frac{n!}{k!(n-2k)!} \mathrm{tr}_b^{\otimes k}\widehat{\otimes} x^{\otimes n-2k} \in \mathcal{N}'^{\widehat{\otimes} n}. \tag{4.11}$$

Hence for elements $f^{(n)}, g^{(n)} \in \mathrm{span}_{\mathbb{C}}\left\{h^{\otimes n} \mid h \in \mathcal{N}\right\} \subseteq \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ it holds via linearity and (4.10)

$$\int_{\mathcal{N}'} \left\langle f^{(n)}, :\cdot^{\otimes n}:_b\right\rangle \overline{\left\langle g^{(n)}, :\cdot^{\otimes n}:_b\right\rangle}\, d\mu = n!\left(f^{(n)}, g^{(n)}\right)_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}}. \tag{4.12}$$

Define for $n \in \mathbb{N}_0$ the subspace $\widetilde{\mathcal{W}}_n(\mathcal{N}')$ of $\mathcal{W}_n(\mathcal{N}')$ by

$$\widetilde{\mathcal{W}}_n(\mathcal{N}') = \left\{\left\langle f^{(n)}, :\cdot^{\otimes n}:_b\right\rangle \mid f^{(n)} \in \mathrm{span}_{\mathbb{C}}\left\{h^{\otimes n} \mid h \in \mathcal{N}\right\}\right\}.$$

By combining the polarization formula with (B.4) and (B.5) we obtain another representation for usual polynomials, i.e.,

$$\mathcal{P}_n(\mathcal{N}') = \sum_{i=0}^{n} \widetilde{\mathcal{W}}_n(\mathcal{N}').$$

From Proposition 4.21 and the orthogonality of the spaces $\mathcal{W}_n(\mathcal{N}')$, $n \in \mathbb{N}_0$ we obtain that

$$\overline{\widetilde{\mathcal{W}}_n(\mathcal{N}')} = \mathcal{W}_n(\mathcal{N}') \text{ for every } n \in \mathbb{N}_0.$$

Via (4.12) and the same reasoning leading to (4.8) we obtain that the closure $\overline{\widetilde{\mathcal{W}}_n(\mathcal{N}')}$ is isometric isomorphic to $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} = \overline{\operatorname{span}_{\mathbb{C}}\{h^{\otimes n} \mid h \in \mathcal{N}\}}$ up to the constant $n!$. The constant $n!$ arises due to the fact that we consider on the symmetric tensor product $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ the norm of the usual tensor product $\mathcal{H}_{\mathbb{C}}^{\otimes n}$, see also [57, Appendix 2]. In the following we denote by $n!\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ the space $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ equipped with the norm $n!\,\|\cdot\|_{\mathcal{H}_{\mathbb{C}}^{\otimes n}}$. As above, we use the formal notation

$$I_n : \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \longrightarrow \mathcal{W}_n(\mathcal{N}'), \quad f^{(n)} \mapsto \left\langle f^{(n)}, :\cdot^{\otimes n}:_b \right\rangle,$$

where $\left\langle f^{(n)}, :\cdot^{\otimes n}:_b \right\rangle$ is understood as a $L^2(\mu)$-limit of elements from $\widetilde{\mathcal{W}}_n(\mathcal{N}')$. In the following we denote the norm and the scalar product of $L^2(\mu)$ by $|\!|\!| \cdot |\!|\!|$ and $(\!(\cdot,\cdot)\!)$, respectively. We recap the previous discussion in the following theorem which is called Wiener-Itô-Segal theorem.

Theorem 4.22. *The space $L^2(\mu)$ is isometric isomorphic to the so-called Boson Fock space over $\mathcal{H}$, i.e.,*

$$L^2(\mu) \cong \oplus_{n \in \mathbb{N}_0} n!\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} =: \Gamma(\mathcal{H}_{\mathbb{C}}).$$

We denote a typical element $F \in L^2(\mu)$ identified with the element $\left(f^{(n)}\right)_{n \in \mathbb{N}_0} \in \Gamma_{\mathbb{C}}(\mathcal{H})$ in the following as

$$F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}:_b \right\rangle \tag{4.13}$$

and for the $L^2(\mu)$-norm $|\!|\!|F|\!|\!|$ it holds

$$|\!|\!|F|\!|\!|^2 = \sum_{n=0}^{\infty} n! \left\| f^{(n)} \right\|^2_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}}.$$

The element $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ is called the n-th kernel of F and the space $\mathcal{W}_n(\mathcal{N}')$ is called the n-th chaos of $L^2(\mu)$, $n \in \mathbb{N}_0$. The representation (4.13) is called the chaos decomposition of F. From the derivation of Theorem 4.22 we obtain the following corollary:

Corollary 4.23. *For the real space $L^2_{\mathbb{R}}(\mu)$ it holds*

$$L^2_{\mathbb{R}}(\mu) \cong \oplus_{n \in \mathbb{N}_0} n!\mathcal{H}^{\widehat{\otimes} n} =: \Gamma(\mathcal{H}).$$

Remark 4.24. *Sometimes the Wick power $:\cdot^{\otimes n}:_b$ is also called Wick renormalization. Let us illustrate that via the following example. We choose $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$, i.e., $b = (\cdot,\cdot)_{L^2(\mathbb{R})}$. In particular, the complexified n-th symmetric tensor power of these spaces are given by $\widehat{S_{\mathbb{C}}(\mathbb{R}^n)}$ and $\widehat{L^2_{\mathbb{C}}(\mathbb{R}^n)}$, where $\widehat{\quad}$ denotes the corresponding subspace of symmetric functions. For simplicity, we omit the index b in the following. Let $f^{(2)} \in \widehat{L^2(\mathbb{R}^2)}$ be given by $f^{(2)}(x,y) = \frac{\exp(-(x^2+y^2))}{|x-y|^{\frac{1}{4}}}$, $x, y \in \mathbb{R}$. Observe that $f^{(2)} = \infty$ on the diagonal $D = \{(x,x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ which has Lebesgue measure zero in $\mathbb{R}^2$. Further let $f_n^{(2)} \in \widehat{S(\mathbb{R}^2)}$, $n \in \mathbb{N}$, be real-valued s.t. $f_n^{(2)} \longrightarrow f^{(2)}$ as $n \to \infty$ in $\widehat{L^2_{\mathbb{C}}(\mathbb{R}^2)}$.*

Moreover, we can choose $(f_n^{(2)})_{n\in\mathbb{N}}$ *s.t.* $\limsup_{n\to\infty}\left\langle f_n^{(2)},\tau\right\rangle = \limsup_{n\to\infty}\int_{\mathbb{R}} f_n^{(2)}(x,x)\,dx = \infty$. *Further, it holds*

$$\int_{S'(\mathbb{R})}\left\langle f_n^{(2)},\cdot^{\otimes 2}\right\rangle^2 d\mu = \left\langle f_n^{(2)},\tau\right\rangle^2.$$

Hence, $\left\langle f_n^{(2)},:\cdot^{\otimes 2}:\right\rangle = \left\langle f_n^{(2)},\cdot^{\otimes 2}\right\rangle - \left\langle f_n^{(2)},\tau\right\rangle$ *is an additive renormalization of* $\left\langle f_n^{(2)},\cdot^{\otimes 2}\right\rangle$ *which makes the renormalized sequence convergent in* $L^2(\mu)$.

Example 4.25. *Let* $f\in\mathcal{N}$. *From the definition of the Hermite polynomials, see (B.1), we obtain that the function*

$$:\!\exp(\langle f,\cdot\rangle)\!: \,:\mathcal{N}'\ni\omega\longrightarrow\exp\left(\langle\omega,f\rangle-\frac{1}{2}\|f\|^2\right)\in\mathbb{R}$$

satisfies

$$:\!\exp(\langle f,\cdot\rangle)\!:=\sum_{n=0}^{\infty}\frac{H_{n,\|f\|^2}(\langle f,\cdot\rangle)}{n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle f^{\otimes n},:\cdot^{\otimes n}:\right\rangle. \tag{4.14}$$

In particular, for $f\in\mathcal{H}$ *the right-hand side of (4.14) defines an element in* $L^2(\mu)$ *which we also denote by* $:\!\exp(\langle f,\cdot\rangle)\!:\in L^2(\mu)$. *The element* $:\!\exp(\langle f,\cdot\rangle)\!:$ *is called the Wick exponential of* $f\in\mathcal{H}$.

Lemma 4.26. *The set of Wick exponentials* $\mathcal{W}:=\{:\!\exp(\langle f,\cdot\rangle)\!:\mid f\in\mathcal{N}\}$ *forms a total set in* $L^2(\mu)$.

Proof. We need to show that $\mathcal{W}^\perp=\{0\}$. Let $F=\sum_{n=0}^{\infty}\left\langle f^{(n)},:\cdot^{\otimes n}:\right\rangle\in\mathcal{W}^\perp$. Hence, it holds for $g\in\mathcal{N}$ and $\lambda\in\mathbb{R}$

$$0=\int_{\mathcal{N}'}F\,:\!\exp(\langle\lambda g,\cdot\rangle)\!:\,d\mu=\sum_{n=0}^{\infty}\lambda^n(f^{(n)},g^{\otimes n})_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}}.$$

Observe that the series on the right converges absolutely for all $\lambda\in\mathbb{R}$. Hence, we can differentiate term-by-term and obtain $(f^{(n)},g^{\otimes n})_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}}=0$ for all $n\in\mathbb{N}_0$. Since $f^{(n)}$ is symmetric and $g\in\mathcal{N}$ was arbitrary we conclude $f^{(n)}=0$ for all $n\in\mathbb{N}_0$. □

Before we proceed, we present a useful formula for the product of two typical elements from $L^2(\mu)$. To this end let $g^{(m)}\in\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}m}$, $f^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$, $m,n\in\mathbb{N}_0$ and $k\in\mathbb{N}_0$ s.t. $0\le k\le\min\{m,n\}$. Furthermore, let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. For a multi index $\alpha\in\mathbb{N}^l$, $l\in\mathbb{N}$, we denote by $e_\alpha\in\mathcal{H}_{\mathbb{C}}^{\otimes l}$ the element $e_\alpha=\otimes_{s=1}^{l}e_{\alpha_s}$. We call $g^{(m)}\otimes_k f^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\otimes m+n-2k}$ the tensor contraction of $g^{(m)}$ and $f^{(n)}$ of order k which is defined by

$$g^{(m)}\otimes_k f^{(n)}=\sum_{\substack{\alpha\in\mathbb{N}^{m-k}\\ \beta\in\mathbb{N}^{n-k}}}e_\alpha\otimes e_\beta\sum_{\gamma\in\mathbb{N}^k}(g^{(m)},e_\gamma\otimes e_\alpha)_{\mathcal{H}_{\mathbb{C}}^{\otimes m}}(f^{(n)},e_\gamma\otimes e_\beta)_{\mathcal{H}_{\mathbb{C}}^{\otimes n}}. \tag{4.15}$$

It is straightforward to check that this definition is independent of the choice of the orthogonal basis $(e_k)_{k\in\mathbb{N}}$ of the Hilbert space $\mathcal{H}$. The symmetrization of $g^{(m)}\otimes_k f^{(n)}$ we denote by $g^{(m)}\widehat{\otimes}_k f^{(n)}$.

We see that for m, n, k as above the bilinear map

$$\widehat{\otimes}_k : \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} m} \times \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} \longrightarrow \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} m+n-2k}, (g^{(m)}, f^{(n)}) \mapsto g^{(m)} \widehat{\otimes}_k f^{(n)}$$

is continuous. In the special case $k = m$ or $k = n$ we denote $g^{(m)} \widehat{\otimes}_k f^{(n)}$ also by $\langle g^{(m)}, f^{(n)} \rangle$.

Lemma 4.27. *Let $g^{(m)} \in \mathcal{H}^{\widehat{\otimes} m}$, $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}$, $n, m \in \mathbb{N}_0$. Then, it holds*

$$\left\langle g^{(m)}, :\cdot^{\otimes n}:_b \right\rangle \left\langle f^{(n)}, :\cdot^{\otimes n}:_b \right\rangle = \sum_{k=0}^{\min\{m,n\}} k! \binom{m}{k} \binom{n}{k} \left\langle g^{(m)} \widehat{\otimes}_k f^{(n)}, :\cdot^{\otimes m+n-2k}:_b \right\rangle. \tag{4.16}$$

Proof. Observe that both sides of (4.16) are linear in $g^{(m)}$ and $f^{(n)}$, respectively. Furthermore, if $g_l^{(m)} \xrightarrow{l\to\infty} g^{(m)}$ in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} m}$ and $f_l^{(n)} \xrightarrow{l\to\infty} f^{(n)}$ in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, then by switching to corresponding subsequences, which we denote again by $(g_l^{(m)})_{l\in\mathbb{N}}$ and $(f_l^{(n)})_{l\in\mathbb{N}}$, respectively, we obtain

$$\begin{aligned}
\left\langle g^{(m)}, :\cdot^{\otimes n}:_b \right\rangle \left\langle f^{(n)}, :\cdot^{\otimes n}:_b \right\rangle &= \lim_{l\to\infty} \left\langle g_l^{(m)}, :\cdot^{\otimes n}:_b \right\rangle \left\langle f_l^{(n)}, :\cdot^{\otimes n}:_b \right\rangle, \\
\left\langle g^{(m)} \widehat{\otimes}_k f^{(n)}, :\cdot^{\otimes m+n-2k}:_b \right\rangle &= \lim_{l\to\infty} \left\langle g_l^{(m)} \widehat{\otimes}_k f_l^{(n)}, :\cdot^{\otimes m+n-2k}:_b \right\rangle, \text{ for } 0 \le k \le \min\{m, n\},
\end{aligned}$$

where equality and the limits are understood in the μ-a.e. sense. Hence, it suffices to that (4.16) holds for $g^{(m)} = g^{\otimes m}$ and $f^{(n)} = f^{\otimes n}$ for $g, f \in \mathcal{N}$. We assume w.l.o.g. $m \le n$. Due to (B.6) we can further assume that $\|f\| = \|g\| = 1$ and write $g = (g, f)f + rf^{\perp}$ where $r \in \mathbb{R}$, $(f, f^{\perp}) = 0$ and $\|f^{\perp}\| = 1$. By using the formula (B.3) and (B.7) we obtain

$$\begin{aligned}
&H_{m,1}(\langle g, \cdot\rangle) H_{n,1}(\langle f, \cdot\rangle) \\
&= \sum_{s=0}^{m} \binom{m}{s} (g, f)^s r^{m-s} \sum_{k=0}^{s} k! \binom{s}{k} \binom{n}{k} H_{n+s-2k,1}(\langle f, \cdot\rangle) H_{m-s,1}(\langle f^{\perp}, \cdot\rangle) \\
&= \sum_{k=0}^{m} k! \binom{m}{k} \binom{n}{k} (g, f)^k \sum_{s=0}^{m-k} \binom{m-k}{s} (g, f)^s r^{m-s} H_{n+s-k,1}(\langle f, \cdot\rangle) H_{m-k-s,1}(\langle f^{\perp}, \cdot\rangle).
\end{aligned}$$

Observe that $g^{\otimes m} \widehat{\otimes}_k f^{\otimes n} = (g, f)^k g^{\otimes m-k} \widehat{\otimes} f^{\otimes n-k}$ for all $0 \le k \le m$. Hence, it suffices to show that for all $0 \le k \le m$ it holds

$$\sum_{s=0}^{m-k} \binom{m-k}{s} (g, f)^s r^{m-s} H_{n+s-k,1}(\langle f, \cdot\rangle) H_{m-k-s,1}(\langle f^{\perp}, \cdot\rangle) = \left\langle g^{\otimes m-k} \widehat{\otimes} f^{\otimes n-k}, :\cdot^{\otimes m+n-2k}:_b \right\rangle. \tag{4.17}$$

To prove the equality in (4.17) we use Lemma 4.26 and integrate both sides against a Wick exponential $:\exp(\langle h, \cdot\rangle):$, $h \in \mathcal{N}$, w.r.t. μ. One easily concludes that both integrals coincide by splitting $:\exp(\langle h, \cdot\rangle):$ into

$$:\exp(\langle h, \cdot\rangle):=:\exp(\langle (h, f)f, \cdot\rangle):\, :\exp(\langle (h, f^{\perp})f^{\perp}, \cdot\rangle):$$

and applying Proposition 4.15. □

4.2.3 Hida test functions and Hida distributions

I. Construction

Recall the situation from the previous section. We started with a real nuclear space $\mathcal{N}$ with inner products $\{(\cdot,\cdot)_p \mid p \in \mathbb{N}\}$ and a bilinear form $b : \mathcal{N} \times \mathcal{N} \longrightarrow \mathbb{C}$ which is assumed to be symmetric, continuous, positive definite and non-degenerated. We denote the completion of $\mathcal{N}$ w.r.t. $\|\cdot\|_{\mathcal{H}} = b^{\frac{1}{2}}(\cdot,\cdot)$ by $\mathcal{H}$. Note that we can assume w.l.o.g. $\|\cdot\|_{\mathcal{H}} \leq \|\cdot\|_1$. Additionally, we assume from now on that $(\mathcal{N}, (\cdot,\cdot)_1)$ is closable on $\mathcal{H}$ or equivalently that $\|\cdot\|_1$ and $\|\cdot\|_0$ are compatible. Then, we obtain a chain of continuous and dense embeddings for $1 \leq p \leq q$

$$\mathcal{N} \subseteq \mathcal{H}_q \subseteq \mathcal{H}_p \subseteq \mathcal{H} \cong \mathcal{H}' \subseteq \mathcal{H}_{-p} \subseteq \mathcal{H}_{-q} \subseteq \mathcal{N}', \tag{4.18}$$

where $\cong$ denotes the Riesz isomorphism of $\mathcal{H}$. In the following we make an additional assumption on the family of norms $\left(\|\cdot\|_p\right)_{p\in\mathbb{N}}$ defined on $\mathcal{N}$. We assume there exists a $0 < C < 1$ s.t. for all $p, q \in \mathbb{N}_0$ with $p \leq q$ it holds

$$\|f\|_p \leq C^{q-p} \|f\|_q, \quad f \in \mathcal{N}. \tag{4.19}$$

The assumptions is only made for convenience to simplify the consideration below. In particular, all concrete choices of $\mathcal{N}$ below are given by $\mathcal{N} = S(\mathbb{R}^d)$, $d \in \mathbb{N}$. The norms defined in Example 4.4 satisfy the assumption (4.19). All considerations below generalize to an arbitrary nuclear space $\left(\mathcal{N}, \{(\cdot,\cdot)_p \mid p \in \mathbb{N}\}\right)$, see e.g. [109, 64] and [7, Section 7]. In Theorem 4.22 we saw that for the Gaussian measure $\mu := \mu_b$ given by (4.7) the space $L^2(\mu)$ admits a unitary representation via $L^2(\mu) \cong \Gamma(\mathcal{H}_{\mathbb{C}})$. Our aim in this section is to construct a chain of embeddings as in (4.18) with the central Hilbert space $\mathcal{H}$ given by $L^2(\mu)$. Instead of the space $\mathcal{N}$ we obtain a space $(\mathcal{N})$ with similar properties.

First, we lift the chain (4.18) to arbitrary tensor powers of the complexified spaces. To this end, we have to show that for $n \in \mathbb{N}$ the scalar products of the Hilbert spaces $\mathcal{H}_{q,\mathbb{C}}^{\hat{\otimes}n}$ and $\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n}$ are compatible. But this is shown for example in [57, Chapter 3.B], therefore we obtain the following continuous embeddings

$$\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n} \subseteq \mathcal{H}_{q,\mathbb{C}}^{\hat{\otimes}n} \subseteq \mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n} \subseteq \mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n} \subseteq \mathcal{H}_{-p,\mathbb{C}}^{\hat{\otimes}n} \subseteq \mathcal{H}_{-q,\mathbb{C}}^{\hat{\otimes}n} \subseteq \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes}n}. \tag{4.20}$$

We define for $p \in \mathbb{N}_0$ the subspace $(\mathcal{H}_p)$ of $L^2(\mu)$ as

$$(\mathcal{H}_p) = \left\{ F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}:_b \right\rangle \mid f^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=0}^{\infty} n! \left\| f^{(n)} \right\|^2_{\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n}} < \infty \right\}$$

and for an element $F \in (\mathcal{H}_p)$ we define the norm $|||\cdot|||_p$ as

$$|||F|||_p^2 = \sum_{n=0}^{\infty} n! \left\| f^{(n)} \right\|^2_{\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n}}.$$

Note that for $p = 0$ the norm $|||\cdot|||_0$ is just the normal $L^2(\mu)$ norm $|||\cdot|||$. One directly sees that the norm $|||\cdot|||_p$ is induced by a scalar product which we denote by $((\cdot,\cdot))_p$. Further, we directly see that $(\mathcal{H}_p)$ is isometrically isomorphic to the weighted direct sum of the spaces $\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{N}_0$, i.e.,

$$(\mathcal{H}_p) \cong \oplus_{n\in\mathbb{N}_0} n! \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}. \tag{4.21}$$

In the theory of Dirichlet forms it is well known that the sum of closable forms is again closable, see e.g. [70, Proposition I.3.7]. Equivalently, one could say that the sum of compatible forms are again compatible. The next lemma follows by similar arguments and we refer for a proof to the last mentioned reference.

Lemma 4.28. *Let $0 \le p \le q$. Then it holds that the norms $|||\cdot|||_p$ and $|||\cdot|||_q$ are compatible on the space of polynomials $\mathcal{P}(\mathcal{N}')$.*

Hence we obtain the decreasing family of Hilbert spaces $\left((\mathcal{H}_p), (\!(\cdot,\cdot)\!)_p\right)$, $p \in \mathbb{N}_0$, in the sense that for $p \le q$ it holds we have a continuous embedding $(\mathcal{H}_q) \subseteq (\mathcal{H}_p)$.

In the following we need a technical lemma which can be proven in a straightforward way. Therefore, we skip the proof.

Lemma 4.29. *Let X_n, $n \in \mathbb{N}_0$, be Banach spaces over $\mathbb{C}$ and $w_n > 0$, $n \in \mathbb{N}_0$. Let $X = \oplus_{n\in\mathbb{N}_0} w_n X_n$ be the weighted direct sum of the spaces X_n with the weights $w_n > 0$, $n \in \mathbb{N}_0$, i.e.,*

$$X = \left\{ (x_n)_{n\in\mathbb{N}_0} \in \prod_{n\in\mathbb{N}_0} X_n \mid \sum_{n=0}^{\infty} w_n \|x_n\|_{X_n}^2 < \infty \right\}.$$

We equip the space X with the norm

$$\|x\|_X := \left(\sum_{n=0}^{\infty} w_n \|x_n\|_{X_n}^2 \right)^{\frac{1}{2}}, \quad x = (x_n)_{n\in\mathbb{N}_0} \in X.$$

It holds that the dual space X' of X is isometrically isomorphic to

$$X' \cong \left\{ (x_n)_{n\in\mathbb{N}_0} \in \prod_{n\in\mathbb{N}_0} X_n' \mid \sum_{n=0}^{\infty} w_n^{-1} \|x_n\|_{X_n'}^2 < \infty \right\}.$$

We obtain from Lemma 4.29 that the dual space of $\left((\mathcal{H}_p), (\!(\cdot,\cdot)\!)_p\right)$ is given by

$$(\mathcal{H}_{-p}) := (\mathcal{H}_p)' \cong \oplus_{n\in\mathbb{N}_0} n! \mathcal{H}_{-p,\mathbb{C}}^{\widehat{\otimes} n}.$$

A typical element $\Phi \in (\mathcal{H}_{-p})$ associated with a sequence $\left(\Phi^{(n)}\right)_{n\in\mathbb{N}_0} \in \oplus_{n\in\mathbb{N}_0} n! \mathcal{H}_{-p,\mathbb{C}}^{\widehat{\otimes} n}$ is denoted by

$$\Phi = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}:_b \right\rangle. \tag{4.22}$$

Obviously, the dual pairings in (4.22) are only formal notations. The representation (4.22) is called the generalized chaos decomposition of Φ.

We further define the space

$$(\mathcal{N}) := \cap_{p\in\mathbb{N}}(\mathcal{H}_p). \tag{4.23}$$

We endow the space $(\mathcal{N})$ with the restrictions of the norms $\left(|||\cdot|||_{p,r}\right)_{p\in\mathbb{N}_0}$ and the topology induced by them, analogue as in (4.1) above. By the definition given in (4.23) we obtain that $(\mathcal{N})$ is complete. The topological dual space $(\mathcal{N})'$ of $(\mathcal{N})$ satisfies

$$(\mathcal{N})' = \cup_{p\in\mathbb{N}}(\mathcal{H}_{-p}),$$

which is understood in the way described in the discussion after (4.4). We equip the space $(\mathcal{N})'$ with the inductive limit topology. We denote the dual pairing between $\varphi \in (\mathcal{N})$ and $\Phi \in (\mathcal{N})'$ by $\langle\langle \varphi, \Phi\rangle\rangle := \Phi(\varphi)$.

Eventually, we can lift the chain in (4.18) and obtain the following continuous and dense embeddings

$$(\mathcal{N}) \subseteq (\mathcal{H}_q) \subseteq (\mathcal{H}_p) \subseteq L^2(\mu) \subseteq (\mathcal{H}_{-p}) \subseteq (\mathcal{H}_{-q}) \subseteq (\mathcal{N})'.$$

The next lemma follows easily by the construction of the spaces $(\mathcal{N}), (\mathcal{H}_{p,r}), (\mathcal{H}_{-p,-r})$ and $(\mathcal{N})'$.

Lemma 4.30. *The polynomials $\mathcal{P}(\mathcal{N}')$ are dense in $(\mathcal{N}), (\mathcal{H}_p), (\mathcal{H}_{-p})$ and $(\mathcal{N})'$ w.r.t. their corresponding topology, respectively.*

The following theorem is taken from [109, Theorem 21]. It emphasizes the role of the topological space $(\mathcal{N}, \mathcal{T})$ in the construction of $(\mathcal{N})$.

Theorem 4.31. *$(\mathcal{N})$ is a nuclear space. The topology on $(\mathcal{N})$ is uniquely defined by the topology $\mathcal{T}$ on $\mathcal{N}$: It does not depend on the choice of the family of norms $\left(\|\cdot\|_p\right)_{p\in\mathbb{N}_0}$ on $\mathcal{N}$.*

Since $(\mathcal{N})$ is a nuclear space the results form the previous section apply to $(\mathcal{N})$ and its dual space $(\mathcal{N})'$. In particular, the strong topology on $(\mathcal{N})'$ coincides with the inductive limit topology, which we defined on $(\mathcal{N})'$ in the first place.

Remark 4.32. *If the norms $(\|\cdot\|_p)_{p\in N}$ do not satisfy the assumption in (4.19) then one defines instead of $(\mathcal{H}_p)$, $p \in \mathbb{N}$, the space $(\mathcal{H}_{p,r})$, $p, r \in \mathbb{N}$, given by*

$$(\mathcal{H}_{p,r}) = \left\{ F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}:_b \right\rangle \mid f^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes} n} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=0}^{\infty} n! 2^{nr} \left\| f^{(n)} \right\|^2_{\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes} n}} < \infty \right\}.$$

Then one can proceed in an analog way as above.

Definition 4.33. *We call the topological space $(\mathcal{N})$ and its dual space $(\mathcal{N})'$ the Hida test function space and the space of Hida distributions, respectively. The dual pairing between $F \in (\mathcal{N})$ and $\Phi \in (\mathcal{N})'$ is denoted by $\langle\langle F, \Phi\rangle\rangle := \Phi(F)$.*

Example 4.34. *(i) Let $f = f_1 + i f_2, g = g_1 + i g_2 \in \mathcal{N}_{\mathbb{C}}$, $f_1, f_2, g_1, g_2 \in \mathcal{N}$ and denote by*

$$b(f, g) := b(f_1, g_1) - b(f_2, g_2) + i(b(f_1, g_2) + b(f_2, g_1))$$

the $\mathbb{C}$-bilinear extension of b to $\mathcal{N}_{\mathbb{C}}$. The function

$$\mathcal{N}' \ni \omega \mapsto :\exp(\langle f, \cdot \rangle):_b (\omega) = \exp\left(\langle f, \omega\rangle - \frac{1}{2} b(f,f)\right)$$

determines an element in $L^2(\mu)$, which can be seen by Proposition 4.15. From the generating function of the Hermite polynomials, we conclude that

$$:\exp(\langle f, \cdot \rangle):_b := \sum_{n=0}^{\infty} \frac{1}{n!} H_{n,b(f,f)}(\langle f, \cdot \rangle) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle f^{\otimes n}, :\cdot^{\otimes n}:_b \right\rangle .$$

Hence, it holds

$$\left|\left|\left| :\exp(\langle f, \cdot \rangle):_b \right|\right|\right|_p^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \|f\|_p^{2n} = \exp(\|f\|_p^2) < \infty,$$

which implies $:\exp(\langle f, \cdot \rangle):_b \in (\mathcal{N})$. Since $(\mathcal{N})$ is a linear space, we also obtain that $\exp(i \langle f, \cdot \rangle) \in (\mathcal{N})$ for all $f \in \mathcal{N}_{\mathbb{C}}$.

(ii) *Let $(A, D(A))$ be a self-adjoint linear operator on $\mathcal{H}$ s.t. $\mathcal{N} \subseteq D(A)$ and $A : \mathcal{N} \longrightarrow \mathcal{H}$ is continuous. Define $tr_A \in \mathcal{N}'^{\hat{\otimes} 2}$ by $tr_A(f^{\otimes 2}) = (Af, f)_{\mathcal{H}}$, $f \in \mathcal{N}$. Then, we define $\Phi^{(2n)} = \frac{1}{2n!!} tr_A^{\otimes n}$ and $\Phi^{(2n-1)} = 0$, $n \in \mathbb{N}$, where $2n!!$ denotes the double factorial. The element Φ, given by its chaos decomposition*

$$\Phi = \sum_{n=0}^{\infty} \left\langle \Phi^{(2n)}, :\cdot^{\otimes 2n}:_b \right\rangle ,$$

constitutes an element of $(\mathcal{N})'$. If $(A, D(A))$ is not Hilbert-Schmidt then $\Phi \notin L^2(\mu)$.

II. Characterization Theorem

Definition 4.35. *Let $\Phi \in (\mathcal{N})'$. The S- and T-transform of Φ are defined by*

$$S\Phi : \mathcal{N} \to \mathbb{C}, f \mapsto \langle\!\langle :\exp(\langle f, \cdot \rangle):_b, \Phi \rangle\!\rangle, \tag{4.24}$$

$$T\Phi : \mathcal{N} \to \mathbb{C}, f \mapsto \langle\!\langle \exp(i \langle \cdot, f \rangle), \Phi \rangle\!\rangle, \tag{4.25}$$

where $:\exp(\langle f, \cdot \rangle):_b$, $\exp(i \langle f, \cdot \rangle) \in (\mathcal{N})$, for $f \in \mathcal{N}$.

If $\Phi \in (\mathcal{N})'$ has the generalized chaos decomposition $\Phi = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}: \right\rangle$ then for its S-transform $S\Phi$ it holds via Example 4.34

$$S\Phi(f) = \sum_{n=0}^{\infty} \left\langle f^{\otimes n}, \Phi^{(n)} \right\rangle, \quad f \in \mathcal{N}. \tag{4.26}$$

Definition 4.36. *A map $U : \mathcal{N} \to \mathbb{C}$ is called a U-functional if the following conditions are fulfilled*

(U1) *U is ray-analytic, i.e., for all $f_1, f_2 \in \mathcal{N}$, the function*

$$\mathbb{R} \ni \lambda \mapsto U(f_1 + \lambda f_2)$$

is analytic and extends to an entire function on $\mathbb{C}$.

(U2) U is uniformly bounded of exponential order 2, i.e., there exist $A, B \geq 0$ and $p \in \mathbb{N}_0$ s.t. for all $f \in \mathcal{N}$ and $z \in \mathbb{C}$ it holds

$$|U(zf)| \leq A \exp\left(B\,|z|^2\,\|f\|_p^2\right), \tag{4.27}$$

Now, we can state the famous characterization theorem for the distributions $(\mathcal{N})'$. For a proof see [63, Theorem 11., Corollary 12.] or [65].

Theorem 4.37. *The S-transform is a bijection between $(\mathcal{N})'$ and the set of U-functionals. In particular, let U be an U-functional satisfying (4.27) and let $\Phi \in (\mathcal{N})'$ s.t. $S\Phi = U$. Let $q \in \mathbb{N}$ s.t. $I_{q,p} : \mathcal{H}_q \longrightarrow \mathcal{H}_p$ is Hilbert-Schmidt and $\rho := e^2 2B \left\|I_{q,p}\right\|_{H.-S.}^2 < 1$, where $\left\|I_{q,p}\right\|_{H.-S.}$ denotes the Hilbert-Schmidt norm of $I_{q,p}$. Then, it holds that $\Phi \in (\mathcal{H}_{-q})$ and*

$$\|\Phi\|_{-q} \leq A(1-\rho)^{\frac{1}{2}}. \tag{4.28}$$

Remark 4.38. *(i) Due to property (U1) in Definition 4.36 every U-functional $U : \mathcal{N} \longrightarrow C$ admits a natural extension to $\mathcal{N}_{\mathbb{C}}$. In particular, for $\Phi \in (\mathcal{N})'$ s.t. $S\Phi = U$ then this extension is given by*

$$U(f_1 + if_2) = \sum_{n=0}^{\infty} \left\langle (f_1 + if_2)^{\otimes n}, \Phi^{(n)} \right\rangle = \langle\!\langle :\exp\left(\langle f_1 + if_2, \cdot\rangle\right):_b, \Phi \rangle\!\rangle, \quad f_1, f_2 \in \mathcal{N}. \tag{4.29}$$

(ii) Observe that for $\Phi \in (\mathcal{N})'$ the S- and T-transform are related via

$$T\Phi(f) = S\Phi(if) \exp\left(-\frac{1}{2} b(f,f)\right), \text{ for all } f \in \mathcal{N}_{\mathbb{C}}. \tag{4.30}$$

Hence, the T-transform is also a bijection between $(\mathcal{N})'$ and the set of U-functionals.

Using Lemma 4.7 we obtain a useful Corollary of Theorem 4.37.

Corollary 4.39. *Let $(\Phi_n)_{n\in\mathbb{N}} \in (\mathcal{N})'$ s.t.*

(i) For every $f \in \mathcal{N}$, $(S\Phi_n(f))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$,

(ii) there exist $A, B \geq 0$ and $p \in \mathbb{N}_0$ s.t. for all $n \in \mathbb{N}$, $f \in \mathcal{N}$ and $\lambda \in \mathbb{C}$ it holds

$$|S\Phi_n(\lambda f)| \leq A \exp\left(B\,|\lambda|^2\,\|f\|_p^2\right), \tag{4.31}$$

.

Then, there exists $\Phi \in (\mathcal{N})$ s.t. $\lim_{n\to\infty} \Phi_n = \Phi$ in $(\mathcal{N})'$.

Proof. Due to (4.31) and (4.28) we obtain that the sequence $(\Phi_n)_{n\in\mathbb{N}}$ is bounded in $(\mathcal{H}_{-q})$ for some $q \in \mathbb{N}$. Hence, due to the Banach–Alaoglu theorem we obtain that every subsequence of $(\Phi_n)_{n\in\mathbb{N}}$ admits a further subsequence which is weakly convergent. Due to (i) each of these limits has the same S-transform, therefore they must coincide. Hence, $(\Phi_n)_{n\in\mathbb{N}}$ is weakly convergent in $(\mathcal{H}_{-q})$ and therefore weakly convergent in $(\mathcal{N})'$. Due to Lemma 4.7 we obtain that $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{N})'$ which finishes the proof. □

Observe that the conditions (i) and (ii) in the previous Corollary are also necessary for a convergent sequence $(\Phi_n)_{n\in\mathbb{N}} \subseteq (\mathcal{N})'$.

Corollary 4.40. *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and $\Phi(\cdot) : \Omega \longrightarrow (\mathcal{N})'$. Assume that $\Phi(\cdot)$ is weakly measurable, i.e., for every $F \in (\mathcal{N})$ it holds $\Omega \ni \omega \mapsto \langle\langle F, \Phi(\omega)\rangle\rangle \in \mathbb{C}$ is measurable. Assume further there exists $p \in \mathbb{N}$, $A \in L^1(\Omega, \nu)$ and $B \in L^\infty(\Omega, \nu)$ s.t.*

$$|S(\Phi(\omega))(zf)| \leq A(\omega) \exp\left(B(\omega)\, |z|^2 \, \|f\|_p\right), \text{ for all } z \in \mathbb{C}, f \in \mathcal{N}.$$

Then there exists $q \in \mathbb{N}$ s.t. the map $\Phi(\cdot)$ is Bochner integrable in $(\mathcal{H}_{-q})$.

