

# Explicit and effective Mather–Yau correspondence in view of analytic gradings

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# Contents

<b>Introduction</b>	<b>1</b>
<b>Conventions and Notations</b>	<b>9</b>
<b>1 Complex Spaces and Singularities</b>	<b>11</b>
1.1 Analytic Algebras	11
1.2 Complex Spaces	14
1.3 Complex Space Germs	18
1.4 Derivations on Analytic Algebras	20
1.4.1 Derivations on Analytic Algebras I: Basics	20
1.4.2 Derivations on Analytic Algebras II: Singularities	24
1.4.3 Derivations on Analytic Algebras III: Triviality	25
1.5 Hypersurface Singularities	26
1.5.1 Isolated Hypersurface Singularities	27
1.5.2 Quasi-Homogeneous Isolated Hypersurface Singularities	29
1.5.3 A General Mather–Yau Theorem	32
<b>2 Gradings of Analytic Algebras and Modules</b>	<b>41</b>
2.1 Gradings of Rings and Modules	42
2.2 $(\mathbb{C}^m, +)$ Gradings of Analytic Algebras and Derivations	44
2.3 Linear Algebraic Subgroups of $\text{Aut}(A)$	48
2.3.1 Linear Reductive Subgroups of $\text{Aut}_I(A)$	49
2.3.2 $(\mathbb{Z}^m, +)$ Gradings, Algebraic Tori and Derivations	52
2.3.3 Maximal Multihomogeneity	54
2.4 The Structure Theorem for Analytic Algebras	57
2.4.1 Lie–Rinehart Algebras	57
2.4.2 The Structure Theorem	60
<b>3 Positive Gradings and Monomial Ideals</b>	<b>63</b>
3.1 Positively Graded Analytic Algebras	63
3.2 Positively Graded Tjurina Algebra	66
3.3 Stanley–Reisner Monomial Singular Loci	71
3.4 Proof of Theorem 3.25	72

<b>4</b>	<b>Free Singularities and Generalized Normal Crossing Divisor</b>	<b>81</b>
4.1	Preliminaries . . . . .	81
4.2	Generic Subspace Arrangements and Freeness . . . . .	84
4.3	Constructing Free Singularities via Products . . . . .	87
4.4	Properties of Generalized Normal Crossing divisors . . . . .	91
4.4.1	Freeness and Generalized Normal Crossing divisors . . . . .	91
4.4.2	Saito Holonomicity and Generalized Normal Crossing Divisors . . . . .	92
<b>5</b>	<b>Algorithms and Examples</b>	<b>99</b>
5.1	Linear Algebra for Vector Fields . . . . .	99
5.2	Computing Weights and Monomiality . . . . .	106
5.3	Classifying Non-Reduced and Generalized Normal Crossing Divisors . . . . .	108
5.4	Reconstructing QHIS from their Milnor Algebra . . . . .	110
5.4.1	More about QHIS . . . . .	110
5.4.2	The Reconstruction Algorithm . . . . .	112
<b>6</b>	<b>On Singularities of Sebastiani–Thom type</b>	<b>119</b>
6.1	Proof of Theorem 6.2 (b) and (c) . . . . .	120
6.2	Proof of Theorem 6.2 (d) . . . . .	123
6.3	The case $w_1 = w_2 = w_3 = 1$ . . . . .	123
6.3.1	The case $w_1 > w_2 > w_3$ . . . . .	127
6.4	The case $w_1 > w_2 = w_3$ . . . . .	144
6.5	The case $w_1 = w_2 > w_3$ . . . . .	162

# Introduction

In 1982 J. Mather and S. Yau published the famous Mather–Yau theorem in [MY82]. The theorem yields a one-to-one correspondence between isomorphism classes of germs of isolated hypersurface singularities and isomorphism classes of their respective so-called Tjurina algebras. Given the defining equation  $f \in \mathbb{C}\{\mathbf{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$  of a hypersurface singularity, the Tjurina algebra  $T_f$  is defined as  $T_f = \mathbb{C}\{\mathbf{x}\}/\langle f, J_f \rangle$ , where  $J_f = \langle \partial_{x_1} f, \dots, \partial_{x_n} f \rangle$  is the Jacobian ideal. A first version of the Mather–Yau theorem in case of quasi-homogeneous isolated hypersurface singularities has been shown six years earlier by A.N. Shoshitaishvili in [Sho76]. The result in the quasi-homogeneous case was also announced by G.-M. Greuel in [Gre77], but has never been published. Three years later, the result has been generalized to singularities of isolated singularity type by T. Gaffney and H. Hauser in [GH85]. One year later, a further generalization to so-called harmonic singularities has been shown by H. Hauser and G. Müller in [HM86]. The purpose of the aforementioned theorems is to reduce the classification of singularities to the classification of  $\mathbb{C}$ -algebras. In the particular case of isolated hypersurface singularities, the problem is reduced to the isomorphy problem of finite-dimensional  $\mathbb{C}$ -algebras. A classical example of an invariant associated to isolated hypersurface singularities is the  $\mathbb{C}$ -dimension of the Tjurina algebra, the so-called Tjurina number. In general the Mather–Yau theorem does not hold in positive characteristic. G.-M. Greuel and T. H. Pham (see [GP17]) stated an analogous result to the Mather–Yau theorem in positive characteristic, which replaces the Tjurina algebra by so-called higher Tjurina algebras  $T_{f,k} = K[[\mathbf{x}]]/\langle f, \mathfrak{m}^k J_f \rangle$ , where  $K$  is an algebraically closed field of positive characteristic and where  $k \in \mathbb{N}$  has to satisfy  $\mathfrak{m}^{\lfloor \frac{k + \text{ord}(f) + \text{ord}(J_f) + 1}{2} \rfloor} \subseteq \mathfrak{m}^2 J_f$ .

In the present thesis we focus on the case of complex singularities. The results by J. Mather and S. Yau, as well as by T. Gaffney and H. Hauser indicate that, for certain classes of singularities, all the information about the singularity is encoded in the Tjurina algebra. Unfortunately, neither the proof in [MY82] nor the proof in [GH85] is constructive. This gives rise to two problems: the **recognition problem** and the **reconstruction problem**. The recognition problem is to decide whether a given  $\mathbb{C}$ -algebra is isomorphic to the Tjurina algebra of a hypersurface singularity, whereas the reconstruction problem is to reconstruct a defining equation for the hypersurface singularity from which a given Tjurina algebra arises.

In case of a quasi-homogeneous isolated hypersurface singularity, S. Yau gave the following theoretical answer to the recognition problem in [Yau87]:

**Theorem.** *Let  $I = \langle f_1, \dots, f_k \rangle \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal with generators  $f_1, \dots, f_k$ , where  $1 \leq k \leq n$ . Then there exists a  $g \in \mathbb{C}\{\mathbf{x}\}$  with  $J_g = I$  if, and only if there exist quasi-homogeneous*

polynomials  $F_1, \dots, F_n \in \mathbb{C}[\mathbf{x}]$  and a matrix  $B \in \mathbb{C}^{k \times n}$  of rank  $k$ , such that

$$\begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = B \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$$

and such that

$$\partial_{x_i} F_j = \partial_{x_j} F_i \text{ for all } 1 \leq i, j \leq n.$$

From a computational point of view the given answer by S. Yau is hard to verify if the singularity is not homogeneous.

In the present thesis, we investigate the aforementioned problems by using the theory of analytic gradings introduced by G. Scheja and H. Wiebe in [SW73]. Their theory is a generalization of the classical theory of gradings of rings, see for example [Eis95, Chapter 1], to analytic algebras. Their work yields a one-to-one relation between  $\mathbb{Z}$ -gradings of an analytic algebra  $A = \mathbb{C}\{\mathbf{x}\}/I$  and semi-simple logarithmic derivations  $\delta \in \text{Der}_I(\mathbb{C}\{\mathbf{x}\})$  (see Definitions 1.49 and 1.46).

## Overview

The thesis is structured as follows:

### Chapter 1

We present the basic theory of complex analytic spaces and we restate the proof of the main theorem in [GH85] for hypersurface singularities. It leads to the following mild generalization, involving so-called strongly Euler-homogeneous singularities:

**Definition 1.100.** Let  $X \subseteq \mathbb{C}^n$  be a hypersurface singularity. Denote by  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  the local equation of  $X$  in  $p \in X$ . We call  $X$  **Euler-homogeneous at**  $p \in X$  if, and only if, there exists a derivation  $\chi_p \in \text{Der}(\mathcal{O}_{X,p})$ , such that  $\chi_p(f_p) = f_p$ , where  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  is the local equation of  $X$  at  $p \in X$ . A derivation  $\chi_p$  is called **Euler-derivation of  $f$  at  $p$** . We call  $X$  **strongly Euler-homogeneous at**  $p \in X$  if, and only if, there exists an Euler derivation  $\chi_p$  satisfying  $\chi_p(p) = 0$ . We call  $X$  **(strongly) Euler-homogeneous**, if  $X$  is (strongly) Euler-homogeneous at all  $p \in X$ . Let  $f \in \mathbb{C}\{\mathbf{x} - p\}$  be holomorphic on  $U \subseteq \mathbb{C}^n$ . We say  $f$  is (strongly) Euler-homogeneous, if  $X = V(f) \subseteq U$  is (strongly) Euler-homogeneous at  $p$ . We call a complex space germ  $(X, p)$  (strongly) Euler homogeneous (at  $p$ ), if there exists a representant which is (strongly) Euler-homogeneous (at  $p$ ).

We obtain the following result:

**Theorem 1.101.** Let  $f, g \in \mathbb{C}\{\mathbf{x}\}$  define singularities  $(X, \mathbf{0})$ , respectively  $(Y, \mathbf{0})$ , which are strongly Euler-homogeneous at  $\mathbf{0}$ . Then the following are equivalent:

- (1)  $(X, \mathbf{0}) \cong (Y, \mathbf{0})$ .
- (2)  $T_f \cong T_g$  as  $\mathbb{C}$ -algebras.

Theorem 1.101 combined with [HM86, Theorem 4] implies the following:



**Corollary 1.105.** *Let  $f, g \in \mathbb{C}\{\mathbf{x}\}$  define weighted-homogeneous hypersurface singularities with weight-vector  $\mathbf{0} \neq \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n$  and with weighted degree  $d := \deg_{\mathbf{w}}(f) = \deg_{\mathbf{w}}(g)$ . Denote the singularities defined by  $f$  and  $g$  by  $(X, \mathbf{0})$ , respectively  $(Y, \mathbf{0})$ .*

*Assume that either*

- (i)  $d \neq 0$ , or
- (ii)  $d - w_i \neq 0$  for  $1 \leq i \leq n$ .

*Then the following are equivalent:*

- (1)  $(X, \mathbf{0}) \cong (Y, \mathbf{0})$ .
- (2)  $\mathbb{T}_f \cong \mathbb{T}_g$  as  $\mathbb{C}$ -algebras.