Proof. See e.g. [65] □

Remark 4.41. *The statements in Corollary 4.39 Corollary 4.40 and are also true if we replace the S-transform by the T-transform, respectively. Observe that the pointwise product of two U-functionals U_1 and U_2 is again a U-functional. Hence, we can formulate the following definition.*

Definition 4.42. *Let $\Phi, \Psi \in (\mathcal{N})'$. Then $\Phi \diamond \Psi := S^{-1}(S\Phi S\Psi) \in (\mathcal{N})'$ is called the Wick product of Φ and Ψ. Similar we can define $\Phi * \Psi := T^{-1}(T\Phi T\Psi)$. The product $\Phi * \Psi$ is called the convolution of Φ and Ψ.*

From Corollary 4.39 one obtains that $\diamond : (\mathcal{N})' \times (\mathcal{N})' \longrightarrow (\mathcal{N})'$ is separately continuous, hence jointly continuous.

Example 4.43. *Let $\Psi = \left\langle \Psi^{(m)}, :\cdot^{\otimes m}:\right\rangle, \Phi = \left\langle \Phi^{(n)}, :\cdot^{\otimes n}:\right\rangle, \in (\mathcal{N})', \Psi^{(m)} \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} m}$ and $\Phi^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} n}$, $n, m \in \mathbb{N}$. Then the Wick product of Φ and Ψ is given by $\Phi \diamond \Psi = \left\langle \Phi^{(n)} \widehat{\otimes} \Psi^m, :\cdot^{\otimes n+m}:\right\rangle$, with $\Phi^{(n)} \widehat{\otimes} \Psi^m \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} m+n}$.*

Remark 4.44. *The Wick product can be considered as a renormalization of the pointwise product. To make this more precise, let us chose $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R}) := L^2(\mathbb{R}, dx)$. Further, denote by $\delta_0 \in S'(\mathbb{R})$ the Dirac distribution in zero. Then it holds $\langle \delta_0, \cdot \rangle \in (\mathcal{N})' \setminus L^2(\mu)$. In particular, $\langle \delta_0, \cdot \rangle \langle \delta_0, \cdot \rangle$ has no meaning even as an element in $(\mathcal{N})'$. Now define $f_n(x) := \frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. In particular, $f_n \in S(\mathbb{R})$ and $f_n \xrightarrow{n\to\infty} \delta_0$ in $S'(\mathbb{R})$. From Lemma 4.27 we obtain*

$$\begin{aligned} \langle f_n, \cdot \rangle \langle f_n, \cdot \rangle &= \left\langle f_n^{\otimes 2}, :\cdot^{\otimes 2}:\right\rangle + (f_n, f_n)_{L^2(\mathbb{R})} \\ &= \langle f_n, \cdot \rangle \diamond \langle f_n, \cdot \rangle + (f_n, f_n)_{L^2(\mathbb{R})}. \end{aligned}$$

Since $(f_n, f_n)_{L^2(\mathbb{R})} \xrightarrow{n\to\infty} \infty$ one can consider the Wick product as a renormalization to make the pointwise product of distributions well-defined.

Example 4.45.

1. *Let $F \in L^2(\mu)$ be given. Then we obtain by Lemma 4.17 that the S-transform of F is given by*

$$SF(f) = \langle\langle :\exp(\langle f, \cdot \rangle):_b, F \rangle\rangle = \int_{\mathcal{N}'} :\exp(\langle f, \cdot \rangle):_b \, F \, d\mu = \int_{\mathcal{N}'} F(\cdot + f)\, d\mu, \quad f \in \mathcal{N}.$$

2. *Let $a \in \mathbb{R}$, $g \in \mathcal{N}$ and define*

$$U_{a,g}(f) := \frac{1}{\sqrt{2\pi b(g,g)}} \exp\left(-\frac{1}{2b(g,g)}(b(f,g) - a)^2\right), \quad f \in \mathcal{N}. \tag{4.32}$$

One easily checks that $U_{a,g}$ is U-functional. Hence, there exists a element $\Phi \in (\mathcal{N})'$ s.t. $S\Phi = U_{a,g}$. The element Φ is denoted by $\delta_a(\langle g, \cdot\rangle)$ and is called Donsker's delta.

III. Properties of Hida Test functions and Differential Operators

In this part we collect some further properties of $(\mathcal{N})$. So far, we constructed $(\mathcal{N})$ as a subspace of $L^2(\mu)$. In particular, the elements of $(\mathcal{N})$ are equivalence classes w.r.t. μ. We see below that the classes of $(\mathcal{N})$ have a nice representative which is continuous (see, Theorem 4.46) and is moreover infinitely Gateaux differentiable (see, Theorem 4.51).

For this purpose, we need to introduce some technical definitions. Let $n, k \in \mathbb{N}$, $k \leq n$, and $f^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ and $y^{(k)} \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} k}$. Then, we define $\left\langle f^{(n)}, y^{(k)}\right\rangle \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n-k}$ via

$$\left\langle \left\langle f^{(n)}, y^{(k)}\right\rangle, \Phi^{(n-k)}\right\rangle = \left\langle f^{(n)}, y^{(k)} \widehat{\otimes} \Phi^{(n-k)}\right\rangle,$$

where $\Phi^{(n-k)} \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} n-k}$. Observe that this definition is well-defined, since both nuclear spaces and their symmetric tensor powers are reflexive, see e.g. [93, Section IV.5]. Furthermore, if $y^{(k)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} k}$, then $\left\langle f^{(n)}, y^{(k)}\right\rangle$ coincides with the tensor contraction $f^{(n)} \widehat{\otimes}_k y^{(k)}$ given in (4.15).

Theorem 4.46. *Every element $F \in (\mathcal{N})$ has a unique representative $\tilde{F} : \mathcal{N}' \longrightarrow \mathbb{C}$ which is continuous w.r.t. the strong topology β_s on $\mathcal{N}'$. In particular, if $F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}:\right\rangle \in (\mathcal{N})$, then $f^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ for every $n \in \mathbb{N}$ and it holds*

$$\tilde{F}(\omega) = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\omega^{\otimes n}:\right\rangle = \sum_{n=0}^{\infty} \left\langle g^{(n)}, \omega^{\otimes n}\right\rangle, \tag{4.33}$$

where $g^{(n)}$ is given by

$$g^{(n)} = \sum_{k=0}^{\infty} \binom{n+2k}{2k} (2k-1)!!(-1)^k \left\langle f^{(n+2k)}, tr_b^{\otimes k}\right\rangle \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad n \geq 0.$$

Proof. See [65, Theorem 6.5.] and its proof. □

In the following we often don't distinguish between an element $F \in (\mathcal{N})$ and its continuous version $\tilde{F}$ and denote them by the same symbol. Observe that the polarization identity implies that the family $(g^{(n)})_{n \in \mathbb{N}_0}$ satisfying (4.33) is uniquely determined by $F \in (\mathcal{N})$. One can even use the family $(g^{(n)})_{n \in \mathbb{N}_0}$ to obtain an equivalent topological description of $(\mathcal{N})$. Let $q \in \mathbb{N}$, $F \in (\mathcal{N})$ and $(g^{(n)})_{n \in \mathbb{N}_0}$, $g^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \geq 0$, satisfy (4.33). Then, define

$$\widetilde{\left|\!\left|\!\left| F \right|\!\right|\!\right|}_q^2 := \sum_{n=0}^{\infty} n! \left\| g^{(n)} \right\|_q^2,$$

where $\left\| g^{(n)} \right\|_q$ is the $\mathcal{H}_{q,\mathbb{C}}^{\otimes n}$-norm of $g^{(n)}$. The following lemma shows that the family $\left(\widetilde{\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|}_q\right)_{q \in \mathbb{N}}$ induces the original locally convex topology on $(\mathcal{N})$.

Lemma 4.47. *The families $\left(\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|_p\right)_{p \in \mathbb{N}}$ and $\left(\widetilde{\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|}_q\right)_{q \in \mathbb{N}}$ induce the same topology on $(\mathcal{N})$. Indeed,* *for every $p, q \in \mathbb{N}$ there exists $p_1, q_1 \in \mathbb{N}$ and $C_1, C_2 \in (0, \infty)$ s.t.*

$$\widetilde{|||\cdot|||}_q \leq C_2 |||\cdot|||_{p_1}, \tag{4.34}$$

$$|||\cdot|||_p \leq C_1 \widetilde{|||\cdot|||}_{q_1}, . \tag{4.35}$$

Proof. The statement in (4.34) coincides with [65, Theorem 6.2.]. To show (4.35) we use the fact that every element $F \in (\mathcal{N})$ restricted to $\mathcal{H}_{-p}$, $p \in \mathbb{N}$, is entire analytic, see e.g. [57, Chapter 4.D]. Furthermore, by [57, Proposition 4.59.] for every $p \in \mathbb{N}$ there exists a positive finite constant C and $q \in \mathbb{N}$ such that

$$\|F\|_p \leq C \sup_{x \in \mathcal{H}_{-q}} |F(x)| \exp\left(-\frac{1}{2}\|x\|_{-q}^2\right). \tag{4.36}$$

Finally, due to [65, Theorem 6.8.] it holds for all $q \in \mathbb{N}$ and $x \in \mathcal{H}_{-q}$

$$|F(x)| \leq \widetilde{\|F\|}_q \exp\left(\frac{1}{2}\|x\|_{-q}^2\right),$$

which together with (4.36) proves the inequality (4.35). □

Corollary 4.48. *Let $T : \mathcal{N} \longrightarrow \mathcal{N}$ be continuous and linear and denote by T^* its adjoint mapping continuously from $\mathcal{N}'$ into itself w.r.t. the weak topology. Then,*

$$\widetilde{\Gamma}(T) : (\mathcal{N}) \longrightarrow (\mathcal{N}), F \mapsto F \circ T^*$$

is well defined and continuous.

Proof. This follows easily by using the norms $\widetilde{|||\cdot|||}_q$, $q \in \mathbb{N}$. □

Thanks to Theorem 4.46 we can define the pointwise product of two elements $F, G \in (\mathcal{N})$. Furthermore, this product defines a continuous bilinear form on $(\mathcal{N})$. We summarize this in the following theorem. For a proof we refer to [65, 57, 79].

Theorem 4.49. *The pointwise defined product on $(\mathcal{N})$ is a continuous bilinear form with values in $(\mathcal{N})$. Indeed, for every $p \in \mathbb{N}$ there exists a $q \in \mathbb{N}$ s.t. for all $F, G \in (\mathcal{N})$ their pointwise defined product satisfies $|||FG|||_p \leq |||F|||_q |||G|||_q$. Furthermore, $(\mathcal{N})$ is stable under complex conjugation.*

Let $F \in (\mathcal{N})$ and $\Phi \in (\mathcal{N})'$ then we can define $F\Phi \in (\mathcal{N})'$ via $\langle\langle G, F\Phi\rangle\rangle := \langle\langle FG, \Phi\rangle\rangle$, $G \in (\mathcal{N})'$. In particular, we obtain the following corollary.

Corollary 4.50. *Let $F \in (\mathcal{N})$. Then $Q_F : (\mathcal{N})' \longrightarrow (\mathcal{N})', \Phi \mapsto F\Phi$ is linear and continuous.*

In the following we present some result concerning the differentiability of the continuous version of elements $F \in (\mathcal{N})$.

Theorem 4.51. *Let $x, y \in \mathcal{N}'$ and $F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}:\right\rangle \in (\mathcal{N})$. Then it holds that the continuous version of F is Gateaux differentiable at x in the direction of y. Furthermore, if we consider the*

Gateaux derivative of F in the direction of y as a function on $\mathcal{N}'$, then it holds

$$\partial_y F(\cdot) = \sum_{n=1}^{\infty} n \left\langle \left\langle f^{(n)}, y \right\rangle, :\cdot^{\otimes n-1}: \right\rangle \in (\mathcal{N}).$$

Furthermore, ∂_y defines a continuous operator on $(\mathcal{N})$, i.e., for every $p \in \mathbb{N}$ s.t. $y \in \mathcal{H}_{-p}$ there exists a $q \in \mathbb{N}$ and a positive constant $C = C_{p,q}$ (both independent of F) s.t.

$$\left\|\left\| \partial_y F \right\|\right\|_p \leq C \left\| y \right\|_{-p} \left\|\left\| F \right\|\right\|_q .$$

In particular, elements from $(\mathcal{N})$ are infinitely often Gateaux differentiable w.r.t all directions from $\mathcal{N}'$.

Similar as in the theory of tempered distributions one can define the derivative of a distribution $\Phi \in (\mathcal{N})'$. To this end, we need to introduce a technical definition. Let $n \in \mathbb{N}$, $\Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ and $h \in \mathcal{N}$. Then, we define $\left\langle h, \Phi^{(n)} \right\rangle \in \mathcal{N}'^{\widehat{\otimes} n-1}_{\mathbb{C}}$ via $\left\langle f^{(n-1)}, \left\langle h, \Phi^{(n)} \right\rangle \right\rangle = \left\langle f^{(n-1)} \widehat{\otimes} h, \Phi^{(n)} \right\rangle$, for all $f^{(n-1)} \in \mathcal{N}^{\widehat{\otimes} n-1}_{\mathbb{C}}$. Further, recall that the test functions $(\mathcal{N})$ are dense in $(\mathcal{N})'$.

Theorem 4.52. *For $h \in \mathcal{N}$ the operator ∂_h admits a continuous and linear extension to an operator $\widetilde{\partial}_h$ from $(\mathcal{N})'$ into itself, i.e., for $\Phi = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}: \right\rangle \in (\mathcal{N})'$ it holds*

$$\widetilde{\partial}_h \Phi = \sum_{n=1}^{\infty} n \left\langle \left\langle h, \Phi^{(n)} \right\rangle, :\cdot^{\otimes n-1}: \right\rangle \in (\mathcal{N})'. \tag{4.37}$$

In particular, if $\Phi \in (\mathcal{H}_{-p})$ for $p \in \mathbb{N}$ then there exists $q \in \mathbb{N}$ and $C = Cq, p \in (0, \infty)$ (both independent of Φ) s.t. it holds $\widetilde{\partial}_h \Phi \in (\mathcal{H}_{-q})$ and

$$\left\|\left\| \widetilde{\partial}_h \Phi \right\|\right\|_{-q} \leq C \left\| h \right\|_q \left\|\left\| \Phi \right\|\right\|_{-p} .$$

We call the distribution $\widetilde{\partial}_h \Phi$, $\Phi \in (\mathcal{N})'$, $h \in \mathcal{N}$, the generalized Gateaux derivative of Φ in the direction of h. In particular, elements of $(\mathcal{N})'$ are infinitely often Gateaux differentiable in the generalized sense. For $h \in \mathcal{N}$ we have due to Theorem 4.51 a linear and continuous operator $\partial_h : (\mathcal{N}) \longrightarrow (\mathcal{N})$. Hence, its adjoint operator $\partial_h^* : (\mathcal{N})' \longrightarrow (\mathcal{N})', \Phi \mapsto \Phi \circ \partial_h$, is also linear and continuous, see e.g. [65, Theorem 9.11.]. The next theorem shows how the operators ∂_h^* and $\widetilde{\partial}_h$ are related.

Theorem 4.53. *Let $h \in \mathcal{N}$. Then, it holds*

$$\partial_h^* = -\widetilde{\partial}_h + Q_{\langle h, \cdot \rangle}, \tag{4.38}$$

as equality of linear and continuous operators on $(\mathcal{N})'$.

Proof. See e.g. [65, Theorem 9.18.]. □

Observe that (4.38) is simply a generalization of the integration by parts formula for the Gaussian measure μ. Indeed, for $F, G \in (\mathcal{N})$ and $h \in \mathcal{N}$ we have the following equality

$$\int_{\mathcal{N}'} \partial_h F G \, d\mu = - \int_{\mathcal{N}'} F \partial_h G \, d\mu + \int_{\mathcal{N}'} FG \langle h, \cdot \rangle \, d\mu. \tag{4.39}$$

Formula (4.39) can be seen by considering first the case $G = 1$. Then one simply has to calculate the 0-th element in the chaos expansion of $\partial_h F$ and $F \langle h, \cdot \rangle$ which can be done via Lemma 4.27 and Theorem 4.51. For general $G \in (\mathcal{N})$ one can simply use the product rule for Gateaux derivatives. Note that, the integration by parts formula (4.39) can be used to obtain a very easy proof for Wick's theorem. Fur this purpose, recall that a pairing σ of an even numbered set A is a partition of A into disjoint subsets $\sigma_1, \ldots, \sigma_n$, where each subset $\sigma_i = \{\sigma_i^1, \sigma_i^2\}$, $i = 1, \ldots, n$, has cardinality equal to 2.

Theorem 4.54. *Let $f_1, \ldots, f_{2n} \in \mathcal{N}$, $n \in \mathbb{N}$. Then, it holds*

$$\int_{\mathcal{N}'} \prod_{i=1}^{2n} \langle f_i, \cdot \rangle \, d\mu = \sum_{\sigma \, pairing} \prod_{i=1}^{n} (f_{\sigma_i^1}, f_{\sigma_i^2})_{\mathcal{H}}, \tag{4.40}$$

where the sum on the right-hand side extends over all $(2n-1)!! = \frac{(2n)!}{2^n n!}$ pairings σ of $\{1, \ldots, 2n\}$.

We conclude this section with an important subset of $(\mathcal{N})'$, i.e., the positive distributions.

Definition 4.55. *An element $\Phi \in (\mathcal{N})'$ is called positive if for all $\varphi \in (\mathcal{N})$ with continuous version $\tilde{\varphi} \geq 0$ it holds $\langle\langle \Phi, \varphi \rangle\rangle \geq 0$. The set of all positive elements in $(\mathcal{N})'$ is denoted by $(\mathcal{N})'_+$.*

The next theorem is a consequence of the Bochner-Minlos Theorem 4.13. For details see e.g. [57, Theorem 4.26 and Theorem 4.28] and the references therein.

Theorem 4.56. *An element $\Phi \in (\mathcal{N})'$ is positive, if and only if its T-transform $T\Phi : \mathcal{N} \longrightarrow \mathbb{C}$ is positive definite. In that case there exists a finite measure ν_Φ on $(\mathcal{N}', \mathcal{B})$ s.t.*

$$\langle\langle \Phi, \varphi \rangle\rangle = \int_{\mathcal{N}'} \tilde{\varphi} \, d\nu_\Phi, \qquad \text{for all } \varphi \in (\mathcal{N}). \tag{4.41}$$

Definition 4.57. *The set of all finite measure measures ν on $(\mathcal{N}', \mathcal{B})$ which correspond to a Hida distribution in the sense of (4.41) we call the set of Hida measures. We also denote the set of all Hida measures by $(\mathcal{N})'_+$.*

Remark 4.58. *(i) The set $(\mathcal{N})'_+$ is closed in the weak topology of $(\mathcal{N})'$.*

(ii) Let ν be a Hida measure with corresponding distribution $\Phi \in (\mathcal{N})'$. Then there exists $p \in \mathbb{N}$ s.t. $\Phi \in (\mathcal{H}_{-p})$. Hence, the characteristic function of ν is continuous w.r.t. one norm $\|\cdot\|_{p'}$, $p' \in \mathbb{N}$, on $\mathcal{N}$. In particular, from the Theorem 4.13 we obtain that ν is supported by some $\mathcal{H}_{-q}$, $q \in \mathbb{N}$, i.e., $\nu(\mathcal{H}_{-q}) = 1$.

(iii) The map $(\mathcal{N})'_+ \ni \Phi \mapsto \nu_\Phi$ is injective, since the T-transform determines a element in $(\mathcal{N})'$ uniquely.

(iv) The same proof as in Proposition 4.21 shows that for $\nu \in (\mathcal{N})'_+$ the polynomials $\mathcal{P}(\mathcal{N}')$ are dense in $L^2(\nu)$.

Example 4.59. *1. Let $F \in L^2(\mu)$ s.t. $F \geq 0$ μ-a.e.. If we consider F as an element of $(\mathcal{N})'$ it holds $F \in (\mathcal{N})'_+$.*

2. Donsker's Delta $\delta_a(\langle g, \cdot \rangle)$, defined in Example 4.45(ii), is positive. This can be seen via (4.30) and the previous theorem.

Chapter 5

An improved Characterization of regular generalized Functions of White Noise

In this chapter we state and prove an improved version of the characterization of the spaces $(\mathcal{G}_K, \mathcal{G}'_K)$ introduced in [52]. The dual pair $(\mathcal{G}_K, \mathcal{G}'_K)$ constitutes a rigging of the Hilbert space $L^2(\mathcal{N}', \mu)$, i.e., we have continuous embeddings

$$\mathcal{G}_K \subseteq L^2(\mathcal{N}'\mu) \subseteq \mathcal{G}'_K.$$

Here, $\mathcal{N} \subseteq \mathcal{H} \subseteq \mathcal{N}'$ is a nuclear rigging of a real Hilbert space $\mathcal{H}$. The measure μ is the Gaussian measure on $\mathcal{N}'$ with variance given by the scalar product $(\cdot, \cdot)_\mathcal{H}$. The symbols K represents a self-adjoint operator $(K, D(K))$ defined on $\mathcal{H}$. To formulate our result we use the concept of U-functionals and the characterization theorem of Hida distributions given in Theorem 4.37. Therefore, the proof and the statement of our improved version does not use the concept of entire function on a complex infinite dimensional Hilbert space explicitly in contrast to the result in [52]. The elements of $\mathcal{G}'_K$ are characterized via an integrability condition of their respective S-transform, see Definition 4.35. Especially from the point of view of applications this is indeed an improvement. In applications many distributions in Gaussian analysis and in particular White noise analysis are constructed and defined via their corresponding U-functional.

The organization of this chapter is as follows. In Section 5.1 we briefly describe the functional analytic framework we use throughout this chapter. In particular we define the space $\mathcal{G}_K := \cap_{s=1} D(\Gamma(K)^s)$, where $(\Gamma(K), D(\Gamma(K)))$ denotes the second quantization of $(K, D(K))$ defined on $L^2(\mu)$. Consequently, we obtain the dual space $\mathcal{G}'_K$ given as $\mathcal{G}'_K := \cup_{s=1}^\infty D(\Gamma(K)^{-s})$, where for $s \in \mathbb{N}$ the space $D(\Gamma(K)^{-s})$ denotes the completion of $L^2(\mu)$ w.r.t. the weaker norm $\|\Gamma(K)^{-s} \cdot\|_{L^2(\mu)}$. We denote by $\mathcal{G}_{K,s}$ the space $D(\Gamma(K)^s$ for all $s \in \mathbb{Z}$. In Section5.2 we briefly recall some relevant results from complex Gaussian analysis. We also explain the characterization given in [52]. This is necessary to distinguish our results to the ones presented in [52]. Section 5.3 contains the main result including its proof. Finally, in Section 5.4 we show an application of our main result to a stochastic parabolic differential equation with possibly singular coefficients. To this end we consider the pair $(\mathcal{G}, \mathcal{G}')$ introduced in [87], which is given by the choice $K = \sqrt{2}I$ and $\mathcal{H} = L^2(\mathbb{R})$. We use the characterization theorem to determine explicitly the regularity of the solution in terms of the coefficients. Namely, we state a criteria on the coefficients to determine explicitly to which of the space $\mathcal{G}_{K,s}$, $s \in \mathbb{Z}$, the solution belongs.

5.1 Definition of Regular Random Variables and their Dual Space

Throughout this entire chapter we work with the functional analytic framework given in Subsection 4.2.3. For the sake of consistency, we recall some objects introduced above. We fix a real nuclear countable Hilbert space $(\mathcal{N}, \{(\cdot,\cdot)_p \mid p \in \mathbb{N}\})$. For sake of simplicity, we also assume that the norms induced by $\{(\cdot,\cdot)_p \mid p \in \mathbb{N}\}$ satisfy the additional assumption given in (4.19). The results proven below also hold for a general nuclear space $\mathcal{N}$, see Remark 4.32. Additionally, we denote by $b = (\cdot,\cdot)$ a positive definite, symmetric, non-degenerated and continuous bilinear form defined on $\mathcal{N}$ which is compatible with every $(\cdot,\cdot)_p$, $p \in \mathbb{N}$. We can assume w.l.o.g. $(\cdot,\cdot)^{\frac{1}{2}} \le (\cdot,\cdot)_p^{\frac{1}{2}}$ for every $p \in \mathbb{N}$. The separable Hilbert space given as the completion of $\mathcal{N}$ w.r.t the norm $\|\cdot\|$ induced by $(\cdot,\cdot)$ is denoted by $\mathcal{H}$. Further, we denote the norm and scalar product on $\mathcal{H}$ also by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Via the Bochner-Minlos Theorem, we define the measure μ_{σ^2}, $\sigma^2 > 0$, on the Borel σ-field of the dual space $\mathcal{N}'$, see Remark 4.12, through

$$\int_{\mathcal{N}'} \exp(i\,\langle \cdot, f\rangle)\, d\mu_{\sigma^2} = \exp\left(-\frac{\sigma^2}{2}(f,f)\right), \quad f \in \mathcal{N}. \tag{5.1}$$

We simply write μ for the measure μ_1. We also recall the decomposition of the space $L^2(\mu)$ given by the Wiener-Itô-Segal isomorphism as

$$L^2(\mu) \cong \oplus_{n\in\mathbb{N}_0} n!\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n} =: \Gamma_{\mathbb{C}}(\mathcal{H}).$$

In accordance with the notation introduced in the previous chapter, we also denote the norm on $L^2(\mu)$ by $|||\cdot|||$. Throughout this entire chapter we fix also a linear operator $(K, D(K))$ on $\mathcal{H}$ which satisfies the following assumption.

Assumption 5.1. *The linear operator $(K, D(K))$ is self-adjoint and fulfills*

$(K1)$ $\sigma(K) \subseteq [1,\infty)$,

$(K2)$ *$\mathcal{N}$ forms a core for $(K, D(K))$,*

$(K3)$ *$K : (\mathcal{N}, \tau) \longrightarrow (\mathcal{N}, \tau)$ is continuous and bijective.*

Remark 5.2. *The inverse mapping theorem implies that K^s is continuous for every $s \in \mathbb{Z}$, see [92, Corollary I.2.12(b)].*

Example 5.3. *(i) An interesting choice is $K = \lambda Id$, where $\lambda \in (1,\infty)$. This choice leads to the pair of spaces $(\mathcal{G}, \mathcal{G}')$ introduced in [87].*

(ii) For $\mathcal{N} = \mathcal{S}(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$ the operator $1 - \Delta$, where $\Delta = \frac{d^2}{dx^2}$, is closable on $\mathcal{S}(\mathbb{R})$. Its closure $K = \overline{1 - \Delta}$ satisfies $(K1) - (K3)$.

In the following let $s \in \mathbb{Z}$. For the sake of better readability, we simply write K^s for the complexification $(K^s)_{\mathbb{C}}$. Similarly, we denote by $(K^s)^{\otimes n}$ the n-fold tensor power of $(K^s)_{\mathbb{C}}$ defined on the n-fold complex symmetric tensor power with domain $D((K^s)^{\otimes n}) \subseteq \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. Via the concept of second quantization we can lift the operator K^s to an operator $\Gamma(K^s)$ on the space $L^2(\mu)$. More precisely, we define the linear operator $(\Gamma(K^s), D(\Gamma(K^s)))$ via

$$D(\Gamma(K^s)) = \left\{ F = \sum_{n\in\mathbb{N}} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \in L^2(\mu) \middle| f^{(n)} \in D((K^s)^{\otimes n}), \sum_{n\in\mathbb{N}} n! \|(K^s)^{\otimes n} f^{(n)}\|^2_{\mathcal{H}} < \infty \right\},$$
$$\Gamma(K^s) F = \sum_{n\in\mathbb{N}} \left\langle (K^s)^{\otimes n} f^{(n)}, :\cdot^{\otimes n}: \right\rangle, \quad F \in D(\Gamma(K^s)).$$

Remark 5.4. *In the definition of $(\Gamma(K^s), D(\Gamma(K^s)))$ one could also allow $s \in \mathbb{R}$ instead of $s \in \mathbb{Z}$ by using the spectral theorem for self-adjoint operators. Later, we need that K^s maps $\mathcal{N}$ continuously into itself. So far it is not known to the author whether in general fractional powers K^s, $s \in \mathbb{R}$, of an operator K satisfying $(K1) - (K3)$ leave the space $\mathcal{N}$ invariant. For the operators given in Example 5.3 this condition is satisfied. In particular, for the operator in 5.3(ii) it can be seen by using the Fourier transform. Hence, if the operator K satisfy this additional constraint everything presented below generalizes by the exact same argument for $s \in \mathbb{R}$.*

The operator $(\Gamma(K^s), D(\Gamma(K^s)))$ inherits important properties of K, some of them we collect in the following lemma.

Lemma 5.5. *For every $s \in \mathbb{N}$ it holds $(\Gamma(K^s), D(\Gamma(K^s)))$ is self-adjoint and its spectrum satisfies $\sigma(\Gamma(K^s)) \subseteq [1, \infty)$. Additionally it holds that $\mathcal{P}(\mathcal{N}')$ forms a core for $(\Gamma(K^s), D(\Gamma(K^s)))$ and the map $\Gamma(K^s) : (\mathcal{N}) \longrightarrow (\mathcal{N})$ is continuous and bijective.*

Proof. It suffices to show the claim for $s = 1$. Otherwise we replace K by K^s. First we observe that $\left(K^{\otimes n}, D(K^{\otimes n})\right)$ is self-adjoint on $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ and $\sigma\left(K^{\otimes n}\right) \subseteq [1, \infty)$, see also [91, Corollary VIII.10]. Therefore, it follows immediately that $(\Gamma(K), D(\Gamma(K)))$ is symmetric. Using Fatou's lemma we also see that $(\Gamma(K), D(\Gamma(K)))$ is closed. Now let $\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$. Then it holds $\lambda + i\mu \in \rho(K^{\otimes n})$ and the operator norm of $\left(\lambda + i\mu - K^{\otimes n}\right)^{-1}$ can be estimated by $\left\| \left(\lambda + i\mu - K^{\otimes n}\right)^{-1} \right\| \leq \frac{1}{|\mu|}$. Now we define the operator $\oplus_{n=0}^{\infty} \left(\lambda + i\mu - K^{\otimes n}\right)^{-1}$ via

$$\oplus_{n=0}^{\infty} \left(\lambda + i\mu - K^{\otimes n}\right)^{-1} \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle := \sum_{n=0}^{\infty} \left\langle \left(\lambda + i\mu - K^{\otimes n}\right)^{-1} f^{(n)}, :\cdot^{\otimes n}: \right\rangle,$$

where $\sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \in L^2(\mu)$. We obtain that $\oplus_{n=0}^{\infty} \left(\lambda + i\mu - K^{\otimes n}\right)^{-1}$ is the bounded inverse of $\lambda + i\mu - \Gamma(K)$. Hence, $\lambda + i\mu \in \rho(\Gamma(K))$ and in particular, it holds by [89, Theorem X.1(3)] that $(\Gamma(K), D(\Gamma(K)))$ is self-adjoint. Further, for $F \in D(\Gamma(K))$ it holds

$$(\Gamma(K)F, F)_{L^2(\mu)} = \sum_{n=0}^{\infty} n!(K^{\otimes n} f^{(n)}, f^{(n)}) \geq (F, F)_{L^2(\mu)}.$$

Hence, from the spectral theorem, see [91, Theorem VIII.4] for self-adjoint operators it follows that $\sigma(\Gamma(K)) \subseteq [1, \infty)$. The fact that $\mathcal{P}(\mathcal{N}')$ forms a core follows from the fact that $\mathcal{N}^{\widehat{\otimes} n}$ forms a core for $\left(K^{\otimes n}, D(K^{\otimes n})\right)$. The last assertion follows from $\Gamma(K)^{-1} = \Gamma(K^{-1})$ and $(K3)$. □

In particular, from the previous lemma it follows that $\mathcal{G}_{K,s} := D(\Gamma(K^s))$ becomes a Hilbert space with the inner product $((\cdot, \cdot))_{K,s} = ((\Gamma(K^s)\cdot, \Gamma(K^s)\cdot))_{L^2(\mu)}$ and the corresponding norm we denote by $|||\cdot|||_{K,s}$, $s \in \mathbb{N}$. We denote the dual space $\mathcal{G}'_{K,s}$ of $\mathcal{G}_{K,s}$ in the following by $\mathcal{G}_{K,-s}$. Now let $l, s \in \mathbb{N}$ and $l \leq s$. Since the operator $\Gamma(K^{s-l})$ is closed, it follows that the bilinear forms $(\cdot,\cdot)_{K,s}$ and $(\cdot,\cdot)_{K,l}$ are compatible on $D(\Gamma(K^s))$. In particular, we obtain the following chain of continuous embeddings, see also [57, Chapter 3.B],

$$\mathcal{G}_K \subseteq \mathcal{G}_{K,s} \subseteq \mathcal{G}_{K,l} \subseteq L^2(\mu) \subseteq \mathcal{G}_{K,-l} \subseteq \mathcal{G}_{K,-s} \subseteq \mathcal{G}'_K, \quad s \geq l,$$

where $\mathcal{G}_K = \bigcap_{s\in\mathbb{N}} \mathcal{G}_{K,s}$ is equipped with the projective limit topology of the spaces $\left(\mathcal{G}_{K,s}\right)_{s\in\mathbb{N}}$, see also [93, Section II.5.], and $\mathcal{G}'_K = \bigcup_{s\in\mathbb{N}} \mathcal{G}_{K,-s}$ is the dual space of $\mathcal{G}_K$ carrying the inductive limit topology of the spaces $\left(\mathcal{G}_{K,-s}\right)_{s\in\mathbb{N}}$, see also [93, Section II.6.]. Via Lemma 4.29 we obtain an isometric isomorphism between the dual space $\mathcal{G}_{K,-s}$ and the direct sum of the dual spaces of $D((K^s)^{\otimes n})$, i.e.,

$$\mathcal{G}_{K,-s} \cong \oplus_{n\in\mathbb{N}_0} n! D((K^s)^{\otimes n})',$$

where the space $D((K^s)^{\otimes n})$ is equipped with the norm $\left\|(K^s)^{\otimes n}\cdot\right\|_{\mathcal{H}}$. The next proposition shows how the two pairs of spaces $\left(\mathcal{G}_K, \mathcal{G}'_K\right)$ and $((\mathcal{N}),(\mathcal{N})')$ are related.

Proposition 5.6. *The space $(\mathcal{N})$ is continuously and densely embedded into $\mathcal{G}_K$. Hence the following chain of continuous and dense embeddings holds true*

$$(\mathcal{N}) \subseteq \mathcal{G}_K \subseteq L^2(\mu) \subseteq \mathcal{G}'_K \subseteq (\mathcal{N})'.$$

Proof. From Lemma 4.30 and Lemma 5.5 we obtain that the polynomials $\mathcal{P}(\mathcal{N}')$ are dense in $(\mathcal{N})$ and $\mathcal{G}_K$, respectively. Hence, it suffices to show that for every $s \in \mathbb{N}$ there exists $p \in \mathbb{N}$ s.t. $|||F|||_{K,s} \leq |||F|||_p$ for all $F \in \mathcal{P}(\mathcal{N}')$ and that the norms $|||\cdot|||_{K,s}$ and $|||\cdot|||_p$ are compatible on $\mathcal{P}(\mathcal{N}')$. Since K^s was assumed to be continuous on $\mathcal{N}$, there exists a $p \in \mathbb{N}$ s.t. K can be extended to a linear and continuous operator from $\mathcal{H}_p$ to $\mathcal{H}$ with the operator norm $\|K^s\|_p \leq 1$. Then, we obtain $|||F|||_{K,s} \leq |||F|||_p$ for all $F \in \mathcal{P}(\mathcal{N}')$. To show compatibility of the norms $|||\cdot|||_{K,s}$ and $|||\cdot|||_p$ we make two simple observations. First, the norm $|||\cdot|||_p$ and the $L^2(\mu)$-norm $|||\cdot|||$ are compatible, see (4.2.3). Second, from Lemma 5.5 it follows $|||\cdot||| \leq |||\cdot|||_{K,s}$ which implies the compatibility of $|||\cdot|||_{K,s}$ and $|||\cdot|||_p$. □

5.2 Complex Gaussian Analyis

In this part we briefly present a analogon of the orthogonal decomposition of $L^2(\mu)$ for a space of square integrable functions on the complexified space $\mathcal{N}'_{\mathbb{C}}$. Indeed, recall the measure $\mu_{\frac{1}{2}}$ defined on the real space $\mathcal{N}'$ given by (5.1). We equip $\mathcal{N}'_{\mathbb{C}} = \mathcal{N}' \times \mathcal{N}'$ with the product σ-field $\mathcal{B} \times \mathcal{B}$ which we denote by $\mathcal{B}_{\mathbb{C}}$. Now, we define the measure ν on $\mathcal{B}_{\mathbb{C}}$ by $\nu = \mu_{\frac{1}{2}} \otimes \mu_{\frac{1}{2}}$. As usual, the space of square integrable functions on $\mathcal{N}'_{\mathbb{C}}$ is denoted by $L^2(\mathcal{N}'_{\mathbb{C}}, \nu)$. The major difference between the spaces $L^2(\mu)$ and $L^2(\mathcal{N}'_{\mathbb{C}}, \nu)$ is that in the latter case there is no need for using the Hermite polynomials, see Proposition 5.9. The reason behind is that the monomials of different order automatically form an orthogonal system in $L^2(\mathbb{C}, e^{-|z|^2}dz)$.

Remark 5.7. *Note that the product σ-field $\mathcal{B}_{\mathbb{C}}$ does not necessarily coincide with the σ-field generated by the product topology. Indeed, recall that $\mathcal{B} = \sigma(\beta_w) = \sigma(\beta_s) = \sigma(\beta_i)$. Since none of the topologies β_w, β_s and β_i are second countable for infinite-dimensional $\mathcal{N}$, see e.g. [14, Fact 26]. We only know $\mathcal{B}_{\mathbb{C}} = \sigma(\beta_w) \times \sigma(\beta_w) \subset \sigma(\beta_w \times \beta_w)$, where $\beta_w \times \beta_w$ denotes the product topology.*

The proofs of the next two propositions are similar to the corresponding statements for the real case and are therefore omitted.

Proposition 5.8. *Let $\varphi_1, \ldots, \varphi_n \in \mathcal{N}$, $n \in \mathbb{N}$. The image measure of ν under the map*

$$T_{\varphi_1,\ldots,\varphi_n} : \mathcal{N}'_{\mathbb{C}} \longrightarrow \mathbb{C}^n, \eta \mapsto (\langle \varphi_i, \eta \rangle)_{i=1,\ldots,n}$$

is absolutely continuous w.r.t. the Lebesgue measure dz on $\mathbb{C}^n$ and has the Radon-Nikodym derivative given by

$$\frac{d\nu \circ T^{-1}_{\varphi_1,\ldots,\varphi_n}}{dz}(z) = \frac{1}{\pi^n} e^{-\overline{z}^\top C z}, \quad z \in \mathbb{C}^n$$

where $C = \big((\varphi_i, \varphi_j)_{\mathcal{H}}\big)_{1 \le i,j \le n} \in \mathbb{R}^{n \times n}$.

The space of polynomials $\mathcal{P}(\mathcal{N}'_{\mathbb{C}})$ on $\mathcal{N}'_{\mathbb{C}}$ is determined by the collection of all functions $G : \mathcal{N}'_{\mathbb{C}} \to \mathbb{C}$ which are given as $G(\eta) = p(\langle \varphi_1, \eta \rangle, \ldots, \langle \varphi_k, \eta \rangle)$, where p is a polynomial of $k \in \mathbb{N}$ variables with complex coefficients and $\varphi_i \in \mathcal{N}$, for $i = 1, .., k$.

Proposition 5.9. *Let $m, n \in \mathbb{N}$, $\varphi, \psi \in \mathcal{N}$. Then it holds*

$$(\langle \varphi, \cdot \rangle^n, \langle \psi, \cdot \rangle^m)_{L^2(\nu)} = \delta_{m,n} \cdot n! \cdot (\varphi^{\otimes n}, \psi^{\otimes n})_{\mathcal{H}}. \tag{5.2}$$

In particular, it holds $\mathcal{P}(\mathcal{N}'_{\mathbb{C}}) \subseteq L^2(\nu)$.

Similar as in the derivation of Theorem 4.22, for $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ we can define an element in $L^2(\nu)$ denoted by $\langle f^{(n)}, \cdot^{\otimes n} \rangle$ which is given as the $L^2(\nu)$-limit of polynomials, i.e.,

$$\left\langle f^{(n)}, \cdot^{\otimes n} \right\rangle := \lim_{m \to \infty} \sum_{k=1}^{l_m} \alpha_{k,m} \langle \varphi_{k,m}, \cdot \rangle^n \in L^2(\nu),$$

where $l_m \in \mathbb{N}$, $\alpha_{k,m} \in \mathbb{C}$, $\varphi_{k,m} \in \mathcal{N}$ for all $k = 1, \ldots, l_m$, $m \in \mathbb{N}$. Moreover, it holds

$$f^{(n)} = \lim_{m \to \infty} \sum_{k=1}^{l_m} \alpha_{k,m} \varphi_{k,m}^{\otimes n} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}.$$

In particular, the orthogonality relation (5.2) stays valid in the limit case, i.e., for $f^{(n)}, g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ it holds

$$\left(\left\langle f^{(n)}, \cdot^{\otimes n} \right\rangle, \left\langle g^{(n)}, \cdot^{\otimes n} \right\rangle \right)_{L^2(\nu)} = \delta_{m,n} \cdot n! \cdot (f^{(n)}, g^{(n)})_{\mathcal{H}}. \tag{5.3}$$

In contrast to the real case, the polynomials $\mathcal{P}(\mathcal{N}'_{\mathbb{C}})$ are not dense in $L^2(\nu)$. For example, one can use the isometry (5.3) to show that for $g \in \mathcal{N}$ and $n \in \mathbb{N}$ the function $G = \overline{\langle g^{\otimes n}, \cdot^{\otimes n} \rangle}$, where $\bar{\cdot}$ denotes complex conjugation, satisfies $G \in \mathcal{P}(\mathcal{N}'_{\mathbb{C}})^{\perp}$ and $\|G\|^2_{L^2(\nu)} = n! \, \|g\|^{2n}$. The closure $\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ is called Bargmann-Segal space and is given by

$$\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)} = \left\{ \sum_{n \in \mathbb{N}} \left\langle g^{(n)}, \cdot^{\otimes n} \right\rangle \mid g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}, \sum_{n \in \mathbb{N}} n! \left\| g^{(n)} \right\|^2_{\mathcal{H}} < \infty \right\}. \tag{5.4}$$

I.e., the Bargmann-Segal space $\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ is also isometric isomorphic to the Fock space $\Gamma(\mathcal{H}_{\mathbb{C}})$:

$$\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)} \cong \Gamma(\mathcal{H}_{\mathbb{C}}) \cong L^2(\mu). \tag{5.5}$$

Before we proceed we present the characterization of $\mathcal{G}_K$ given in [52]. Therefore we need the following definition.

Definition 5.10. *Let $m \in \mathbb{N}$ and $(\varphi_i)_{i=1}^m \subset \mathcal{N}$ be an orthonormal system in $\mathcal{H}$. We call*

$$P : \mathcal{N}'_{\mathbb{C}} \to \mathcal{N}_{\mathbb{C}}, P\eta := \sum_{i=1}^{m} \langle \varphi_i, \eta \rangle \, \varphi_i$$

an orthogonal projection from $\mathcal{N}'_{\mathbb{C}}$ into $\mathcal{N}_{\mathbb{C}}$. We denote the set consisting of all orthogonal projections from $\mathcal{N}'_{\mathbb{C}}$ into $\mathcal{N}_{\mathbb{C}}$ by $\mathbb{P}$.

Remark 5.11. *In [52] the Bargmann-Segal space is denoted by $E^2(\nu)$ and is introduced differently. There the Bargmann-Segal is given by the space of all entire functions G defined on $\mathcal{H}_{\mathbb{C}}$ s.t.*

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |G(P\cdot)|^2 \, d\nu < \infty.$$

The authors in [52] constructed a isomorphism

$$R : \overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)} \longrightarrow E^2(\nu), F \mapsto RF := F_{|\mathcal{H}_{\mathbb{C}}}.$$

Observe that $\mathcal{H}_{\mathbb{C}}$ is a ν-zero set. Therefore the definition of R is at the first sight not well-defined. One uses the structure of $\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ given in (5.4) to make this definition well-defined. To show that the map R is surjective one needs to use the analyticity of elements $G \in E^2(\nu)$. The chaos decomposition given in (5.4) for $R^{-1}G$ is the collection of the Taylor coefficients of G at zero. To establish the isomorphism R can be understood as the major step in the characterization in [52], since subspace of $L^2(\mu)$, in particular $\mathcal{G}_K$, are mapped under the isomorphisms in (5.5) and R into subspaces of $E^2(\nu)$. In the following we omit the isomorphisms in (5.5) and identify an element $F \in L^2(\mu)$ without further notice as an element of $\overline{\mathcal{P}(\mathcal{N}'_{\mathbb{C}})}^{L^2(\nu)}$ and vice versa. Then, the characterization in [52] of $\mathcal{G}_K$ can be stated as follows, see [52, Theorem 7.1]:

Theorem 5.12. *If $F \in \mathcal{G}_K$ then it holds for $RF \in E^2(\nu)$ that $\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |RF(K^s P\cdot)|^2 \, d\nu < \infty$ for all $s \in \mathbb{N}$. In reverse, if $G \in E^2(\nu)$ satisfies $\sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |G(K^s P\cdot)|^2 \, d\nu < \infty$ for all $s \in \mathbb{N}$ than $R^{-1}G \in \mathcal{G}_K$.*

A similar statement is proven in [52] for the dual space $\mathcal{G}'_K$. We proof an equivalent characterization without introducing the space $E^2(\nu)$ at all. In particular, we don't need to use the concept of holomorphy on infinite dimensional spaces explicitly, see also Remark 5.16(ii) below.

5.3 Main Results

In this section we formulate the main results of this chapter. For this purpose, we need some preliminary technical discussion concerning tensor products of linear operators on nuclear spaces. Recall the chain of continuous embeddings from (4.20).

$$\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{H}_{q,\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{H}_{-q,\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{H}_{-p,\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes}n}, \quad p \geq q,$$

Recall further the operator $(K, D(K))$ which is defined as a self-adjoint operator on $\mathcal{H}$. Additionally, by assumption $(K3)$ we can consider K as a bijective and continuous operator $K : \mathcal{N} \longrightarrow \mathcal{N}$. In the following we fix $s \in \mathbb{Z}$ and $n \in \mathbb{N}$. Our goal is to define tensor powers $(K^s)^{\otimes n}$ of K^s as a linear and continuous operator on $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$. Due to the continuity of K on $\mathcal{N}$, there exists for every $q \in \mathbb{N}$ a $p \in \mathbb{N}$ and a constant C s.t. $\|K^s\varphi\|_q \leq C\|\varphi\|_p$ for all $\varphi \in \mathcal{N}$. Hence, K^s extends to a linear and continuous operator from $\mathcal{H}_p$ to $\mathcal{H}_q$ and furthermore, its tensor product $(K^s)^{\otimes n}$ is well-defined as an element from $L(\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes}n}, \mathcal{H}_{q,\mathbb{C}}^{\widehat{\otimes}n})$. Since $q \in \mathbb{N}$ is arbitrary, we obtain that $(K^s)^{\otimes n}$ maps $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$ continuously into itself, i.e., $(K^s)^{\otimes n} \in L(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n})$. Furthermore, $(K^s)^{\otimes n}$ is invertible with inverse $(K^{-s})^{\otimes n}$. In addition, we can form the tensor powers of the operator $(K, D(K))$ as an operator on the Hilbert space $\mathcal{H}$ and obtain the self-adjoint operator $\big((K^s)^{\otimes n}, D((K^s)^{\otimes n})\big)$ on $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$, where $D((K^s)^{\otimes n}) = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ for $s \leq 0$. One easily sees that this definition is an extension of $(K^s)^{\otimes n} \in L(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n})$. Due to the self-adjointness and the continuity of $(K^s)^{\otimes n}$ on $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$, the following extension to $\mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes}n}$ is consistent

$$(K^s)^{\otimes n} : \mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes}n} \longrightarrow \mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes}n}, \Phi \mapsto (K^s)^{\otimes n}\Phi := \Phi \circ (K^s)^{\otimes n}. \tag{5.6}$$

For the next proposition, recall that every element $\Phi \in (\mathcal{N})'$ has a generalised chaos decomposition $\Phi = \sum_{n\in\mathbb{N}} \langle \Phi^{(n)}, :\cdot^{\otimes n}: \rangle$ where for some $p \in \mathbb{N}$ it holds $\Phi^{(n)} \in \mathcal{H}_{-p,\mathbb{C}}^{\widehat{\otimes}n} \subseteq \mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes}n}$ for all $n \in \mathbb{N}_0$.

The next lemma plays a central role in the proof of Theorem 5.15 below. Hence, we present its proof in detail.

Lemma 5.13. *Let $s \in \mathbb{Z}$. Then it holds*

$$\mathcal{G}_{K,s} = \left\{ \Phi = \sum_{n\in\mathbb{N}} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}: \right\rangle \in (\mathcal{N})' \,\middle|\, (K^s)^{\otimes n}\Phi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}, \sum_{n\in\mathbb{N}} n! \left\| (K^s)^{\otimes n}\Phi^{(n)} \right\|_{\mathcal{H}}^2 < \infty \right\}, \tag{5.7}$$

where $(K^s)^{\otimes n}\Phi^{(n)}$ in (5.7) is defined via (5.6).

Proof. Denote the set on the right-hand side of (5.7) by $\mathcal{A}_s$. We split the proof into two parts. First let s be non-negative. In this case the inclusion $\mathcal{G}_{K,s} \subseteq \mathcal{A}_s$ follows immediately by the definition of $\mathcal{G}_{K,s}$. Now, let $\Phi \in \mathcal{A}_s$, i.e., $\Phi = \sum_{n\in\mathbb{N}} \langle \Phi^{(n)}, :\cdot^{\otimes n}: \rangle \in (\mathcal{N})'$ s.t. $(K^s)^{\otimes n}\Phi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ and $\sum_{n\in\mathbb{N}} n! \|(K^s)^{\otimes n}\Phi^{(n)}\|_{\mathcal{H}}^2 < \infty$. To prove that $\Phi \in \mathcal{G}_{K,s}$ it suffices to show that $\Phi^{(n)} \in D(K^s)^{\otimes n}$ for all $n \in \mathbb{N}$. By assumption, for all $n \in \mathbb{N}$ there exists a $\psi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ s.t.

$$\left(\varphi^{(n)}, \psi^{(n)}\right)_{\mathcal{H}} = \left\langle \varphi^{(n)}, (K^s)^{\otimes n}\Phi^{(n)} \right\rangle = \left\langle (K^s)^{\otimes n}\varphi^{(n)}, \Phi^{(n)} \right\rangle, \quad \forall \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}.$$

By using [91, Theorem VIII.33] we obtain that $(K^s)^{\otimes n} : D((K^s)^{\otimes n}) \longrightarrow \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ is bijective and self-adjoint. Hence, we can find a $\tilde{\psi}^{(n)} \in D((K^s)^{\otimes n})$ s.t. $(K^s)^{\otimes n}\tilde{\psi}^{(n)} = \psi^{(n)}$. From the self-adjointness of $(K^s)^{\otimes n}$ we can conclude $\Phi^{(n)} = \overline{\tilde{\psi}^{(n)}}$, where $\bar{\cdot}$ is the natural complex conjugation on the complexified vector space $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. This finishes the proof for non-negative s.

For the second part we replace s by $-s$, $s \in \mathbb{N}$. Recall that $\mathcal{G}_{K,-s} \cong \Gamma(D(K^s)')$. Denote by $\overline{\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}}^{\|(K^{-s})^{\otimes n}\cdot\|}$ the abstract completion of $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ w.r.t. $\|(K^{-s})^{\otimes n}\cdot\|$. One easily checks via the Riesz isomorphism that

$$\overline{\mathcal{H}}^{\|(K^{-s})^{\otimes n}\cdot\|} \ni (\Phi_k^{(n)})_{k\in\mathbb{N}} \mapsto \lim_{k\to\infty}\left((K^s)^{\otimes n}\cdot, (K^{-s})^{\otimes n}\Phi_k^{(n)}\right)_{\mathcal{H}} \in \left(D((K^s)^{\otimes n})\right)' \tag{5.8}$$

is an isometric complex conjugate linear isomorphism. Hence, the inclusion $\mathcal{G}_{K,-s} \subseteq \mathcal{A}_{-s}$ follows. Now let $\Phi \in \mathcal{A}_{-s}$ with generalised chaos decomposition $\Phi = \sum_{n\in\mathbb{N}} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}:\right\rangle$. It suffices to show $\Phi^{(n)} \in \left(D((K^s)^{\otimes n})\right)'$ for all $n \in \mathbb{N}$. By assumption, for every $n \in \mathbb{N}$ there exists a $\psi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ s.t.

$$\left\langle \varphi^{(n)}, \Phi^{(n)}\right\rangle = \left\langle (K^{-s})^{\otimes n}(K^s)^{\otimes n}\varphi^{(n)}, \Phi^{(n)}\right\rangle = \left((K^s)^{\otimes n}\varphi^{(n)}, \psi^{(n)}\right)_{\mathcal{H}}, \quad \forall \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}.$$

Since $D((K^s)^{\otimes n}) = (K^{-s})^{\otimes n}\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ is dense in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ there exists a sequence $(\chi_k^{(n)})_{k\in\mathbb{N}}$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ s.t. $(K^{-s})^{\otimes n}\chi_k^{(n)} \longrightarrow \psi^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in \mathbb{N}$. Hence, by (5.8) we obtain $\Phi^{(n)} \in \left(D((K^s)^{\otimes n})\right)'$ which completes the proof. □

Before we proceed we need to clarify some technical issues. Let $n \in \mathbb{N}$ and $P \in \mathbb{P}$ be given by $P = \sum_{j=1}^{m} \langle \varphi_j, \cdot\rangle\, \varphi_j$, where $(\varphi_j)_{j=1}^m \subset \mathcal{N}$ is an orthonormal system in $\mathcal{H}$. We consider P also as an orthogonal projection on $\mathcal{H}_{\mathbb{C}}$ onto the closed subspace $\operatorname{span}_{\mathbb{C}}\{\varphi_j, j = 1, \ldots, m\}$. Observe that the n-th tensor power $P^{\otimes n}$ of P defines a orthogonal projection onto the closed subspace $\operatorname{span}_{\mathbb{C}}\left\{\hat{\otimes}_{i=1}^n \varphi_{j_i} \mid j_i \in \{1, \ldots, m\} \text{ for } i = 1, \ldots, n\right\}$ of $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. Further, we can extend $P^{\otimes n}$ to a linear operator from $\mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}$ to $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}$ via

$$P^{\otimes n} : \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n} \longrightarrow \mathcal{N}_{\mathbb{C}}^{\hat{\otimes} n}, \Phi \mapsto \frac{1}{n!}\sum_{j_1,\ldots,j_n=1}^{m} \left\langle \hat{\otimes}_{i=1}^n \varphi_{j_i}, \Phi\right\rangle \hat{\otimes}_{i=1}^n \varphi_{j_i}.$$

Observe that for $n \in \mathbb{N}$, $\Phi^{(n)}, \Psi^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}$ and $P \in \mathbb{P}$ it holds $\left\langle P^{\otimes n}\Phi^{(n)}, \Psi^{(n)}\right\rangle = \left\langle P^{\otimes n}\Psi^{(n)}, \Phi^{(n)}\right\rangle$. The following lemma is stated in [52] without proof. For convenience we give the short proof here.

Lemma 5.14. *Let $\Phi^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\hat{\otimes} n}$, $n \in \mathbb{N}_0$, satisfy $\sup_{P\in\mathbb{P}} \sum_{n=0}^{\infty} n!\|P^{\otimes n}\Phi^{(n)}\|_{\mathcal{H}}^2 < \infty$. Then, it holds*

$$\left(\Phi^{(n)}\right)_{n\in\mathbb{N}_0} \in \Gamma(\mathcal{H}) \text{ and } \sup_{P\in\mathbb{P}}\sum_{n=0}^{\infty} n!\|P^{\otimes n}\Phi^{(n)}\|_{\mathcal{H}}^2 = \sum_{n=0}^{\infty} n!\|\Phi^{(n)}\|_{\mathcal{H}}^2.$$

Proof. We first show that $\Phi^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ for all $n \in \mathbb{N}$. Hence, fix $n \in \mathbb{N}$ and choose $p \in \mathbb{N}$ s.t. $\Phi^{(n)} \in \mathcal{H}_{-p,\mathbb{C}}^{\hat{\otimes} n}$ and let $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{N}$ be an orthonormal basis of $\mathcal{H}_p$. We apply the Gram-Schmidt procedure to $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$ to obtain an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$. Observe that for every $k \in \mathbb{N}$ the element e_k is a linear combination of the elements $f_1, \ldots, f_k$, hence $e_k \in \mathcal{N}$ for all

$k \in \mathbb{N}$. Define for $l \in \mathbb{N}$ the projection $P_l := \sum_{k=1}^{l} \langle e_k, \cdot \rangle e_k$. By assumption it is known that the sequence $\left(P_l^{\otimes n} \Phi^{(n)}\right)_{l \in \mathbb{N}} \subseteq \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ is bounded. Therefore we can find a weakly convergent subsequence $\left(P_{l_m}^{\otimes n} \Phi^{(n)}\right)_{m \in \mathbb{N}}$ with weak limit $g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. For $\alpha \in \mathbb{N}^n$ we define $e_\alpha := \hat{\otimes}_{i=1}^n e_{\alpha_i}$ and obtain $\left\langle e_\alpha, P_l^{\otimes n} \Phi^{(n)} \right\rangle = \left\langle e_\alpha, \Phi^{(n)} \right\rangle$ for $l \geq \max\{\alpha_i \mid 1 \leq i \leq n\}$. Hence, it holds

$$\left\langle e_\alpha, \Phi^{(n)} \right\rangle = \lim_{m \to \infty} \left\langle e_\alpha, P_{l_m}^{\otimes n} \Phi^{(n)} \right\rangle = \lim_{m \to \infty} \left(e_\alpha, \overline{P_{l_m}^{\otimes n} \Phi^{(n)}} \right)_{\mathcal{H}} = \left(e_\alpha, \overline{g^{(n)}} \right)_{\mathcal{H}}.$$

Additionally, by the choice of $(e_k)_{k \in \mathbb{N}}$ it holds that the set $\{e_\alpha \mid \alpha \in \mathbb{N}^n\}$ is total in $\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes} n}$. Thus, we conclude that $\Phi^{(n)} = g^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$. The last part of statement follows from the fact that for $P \in \mathbb{P}$ the restriction of $P^{\otimes n}$ to $\mathcal{H}_{\mathbb{C}}^{\hat{\otimes} n}$ is an orthogonal projection and Fatou's lemma. □

Now we are ready to state and prove the main result.

Theorem 5.15. *Let $\Phi \in (\mathcal{N})'$ or equivalently let U be a U-functional s.t. $U = S\Phi$ with S being the S-transform defined in Definition 4.35. Then the following two statements hold true:*

(i)

$$\Phi \in \mathcal{G}_K \iff \forall s \in \mathbb{N} : \sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |U(K^s P\eta)|^2 \, \nu(d\eta) < \infty.$$

(ii)

$$\Phi \in \mathcal{G}'_K \iff \exists s \in \mathbb{N} : \sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |U(K^{-s} P\eta)|^2 \, \nu(d\eta) < \infty.$$

Proof. Recall that $\mathcal{G}_K = \bigcap_{s \in \mathbb{N}} \mathcal{G}_{K,s}$ and $\mathcal{G}'_K = \bigcup_{s \in \mathbb{N}} \mathcal{G}_{K,-s}$. Hence, it is sufficient to show for all $s \in \mathbb{Z}$ the equivalence

$$\Phi \in \mathcal{G}_{K,s} \iff \sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |S\Phi(K^s P\eta)|^2 \, \nu(d\eta) < \infty.$$

We make some observations which rely on (4.29), (5.3) and Lemma 5.14. Now, let $s \in \mathbb{Z}$ and $\Phi = \sum_{n \in \mathbb{N}} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}: \right\rangle \in \mathcal{G}_{K,s}$. Then, it holds

$$\begin{aligned}
\|\Phi\|_{K,s}^2 &= \sum_{n \in \mathbb{N}} n! \|(K^s)^{\otimes n} \Phi^{(n)}\|_{\mathcal{H}}^2 \\
&= \sup_{P \in \mathbb{P}} \sum_{n \in \mathbb{N}} n! \|P^{\otimes n} (K^s)^{\otimes n} \Phi^{(n)}\|_{\mathcal{H}}^2 = \sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} \left| \sum_{n \in \mathbb{N}} \left\langle P^{\otimes n} (K^s)^{\otimes n} \Phi^{(n)}, \eta^{\otimes n} \right\rangle \right|^2 \nu(d\eta) \\
&= \sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} \left| \sum_{n \in \mathbb{N}} \left\langle (K^s P\eta)^{\otimes n}, \Phi^{(n)} \right\rangle \right|^2 \nu(d\eta) = \sup_{P \in \mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |S\Phi(K^s P\eta)|^2 \, \nu(d\eta).
\end{aligned} \tag{5.9}$$

Now let $\Phi = \sum_{n\in\mathbb{N}} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}: \right\rangle \in (\mathcal{N})'$. The same calculations yield

$$\begin{aligned}
\sup_{P\in\mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |S\Phi(K^s P\eta)|^2 \, \nu(d\eta) &= \sup_{P\in\mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} \left| \sum_{n\in\mathbb{N}} \left\langle (K^s P\eta)^{\otimes n}, \Phi^{(n)} \right\rangle \right|^2 \nu(d\eta) \\
&= \sup_{P\in\mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} \left| \sum_{n\in\mathbb{N}} \left\langle P^{\otimes n} (K^s)^{\otimes n} \Phi^{(n)}, \eta^{\otimes n} \right\rangle \right|^2 \nu(d\eta) \\
&= \sup_{P\in\mathbb{P}} \sum_{n\in\mathbb{N}} n! \| P^{\otimes n} (K^s)^{\otimes n} \Phi^{(n)} \|^2_{\mathcal{H}},
\end{aligned}$$

where we used the definition of $(K^s)^{\otimes n}$ given in (5.6) in the second line. Hence, if the left-hand side is finite we obtain by Lemma 5.14 that $\left((K^s)^{\otimes n}\Phi^{(n)}\right)_{n\in\mathbb{N}} \in \Gamma(\mathcal{H})$ which implies $\Phi \in \mathcal{G}_{K,s}$ by Proposition 5.13. □

Remark 5.16. *(i) The proof of Theorem 5.15 gives some additional insight, i.e., for $s \in \mathbb{Z}$ it holds*

$$\Phi \in \mathcal{G}_{K,s} \iff \sup_{P\in\mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |S\Phi(K^s P\eta)|^2 \, \nu(d\eta) < \infty. \tag{5.10}$$

In particular, it holds that $\Phi \in L^2(\mu)$ if and only if $\sup_{P\in\mathbb{P}} \int_{\mathcal{N}'_{\mathbb{C}}} |S\Phi(P\eta)|^2 \, \nu(d\eta) < \infty$.

(ii) In the proof of Theorem 5.15 we did not use the concept of entire functions on infinite dimensional spaces explicitly. Anyway, the proof of Theorem 4.37 given in [63] relies heavily on this concept. From this point of view one could think of Theorem 5.15 as a modern formulation of the corresponding results in [52] via U-functionals.

5.4 Application to a Stochastic Transport Equation

In this section we present an application for Theorem 5.15. To this end, we fix a choice of our functional analytic setting. Namely, for the rest of this section we set

$$\begin{aligned}
\mathcal{N} &:= S(\mathbb{R}), \\
\mathcal{H} &:= L^2(\mathbb{R}), \\
\mathcal{N}' &:= S'(\mathbb{R}),
\end{aligned}$$

where $S(\mathbb{R})$ denotes the Schwartz space from Example 4.4 and $S'(\mathbb{R})$ denotes it dual space. In particular, the corresponding complexified symmetric tensor powers of $S(\mathbb{R})$ and $L^2(\mathbb{R})$ are given by the spaces of complex valued functions which are invariant under permutations of their arguments

$$\begin{aligned}
\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n} &= \widehat{S_{\mathbb{C}}(\mathbb{R}^n)}, \\
\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n} &= \widehat{L^2_{\mathbb{C}}(\mathbb{R}^n)}.
\end{aligned}$$

Furthermore, we choose the operator $K = \sqrt{2}I$. Henceforth, we skip the subscript K and simply write $\mathcal{G}$ and $\mathcal{G}_s$ instead of $\mathcal{G}_{\sqrt{2}I}$ and $\mathcal{G}_{\sqrt{2}I,s}$ for $s \in \mathbb{R}$, see also Remark 5.4. The multiplier $\sqrt{2}$ is of course arbitrary. Any positive number $\gamma > 1$ leads to the same space $\mathcal{G}$. In particular, the space $\mathcal{G}_s$ is given by

$$\mathcal{G}_s = \left\{ F = \sum_{n\in\mathbb{N}} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \in L^2(\mu) \,\middle|\, \sum_{n\in\mathbb{N}} n! 2^{sn} \|f^{(n)}\|_{\mathcal{H}}^2 < \infty \right\}.$$

In [22], [86] and [44] a parabolic stochastic partial differential equation (SPDE) modeling the transport of a substance in a turbulent medium is treated via white noise analysis. There the authors are searching for a solution $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega :\longrightarrow \mathbb{R}, (t, x, \omega) \mapsto u(t, x, \omega)$ describing the concentration of the substance, where t stands for the time, x for the position and ω for the random parameter which will be suppressed. Moreover, the u satisfies

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\nu(t)\frac{\partial^2 u(t,x)}{\partial x^2} + \sigma(t)\frac{\partial u(t,x)}{\partial x} \circ^{S/I} dB_t, \tag{5.11}$$

$$u(0,\cdot) = \delta_0, \tag{5.12}$$

where ν describes the molecular viscosity of the medium and $\circ^{S/I} dB_t$ denotes the Stratonovich/Itô integral w.r.t. a Brownian motion $(B_t)_{t\geq 0}$ modelling the turbulence in the medium. The initial condition (5.12) is a physical idealisation that at time zero the substance is only concentrated at the point $x = 0$. More realistic and even random initial conditions can be realised via convolution, see Remark 5.24 below. In [22] and [86] the Stratonovich case is treated and existence of an L^2-valued solution $u(t, x)$ is shown. The Itô case is also treated in [22]. In that case the solution is constructed as a generalised Brownian functional, see the last mentioned reference as well as [56] for the precise meaning. We follow the approach in [44] and state the Itô interpretation of (5.11), (5.12) in terms of white noise analysis and give a solution u via the same techniques as in [44] and [86]. Afterwards, we use Remark 5.16 to determine to which of the spaces $(\mathcal{G}_s)_{s\in\mathbb{R}}$ the solution u belongs. In particular, we find an explicit criterion in terms of the coefficients ν and σ to determine whether $u \in \mathcal{G}_s$, $s \in \mathbb{R}$.

To formulate (5.11), (5.12) in terms of white noise analysis we introduce the white noise process $(w_t)_{t\geq 0} \subseteq (\mathcal{N})'$. The element w_t is given by its generalised chaos decomposition $w_t = \langle \delta_t, \cdot \rangle$, where $\delta_t \in S'(\mathbb{R})$ denotes the Dirac delta distribution at $t \geq 0$. A rigorous interpretation of (5.11), (5.12) in terms of white noise analysis is now given as follows. We search for a map $u : \mathbb{R}_+ \times \mathbb{R} \longrightarrow (\mathcal{N})'$ fulfilling

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\nu(t)\frac{\partial^2 u(t,x)}{\partial x^2} + \sigma(t)\frac{\partial u(t,x)}{\partial x} \diamond w_t, t > 0, x \in \mathbb{R}, \tag{5.13}$$

$$(Su(t,\cdot)(\varphi))_{t>0} \text{ is a Dirac sequence for all } \varphi \in S(\mathbb{R}). \tag{5.14}$$

We explain the connection between the Itô term in (5.11) and the so called Hitsuda-Skorokhod term $\sigma(t)\frac{\partial u(t,x)}{\partial x} \diamond w_t$ in (5.13) in Remark 5.24 below. We formulate our existence result in the next theorem:

Theorem 5.17. *Assume that $\nu : [0, \infty) \longrightarrow \mathbb{R}$ is strictly positive and locally integrable and $\sigma : [0, \infty) \longrightarrow \mathbb{R}$ is locally square integrable. If the function $(0, \infty) \ni t \mapsto \kappa(t) := \frac{\int_{[0,t]} \sigma^2(s)\, ds}{\int_{[0,t]} \nu(s)\, ds} \in \mathbb{R}$ is bounded in the vicinity of 0 then for every $T \in \mathbb{N}$ there exists an $s \in \mathbb{R}$ and a map*

$$u : (0, T] \times \mathbb{R} \longrightarrow \mathcal{G}_s$$

satisfying (5.14). Furthermore, for dt-a.e. $t \in (0,T]$ and all $x \in \mathbb{R}$ the map u is once differentiable w.r.t. t and twice differentiable w.r.t. x at (t,x) and satisfies (5.13). In particular, for $s \in \mathbb{R}$ and $t \in (0,\infty)$ satisfying $2^s\kappa(t) < 1$ it holds $u(t,x) \in \mathcal{G}_s$ for all $x \in \mathbb{R}$.

Proof. The same computations as in [86, Section 5] yield a candidate for the S-transform of u:

$$Su(t,x)(\varphi) = \frac{1}{\sqrt{2\pi\vartheta(t)}} \exp\left(-\frac{1}{2\vartheta(t)}\left(x - \int_{[0,t]} \sigma(s)\varphi(s)\,ds\right)^2\right), \quad \varphi \in S(\mathbb{R}), \tag{5.15}$$

where $\vartheta(t) = \int_{[0,t]} \nu(s)\,ds$, $t > 0$. One easily sees that (5.15) defines a U-functional satisfying (5.14). Via Theorem 4.37 we obtain the element $u(t,x) \in (\mathcal{N})'$ having S-transform given by (5.15). If $\int_{[0,t]} \nu(s)\,ds = \int_{[0,t]} \sigma(s)^2\,ds$ the corresponding Hida distribution $u(t,x)$ is given by $\delta_x(\langle 1_{[0,t]}\sigma, \cdot\rangle)$, see Example 4.25(ii)

We divide the proof into two separate parts. In the first part we show that u is differentiable and satisfies (5.13) in the above mentioned sense. In the second part we show that for all $t \in \mathbb{N}$ there exists a $s \in \mathbb{R}$ s.t. $u(t,x) \in \mathcal{G}_s$ for all $(t,x) \in (0,T] \times \mathbb{R}$.
Part 1: To show that u is differentiable and satisfies (5.13) in the above mentioned sense we use Theorem 4.39. We only show that u is differentiable w.r.t. t at every $(t,x) \in D \times \mathbb{R}$, where $(0,T] \setminus D$ is of Lebesgue measure zero. The treatment of the derivatives w.r.t. x is easier and can be done by the same procedure. We make the following observation. Let $\varphi \in S(\mathbb{R})$ and $T \in \mathbb{N}$. Via the fundamental theorem of Lebesgue calculus, the functions ϑ and $t \mapsto \varrho(t) := \int_{[0,t]} \sigma(s)\varphi(s)\,ds$ are absolutely continuous and differentiable at dt-a.e. $t \in (0,T]$ with respective derivatives $\nu(t)$ and $\sigma(t)\varphi(t)$. We denote by A the set of all $t \in (0,T]$ s.t. ϑ and ϱ are differentiable at t. Hence, $Su(\cdot,x)(\varphi)$ is differentiable at $t \in A$ and for a zero sequence $(h_n)_{n\in\mathbb{N}}$, s.t. $|h_n| \le \frac{t}{2}$, it holds

$$\begin{aligned}
&\lim_{n\to\infty} \frac{Su(t+h_n,x)(\varphi) - S(u(t,x)(\varphi)}{h_n} = \frac{\partial Su(t,x)(\varphi)}{\partial t} \\
=&\frac{1}{2\vartheta(t)^{\frac{3}{2}}\sqrt{2\pi}} exp\left(-\frac{1}{2\vartheta(t)}\left(x - \int_{[0,t]} \sigma(s)\varphi(s)\,ds\right)^2\right) \\
&\times\left(-\nu(t) - 2\sigma(t)\varphi(t)\left(x - \int_{[0,t]} \sigma(s)\varphi(s)\,ds\right) - \frac{\nu(t)\left(x - \int_{[0,t]} \sigma(s)\varphi(s)\,ds\right)^2}{\vartheta(t)}\right).
\end{aligned}$$

Hence, for $z \in \mathbb{C}$ we obtain the following estimation

$$\begin{aligned}
&\left|\frac{\partial Su(t,x)(z\varphi)}{\partial t}\right| \\
\leq &\frac{1}{2\vartheta(t)^{\frac{3}{2}}\sqrt{2\pi}} \exp\left(\frac{\|\sigma\|_{L_T^2}^2}{\vartheta(t)} |z|^2 \|\varphi\|_{L_T^2}^2\right) \\
&\times\left(|\nu(t)| + |z|^2 \|\varphi\|_\infty^2 \sigma(t)^2 + \left(|x| + \|\sigma\|_{L_T^2} |z| \|\varphi\|_{L_T^2}\right)^2 + \frac{|\nu(t)| \left(|x| + \|\sigma\|_{L_T^2} |z| \|\varphi\|_{L_T^2}\right)^2}{\vartheta(t)}\right) \\
\leq &\frac{1}{2\vartheta(t)^{\frac{3}{2}}\sqrt{2\pi}} \exp\left(\frac{\|\sigma\|_{L_T^2}^2}{\vartheta(t)} |z|^2 \|\varphi\|_{L_T^2}^2\right) \\
&\times\left(|\nu(t)| + \exp(|z|^2 \|\varphi\|_\infty^2)\sigma(t)^2 + C_1 \exp(\|\sigma\|_{L_T^2}^2 |z|^2 \|\varphi\|_{L_T^2}^2) + |\nu(t)| \frac{C_1 \exp\left(\|\sigma\|_{L_T^2}^2 |z|^2 \|\varphi\|_{L_T^2}^2\right)}{\vartheta(t)}\right) \\
\leq &C_3(t) \exp\left(C_2(t) |z|^2 \|\varphi\|_p^2\right) \left(|\nu(t)| + \sigma^2(t) + 1\right)
\end{aligned}$$

where $\|\cdot\|_{L_T^2}$ denotes the $L^2((0,T))$-norm, $\|\cdot\|_\infty$ is the $L^\infty(\mathbb{R})$-norm, $p \in \mathbb{N}$ is chosen s.t. for all $\varphi \in S(\mathbb{R})$ it holds $\max\left\{\|\varphi\|_{L_T^2}, \|\varphi\|_\infty\right\} \leq \|\varphi\|_p := \|A^p\varphi\|_{L^2(\mathbb{R})}$ and

$$\begin{aligned}
C_1 &= \max\{2|x|, 2\} \\
C_2(t) &= \frac{1}{\vartheta(t)} \|\sigma\|_{L_T^2}^2 + 1 + \|\sigma\|_{L_T^2}^2 \\
C_3(t) &= \frac{\max\left\{C_1, 1 + \frac{C_1}{\vartheta(t)}\right\}}{2\vartheta(t)^{\frac{3}{2}}\sqrt{2\pi}}.
\end{aligned}$$

Observe that C_2 and C_3 are decreasing. Applying the fundamental theorem of Lebesgue calculus to $Su(\cdot,x)(z\varphi)$ yields

$$\begin{aligned}
\left|\frac{Su(t+h_n,x)(z\varphi) - S(u(t,x)(z\varphi)}{h_n}\right| &= \left|\frac{1}{h_n}\int_{[t,t+h_n]} \frac{\partial Su(s,x)(z\varphi)}{\partial s}\, ds\right| \\
&\leq C_3\left(\frac{t}{2}\right) \exp\left(C_2\left(\frac{t}{2}\right) |z|^2 \|\varphi\|_p^2\right) \frac{1}{|h_n|} \int_{[t,t+h_n]} |\nu(s)| + \sigma^2(s) + 1\, ds
\end{aligned}$$

Via the fundamental theorem of Lebesgue calculus it holds for dt-a.e. t

$$C_4(t) := \sup_{n\in\mathbb{N}} \frac{1}{|h_n|} \int_{[t,t+h_n]} |\nu(s)| + \sigma^2(s) + 1\, ds < \infty. \tag{5.16}$$

We denote the set of all $t \in (0,T]$ s.t. (5.16) holds true by B. We conclude that for $t \in A \cap B$ it holds

$$\lim_{n\to\infty} \frac{Su(t+h_n,x)(z\varphi) - S(u(t,x)(z\varphi)}{h_n} \text{ exists,}$$

$$\left|\frac{Su(t+h_n,x)(z\varphi) - S(u(t,x)(z\varphi)}{h_n}\right| \le C_3\left(\frac{t}{2}\right) C_4(t) \exp\left(C_2\left(\frac{t}{2}\right)|z|^2 \|\varphi\|_p^2\right).$$

Now we apply Theorem 4.39 and obtain that u is differentiable w.r.t. t at $(t,x) \in A\cap B\times\mathbb{R}$. The first part is finished.