We show, by adapting [GH85, Example 4], that the statement of Corollary 1.105 is sharp in the following sense:

**Example 1.107.** *For  $t \in \mathbb{C} \setminus \{-1\}$ , consider the family of polynomials*

$F_t = x_1^3 x_2 + x_2^5 x_3 + x_3^5 x_1 + x_1 x_2 x_3 + x_4 x_5 + (1 + z + t) \cdot (y_1^3 y_2 + y_2^5 y_3 + y_3^5 y_1 + y_1 y_2 y_3 + y_4 y_5)$   
*as elements of  $\mathbb{C}\{x_1, \dots, x_5, y_1, \dots, y_5, z\}$ . Define  $(X_t, \mathbf{0}) = (V(F_t), \mathbf{0}) \subseteq (\mathbb{C}^{11}, \mathbf{0})$  for  $t \in \mathbb{C} \setminus \{-1\}$ . The  $F_t$  are weighted-homogeneous with respect to  $\mathbf{w} = (0, 0, 0, 1, -1, 0, 0, 0, 1, -1, 0) \in \mathbb{Z}^{11}$ . Then  $\deg_{\mathbf{w}}(F_t) = 0$  and  $(\text{Sing}(X_0), \mathbf{0}) \cong (\text{Sing}(X_t), \mathbf{0})$  for all  $t \in \mathbb{C} \setminus \{-1\}$ , but  $(X_0, \mathbf{0}) \not\cong (X_t, \mathbf{0})$  for  $t \in V \setminus \{0\}$ , where  $V \subseteq \mathbb{C} \setminus \{-1\}$  is an open neighborhood of 0.*

## Chapter 2

We first present the theory of analytic gradings by G. Scheja and H. Wiebe following [SW73]. Next we investigate the relation between analytic  $\mathbb{Z}^k$ -gradings of an analytic algebra  $A = \mathbb{C}\{\mathbf{x}\}/I$ , toral Lie subalgebras of the module of logarithmic derivations of  $I \text{Der}'_I(\mathbb{C}\{\mathbf{x}\}) = \text{Der}'_I(\mathbb{C}\{\mathbf{x}\}) \cap \text{Der}_{\mathbf{m}}(\mathbb{C}\{\mathbf{x}\})$  and the subgroup  $\text{Aut}_I(\mathbb{C}\{\mathbf{x}\})$  (see Definition 2.38) of the automorphism group  $\text{Aut}(\mathbb{C}\{\mathbf{x}\})$ . We show that the dimension  $s$  of the maximal algebraic tori contained in  $\text{Aut}_I(\mathbb{C}\{\mathbf{x}\})$  is an invariant of  $A$ , and that  $s$  corresponds to the maximal possible value of  $k$  such that  $A$  admits  $\mathbb{Z}^k$ -grading. The integer  $s$  is called rank of maximal multihomogeneity of  $A$  and will be used in the chapters 3 and 6.

We finish the chapter by generalizing [Sch07, Theorem 1] to the case of arbitrary ideals.

**Theorem 2.80.** *Let either*

- (i)  $A = \mathbb{C}[[\mathbf{x}]]$  and  $I \subseteq A$ , or
- (ii)  $A = \mathbb{C}\{\mathbf{x}\}$  and  $I \subseteq A$  be an algebraic ideal.

*Define  $\mathfrak{g} := \text{Der}'_I(A) \subseteq \text{Der}'(A)$  and let  $s \in \mathbb{N}$  be the rank of maximal multihomogeneity. Then there exist  $\delta_1, \dots, \delta_s, \nu_1, \dots, \nu_r \in \mathfrak{g}$ , such that*

- (1)  $\delta_1, \dots, \delta_s, \nu_1, \dots, \nu_r$  is a minimal set of generators of  $\mathfrak{g}$  as an  $A$ -module,
- (2) if  $\sigma \in \mathfrak{g}$  with  $[\delta_i, \sigma] = 0$  for all  $i$ , then  $\sigma_S \in \langle \delta_1, \dots, \delta_s \rangle_{\mathbb{C}}$ ,
- (3)  $\delta_i$  is diagonal with eigenvalues in  $\mathbb{Q}$ ,
- (4)  $\nu_i$  is nilpotent, and
- (5)  $[\delta_i, \nu_j] \in \mathbb{Q} \cdot \nu_j$

### Chapter 3

In Chapter 3 we focus on the following problem given by H. Hauser and J. Schicho in [HS11]:

**Problem.** *Characterize all germs of hypersurface singularities  $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$ , such that  $\langle f, J_f \rangle$  is monomial.*

By modifying the proof of [XY96, Theorem 1.2] and by using the theory of analytic gradings presented in Chapter 2, we show the following:

**Theorem 3.11.** *Let  $f \in \mathfrak{m} \subseteq \mathbb{C}\{\mathbf{x}\}$  and assume that the Tjurina algebra  $\mathbb{T}_f$  admits a positive grading. Then  $f \in \mathfrak{m}J_f$ . Equivalently, the germ  $(V(f), \mathbf{0})$  is strongly Euler-homogeneous at  $\mathbf{0}$ .*

An immediate corollary of Theorem 3.11 is the fact that hypersurface singularities with monomial Tjurina ideal  $\langle f, J_f \rangle$  are strongly Euler-homogeneous and thus also satisfy Theorem 1.101. This indicates that combinatorial properties of the singular locus yield information about the singularity itself. We consider so-called ideals of Stanley–Reisner type:

**Definition 3.23.** *Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $\mathbb{C}[[\mathbf{x}]]$ . Let  $I \subseteq A$  be an ideal. We say  $I$  is an ideal of **monomial type**, if there exists an automorphism  $\varphi \in \text{Aut}(A)$ , such that  $\varphi(I)$  is a monomial ideal. We say  $I$  is an ideal of **Stanley–Reisner type**, if  $I$  is a radical ideal of monomial type.*

The main result of this chapter is the classification of all hypersurface singularities where the ideal  $\langle f, J_f \rangle$  is of Stanley–Reisner type.

**Theorem 3.25.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$ . Then  $\langle f, J_f \rangle$  being of Stanley–Reisner type is equivalent to the existence of an automorphism  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$  and a partition of the  $\mathbf{x}$  variables, denoted by  $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l+1)}$ , such that*

$$\varphi(f) = \sum_{j=1}^{r_1} \left(x_j^{(0)}\right)^2 + \sum_{i=1}^l g_i,$$

where  $g_i \in \mathbb{C}[\mathbf{x}^{(i)}]$  is a normal crossing divisor for  $1 \leq i \leq l$ . This means that all singularities with Stanley–Reisner singular locus are of Sebastiani–Thom type where the summands are  $A_1$ -singularities or normal crossing divisors. In particular, the Sebastiani–Thom components are unique up to isomorphism and permutation.

Due to the shape of the defining equation, we call the singularities with analytical Stanley–Reisner Tjurina ideal **generalized normal crossing divisors**. Theorem 3.25 shows that every generalized normal crossing divisor is of Sebastiani–Thom type, that is:

**Definition 3.24.** *Let  $f \in \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ . We say  $f$  is of **Sebastiani–Thom type**, if there exist  $g \in \mathbb{C}\{\mathbf{x}\}$  and  $h \in \mathbb{C}\{\mathbf{y}\}$ , such that  $f = g + h$ . We say that a hypersurface singularity  $X \subseteq \mathbb{C}^{n+m}$  is of Sebastiani–Thom type at  $p = (p_1, p_2) \in X$  if there exists an isomorphism such that  $(X, p) \cong (V(f), p)$ , where  $f \in \mathbb{C}\{\mathbf{x} - p_1, \mathbf{y} - p_2\}$  is of Sebastiani–Thom type. We call  $X$  a **Sebastiani–Thom type hypersurface singularity**, if it is of Sebastiani–Thom type for all  $p \in X$ . We say a complex space germ  $(X, \mathbf{0})$  is of Sebastiani–Thom type, if there exists a representant which is of Sebastiani–Thom type. Consider the germ  $(X, \mathbf{0}) \cong (V(f), \mathbf{0})$  with  $f = g + h$  and  $g \in \mathbb{C}\{\mathbf{x}\}, h \in \mathbb{C}\{\mathbf{y}\}$ . We call the germs  $(X_1, \mathbf{0}) = (V(g), \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$  and  $(X_2, \mathbf{0}) = (V(h), \mathbf{0}) \subseteq (\mathbb{C}^m, \mathbf{0})$  the **Sebastiani–Thom components** of  $(X, \mathbf{0})$ .*

## Chapter 4

This chapter is joint work with D. Pol (see [EP20]). In Chapter 2 we work with the logarithmic derivation modules  $\text{Der}_I(\mathbb{C}\{\mathbf{x}\})$ . In case  $I = \langle f \rangle$  is a principal ideal, it coincides with the notion of logarithmic derivations considered by K. Saito in [Sai75] and [Sai80]. In these papers, K. Saito investigates a particular family of hypersurfaces called free divisors. A hypersurface defined by  $I = \langle f \rangle$  is called free if  $\text{Der}_I(\mathbb{C}\{\mathbf{x}\})$  is a free  $\mathbb{C}\{\mathbf{x}\}$ -module. Several characterizations of freeness are known:  $X$  is free if and only if the module of logarithmic 1-forms is free, where a form  $\omega$  is logarithmic if  $\omega$  and  $d\omega$  have simple poles along  $X$  (see [Sai80]). Another characterization of freeness is given by H. Terao (see [Ter80]) in case of quasi-homogeneous hypersurface and A. G. Aleksandrov ([Ale88]) in general: a hypersurface  $(X, 0) \subseteq \mathbb{C}^n$  defined by  $f \in \mathbb{C}\{\mathbf{x}\}$  is free if and only if  $(X, 0)$  is smooth or  $T_f$  is Cohen–Macaulay of dimension  $n - 2$ .

Generalizations of logarithmic forms along complete intersections are introduced in [AT01] and [Ale12]. The definition of multi-logarithmic forms in [Ale12] is then extended to Cohen–Macaulay spaces in [Ale14], inspired by a characterization of regular meromorphic forms given by M. Kersken in [Ker84]. Analogously, a generalization of logarithmic vector fields is introduced in [GS12] for complete intersections by M. Granger and M. Schulze, and then in [Pol16] for Cohen–Macaulay spaces by D. Pol. These definitions extend verbatim to equidimensional subspaces (see [Pol20]). In this chapter, we use the equivalent definition given by M. Schulze and L. Tozzo in [ST18]:

**Definition 4.1.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$  be a Cohen–Macaulay subspace of codimension  $k$  defined as the vanishing set of the radical ideal  $I_X \subseteq \mathbb{C}\{\mathbf{x}\}$ . The module of multi-logarithmic  $k$ -vector fields along  $X$  is defined by*

$$\text{Der}^k(-\log X) = \left\{ \delta \in \bigwedge^k \text{Der}(\mathbb{C}\{\mathbf{x}\}) \mid \forall (f_1, \dots, f_k) \in I_X^k, \langle \delta, df_1 \wedge \dots \wedge df_k \rangle \in I_X \right\}.$$

A generalization of freeness is suggested in [GS12] for complete intersection, which is inspired by the characterization of H. Terao and A.G. Aleksandrov mentioned before, and afterwards extended by M. Schulze in [Sch16] to Gorenstein singularities. We use the generalization of freeness for an equidimensional subspace  $(X, \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$  of codimension  $k$  given in [Pol20]:  $X$  is free if

$$\text{projdim } \text{Der}^k(-\log X) = k - 1.$$

Using a perfect pairing between the multi-logarithmic  $k$ -forms and  $\text{Der}^k(-\log X)$ , one can show a characterization of freeness involving multi-logarithmic  $k$ -forms (see [Pol16]). These results have been translated in terms of general commutative algebra in [ST18], removing the singularity theoretical context.

The first main result in this chapter is the following:

**Theorem 4.30.** *Let  $(X_1, \mathbf{0}) \subseteq (\mathbb{C}^{n_1}, \mathbf{0})$  and  $(X_2, \mathbf{0}) \subseteq (\mathbb{C}^{n_2}, \mathbf{0})$  be reduced Cohen–Macaulay subspaces and  $(X, \mathbf{0}) = (X_1, \mathbf{0}) \times (X_2, \mathbf{0}) \subseteq (\mathbb{C}^{n_1}, \mathbf{0}) \times (\mathbb{C}^{n_2}, \mathbf{0})$ . Then  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0})$  are free if and only if  $(X, \mathbf{0})$  is free.*

Applying this result to the generalized normal crossing divisors considered in the previous chapter gives us the following:

**Proposition 4.40.** *Let  $(X, \mathbf{0}) \subseteq \mathbb{C}^n$  be a generalized normal crossing divisor. Then  $(\text{Sing}(X), \mathbf{0})$  is a free singularity.*

We also give another approach to the following conjecture by E. Faber (see [Fab15]):

**Conjecture.** *Let  $(X, \mathbf{0}) \subseteq \mathbb{C}^n$  be the germ of a hypersurface singularity. Denote by  $f \in \mathbb{C}\{\mathbf{x}\}$  a local equation of  $(X, \mathbf{0})$ . Then the following are equivalent:*

- (1)  $(X, \mathbf{0})$  is a normal crossing divisor.
- (2)  $(X, \mathbf{0})$  is free and  $J_f$  is a radical ideal.