Part 2: It is left to show that for every finite time $T \in \mathbb{N}$ we can find an $s \in \mathbb{R}$ s.t. $u(t,x) \in \mathcal{G}_{-s}$ for all $t \in [0,T]$ and $x \in \mathbb{R}$. We prove this via the statement in Remark 5.16. To this end let $P = \sum_{l=1}^{m} \langle e_l, \cdot\rangle\, e_l$ be a projection as in Definition 5.10 and $\varepsilon > 0$. We define for $\eta_1 + i\eta_2 = \eta \in S'_{\mathbb{C}}(\mathbb{R})$ the complex vector $z = (z_l)_{l=1,..,m} \in \mathbb{C}^m$ by $z_l = z_{l,1} + iz_{l,2}$, $z_{l,j} = \langle e_l, \eta_j\rangle \in \mathbb{R}$, $j = 1,2$, and the real vector $\rho = (\rho_l)_{l=1,..,m} \in \mathbb{R}^m$ via $\rho_l = \rho_l(t) = \langle 1_{[0,t]}\sigma, e_l\rangle$. Hence, we obtain

$$\begin{aligned}|Su(t,x)(\varepsilon P\eta)|^2 &= \left|\frac{1}{\sqrt{2\pi\vartheta(t)}} \exp\left(-\frac{1}{2\vartheta(t)}\left(x - \varepsilon\int_{[0,t]} \sigma(s)P\eta(s)\,ds\right)^2\right)\right|^2 \\ &= \frac{1}{2\pi\vartheta(t)} \exp\left(-\frac{\varepsilon^2}{\vartheta(t)}\left(\frac{x}{\varepsilon} - \sum_{l=1}^{m} z_{l,1}\rho_l\right)^2\right) \exp\left(\frac{\varepsilon^2}{\vartheta(t)}\left(\sum_{l=1}^{m} z_{l,2}\rho_l\right)^2\right)\end{aligned}$$

Now we calculate the integral of $|S(u(t,x))(\varepsilon P\cdot)|^2$ w.r.t. the measure $\nu = \mu_{\frac{1}{2}} \otimes \mu_{\frac{1}{2}}$. We denote by I_m the $m\times m$ unit matrix. Using Proposition 5.8 we conclude

$$\begin{aligned}&\int_{S_{\mathbb{C}}(\mathbb{R})} |S(u(t,x))(\varepsilon P\eta)|^2\, d\nu(\eta) \\ =& \frac{1}{\sqrt{2\pi\vartheta(t)\pi^m}} \int_{\mathbb{R}^m} \exp\left(-\frac{\varepsilon^2}{\vartheta(t)}\left(\frac{x}{\varepsilon} - y_1^\top\rho\right)^2\right) \exp\left(-|y_1|^2\right)\, dy_1 \\ &\times \frac{1}{\sqrt{2\pi\vartheta(t)\pi^m}} \int_{\mathbb{R}^m} \exp\left(\frac{\varepsilon^2}{\vartheta(t)}(y_2^\top\rho)^2\right) \exp\left(-|y_2|^2\right)\, dy_2 \\ =& \frac{1}{\sqrt{2\pi\vartheta(t)\pi^m}} \int_{\mathbb{R}^m} \exp\left(-y_1^\top\left(I_m + \frac{\varepsilon^2}{\vartheta(t)}\rho\rho^\top\right)y_1 + y_1^\top\frac{2\varepsilon x}{\vartheta(t)}\rho - \frac{x^2}{\vartheta(t)}\right) dy_1 \\ &\times \frac{1}{\sqrt{2\pi\vartheta(t)\pi^m}} \int_{\mathbb{R}^m} \exp\left(-y_2^\top\left(I_m - \frac{\varepsilon^2}{\vartheta(t)}\rho\rho^\top\right)y_2\right) dy_2. \end{aligned} \tag{5.17}$$

Define the matrices $A_\pm = I_m \pm \frac{\varepsilon^2}{\vartheta(t)}\rho\rho^\top$. From this point we see that a necessary condition for $\sup_{P\in\mathbb{P}} \int_{S_{\mathbb{C}}(\mathbb{R})} |S(u(t,x))(\varepsilon P\eta)|^2\,\nu(d\eta)$ to be finite for some $\varepsilon > 0$ is the positive definiteness of $A_\pm$. From Sylvester's determinant identity we obtain $\det(A_\pm) = 1 \pm \frac{\varepsilon^2}{\vartheta(t)}|\rho|^2$. Furthermore, it holds $|\rho|^2 \le \int_{[0,t]} \sigma^2(s)\,ds$. To ensure the positive definiteness of $A_\pm$ we choose $\varepsilon > 0$ s.t. $0 < \varepsilon^2\kappa(t) < 1$.

In particular, for this choice of ε it holds

$$A_+^{-1} = I_m - \frac{1}{1 + \frac{\varepsilon^2 |\rho|^2}{\vartheta(t)}} \frac{\varepsilon^2}{\vartheta(t)} \rho \rho^\top .$$

Evaluating the Gaussian integrals in (5.17) yields

$$\int_{S_{\mathbb{C}}(\mathbb{R})} |S(u(t,x))(\varepsilon P \eta)|^2 \, d\nu(\eta) = \frac{\exp\left(-\frac{x^2}{\vartheta(t)}\right)}{2\pi\vartheta(t)\sqrt{\det(A_+ A_-)}} \exp\left(\frac{\varepsilon^2 x^2}{\vartheta(t)^2} \rho^\top A_+^{-1} \rho\right)$$

$$= \frac{\exp\left(-\frac{x^2}{\vartheta(t)}\right) \exp\left(\frac{x^2}{\vartheta(t)(\frac{\vartheta(t)}{\varepsilon^2|\rho|^2}+1)}\right)}{2\pi\vartheta(t)\sqrt{1 - \left(\frac{\varepsilon^2|\rho|^2}{\vartheta(t)}\right)^2}} < \infty.$$

We conclude that for $0 < \varepsilon^2 \kappa(t) < 1$ it holds

$$\sup_{P \in \mathbb{P}} \int_{S_{\mathbb{C}}(\mathbb{R})} |S(u(t,x))(\varepsilon P \eta)|^2 \, \nu(d\eta) \leq \frac{\exp\left(-\frac{x^2}{2\vartheta(t)}\right)}{2\pi\vartheta(t)\sqrt{1 - (\varepsilon^2\kappa(t))^2}} < \infty.$$

Observe, the assumptions on the coefficients ν and σ imply that $(0,\infty) \ni t \mapsto \kappa(t) \in \mathbb{R}$ is continuous. Consequently, by assumption κ is bounded on finite intervals $(0,T]$, $T \in \mathbb{N}$. Hence, for every $T \in \mathbb{N}$ we can choose $\varepsilon > 0$ s.t. $0 < \varepsilon^2\kappa(t) < 1$ for all $t \in (0,T]$. Eventually, we conclude by (5.10) that for $s \in \mathbb{R}$ fulfilling $2^{\frac{s}{2}} = \varepsilon$ it holds

$$u(t,x) \in \mathcal{G}_s, \ \text{for all } t \in (0,T], x \in \mathbb{R}.$$

In the case $\kappa(t) < 1$ we can choose $\varepsilon > 1$ and hence $s > 0$ which implies in particular $u(t,x) \in L^2(\mu)$, see Remark 5.16. □

Remark 5.18. *(i) The calculation in the proof of Theorem 5.17 above shows that Donsker's delta $\delta_x(\langle f,\cdot\rangle)$, $x \in \mathbb{R}$, $f \in L^2(\mathbb{R})$, is an element of $\mathcal{G}_{-\varepsilon}$ for all $\varepsilon > 0$.*

(ii) By using for example the generating function of the Hermite polynomials, see (B.1), one finds the generalized chaos expansion of the element $u(t,x)$ given in (5.15), i.e.,

$$S(u(t,x))(\varphi) = \frac{1}{\sqrt{2\pi\vartheta(t)}} \exp\left(-\frac{x^2}{2\vartheta(t)}\right) \sum_{n=0}^{\infty} \frac{1}{n!} H_{n,\vartheta(t)}(x) \left(\frac{\int_{[0,t]} \sigma\varphi \, ds}{\vartheta(t)}\right)^n$$

$$\Longrightarrow u(t,x) = \frac{1}{\sqrt{2\pi\vartheta(t)}} \exp\left(-\frac{x^2}{2\vartheta(t)}\right) \sum_{n=0}^{\infty} \frac{1}{n!\vartheta(t)^n} H_{n,\vartheta(t)}(x) \left\langle (1_{[0,t]}\sigma)^{\otimes n}, :\cdot^{\otimes n}: \right\rangle . \quad (5.18)$$

Hence, one could also determine $s \in \mathbb{R}$ s.t. the solution $u(t,x)$ belongs to $\mathcal{G}_s$ by considering the single elements of the chaos expansion in detail. To obtain a sharp $s \in \mathbb{R}$ one needs lower and upper estimates for the growth of Hermite polynomials. This seems to be more involved than the elementary calculation of Gaussian integrals in the proof of Theorem 5.17.

Finally we want to give some remarks concerning the solution $(u(t,x))_{t,x}$ constructed in Theorem 5.17 and the Itô equation (5.11). In the theory of stochastic differential equations one is interested in solutions which are adapted to the driving stochastic process of the equation under consideration. Recall that in the White noise framework one is dealing merely with equivalence classes of functions defined on the measure space $(S'(\mathbb{R}), \mathcal{B}, \mu)$. But measurability, hence adaptedness, of a class of functions is in general not well-defined. Therefore we explain in the following that there is an adapted modification of the process $(u(t,x))_{t\geq 0} \subseteq \mathcal{G}'$ w.r.t. the standard Brownian motion in White noise analysis, see Definition 5.21.

In the following let $\Phi = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}: \right\rangle \in \mathcal{G}'$. Recall that the S-transform of Φ is given by

$$S\Phi(\varphi) = \sum_{n=0}^{\infty} \left\langle \varphi^{\otimes n}, \Phi^{(n)} \right\rangle \tag{5.19}$$

and the dual pairing $\left\langle \varphi^{\otimes n}, \Phi^{(n)} \right\rangle$ is given by the scalar product $\left(\varphi^{\otimes n}, \overline{\Phi^{(n)}}\right)_{L^2(\mathbb{R}^n)}$. Hence, the S-transform of Φ admits a natural extension to elements $f \in L^2(\mathbb{R})$ which is also given by (5.19). Of course, this extension is nothing else than the dual pairing of Φ with $:\exp(\langle \cdot, f\rangle):$, $f \in L^2(\mathbb{R})$. The next lemma follows immediately from the polarization identity.

Lemma 5.19. *Let $\Phi = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, :\cdot^{\otimes n}: \right\rangle \in \mathcal{G}'$ and $I \subseteq \mathbb{R}$ be measurable. Then the following are equivalent:*

(i) $supp(\Phi^n) \subseteq I^n$ for all $n \in \mathbb{N}$.

(ii) $S\Phi(\varphi) = S\Phi(1_I \varphi)$ for all $\varphi \in S(\mathbb{R})$.

Lemma 5.20. *Let $F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \in L^2(\mu)$ and $I \subseteq \mathbb{R}$ be measurable. Then there exists a version $\tilde{F}$ of F which is measurable w.r.t. $\sigma(\langle \xi, \cdot \rangle, \xi \in S(\mathbb{R}), supp(\xi) \subseteq I)$ if and only if $supp(f^{(n)}) \subseteq I^n$ for all $n \in \mathbb{N}$.*

Proof. It suffices to prove the statement for $F = \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle$, $n \in \mathbb{N}$. Necessity can be proven as in [56, Proposition 4.5.]. To show sufficiency recall the construction of the element $\left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \in L^2(\mu)$. Namely, $\left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle$ is given as the $L^2(\mu)$-limit of functions given by $\sum_{k=1}^{m} \alpha_k H_{n,\sigma_k}(\langle \cdot, f_k \rangle)$, where $m \in \mathbb{N}$, $\alpha_k \in \mathbb{C}$, $f_k \in S(\mathbb{R})$ and $\sum_{k=1}^{m} \alpha_k f_k^{\otimes n} \longrightarrow f^{(n)}$ in $\widehat{L^2(\mathbb{R}^n)}$. If $\text{supp}(f^{(n)}) \subseteq I^n$, we can choose f_k s.t. $\text{supp}(f_k) \subseteq I$. Hence, F is the limit of functions which are measurable w.r.t. $\sigma(\langle \xi, \cdot \rangle, \xi \in S(\mathbb{R}), \text{supp}(\xi) \subseteq I)$ which finishes the proof. □

Definition 5.21. *Observe that via the Kolmogorov continuity theorem one can find a modification $(B_t)_{t\geq 0}$ of the family $\left(\left\langle 1_{[0,t]}, \cdot \right\rangle\right)_{t\geq 0} \subseteq L^2(\mu)$ s.t. $(B_t)_{t\geq 0}$ is a Brownian motion. We denote the natural filtration of the Brownian motion $(B_t)_{t\geq 0}$ by $(\mathcal{F}_t)_{t\geq 0}$, i.e., $\mathcal{F}_t = \sigma(B_s \mid s \in [0,t])$, $t \geq 0$.*

Lemma 5.22. *Assume that $F : S'(\mathbb{R}) \longrightarrow \mathbb{C}$ is measurable w.r.t. $\mathcal{F}_t$, $t \in [0, \infty)$. Then, there exists $G : S'(\mathbb{R}) \longrightarrow \mathbb{C}$ which is measurable w.r.t. $\mathcal{A}_t := \sigma(\langle \xi, \cdot \rangle \mid \xi \in S(\mathbb{R}), supp(\xi) \subseteq [0,t])$ and it holds $F = G$ μ-a.e. and vice versa.*

Proof. By using [20, Corollary 2.9] it suffices to show that a version of B_s is measurable w.r.t. $\mathcal{A}_t$ for $s \in [0, t]$ and that a version of $\langle \xi, \cdot \rangle$ is measurable w.r.t. $\mathcal{F}_t$ for $\xi \in S(\mathbb{R})$ with $\mathrm{supp}(\xi) \subseteq [0, t]$. Both statements follow by the same argument as in the proof of Lemma 5.20. □

Combining the previous lemmas we obtain the following corollary.

Corollary 5.23. *Denote by $(u(t, x))_{t,x}$ the solution of (5.13), (5.14) given in Theorem 5.17. Let $t \in [0, \infty)$ and $x \in \mathbb{R}$ s.t. $u(t, x) \in L^2(\mu)$ then there exists a version of $u(t, x)$ which is measurable w.r.t. $\mathcal{F}_t$.*

Several remarks are in order.

Remark 5.24. (i) Let $f : \mathbb{R} \longrightarrow \mathcal{G}'$ and u the element defined in (5.15). Assume that for all $(t, x) \in (0, \infty) \times \mathbb{R}$ the map

$$\mathbb{R} \ni y \mapsto f(y) \diamond u(t, x - y) \in \mathcal{G}'$$

is weakly in $L^1(\mathbb{R})dx)$, i.e., for every $F \in \mathcal{G}$ it holds that $\mathbb{R} \ni y \mapsto \langle F, f(y) \diamond u(t, x - y)\rangle \in \mathbb{C}$ is in $L^1(\mathbb{R}, \mathcal{B}, dx)$. Then we can define the Pettis-integral given as

$$u_f(t, x) := \int_{\mathbb{R}} f(y) \diamond u(t, x - y)\, dy \in \mathcal{G}', \tag{5.20}$$

see also [57, Proposition 8.1] and [87, Proposition 2.6.]. Under additional assumptions on f we obtain that u_f satisfies the initial condition $u_f(0, x) = \lim_{t \to 0} u_f(t, x) = f(x)$ for all $x \in \mathbb{R}$. If we assume that the time and space derivatives $\frac{d}{dt}$, $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ commute with the Pettis-integral (5.20) we obtain that $u_f : \mathbb{R}_{\geq 0} \times \mathbb{R} \longrightarrow \mathcal{G}'$ satisfies (5.13) with the initial condition $u_f(0, x) = f(x)$.

(ii) If we assume for the sake of simplicity that the initial data f in (i) is deterministic and an element of $S(\mathbb{R})$, then all steps in (i) are justified. This can be seen by the estimates in the first part of the proof of Theorem 5.17. Thus, we obtain that u_f given by (5.20) is a solution to (5.13) with initial condition $u_f(0, x) = f(x)$ for all $x \in \mathbb{R}$. From Theorem 5.17 and Lemma 5.19 we can conclude that $(u_f(t, x))_{t\in[0,T]}$, $x \in \mathbb{R}$, is a generalised stochastic process and adapted in the sense of [15, Definition 1]. In particular, if $u_f(t, x) \in L^2(\mu)$, then it holds by Lemma 5.20 and Lemma 5.22 that a version of $u_f(t, x)$ is measurable w.r.t. $\mathcal{F}_t$ for all $x \in \mathbb{R}$. It is well-known that for such a process the Itô integral and the Hitsuda-Skorokhod integral coincide, see e.g. [57, Theorem 8.7.]. Indeed, if for $x \in \mathbb{R}$ and $T \in [0, \infty)$ the process $\left(\sigma(s)\frac{\partial u_f(s,x)}{\partial x}\right)_{s\in[0,T]}$ is in $L^2([0, T]; L^2(\mu))$, then it holds for $t \in [0, T]$

$$\int_0^t \sigma(s)\frac{\partial u_f(s, x)}{\partial x} \diamond w_s\, ds = \int_0^t \sigma(s)\frac{\partial u_f(s, x)}{\partial x}\, dB_s \quad \mu\text{-a.e.},$$

where the term on the right-hand side is the Itô integral of an $(\mathcal{F}_s)_{s\in[0,T]}$-adapted modification of $\left(\sigma(s)\frac{\partial u_f(s,x)}{\partial x}\right)_{s\in[0,T]}$ w.r.t. the Brownian motion $(B_s)_{s\in[0,T]}$, see e.g. [57, Theorem 8.7.], [15, Proposition 4].

To conclude this section we proof that elements $\langle f^{(n)}, :\cdot^{\otimes n}: \rangle$ have a representation as iterated Itô integrals. Although this seems to be folklore wisdom in White Noise analysis, the standard literature like [57, 79, 65] lacks a proof for that fact. In the context of Malliavin calculus a similar statement for multiple Itô integrals is known, see e.g. [78, Section 1.1.2]. Fortunately, there is no need to introduce multiple Itô integrals to proof the statement. Recall that by $(B_t)_{t\geq 0}$ we denote the Brownian Motion from Definition 5.21.

Theorem 5.25. *Let $n \in \mathbb{N}$ and $f^{(n)} \in \widehat{L^2(\mathbb{R}^n)}$. Then it holds*

$$\left\langle 1_{[0,t]}^{\otimes n} f^{(n)}, :\cdot^{\otimes n}: \right\rangle = n! \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} f^{(n)}(t_1, \dots, t_n)\, dB_{t_n} \dots dB_{t_2}\, dB_{t_1}, \text{ for all } t \geq 0. \tag{5.21}$$

Remark 5.26. *Before we prove Theorem 5.25 several remarks are in order:*

1. *The statement in Theorem 5.25 is only true for the particular Brownian Motion $(B_t)_{t\geq 0}$ stated in Definition 5.21.*

2. *The integral in (5.21) is well-defined. Indeed, due to the Itô isometry two different version of $f^{(n)}$ lead to the same μ-class. Furthermore, by performing the integrals on the right-hand side of (5.21) successively we obtain after each step, by fixing the remaining variables of $f^{(n)}$, an adapted and square integrable process. Hence, the Itô integral in the next iteration step is well-defined.*

Proof of Theorem 5.25. Throughout the entire proof we fix $t \geq 0$. We prove the statement via induction on n. Let $n = 1$. Due to the Itô isometry and the isometry in (4.12) it suffices to show the statement for $f \in S(\mathbb{R})$. To show (5.21) it suffices to prove that the corresponding S-transforms coincide. To this end let $\varphi \in S(\mathbb{R})$. Recall that the Itô integral is defined as a $L^2(\mu)$-limit of Itô integrals of simple processes. Then it holds

$$\begin{aligned}
S\left(\int_0^t f(s)\, dB_s\right)(\varphi) &= \lim \sum_{i=0}^{m-1} f(s_i) S\left((B_{s_{i+1}} - B_{s_i})\right)(\varphi) \\
&= \lim \sum_{i=0}^{m-1} f(s_i) \int_{s_i}^{s_{i+1}} \varphi(s)\, ds \\
&= \int_0^t f(s)\varphi(s)\, ds \\
&= S\left(\left\langle 1_{[0,t]} f, :\cdot^{\otimes 1}: \right\rangle\right)(\varphi),
\end{aligned}$$

where the limit in the first line is taken in the $L^2(\mu)$-sense over all partitions of $[0, t]$ s.t. the maximum distance between two points of the partition tends to zero. Now let $n > 1$ and $f^{(n)} \in \widehat{L^2(\mathbb{R}^{n+1})}$. Both sides of (5.21) are linear in $f^{(n+1)}$. Hence, via the Itô isometry it suffices to show the statement for $f^{(n+1)} = f^{\otimes(n+1)}$ for $f \in S(\mathbb{R})$. We show again that both sides have the same S-transform. By Induction hypothesis we know that the right-hand side of (5.21) fulfills for all

$t \geq 0$

$$n! \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} f^{(n)}(t_1, \dots, t_n)\, dB_{t_n} \dots dB_{t_2}\, dB_{t_1} = n \int_0^t f(t_1) X_{t_1}\, dB_{t_1}, \tag{5.22}$$

where $X_{t_1} = \langle (1_{[0,t_1]} f)^{\otimes n-1}, :\cdot^{\otimes n-1}:\rangle$. Hence, we obtain for $\varphi \in S(\mathbb{R})$ as in the case $n = 1$

$$S\left(n! \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} f^{(n)}(t_1, \dots, t_n)\, dB_{t_n} \dots dB_{t_2}\, dB_{t_1} \right)(\varphi) = n \lim \sum_{i=0}^{m-1} f(s_i) S\left(X_{s_i}(B_{s_{i+1}} - B_{s_i}) \right)(\varphi). \tag{5.23}$$

From Lemma 4.27 we obtain $X_{s_i}(B_{s_{i+1}} - B_{s_i}) = \langle (1_{[0,s_i]} f)^{\otimes n-1} \hat{\otimes} 1_{[s_i, s_{i+1}]}, :\cdot^{\otimes n}:\rangle$ and therefore

$$S\left(X_{s_i}(B_{s_{i+1}} - B_{s_i}) \right)(\varphi) = \left(\int_0^{s_i} f(r)\varphi(r)\, dr \right)^{n-1} \int_{s_i}^{s_{i+1}} \varphi(r)\, dr.$$

Due to the continuity of f and φ we obtain in (5.23)

$$\begin{aligned} n \lim \sum_{i=0}^{m-1} f(s_i) S\left(X_{s_i}(B_{s_{i+1}} - B_{s_i}) \right)(\varphi) &= \int_0^t n f(t_1)\varphi(t_1) \left(\int_0^{t_1} f(r)\varphi(r)\, dr \right)^{n-1} dt_1 \\ &= \left(\int_0^t f(r)\varphi(r)\, dr \right)^n, \end{aligned}$$

which equals $S\left(\langle (1_{[0,t]} f)^{\otimes n}, :\cdot^{\otimes n}:\rangle \right)(\varphi)$ and therefore finishes the proof. □

Chapter 6

The Representation of the $(\Phi)^4_3$-Schwinger functions as moments of a positive Hida Distribution

In this chapter we deal with the so-called $(\Phi)^4_3$ model from axiomatic quantum field theory. The symbol $(\Phi)^4_3$ refers to a polynomial self-interacting field of degree 4 in space-time dimension $d = 3$. The importance of the $(\Phi)^4_3$ field is given by the following fact. So far, there is no other non-trivial model of polynomial interaction successfully constructed in space-time dimension $d \geq 3$. The $(\Phi)^4_3$ model is given by the collection of the corresponding Schwinger functions $(S_n)_{n\in\mathbb{N}_0}$. For every $n \in \mathbb{N}_0$, S_n is a distribution in $S'(\mathbb{R}^{3n})$. Additionally, S_n, $n \in \mathbb{N}$, is defined as the limit of distributions $S_n^{t,g} \in S'(\mathbb{R}^{3n})$ which are the moments of a positive measures $\mu_{t,g}$ on $(S'(\mathbb{R}^3), \mathcal{B})$, where t and g are appropriate cutoff parameters. Indeed, the measures $\mu_{t,g}$ are absolutely continuous w.r.t. the Euclidean free field measure μ_C on $(S'(\mathbb{R}^3), \mathcal{B})$, which is defined through its characteristic function $\hat{\mu}_C$ given by

$$\hat{\mu}_C(f) := \int\limits_{S'(\mathbb{R}^d)} \exp(i\langle f, \omega\rangle)\mu_C(d\omega) = \exp\left(-\frac{1}{2}(f, Cf)_{L^2(\mathbb{R}^d)}\right), \tag{6.1}$$

where $f \in S(\mathbb{R}^d)$ and $C = (-\Delta + m^2)^{-1}$ for a positive number m. The density of $\mu_{t,g}$ w.r.t. μ_C is given by

$$\frac{d\mu_{t,g}}{d\mu_C} = \frac{1}{Z_{t,g}} \exp\left(-V(t,g)\right), \tag{6.2}$$

where the cutoff interaction term $V(t,g)$ is bounded from below and satisfies $V(t,g) \in L^2(\mu_C)$ and $Z_{t,g}$ normalizes the measure $\mu_{t,g}$. At this point we also want to refer to the idea presented in the Introduction at the beginning of this thesis. We give the precise definition of $V(t,g)$ in Section 1 below. A milestone in axiomatic quantum field theory was achieved by showing that the measures $\mu_{t,g}$ converge in the sense of moments as the cutoff parameters are removed. This was established in a series of papers [34, 36, 72, 45]. We focus on the results given in [36]. There, the authors managed to prove that the moments $S_n^{t,g}$, $n \in \mathbb{N}$, of the measures $\mu_{t,g}$ satisfy a uniform bound (uniform w.r.t. the cutoff parameters) and converge pointwise as the cutoffs are removed, see also Theorem 6.3 below. Furthermore, in [36] it is shown that the Schwinger functions $(S_n)_{n\in\mathbb{N}_0}$ are the moments of a measure on $(S'(\mathbb{R}^3), \mathcal{B})$ which we denote by ν^4_3. This is our starting point. Since μ_C is a Gaussian measure we can use the results from Section 4.2 to obtain a nuclear triplet $(\mathcal{N}) \subseteq L^2(\mu_C) \subseteq (\mathcal{N})'$. We show in Section 1 that the measures $\mu_{t,g}$ correspond to Hida distributions $\Phi_{t,g} \in (\mathcal{N})'$ for every cutoff parameter t and g in the sense of Theorem 4.56. We use the results from [36] to show that the distribution $\Phi_{t,g}$ converge in the

space $(\mathcal{N})'$ to an element we denote by Φ^4_3 as the cutoffs are removed. In particular, the Schwinger functions $(S_n)_{n\in\mathbb{N}_0}$ are the moments of the positive distribution Φ^4_3. This can be understood as the main result in Section 1. In Section 2 we use the differential calculus of Hida distributions, see Subsection 4.2.3, to obtain convergence of the logarithmic derivatives of the measures $\mu_{t,g}$ in the distributional sense. Additionally, in Remark 6.17 we explain how this could be used as a general strategy to obtain field equations as in [35].
Before we start, we mention the concept of stochastic quantization of the measure ν^4_3, which is also part of our motivation in the study of the $(\Phi)^4_3$ model. The idea of the stochastic quantization of a measure ν consists basically of the construction of a Markov process $(X_t)_{t\geq 0}$ which attains ν as its invariant measure. Then one can study the measure ν via the stochastic process $(X_t)_{t\geq 0}$, see [82, 60]. One approach of constructing such a stochastic process is given by Dirichlet form methods. See e.g. the monographs [70] and [39] and the references therein for an introduction to Dirichlet forms. See also [5, 6] for in introduction to Dirichlet forms in White Noise analysis. The Dirichlet form approach for a stochastic quantization of ν^4_3 can be briefly described as follows. We consider on the Hilbert space $L^2(S'(\mathbb{R}^3), \nu^4_3)$ the densely defined quadratic form $(\mathcal{E}, (\mathcal{N}))$ given by

$$\mathcal{E}(F,G) = \int\limits_{S'(\mathbb{R}^3)} (\nabla F, \nabla G)_{H^{-1}(\mathbb{R}^3)} d\nu^4_3, \quad F, G \in (\mathcal{N}), \tag{6.3}$$

where ∇ denotes the gradient along $H^{-1}(\mathbb{R}^3)$. Here $H^{-1}(\mathbb{R}^3)$ is the Sobolev space of order -1, i.e., the completion of $L^2(\mathbb{R}^3)$ w.r.t. the norm $||C^{\frac{1}{2}} \cdot ||_{L^2(\mathbb{R}^3)}$. Here we face the first difficulty. We do not know if ν^4_3 has full topological support. Thus, the gradient ∇ might be ill-defined as an linear operator on $L^2(S'(\mathbb{R}^3), \nu^4_3)$ with domain $(\mathcal{N})$. Apart from this fact, it is unknown whether the form $(\mathcal{E}, (\mathcal{N}))$ is closable, see [70, Definition I.3.1]. The result in [9] indicates that there is no integration by parts formula available for the measure ν^4_3. One way to show well-definedness and closability of the form $(\mathcal{E}, (\mathcal{N}))$ is to decomposes $\mathcal{E}$ along an orthonormal basis $(h_k)_{k\in\mathbb{N}} \subseteq S(\mathbb{R}^3)$ of the Hilbert space $H^{-1}(\mathbb{R}^3)$, i.e.,

$$\begin{aligned} \mathcal{E}(F,G) &= \sum_{k=1}^{\infty} \mathcal{E}_{h_k}(F,G), \\ \mathcal{E}_{h_k}(F,G) &= \int\limits_{S'(\mathbb{R}^3)} (\nabla F, h_k)_{H^{-1}(\mathbb{R}^3)} (h_k, \nabla G)_{H^{-1}(\mathbb{R}^3)} d\nu^4_3, \quad F, G \in (\mathcal{N}), k \in \mathbb{N}. \end{aligned} \tag{6.4}$$

A sufficient condition for the well-definedness and the closability of $(\mathcal{E}, (\mathcal{N}))$ is given by the well-definedness and the closability of the single summands $\left(\mathcal{E}_{h_k}, (\mathcal{N})\right)$, $k \in \mathbb{N}$. A necessary and sufficient condition for the latter is given in terms of a disintegration of the measure ν^4_3, see [4] as well as the discussion at the end of Section 6.2. If one can show that the form $(\mathcal{E}, (\mathcal{N}))$ is closable on $L^2(S'(\mathbb{R}^3), \nu^4_3)$ we obtain a quasi-regular Dirichlet form, see [70, Definition IV.3.1., Section IV.4.]. Then, the result in [70, Theorem IV.3.5.] implies the existence of the a Markov process in the sense of Definition 2.27.

6.1 Construction of the $(\Phi)^4_3$-Field and its Representation as a positive Hida Distribution

In the beginning of this section we briefly explain the construction of the Schwinger functions $(S_n)_{n\in\mathbb{N}}$ of the $(\Phi)^4_3$-field given in [36]. Eventually we show that for $n \in \mathbb{N}$ the n-th Schwinger function S_n is given as the n-th moment of a positive Hida distribution $\Phi^4_3 \in (\mathcal{N})'$.

To apply the concepts from Chapter 4.2 let us fix the functional analytic framework we work in. We define the nuclear space $\mathcal{N}$ as the space of Schwartz functions over $\mathbb{R}^3$, i.e., $\mathcal{N} = S(\mathbb{R}^3)$. On $S(\mathbb{R}^3)$ we consider the family of seminorm $\|\cdot\|_p = \|H^p\cdot\|_{L^2(\mathbb{R}^3)}$, where H is given in Example 4.4. The completion of $S(\mathbb{R}^3)$ w.r.t. $\|\cdot\|_p$ we denote as usual by $\mathcal{H}_p$. As our central Hilbert space we choose $\mathcal{H} = H^{-1}(\mathbb{R}^3)$ the Sobolev space of order -1. Namely, $H^{-1}(\mathbb{R}^3)$ is given as the completion of $L^2(\mathbb{R}^3)$ w.r.t. the norm $||C^{\frac{1}{2}}\cdot||_{L^2(\mathbb{R}^3)}$ and the operator $C^{\frac{1}{2}}$ is defined through the Fourier transform $\mathcal{F}$ via

$$\mathcal{F}(C^{\frac{1}{2}}f)(\cdot) = \frac{1}{(|\cdot|^2+m^2)^{\frac{1}{2}}}\mathcal{F}f(\cdot), \quad f \in L^2(\mathbb{R}^3),$$

where m is a fixed positive number. Hence, we have the chain of continuous and dense embeddings given by

$$S(\mathbb{R}^3) \subseteq H^{-1}(\mathbb{R}^3) \subseteq S'(\mathbb{R}^3). \tag{6.5}$$

In particular, for $f, g \in S(\mathbb{R}^3)$ the dual pairing between f and g is given by $\langle f, g\rangle = (Cf, g)_{L^2(\mathbb{R}^3)}$. From the Bochner-Minlos theorem we obtain the Euclidean free field measure μ_C of mass m on $S'(\mathbb{R}^3)$ given by

$$\int_{S'(\mathbb{R}^d)} \exp(i\langle f, \omega\rangle)\mu_C(d\omega) = \exp\left(-\frac{1}{2}(f, Cf)_{L^2(\mathbb{R}^d)}\right), \quad f \in S(\mathbb{R}^d). \tag{6.6}$$

Recall that an element $F \in L^2(\mu_C)$ admits a chaos decomposition $F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}:\right\rangle$ where the Wick power $:\cdot^{\otimes n} :=: \cdot^{\otimes n} :_b$ is defined w.r.t. the bilinear form $b(\cdot,\cdot) = (C\cdot,\cdot)_{L^2(\mathbb{R}^3)}$, see Equation (4.11). If not mentioned otherwise, we only work with the Wick powers w.r.t. b, hence, for the sake of simplicity we omit the index b throughout this chapter. From Subsection 4.2.3 we obtain the spaces $(\mathcal{N})$ and $(\mathcal{N})'$ together with the embeddings

$$(\mathcal{N}) \subseteq L^2(\mu_C) \subseteq (\mathcal{N})'.$$

Note that the spaces $(\mathcal{N})$ and $(\mathcal{N})'$ depend on the bilinear form b, too. We start with a technical lemma.

Lemma 6.1. *Let $f \in S(\mathbb{R}^3)$, $n \in \mathbb{N}$ and $g \in L^1(\mathbb{R}^3)$ be given s.t. g vanishes $dx-$a.e. outside a compact set. Define for $x \in \mathbb{R}^3$ the function $\tau_x f(y) := f(y-x)$, $y \in \mathbb{R}^d$ and the map*

$$G : \mathbb{R}^3 \longrightarrow S(\mathbb{R}^3)^{\hat{\otimes} n}, x \mapsto g(x)\,(\tau_x f)^{\otimes n}.$$

It holds that G is Bochner-integrable and $\int_{\mathbb{R}^d} G(x)\,dx \in S(\mathbb{R}^d)^{\hat{\otimes} n}$, i.e., for every $p \in \mathbb{N}$ the map G is Bochner-integrable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$ with values in $\mathcal{H}_p^{\hat{\otimes} n}$.

Proof. It suffices to show that the map G is continuous. Indeed, if G is continuous, it is also continuous as a function with range in $\mathcal{H}_p^{\hat{\otimes} n}$. Hence, by Pettis measurability theorem, see [85], G is measurable and further by the continuity and the compact support of g we obtain integrability. For this purpose one can show that the map $\mathbb{R}^3 \ni x \mapsto \tau_x f \in S(\mathbb{R}^3)$ is continuous, but this is well-known, see e.g. [91, Exercise V.16]. □

In the following we fix a non-negative $g \in C_c^\infty(\mathbb{R}^3)$ and $t > 0$. We further define $f_t \in S(\mathbb{R}^3)$ by

$$f_t : \mathbb{R}^3 \longrightarrow \mathbb{R}, y \mapsto \frac{e^{-\frac{1}{2}m^2 t}}{\sqrt{2\pi t}^3} \exp\left(-\frac{|y|^2}{2t}\right). \tag{6.7}$$

The function f_t is a smeared version of the Dirac delta in 0 and corresponds to a smooth cut-off in momentum space, see (6.9) For $x \in \mathbb{R}^3$ we define a Gaussian random variable on $(S'(\mathbb{R}^3), \mathcal{B}, \mu_C)$ by

$$\Phi(x,t) : S'(\mathbb{R}^3) \longrightarrow \mathbb{R}, \omega \mapsto \langle \omega, \tau_x f_t \rangle \tag{6.8}$$

and obtain by the definition of μ_C it holds for $x, y \in \mathbb{R}^3$

$$\begin{aligned} \mathbb{E}_C\left[\Phi(x,t)\Phi(y,t)\right] &= \int_{S'(\mathbb{R}^3)} \Phi(x,t)\Phi(y,t) d\mu_C \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \exp\left(-ip^\top(x-y)\right) \frac{\exp\left(-t(|p|^2+m^2)\right)}{|p|^2+m^2}\, dp \end{aligned} \tag{6.9}$$

Due to Lemma 6.1 we obtain that the map

$$G_t : \mathbb{R}^d \longrightarrow S(\mathbb{R}^d)^{\hat{\otimes} n}, x \mapsto g(x)\left(\tau_x f_t\right)^{\otimes n}$$

is Bochner integrable in $\mathcal{H}_p$ for every $p \in \mathbb{N}$ and, hence, we can define for $\lambda > 0$ the interaction term as

$$V(t,g) = V_i(t,g) + V_c(t,g),$$

where

$$\begin{aligned} &V_i(t,g) : S'(\mathbb{R}^3) \longrightarrow \mathbb{R}, \\ &\qquad \omega \mapsto \lambda \int_{\mathbb{R}^3} g(x) : \Phi(x,t)^4(\omega) : \, dx := \lambda \left\langle \int_{\mathbb{R}^3} g(x)(\tau_x f_t)^{\otimes 4}\, dx, : \omega^{\otimes 4} :_C \right\rangle, \\ &V_c(t,g) : S'(\mathbb{R}^3) \longrightarrow \mathbb{R}, \\ &\qquad \omega \mapsto \frac{\lambda^2 \delta m_t^2}{2} \int_{\mathbb{R}^3} g^2(x) : \Phi(x,t)^2(\omega) : \, dx := \frac{\lambda^2 \delta m_t^2}{2} \left\langle \int_{\mathbb{R}^3} g(x)(\tau_x f_t)^{\otimes 2}\, dx, : \omega^{\otimes 2} :_C \right\rangle, \end{aligned}$$

$$\text{and } \delta m_t^2 := 4^2 6 \int_{\mathbb{R}^3} \mathbb{E}_C\left[\Phi(0,t)\Phi(y,t)\right]^3 \, dy \in \mathbb{R}.$$

Observe that $E_C\left[\Phi(0,t)\Phi(y,t)\right]$ is the Fourier transform of a rotation invariant function and, therefore, δm_t^2 is real valued. Due to Lemma 6.1 the function $V(t,g)$ is continuous w.r.t. the weak topology on $S'(\mathbb{R}^3)$ and, hence, measurable. By using the definition of the Wick ordering $:\cdot^{\otimes n}:$, $n \in \mathbb{N}$, or equivalently the definition of the Hermite polynomials and the fact that $(\tau_x f_t, C\tau_x f_t)_{L^2(\mathbb{R}^3)}$ is independent of $x \in \mathbb{R}^3$, we obtain for $\omega \in S'(\mathbb{R}^3)$

$$\begin{aligned} V_i(t,g)(\omega) &= \lambda \int_{\mathbb{R}^3} g(x) \left(\langle \tau_x f_t, \omega\rangle^4 - 6(\tau_x f_t, C\tau_x f_t)_{L^2(\mathbb{R}^3)} \langle \tau_x f_t, \omega\rangle^2 + 3(\tau_x f_t, C\tau_x f_t)^2_{L^2(\mathbb{R}^3)} \right) dx \\ &= \lambda \int_{\mathbb{R}^3} g(x) \left(\langle \tau_x f_t, \omega\rangle^2 - 3(f_t, Cf_t)_{L^2(\mathbb{R}^3)} \right)^2 - 6(f_t, Cf_t)^2_{L^2(\mathbb{R}^3)} \, dx \\ &\geq -6\lambda (f_t, Cf_t)^2_{L^2(\mathbb{R}^3)} \int_{\mathbb{R}^3} g(x)\, dx, \end{aligned} \tag{6.10}$$

Similarly, we obtain for $V_c(t,g)$

$$V_c(t,g)(\omega) \geq -\frac{\lambda^2 \delta m_t^2}{2} (f_t, Cf_t)_{L^2(\mathbb{R}^3)} \int_{\mathbb{R}^3} g(x)\, dx. \tag{6.11}$$

In particular, the interaction term $V(t,g) : S'(\mathbb{R}^3) \longrightarrow \mathbb{R}$ is bounded from below and it holds $V(t,g) \in (\mathcal{N})$. Observe that the lower bounds in (6.10) and (6.11) tend to minus infinite as t tends to zero, see also Section B.2. The term V_i incorporates the actual interaction one has in mind, i.e., a polynomial self-interaction of order 4, see also the explanation in the Introduction 0. The term V_c represents a so-called counter terms or renormalization term which is needed to obtain a well-defined limit.

Since μ_C is a probability measure, we obtain that $\exp(-V(t,g))$ is integrable and we can define the following normalization constant

$$Z_{t,g} := \int_{S'(\mathbb{R}^3)} \exp(-V(t,g))\, d\mu_C.$$

Furthermore, we can define a probability measure $\mu_{t,g}$, which is absolutely continuous w.r.t. μ_C, by

$$\frac{d\mu_{t,g}}{d\mu_C} = \frac{1}{Z_{t,g}} \exp(-V(t,g)).$$

The density of $\mu_{t,g}$ w.r.t. μ_C we also denote by $\Phi_{t,g} \in L^2(\mu_C)$. In particular, $\mu_{t,g}$ is a Hida measure, see Definition 4.57. The corresponding distribution is of course just the element $\Phi_{t,g}$, i.e., $\langle\langle F, \Phi_{t,g}\rangle\rangle = (F, \Phi_{t,g})_{L^2(\mu_C)}$.

Remark 6.2. *(i) In the original proof for the convergence of the measures $\mu_{t,g}$ one has to introduces additional counterterms in the function $V_c(t,g)$. These terms also diverge as t tends to zero but are constant w.r.t. the variable $\omega \in S'(\mathbb{R}^3)$ and, therefore, we collect them in the normalization constant $Z_{t,g}$.*

(ii) A detailed explanation how these counter terms arise from formal perturbation theory can be found in [12, Section 1.13].

For $n \in \mathbb{N}$ we define the n-th moment of $\mu_{t,g}$ as the n-linear functional given by

$$S^{t,g}_n : S(\mathbb{R}^3)^n \longrightarrow \mathbb{R}, (f_1, \ldots, f_n) \mapsto \int_{S'(\mathbb{R}^3)} \prod_{i=1}^n \langle f_i, \cdot \rangle \, d\mu_{t,g} = \langle\langle \prod_{i=1}^n \langle f_i, \cdot \rangle , \Phi_{t,g} \rangle\rangle. \tag{6.12}$$

We infer that $S^{t,g}_n$ is well-defined, i.e., the integral in (6.12) is finite, since the density of $\mu_{t,g}$ w.r.t. μ_C is bounded and μ_C, being a Gaussian measure, has moments of any order. Observe that the density $\frac{1}{Z_{t,g}} \exp(-V(t,g))$ and the measure μ_C are even, i.e., it holds for all $\omega \in S'(\mathbb{R}^3)$

$$\exp(-V(t,g))(R\omega) = \exp(-V(t,g))(\omega),$$
$$\mu_C \circ R^{-1} = \mu_C,$$

where $R\omega = -\omega$ for $\omega \in S'(\mathbb{R}^3)$. Hence, we immediately obtain that the odd moments are identical to zero

$$S^{t,g}_{2n-1} = 0, \text{ for all } n \in \mathbb{N}.$$

The moments $(S^{t,g}_n)_{n\in\mathbb{N}}$ are called the Schwinger functions of the corresponding cutoff measures $\mu_{t,g}$. Note that all objects $V(t,g)$, $\mu_{t,g}$, $\Phi_{t,g}$, $S^{t,g}_n$, $n \in \mathbb{N}$ depend additionally on the parameter λ. We only suppress this parameter to keep the notation manageable. Now we are ready to state the main result from [36].

Theorem 6.3. *Assume that λ is sufficiently small and the mass m is sufficiently large. Then there exists a uniform constant $p \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$ and $f_i \in S(\mathbb{R}^3)$, $i \in \{1, \ldots, 2n\}$, it holds*

$$\left| S^{t,g}_{2n}(f_1, \ldots, f_{2n}) \right| \leq K(2n!)^{\frac{1}{2}} \prod \|f_i\|_p . \tag{6.13}$$

Furthermore, the double limit

$$S_{2n}(f_1, \ldots, f_{2n}) := \lim_{g\to 1} \lim_{t\to 0} S^{t,g}_{2n}(f_1, \ldots, f_{2n}) \tag{6.14}$$

exists and also satisfies the bound from (6.13). Furthermore, $(S_n)_{n\in\mathbb{N}}$ satisfy all axioms of Osterwalder-Schrader (E0) − (E4), see also Chapter 7, where $S_{2n-1} = 0$ for all $n \in \mathbb{N}$.

Proof. See [36], in particular, Lemma 1.3 and 1.4, as well as the remark following Lemma 4.5 in the last mentioned reference. □

Remark 6.4. *The limit $\lim_{g\to 1}$ in (6.14) means that $\lim_{t\to 0} S^{t,g}_{2n}(f_1, \ldots, f_{2n})$ converges as the distance between the support of $1 - g$ and $0 \in \mathbb{R}^3$ tends to infinity. Note again that $(S_n)_{n\in\mathbb{N}}$ also depends on λ.*

For the rest of this chapter we fix the parameters λ and m s.t. the statement of Theorem 6.3 is valid. As a direct consequence of the Kernel Theorem 4.9 we obtain:

Corollary 6.5. *For every $n \in \mathbb{N}$ there exists $\Theta^{(2n)}, \in S'(\mathbb{R}^3)^{\hat{\otimes}2n}$ s.t. for all $f_i \in S(\mathbb{R}^3)$, $i \in \{1, \ldots, 2n\}$, it holds*

$$\left\langle \hat{\otimes}^{2n}_{i=1} f_i, \Theta^{(2n)} \right\rangle = S_{2n}(f_1, \ldots, f_{2n}).$$

Furthermore, for $s > \frac{1}{2}$ it holds

$$\left\|\Theta^{(2n)}\right\|_{-(p+s)} \leq (2n!)^{\frac{1}{2}} \left\|H^{-s}\right\|_{H.S.}^{2n}. \tag{6.15}$$

In the following we fix the two families of distributions $(\Theta^{(2n)})_{n\in\mathbb{N}}$, $(\Theta^{(2n)}_{t,g})_{n\in\mathbb{N}}$ given in the previous corollary. Occasionally, we also write $(\Theta^{(n)})_{n\in\mathbb{N}}$, $(\Theta^{(n)}_{t,g})_{n\in\mathbb{N}}$, where $\Theta^{(n)} = \Theta^{(n)}_{t,g} = 0$ if n is odd. The next theorem is taken from [5, Theorem 3.1.]. However, we give a proof which uses the characterization theorem given in 4.37, in contrast to the last mentioned reference. Obvious modifications of the proof given below show the result for general space-time dimension $d \in \mathbb{N}$.

Theorem 6.6. *Let $\Xi^{(n)} \in S'(\mathbb{R}^3)^{\hat{\otimes} n}$, $n \in \mathbb{N}$, and assume there exist $q \in \mathbb{N}_0$ and $K \in (0, \infty)$ s.t. for all $n \in \mathbb{N}$ it holds*

$$\Xi^{(n)} \in \mathcal{H}_{-q}^{\hat{\otimes} n}, \tag{6.16}$$

$$\left\|\Xi^{(n)}\right\|_{-q} \leq (n!)^{\frac{1}{2}} K^n. \tag{6.17}$$

Then there exists $\Psi \in (S)'$ s.t. for all $n \in \mathbb{N}$, $f_i \in S(\mathbb{R}^3)$, $i \in \{1, \ldots, n\}$, it holds

$$\langle\langle \prod_{i=1}^{n} \langle \cdot, f_i \rangle, \Psi \rangle\rangle = \left\langle \hat{\otimes}_{i=1}^{n} f_i, \Xi^{(n)} \right\rangle. \tag{6.18}$$

Furthermore, it holds for $s > \frac{1}{2}$ that

$$\|\Psi\|_{-(q+s)} \leq \sqrt{2} \left(1 - 2e^2 K \left\|H^{-s}\right\|_{H.S.}^2\right)^{-\frac{1}{2}}. \tag{6.19}$$

Proof. Consider the following map

$$U : S(\mathbb{R}^3) \longrightarrow \mathbb{C}, f \mapsto \sum_{n=0}^{\infty} \frac{i^n}{n!} \left\langle f^{\otimes n}, \Xi^{(n)} \right\rangle. \tag{6.20}$$

We show that U is a U-functional. Observe that, due to (6.17), we obtain for $f \in S(\mathbb{R}^3)$ and $z \in \mathbb{C}$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left| \left\langle (zf)^{\otimes n}, \Xi^{(n)} \right\rangle \right| \leq \sum_{n=0}^{\infty} \frac{(|z|\, K)^n}{\sqrt{n!}} \|f\|_p^n \leq \sqrt{2} \exp\left(4K^2 |z|^2 \|f\|_p^2\right). \tag{6.21}$$

Namely, the series in (6.20) is absolutely convergent, hence, U is well-defined. Next, we show the property (U1) from Definition 4.36. Now let $\lambda \in \mathbb{R}$ and $f, g \in S(\mathbb{R}^3)$. By using a similar estimate as in (6.21) we see that we can reorder the following series and obtain

$$\begin{aligned} U(f+g) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \left\langle (f+g)^{\otimes n}, \Xi^{(n)} \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda^k \left\langle f^{\otimes(n-k)} \hat{\otimes} g^{\otimes k}, \Xi^{(n)} \right\rangle \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!} \lambda^k, \end{aligned}$$

where $a_k = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left\langle f^{\otimes n} \hat{\otimes} g^{\otimes k}, \Xi^{(n+k)} \right\rangle$. Hence, $\mathbb{R} \ni \lambda \mapsto U(f+g) \in \mathbb{C}$ coincides with an everywhere convergent power series and can, therefore, be extended to an entire function on $\mathbb{C}$. We conclude that U is a U-functional. By Theorem 4.37 we obtain an unique element $\Psi \in (\mathcal{N})'$ s.t. $T\Psi = U$ and Ψ satisfies (6.19). It is left to show that (6.18) holds true. Due to the polarization identity it suffices to show

$$\sum_{n=0}^{m} \frac{i^n \langle \cdot, f \rangle^n}{n!} \xrightarrow{m\to\infty} \exp\left(i \langle f, \cdot \rangle\right) \text{ in } (\mathcal{N}) \text{ for all } f \in S(\mathbb{R}^3). \tag{6.22}$$

The convergence in (6.22) follows immediately by using the system of norms $\widetilde{|||\cdot|||}_q$, $q \in \mathbb{N}$, defined in (4.2.3) and Lemma 4.47. □

From Corollary 6.5 and Theorem 6.6 we obtain the following Corollary.

Corollary 6.7. *There exists $\Phi^4_3 \in (\mathcal{N})'$, which is in correspondence with the family $\left(\Theta^{(n)}\right)_{n\in N}$, i.e., for all $n \in \mathbb{N}$ and $f_i \in S(\mathbb{R}^3)$, $i \in \{1, \dots, 2\}$ it holds*

$$\langle\langle \prod_{i=1}^{n} \langle \cdot, f_i \rangle, \Phi^4_3 \rangle\rangle = \left\langle \hat{\otimes}_{i=1}^{n} f_i, \Theta^{(n)} \right\rangle,$$

Furthermore, it holds

$$\Phi^4_3 = \lim_{g\to 1} \lim_{t\to 0} \Phi_{t,g} \text{ in } (\mathcal{N})'. \tag{6.23}$$

In particular, $\Phi^4_3 \in (\mathcal{N})'_+$.

Proof. Only the statement in (6.23) needs clarification. Due to linearity we obtain from (6.14) that for every polynomial $P \in \mathcal{P}$ we obtain

$$\langle\langle P, \Phi^4_3 \rangle\rangle = \lim_{g\to 1} \lim_{t\to 0} \langle\langle P, \Phi_{t,g} \rangle\rangle.$$

Since the polynomials are dense in $(\mathcal{N})$ it suffices to show that the distributions $\left(\Phi_{t,g}\right)_{t>0,g}$ are uniformly bounded in $(\mathcal{H}_{-q})$ for some $q \in \mathbb{N}$. To this end we can proceed as we did to construct the distribution Φ^4_3. Indeed, due to Corollary 6.5 we obtain for every cutoff parameters t and g and every $n \in \mathbb{N}$ distributions $\Theta^{(n)}_{t,g}, \in S'(\mathbb{R}^3)^{\hat{\otimes} n}$ s.t. for all $f_i \in S(\mathbb{R}^3)$, $i \in \{1, \dots, n\}$, it holds

$$\left\langle \hat{\otimes}_{i=1}^{n} f_i, \Theta^{(n)}_{t,g} \right\rangle = S^{t,g}_n(f_1, \dots, f_n).$$

Since $\left(\Theta^{(n)}_{t,g}\right)_{n\in\mathbb{N}_0}$ satisfies the assumptions (6.16) and (6.17) of Theorem 6.6 uniformly in t and g we obtain uniform norm bounds of the distributions $\Phi_{t,g}$, which finishes the proof.

□

In the following we denote the probability measure on $(S'(\mathbb{R}^3), \mathcal{B})$ corresponding to Φ^4_3 by ν^4_3. Recall that the Schwinger functions S_{2n} still depend on λ and m which we suppress to keep the notation simple. Therefore, the measure ν^4_3 and the distribution Φ^4_3 also depend on λ and m.

Remark 6.8. *Of course one could construct the generalized chaos expansion of Φ^4_3 and $\Phi_{t,g}$ via the* *families* $(\Theta^{(n)})_{n\in N}$, $\left(\Theta^{(n)}_{t,g}\right)_{n\in\mathbb{N}}$. *This could be done for example via the relation (4.30). Actually, this reflects the logic for proof of Theorem 6.6 given in [5, Theorem 3.1.]. Since the elements* $\Theta^{(n)}$ *and* $\Theta^{(n)}_{t,g}$, $n \in \mathbb{N}$, *are merely implicitly known, an explicit representation of the chaos decomposition of* Φ^4_3 *and* $\Phi_{t,g}$ *are difficult to achieve.*

Corollary 6.9. *The measures* $\mu_{t,g}$ *converge weakly to* ν^4_3.

Proof. We know that the corresponding distributions converge in $(\mathcal{N})'$. The characteristic functions, which are the T-transforms of the corresponding distributions converge. The claim follows by the generalization of Lévys continuity theorem for nuclear spaces, see e.g. [74]. □

Remark 6.10. *(i) By definition the approximations* $\mu_{t,g}$ *of* ν^4_3 *are absolutely continuous w.r.t.* μ_C. *For the measure* ν^4_3 *this is not the case. One has that* ν^4_3 *and* μ_C *are singular w.r.t. each other, see [36, Theorem 4] and [38, Theorem 4.3.]. In Section 7.3 we present the general idea of [38, Theorem 4.3.] in detail to show singularity of a large class of measures on* $S'(\mathbb{R}^d)$, *see Corollary 7.27 and Corollary 7.28. The singularity of* ν^4_3 *and* μ_C *can be understood as an Euclidean version of Haag's theorem, see e.g. [102] as well as [62]. For the* $P(\Phi)_2$*-models a corresponding result was proven by Schrader in [94, 95]. We want to point out that the method presented in Section 7.3 applies for a large class of probability measures including the measures from the* $P(\Phi)_2$*-models.*

(ii) Recall that the central Hilbert space in the chain (6.5) was the negative Sobolev space $H^{-1}(\mathbb{R}^3)$ *which leads to the measure* μ_C *and also determines the Wick-ordering we consider. In the mathematical community and in particular in White noise analysis it is common to work with the central Hilbert space* $L^2(\mathbb{R}^3)$ *and the White noise measure* μ *defined by*

$$\int_{S'(\mathbb{R}^3)} \exp(i\langle f,\omega\rangle)\mu(d\omega) = \exp\left(-\frac{1}{2}(f,f)_{L^2(\mathbb{R}^3)}\right), \quad f \in S(\mathbb{R}^3), \tag{6.24}$$

instead of the measure μ_C. *One could equivalently start with this choice and obtain similar results. Another way to formulate the result above in the White noise setting is the following. Observe that the operator* $C^{\frac{1}{2}}$ *defined on* $L^2(\mathbb{R}^3)$ *is self-adjoint and maps* $S(\mathbb{R}^3)$ *continuously into itself, see e.g. [8, Example 2.2]. We also denote by* $C^{\frac{1}{2}}$ *the adjoint defined on* $S'(\mathbb{R}^3)$. *From their respective characteristic functions one obtains*

$$\mu \circ (C^{\frac{1}{2}})^{-1} = \mu_C.$$

At this point one should observe that the two measures μ_C *and* μ *are mutually singular, see Corollary 7.27. To distinguish between the two measures, we denote in the following by* $(\mathcal{N})_\mu$ *and* $(\mathcal{N})_{\mu_C}$ *the nuclear subspaces of* $L^2(\mu)$ *and* $L^2(\mu_C)$, *respectively. Recall that the elements of* $(\mathcal{N})_\mu$ *and* $(\mathcal{N})_{\mu_C}$ *have a unique continuous representative, see Theorem 4.46. Hence, we can define the following map*

$$\widetilde{\Gamma}\left(C^{-\frac{1}{2}}\right) : (\mathcal{N})_\mu \longrightarrow (\mathcal{N})_{\mu_C}, F \mapsto F \circ C^{-\frac{1}{2}}, \tag{6.25}$$

where we identified $F \in (\mathcal{N})_{\mu_C}$ *with its continuous version and* $C^{-\frac{1}{2}}$ *is the inverse operator of* $C^{\frac{1}{2}}$ *on* $S(\mathbb{R}^3)$. *Using the system of norms* $\left(\widetilde{|||\cdot|||_q}\right)_{q\in\mathbb{N}}$ *defined in (4.2.3) and the continuity* *of $C^{\frac{1}{2}}$ one obtains that the map in (6.25) is well-defined and continuous. In particular, the adjoint of $\widetilde{\Gamma}\left(C^{-\frac{1}{2}}\right)$ satisfies $\widetilde{\Gamma}\left(C^{-\frac{1}{2}}\right)^* \in L((\mathcal{N})'_{\mu_C}, (\mathcal{N})'_\mu)$ and preserves positiveness of Hida distributions, i.e., if $\Psi \in (\mathcal{N})'_{\mu_C,+}$ then it holds $\widetilde{\Gamma}\left(C^{-\frac{1}{2}}\right)^* \Psi \in (\mathcal{N})'_{\mu,+}$. thus, for all $n \in \mathbb{N}$ and all $f_1, \dots, f_n \in S(\mathbb{R}^3)$ it holds*

$$S_n(f_1, \dots, f_n) = \langle\!\langle \prod_{i=1}^n \left\langle C^{\frac{1}{2}} f_i, \cdot \right\rangle, \widetilde{\Gamma}\left(C^{-\frac{1}{2}}\right)^* \Phi_3^4 \rangle\!\rangle,$$

where the dual pairing on the right-hand side is the dual pairing between $(\mathcal{N})_\mu$ and $(\mathcal{N})'_\mu$. Hence, we can also represent the Schwinger functions of the $(\Phi)^4_3$ theory as the moments of a positive Hida distribution in the White Noise setting.

6.2 Remarks on the Logarithmic Derivative of ν_3^4

In the previous section we established that the Schwinger functions of the $(\Phi)^4_3$-theory are given as the moments of a positive Hida distribution Φ_3^4 with associated measure ν_3^4 on $(S'(\mathbb{R}^3), \mathcal{B})$. In this section we aim to give some remarks on the logarithmic derivative of the measure ν_3^4. We neither prove nor disprove that the logarithmic derivative of ν_3^4 exists as an integrable function on $S'(\mathbb{R}^3)$. Let us first remark some immediate consequences of the fact $\Phi_3^4 \in (\mathcal{N})'$ and the theory of differential operators on $(\mathcal{N})'$, see Subsection 4.2.3. For this purpose, let $F \in (\mathcal{N})$ and $h \in \mathcal{N}$. In the following, we denote the continuous version of F also by $\widetilde{F}$. Recall the linear operators ∂_h^* and $\widetilde{\partial}_h$ defined on $(\mathcal{N})'$. Then, it holds

$$\begin{aligned}\int_{S'(\mathbb{R}^3)} \partial_h \widetilde{F}\, d\nu_3^4 &= \langle\!\langle \partial_h F, \Phi_3^4 \rangle\!\rangle \\ &= \langle\!\langle F, \partial_h^* \Phi_3^4 \rangle\!\rangle \\ &= -\langle\!\langle F, \widetilde{\partial}_h \Phi_3^4 \rangle\!\rangle + \langle\!\langle \langle h, \cdot \rangle F, \Phi_3^4 \rangle\!\rangle.\end{aligned}$$

Note that the operator $\widetilde{\partial}_h$ is continuous on $(\mathcal{N})'$ and, therefore, we obtain $\widetilde{\partial}_h \Phi_3^4 = \lim_{t,g} \widetilde{\partial}_h \Phi_{t,g}$ where $\lim_{t,g} := \lim_{g \to 1} \lim_{t \to 0}$. Hence, we obtain

$$\int_{S'(\mathbb{R}^3)} \partial_h \widetilde{F}\, d\nu_3^4 = -\lim_{t,g} \langle\!\langle F, \widetilde{\partial}_h \Phi_{t,g} \rangle\!\rangle + \langle\!\langle \langle h, \cdot \rangle F, \Phi_3^4 \rangle\!\rangle. \tag{6.26}$$

The operator $\widetilde{\partial}_h$ is an extension of the Gateaux derivative defined on $(\mathcal{N})$. We know that $\Phi_{t,g} = \frac{1}{Z_{t,g}} \exp(-V(t,g))$, $V(t,g) \in (\mathcal{N})$, is a function defined on $S'(\mathbb{R}^3)$. It is necessary to address that the elements $\Phi_{t,g}$ are not elements of $(\mathcal{N})$, see Lemma 6.18 below. However, we still expect that $\widetilde{\partial}_h \Phi_{t,g}$ and the classical Gateaux derivative $\partial_h \Phi_{t,g}$ of $\Phi_{t,g}$ coincide, since $\partial_h \Phi_{t,g} \in L^2(\mu_C)$.

Indeed, the Gateaux derivative $\partial_h \Phi_{t,g}$ is given by

$$\begin{aligned}\partial_h \Phi_{t,g} =& -\frac{1}{Z_{t,g}} \partial_h V(t,g) \exp(-V(t,g)),\\ \partial_h V(t,g) =& 4\lambda \left\langle \int_{\mathbb{R}^3} g(x)(h, \tau_x f_t)_{H^{-1}(\mathbb{R}^3)} (\tau_x f_t)^{\otimes 3}\, dx, :\cdot^{\otimes 3}: \right\rangle\\ &+ \lambda^2 \delta m_t^2 \left\langle \int_{\mathbb{R}^3} g^2(x)(h, \tau_x f_t)_{H^{-1}(\mathbb{R}^3)} \tau_x f_t\, dx, \cdot \right\rangle.\end{aligned}$$

To show that $\widetilde{\partial}_h \Phi_{t,g}$ and $\partial_h \Phi_{t,g}$ coincide we show that the chaos decomposition of $\partial_h \Phi_{t,g}$ coincides with the chaos decomposition of $\widetilde{\partial}_h \Phi_{t,g}$, see (4.37).

In that case, it holds in (6.26)

$$\int_{S'(\mathbb{R}^3)} \partial_h \widetilde{F}\, d\nu_3^4 = \lim_{t,g} \int_{S'(\mathbb{R}^3)} F\left(\partial_h V(t,g) + \langle h, \cdot\rangle\right) \Phi_{t,g}\, d\mu_C, \text{ for all } F \in (\mathcal{N}). \tag{6.27}$$

To derive the chaos decomposition of $\partial_h \Phi_{t,g}$ we use a well-known result from the Malliavin calculus. First, we define the Gateaux derivative on a different domain $\mathcal{F}C_p^\infty$, the smooth cylinder functions on $S'(\mathbb{R}^3)$ with polynomially bounded derivatives. Denote by $C_b^\infty(\mathbb{R}^m)$ ($C_p^\infty(\mathbb{R}^m)$) the space of all complex valued smooth functions f defined on $\mathbb{R}^m$ s.t. f and all its partial derivatives are bounded (by some polynomial). Now, define

$$\mathcal{F}C_p^\infty := \left\{ F = f(\langle \xi_1, \cdot\rangle, \ldots, \langle \xi_m, \cdot\rangle) \mid m \in \mathbb{N}, f \in C_p^\infty(\mathbb{R}^m), \xi_j \in S(\mathbb{R}^3), j = 1, \ldots, m \right\}. \tag{6.28}$$

It holds that $F \in \mathcal{F}C_p^\infty$ is Gateaux differentiable in the direction of $h \in S(\mathbb{R}^3)$ and the derivative is given by

$$\partial_h F = \sum_{j=1}^m \partial_{x_j} f(\langle \xi_1, \cdot\rangle, \ldots, \langle \xi_m, \cdot\rangle)(h, \xi_j)_{H^{-1}(\mathbb{R}^3)},$$

where F is given as in (6.28). Observe that ∂_h leaves $\mathcal{F}C_p^\infty$ invariant. Furthermore, the set $\mathcal{F}C_p^\infty$ determines a dense subspace of $L^2(\mu_C)$, see e.g. [71, Section II.3.a)]. In addition, for elements $F, G \in \mathcal{F}C_p^\infty$ it holds

$$\int_{\mathcal{N}'} \partial_h F G\, d\mu_C = -\int_{\mathcal{N}'} F \partial_h G\, d\mu_C + \int_{\mathcal{N}'} FG\, \langle h, \cdot\rangle\, d\mu_C.$$

This follows of course from the integration by parts formula of Gaussian laws on $\mathbb{R}^m$, $m \in \mathbb{N}$. Hence, we obtain that an arbitrary power ∂_h^k, $k \in \mathbb{N}$, of ∂_h is closable on $L^2(\mu_C)$ since the adjoint operator is densely defined, see e.g. [91, Theorem VIII.1(b)]. We denote the closure of $\left(\partial_h^k, \mathcal{F}C_p^\infty\right)$, $k \in \mathbb{N}$, by $\left(\partial_h^k, D(\partial_h^k)\right)$.

Theorem 6.11. *Let $h \in \mathcal{N}$ and $k \in \mathbb{N}$. Then, the operator $\left(\partial_h^k, D(\partial_h^k)\right)$ on $L^2(\mu_C)$ is given by*

$$D(\partial_h^k) = \left\{ F = \sum_{n=0}^{\infty} \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \in L^2(\mu_C) \mid \sum_{n=k}^{\infty} n! \frac{n!}{(n-k)!} \left\| \left\langle h^{\otimes k}, f^{(n)} \right\rangle \right\|^2_{H^{-1}(\mathbb{R}^3)^{\otimes n}} < \infty \right\},$$

$$\partial_h^k F = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \left\langle \left\langle h^{\otimes k}, f^{(n)} \right\rangle, :\cdot^{\otimes (n-k)}: \right\rangle, \quad F \in D(\partial_h^k).$$

where $\left\langle h^{\otimes k}, f^{(n)} \right\rangle := h^{\otimes k} \otimes_k f^{(n)}$ is the tensor contraction defined in (4.15).

Proof. It suffices to show the claim for $k = 1$. The general case follows by inductive application of the statement for $k = 1$. The case $k = 1$ can be proven by using the exact same arguments as in [78, Proposition 1.2.1]. □

Corollary 6.12. *Let $h \in \mathcal{N}$ and $k \in \mathbb{N}$. For $F \in D(\partial_h^k)$ it holds $\partial_h^k F = \widetilde{\partial}_h^k F$.*

Remark 6.13. *A similar statement for $k = 1$ is well-known in the context of Dirichlet forms in White Noise analysis, see [6].*

In the following we show that $\Phi_{t,g} \in D(\partial_h^k)$ for all $h \in \mathcal{N}$ and $k \in \mathbb{N}$. To this end we need a proposition.

Proposition 6.14. *For every $n, k \in \mathbb{N}$ there exists a function $g_{n,k} : [0, \infty) \longrightarrow \mathbb{R}$, which is continuous, non-decreasing and satisfies $g_{n,k}(0) = 0$ s.t. for all $f^{(n)} \in H^{-1}(\mathbb{R}^3)^{\hat{\otimes} n}$ it holds*

$$\int_{S'(\mathbb{R}^3)} \left| \left\langle f^{(n)}, :\cdot^{\otimes n}: \right\rangle \right|^{2^k} d\mu_C \le g_{n,k}\left(\left\| f^{(n)} \right\|^2_{H^{-1}(\mathbb{R}^3)^{\hat{\otimes} n}} \right).$$

Proof. We prove the statement via induction on $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary and $f^{(n)}$ be given as above. For $k = 1$ the statement is nothing but the well-known Itô isometry, see (4.12). Now, let $k > 1$ and assume the statement is correct for all $\widetilde{k} \in \{1, ..., k-1\}$. Then we obtain from Lemma

4.27 and the induction hypothesis that

$$\begin{aligned}
\int\limits_{S'(\mathbb{R}^3)} \left|\left\langle f^{(n)}, :\cdot^{\otimes n}:\right\rangle\right|^{2^k} d\mu_C &= \int\limits_{S'(\mathbb{R}^3)} \left|\left\langle f^{(n)}, :\cdot^{\otimes n}:\right\rangle^2\right|^{2^{k-1}} d\mu_C \\
&= \int\limits_{S'(\mathbb{R}^3)} \left|\sum_{l=0}^{n} l!\binom{n}{l}^2 \left\langle f^{(n)}\hat{\otimes}_l f^{(n)}, :\cdot^{\otimes 2n-2l}:\right\rangle\right|^{2^{k-1}} d\mu_C \\
&\leq C_{n,k} \sum_{l=0}^{n} \int\limits_{S'(\mathbb{R}^3)} \left|\left\langle f^{(n)}\hat{\otimes}_l f^{(n)}, :\cdot^{\otimes 2n-2l}:\right\rangle\right|^{2^{k-1}} d\mu_c \\
&\leq C_{n,k} \sum_{l=0}^{n} g_{2n-2l,k-1}\left(\left\|f^{(n)}\hat{\otimes}_l f^{(n)}\right\|^2_{H^{-1}(\mathbb{R}^3)^{\hat{\otimes} n}}\right) \\
&\leq C_{n,k} \sum_{l=0}^{n} g_{2n-2l,k-1}\left(\left\|f^{(n)}\right\|^4_{H^{-1}(\mathbb{R}^3)^{\hat{\otimes} n}}\right),
\end{aligned}$$

where $C_{n,k}$ is a suitably chosen constant depending only on n and k. Hence, if the choice of the function $g_{n,k} := C_{n,k} \sum_{l=0}^{n} g_{2n-2l,k-1}\left(\cdot^2\right)$ completes the proof. □

Remark 6.15. *(i) The statement and the proof of Proposition 6.14 are also valid for a general Gaussian measure on the dual $\mathcal{N}'$ of a nuclear space.*

(ii) The result in Proposition 6.14 far from being optimal. Nevertheless, its proof is elementary and self-contained. Moreover, the result suffices to prove the next lemma. For more sophisticated results in this direction like hypercontractive bounds see e.g. [99, Section I.5.] and [77].