**Proposition 4.42.** *Let  $(X, \mathbf{0}) \subseteq \mathbb{C}^n$  be the germ of a hypersurface singularity. Denote by  $f \in \mathbb{C}\{\mathbf{x}\}$  a local equation of  $(X, \mathbf{0})$ . Then the following are equivalent:*

- (1)  $(X, \mathbf{0})$  is a normal crossing divisor.
- (2)  $(X, \mathbf{0})$  is free and  $J_f$  is of Stanley–Reisner type.

The last part of this chapter is devoted to another property related to logarithmic derivations, that is the holonomicity in the sense of K. Saito (see [Sai80]). The main result of this part is:

**Theorem 4.43.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^{n+m}, \mathbf{0})$  be a strongly Euler-homogeneous singularity of Sebastiani–Thom type. We denote the Sebastiani–Thom components of  $(X, \mathbf{0})$  by  $(X_1, \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$  and  $(X_2, \mathbf{0}) \subseteq (\mathbb{C}^m, \mathbf{0})$ . Then the following hold:*

- (1)  $(Y, \mathbf{0}) \subseteq (\text{Sing}(X), \mathbf{0})$  is a logarithmic stratum if, and only if, there exists a logarithmic stratum  $(X_{1,\alpha}, \mathbf{0}) \subseteq (\text{Sing}(X_1), \mathbf{0})$  and a logarithmic stratum  $(X_{2,\beta}, \mathbf{0}) \subseteq (\text{Sing}(X_2), \mathbf{0})$ , such that

$$(Y, \mathbf{0}) = (X_{1,\alpha}, \mathbf{0}) \times (X_{2,\beta}, \mathbf{0}) =: (X_{(\alpha,\beta)}, \mathbf{0}).$$

- (2)  $(X, \mathbf{0})$  is holonomic if, and only if,  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0})$  are holonomic.

In particular, Theorem 4.43 implies that generalized normal crossing divisors are Saito holonomic.

## Chapter 5

In this chapter we present an algorithmic approach to the questions considered in the previous chapters. In particular, we provide a Las Vegas algorithm solving the recognition and reconstruction problem for quasi-homogeneous isolated hypersurface singularities. A similar algorithm for the homogeneous case has been presented in [IK14]. Our approach has been announced in [ERS17]. The algorithm is implemented in the computer algebra system OSCAR (see [Tea20]) and can be downloaded at <https://github.com/raulepure/reconstruction.jl>.

## Chapter 6

Computations using the algorithms from Chapter 5 give rise to the following conjecture for quasi-homogeneous isolated hypersurface singularities.

**Conjecture 6.1.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  define a quasi-homogeneous isolated hypersurface singularity. Then  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components if, and only if, the maximal multihomogeneity of  $J_f$  is at least 2.*

We are able to show Conjecture 6.1 in some particular cases. The main result is the following:

**Theorem 6.2.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  be a quasi-homogeneous isolated hypersurface singularity with respect to the weight-vector  $\mathbf{w} \in \mathbb{N}_{>0}^n$ . Assume that  $J_f$  is multihomogeneous with respect to  $\mathbf{w}$  and  $\mathbf{v} \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$ , where  $\mathbf{w}$  and  $\mathbf{v}$  are linearly independent, and that one of the following properties holds:*

(a)  $J_f$  is of monomial type.

(b)  $\mathbf{w}$  satisfies, after possibly permuting the variables,

$$w_1 > \dots > w_n > \frac{w_1}{2}.$$

(c)  $\mathbf{w}$  satisfies, after possibly permuting the variables,

$$w_1 \geq \dots \geq w_n > \frac{w_1}{2}$$

$$\text{and } \mathbf{v} = (1, \dots, 1).$$

(d)  $n \leq 3$ .

*Then  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components.*



# Conventions and Notations

In the following  $X$  denotes a complex space,  $A$  an analytic algebra and  $I \subseteq A$  an ideal. The germ of  $X$  at  $p \in X$  is denote by  $(X, p)$ . Boldface letters, for example  $\mathbf{x}$ , represent vectors  $\mathbf{x} = (x_1, \dots, x_n)$  for some  $n \in \mathbb{N}$ .

$\mathbb{C}[\mathbf{x}]$	The polynomial ring in $\mathbf{x}$ over $\mathbb{C}$ .
$\mathbb{C}\{\mathbf{x}\}$	The ring of convergent power series in $\mathbf{x}$ over $\mathbb{C}$ .
$K[[\mathbf{x}]]$	The ring of formal power series in $\mathbf{x}$ over $K$ .
$\text{ord}(f)$	The order of a power series $f$ .
$J_f$	The Jacobian ideal of $f$ .
$M_f$	The Milnor algebra of $f$ , that is $\mathbb{C}\{\mathbf{x}\}/J_f$ .
$T_f$	The Tjurina algebra of $f$ , that is $\mathbb{C}\{\mathbf{x}\}/\langle f, J_f \rangle$ .
$\text{Der}(A, B)$	The set of all $\mathbb{C}$ -linear derivations on $A$ with values in $B$ .
$\text{Der}(A)$	The set of all $\mathbb{C}$ -linear derivations on $A$ with values in $A$ .
$\text{Der}_I(A)$	The set of logarithmic vector fields of $I$ .
$\text{Der}'_I(A)$	The set of logarithmic vector fields of $I$ and $\mathfrak{m}$ .
$\mathcal{O}_X$	The sheaf of holomorphic functions on $X$ .
$\dim_p(X)$	The dimension of $X$ at $p$ .
$\dim(A)$	The Krull dimension of $A$ .
$\text{Sing}(X)$	The singular locus of $X$ .
$V(I)$	The vanishing set of $I$ .





# Chapter 1

## Complex Spaces and Singularities

The following chapter serves as an introduction to the underlying topic of this thesis, namely singularities. We begin with the basic definitions of analytic algebras, complex spaces and singularities. The chapter will be finished by a proof of the analogous result to the Mather–Yau theorem by Gaffney and Hauser (see [GH85]) in case of hypersurfaces of isolated singularity type and strongly Euler-homogeneous hypersurface singularities. We will follow the outline of this topic as presented in the literature, as for example in [JP00], [Fis76], [GLS07] and [GR71]. Since we intend to focus on the most important definitions and results, we omit an introduction to category theory and sheaf theory at this point.

### 1.1 Analytic Algebras

In the following we will present the theory of analytic algebras exclusively over the field of complex numbers  $\mathbb{C}$  endowed with the standard absolute value. Although every construction in this section would also work over complete real valued fields of characteristic 0 (see for example [GLS07, Chapter I]), we omit it at this point, since we want to consider singularities over the complex numbers. To keep notation short we will write vectors using bold letters. For example  $\mathbf{x} = (x_1, \dots, x_n)$ . Using this notation we can write any **formal power series**  $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$  shorthand as  $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{x}^\alpha$ . We denote by  $\mathbb{C}[[x_1, \dots, x_n]] = \mathbb{C}[[\mathbf{x}]]$  the **formal power series ring**, where the addition and multiplication are as usual. The first object we consider are convergent power series.

**Definition 1.1.** Let  $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{x}^\alpha$  be a formal power series. We call  $f$  a **convergent power series**, if there exists a vector  $\mathbf{v} \in \mathbb{R}_{>0}^n$ , such that  $\sum_{\alpha \in \mathbb{N}^n} |c_\alpha| \mathbf{v}^\alpha$  is a convergent series. We denote the **convergent power series ring** by  $\mathbb{C}\{\mathbf{x}\}$ .

The main objects of interest in this thesis are quotients of the convergent or formal power series ring.

**Definition 1.2.** Let  $A$  be a  $\mathbb{C}$ -algebra.  $A$  is an **analytic algebra**, if  $A$  is isomorphic (as a  $\mathbb{C}$ -algebra) to  $\mathbb{C}\{\mathbf{x}\}/I$  for some ideal  $I \subset \mathbb{C}\{\mathbf{x}\}$ . We call  $A$  a **formal analytic algebra**, if  $A$  is isomorphic to  $\mathbb{C}[[\mathbf{x}]]/I$  for some ideal  $I \subseteq \mathbb{C}[[\mathbf{x}]]$ .

**Remark 1.3.** From an algebraic point of view (formal) analytic algebras are interesting since they are Noetherian local rings with maximal ideal  $\mathfrak{m} := \langle \mathbf{x} \rangle$ . In particular the units of  $\mathbb{C}[[\mathbf{x}]]$  are the elements with non-zero constant term.

The definition of convergence for convergent power series rings reduces to the analytic notion of convergence. Next we will see an algebraic definition of convergence, which turns the formal power series ring into a complete ring with respect to this notion.

**Definition 1.4.** Let  $A$  be either  $\mathbb{C}\{\mathbf{x}\}$  or  $\mathbb{C}[[\mathbf{x}]]$ . A sequence  $(f_k)_{k \in \mathbb{N}} \subset R$  is called **convergent in the  $\mathfrak{m}$ -adic topology** to  $f \in A$ , if for each  $l \in \mathbb{N}$  there exists a number  $K$ , such that  $f_k - f \in \mathfrak{m}^l$  for all  $k \geq K$ . It is called a **Cauchy sequence** if for each  $l \in \mathbb{N}$  there exists a number  $K$ , such that  $f_k - f_m \in \mathfrak{m}^l$  for all  $k, m \geq K$ .

One can show that  $\mathbb{C}[[\mathbf{x}]]$  is the completion of  $\mathbb{C}\{\mathbf{x}\}$  with respect to this notion of convergence. Using this it is fairly easy to prove the following lemma.

**Lemma 1.5.** Let  $A$  be a (formal) analytic algebra and  $M$  a finite  $A$ -module. Then

$$\bigcap_{i \geq 0} \mathfrak{m}_A^i M = 0.$$

*Proof.* See [GLS07, Chapter I, Lemma 1.3]. □

The next step is to define morphisms of analytic algebras. With these we obtain the **category of analytic algebras**  $\mathfrak{A}$ . For more details on this topic see [GR71, Kapitel 2, §0].

**Definition 1.6.** Let  $A$  and  $B$  be (formal) analytic algebras. We call  $\varphi : A \rightarrow B$  a **morphism of (formal) analytic algebras**, if  $\varphi$  is a  $\mathbb{C}$ -algebra morphism.

**Remark 1.7.** Let  $\varphi : \mathbb{C}\{x_1, \dots, x_n\} \rightarrow \mathbb{C}\{y_1, \dots, y_m\}$  be a morphism of analytic algebras. Then  $\varphi$  is determined by the values on a minimal generating set of the maximal ideal  $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ . This means that  $\varphi(x_i) = f_i$  for certain  $f_i \in \mathbb{C}\{y_1, \dots, y_m\}$  already determines the morphism.

The most important feature of analytic algebras is the fact that we can apply tools from analysis and obtain algebraic results.

**Theorem 1.8 (Implicit Function Theorem).** Let  $A = \mathbb{C}\{x_1, \dots, x_n, y_1, \dots, y_m\}$ . Furthermore, let  $f_i \in A$  for  $i = 1, \dots, m$  satisfy  $f_i(\mathbf{0}) = 0$  and

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\mathbf{0}) & \dots & \frac{\partial f_1}{\partial y_m}(\mathbf{0}) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial y_1}(\mathbf{0}) & \dots & \frac{\partial f_m}{\partial y_m}(\mathbf{0}) \end{pmatrix} \neq 0.$$

Then  $A/\langle f_1, \dots, f_m \rangle \cong \mathbb{C}\{x_1, \dots, x_n\}$  and there exist unique power series  $Y_1, \dots, Y_m \in \langle x_1, \dots, x_n \rangle$  solving the implicit system of equations

$$f_i(\mathbf{x}, Y_1(\mathbf{x}), \dots, Y_m(\mathbf{x})) = 0, \quad i = 1, \dots, m.$$

Moreover,  $\langle f_1, \dots, f_m \rangle = \langle y_1 - Y_1, \dots, y_m - Y_m \rangle$ .

*Proof.* See [GLS07, Chapter I, Theorem 1.18].  $\square$

Using the implicit function theorem we can prove the inverse function theorem for analytic algebras, which allows us to check whether a given morphism is an isomorphism. It even allows us to lift the information to a corresponding holomorphic map. This is useful when we are dealing with complex spaces.