Lemma 6.16. *For every $h \in \mathcal{N}$ and $k \in \mathbb{N}$ it holds $\Phi_{t,g} \in D(\partial_h^k)$. In particular, it holds $\partial_h^k \Phi_{t,g} = \widetilde{\partial}_h^k \Phi_{t,g}$.*

Proof. Recall that $\Phi_{t,g} = \frac{1}{Z_{t,g}} \exp(-V(t,g))$ and $V(t,g) \in (\mathcal{N})$, where $\Phi_{t,g}$ is infinitely often Gateaux differentiable. Furthermore, the derivatives are elements of $L^2(\mu_C)$, which follows by Theorem 4.49 and Theorem 4.51. To show the claim we approximate the Bochner integrals involved in the definition of $V(t,g)$ by Riemann sums. Observe that this is possible since the integrands are continuous, i.e.,

$$\begin{aligned}
V(t,g) &= \lambda\left\langle \int\limits_{\mathbb{R}^3} g(x)(\tau_x f_t)^{\otimes 4}\, dx, :\cdot^{\otimes 4}:\right\rangle + \frac{\lambda^2 \delta m_t^2}{2}\left\langle \int\limits_{\mathbb{R}^3} g^2(x)(\tau_x f_t)^{\otimes 2}\, dx, :\cdot^{\otimes 2}:\right\rangle \\
&= \lim_{\varepsilon\to 0} \underbrace{\lambda\varepsilon^3 \sum_{x\in\varepsilon\mathbb{Z}^3} \left\langle g(x)(\tau_x f_t)^{\otimes 4}, :\cdot^{\otimes 4}:\right\rangle + \frac{\lambda^2\delta m_t^2}{2}\varepsilon^3 \sum_{x\in\varepsilon\mathbb{Z}^3}\left\langle g^2(x)(\tau_x f_t)^{\otimes 2}, :\cdot^{\otimes 2}:\right\rangle}_{V_\varepsilon(t,g)},
\end{aligned}$$

where the convergence takes place in $L^2(\mu_C)$. In particular, by Hölder's inequality and Proposition 6.14 we obtain that $V_\varepsilon(t,g)$ and its derivatives $\partial_h^l V_\varepsilon(t,g)$, $l \in \{1,2,3,4\}$, converge in every $L^p(\mu)$, $p \in [1,\infty)$, to $V(t,g)$ and $\partial_h^l V(t,g)$, $l \in \{1,2,3,4\}$, respectively. As in (6.10) and (6.11), we obtain

$$V_\varepsilon(t,g) \geq -\left(6\lambda(f_t, Cf_t)^2_{L^2(\mathbb{R}^3)} + \frac{\lambda^2 \delta m_t^2}{2}(f_t, Cf_t)_{L^2(\mathbb{R}^3)}\right)\varepsilon^3 \sum_{x\in\varepsilon\mathbb{Z}^3} g(x).$$

Hence we can find a lower bound $x_{t,g} \in \mathbb{R}$ uniformly in $\varepsilon \in (0,\delta)$ for a sufficiently small $\delta > 0$ s.t. $V(t,g), V_\varepsilon(t,g) \geq x_{t,g}$. Now, choose a function $\rho \in C_b^\infty(\mathbb{R})$, s.t. $\rho = e^{-x}$ for $x \geq x_{t,g}$. We conclude by Hölder's inequality that, for all $k \in \mathbb{N}$ it holds

$$\begin{aligned} \Phi_{t,g} &= \rho\left(V(t,g)\right) = \lim_{\varepsilon\to 0} \rho\left(V_\varepsilon(t,g)\right) \\ \partial_h^k \Phi_{t,g} &= \partial_h^k \rho\left(V(t,g)\right) = \lim_{\varepsilon\to 0} \partial_h^k \rho\left(V_\varepsilon(t,g)\right), \end{aligned}$$

in $L^2(\mu_C)$. □

Remark 6.17. *In [35] the authors derive a field equation for the so-called Wightman field of the $(\Phi)^4_3$-theory. For this purpose, the authors introduce the generalized cutoff Schwinger functions. These are given as a family of distributions $(S^{t,g}_{(n,k)})_{n,k\in\mathbb{N}_0}$ defined through*

$$S^{t,g}_{n,k}(f_1,\dots,f_n,h_1,\dots,h_k) = \frac{1}{Z_{t,g}} \int_{S'(\mathbb{R}^3)} \prod_{i=1}^{n} \langle f_i, \cdot\rangle \prod_{j=1}^{k} \partial_{h_j} \exp(-V(t,g))\, d\mu_C,$$

where $f_1,\dots,f_n,h_1,\dots,h_k \in S(\mathbb{R}^3)$ for $n,k \in \mathbb{N}_0$. To derive the field equation the authors need to show uniform bounds in t and g of the distributions $S^{t,g}_{n,k}$. This is done via a very technical proof. By using the fact that the derivatives $\frac{1}{Z_{t,g}} \prod_{j=1}^k \partial_{h_j} \exp(-V(t,g))$ converge in the space $(\mathcal{N})'$ we directly get the desired uniform estimates without further calculations. This procedure is not restricted to the $(\Phi)^4_3$ model and might apply to other models from constructive quantum field theory as well. Basically, if one establishes the convergence of the cutoff Schwinger function in the sense of convergence in the distribution space $(\mathcal{N})'$, one gets the convergence of the generalized Schwinger functions for free.

We conclude this section by some further comments on the measure ν_3^4 and the distribution $\widetilde{\partial}_h \Phi^4_3$. So far, it is an unanswered question whether $\widetilde{\partial}_h \Phi^4_3$ is given by a finite signed measure on $\left(S'(\mathbb{R}^3), \mathcal{B}\right)$. In particular, it is not known, if such a measure has a density w.r.t. ν_3^4. In terms of a stochastic quantization of the measure ν_3^4 via Dirichlet forms as described in the introduction of this chapter, this property is advantageous, see e.g. [70, Chapter II.]. At this point it is worth to briefly recall the results from [9]. These indicate that an answer to the questions above are quite challenging. There the authors showed that the $L^2(\mu_{t,g})$-norms of the logarithmic derivatives $\partial_h V(t,g)$ are unbounded as t shrinks to zero. Since $\partial_h V(t,g) \in (\mathcal{N})$, we can use Lemma 6.16 to extend the integration by parts formula for μ_C in (4.39) to $G = \Phi_{t,g}$ and obtain that

$$\begin{aligned}
&\int_{S'(\mathbb{R}^3)} (\partial_h V(t,g))^2 \, d\mu_{t,g} \\
=&\frac{1}{Z_{t,g}} \int_{S'(\mathbb{R}^3)} (\partial_h V(t,g))^2 \exp(-V(t,g)) \, d\mu_C \\
=&-\frac{1}{Z_{t,g}} \int_{S'(\mathbb{R}^3)} \partial_h V(t,g) \partial_h \exp(-V(t,g)) \, d\mu_C \\
=&\frac{1}{Z_{t,g}} \int_{S'(\mathbb{R}^3)} \partial_h^2 V(t,g) \exp(-V(t,g)) \, d\mu_C - \frac{1}{Z_{t,g}} \int_{S'(\mathbb{R}^3)} \langle h, \cdot \rangle \, \partial_h \exp(-V(t,g)) \, d\mu_C
\end{aligned} \tag{6.29}$$

The second term in (6.29) converges as $t \longrightarrow 0$. Furthermore, the second order derivative $\partial_h^2 V(t,g)$ is given by

$$\partial_h^2 V(t,g) = 12\lambda \int_{\mathbb{R}^3} g(x)(h, \tau_x f_t)^2_{H^{-1}(\mathbb{R}^3)} \left(\left\langle (\tau_x f_t)^{\otimes 2}, :\cdot^{\otimes 2}: \right\rangle + \frac{\lambda^2 \delta m_t^2}{12} \right) dx. \tag{6.30}$$

Hence, for fixed g the first summand on the right-hand side in (6.30) converges in $L^2(\mu_C)$ and the limit we denote by $\left\langle \int_{\mathbb{R}^3} g(x) C h^2(x) \delta_x^{\otimes 2} \, dx, :\cdot^{\otimes 2}: \right\rangle \in L^2(\mu_C)$. Indeed, for $x, y \in \mathbb{R}^3$ and $t, t' > 0$ we obtain

$$\begin{aligned}
(\tau_x f_t, \tau_y f_{t'})_{H^{-1}(\mathbb{R}^3)} &= \sqrt{2\pi}^3 \mathcal{F} \left(\frac{\exp(-(t+t')(|\cdot|^2 + m^2))}{|\cdot|^2 + m^2} \right) (y - x) \\
&\xrightarrow{t,t' \to 0} \sqrt{2\pi}^3 \mathcal{F} \left(\frac{1}{|\cdot|^2 + m^2} \right) (y - x),
\end{aligned} \tag{6.31}$$

where $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R}^3)$. Note that the convergence takes place in $L^2(\mathbb{R}^3)$ in the difference variable $z = y - x$. From (6.31) we obtain that any zero sequence $(t_n)_{n \in \mathbb{N}}$ of positive real numbers yields that $\left(\int_{\mathbb{R}^3} g(x) C h(x)^2 (\tau_x f_{t_n})^{\otimes 2} \, dx \right)_{n \in \mathbb{N}}$ is Cauchy sequence in $H^{-1}(\mathbb{R}^3)^{\hat{\otimes} 2}$. The corresponding limit is denoted by $\int_{\mathbb{R}^3} g(x) C h^2(x) \delta_x^{\otimes 2} \, dx$. Further, one sees that this limit is not an element of $L^2(\mathbb{R}^3)^{\hat{\otimes} 2}$. In particular, it holds

$$\left\langle \int_{\mathbb{R}^3} g(x) C h^2(x) \delta_x^{\otimes 2} \, dx, :\cdot^{\otimes 2}: \right\rangle \notin (\mathcal{N}). \tag{6.32}$$

Furthermore, due to the Wick renormalization $:\cdot^{\otimes 2}:$, we see that $\left\langle \int_{\mathbb{R}^3} g(x) C h^2(x) \delta_x^{\otimes 2} \, dx, :\cdot^{\otimes 2}: \right\rangle$ can be negative of arbitrary magnitude on a non negligible set. The authors in [9] claim that the term

$$\frac{1}{Z_{t,g}} \int\limits_{S'(\mathbb{R}^3)} \left\langle \int\limits_{\mathbb{R}^3} g(x)Ch(x)^2(\tau_x f_{t_n})^{\otimes 2}\, dx, :\cdot^{\otimes 2}: \right\rangle \exp(-V(t,g))\, d\mu_C \tag{6.33}$$

converges as $t \longrightarrow 0$. In the light of previously established results we argue that the convergence can not be immediately deduced and, hence, an additional argument is required. This can be found in [35, Theorem 3.] which basically says that the term in (6.33) converges as $t \longrightarrow 0$. Since the second term in (6.30) is constant and diverges as $t \longrightarrow 0$, see (B.13), we obtain the divergence of the $L^2(\mu_{t,g})$-norms of the logarithmic derivative of $\mu_{t,g}$. One should observe that this result does not rule out the possibility of a stochastic quantization of ν^4_3 via Dirichlet form techniques. It basically just shows that it is difficult to calculate a possible generator of the Dirichlet form (if the form exists). So far we also don't know how the topological support of the measure ν^4_3 looks like. Hence, one runs into trouble while defining a gradient Dirichlet form on $L^2(\nu^4_3)$ as in (6.3), since derivative operators are not necessarily well-defined on ν^4_3-classes of functions from $\mathcal{F}C^\infty_b$ or $(\mathcal{N})$. To investigate 'merely' well-definedness and closability of the forms $(\mathcal{E}_k, (\mathcal{N}))$ defined in (6.4), one should rather try to find the disintegration measures of the measure ν^4_3, see [4]. Here, several difficulties arise. First of all, from a measure theoretical point of view we have established ν^4_3 as the weak limit of the cutoff measures $\mu_{t,g}$. But a disintegration is in general not stable under weak convergence, see e.g. [73] for a counterexample[1]. Nevertheless, one can start calculating the disintegration measures of the approximations $\mu_{t,g}$. Here, one faces even more severe so-called 'infinities' than above. In the following we describe this in more detail. We fix $h \in S(\mathbb{R}^3)$ to be real and satisfies $\|h\|_{H^{-1}(\mathbb{R}^3)} = 1$. Denote by π_h the map

$$\pi_h : S'(\mathbb{R}^3) \longrightarrow S'(\mathbb{R}^3), x \mapsto x - \langle h, x\rangle\, h.$$

To disintegrate a probability measure ν on $(S'(\mathbb{R}^3), \mathcal{B})$ w.r.t. π_h means to find a family of probability measures $(\nu_x)_{x\in S'(\mathbb{R}^3)}$ which live on the measurable space $(S'(\mathbb{R}^3), \mathcal{B})$ and fulfill

(D1) $\nu_x(\pi_h^{-1}(\{x\})) = 1$ for all $x \in S'(\mathbb{R}^3)$,

(D2) $S'(\mathbb{R}^3) \ni x \mapsto \nu_x(A)$ is measurable for all $A \in \mathcal{B}$,

(D3) $\nu(A) = \int\limits_{S'(\mathbb{R}^3)} \nu_x(A)\, \nu \circ \pi_h^{-1}(dx)$ for all $A \in \mathcal{B}$.

Such a family $(\nu_x)_{x\in S'(\mathbb{R}^3)}$ is essentially unique, i.e., if $\left(\nu'_x\right)_{x\in S'(\mathbb{R}^3)}$ is another family satisfying (D1)-(D3) then it holds $\nu_x = \nu'_x$ for $\nu \circ \pi_h^{-1}$-a.e. $x \in S'(\mathbb{R}^3)$. Furthermore, in the case of the measure space $(S'(\mathbb{R}^3), \mathcal{B})$ such a family of measures $(\nu_x)_{x\in S'(\mathbb{R}^3)}$ always exists, see e.g. [29] and [21]. In particular, for a bounded measurable function $F : S'(\mathbb{R}^3) \longrightarrow \mathbb{R}$ it holds

$$\begin{aligned}\int\limits_{S'(\mathbb{R}^3)} F(y)\, \nu(dy) &= \int\limits_{S'(\mathbb{R}^3)} \int\limits_{S'(\mathbb{R}^3)} F(y)\, \nu_x(dy)\, \nu \circ \pi_h^{-1}(dx) \\ &= \int\limits_{S'(\mathbb{R}^3)} \int\limits_{\mathbb{R}} F(x+sh)\, \rho(x,ds)\, \nu \circ \pi_h^{-1}(dx),\end{aligned}$$

[1] Another useful counterexample can be found online in https://math.stackexchange.com/q/2648121 (version: 2018-02-13)

where $\rho(x, ds) = \nu_x \circ \langle h, \cdot \rangle^{-1}(ds)$. We also call the transition kernel ρ the disintegration kernel of ν w.r.t. π_h. In [4] Albeverio and Röckner established a characterization for well-definedness and closability for bilinear forms $\left(\mathcal{E}_h, \mathcal{F}C_b^\infty\right)$, $h \in S(\mathbb{R}^3)$, defined on the Hilbert space $L^2(\nu)$ by

$$\mathcal{E}_h(F, G) = \int_{S'(\mathbb{R}^3)} \partial_h F \partial_h G \, d\nu.$$

Recall from the introduction of this chapter, that it suffices to show well-definedness and closability for the bilinear forms $(\mathcal{E}_h, \mathcal{F}C_b^\infty)$, $h \in S(\mathbb{R}^3)$, to obtain the corresponding result for the full gradient form $(\mathcal{E}, \mathcal{F}C_b^\infty)$, defined in (6.3). This characterization, [4, Theorem 3.2], is formulated in terms of the transition kernel $\rho(x, ds)$ only. An affirmative answer to the question of well-definedness and closability can be given, if the measure $\rho(x, \cdot)$ satisfies the Hamza-condition for $\nu \circ \pi_h^{-1}$-a.e. $x \in S'(\mathbb{R}^3)$, see [4] for the precise statement and more details. A positive measure m on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to satisfy the Hamza condition, if it is absolutely continuous w.r.t. the Lebesque measure ds and further, its density κ fulfills the property (H)

$$(H) \qquad \begin{gathered} \kappa = 0 \; ds\text{-a.e. on } \mathbb{R} \setminus R(\kappa) \text{ where} \\ R(\kappa) = \left\{ t \in \mathbb{R} \mid \int_{[t-\varepsilon, t+\varepsilon]} \kappa^{-1} ds < \infty, \text{ for some } \varepsilon > 0 \right\}. \end{gathered}$$

For the Gaussian measure μ_C it is particularly easy to find the family of disintegration measures $(\mu_{C,x})_{x \in S'(\mathbb{R}^3)}$ w.r.t. π_h and accordingly the transition kernel $\rho_C(x, ds)$. Indeed, define for $x \in S'(\mathbb{R}^3)$ the measure $\mu_{C,x}$ via its characteristic function

$$\hat{\mu}_{C,x}(\varphi) = \exp\left(i \langle \varphi, \pi_h x \rangle - \frac{1}{2} (C\varphi, h)^2_{L^2(\mathbb{R}^3)} \right), \quad \varphi \in S(\mathbb{R}^3). \tag{6.34}$$

It is not difficult to check that $\left(\mu_{C,x}\right)_{x \in S'(\mathbb{R}^3)}$ satisfies (D1)-(D3) for the measure μ_C. From (6.34) we obtain $\rho_C(x, \cdot) = \mathcal{N}(0,1)(\cdot)$, i.e., $\rho_C(x, \cdot)$ is the standard normal distribution with mean 0 and variance 1. Recall that the approximations $\mu_{t,g}$ are absolutely continuous w.r.t. μ_C with density $\frac{1}{Z_{t,g}} \exp(-V(t,g))$. Hence, for bounded measurable F as above it holds

$$\begin{aligned} \int_{S'(\mathbb{R}^3)} F(y) \, \mu_{t,g}(dy) &= \int_{S'(\mathbb{R}^3)} \int_{\mathbb{R}} F(x + sh) \frac{1}{Z_{t,g}\sqrt{2\pi}} \exp\left(-V(t,g)(x + sh) - \frac{1}{2} s^2 \right) ds \, \mu_C \circ \pi_h^{-1}(dx) \\ &= \int_{S'(\mathbb{R}^3)} \int_{\mathbb{R}} F(x + sh) \, \rho_{t,g}(x, ds) \, N_{t,g}(x) \mu_C \circ \pi_h^{-1}(dx), \end{aligned}$$

where

$$\rho_{t,g}(x,ds) = \frac{1}{N_{t,g}(x)Z_{t,g}\sqrt{2\pi}} \exp\left(-V(t,g)(x+sh) - \frac{s^2}{2}\right) ds,$$

$$N_{t,g}(x) = \int_{S'(\mathbb{R}^3)} \frac{1}{Z_{t,g}} \exp\left(-V(t,g)\right) d\mu_{C,x}$$

$$= \int_{\mathbb{R}} \frac{1}{Z_{t,g}} \exp\left(-V(t,g)(x+sh) - \frac{s^2}{2}\right) ds.$$

Due to [21, Theorem 3] we obtain that $\rho_{t,g}(x,ds)$ is indeed the disintegration kernel of $\mu_{t,g}$ w.r.t. π_h and it holds

$$\mu_{t,g} \circ \pi_h^{-1} = N_{t,g}(x)\mu_C \circ \pi_h^{-1}.$$

Further, one sees that $\rho_{t,g}$ satisfies the Hamza condition (H), i.e.

$$\frac{d\rho_{t,g}(x,\cdot)}{ds}(s) = \frac{1}{\widetilde{N}_{t,\lambda,g}(x)} \exp\left(-V(t,g)(x+sh) - \frac{s^2}{2}\right), \tag{6.35}$$

$$\widetilde{N}_{t,\lambda,g}(x) = Z_{t,g}N_{t,\lambda,g}(x).$$

Observe that π_h is a continuous map on $S'(\mathbb{R}^3)$ equipped with the strong topology and therefore we obtain that $\mu_{t,g} \circ \pi_h^{-1}$ converges weakly to $\nu_3^4 \circ \pi_h^{-1}$ as the cutoffs are removed. One even obtains the stronger result stating that the convergence takes place in the strong topology of $(\mathcal{N})'$. Indeed, from Corollary 4.48 we obtain that

$$(\mathcal{N}) \ni F \mapsto \int_{S'(\mathbb{R}^3)} F \, d\mu_{t,g} \circ \pi_h^{-1} = \int_{S'(\mathbb{R}^3)} F \circ \pi_h \, d\mu_{t,g} = \langle\!\langle \widetilde{\Gamma}(\pi_h)F, \Phi_{t,g} \rangle\!\rangle \in \mathbb{C}$$

is an element of $(\mathcal{N})'$. The measure $\mu_{t,g} \circ \pi_h^{-1}$ corresponds to the distribution $\widetilde{\Gamma}(\pi_h)^*\Phi_{t,g} \in (\mathcal{N})'$. The assertion follows now by Lemma 4.7. In the following we denote by ρ_3^4 the disintegration kernel of ν_3^4 w.r.t. π_h. Observe that $\rho_3^4(x,\cdot)$ is only $\nu_3^4 \circ \pi_h^{-1}$-a.e. determined. But we don't have any knowledge about a measurable set of ν_3^4-measure 1 except $\mathcal{H}_{-p}$, for some $p \in \mathbb{N}$, where $\mathcal{H}_{-p}$ is the completion of $S(\mathbb{R}^3)$ w.r.t. the scalar product $(\cdot,\cdot)_{-p}$ defined in Example 4.4.

The above mentioned counterexamples point out that it might be impossible to gain any information about ρ_3^4 through the kernels $\rho_{t,g}$. In any case, we can simplify the density in (6.35) a little further via a Taylor expansion of the element $V(t,g) \in (\mathcal{N})$, see e.g. (B.2). Thus for $x \in S'(\mathbb{R}^3)$ and $s \in \mathbb{R}$ it holds

$$V(t,g)(x+sh) = \sum_{n=0}^{4} \frac{s^n}{n!} \partial_h^n V(t,g)(x).$$

Observe that for $n = 3,4$ the derivative $\partial_h^n V(t,g)(x)$ converges as the cutoffs are removed for every $x \in S'(\mathbb{R}^3)$ and $\partial_h^4 V(t,g)(x) \geq 0$. The term of order $n = 0$ vanishes via the normalization

$\widetilde{N}_{t,g}(x)$, i.e.,

$$\frac{d\rho_{t,g}(x,\cdot)}{ds}(s) = \frac{1}{\widehat{N}_{t,\lambda,g}(x)} \exp\left(-\sum_{n=1}^{4} \frac{s^n}{n!}\partial_h^n V(t,g)(x) - \frac{s^2}{2}\right), \tag{6.36}$$

$$\widehat{N}_{t,\lambda,g}(x) = \int_{\mathbb{R}} \exp\left(-\sum_{n=1}^{4} \frac{s^n}{n!}\partial_h^n V(t,g)(x) - \frac{1}{2}s^2\right) ds. \tag{6.37}$$

We have seen above that for $n = 1, 2$ the terms $\partial_h^n V(t,g)(x)$ are difficult to control. Fortunately, at this point everything is reduced to tackle the real integral in (6.37). It may be possible to show that due to the renormalization of $\widehat{N}_{t,g}(x)$, one obtains a well-defined limit of $\rho_{t,g}(x,\cdot)$ in an appropriate sense which inherits (H) as the cutoffs are removed. If one tries to show quasi-invariance of ν_3^4 w.r.t. a shift by $h \in S(\mathbb{R}^3)$ one faces similar difficulties as in (6.36) and (6.37).

Eventually we prove that $\Phi_{t,g}$ is not a element of $(\mathcal{N})$.

Lemma 6.18. *For every $t > 0$ and non-negative $g \in C_c^\infty(\mathbb{R}^3)$ it holds $\Phi_{t,g} \in L^2(\mu_C) \setminus (\mathcal{N})$.*

Proof. Assume for the sake of contradiction $\Phi_{t,g} \in (\mathcal{N})$ for some t and g. We use the fact that elements from $(\mathcal{N})$ admit an analytic extension to $S'_{\mathbb{C}}(\mathbb{R}^3)$ and a quadratic growth bound, see e.g. [65, Theorem 6.13.]. Indeed, there exists an extension of $\Phi_{t,g}$ to $S'_{\mathbb{C}}(\mathbb{R}^3)$, which we also denote by $\Phi_{t,g}$ s.t. for every $p \in \mathbb{N}$ the extension $\Phi_{t,g}$ is analytic on the complexification $\mathcal{H}_{-p,\mathbb{C}}$ of $\mathcal{H}_{-p}$. Furthermore, there exist $q \in \mathbb{N}$ and a constant $C = C_{p,q}$ s.t. it holds

$$\left|\Phi_{t,g}(\omega)\right| \le C \left\|\!\left|\Phi_{t,g}\right|\!\right\|_{(p+q)} \exp\left(\frac{1}{2}\,|\omega|_{-p}^2\right), \quad \omega \in \mathcal{H}_{-p,\mathbb{C}}. \tag{6.38}$$

Recall that $\Phi_{t,g}$ is given by $\Phi_{t,g} = \frac{1}{Z_{t,g}} \exp(-V(t,g))$. For $\alpha \in \mathbb{R}$ and $\omega \in S'(\mathbb{R}^3)$ it holds

$$\begin{aligned} V(t,g)(\alpha\omega) =& \lambda \left\langle \int_{\mathbb{R}^3} g(x)(\tau_x f_t)^{\otimes 4}\, dx, :(\alpha\omega)^{\otimes 4}:_C \right\rangle + \frac{\lambda^2 \delta m_t^2}{2} \left\langle \int_{\mathbb{R}^3} g(x)(\tau_x f_t)^{\otimes 2}\, dx, :(\alpha\omega)^{\otimes 2}:_C \right\rangle \\ =& \alpha^4 \lambda \int_{\mathbb{R}^3} g(x) \left\langle \tau_x f_t, \omega\right\rangle^4 \, dx + \alpha^2 \lambda \int_{\mathbb{R}^3} g(x) \left\langle \tau_x f_t, \omega\right\rangle \left(\frac{\lambda \delta m_t^2}{2} - (f_t, f_t)_{H^{-1}(\mathbb{R}^3)}\right) \\ &+ \lambda (f_t, f_t)_{H^{-1}(\mathbb{R}^3)} \left((f_t, f_t)_{H^{-1}(\mathbb{R}^3)} - \frac{\lambda \delta m_t^2}{2}\right) \int_{\mathbb{R}^3} g(x)\, dx. \end{aligned} \tag{6.39}$$

Hence, for $\alpha \in \mathbb{C}$ and $\omega \in S'(\mathbb{R}^3) \setminus \{0\}$ the analytic extension of $\Phi_{t,g}$ at $\alpha\omega$ is given by $\Phi_{t,g}(\alpha\omega) = \frac{1}{Z_{t,g}} \exp(-V(t,g)(\alpha\omega))$ where $V(t,g)(\alpha\omega)$ is given by (6.39), too. In particular, for $\alpha = \beta\sqrt{i}$, where $\beta \in \mathbb{R}$ and $\sqrt{i}$ is a square root of the imaginary unit i, and $\omega \in S'(\mathbb{R}^3)$ s.t. $\langle \tau_x f_t, \omega\rangle \neq 0$ for some $x \in \operatorname{supp}(g)$ we obtain that the function

$$\mathbb{R} \ni \beta \mapsto \left|\Phi_{t,g}(\sqrt{i}\beta\omega)\right| \in \mathbb{R}$$

grows like $\exp(C_\omega \beta^4)$, $C_\omega > 0$, as $\beta \longrightarrow \infty$. This contradicts the bound in (6.38) which finishes the proof. □

Chapter 7

The Existence of Non-trivial Relativistic Quantum Fields in arbitrary Space-Time Dimension

In this chapter we construct a relativistic quantum field in terms of corresponding Schwinger functions, which is not a generalized free field. The Schwinger functions are given as moments of a probability measure μ_ϱ on the measure space $\big(S'(\mathbb{R}^d), \mathcal{B}\big)$ where $d \in \mathbb{N}$ denotes the space-time dimension. The construction is different from the usual renormalization approach in the Euclidean strategy as in the $P(\Phi)_2$ or $(\Phi)^4_3$ models. The measure μ_ϱ, which we consider, is given as the superposition of Euclidean free field measures μ_m of mass $m > 0$. The symbol ϱ denotes a measure on the positive real line describing which masses $m > 0$ contribute to the superposition. In particular, the approach chosen here works for arbitrary space-time dimension $d \in \mathbb{N}$. The construction of the Schwinger functions is heavily inspired by the Källen-Lehmann representation of the two point function of a relativistic quantum field, see e.g. [89, Theorem IX.34].

To explain the underlying idea of our construction let us make the following observation. All Osterwalder-Schrader axioms, except the cluster property, are are linear constraints in a family of distributions $(S_n)_{n\in\mathbb{N}_0}$, see (E0)-(E4) below. Indeed, if $(S^1_n)_{n\in\mathbb{N}_0}$ and $(S^2_n)_{n\in\mathbb{N}_0}$ are two families of distributions satisfying the conditions (E0)-(E3) given below, then we obtain immediately that their sum $(S_n)_{n\in\mathbb{N}_0} := (S^1_n + S^2_n)_{n\in\mathbb{N}_0}$ satisfies (E0)-(E3), too. Only the cluster property is a non-linear condition. Thus, it is not directly clear, whether $(S_n)_{n\in\mathbb{N}_0}$ satisfies (E4),provided that $(S^1_n)_{n\in\mathbb{N}_0}$ and $(S^2_n)_{n\in\mathbb{N}_0}$ do. Further, from the construction of generalized free fields (g.f.f.), see e.g. [99, 46], one sees that the truncated vacuum expectation values corresponding to g.f.f. equal zero. See e.g. [90, Section XI.16] for the definition of truncated vacuum expectation values. Equivalently, the truncated Schwinger functions of g.f.f. are equal to zero. Observe that truncated Schwinger functions are given as the image of the respective Schwinger functions under a non-linear transformation. In particular, the truncated Schwinger functions of a superposition of Schwinger functions are in general not zero, even if this holds for the single summands. Consequently, if one can show that the cluster property holds for a superposition of Schwinger functions, one automatically obtains a non-generalized free field.

The content of this chapter is as follows. We start in Section 1 by introducing the complete list of Osterwalder-Schrader axioms and some related notations. Subsequently, we show for a certain class of probability measures $\varrho(dm)$ that the superposition of Schwinger functions corresponding to the Euclidean free field of mass m satisfies (E0)-(E4). The major challenge here is to prove the cluster property as explained above. We conclude the first section by showing that in general the corresponding truncated Schwinger functions don't vanish identically. In Section 2 we show that for every measurable set $A \in \mathcal{B}$ the map $m \mapsto \mu_m(A)$ is Borel measurable. Thus, we obtain what we intuitively expect, i.e., the measure μ_ϱ satisfies $\mu_\varrho(A) = \int_{\mathbb{R}_{\geq 0}} \mu_m(A)\varrho(dm)$ for all $A \in \mathcal{B}$. From this property many properties of μ_m, $m > 0$, directly lift to the measure μ_ϱ. In the last section of this chapter we don't prove any new result. We state a general concept known from ergodic theory which provides additional information of the measure μ_ϱ.

7.1 Construction of Schwinger Functions as Moments of a Superposition of Gaussian Measures

In the following we fix an arbitrary space-time dimension $d \in \mathbb{N}$. For sake of simplicity we occasionally don't distinguish between a continuous n-linear mapping on $S(\mathbb{R}^d)^n$ and a tempered distribution in $S'(\mathbb{R}^{dn})$, $n \in \mathbb{N}$, which is legitimated by Theorem 4.9. We denote by $(\|\cdot\|_p)_{p\in\mathbb{N}}$ the family of semi-norms on $S(\mathbb{R}^d)$ which are induced by the scalar products given in Example 4.4. In the following, let $n \in \mathbb{N}$. A function $f \in S_{\mathbb{C}}(\mathbb{R}^{dn})$ is also considered as a function in n variables $x_1, ..., x_n \in \mathbb{R}^d$. The first component of a vector $x \in \mathbb{R}^d$ is called the time component of x and we usually write $x = (x^0, \vec{x})$ where $x^0 \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^{d-1}$. By $S_+(\mathbb{R}^{dn})$ we denote the subspace of $S_{\mathbb{C}}(\mathbb{R}^{dn})$ consisting of the functions which vanish together with their partial derivatives of any order at $(x_1, ..., x_n) \in \mathbb{R}^{dn}$, unless $0 < x_1^0 < x_2^0 < ... < x_n^0$. In the following, let $a \in \mathbb{R}^d$, $\Lambda \in SO(d)$ and $\pi \in \Sigma_n$ a permutation of n elements. We introduce the following linear operators $\cdot^*$, Θ, $\cdot_{(a,\Lambda)}$ and $\cdot^\pi$ on $S(\mathbb{R}^{dn})$. To this end, let $f \in S_{\mathbb{C}}(\mathbb{R}^{dn})$ and $(x_1, ..., x_n) \in \mathbb{R}^d$ be arbitrary and define

$$
\begin{aligned}
f^*(x_1, ..., x_n) &:= \overline{f}(x_n, ..., x_1),\\
\Theta f(x_1, ..., x_n) &:= f((-x_1^0, \vec{x}_1), (-x_2^0, \vec{x}_2), ..., (-x_n^0, \vec{x}_n))\\
f_{(a,\Lambda)}(x_1, ..., x_n) &:= f(\Lambda x_1 + a, \Lambda x_2 + a, ..., \Lambda x_n + a),\\
f^\pi(x_1, ..., x_n) &:= f(x_{\pi(1)}, ..., x_{\pi(n)}).
\end{aligned}
$$

In particular, for $t \geq 0$ we denote by T_t the operator $\cdot_{(\vec{t},I)}$, where $\vec{t} \in \mathbb{R}^d$ is given by $\vec{t}_i = t\delta_{1,i}$, $i = 1, ..., d$. $(T_t)_{t\geq 0}$ is called the time translation semigroup. The space of finite sequences $\vec{f} = (f_0, f_1, f_2, ...)$ with $f_i \in S_{\mathbb{C}}(\mathbb{R}^{di})$ $\big(f_i \in S_+(\mathbb{R}^{di})\big)$, $i = 0, 1, 2, ..$, we denote by S (S_+). The operators $\cdot^*$, Θ and $\cdot_{(a,\Lambda)}$ extend to operators on S via componentwise application and their extensions are denoted by the same symbols. We say that a sequence of real numbers $(\sigma_k)_{k\in\mathbb{N}}$ is of factorial growth, if there are positive constants α and β s.t. $\sigma_k \leq \alpha(k!)^\beta$ for all $k \in \mathbb{N}$. Now, we are ready to state the Osterwalder-Schrader axioms, see also [80, 81].

Definition 7.1. *Let $S_0 = 1$ and $(S_n)_{n\in\mathbb{N}}$ be a sequence of distributions s.t. $S_n \in S'_{\mathbb{C}}(\mathbb{R}^{dn})$.*

(E0) *[Distribution property] There exists a number $p \in \mathbb{N}$ and a sequence of real numbers $(\sigma_k)_{k\in\mathbb{N}}$ of factorial growth s.t. for every $n \in \mathbb{N}$ and $f_1, ..., f_n \in S_{\mathbb{C}}(\mathbb{R}^d)$ it holds*

$$|S_n(f_1 \otimes f_2 \otimes ... \otimes f_n)| \leq \sigma_n \prod_{i=1}^{n} \|f_i\|_p \,.$$

(E1) *[Euclidean invariance] For every $n \in \mathbb{N}$, $a \in \mathbb{R}^d$, $\Lambda \in SO(d)$ and $f \in S(\mathbb{R}^{dn})$ it holds*

$$S_n(f) = S_n(f_{(a,\Lambda)}).$$

(E2) *[Reflection positivity] For every* $\vec{f} \in \mathrm{S}_+$, *it holds*

$$\sum_{n,k=0}^{\infty} S_{n+k}(\Theta f_n^* \otimes f_k) \geq 0.$$

(E3) *[Symmetry] For* $n \in \mathbb{N}$, $f \in S_{\mathbb{C}}(\mathbb{R}^{dn})$ *and every permutation* $\pi \in \Sigma_n$ *it holds*

$$S_n(f) = S_n(f^\pi).$$

(E4) *[Cluster property] For every* $\vec{f}, \vec{g} \in \mathrm{S}_+$ *it holds*

$$\lim_{t\to\infty} \sum_{n,k=0}^{\infty} S_{n+k}(\Theta f_n^* \otimes T_t g_m) = \sum_{n=0}^{\infty} S_n(\Theta f_n^*) \sum_{k=0}^{\infty} S_k(g_k).$$

Remark 7.2. *(i) Observe that the sums in* (E2) *and* (E4) *extend only over finitely many indices.*

(ii) Note that, here we formulated the axioms (E0) – (E4) *in a slightly strict manner. In particular, the distribution property can be weakened. For more details we refer to the original papers [80, 81], see also [99].*

(iii) The cluster property (E4) *is not the original condition of [80]. Under the assumption of* (E1) *the formulation used here and the one in [80] are obviously equivalent.*

Assumption 7.3. *Let* ϱ *be a probability measure on* $((0,\infty), \mathcal{B}(0,\infty))$ *s.t. for some* $m_0 \in (0,\infty)$ *it holds* $supp(\varrho) \subseteq [m_0, \infty)$.