**Theorem 1.9** (Inverse Function Theorem). *Let  $f_1, \dots, f_n \in \mathbb{C}\{x_1, \dots, x_n\}$  such that  $f_i(\mathbf{0}) = 0$  for  $i = 1, \dots, n$ . Then  $\det \left( \frac{\partial f_i}{\partial x_j}(\mathbf{0}) \right) \neq 0$  if and only if the  $\mathbb{C}$ -algebra morphism*

$$\begin{aligned} \mathbb{C}\{x_1, \dots, x_n\} &\longrightarrow \mathbb{C}\{x_1, \dots, x_n\} \\ x_i &\longmapsto f_i \end{aligned}$$

*is an isomorphism. This again holds if and only if there exist open neighborhood  $U$  and  $W$  of  $\mathbf{0}$  such that  $F := (f_1, \dots, f_n)$  defines a holomorphic map  $F : U \rightarrow W$  with and this map has a holomorphic inverse.*

The inverse function theorem is the main ingredient in the proof of the next result. Before we state it we need to define the notion of lift of a morphism.

**Definition 1.10.** *Let  $\varphi : \mathbb{C}\{x_1, \dots, x_n\}/I \rightarrow \mathbb{C}\{y_1, \dots, y_m\}/J$ . We call  $\tilde{\varphi} : \mathbb{C}\{x_1, \dots, x_n\} \rightarrow \mathbb{C}\{y_1, \dots, y_m\}$  a **lift** of  $\varphi$  if  $\tilde{\varphi}(I) \subseteq J$  or equivalently, if the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{C}\{x_1, \dots, x_n\} & \xrightarrow{\tilde{\varphi}} & \mathbb{C}\{y_1, \dots, y_m\} \\ \downarrow & & \downarrow \\ \mathbb{C}\{x_1, \dots, x_n\}/I & \xrightarrow{\varphi} & \mathbb{C}\{y_1, \dots, y_m\}/J. \end{array}$$

**Lemma 1.11** (Lifting Lemma). *Let  $\varphi$  be a morphism of analytic  $\mathbb{C}$ -algebras, i.e.  $\varphi : \mathbb{C}\{x_1, \dots, x_n\}/I \rightarrow \mathbb{C}\{y_1, \dots, y_m\}/J$ . Then there exists a lift  $\tilde{\varphi} : \mathbb{C}\{\mathbf{x}\} \rightarrow \mathbb{C}\{\mathbf{y}\}$  of  $\varphi$  which can be chosen as an isomorphism in the case that  $\varphi$  is an isomorphism and  $n = m$ , respectively as an epimorphism in the case that  $\varphi$  is an epimorphism and  $n \geq m$ .*

*Proof.* See for example [GLS07, Chapter I, Lemma 1.23].  $\square$

**Remark 1.12.** *Theorem 1.8, Theorem 1.9 and Lemma 1.11 also hold, if we replace the convergent power series ring by the formal power series ring.*

We finish this section by defining the so-called analytic tensor product. It generalizes the classical tensor product of rings to the context of power series rings.

**Definition 1.13.** *Let  $A = \mathbb{C}\{\mathbf{x}\}/I$ , respectively  $A = \mathbb{C}[[\mathbf{x}]]/I$ , and  $B = \mathbb{C}\{\mathbf{y}\}/J$ , respectively  $B = \mathbb{C}[[\mathbf{y}]]/J$ . Then we define*

$$A \hat{\otimes} B := \mathbb{C}\{\mathbf{x}, \mathbf{y}\} / (IC\{\mathbf{x}, \mathbf{y}\} + JC\{\mathbf{x}, \mathbf{y}\}),$$

*respectively*

$$A \hat{\otimes} B := \mathbb{C}[[\mathbf{x}, \mathbf{y}]] / (IC[[\mathbf{x}, \mathbf{y}]] + JC[[\mathbf{x}, \mathbf{y}]]) .$$

*We call  $A \hat{\otimes} B$  the **analytic tensor product**.*

The analytic tensor product serves as the product in the category  $\mathfrak{A}$ . It will be useful to us in the next section, when we deal with germs of complex spaces and products of them.

**Remark 1.14.** *One can show (see for example [GR71, Kapitel III, §5]) that the analytic tensor product satisfies the expected universal property. Let  $A, B, T \in \mathfrak{A}$  and assume there exists morphisms  $\pi_1 : T \rightarrow A$  and  $\pi_2 : T \rightarrow B$ . Then  $T \cong A \hat{\otimes} B$  if and only if for every  $C \in \mathfrak{A}$  for every morphism  $\varphi_1 : A \rightarrow C$  and  $\varphi_2 : B \rightarrow C$  there exists a unique morphism  $\gamma : T \rightarrow C$ , such that the following diagram commutes*

$$\begin{array}{ccc} T & \xrightarrow{\pi_1} & A \\ \pi_2 \downarrow & \searrow \gamma & \downarrow \varphi_1 \\ B & \xrightarrow{\varphi_2} & C. \end{array}$$

As usual for objects satisfying universal properties, the analytic tensor product is unique up to unique isomorphism.

## 1.2 Complex Spaces

In this section we are going to define the basic geometric objects of interest in complex analytic geometry, namely complex spaces. To be able to define them, we need elementary results from sheaf theory. We intend to give only the most basic definitions, as far as we need them in the following. For more details see for example [JP00] and [GLS07]. The main difference between analytic geometry and algebraic geometry, is the fact that we use the euclidean topology and not the Zariski topology, which allows us to consider "small" open neighborhoods of points. This difference leads to the so-called singularity theory. Using the same approach through ringed spaces shows that nilpotent elements of the structure sheaf can tell us more about the geometric object, than the reduced structure. See for example Theorem 1.83.

**Definition 1.15.** *Let  $D \subseteq \mathbb{C}^n$  be an open subset. Define  $\mathcal{O}_{\mathbb{C}^n}$  by  $\mathcal{O}(U) := \{f : U \rightarrow \mathbb{C} \text{ holomorphic}\}$  for every open subset  $U \subseteq \mathbb{C}^n$ . We call  $\mathcal{O}_{\mathbb{C}^n}$  the **sheaf of holomorphic functions on  $\mathbb{C}^n$** . Denote by  $\iota : D \hookrightarrow \mathbb{C}^n$  the canonical inclusion map. Then the **sheaf of holomorphic functions on  $D$**  is defined by  $\mathcal{O}_D = \iota^{-1}\mathcal{O}_{\mathbb{C}^n}$ . We denote the **stalk of  $\mathcal{O}_D$  at  $p \in D$**  by  $\mathcal{O}_{D,p}$ . The elements of  $\mathcal{O}_{D,p}$  are called **germs of holomorphic functions**.*

Before we can define general complex spaces, we start with so-called complex model spaces, which is just the special case of subsets of  $\mathbb{C}^n$ .

**Definition 1.16 (Complex Model Spaces).** *Let  $D \subseteq \mathbb{C}^n$  be an open subset and let  $\mathcal{I} \subseteq \mathcal{O}_D$  be an ideal sheaf of finite type. Then  $\mathcal{O}_D/\mathcal{I}$  is a sheaf of rings on  $D$ , and we define*

$$V(\mathcal{I}) := \{p \in D \mid \mathcal{I}_p \neq \mathcal{O}_{D,p}\} = \{p \in D \mid (\mathcal{O}_D/\mathcal{I})_p \neq 0\}$$

*to be the **analytic set in  $D$  defined by  $\mathcal{I}$** . Let  $X := V(\mathcal{I})$  and set  $\mathcal{O}_X := (\mathcal{O}_D/\mathcal{I})|_X$ . Then  $(X, \mathcal{O}_X) = (V(\mathcal{I}), (\mathcal{O}_D/\mathcal{I})|_X)$  is an analytic ringed space, called a **complex model space (defined by  $\mathcal{I}$ )**.*

**Remark 1.17.** *The definition of  $V(\mathcal{I})$  can be reformulated as*

$$V(\mathcal{I}) = \text{Supp}(\mathcal{O}_D/\mathcal{I}).$$

*Using this one can easily see that the  $V(\mathcal{I})$  are defined as vanishing sets in algebraic geometry. Let  $f_p \in \mathcal{O}_{D,p}$  be a germ. Then there exists an open neighborhood  $U$  of  $p$  and  $f_p$  lifts to a holomorphic function  $f : U \rightarrow \mathbb{C}$ . Then  $\mathcal{I}_p \neq \mathcal{O}_{D,p}$  if and only if  $f(p) = 0$  for all  $f_p \in \mathcal{I}_p$ . This means that there exist holomorphic functions  $f_1, \dots, f_k$  defined on  $U$ , such that*

$$V(\mathcal{I}) \cap U = \{p \in U \mid f_1(p) = \dots = f_k(p) = 0\} = V(f_1, \dots, f_k).$$

Next we need to define morphisms of complex model spaces.

**Definition 1.18.** *Let  $(X, \mathcal{O}_X)$  and  $(Y, \mathcal{O}_Y)$  be complex model spaces. A **morphism of complex model spaces**  $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is just a morphism of analytic ringed spaces.*

Now we can define complex spaces.

**Definition 1.19 (Complex Spaces).** *Let  $(X, \mathcal{O}_X)$  be an analytic ringed space with  $X$  Hausdorff. We call  $(X, \mathcal{O}_X)$  a **complex space**, if for every  $p \in X$  there exists an open neighborhood  $U$ , such that  $(U, \mathcal{O}_X|_U)$  is isomorphic to a complex model space. A **closed complex subspace** of  $X$  is an analytic ringed space  $(Y, \mathcal{O}_Y)$ , given by an ideal sheaf of finite type  $\mathcal{I}_Y \subseteq \mathcal{O}_X$  such that  $Y = V(\mathcal{I}_Y) := \text{Supp}(\mathcal{O}_X/\mathcal{I}_Y)$  and  $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{I}_Y)|_Y$ . Analogously, an **open complex subspace**  $(U, \mathcal{O}_U)$  of  $(X, \mathcal{O}_X)$  is given by an open subset  $U \subseteq X$  and  $\mathcal{O}_U = \mathcal{O}_X|_U$ . A subset  $A \subseteq X$  is called **analytic at a point**  $p \in X$ , if there exists an open neighborhood  $U$  of  $p$  and  $f_1, \dots, f_k \in \mathcal{O}(U)$  such that*

$$A \cap U = V(f_1, \dots, f_k) := \text{Supp}(\mathcal{O}_U/\mathcal{I}),$$

*where  $\mathcal{I} := f_1\mathcal{O}_U + \dots + f_k\mathcal{O}_U$ . If  $A$  is analytic at every point  $p \in A$ , then it is called **locally closed analytic set in  $X$** . If  $A$  is analytic at every  $p \in X$ , then it is called a **(closed) analytic set in  $X$** . To keep notation short, we usually write  $X$  instead of  $(X, \mathcal{O}_X)$ .*

In case we deal with coherent sheaves we can easily obtain analytic sets.

**Proposition 1.20.** *Let  $X$  be a complex space. Then a closed subset  $A \subseteq X$  is analytic if and only if there exists a coherent sheaf  $\mathcal{F}$  such that  $A = \text{Supp}(\mathcal{F})$ .*

*Proof.* See for example [GLS07, Chapter I, Corollary 1.64]. □

**Remark 1.21.** *The connection between complex spaces and analytic algebras lies in the fact that every stalk of the structure sheaf  $\mathcal{O}_X$  is an analytic algebra and, conversely, every analytic algebra can be obtained as the stalk of a structure sheaf of a certain complex space  $X$ . We make this more precise. Let  $X$  be a complex space and  $p \in X$ . It follows from the definition that there exist  $f_1, \dots, f_k \in \mathbb{C}\{\mathbf{x}\}$ , such that*

$$\mathcal{O}_{X,p} \cong \mathcal{O}_{\mathbb{C}^n, \mathbf{0}}/\mathcal{I}_0 \cong \mathbb{C}\{\mathbf{x}\}/\langle f_1, \dots, f_k \rangle.$$

*In this case we call  $\mathbf{x} = (x_1, \dots, x_n)$  **local coordinates** and  $f_1, \dots, f_k$  **local equations** for  $X$  at  $p$ . On the other hand, given convergent power series  $f_1, \dots, f_k \in \mathbb{C}\{\mathbf{x}\}$ , there is an open neighborhood  $U \subseteq \mathbb{C}^n$  of  $\mathbf{0}$ , such that each  $f_i$  defines a holomorphic map  $f_i : U \rightarrow \mathbb{C}$ . Set  $\mathcal{I} = f_1\mathcal{O}_U + \dots + f_k\mathcal{O}_U$ , the complex model space*

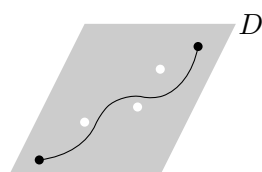
$$(X, \mathcal{O}_X) := (V(\mathcal{I}), (\mathcal{O}_U/\mathcal{I})|_{V(\mathcal{I})})$$

*satisfies  $\mathcal{O}_{X, \mathbf{0}} \cong \mathbb{C}\{\mathbf{x}\}/\langle f_1, \dots, f_k \rangle$ .*

Let us have a look at two examples for complex spaces.