Remark 7.4. *In the following we implicitly work with the completion* $\mathcal{B}^\varrho((0,\infty))$ *of* $\mathcal{B}((0,\infty))$ *w.r.t.* ϱ *and denote its extension to* $\mathcal{B}^\varrho((0,\infty))$ *by* ϱ*, too.*

Let $m \in (0,\infty)$ and denote by μ_m the Euclidean free field measure on $(S'(\mathbb{R}^d), \mathcal{B})$, which is given by the Bochner-Minlos theorem via its characteristic function

$$\hat{\mu}_m(f) = \exp\left(-\frac{1}{2}(C_m f, f)_{L^2(\mathbb{R}^d)}\right), \quad f \in S(\mathbb{R}^d), \tag{7.1}$$

where the linear operator $C_m = (-\Delta + m^2)^{-1}$ on $L^2(\mathbb{R}^d)$ is defined through the Fourier transform

$$\mathcal{F}(C_m f)(p) = \frac{1}{|p|^2 + m^2}\mathcal{F}f(p), \quad p \in \mathbb{R}^d, f \in S(\mathbb{R}^d).$$

In the following, we denote by $S_{n,m}$, $n \in \mathbb{N}_0$, the n–th Schwinger function of the free field of mass m, i.e., $S_{n,m}$ denotes the n-th moment of μ_m for all $n \in \mathbb{N}$. Namely, the second Schwinger function $S_{2,m}$ is given by

$$S_{2,m}(f_1, f_2) = \int_{\mathbb{R}^d} \frac{1}{|p|^2 + m^2}\mathcal{F}f_1(p)\mathcal{F}f_2(p)\,dp, \quad f_1, f_2 \in S(\mathbb{R}^d).$$

And, further, for $n \in \mathbb{N}_0$ and $f_1, \ldots, f_{2n}, f_{2n+1} \in S_{\mathbb{C}}(\mathbb{R}^d)$ we obtain via Wick's Theorem 4.54 that it holds

$$S_{2n,m}(f_1,\dots,f_{2n}) = \sum_{\sigma \text{pairing}} \prod_{i=1}^{n} S_{2,m}(f_{\sigma_i^1}, f_{\sigma_i^2}), \tag{7.2}$$
$$S_{2n+1,m}(f_1,\dots,f_{2n+1}) = 0,$$

where the sum $\sum\limits_{\text{pairings}}$ in (7.2) extends over all $(2n-1)!! = \frac{(2n)!}{2^n n!}$ pairings of the set $\{1,\dots,2n\}$.

Proposition 7.5. *For every $m > 0$ the Schwinger functions $(S_{n,m})_{n\in\mathbb{N}_0}$ satisfy (E0) – (E4).*

Proof. See e.g. [99]. □

Observe that for $f_1, f_2 \in S(\mathbb{R}^d)$ the map

$$R_{f_1,f_2} : (0,\infty) \longrightarrow \mathbb{C},\, m \mapsto S_{2,m}(f_1,f_2)$$

is analytic and satisfies the estimate $\left|R_{f_1,f_2}(m)\right| \le \frac{1}{m^2} \|f_1\|_{L^2(\mathbb{R}^d)} \|f_2\|_{L^2(\mathbb{R}^d)}$. In particular, it holds for $f_1,\dots,f_{2n} \in S_{\mathbb{C}}(\mathbb{R}^d)$

$$\left|S_{2n,m}(f_1,\dots,f_{2n})\right| \le \frac{(2n-1)!!}{m^{2n}} \prod_{i=1}^{2n} \|f_i\|_{L^2(\mathbb{R}^d)}\,. \tag{7.3}$$

Hence, if ϱ satisfies the Assumption 7.3, then, for $n \in \mathbb{N}_0$ and $f_1,\dots,f_{2n},f_{2n+1} \in S_{\mathbb{C}}(\mathbb{R}^d)$ we can define the multilinear maps given by

$$S_{2n,\varrho}(f_1,\dots,f_{2n}) := \int_{[m_0,\infty)} S_{2n,m}(f_1,\dots,f_{2n})\,\varrho(dm), \tag{7.4}$$
$$S_{2n+1,\varrho}(f_1,\dots,f_{2n+1}) := 0.$$

Remark 7.6. *Observe that for a general relativistic quantum field the corresponding second Schwinger function is determined by a polynomially bounded measure on the positive real axis. This is basically the content of the Källen-Lehmann representation, see [89, Theorem IX.34].*

Theorem 7.7. *Let ϱ satisfy the Assumption 7.3. Then, it holds*

(i) *The family $(S_{n,\varrho})_{n\in\mathbb{N}_0}$ fulfills the axioms (E0) – (E4).*

(ii) *The functions $S_{n,\varrho}$, $n \in \mathbb{N}_0$, are the moments of a unique probability measure μ_ϱ on $\left(S'(\mathbb{R}^d), \mathcal{B}\right)$.*

Remark 7.8. *For $\varrho = \delta_m$, $m \in (0,\infty)$, we simply obtain $S_{n,\varrho} = S_{n,m}$ for all $n \in \mathbb{N}_0$.*

In the following, for a linear operator $(A, D(A))$ on a Hilbert space $\mathcal{H}$ and a complex number $\lambda \in \mathbb{C}$, we denote the eigenspace of A with the corresponding eigenvalue λ by $\mathrm{Eig}(A,\lambda)$. We need a two results from the theory of symmetric semigroups.

Proposition 7.9. *Let $(S_t)_{t\ge 0}$ be a strongly continuous contraction semigroup of symmetric operators with the corresponding generator $(L, D(L))$ on a Hilbert space $\mathcal{H}$.*

(i) *The orthogonal projection P_0 onto $\mathrm{Eig}(L,0)$ is given by $P_0 = \lim\limits_{t\to\infty} S_t$, where the limit is taken* *in the strong operator topology.*

(ii) It holds that $\cap_{t\geq 0} Eig(S_t, 1) = Eig(L, 0)$.

Proof. We first proof (ii): Let $x \in \mathrm{Eig}(L,0)$. The orthogonal projection P_0 onto $\mathrm{Eig}(L,0)$ is given via the spectral theorem for self-adjoint operators by $\chi_{\{0\}}(L)$, where $\chi_{\{0\}}$ is the indicator function of the set $\{0\}$, see e.g. [91, Section VIII.3]. Hence, for $t \geq 0$ it holds by [91, Theorem VIII.5(a)]

$$S_t x = \exp(-tL)\chi_{\{0\}}(L)x = \left(\exp(-t\cdot)\chi_{\{0\}}\right)(L)x = \chi_{\{0\}}(L)x = x.$$

The second inclusion is trivial.
Now, let us show (i): Denote by $E_\cdot$ the spectral measure of $(L, D(L))$. Let $x \in \mathcal{H}$ be arbitrary. We need to show $\|P_0 x - S_t x\|_{\mathcal{H}} \longrightarrow 0$ as $t \to \infty$. By the spectral theorem it holds

$$\|P_0 x - S_t x\|_{\mathcal{H}}^2 = \int\limits_{[0,\infty)} \left|\chi_{\{0\}}(\lambda) - \exp(-t\lambda)\right|^2 \, d(E_\lambda x, x)$$

Hence, the claim follows from the dominated convergence theorem. □

Proof of Theorem 7.7. We first show (i): The distribution property $(E0)$ follows immediately from (7.3) and the definition of $S_{n,\varrho}$, $n \in \mathbb{N}$. In particular, it holds

$$\left|S_{2n,\varrho}(f_1, \dots, f_{2n})\right| \leq \frac{(2n-1)!!}{m_0^{2n}} \prod_{i=1}^{2n} \|f_i\|_{L^2(\mathbb{R}^d)}, \quad f_1, \dots, f_{2n} \in S(\mathbb{R}^d). \tag{7.5}$$

We note that the properties $(E1) - (E3)$ are linear in the family $(S_n)_{n\in\mathbb{N}_0}$. Hence, by Proposition 7.5 the properties $(E1) - (E3)$ are satisfied by $(S_{n,\varrho})_{n\in\mathbb{N}_0}$. Moreover, the cluster property $(E4)$ is non-linear in $(S_n)_{n\in\mathbb{N}_0}$. To show the cluster property we use ideas from the proof of the Osterwalder-Schrader reconstruction theorem and translate the cluster property into the corresponding property for the Gårding-Wightman theory, i.e., into the uniqueness of the vaccum vector. Hence, we introduce some objects from the original proof of Osterwalder and Schrader from [80, 81] and we refer the reader to the last mentioned references for more details. Let S_+ be given as above. We equip the space S_+ with several semi-definite inner products. For $\vec{f}, \vec{g} \in \mathrm{S}_+$ and $m \in [m_0, \infty)$ we define

$$\left(\vec{f}, \vec{g}\right)_m := \sum_{l,k=0}^{\infty} S_{l+k,m}(\Theta f_l^* \otimes g_k) \tag{7.6}$$

$$\left(\vec{f}, \vec{g}\right)_\varrho := \sum_{l,k=0}^{\infty} S_{l+k,\varrho}(\Theta f_l^* \otimes g_k) \tag{7.7}$$

$$= \int\limits_{[m_0,\infty)} \left(\vec{f}, \vec{g}\right)_m \varrho(dm). \tag{7.8}$$

Observe that the sums in (7.6) and (7.7) are finite. Next, we define the subspaces

$$\mathcal{N}_m := \left\{\vec{f} \in \mathrm{S}_+ \mid (\vec{f}, \vec{f})_m = 0\right\},$$

$$\mathcal{N}_\varrho := \left\{\vec{f} \in \mathrm{S}_+ \mid (\vec{f}, \vec{f})_\varrho = 0\right\}.$$

Now we form the quotient spaces $\widetilde{\mathcal{H}}_m := {}^{S_+}\!/\!_{\mathcal{N}_m}$ and $\widetilde{\mathcal{H}}_\varrho := {}^{S_+}\!/\!_{\mathcal{N}_\varrho}$ and define the Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_\varrho$ as the abstract completions of $\widetilde{\mathcal{H}}_m$ and $\widetilde{\mathcal{H}}_\varrho$, respectively. We denote the extensions of $(\cdot,\cdot)_m$ and $(\cdot,\cdot)_\varrho$ to scalar products on $\widetilde{\mathcal{H}}_m$ ($\mathcal{H}_m$) and $\widetilde{\mathcal{H}}_\varrho$ ($\mathcal{H}_\varrho$) by the same symbol and the induced norms by $\|\cdot\|_m$ and $\|\cdot\|_\varrho$, respectively. For $\vec{f} \in S_+$ we denote by $[\vec{f}]_m$ and $[\vec{f}]_\varrho$ the respective equivalence class in $\widetilde{\mathcal{H}}_m$ ($\mathcal{H}_m$) and $\widetilde{\mathcal{H}}_\varrho$ ($\mathcal{H}_\varrho$). We also define $\Omega := (1, 0, 0, \ldots) \in S_+$. The time translation operators T_t, $t \geq 0$, defined above, lift to continuous and symmetric linear operators T_t^m and T_t^ϱ on the Hilbert spaces $\mathcal{H}_m$ and $\mathcal{H}_\varrho$, respectively, see [80]. Furthermore, $\left(T_t^m\right)_{t\geq 0}$ and $\left(T_t^\varrho\right)_{t\geq 0}$ form strongly continuous semigroups of contractions. Their respective self-adjoint generator are denoted by $(H^m, D(H^m))$ and $(H^\varrho, D(H^\varrho))$. Observe that H^m and H^ϱ are positive. Now we reformulate the cluster property in the following way. Denote by P_0^ϱ the orthogonal projection onto the eigenspace $\mathrm{Eig}(H^\varrho, 0)$. Then, the assertions in (7.9)-(7.11) are equivalent:

$$(E4) \text{ holds for } (S_{n,\varrho})_{n\in\mathbb{N}_0}, \tag{7.9}$$

$$P_0^\varrho = ([\Omega]_\varrho, \cdot)_\varrho[\Omega]_\varrho, \text{ i.e., } \mathrm{Eig}(H^\varrho, 0) = \mathrm{span}_{\mathbb{C}}\{[\Omega]_\varrho\}, \tag{7.10}$$

$$\text{If } \Psi \in \mathcal{H}_\varrho \text{ satisfies } T_t^\varrho\Psi = \Psi \text{ for all } t \geq 0 \text{ then it holds } \Psi \in \mathrm{span}_{\mathbb{C}}\{[\Omega]_\varrho\}. \tag{7.11}$$

The equivalence of (7.9) and (7.10) is now a direct consequence of Proposition 7.9(i) and the continuity of P_0^ϱ and $([\Omega]_\varrho, \cdot)_\varrho[\Omega]_\varrho$. The equivalence of (7.10) and (7.11) follows directly from Proposition 7.9(ii).

For the choice $\varrho = \delta_m$, $m \in (0, \infty)$, we obtain by Proposition 7.5 that the equivalent statements (7.9)-(7.11) for the Schwinger functions $(S_{n,m})_{n\in\mathbb{N}_0}$ hold true. Our goal is to show that (7.11) holds true for $(T_t^\varrho)_{t\geq 0}$. The idea is to use that the operator T_t^ϱ, $t \geq 0$, *factorizes* along the operators $\left(T_t^m\right)_{m\in[m_0,\infty)}$ and use the corresponding result for T_t^m. We elaborate the idea below. The formula (7.8) for the scalar product $(\cdot,\cdot)_\varrho$ indicates that $\mathcal{H}_\varrho$ is isometric isomorphic to a subspace of the direct integral of Hilbert spaces $\int_{[m_0,\infty)}^{\oplus} \mathcal{H}_m\, \varrho(dm)$. Indeed, the spaces $(\mathcal{H}_m)_{m\in[m_0,\infty)}$ form a measurable field of Hilbert spaces in the sense of [1, Definition 1.], see also [42, Chapter I.]. Hence, we can define the direct integral of Hilbert spaces $\mathcal{H} := \int_{[m_0,\infty)}^{\oplus} \mathcal{H}_m\, \varrho(dm)$, see also [1, Definition 5.]. Further, we define the map

$$\widetilde{U} : \widetilde{H}_\varrho \longrightarrow \mathcal{H}, \quad [\vec{f}]_\varrho \mapsto \left([\vec{f}]_m\right)_{m\in[m_0,\infty)}.$$

One obtains just by considering the definitions of the involved spaces that $\widetilde{U}$ is well-defined, linear and an isometry. Hence, $\widetilde{U}$ extends to an isometry U from $\mathcal{H}_\varrho$ to $\mathcal{K} := \overline{\mathrm{Im}(\widetilde{U})}$, where the closure is understood in $\mathcal{H}$.

Next, we claim that for every $\Psi \in \mathcal{H}_\varrho$ and $t \geq 0$ there exists a ϱ-negligible set $N_{t,\Psi}$, s.t. it holds

$$T_t^m(U\Psi(m)) = (UT_t^\varrho\Psi)(m) \text{ for all } m \in N_{t,\Psi}^c.$$

We prove the claim in two steps. First, let $\Psi = [\vec{f}]_\varrho \in \widetilde{H}_\varrho$. Then, the statement follows from the definition of the operators T_t^m, T_t^ϱ and the definition of U as an extension of $\widetilde{U}$. For an arbitrary Ψ choose $\Psi_n \in \widetilde{H}_\varrho$ s.t. $\Psi_n \xrightarrow{n\to\infty} \Psi$ in $\mathcal{H}_\varrho$. Now, we define $N_{t,\Psi}^1 := \cup_{n\in\mathbb{N}} N_{t,\Psi_n}$. Via [1, Proposition 5.(ii)], we can switch to a subsequence, which we also denote by $(\Psi_n)_{n\in\mathbb{N}}$, s.t. $\lim_{n\to\infty} U\Psi_n(m) =$

$U\Psi(m)$ and $\lim_{n\to\infty} UT_t^{\varrho}\Psi_n(m) = UT_t^{\varrho}\Psi(m)$ for all m outside a ϱ-negligible set $N^2_{t,\Psi}$. Then, for $m \notin N_{t,\Psi} := N^1_{t,\Psi} \cup N^2_{t,\Psi}$ we obtain by the continuity of the operators T_t^m, $m \in [m_0, \infty)$,

$$\begin{aligned}(UT_t^{\varrho}\Psi)(m) &= \lim_{n\to\infty} (UT_t^{\varrho}\Psi_n)(m)\\ &= \lim_{n\to\infty} T_t^m(U\Psi_n(m))\\ &= T_t^m(U\Psi(m)).\end{aligned}$$

Now, let us prove the property (7.11). Let $\Psi \in \mathcal{H}_\varrho$ s.t. $T_t^{\varrho}\Psi = \Psi$ for all $t \geq 0$. We define the ϱ-negligible set $N_\Psi := \cup_{t\in[0,\infty)\cap\mathbb{Q}} N_{t,\Psi}$. For arbitrary $t \geq 0$ we choose a sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ s.t. $t_n \xrightarrow{n\to\infty} t$. Then, for $m \notin N_\Psi$ it holds by the strong continuity of $\left(T_t^m\right)_{t\geq 0}$, $m \in [m_0, \infty)$ that

$$T_t^m(U\Psi)(m) = \lim_{n\to\infty} T_{t_n}^m(U\Psi)(m) = \lim_{n\to\infty} (UT_{t_n}^{\varrho}\Psi)(m) = (U\Psi)(m).$$

Since the Schwinger functions $(S_{n,m})_{n\in\mathbb{N}}$ satisfy $(E4)$ for every $m > 0$, we conclude that for $m \in N_\Psi^c$ it holds $(U\Psi)(m) = [\Omega]_m$. Eventually, we obtain $\Psi = [\Omega]_\varrho$ since U is injective. This finishes the proof of part (i). For (ii) we simply define the characteristic function of the measure μ_ϱ via the Schwinger functions, i.e.,

$$\hat{\mu}_\varrho(f) := \sum_{n=0}^{\infty} \frac{i^n}{n!} S_{n,\varrho}(f^{\otimes n}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{[m_0,\infty)} S_{n,m}(f^{\otimes n})\varrho(dm), \quad f \in S(\mathbb{R}^d). \tag{7.12}$$

Due to (7.5) the series in (7.12) is absolutely convergent and we can interchange the sum and the integral, which yields for all $f \in S(\mathbb{R}^d)$

$$\hat{\mu}_\varrho(f) = \int_{[m_0,\infty)} \hat{\mu}_m(f)\varrho(dm). \tag{7.13}$$

Hence, from the Bochner-Minlos theorem we obtain a measure μ_ϱ with characteristic function given by $\hat{\mu}_\varrho$. By differentiating $\hat{\mu}_\varrho$ we obtain that $S_{n,\varrho}$, $n \in \mathbb{N}$ is the n-th moment of μ_ϱ. □

Remark 7.10. *(i) The assumption supp(ϱ) $\subseteq [m_0, \infty)$ with $m_0 > 0$ in Theorem 7.7 is made to ensure that the integral in (7.8) is convergent, see also the estimate (7.3). It is clear that the result in Theorem 7.7 can be generalized in various ways to a more general class of measures ϱ. In particular, one could replace the assumption supp(ϱ) $\subseteq [m_0, \infty)$ by some growth condition of ϱ near zero. Furthermore, one could also allow ϱ to be non-finite, since the integrand in (7.8) has polynomial decay, see again (7.3).*

(ii) The general idea of constructing a new family of Schwinger functions via a superposition does not restrict to start with Schwinger functions, which correspond to the free fields of different masses. The proof shows that one could also superpose Schwinger functions which correspond to models with some interaction. Moreover, one could also consider Schwinger functions which are moments of a non-Gaussian measure or only moments of a Kondratiev distribution, see [8] and [49], repsectively. One only has to guarantee that the integral in (7.4) does converge and satisfies the distribution property (E0). Of course in this way, we could also construct Schwinger functions S_n, $n \in \mathbb{N}$, which do vanish for odd n.

(iii) Recall the Källen-Lehmann representation, which in general holds true for the two point function or equivalently the second Schwinger function. The measure ϱ of our approach is in terms

of the two point function by definition the spectral measure of the Källen-Lehmann representation of the field corresponding to $(S_{n,\varrho})_{n\in\mathbb{N}_0}$*. Hence, this approach allows us to construct a large class of fields with Källen-Lehmann measure given through* ϱ*.*

The next lemma shows that the map $\varrho \mapsto \mu_\varrho$ is injective. For $A \subseteq \mathbb{R}$ and a subset $C \subseteq C_b(A)$ we denote by $\langle C\rangle$ the subalgebra of $C_b(A)$ generated by $C \cup \{1\}$, where 1 denotes the function which is constantly one.

Lemma 7.11. *Let* ϱ_1 *and* ϱ_2 *be two probability measures satisfying Assumption 7.3 s.t.* $\mu_{\varrho_1} = \mu_{\varrho_2}$*. Then,* $\varrho_1 = \varrho_2$*. Moreover, if* $\varrho_1 \neq \delta_m$ *for all* $m > 0$*, then the measure* μ_{ϱ_1} *is non-Gaussian.*

Proof. Throughout the entire proof we fix $f \in S(\mathbb{R}^d)\setminus\{0\}$ and let $m_0 \in (0,\infty)$ s.t. both sets $\operatorname{supp}(\varrho_1)$ and $\operatorname{supp}(\varrho_2)$ are contained in $[m_0,\infty)$. Observe that the bounded and continuous function

$$\varphi : [m_0,\infty) \longrightarrow \mathbb{R},\, m \mapsto \hat{\mu}_m(f) = \exp\left(-\frac{1}{2}(C_m f, f)_{L^2(\mathbb{R}^d)}\right)$$

is strictly increasing. Hence, the elements of the subalgebra $M := \langle\{\varphi\}\rangle$ of $C_b([m_0,\infty))$ separate points on $[m_0,\infty)$. Furthermore, by (7.13) and by assumption it holds for $k \in \mathbb{N}_0$

$$\begin{aligned}\int_{[m_0,\infty)} \varphi^k(m)\,\varrho_1(dm) &= \int_{[m_0,\infty)} \hat{\mu}_m(\sqrt{k}f)\,\varrho_1(dm)\\ &= \hat{\mu}_1(\sqrt{k}f)\\ &= \hat{\mu}_2(\sqrt{k}f)\\ &= \int_{[m_0,\infty)} \hat{\mu}_m(\sqrt{k}f)\,\varrho_2(dm)\\ &= \int_{[m_0,\infty)} \varphi^k(m)\,\varrho_2(dm).\end{aligned}$$

Hence, the measures ϱ_1 and ϱ_2 coincide on a separating algebra M. We conclude the proof by using [32, Theorem 3.4.5.(a)]. To prove the last assertion we assume for the sake of a contradiction that the measure μ_ϱ is Gaussian. It is clear that μ_{ϱ_1} has mean zero, since $\int_{S'(\mathbb{R}^d)} \langle g,\cdot\rangle\, d\mu_{\varrho_1} = 0$ for all $g \in S(\mathbb{R}^d)$. We denote in the following by χ the continuous and strictly decreasing function $\chi(m) = (C_m f, f)_{L^2(\mathbb{R}^d)}$, $m > 0$. If μ_{ϱ_1} is Gaussian, then there exists a non-negative number

$\sigma^2 = \sigma_f^2$, s.t. for all $\lambda \in \mathbb{R}$ it holds by the dominated convergence theorem

$$\begin{aligned} e^{-\frac{1}{2}\lambda^2\sigma^2} &= \hat{\mu}_{\varrho_1}(\lambda f) \\ &= \int_{[m_0,\infty)} \hat{\mu}_m(\lambda f)\varrho_1(dm) \\ &= \int_{[m_0,\infty)} e^{-\frac{1}{2}\lambda^2\chi(m)}\varrho_1(dm) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}(-\frac{1}{2}\lambda^2)^k \int_{[m_0,\infty)} \chi^k(m)\varrho_1(dm). \end{aligned} \tag{7.14}$$

Thus, for every $k \in \mathbb{N}$ it holds $\sigma^{2k} = \int_{[m_0,\infty)} \chi^k(m)\varrho_1(dm)$, since λ in (7.14) was arbitrary. Now, we define $K := \max\{\sigma^2, \frac{1}{m_0^2}\|f\|^2_{L^2(\mathbb{R}^d)}\} + 1$. Then, we conclude that for every $k \in \mathbb{N}$ it holds

$$\int_{[0,K]} x^k\,\delta_{\sigma^2}(dx) = \sigma^{2k} = \int_{[m_0,\infty)} \chi^k(m)\varrho_1(dm) = \int_{\chi[m_0,\infty)} x^k\varrho_1\circ\chi^{-1}(dx) = \int_{[0,K]} x^k\varrho_1\circ\chi^{-1}(dx).$$

Once more, we conclude by [32, Theorem 3.4.5.(a)] that $\delta_{\sigma^2} = \varrho_1\circ\chi^{-1}$ which contradicts the assumption $\varrho_1 \neq \delta_m$ for all $m > 0$, since χ is injective. This completes the proof. □

Corollary 7.12. *If $\varrho \neq \delta_m$ for all $m > 0$, then the truncated moments (Ursell functions) $S^T_{n,\varrho}$, $n \in \mathbb{N}$, of the measure μ_ϱ do not vanish for all $n \geq 4$. In particular, the Wightman-theory corresponding to $(S_{n,\varrho})_{n\in\mathbb{N}}$ is non-trivial in the sense that it is not a generalized free field.*

Proof. Recall that the truncated moments $S^T_{n,\varrho}$, $n \in \mathbb{N}$, are recursively defined by

$$\begin{aligned} S^T_{n,\varrho} &= 1, \\ S_{n,\varrho}(f_1,\dots,f_n) &= \sum_{I\in P^{(n)}} \prod_{\{i_1,\dots,i_k\}\in I} S^T_{k,\varrho}(f_{i_1},\dots,f_{i_k}), \end{aligned} \tag{7.15}$$

where $f_1,\dots,f_n \in S_{\mathbb{C}}(\mathbb{R}^d)$, $n \in \mathbb{N}$, and $P^{(n)}$ denotes the set of all partitions of the set $\{1,\dots,n\}$, see e.g. [103]. In case $S^T_{n,\varrho} = 0$ for all $n \geq 4$ we directly obtain from (7.15) that for $n \in \mathbb{N}_0$ and $f_1,\dots,f_{2n} \in S_{\mathbb{C}}(\mathbb{R}^d)$ it holds

$$S_{2n,\varrho}(f_1,\dots,f_{2n}) = \sum_{\sigma\,\text{pairing}} \prod_{i=1}^{n} S_{2,\varrho}(f_{\sigma_i^1}, f_{\sigma_i^2}),. \tag{7.16}$$

Due to Wick's Theorem 4.54, the right-hand side of (7.16) equals the $2n$-th moment of the Gaussian measure μ_{b_ϱ} given by the covariance functional

$$b_\varrho(f_1,f_2) := \int_{\mathbb{R}^d} \hat{C}_\varrho(p)\mathcal{F}f_1(p)\mathcal{F}f_2(p)\,dp, \quad f_1,f_2 \in S(\mathbb{R}^d),$$

where

$$\hat{C}_\varrho(p) = \int\limits_{(0,\infty)} \frac{1}{|p|^2 + m^2}\,\varrho(dm), \qquad p \in \mathbb{R}^d.$$

Hence, μ_ϱ and μ_{b_ϱ} have the same moments. Since μ_{b_ϱ} is Gaussian we also obtain that the characteristic functions of μ_ϱ and μ_{b_ϱ} coincide. In the light of Lemma 7.11, this yields a contradiction. The last assertion follows from the definition of the generalized free fields given in [99, Section II.5]. □

Remark 7.13. *The idea of constructing non-Gaussian measures from Gaussian ones via superposition is also used in [53, 48] in the context of White noise and Mittag-Leffler analysis.*

7.2 Properties of the Measure μ_ϱ

In this section we prove certain properties of the measure μ_ϱ constructed in the previous section. Therefore, let us fix throughout this section a probability measure ϱ on $((0,\infty),\mathcal{B}(0,\infty))$ which satisfies Assumption 7.3. In particular, we fix a number $m_0 > 0$ s.t. $\operatorname{supp}(\varrho) \subseteq [m_0,\infty)$. Assume also that $\varrho \neq \delta_m$ for all $m > 0$. We denote the unique probability measure given in Theorem 7.7 by μ_ϱ.

From the construction of μ_ϱ one might think that $\mu_\varrho(A) = \int\limits_{[m_0,\infty)} \mu_m(A)\varrho(dm)$ for all $A \in \mathcal{B}$. The crucial point is that we did not establish yet, that $m \mapsto \mu_m(A)$ is measurable for all $A \in \mathcal{B}$. The next lemma clarifies this in an affirmative way. All subsets of $B \subseteq \mathbb{R}$ are equipped with the trace σ-algebra induced by the Borel σ-algebra on $\mathbb{R}$.

Lemma 7.14. *For all $A \in \mathcal{B}$ the map $[m_0,\infty) \ni m \mapsto \mu_m(A) \in [0,1]$ is measurable.*

Proof. We use a monotone class argument. We denote by $\mathcal{B}_b$ the set of all bounded, real-valued and $\mathcal{B}$-measurable functions. We also write $\nu(F)$ for the integral $\int\limits_{S'(\mathbb{R}^d)} F\,d\nu$ for $F \in \mathcal{B}_b$ and a probability measure ν on $(S'(\mathbb{R}^d),\mathcal{B})$. Define the vector space

$$V := \{F \in \mathcal{B}_b \mid [m_0,\infty) \ni m \mapsto \mu_m(F) \in [0,1] \text{ is measurable}\}.$$

Due to the monotone convergence theorem V is a monotone vector space, see e.g. [98, Appendix A0.]. Similar as above, we denote by $\mathcal{F}C_b$ the space of real-valued functions F on $S'(\mathbb{R}^d)$, i.e,

$$\mathcal{F}C_b := \left\{F = f(\langle \xi_1,\cdot\rangle,\dots,\langle \xi_n,\cdot\rangle) \mid n \in \mathbb{N}, f \in C_b(\mathbb{R}^n), \xi_j \in S(\mathbb{R}^d), j = 1,\dots,n\right\} \tag{7.17}$$

Observe that $\mathcal{F}C_b$ generates the Borel σ-algebra $\mathcal{B}$, i.e., $\sigma(\mathcal{F}C_b) = \mathcal{B} = \sigma(\beta_w)$. Due to [98, Theorem (A0.6)] it suffices to show that $\mathcal{F}C_b \subseteq V$. Let $F = f(\langle \xi_1,\cdot\rangle,\dots,\langle \xi_n,\cdot\rangle) \in \mathcal{F}C_b$ as in (7.17). We claim that $m \mapsto \mu_m(F)$ is continuous. Observe that $\mu_m(F) = \int\limits_{\mathbb{R}^n} f(x_1,\dots,x_n)\,\mathcal{N}(0,A(m))(dx)$, where $\mathcal{N}(0,A(m))$ is the Gaussian law on $(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n))$ with mean zero and covariance matrix $A(m)$, $A(m)_{ij} = (C_m\xi_i,\xi_j)_{L^2(\mathbb{R}^d)}$. Since $m \mapsto (C_m\xi_i,\xi_j)_{L^2(\mathbb{R}^d)}$ is continuous we have that $m \mapsto A(m)$ is continuous. Hence, by Levy's continuity theorem, see e.g. [19, Theorem 26.3], it holds $m \mapsto \mu_m(F)$ is continuous which implies $F \in V$. □

Corollary 7.15. *The map* $\widetilde{\mu}_\varrho : \mathcal{B} \longrightarrow [0,1], A \mapsto \int_{[m_0,\infty)} \mu_m(A)\,\varrho(dm)$ *is a probability measure and* $\widetilde{\mu}_\varrho$ *coincides with* μ_ϱ.

Proof. The countable additivity of $\tilde{\mu}_\varrho$ follows from the monotone convergence theorem. By using a monotone class argument, as in the proof of Lemma 7.14, we obtain

$$\widetilde{\mu}_\varrho(F) = \int_{[m_0,\infty)} \mu_m(F)\,\varrho(dm), \text{ for all } F \in \mathcal{B}_b.$$

Hence, the last assertion holds true by the definition of μ_ϱ given in (7.13) and the fact that a characteristic function determines a measure uniquely. □

Remark 7.16. *Observe that, one could also use Corollary 7.15 as an alternative definition of* μ_ϱ*, see also Remark 7.25.*

Let $f \in S(\mathbb{R}^d)$ and recall the shift operator τ_f on $S'(\mathbb{R}^d)$ defined before Lemma 4.15, i.e., $\tau_f : S'(\mathbb{R}^d) \longrightarrow S'(\mathbb{R}^d), x \mapsto x + f$. Here $f \in S(\mathbb{R}^d)$ is considered as an element of $S'(\mathbb{R}^d)$ via the scalar product of $L^2(\mathbb{R}^d)$, i.e., $\langle g, f\rangle = (g,f)_{L^2(\mathbb{R}^d)}, g \in S(\mathbb{R}^d)$.

Corollary 7.17. *The measure* μ_ϱ *is quasi shift-invariant w.r.t. the directions of* $S(\mathbb{R}^d)$*, i.e., for* $\xi \in S(\mathbb{R}^d)$ *it holds that* $\mu_\varrho \circ \tau_f^{-1}$ *is absolutely continuous w.r.t.* μ_ϱ*. In particular,* μ_ϱ *has full topological support, i.e., for every strongly open set* $\mathcal{O} \in \beta_s$ *it holds* $\mu_\varrho(\mathcal{O}) > 0$.

Proof. Let $A \in \mathcal{B}$ s.t. $\mu_\varrho(A) = 0$. Due to Corollary 7.15 it holds that $\mu_m(A) = 0$ for ϱ-a.e. $m \in [m_0,\infty)$. Since μ_m is quasi shift invariant along directions from $S(\mathbb{R}^d)$ it holds $\mu_m \circ \tau_f^{-1}(A) = 0$ for ϱ-a.e. $m \in [m_0,\infty)$ which implies $\mu_\varrho \circ \tau_f^{-1}(A) = 0$. The last assertion follows as in [79, Proof of Proposition 3.2.2]. □

The last assertion of the previous corollary can also be proven by using the corresponding statement for the components μ_m, $m \in [m_0,\infty)$, see Corollary 4.18.

In the following, we show that the measure μ_ϱ is in fact a Hida measure, see Definition 4.57. For this purpose, let us consider the standard White noise setting. Namely, we consider the chain of continuous embeddings of the real spaces

$$S(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \subseteq S'(\mathbb{R}^d).$$

By μ we denote the White noise measure on $(S'(\mathbb{R}^d), \mathcal{B})$ given by its characteristic function $\hat{\mu}$ defined by

$$\hat{\mu}(\varphi) = \exp\left(-\frac{1}{2}(\varphi,\varphi)_{L^2(\mathbb{R}^d)}\right), \quad \varphi \in L^2(\mathbb{R}^d).$$

Hence, due to Theorem 4.22 the space $L^2(\mu)$ is isometrically isomorphic to $\Gamma(L^2_{\mathbb{C}}(\mathbb{R}^d))$. We obtain the nuclear triplet consisting of the Hida test functions, the central space $L^2(\mu)$ and the Hida distributions

$$(\mathcal{N}) \subseteq L^2(\mu) \subseteq (\mathcal{N})',$$

see Subsection 4.2.3. Now, we show that μ_ϱ corresponds to an element $\Phi_\varrho \in (\mathcal{N})'_+$ in the sense of Theorem 4.56. To this end, we observe that the corresponding statement holds true for the single components μ_m, $m \in [m_0, \infty)$. Indeed, recall the characteristic function of μ_m which is given by

$$\hat{\mu}_m(\varphi) = \exp\left(-\frac{1}{2}(C_m\varphi, \varphi)_{L^2(\mathbb{R}^d)}\right), \quad \varphi \in S(\mathbb{R}^d).$$

One easily sees that $\hat{\mu}_m$ is a U-functional and, therefore, there exists $\Phi_m \in (\mathcal{N})'_+$, s.t. $T\Phi_m = \hat{\mu}_m$. The distribution Φ_m is called the Gauss kernel corresponding to $-\Delta + m^2 - 1$, see [57, Example 4.23, Theorem 4.24]. From Corollary 4.40 we obtain that $[m_0, \infty) \ni m \mapsto \Phi_m \in (\mathcal{N})'$ is Bochner integrable in $(\mathcal{H}_{-q})$ for some $q \in \mathbb{N}$. Therefore, we end up with the following theorem.