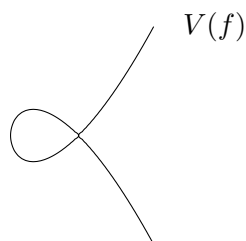
**Example 1.22.**

- (1) Let  $D \subseteq \mathbb{C}$  be defined by  $D = V(f)$ , where  $f \in \mathbb{C}\{x\}$ . We want to see that closed complex subspaces of  $\mathbb{C}$  can have only a certain shape. In case  $f = 0$ , we get  $D = \mathbb{C}$ . So let us assume that  $f$  is not identically zero on  $D$ . By the identity theorem for holomorphic functions in one dimension (see for example [JP00, Remark 3.1.10]), we have that the zeros of  $f$  are isolated, since otherwise  $f$  would be identically zero on  $D$ . Thus closed complex subspaces of  $\mathbb{C}$  are path-connected. This simple statement will be crucial for the proof of Theorem 1.101. We visualize this in the following picture. The white points correspond to the zeros of the given holomorphic function  $f$  and the black dots are arbitrary points on  $D$ .



**Figure 1.1:** Sketch of the path-connectedness of the non-zero locus of a one-dimensional holomorphic function.

- (2) The next object we want to visualize is the complex space defined by the polynomial  $f = y^2 - x^3 - x^2 \in \mathbb{C}[x, y]$ . The zero-set looks as follows:



**Figure 1.2:** Real picture of  $V(f)$ .

We are going to use this curve in the following section to show that analytic geometry and algebraic geometry yield different results, even if the input is the same.

Due to the fact that we are working with geometric objects we would like a notion of dimension in order to have a simple invariant which allows us to distinguish complex spaces.

**Definition 1.23.** Let  $X$  be a complex space,  $p \in X$  and  $\mathfrak{m}_p$  the maximal ideal of  $\mathcal{O}_{X,p}$ . Then we define

$$\begin{aligned} \dim_p X &:= \text{Krull dimension of } \mathcal{O}_{X,p}, \text{ the } \mathbf{dimension} \text{ of } X \text{ at } p, \\ \dim X &:= \sup_{p \in X} \dim_p X, \text{ the } \mathbf{dimension} \text{ of } X, \\ \text{edim}_p X &:= \dim_{\mathbb{C}} \mathfrak{m}_p / \mathfrak{m}_p^2, \text{ the } \mathbf{embedding dimension} \text{ of } X \text{ at } p. \end{aligned}$$

**Example 1.24.** *The dimension of a complex space  $X$  at a point  $p$  and the embedding dimension at the same point do not necessarily need to coincide. Consider the complex space  $X$  defined by the vanishing set of  $f = y^2 - x^3 - x^2 \in \mathbb{C}[x, y]$  and the points  $p = (0, 0)$  and  $q = (1, \sqrt{2})$ . Using Taylor expansion we obtain*

$$\mathcal{O}_{X,p} = \mathbb{C}\{x, y\}/\langle y^2 - x^3 - x^2 \rangle \text{ and } \mathcal{O}_{X,q} = \mathbb{C}\{u, v\}/\langle v^2 + 2\sqrt{2}v - u^3 - 4u^2 - 5u \rangle.$$

A simple computation shows

$$\dim_p X = 1 < 2 = \text{edim}_p X \text{ and } \dim_q X = 1 = \text{edim}_q X.$$

The observation from Example 1.24 leads to the following definition.

**Definition 1.25.** *Let  $X$  be a complex space and  $p \in X$  a point. We say  $X$  is **regular** at  $p$ , if  $\dim_p X = \text{edim}_p X$ . Otherwise we say  $X$  is **singular** at  $p$ . We call  $p$  a **regular point**, respectively a **singular point**. The set of all singular points of  $X$  will be denoted by  $\text{Sing}(X)$ , the so-called **singular locus** of  $X$ .*

**Proposition 1.26.** *Let  $X$  be a complex space. Then  $\text{Sing}(X)$  is a closed analytic set in  $X$ .*

*Proof.* See for example [GLS07, Chapter I, Corollary 1.111].  $\square$

**Remark 1.27.** *We are going to see in Section 1.4.2 that we can use differential methods to compute singular locus of given complex space  $X$ .*

The next definition is the definition of morphisms of complex spaces.

**Definition 1.28.** *Let  $(X, \mathcal{O}_X)$  and  $(Y, \mathcal{O}_Y)$  be complex spaces. A **morphism**  $(f, f^\sharp) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  **of complex spaces** is a morphism of analytic ringed spaces. Such a morphism is also-called **holomorphic map**. We write  $\text{Mor}(X, Y)$  for the set of morphisms  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ . An isomorphism of complex spaces is also-called **biholomorphic map**. To keep notation short, we usually write  $f$  instead of  $(f, f^\sharp)$ .*

Using morphisms of complex spaces we can define products of the latter.

**Definition 1.29.** *Let  $f : X \rightarrow T$  and  $g : Y \rightarrow T$  be two morphisms of complex spaces. Then the **analytic fibre product of  $X$  and  $Y$  over  $T$**  is a triple  $(X \times_T Y, \pi_X, \pi_Y)$  consisting of a complex space  $X \times_T Y$  and two morphisms  $\pi_X : X \times_T Y \rightarrow X, \pi_Y : X \times_T Y \rightarrow Y$  such that  $f \circ \pi_X = g \circ \pi_Y$ , satisfying the following universal property: for any complex space  $Z$  and any two morphisms  $h : Z \rightarrow X, h' : Z \rightarrow Y$  satisfying  $f \circ h = g \circ h'$  there exists a unique morphism  $\varphi : Z \rightarrow X \times_T Y$  such that the following diagram commutes:*

$$\begin{array}{ccccc} Z & & & & \\ & \searrow h & & & \\ & & X \times_T Y & \xrightarrow{\pi_X} & X \\ & \searrow \varphi & \downarrow \pi_Y & & \downarrow f \\ & & Y & \xrightarrow{g} & T \\ & \searrow h' & & & \end{array}$$

**Lemma 1.30.** *Consider the setup of Definition 1.29. Then the analytic fibre product  $X \times_T Y$  exists and is unique up to unique isomorphism.*



*Proof.* For the existence see for example [Fis76, Proposition 0.29]. The uniqueness follows from immediately from the universal property.  $\square$

**Remark 1.31.** *As a topological space the analytic fibre product is nothing more than  $X \times_T Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ . The proof of existence relies on the observation, that if  $T$  is a single point, then  $X \times_T Y = X \times Y$  and  $\mathcal{O}_{(X \times Y, (x, y))} \cong \mathcal{O}_{X, x} \hat{\otimes} \mathcal{O}_{Y, y}$ .*

### 1.3 Complex Space Germs

In this section we are going to define complex space germs. They turn out to be useful in the study of the local behavior of complex spaces in the neighborhood of a fixed point. In order to define these objects we need the notion of pointed complex spaces and morphisms of the latter.

**Definition 1.32.** *Let  $X$  be a complex space and  $x \in X$  a point. The pair  $(X, x)$  is called **pointed complex space**. A **morphism  $f : (X, x) \rightarrow (Y, y)$  of pointed complex spaces** is a morphism  $f : X \rightarrow Y$  of complex spaces, such that  $f(x) = y$ .*

Next we need to define germs of morphisms of complex spaces.

**Definition 1.33.** *Let  $(X, x)$  and  $(Y, y)$  be pointed complex spaces,  $U, V \subseteq X$  open neighborhoods of  $x$  and  $f : (U, x) \rightarrow (Y, y)$  and  $g : (V, x) \rightarrow (Y, y)$  morphisms of pointed complex spaces. We say  $f$  and  $g$  are equivalent, if there exists an open neighborhood  $W \subseteq U \cap V$  of  $x$ , such that  $f|_W = g|_W$ . We call the equivalence class of a morphism with respect to this equivalence relation **holomorphic map germs**. By abuse of notation we denote the holomorphic map germs by  $f : (X, x) \rightarrow (Y, y)$ . We can define the **composition of two holomorphic map germs**  $f : (X, x) \rightarrow (Y, y)$  and  $g : (Y, y) \rightarrow (Z, z)$  as follows: Consider representatives  $f : (U, x) \rightarrow (Y, y)$  and  $g : (V, y) \rightarrow (Z, z)$ . Then  $g \circ f$  is the equivalence class of the morphism  $g \circ f|_{f^{-1}(V) \cap U}$ . A map germ  $f : (X, x) \rightarrow (Y, y)$  is called an **isomorphism**, if there exists a map germ  $h : (Y, y) \rightarrow (X, x)$  such that  $f \circ h = \text{id}_{(Y, y)}$  and  $h \circ f = \text{id}_{(X, x)}$ .*

**Remark 1.34.** *From the definition of an isomorphic holomorphic map germ, we obtain that any pointed complex space  $(X, x)$  is in this sense isomorphic to any pointed complex space of type  $(U, x)$ , where  $U$  is an open neighborhood of  $x$ , where the map germ is given by the canonical inclusion  $\iota : U \hookrightarrow X$ . We call  $U$  in this case a **representative of the germ  $(X, x)$** . Let  $(Y, y)$  be another complex space and  $f : (X, x) \rightarrow (Y, y)$  a map germ. Assume  $V \subseteq Y$  is an open neighborhood of  $y$ , with  $f(U) \subseteq V$ . Then we call  $f : U \rightarrow V$  a **representative of the map germ  $f$** .*

Since we are able to compose map germs, we can build the following category.

**Definition 1.35.** *The category whose objects are pointed complex spaces and as morphisms holomorphic map germs is called **category of complex space germs** and is denoted by  $\mathfrak{G}$ . We call the objects of  $\mathfrak{G}$  **complex space germs** or **singularities**.*

**Definition 1.36.** *Let  $(X, x)$  be a complex space germ represented by the complex space  $X$  with structure sheaf  $\mathcal{O}_X$ , then the stalk  $\mathcal{O}_{X, x}$  is called the **(analytic) local ring** of the germ  $(X, x)$  and also denoted by  $\mathcal{O}_{(X, x)}$ .*

Next we need the notion of closed analytic subgerms.



**Definition 1.37.** Let  $(X, x)$  be a complex space germ and  $I \subseteq \mathcal{O}_{X,x}$  be an ideal. Furthermore, let  $(U, \mathcal{O}_U)$  be a representative of  $(X, x)$  and  $f_1, \dots, f_k \in \mathcal{O}_U(U)$  such that  $I$  is generated by the germs of  $f_1, \dots, f_k$  at  $x$ . The closed complex subspace of  $U$  defined by  $\mathcal{I} = f_1\mathcal{O}_U + \dots + f_k\mathcal{O}_U$  defines a closed analytic subgerm

$$(V(I), x) := (V(\mathcal{I}), x) \subseteq (U, x) = (X, x)$$

of  $(X, x)$ , called the **closed analytic subgerm defined by  $I$** . In case  $I = \langle f \rangle \subseteq \mathcal{O}_{\mathbb{C}^n, p}$  with  $f \neq 0$  the germ

$$(V(f), p) := (V(I), p) \subseteq (\mathbb{C}^n, p)$$

is called a **hypersurface singularity**.

The last notions that pass on from complex spaces to germs are the notion of dimension and regularity.

**Definition 1.38.** Let  $(X, x)$  be a complex space germ represented by the complex space  $X$ . We define

$$\begin{aligned} \dim(X, x) &:= \dim_x X, \text{ the } \mathbf{dimension} \text{ of } (X, x), \\ \text{edim}(X, x) &:= \text{edim}_x X, \text{ the } \mathbf{embedding dimension} \text{ of } (X, x). \end{aligned}$$

We call a complex space germ **regular**, if  $\dim(X, x) = \text{edim}(X, x)$ , otherwise we call it **singular**.

After all these definitions we are able to state the most important result connecting complex space germs and analytic algebras.

**Proposition 1.39.** *The functor*

$$\begin{aligned} \Phi : \mathfrak{G} &\longrightarrow \mathfrak{A} \\ (X, x) &\longmapsto \mathcal{O}_{X,x} \\ f_x : (X, x) \rightarrow (Y, y) &\longmapsto f_x^\sharp : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x} \end{aligned}$$

is an antiequivalence of categories.

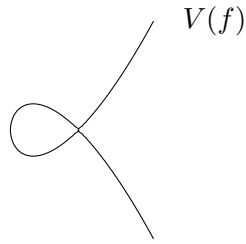
*Proof.* See for example [Fis76, Proposition 0.21]. □

Proposition 1.39 is crucial for the work in this thesis. We want to classify certain hypersurface singularities up to isomorphism. The proposition now tells us, that it is equivalent to study the corresponding analytic algebras. Due to this we can apply as well methods from complex analysis as methods from abstract algebra.

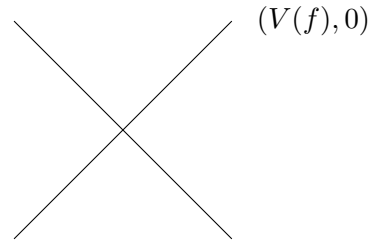
We conclude this section by showing that analytic and algebraic geometry behave differently.

**Example 1.40.** We consider the same equation as in Example 1.22, namely  $f = y^2 - x^3 - x^2$ . From the point of view of classic algebraic geometry, we see  $f$  as an element of the polynomial ring  $\mathbb{C}[x, y]$ . In this case  $f$  is an irreducible polynomial and we obtain the real picture of  $V(f)$  as in Figure 1.3. From the point of view of analytic geometry we see  $f$  as a power series in  $\mathbb{C}\{x, y\}$ . In this case  $f$  is reducible, since it can be written as  $f = (y - x\sqrt{x+1}) \cdot (y +$

$x\sqrt{x+1}$ ). This yields the real picture of the complex space germ  $(V(f), 0)$  in a small neighborhood of 0 as in Figure 1.4.



**Figure 1.3:** Real picture of the vanishing set  $V(y^2 - x^3 - x^2) \subseteq \mathbb{C}^2$ .



**Figure 1.4:** Real picture of the complex space germ  $(V(f), 0) \subseteq (\mathbb{C}^2, 0)$ .

We can see, not only in the equations, but also in the (real) picture, that we obtain an irreducible curve in algebraic case, but an intersection of two lines in the analytic case. This is a small example where we obtain different results, although we start with the same equation.

## 1.4 Derivations on Analytic Algebras

This section deals with derivations and their applications to analytic algebras. We start by presenting basic results. Afterwards we show how to use derivations in the computation of singular points. The final subsection is concerned with detecting local analytic triviality, this means we want to decide whether a complex space can be considered locally as a product of a smaller complex space with  $\mathbb{C}^k$  for some  $k \in \mathbb{N}$

### 1.4.1 Derivations on Analytic Algebras I: Basics

The following subsection is concerned with derivations and their relation to analytic algebras, respectively complex space germs. We restrict our setup to the case of derivations between analytic algebras. The general case is treated in [Kun86].

**Definition 1.41.** Let  $A$  and  $B$  be (formal) analytic algebras. A  $\mathbb{C}$ -linear map  $\delta : A \rightarrow B$  satisfying the Leibniz rule, that is

$$\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(g)$$

is called a **derivation of  $A$  with values in  $B$** . The set

$$\text{Der}(A, B) := \{\delta : A \rightarrow B \mid \delta \text{ is a derivation}\}$$

is via  $(a \cdot \delta)(f) := a \cdot \delta(f)$  an  $A$ -module, the **module of derivations of  $A$  with values in  $B$** . In case  $A = B$  we define  $\text{Der}(A) := \text{Der}(A, A)$ .

**Remark 1.42.** Let  $A$  be a (formal) analytic algebra. Then  $\text{Der}(A)$  is a vector space over  $\mathbb{C}$  and it is also a Lie algebra, if we define the multiplication as follows:

$$[\delta, \sigma](f \cdot g) := (\delta \circ \sigma - \sigma \circ \delta)(f \cdot g),$$

with  $\delta, \sigma \in \text{Der}(A)$ ,  $f, g \in A$ . A simple computation yields

$$[\delta, \sigma](f \cdot g) = [\delta, \sigma](f) \cdot g + f \cdot [\delta, \sigma](g),$$

hence the multiplication is closed. The other properties of a Lie algebra can also be verified by simple computations.

The most relevant case in our considerations will be the cases  $A = \mathbb{C}\{\mathbf{x}\}$  respectively  $A = \mathbb{C}[[\mathbf{x}]]$ . In these cases the derivation module is free and its generators can be stated explicitly.

**Theorem 1.43.** *Let  $A = \mathbb{C}\{\mathbf{x}\}$  respectively  $A = \mathbb{C}[[\mathbf{x}]]$ . Then every  $\delta \in \text{Der}(A)$  can be uniquely written as*

$$\delta = \sum_{i=1}^n \delta(x_i) \partial_{x_i},$$

where  $\partial_{x_i}$  denotes the partial derivation with respect to  $x_i$ .

*Proof.* This follows for example from [GR71, Kapitel III, §4, Satz 2] □

Our goal is to investigate complex space germs  $(X, x)$  using derivations on the corresponding analytic algebras  $\mathcal{O}_{X,x}$ , which are isomorphic to  $\mathbb{C}\{\mathbf{x}\}/I$  for some ideal  $I \subseteq \mathbb{C}\{\mathbf{x}\}$ . Therefore we need to understand the derivation module  $\text{Der}(\mathbb{C}\{\mathbf{x}\}/I)$ . A first step towards this is the following proposition.

**Proposition 1.44.** *Let  $A = \mathbb{C}\{\mathbf{x}\}/I$  be an analytic algebra. Then every derivation  $\delta \in \text{Der}(A)$  lifts to a derivation  $\tilde{\delta} \in \text{Der}(\mathbb{C}\{\mathbf{x}\})$  with the property  $\tilde{\delta}(I) \subseteq I$ .*

*Proof.* The result can be found in the standard literature. A specific proof of this result for analytic algebras can be found in [SW73, (2.1)]. □

**Remark 1.45.** *The result of Proposition 1.44 holds also in the case of formal analytic algebras.*

It follows from Proposition 1.44 that it is helpful to consider the following set of derivations.

**Definition 1.46.** *Let  $A$  be a (formal) analytic algebra and  $I \subseteq A$  an ideal. We call the module*

$$\text{Der}_I(A) := \{\delta \in \text{Der}(A) \mid \delta(I) \subseteq I\}$$

*the module of logarithmic derivations of  $I$ . In the case  $I = \mathfrak{m}$  we write  $\text{Der}'(A)$  instead of  $\text{Der}_{\mathfrak{m}}(A)$ . We denote the module  $\text{Der}_I \cap \text{Der}'$  by  $\text{Der}'_I$ .*

**Corollary 1.47.** *Let  $A = \mathbb{C}\{\mathbf{x}\}/I$  be an analytic algebra. Then*

$$\text{Der}(A) \cong \text{Der}_I(\mathbb{C}\{\mathbf{x}\})/I \text{Der}(\mathbb{C}\{\mathbf{x}\}).$$

*Proof.* Let  $\pi : \mathbb{C}\{\mathbf{x}\} \rightarrow A$  denote the canonical projection. We consider the sequence of  $\mathbb{C}\{\mathbf{x}\}$ -modules

$$0 \longrightarrow I \text{Der}(\mathbb{C}\{\mathbf{x}\}) \xleftarrow{\iota} \text{Der}_I(\mathbb{C}\{\mathbf{x}\}) \xrightarrow{\varphi} \text{Der}(A) \longrightarrow 0,$$

where  $\iota$  denotes the canonical inclusion and  $\varphi(\delta) = \pi \circ \delta$ . We want to show that this sequence is exact. It is clear by construction that  $\iota$  is injective. Proposition 1.44 implies the surjectivity of  $\varphi$ . Now it only remains to show that  $\text{im}(\iota) = I \text{Der}(\mathbb{C}\{\mathbf{x}\}) = (\varphi)$ . The inclusion  $\text{im}(\iota) \subseteq \ker(\varphi)$  is clear.

Let  $\delta \in \ker(\varphi)$ . Then  $\pi(\delta(x_i)) = 0$  for all  $i = 1, \dots, n$ . By Theorem 1.43 we know that  $\delta = \sum_{i=1}^n \delta(x_i) \partial_{x_i}$ , hence  $\delta(x_i) \in I$  for all  $i = 1, \dots, n$ . Thus  $\delta \in I \text{Der}(\mathbb{C}\{\mathbf{x}\})$ . The claim follows from the isomorphism theorem. □

**Remark 1.48.** Corollary 1.47 also holds, if we replace  $\mathbb{C}\{\mathbf{x}\}$  by  $\mathbb{C}[[\mathbf{x}]]$ .

So far we have dealt with basic properties of derivation modules. Next we consider properties of derivations which can be extended from classical linear algebra. Let  $A$  be an analytic algebra. Then we have for every  $k \in \mathbb{N}$  natural projections  $\pi_k : \text{Der}(A) \rightarrow \text{Der}(A/\mathfrak{m}_A^k) \subseteq \text{End}_{\mathbb{C}}(A)$ . Using these projections we start with the classical properties of being diagonalizable and being nilpotent adapted to derivations.

**Definition 1.49.** Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$ . We call  $\delta$  **semi-simple**, if the linear operator induced by  $\pi_k(\delta)$  in  $\text{Der}(A/\mathfrak{m}_A^k)$  is semi-simple on  $A/\mathfrak{m}_A^k$  for all  $k \in \mathbb{N}$ .  $\delta$  is called **nilpotent**, if the linear operator induced by  $\pi_k(\delta)$  in  $\text{Der}(A/\mathfrak{m}_A^k)$  is nilpotent on  $A/\mathfrak{m}_A^k$  for all  $k \in \mathbb{N}$ .  $\delta$  is called **diagonalizable**, if  $\mathfrak{m}_A$  has a system of generators containing only eigenvectors of  $\delta$ .

**Remark 1.50.** Since we work over an algebraically closed field, semi-simple derivations are diagonalizable.

**Lemma 1.51.** Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$ . Then  $\delta$  is nilpotent if and only if the  $\mathbb{C}$ -linear operator induced by  $\pi_2(\delta)$  on  $\mathfrak{m}_A/\mathfrak{m}_A^2$  is nilpotent.