Theorem 7.18. *There exists $\Phi_\varrho \in (\mathcal{N})'_+$, which corresponds to μ_ϱ in the sense of Theorem 4.56. Furthermore, Φ_ϱ is given by*

$$\Phi_\varrho := \int_{[m_0,\infty)} \Phi_m \, \varrho(dm).$$

Since μ_ϱ is a Hida measure, we obtain that the polynomials $\mathcal{P}(S'(\mathbb{R}^d))$ are dense in $L^2(\mu_\varrho)$, see Remark 4.58(iv). For an alternative argument see also Remark 7.29(ii). A similar Hida distribution as Φ_ϱ was already considered in the context of Dirichlet forms and White Noise analysis, see [5, Proposition 4.9.]. There the authors considered instead of a probability measure ϱ the Lebesgue measure over a finite interval on the positive real axes. In the last mentioned reference only closability of Dirichlet forms is considered. Our motivation is entirely different. Here we are concerned with the construction of a non-trivial field by verifying the Osterwalder-Schrader axioms for the corresponding moments of μ_ϱ. Nothing in this direction was shown in [5].

Theorem 7.19. *For every $m \in (0, \infty)$ the measures μ_ϱ, μ_m and μ are pairwise mutually singular.*

Proof. See Corollary 7.28 in Section 7.3 below. □

7.3 Singularity of Typical Measures in Quantum Field Theory

In the Euclidean formulation of quantum field theory probability measures on $(S'(\mathbb{R}^d), \mathcal{B})$ are the central objects under consideration, see e.g. [46, Chapter 6] and in particular the measure μ_ϱ constructed in Section 7.1. In this section, we aim to present a general strategy known from ergodic theory to establish the singularity of two probability measures. This idea is already used in [38, Theorem 4.3.] for the $P(\Phi)_2$-models of quantum field theory. In the context of quantum field theory singularity of probability measures can be considered as a formulation of Haag's theorem, see [102]. The general method is based on [100, Theorem 1.4], which basically says that, if one has two distinct measures attaining a common ergodic family of measurable transformations, then they must be mutually singular. To prove this theorem in a self-contained way we employ Hilbert space methods, in particular, the von Neumann's ergodic theorem, and follow a similar approach given in [91, Section II.5]. At the end of this section we explain how these results can be applied to the measures from quantum field theory, with their moments satisfying the axioms of Osterwalder and Schrader (E0)-(E4).

First, we prove the von Neumann's ergodic theorem.

Theorem 7.20. *Let $\mathcal{H}$ be a complex Hilbert space and $(U_t)_{t\in\mathbb{R}}$ a u.s.c.g., see Definition 1.34. Denote by $(A, D(A))$ the generator of $(U_t)_{t\in\mathbb{R}}$ and by P the orthogonal projection onto the eigenspace $Eig(A,0)$. Then for all $f \in \mathcal{H}$ it holds*

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T U_t f \, dt = Pf.$$

Proof. Since $(A, D(A))$ is skew-adjoint, the operator $B := -iA$ with domain $D(B) = D(A)$ is self-adjoint. Via the spectral theorem for self-adjoint operators, we define the operator $e^{itB} \in L(\mathcal{H})$, $t \in \mathbb{R}$. One obtains that $\left(e^{itB}\right)_{t\in\mathbb{R}}$ is a u.s.c.g. with generator A, see [91, Theorem VIII.7]. Hence, $U_t = e^{itB}$ for all $t \in \mathbb{R}$. Since $\mathrm{Eig}(A,0) = \mathrm{Eig}(B,0)$, it holds by the spectral theorem $P = \chi_{\{0\}}(B)$, where $\chi_{\{0\}}$ is the indicator function of $\{0\}$. Denote by E_λ the spectral measure of $(B, D(B))$. Then, for $f \in \mathcal{H}$ it holds

$$\begin{aligned}
\left\| \frac{1}{T}\int_0^T U_t f\, dt - Pf \right\|_{\mathcal{H}}^2 &= \left\| \frac{1}{T}\int_0^T e^{itB} f - \chi_{\{0\}} f \, dt \right\|_{\mathcal{H}}^2 \\
&= \frac{1}{T^2}\int_0^T\int_0^T \Big((e^{it_1B} - \chi_{\{0\}})f, (e^{it_2B} - \chi_{\{0\}})f \Big)_{\mathcal{H}} \, dt_2\, dt_1 \\
&= \int_{\sigma(B)} \frac{1}{T^2}\int_0^T\int_0^T (e^{it_1\lambda} - \chi_{\{0\}}(\lambda))(e^{-it_2\lambda} - \chi_{\{0\}}(\lambda))\, dt_2\, dt_1\, d(E_\lambda f, f)_{\mathcal{H}},
\end{aligned}$$

where we use Tonelli's theorem in the last equality. Via an explicit computation, we obtain that the integrand $\lambda \mapsto \frac{1}{T^2}\int_0^T\int_0^T (e^{it_1\lambda} - \chi_{\{0\}}(\lambda))(e^{-it_2\lambda} - \chi_{\{0\}}(\lambda))\, dt_2\, dt_1$ is bounded by 4 and converges pointwise to zero as $T \longrightarrow \infty$. Hence, by the dominated convergence theorem the proof is completed. □

The proof of the von Neumann's ergodic theorem can also be given without the usage of the spectral theorem for self-adjoint operators, see e.g. [91, Theorem II.11, Problem II.18.].

Let us fix throughout this section a probability space $(\Omega, \mathcal{B}, \nu)$ and a group of measurable transformations $T_t : \Omega \longrightarrow \Omega$, $t \in \mathbb{R}$. Namely, for $t, s \in \mathbb{R}$ it holds $T_t \circ T_{-t} = T_0 = Id$ and $T_{t+s} = T_t \circ T_s$. For two sets $A, B \in \mathcal{B}$ denote by $A \,\Delta\, B$ their symmetric difference, i.e., $A \,\Delta\, B := A \setminus B \cup B \setminus A$.

Definition 7.21. *Let us state the following definitions concerning the family $(T_t)_{t\geq 0}$.*

(i) We call $(T_t)_{t\geq 0}$ measure preserving (for ν), if for every $t \in \mathbb{R}$ it holds $\nu \circ T_t^{-1} = \nu$.

(ii) A measure preserving family $(T_t)_{t\geq 0}$ (w.r.t. ν) we call ergodic (for ν), if for $A \in \mathcal{B}$ satisfying $\nu(T_t^{-1} A \,\Delta\, A) = 0$ for all $t \in \mathbb{R}$, it holds $\nu(A) \in \{1, 0\}$.

If $(T_t)_{t\geq 0}$ is measure preserving for ν, we also denote by U_t the linear operator in $L\left(L^2(\Omega,\nu)\right)$ defined by $U_t f(\cdot) := f(T_t \cdot)$, $f \in L^2(\Omega,\nu)$. If for all $f \in L^2(\Omega,\nu)$ the map $\mathbb{R} \ni t \mapsto U_t f \in L^2(\Omega,\nu)$ is continuous, then $(U_t)_{t\in\mathbb{R}}$ forms a u.s.c.g. on $L^2(\Omega,\nu)$. A sufficient condition for strong continuity can be found in [91, Theorem VIII.9]. Observe that a measure preserving family $(T_t)_{t\in\mathbb{R}}$ is ergodic for ν if and only if for $f\in L^2(\Omega,\nu)$ satisfying $U_t f = f$ for all $t\in\mathbb{R}$, it holds f is constant ν-a.e.. If $(U_t)_{t\in\mathbb{R}}$ is strongly continuous then it holds $(T_t)_{t\in\mathbb{R}}$ is ergodic for ν, if and only if for the generator $(A, D(A))$ of $(U_t)_{t\in\mathbb{R}}$ it holds $\mathrm{Eig}(A,0) = \mathrm{span}_{\mathbb{C}}\{1\}$. The following theorem is taken from [100, Theorem 1.4]. There, a stronger statement is proven by using the Birkhoff-Khinchin ergodic theorem.

Theorem 7.22. *Let ν_1 and ν_2 be probability measures on $(\Omega,\mathcal{B})$ and assume that $(T_t)_{t\geq 0}$ is ergodic for ν_1 and ν_2. Further, assume that $(T_t)_{t\geq 0}$ induces u.s.c. groups $\left(U_t^1\right)_{t\in\mathbb{R}}$ and $\left(U_t^2\right)_{t\in\mathbb{R}}$ on $L^2(\Omega,\nu_1)$ and $L^2(\Omega,\nu_1)$, respectively. If $\nu_1\neq\nu_2$, then ν_1 and ν_2 are mutually singular.*

Proof. Let $B\in\mathcal{B}$ s.t. $\nu_1(B)\neq\nu_2(B)$. Observe that the orthogonal projections onto $\mathrm{Eig}(A_i,0)$ are given by $(\cdot,1)_{L^2(\Omega,\nu_i)}1$, where A_i is the generator of $\left(U_t^i\right)_{t\in\mathbb{R}}$, $i=1,2$. Now, we apply Theorem 7.20 to the indicator function $f = 1_B$. Hence, there exists positive numbers T_n, $n\in\mathbb{N}$, increasing to infinity s.t. for $i\in\{1,2\}$ it holds ν_i-a.e.

$$\frac{1}{T_n}\int_0^{T_n} U_t^i 1_B\,dt \xrightarrow{n\to\infty} \nu_i(B).$$

Hence, the sets $\Omega_i := \left\{\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} U_t^i 1_B\,dt = \nu_i(B)\right\}$, satisfy $\nu_i(\Omega_i) = 1$, $i=1,2$ and it holds $\Omega_1\cap\Omega_2=\emptyset$. □

Basically, the previous theorem implies that it suffices to find a common family of ergodic transformations $(T_t)_{t\in\mathbb{R}}$ for two distinct probability measures to show that they are mutually singular. Thus, we state in the following two criteria for ergodicity of a family of measurable transformations. First, we formulate a definition.

Definition 7.23. *Assume that $(T_t)_{t\geq 0}$ is measure preserving for ν.*

(i) We call $(T_t)_{t\geq 0}$ (strongly) mixing (for ν), if for every $A, B\in\mathcal{B}$ it holds

$$\lim_{t\to\infty}\nu(T_t^{-1}A\cap B) = \nu(A)\nu(B).$$

(ii) We call $(T_t)_{t\geq 0}$ weakly mixing (for ν), if for every $A,B\in\mathcal{B}$ it holds

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \nu(T_t^{-1}A\cap B)\,dt = \nu(A)\nu(B).$$

One directly sees that the strong mixing property is equivalent to

$$\lim_{t\to\infty}(U_t f, g)_{L^2(\Omega,\nu)} = (f,1)_{L^2(\Omega,\nu)}(1,g)_{L^2(\Omega,\nu)},\quad \text{for all } f,g\in D \tag{7.18}$$

where $\mathrm{span}(D)\subseteq L^2(\Omega,\nu)$ is dense. Observe that this is very similar to the statement in Proposition 7.9 for a symmetric s.c.c.s. $(T_t)_{t\geq 0}$. The next lemma and its proof are well-known in ergodic theory and can be found in [91, Section VII.4] and [100].

Lemma 7.24. *Let $(T_t)_{t\geq 0}$ be a measure preserving family. Then, it holds*

(i) If $(T_t)_{t\geq 0}$ is strongly mixing, then it is weakly mixing.

(ii) If $(T_t)_{t\geq 0}$ is weakly mixing, then it is ergodic.

Now, let us specify $(\Omega, \mathcal{B}) = (S'(\mathbb{R}^d), \mathcal{B})$, where $\mathcal{B}$ denotes again the Borel σ-field of the weak topology of $S'(\mathbb{R}^d)$. Let $t \in \mathbb{R}$ and $a \in \mathbb{R}^d$ be a unit vector. First, we define the following continuous map

$$T_t : S(\mathbb{R}^d) \longrightarrow S(\mathbb{R}^d), f \mapsto T_t f(\cdot) = f(ta + \cdot). \tag{7.19}$$

See also Lemma 6.1. We extend T_t to $S'(\mathbb{R}^d)$ via

$$\langle f, T_t\omega\rangle := \langle T_{-t}f, \omega\rangle, \quad f \in S(\mathbb{R}^d), \omega \in S'(\mathbb{R}^d). \tag{7.20}$$

Hence, $(T_t)_{t\in\mathbb{R}}$ forms a group of measurable transformations on $(S'(\mathbb{R}^d), \mathcal{B})$.

Remark 7.25. *Let ν be a probability measure on $(\Omega, \mathcal{B}) = (S'(\mathbb{R}^d), \mathcal{B})$ s.t. its moments $(S_n)_{n\in\mathbb{N}_0}$ satisfy the Osterwalder-Schrader axioms (E0)-(E4). Assume additionally the polynomials $\mathcal{P}(S'(\mathbb{R}^d))$ to be dense in $L^2(S'(\mathbb{R}^d), \nu)$. Observe that in this case the Cluster property (E4) is equivalent to the strong mixing property of $(T_t)_{t\in\mathbb{R}}$ w.r.t. ν. In particular, in Theorem 7.7 we could have proven the cluster property of $(S_{n,\varrho})_{n\in\mathbb{N}_0}$ also by first constructing the measure μ_ϱ via Corollary 7.15. Proceeding this way the strong mixing property of $(T_t)_{t\in\mathbb{R}}$ w.r.t. μ_ϱ follows immediately from the dominated convergence theorem.*

Before we proceed let us consider two important classes of probability measures on $(S'(\mathbb{R}^d), \mathcal{B})$ s.t. $(T_t)_{t\in\mathbb{R}}$ is strongly mixing for them. To this end we recall the White noise measure μ_{σ^2} with variance $\sigma^2 > 0$ on $(S'(\mathbb{R}^d), \mathcal{B})$ given via its characteristic function

$$\hat{\mu}_{\sigma^2}(f) = \int_{S'(\mathbb{R}^d)} e^{i\langle f,\cdot\rangle}\, d\mu_{\sigma^2} = \exp\left(-\frac{1}{2}\sigma^2(f,f)_{L^2(\mathbb{R}^d)}\right), \quad f \in S(\mathbb{R}^d).$$

Recall also the Euclidean free field measure μ_m, $m > 0$, defined in (7.1). Here, it is necessary to point out that the notation concerning the measures μ_{σ^2} and μ_m is not well-chosen. Since we used this notation already in the previous chapters, for the sake of consistency we still stick to this notation. However, it should be noted that we never plug in concrete numbers for σ^2 and m. Thus, the authors expects that this will not cause any confusion for the reader. Indeed, σ^2 as a subscript always refers to the White noise measure μ_{σ^2} and a subscript m to the Euclidean free field measure μ_m. By considering their respective characteristic functions, we obtain $\mu_{\sigma^2} \circ T_t^{-1} = \mu_{\sigma^2}$ and $\mu_m \circ T_t^{-1} = \mu_m$ for all $t \in \mathbb{R}$, i.e., $(T_t)_{t\in\mathbb{R}}$ is measure preserving for μ_{σ^2} and μ_m, $\sigma^2, m > 0$.

Lemma 7.26. *Let $\sigma^2, m > 0$. The transformations $(T_t)_{t\geq 0}$ defined in (7.20) are strongly mixing for μ_{σ^2} and μ_m. In particular, $(T_t)_{t\geq 0}$ is ergodic for μ_{σ^2} and μ_m. Furthermore, $(U_t)_{t\in\mathbb{R}}$ extends to a u.s.c.g. on $L^2(S'(\mathbb{R}^d, \mu_{\sigma^2})$ and $L^2(S'(\mathbb{R}^d, \mu_m)$, respectively.*

Proof. Observe that $D := \left\{e^{i\langle f,\cdot\rangle} \mid f \in S(\mathbb{R}^d)\right\}$ is a total set in $L^2(S'(\mathbb{R}^d, \mu_{\sigma^2})$ and $L^2(S'(\mathbb{R}^d, \mu_m)$, respectively. So (7.18) can be easily checked on D. Furthermore, $\mathbb{R} \ni t \mapsto U_t F \in L^2(S'(\mathbb{R}^d, \mu_{\sigma^2})$ is continuous for $F \in D$ which proves the last claim for μ_{σ^2}. The same argument also works for

μ_m.

□

From Lemma 7.22 and Lemma 7.26 we state the following corollary.

Corollary 7.27. *The set* $\mathcal{M} := \{\mu_{\sigma^2}, \mu_m \mid \sigma^2, m > 0\}$ *is mutually singular, i.e., any two distinct elements* $\nu_1, \nu_2 \in \mathcal{M}$ *are mutually singular.*

Let us briefly relate what we've done so far to axiomatic quantum field theory. Assume for the rest of this section that ν is a probability measure on $(S'(\mathbb{R}^d), \mathcal{B})$, s.t. its moments $(S_n)_{n\in\mathbb{N}_0}$ defined by

$$S_n : S_{\mathbb{C}}(\mathbb{R}^d)^n \longrightarrow \mathbb{C}, (f_1, \dots, f_n) \mapsto \int_{S'(\mathbb{R}^d)} \prod_{i=1}^{n} \langle f_i, \cdot \rangle \, d\nu, \quad n \geq 1$$

satisfy the Osterwalder-Schrader axioms (E0)-(E4), see Section 7.1. In particular, due to the distribution property (E0), the Euclidean invariance property (E1) and the cluster property (E4) it holds for $n, k \in \mathbb{N}$ and $f_1, .., f_n, g_1, \dots, g_k \in S(\mathbb{R}^d)$

(i)

$$\left| \int_{S'(\mathbb{R}^d)} \prod_{i=1}^{n} \langle f_i, \cdot \rangle \, d\nu \right| = |S_n(f_1, \dots, f_n)| \leq \alpha (n!)^{\beta} \prod_{i=1}^{n} \|f_i\|_p \,, \tag{7.21}$$

where α, β are positive constants and $\|\cdot\|_p$ is a continuous seminorm on $S(\mathbb{R}^d)$,

(ii)

$$\begin{aligned} \int_{S'(\mathbb{R}^d)} \prod_{i=1}^{n} \langle f_i, \cdot \rangle \, d\nu &= S_n(f_1, \dots, f_n) \\ &= S_n(T_t f_1, \dots, T_t f_n) \\ &= \int_{S'(\mathbb{R}^d)} \prod_{i=1}^{n} \langle f_i, \cdot \rangle \, d\nu \circ T_t^{-1}, \end{aligned} \tag{7.22}$$

where T_t is defined in (7.20),

(iii)

$$\begin{aligned} \lim_{t\to\infty} \int_{S'(\mathbb{R}^d)} \prod_{i=1}^{n} \langle f_i, \cdot \rangle \prod_{j=1}^{k} \langle T_t g_j, \cdot \rangle \, d\nu &= \lim_{t\to\infty} S_{n+k}(f_1, \dots, f_n, T_t g_1, \dots, T_t g_k) \\ &= S_n(f_1, \dots, f_n) S_k(g_1, \dots, g_k) \\ &= \int_{S'(\mathbb{R}^d)} \prod_{i=1}^{n} \langle f_i, \cdot \rangle \, d\nu \int_{S'(\mathbb{R}^d)} \prod_{j=1}^{k} \langle g_j, \cdot \rangle \, d\nu. \end{aligned} \tag{7.23}$$

For a polynomial $F \in \mathcal{P}(S'(\mathbb{R}^d))$ it holds, due to the estimate (7.21), that $U_t F = F \circ T_t \xrightarrow{t \to 0} F$ in $L^2(S'(\mathbb{R}^d), \nu)$. If the polynomials $\mathcal{P}(S'(\mathbb{R}^d))$ are dense in $L^2(S'(\mathbb{R}^d), \nu)$, then (7.22) and (7.23) imply that $(T_t)_{t \geq 0}$ is measure preserving and strongly mixing for ν. Hence, in that case $(T_t)_{t \geq 0}$ gives rise to a u.s.c.g. $(U_t)_{t \in \mathbb{R}}$ on $L^2(S'(\mathbb{R}^d), \nu)$. Due to Lemma 7.22, we obtain the next corollary.

Corollary 7.28. *Assume that $\mathcal{P}(S'(\mathbb{R}^d))$ is dense in $L^2(S'(\mathbb{R}^d), \nu)$ and $\nu \notin \mathcal{M}$. Then, the measure ν is singular w.r.t. every element from $\mathcal{M}$.*

We conclude this section with two remarks concerning the previous corollary.

Remark 7.29. *(i) To check that the polynomials are dense, one can use the procedure given in Chapter 6. Indeed, assume $\beta = \frac{1}{2}$ in (7.21). Then, one can proceed as in Theorem 6.6 and construct a positive distribution $\Phi_\nu \in (\mathcal{N})'_+$ corresponding to ν. Eventually, by Remark 4.58(iv) one concludes that the polynomials $\mathcal{P}(S'(\mathbb{R}^d))$ are dense in $L^2(S'(\mathbb{R}^d), \nu)$. If $\beta \in (\frac{1}{2}, 1)$, one can still proceed similar by replacing the spaces $((\mathcal{N}), (\mathcal{N})')$ with the Kondratiev test functions and distributions, see e.g. [65], [64].*

(ii) To show that the polynomials $\mathcal{P}(S'(\mathbb{R}^d))$ are dense in $L^2(S'(\mathbb{R}^d), \nu)$ and that $(T_t)_{t \geq 0}$ is measure preserving for ν, one can also argue similar as in the proof of Proposition 4.21. Indeed, due to the Bochner-Minlos theorem 4.13 ν and $\nu \circ T_t^{-1}$ are uniquely determined by their respective one dimensional distributions $\nu \circ \langle f, \cdot \rangle^{-1}$ and $\nu \circ T_t^{-1} \circ \langle f, \cdot \rangle^{-1}$, $f \in S(\mathbb{R}^d)$. Hence, one can use results from the Hamburger moment problem, as for instance, the Carleman condition, see e.g. [40]. In particular, for $\beta \in [0, 1]$ in (7.21) we obtain $\nu = \nu \circ T_t^{-1}$ for all $t \geq 0$. Density of the polynomials follows by the exact same arguments as in the proof of Proposition 4.21 by using the Carleman condition instead of the Cramér condition.

(iii) Corollary 7.28 basically implies that a large class of measures in constructive quantum field theory, whose moments satisfy the Osterwalder-Schrader axioms are already mutually singular. This can be considered as a formulation of Haag's theorem in quantum field theory, see e.g. [102] and [62].

Appendix A

Appendix to Part I

A.1 Supplementary comments and proofs for Remark 2.15

This section is intended to give some additional details concerning Remark 2.15. The reader who is not familiar with filters and neighborhood filter should consult [107, Chapter 1]. We briefly recall the situation of Remark 2.15. Let (F, r) be a metric space and denote by $C([0,\infty), F)$ the space of continuous functions from $[0,\infty)$ to F equipped with the metric

$$d(x,y) := \sum_{T=1}^{\infty} 2^{-T} \sup_{0\le t\le T} (r(x(t),y(t)) \wedge 1), \quad x,y \in C([0,\infty),F). \tag{A.1}$$

For $f \in C([0,\infty),F)$ we denote by $\widetilde{\mathcal{N}}(f)$ the filter of neighborhoods of f in the topology induced by d. Indeed, $\widetilde{\mathcal{N}}(f)$ is generated by the filter base

$$\widetilde{B}_f := \left\{ B_{\varepsilon,d}(f) \mid \varepsilon > 0 \right\},$$

where $B_{\varepsilon,d}(f)$ denotes the open ball w.r.t. d around f with radius ε. Another system of neighborhood filters $\mathcal{N}(f)$, $f \in C([0,\infty),F)$, arises in the following way. Let $(T, \mathbf{P}, \mathcal{U})$ be a triple consisting of $T \in \mathbb{N}$, a partition $\mathbf{P} = \{t_0, \dots, t_n\}$ of $[0,T]$, $n \in \mathbb{N}$, and a family $\mathcal{U} = \{U_i\}_{i=1,\dots,n}$ of open sets in F. Define the set

$$N(T,P,\mathcal{U}) := \{g \in C([0,\infty),F) \mid g(t) \in U_{i+1} \text{ if } t \in [t_i, t_{i+1}], i = 0,..,n-1\}.$$

Define for $f \in C([0,\infty),F)$ the filter $\mathcal{N}(f)$ as the filter generated by the filter base

$$B_f := \{N(T,P,\mathcal{U}) \mid T,P,\mathcal{U} \text{ as above}, f \in N(T,P,\mathcal{U})\}.$$

It is clear that B_f is indeed a filter base, i.e., for any two elements $N(T,P,\mathcal{U})$ and $N(T',P',\mathcal{U}')$ from B_f there exists an element $N(T'',P'',\mathcal{U}'') \in B_f$ s.t.

$$N(T'',P'',\mathcal{U}'') \subseteq N(T,P,\mathcal{U}) \cap N(T',P',\mathcal{U}').$$

Lemma A.1. *For all $f \in C([0,\infty),F)$ it holds $\mathcal{N}(f) = \widetilde{\mathcal{N}}(f)$.*

Proof. It suffices to show $B(f) \subseteq \widetilde{\mathcal{N}}(f)$ and $\widetilde{B}(f) \subseteq \mathcal{N}(f)$ for all $f \in C([0,\infty),F)$. In the following we fix $f \in C([0,\infty),F)$. First let $N(T,P,\mathcal{U}) \in B_f$ be arbitrary, where $T \in \mathbb{N}$, $\mathbf{P} = \{t_0,\dots,t_n\}$ a partition of $[0,T]$, $n \in \mathbb{N}$, and a family $\mathcal{U} = \{U_i\}_{i=1,\dots,n}$ of open sets in F. Now, let $i \in \{0,\dots,n-1\}$. As f is continuous, it maps compact sets to compact sets. Hence, $f([t_i,t_{i+1}]) \subseteq U_{i+1}$ is compact. Further, since U_i is open, there exists for every $t \in [t_i,t_{i+1}]$ a $\varepsilon_t > 0$ s.t. $B_{\varepsilon_t,r}(f(t)) \subseteq U_i$. By the compactness of $f([t_i, t_{i+1}])$ there exists finitely many $t_1^i, \dots, t_{n_i}^i \in [t_i, t_{i+1}]$, $n_i \in \mathbb{N}$, s.t.

$$f([t_i, t_{i+1}]) \subseteq \cup_{j=1}^{n_i} B_{\varepsilon_{t_j^i}, r}(f(t_j^i)).$$

Now define $\varepsilon := 2^{-T} \min\{\varepsilon_{t_j^i} \mid i = 0, \dots, n-1, j = 1, \dots, n_i\}$. Then by the choice of ε it holds $B_{\varepsilon,d}(f) \subseteq N(T, P, \mathcal{U})$. Hence, it also holds $N(T, P, \mathcal{U}) \in \widetilde{\mathcal{N}}(f)$. Now, let $\varepsilon > 0$ be arbitrary. We need to show that there exists $(T, \mathbf{P}, \mathcal{U})$ as above s.t. $f \in N(T, \mathbf{P}, \mathcal{U})$ and $N(T, \mathbf{P}, \mathcal{U}) \subseteq B_{\varepsilon,d}(f)$. Let $T \in \mathbb{N}$ s.t. $2^{-T+1} < \frac{\varepsilon}{2}$. Since f is uniformly continuous on $[0, T]$ there exists an $n \in \mathbb{N}$ s.t. $r(f(t), f(s)) < \frac{\varepsilon}{2}$ if $|t - s| \leq \frac{1}{n}$ for $t, s \in [0, T]$. Define a partition of $[0, T]$ by $\mathbf{P} := \{t_0, \dots, t_{nT}\}$ via $t_i = \frac{i}{n}$, $i = 0, \dots, nT$. Further, define the collection of open sets $\mathcal{U} := \{U_1, \dots, U_{nT}\}$ via $U_i := B_{\frac{\varepsilon}{4}, r}(f(t_i))$. Now, by construction it holds $f \in N(T, \mathbf{P}, \mathcal{U})$ and $N(T, \mathbf{P}, \mathcal{U}) \subseteq B_{\varepsilon,d}(f)$, which finishes the proof. □

Appendix B

Appendix to Part II

B.1 Hermite Polynomials

Since there are several different definitions of Hermite polynomials in the literature, we give the definition we work with and collect facts and useful formulas here. All formulas below can be proven by using the definition (B.1). We define the Hermite polynomials with parameter $\sigma^2 > 0$ $\left(H_{n,\sigma^2}\right)_{n\in\mathbb{N}}$ via their generating function,

$$\exp\left(-\sigma^2\frac{t^2}{2}+tx\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n,\sigma^2}(x), \tag{B.1}$$

i.e.,

$$H_{n,\sigma^2}(x)=\frac{d^n}{dt^n}\exp\left(-\sigma^2\frac{t^2}{2}+tx\right)_{|t=0}.$$

Let $x,y\in\mathbb{R}$, $m,n\in\mathbb{N}$ and $\sigma^2>0$ then it holds

$$H_{n,\sigma^2}(x+y)=\sum_{k=0}^{n}\binom{n}{k}y^kH_{n-k,\sigma^2}(x), \tag{B.2}$$

$$H_{n,\sigma^2}(\alpha x+\beta y)=\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^kH_{n-k,\sigma^2}(x)H_{k,\sigma^2}(y),\quad \alpha^2+\beta^2=1, \tag{B.3}$$

$$H_{n,\sigma^2}(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left(-\frac{1}{2}\sigma^2\right)^k\frac{n!}{k!(n-2k)!}x^{n-2k}, \tag{B.4}$$

$$x^n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left(\frac{1}{2}\sigma^2\right)^k\frac{n!}{k!(n-2k)!}H_{n-2k,\sigma^2}(x), \tag{B.5}$$

$$H_{n,\sigma^2}(\lambda x)=\lambda^nH_{n,(\frac{\sigma}{\lambda})^2}(x),\quad \lambda\in\mathbb{R}\setminus\{0\}, \tag{B.6}$$

$$H_{n,\sigma^2}(x)H_{m,\sigma^2}(x)=\sum_{k=0}^{\min\{m,n\}}k!\binom{m}{k}\binom{n}{k}\sigma^{2k}H_{m+n-2k,\sigma^2}(x), \tag{B.7}$$

$$H_{n+1,\sigma^2}(x)=xH_{n,\sigma^2}(x)-n\sigma^2H_{n-1,\sigma^2}(x), \tag{B.8}$$

$$0=\left(\sigma^2\frac{d^2}{dx^2}-x\frac{d}{dx}+n\right)H_{n,\sigma^2}(x) \tag{B.9}$$

The Hermite polynomials satisfy the following orthogonality relation

$$\int_{\mathbb{R}} H_{n,\sigma^2}(x) H_{m,\sigma^2}(x) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \sqrt{2\pi\sigma^2}\delta_{n,m} n! \sigma^{2n}$$

B.2 Divergence of the Renormalization terms in the $(\Phi)^4_3$ model

In this section we show that the renormalization terms in the interaction potential of the $(\Phi)^4_3$ model tend to infinity, as the cutoff parameter t tends to 0. In particular, this shows that the lower bounds in (6.10) and (6.11) tend to minus infinity as t goes to zero. Furthermore, we obtain also upper bounds which describe the order of the divergence exactly.

To this end, recall the function $f_t \in S(\mathbb{R}^3)$ defined in (6.7) as well as the random variable $\Phi(x,t)$, $x \in \mathbb{R}^3$, $t > 0$ defined in (6.8). We define the functions

$$\alpha(t) := (f_t, Cf_t)_{L^2(\mathbb{R}^3)} = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \frac{\exp\left(-t(|p|^2 + m_0^2)\right)}{|p|^2 + m_0^2}\, dp$$

$$\beta(t) := \delta m_t^2 = 4^2 6 \int\limits_{\mathbb{R}^3} \mathbb{E}_C\left[\Phi(0,t)\Phi(y,t)\right]^3\, dy.$$

In the following, we determine upper and lower bounds for the functions α and β. The function α is continuously differentiable at $t \in (0,\infty)$ with derivative

$$\alpha'(t) = -\frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}^3} \exp\left(-t(|p|^2 + m_0^2)\right)\, dp = -\gamma \frac{\exp\left(-tm_0^2\right)}{t^{\frac{3}{2}}},$$

where γ is a positive constant. From the fundamental theorem of calculus we obtain for $t \in (0,1)$

$$\alpha(1) + \gamma \int\limits_t^1 \frac{\exp\left(-sm_0^2\right)}{s^{\frac{3}{2}}}\, ds = \alpha(t). \tag{B.10}$$

The integral in (B.10) can be estimated from above and from below by

$$\exp\left(-m_0^2\right) \int\limits_t^1 s^{-\frac{3}{2}}\, ds \le \int\limits_t^1 \frac{\exp\left(-sm_0^2\right)}{s^{\frac{3}{2}}}\, ds \le \int\limits_t^1 s^{-\frac{3}{2}}\, ds.$$

This implies that we can find two positive constants γ_1, γ_2 and a real number γ_3 s.t.

$$(\gamma_3 + \gamma_1 t^{-\frac{1}{2}}) \le \alpha(t) \le \gamma_2(1 + t^{-\frac{1}{2}}). \tag{B.11}$$

To establish the order of divergence of β one proceeds similarly. Observe that the integrand in the definition of β is given by

$$E_C\left[\Phi(0,t)\Phi(y,t)\right]^3 = (2\pi)^{\frac{3}{2}} \mathcal{F}\left[\frac{\exp\left(-t(|p|^2 + m_0^2)\right)}{|p|^2 + m_0^2}\right]^3 (y).$$

An elementary calculation shows that the map

$$\mathbb{R} \ni t \mapsto \mathcal{F}\left[\frac{\exp\left(-t(|p|^2 + m_0^2)\right)}{|p|^2 + m_0^2}\right]^3 \in S(\mathbb{R}^3)$$

is differentiable, which implies that we can differentiate under the integral sign in the definition of $\beta(t)$ and obtain

$$\beta'(t) = -\gamma \int_{\mathbb{R}^3} \mathcal{F}\left[\frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2}\right]^2 (y) \mathcal{F}\left[\exp\left(-t(|p|^2+m_0^2)\right)\right](y)\, dy,$$

where γ is again a positive constant. By using Fubinis theorem and (B.11) we obtain for $t \in (0,1)$

$$\begin{aligned}
\beta'(t) &= -\gamma \left(\frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2}, \frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2} * \exp\left(-t(|p|^2+m_0^2)\right)\right)_{L^2(\mathbb{R}^3)} \qquad \text{(B.12)}\\
&\geq -\gamma \left\|\frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2}\right\|_{L^1(\mathbb{R}^3)} \left\|\frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2} * \exp\left(-t(|p|^2+m_0^2)\right)\right\|_{L^\infty(\mathbb{R}^3)}\\
&\geq -\gamma \left\|\frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2}\right\|^2_{L^1(\mathbb{R}^3)}\\
&\geq -2\gamma\gamma_2^2(1+t^{-\frac{1}{2}})^2 \geq -\gamma\gamma_2^2(1+t^{-1}),
\end{aligned}$$

Finally, by integrating the inequality above we obtain for $t \in (0,1)$

$$\beta(t) \leq \gamma\left(1+\ln\left(\frac{1}{t}\right)\right).$$

To obtain a lower bound for β we find again a suitable bound for β' starting again from (B.12). Let $t \in (0,1)$. Then, it holds

$$\begin{aligned}
-\gamma^{-1}\beta'(t) &= \left(\frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2}, \frac{\exp\left(-t(|p|^2+m_0^2)\right)}{|p|^2+m_0^2} * \exp\left(-t(|p|^2+m_0^2)\right)\right)_{L^2(\mathbb{R}^3)}\\
&\geq \int_{B_{\frac{1}{\sqrt{t}}}(0)} \frac{\exp\left(-t(|p_1|^2+m_0^2)\right)}{|p_1|^2+m_0^2} \int_{B_{\frac{1}{\sqrt{t}}}(0)} \frac{\exp\left(-t(|p_1-p_2|^2+m_0^2)\right)}{|p_1-p_2|^2+m_0^2} \exp\left(-t(|p_2|^2+m_0^2)\right)\, dp_2\, dp_1\\
&\geq \exp(-3tm_0^2-6)\frac{1}{\frac{1}{t}+m_0^2}\int_{B_{\frac{1}{\sqrt{t}}}(0)} dp_1 \frac{1}{\frac{4}{t}+m_0^2}\int_{B_{\frac{1}{\sqrt{t}}}(0)} dp_2\\
&\geq \left(\frac{4}{3}\pi\right)^2 \exp(-3tm_0^2-6)\frac{1}{\sqrt{t}+t^{\frac{3}{2}}m_0^2}\frac{1}{4\sqrt{t}+t^{\frac{3}{2}}m_0^2}\\
&\geq \frac{1}{t}\left(\frac{4}{3}\pi\right)^2 \frac{\exp(-3m_0^2-6)}{(1+m_0^2)(4+m_0^2)}.
\end{aligned}$$

Via integration, we obtain for some positive constant $\hat{\gamma}$ that.

$$\beta(t) \geq \hat{\gamma}\left(1+\ln\left(\frac{1}{t}\right)\right). \qquad \text{(B.13)}$$

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Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit mit dem Titel

Operator Semigroups and Infinite Dimensional Analysis Applied to Problems from Mathematical Physics

selbstständig und ohne unerlaubte fremde Hilfe angefertigt, keine anderen als die angegebenen Quellen und Hilfsmittel verwendet und die den verwendeten Quellen und Hilfsmitteln wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe. Darüber hinaus wurde diese Dissertation ausschließlich dem Fachbereich Mathematik der TU Kaiserslautern als Prüfungsleistung vorgelegt.

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Typeset October 26, 2020