*Proof.* Assume  $\delta$  is nilpotent, then it induces a nilpotent  $\mathbb{C}$ -linear operator on  $\mathfrak{m}_A/\mathfrak{m}_A^2$  by definition. Now assume  $\delta$  induces a nilpotent  $\mathbb{C}$ -linear operator on  $\mathfrak{m}_A/\mathfrak{m}_A^2$ . This means, there exists some  $n \in \mathbb{N}$ , such that  $\delta^n(\mathfrak{m}_A) \subseteq \mathfrak{m}_A^2$ . Assume that we have an  $n$ , such that  $\delta^n(\mathfrak{m}_A^{k-1}) \subseteq \delta(\mathfrak{m}_A^k)$ , for some  $k \in \mathbb{N}$ . Our result for  $k + 1$  follows by a application of the Leibniz rule:

$$\delta^n(\mathfrak{m}_A^k) = \delta^n(\mathfrak{m}_A^{k-1}\mathfrak{m}_A) = \underbrace{\delta^n(\mathfrak{m}_A^{k-1})\mathfrak{m}_A}_{\subseteq \mathfrak{m}_A^{k+1}} + \underbrace{\mathfrak{m}_A^{k-1}\delta^n(\mathfrak{m}_A)}_{\subseteq \mathfrak{m}_A^{k+1}} \subseteq \mathfrak{m}_A^{k+1}.$$

Thus,  $\delta$  induces a nilpotent  $\mathbb{C}$ -linear operator on  $\mathfrak{m}_A/\mathfrak{m}_A^k$  for all  $k \in \mathbb{N}$ . As  $\delta(\mathbb{C}) = 0$  and  $A = \mathbb{C} \oplus \mathfrak{m}_A$ , we get that it induces a nilpotent operator on  $A/\mathfrak{m}_A^k$  for all  $k \in \mathbb{N}$ . Finally,  $\delta$  is nilpotent, as we can always take  $m := n \cdot k$  and get that  $\delta^m(A) \subseteq \mathfrak{m}_A^k$ .  $\square$

**Definition 1.52.** Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$ . We say that  $\delta$  has a **Chevalley decomposition**, if  $\delta$  can be written as  $\delta = \delta_S + \delta_N$  with  $[\delta_S, \delta_N] = 0$ , where  $\delta_S$  is a semi-simple derivation,  $\delta_N$  is a nilpotent derivation and  $\delta_S, \delta_N \in \text{Der}'(A)$ .

Obviously the Chevalley decomposition from Definition 1.52 is analogous to the Jordan decomposition known from linear algebra (see for example [Lan02, Chapter XIV, Theorem 2.4]).

As in the linear algebra case, we cannot expect the Chevalley decomposition to exist without any restrictions to the analytic algebra. The following three theorems are the most important results regarding the linear algebra of derivations, which we are going to use.

**Theorem 1.53.** Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$  admitting a Chevalley decomposition  $\delta = \delta_S + \delta_N$ . Then the Chevalley decomposition of  $\delta$  is unique, that is, if  $\delta = \delta_S + \delta_N = \delta'_S + \delta'_N$  with  $[\delta_S, \delta_N] = [\delta'_S, \delta'_N] = 0$ , then  $\delta_S = \delta'_S$  and  $\delta_N = \delta'_N$ .

*Proof.* See [SW81, Remark after (1.1)].

$\square$

**Theorem 1.54.** *Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$  admitting a Chevalley decomposition  $\delta = \delta_S + \delta_N$ . Furthermore let  $I \subseteq A$  be an ideal with  $\delta \in \text{Der}_I(A)$ , then  $\delta_S, \delta_N \in \text{Der}_I(A)$ .*

*Proof.* See [SW81, Remark after (1.1)] □

**Theorem 1.55.** *Let  $A$  be a complete analytic algebra. Then every  $\delta \in \text{Der}'(A)$  admits a Chevalley decomposition.*

*Proof.* See [SW81, (1.2)]. For a constructive approach, which will be turned to an algorithm in Chapter 5, see [Sai71, Satz 3.1]. □

Let us take a look at an example for the Chevalley decomposition.

**Example 1.56.** *Let  $A := \mathbb{C}[[x, y]]$ . Consider the derivation  $\delta := (x + y)\partial_x + y\partial_y$ . Then  $\delta_S = x\partial_x + y\partial_y$  is the semi-simple part of  $\delta$  and  $\delta_N = y\partial_x$  is the nilpotent part of  $\delta$ . The first statement follows, as  $\delta_S(x) = x$  and  $\delta_S(y) = y$ . The second statement follows from the fact that  $\delta_N^2 = 0$ .*

*Now consider  $\delta := (x + y + xy)\partial_x + y\partial_y$ . We want to show that the semi-simple part of the linear part of our derivation is not necessarily the semi-simple part of our derivation. Assume that  $\delta_S = x\partial_x + y\partial_y$ , then  $\delta_N = (y + xy)\partial_x$ . Using the same argument as before,  $\delta_S$  is semi-simple, but  $[\delta_S, \delta_N] = xy\partial_x \neq 0$ , hence  $\delta_S$  cannot be the semi-simple part of  $\delta$ . This example shows that it is a non-trivial task to compute the semi-simple part of a derivation.*

**Proposition 1.57.** *Let  $A$  be a complete analytic algebra and  $\delta, \epsilon \in \text{Der}'(A)$ . If  $[\epsilon, \delta] = 0$ , then we have  $[\epsilon, \delta_S] = 0$  and  $[\epsilon, \delta_N] = 0$ .*

*Proof.* Denote by  $\bar{\delta}$  and  $\bar{\epsilon}$  the images of  $\delta$  and  $\epsilon$  to  $\text{Der}(A/\mathfrak{m}_A^k)$ , for any  $k \in \mathbb{N}$ . As in the proof of Theorem 1.54, we can write  $\bar{\delta}_S$  as a polynomial in  $\bar{\delta}$ . Due to the fact that  $[\bar{\epsilon}, \bar{\delta}] = 0$ , we get that  $\bar{\epsilon}$  commutes with any polynomial expression in  $\bar{\delta}$ , hence with  $\bar{\delta}_S$ . The analogous result follows for  $\bar{\delta}_N$ . The result follows, as  $\delta_S$  and  $\delta_N$  can be considered as sequences of the  $\bar{\delta}_S$  respectively  $\bar{\delta}_N$ . □

The final theorem of this section is the analog to a classical result from linear algebra.

**Theorem 1.58.** *Let  $A$  be a complete analytic algebra and let  $\delta_1, \dots, \delta_k \in \text{Der}'(A)$ ,  $k \in \mathbb{N}_{\geq 2}$ , be pairwise commuting diagonalizable derivations. Then there exists a minimal generating system  $x_1, \dots, x_n$  of  $\mathfrak{m}_A$  consisting of common eigenvectors of the  $\delta_i$ .*

*Proof.* See [SW81, (2.1)]. □

We conclude this section with a criterion using commutators to check if a derivation is nilpotent.

**Lemma 1.59.** *Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$  diagonalizable. If  $\epsilon \in \text{Der}'(A)$  satisfies  $[\delta, \epsilon] = \lambda \cdot \epsilon$  for some  $\lambda \in \mathbb{C}^*$ , then  $\epsilon$  is nilpotent.*

*Proof.* See [Epu15, Lemma 4.38]. □

## 1.4.2 Derivations on Analytic Algebras II: Singularities

In this subsection we want to present one of the main application of derivations, namely checking whether a point in a complex space is regular or not. We start with an algebraic lemma motivating the following definitions.

**Lemma 1.60.** *Let  $A$  be a (formal) analytic algebra. Then there is a canonical isomorphism*

$$\mathrm{Der}(A, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_A/\mathfrak{m}_A^2, \mathbb{C}).$$

*In particular,  $\mathrm{edim}(A) = \dim_{\mathbb{C}}(\mathrm{Der}(A, \mathbb{C}))$ .*

*Proof.* See for example [GLS07, Chapter I, Lemma 1.107].  $\square$

Lemma 1.60 motivates the following definition.

**Definition 1.61.** *Let  $X$  be a complex space. We call the  $\mathbb{C}$ -vector space  $T_x X := \mathrm{Der}(\mathcal{O}_{X,x}, \mathbb{C})$  the *tangent space of  $X$  at  $x$* .*

We obtain the following corollary.

**Corollary 1.62.** *Let  $X$  be a complex space and  $x \in X$ . Then  $X$  is regular at  $x$  if and only if  $\dim_{\mathbb{C}} T_x X = \dim \mathcal{O}_{X,x}$ .*

Using methods from computer algebra we can compute  $\dim \mathcal{O}_{X,x}$ . Our next result will tell us how to compute  $T_x X$  in case of closed complex subspaces of complex spaces.

**Proposition 1.63.** *Let  $D \subseteq \mathbb{C}^n$  be a complex model space and  $X$  a closed complex subspace of  $D$  defined by the ideal sheaf  $\mathcal{I}$ . Denote the local coordinates at a point  $p \in X \subseteq D$  by  $\mathbf{x} = (x_1, \dots, x_n)$ . Then there is a  $\mathbb{C}$ -vector space isomorphism*

$$T_p X \longrightarrow \{(s_1, \dots, s_n) \in \mathbb{C}^n \mid s_1(\partial_{x_1} f)(p) + \dots + s_n(\partial_{x_n} f)(p) = 0 \text{ for all } f \in \mathcal{I}_p\}.$$

*Proof.* See for example [Fis76, Chapter 2, Section 2].  $\square$

Let us have a look at an example.

**Example 1.64.** *We consider the same equation as in Example 1.22, namely  $f = y^2 - x^3 - x^2$ , and check the results from Example 1.24. The complex space is  $X = V(f) \subseteq \mathbb{C}^2$ . We consider the points  $p = (0, 0)$  and  $q = (1, \sqrt{2})$ . A simple computation shows that  $T_p X = \{(s_1, s_2) \in \mathbb{C}^2 \mid s_1 \cdot 0 + s_2 \cdot 0 = 0\} = \mathbb{C}^2$  and  $T_q X = \{(s_1, s_2) \in \mathbb{C}^2 \mid s_1 \cdot (-5) + s_2 \cdot 2\sqrt{2} = 0\} \cong \mathbb{C}$ . We visualize the real pictures of  $T_p X$  and  $T_q X$  by dashed lines where  $T_p X$  is visualized through two dashed lines, since it would cover the whole picture.*

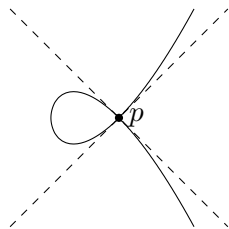


Figure 1.5: Real picture of  $T_p X$ .

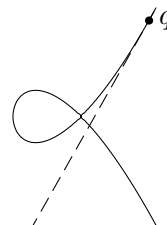


Figure 1.6: Real picture of  $T_q X$ .

From the picture we obtain a result we already know, namely that  $p$  is a singular point and  $q$  a regular point.

One can show that the defining equations for the tangent spaces can be checked on generators of the given ideal. This yields the following criterion to check whether a point is regular or not.

**Definition 1.65.** Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $A = \mathbb{C}[[\mathbf{x}]]$  and  $\mathbf{f} := (f_1, \dots, f_k) \in A^k$ . We call the matrix

$$J_{\mathbf{f}} := \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_1} f_k \\ \vdots & \ddots & \vdots \\ \partial_{x_n} f_1 & \cdots & \partial_{x_n} f_k \end{pmatrix}$$

the **Jacobian matrix of  $\mathbf{f}$** .

**Theorem 1.66** (Jacobian Criterion). Let  $D \subseteq \mathbb{C}^n$  be a complex model space,  $X$  a closed complex subspace of  $D$  defined by the ideal sheaf  $\mathcal{I}$  and  $x \in X$  a point. Assume  $\mathcal{I}_x = \langle f_1, \dots, f_k \rangle \subseteq \mathcal{O}_{X,x}$ . Then the complex space germ  $(X, x)$  is regular if and only if  $\text{rk}(J_{\mathbf{f}})(x) = n - \dim \mathcal{O}_{X,x}$ .

*Proof.* See for example [JP00, Theorem 4.3.6]. □

### 1.4.3 Derivations on Analytic Algebras III: Triviality

This final section about derivations is concerned with the question whether a given complex space (germ) is a product or not. Let us make this more precise.

**Definition 1.67.** Let  $X$  be a complex space and  $x \in X$  a point. We say  $X$  is **locally trivial in  $x$** , if there exists an open neighborhood  $U$  of  $x$ , a complex space  $X'$  and an open set  $V \subseteq \mathbb{C}^k$  for some  $k \in \mathbb{N}$  together with a biholomorphic map

$$\varphi : U \longrightarrow X' \times V.$$

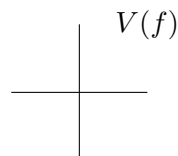
In case  $\varphi(x) = (x', \mathbf{0})$  it is equivalent to say that there exists an biholomorphic map

$$\varphi : (X, x) \longrightarrow (X' \times \mathbb{C}^k, (x', \mathbf{0})).$$

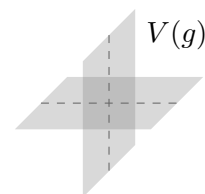
We call  $(X, x)$  a **suspension of  $(X', x')$** .

Before we state a criterion to decide whether a complex space is locally trivial, we have a look at an example.

**Example 1.68.** We consider the complex space  $X \subseteq \mathbb{C}^2$  defined by the polynomial  $f = xy \in \mathbb{C}[x, y]$  and the complex space  $Y \subseteq \mathbb{C}^3$  defined by  $g = xy \in \mathbb{C}[x, y, z]$ .



**Figure 1.7:** Real picture of the vanishing set  $V(xy) \subseteq \mathbb{C}^2$ .



**Figure 1.8:** Real picture of the complex space  $V(xy) \subseteq \mathbb{C}^3$ .

From the pictures it easy to see, that  $(Y, \mathbf{0})$  is a suspension of  $(X, \mathbf{0})$ .



The equations in Example 1.68 are very simple, so we can immediately see that we have a suspension. Our next results states a differential criterion to decide local triviality for hypersurfaces.

**Theorem 1.69** (Local Analytic Triviality I). *Let  $f \in \mathbb{C}\{x_1, \dots, x_n, y_1, \dots, y_m\}$  and  $c \in \mathbb{N}$ . The following conditions are equivalent:*

- (1)  $\partial_{y_i} f \in \langle x_1, \dots, x_n \rangle^c \langle \partial_{x_1} f, \dots, \partial_{x_n} f \rangle + \langle f \rangle$  for  $i = 1, \dots, m$ .
- (2) There exist  $\varphi_1, \dots, \varphi_n, u \in \mathbb{C}\{x_1, \dots, x_n, y_1, \dots, y_m\}$  such that
  - (a)  $u(x_1, \dots, x_n, 0, \dots, 0) = 1$ ,
  - (b)  $\varphi_i(x_1, \dots, x_n, 0, \dots, 0) = x_i$ ,
  - (c)  $\varphi_i - x_i \in \langle x_1, \dots, x_n \rangle^c$ ,
  - (d)  $f(x_1, \dots, x_n, y_1, \dots, y_m) = u \cdot f(\varphi_1, \dots, \varphi_n, 0, \dots, 0)$ .

If moreover  $\partial_{y_i} f \in \langle x_1, \dots, x_n \rangle^c \langle \partial_{x_1} f, \dots, \partial_{x_n} f \rangle$  for all  $i$ , then we can choose  $u = 1$ .

*Proof.* The proof follows by induction from [JP00, Theorem 9.1.5]. □

A more general version is the following.

**Theorem 1.70** (Local Analytic Triviality II). *Let  $X$  be a complex space and  $x \in X$ . Then  $X$  is locally trivial in  $x$  if and only there exists derivations  $\delta_1, \dots, \delta_m \in \text{Der}(\mathcal{O}_{X,x})$ , such that  $\delta_1(x), \dots, \delta_m(x)$  are linearly independent. In this case  $(X, x) \cong (X' \times \mathbb{C}^m, (x', \mathbf{0}))$  for some complex space germ  $(X', x')$ .*

*Proof.* See [Fis76, Theorem 2.12]. □

We stated both versions of local analytic triviality, since we are going to need both separately. We conclude this section by stating a lemma concerning isomorphisms of suspensions.

**Lemma 1.71** (Cancellation Lemma). *Let  $(X, x) \cong (X' \times \mathbb{C}^k, (x', \mathbf{0}))$  and  $(Y, y) \cong (Y' \times \mathbb{C}^m, (y', \mathbf{0}))$  be complex space germs with  $k$  and  $m$  maximal. Then  $(X, x) \cong (Y, y)$  if and only if  $k = m$  and  $(X', x') \cong (Y', y')$ .*

*Proof.* The if part follows from [Eph78, Lemma 1.5] and the only if part from the maximality of  $k$  and  $m$ , the fact that  $\text{Der}(\mathcal{O}_{X,x}) \cong \text{Der}(\mathcal{O}_{Y,y})$  if  $(X, x) \cong (Y, y)$  and Theorem 1.70. □

## 1.5 Hypersurface Singularities

This section deals with the basics of hypersurface singularities. After introducing basic invariants of general hypersurface singularities, we define the notion of isolated hypersurface singularities and state their properties. Next we consider basic results regarding quasi-homogeneous isolated hypersurface singularities. We finish this section by presenting harmonic hypersurface singularities and by showing that they are determined, as well as isolated hypersurface singularities, by their singular locus following [GH85].



### 1.5.1 Isolated Hypersurface Singularities

For a given holomorphic function  $f : U \rightarrow \mathbb{C}$ ,  $U \subseteq \mathbb{C}^n$  open, the singular locus is defined by  $\text{Sing}(X) = \{x \in U \mid f(x) = \partial_{x_1}f(x) = \dots = \partial_{x_n}f(x) = 0\}$ . Locally the information is stored in the ideal generated by the partial derivatives of  $f$  and  $f$ .

**Definition 1.72.** Let  $f : U \rightarrow \mathbb{C}$  be a holomorphic function and  $x \in U$ . We call  $(V(f), x)$  an *isolated hypersurface singularity* if there exists an open neighborhood  $V$  of  $x$  such that  $(\text{Sing}(X) \cap V) \setminus \{x\} = \emptyset$ .

**Remark 1.73.** For simplicity, all germs we consider will have base point  $x = \mathbf{0}$ .

This definition of isolated singularity is not easy to check, so we want to have an algebraic argument to verify this property. To do so, we need some definitions

**Definition 1.74.** Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $A = \mathbb{C}[[\mathbf{x}]]$  and  $f \in A$ .

(1) We call the ideal

$$J_f := \langle \partial_{x_1}f, \dots, \partial_{x_n}f \rangle$$

the *Jacobian ideal of  $f$* .

(2) The analytic algebras

$$M_f := A/J_f \text{ and } T_f := A/\langle f, J_f \rangle$$

are called *Milnor and Tjurina algebra of  $f$* , respectively.

(3) The numbers

$$\mu_f := \dim_{\mathbb{C}} M_f \text{ and } \tau_f := \dim_{\mathbb{C}} T_f$$

are called *Milnor and Tjurina number of  $f$*  respectively.

**Example 1.75.** We continue Example 1.68. From the pictures it is clear that  $(V(f), \mathbf{0})$  defines an isolated singularity and  $(V(g), \mathbf{0})$  does not.

Pictures can be tricky, since we cannot picture the two or three dimensional complex space. So we have develop an algebraic method to check this property.

**Lemma 1.76.** Let  $(X, \mathbf{0})$  be a hypersurface singularity defined by  $f \in \mathbb{C}\{\mathbf{x}\}$ . Then the following are equivalent:

(1)  $(X, \mathbf{0})$  is an isolated hypersurface singularity.

(2)  $\mu_f < \infty$ .

(3)  $\tau_f < \infty$ .

*Proof.* See for example [GLS07, Chapter I, Lemma 2.3]. □

Using methods from computer algebra we can compute  $\mu_f$  and  $\tau_f$  in case  $f$  is a polynomial. As it turns out next, we can always change coordinates in such a way, that an isolated hypersurface singularity is defined by a polynomial. Let us make this more precise.

**Definition 1.77.** Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $A = \mathbb{C}[[\mathbf{x}]]$  and  $f, g \in A$ .

- (1) We say  $f$  is **contact equivalent** to  $g$ , denoted by  $f \sim g$ , if there exists an automorphism  $\varphi \in \text{Aut}(A)$  and a unit  $u \in A^*$ , such that  $f = u \cdot \varphi(g)$ .
- (2) Denote by  $f^{(k)}$  the truncation of  $f$  up to degree  $k$ . Then we say  $f$  is  **$k$ -determined** if  $f \sim f^{(k)}$  for some  $k \in \mathbb{N}$ .

**Remark 1.78.** It is easy to see that  $f \sim g$  is equivalent to  $(V(f), \mathbf{0}) \cong (V(g), \mathbf{0})$ . There is also a notion of so-called right equivalence, but we omit it at this point, since we do not use it in the further course of the thesis. For more details see [GLS07, Chapter I, Definition 2.9].

For isolated hypersurface singularities we have the following statement.

**Proposition 1.79.** Let  $f \in \mathbb{C}\{\mathbf{x}\}$ , such that  $(V(f), 0)$  is an isolated hypersurface singularity. Then  $f$  is  $\tau_f$ -determined.

*Proof.* See for example [GLS07, Chapter I, Corollary 2.24]. □

Using Proposition 1.79 one can prove the following lemma.

**Lemma 1.80 (Splitting Lemma).** Let  $f \in \mathbb{C}\{x_1, \dots, x_n\}$  define an isolated hypersurface singularity and let  $H := (\partial_{x_i} \partial_{x_j} f)$  denote the Hessian matrix of  $f$ . Assume  $\text{rk } H(0) = k$ , then there exists a polynomial  $g \in \mathbb{C}[x_{k+1}, \dots, x_n]$  with  $\text{ord}(g) \geq 3$ , such that

$$f \sim x_1^2 + \dots + x_k^2 + g.$$

*Proof.* See for example [GLS07, Chapter I, Theorem 2.47]. □

Singularities which are sums of squares are a special type of singularities.

**Definition 1.81.** Let  $f \in \mathbb{C}\{\mathbf{x}\}$ . We call  $f$  an  $A_1$ -singularity, if

$$f \sim \sum_{i=1}^k x_i^2$$

for some  $k \leq n$ .

**Remark 1.82.** The splitting lemma basically tells us that every isolated hypersurface singularity is tight-equivalent to the sum of an  $A_1$ -singularity and an isolated hypersurface singularity  $g$  of order greater equal to 3.

Next to the fact that the dimension of the Tjurina algebra determines the determinacy of an isolated hypersurface singularity, it also determines the isomorphism class as follows:

**Theorem 1.83 (Mather–Yau Theorem).** Let  $f, g \in \mathbb{C}\{\mathbf{x}\}$  define isolated hypersurface singularities  $(X, \mathbf{0})$  respectively  $(Y, \mathbf{0})$ . Then the following are equivalent:

- (1)  $f \sim g$ .

- (2)  $(X, \mathbf{0}) \cong (Y, \mathbf{0})$ .
- (3)  $T_f \cong T_g$  as  $\mathbb{C}$ -algebras.
- (4)  $(\text{Sing}(X), \mathbf{0}) \cong (\text{Sing}(Y), \mathbf{0})$ .

*Proof.* See for example [GLS07, Chapter I, Theorem 2.26]. □

As mentioned in the introduction the goal of this thesis is to get a better understanding of the explicit correspondence between the isomorphism class of the Tjurina algebra and the isomorphism class of the complex space germ of isolated hypersurface singularities.

We conclude this section with an example that shows the limits of the Mather–Yau theorem.

**Example 1.84.** Consider the hypersurface singularities defined by  $f = x^2 + y^2 \in \mathbb{C}\{x, y\}$  and  $g = x^2 - y^2 \in \mathbb{C}\{x, y\}$ . One can easily see that

$$J_f = \langle x^2 + y^2, 2x, 2y \rangle = \langle x, y \rangle = \langle x^2 - y^2, 2x, -2y \rangle = J_g.$$

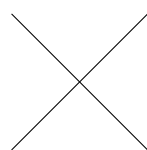
We see that  $\dim_{\mathbb{C}} T_f = \dim_{\mathbb{C}} T_g = 1$ , hence  $f$  and  $g$  define isolated hypersurface singularities. With  $J_f = J_g$ , we obtain  $T_f \cong T_g$  and the Mather–Yau theorem yields  $(V(f), \mathbf{0}) \cong (V(g), \mathbf{0})$ . Next we compare the real pictures.

$(V(f), \mathbf{0})$



**Figure 1.9:** Real picture of the hypersurface singularity  $(V(f), \mathbf{0})$ .

$(V(g), \mathbf{0})$



**Figure 1.10:** Real picture of the hypersurface singularity  $(V(g), \mathbf{0})$ .

From the real picture we can see that the hypersurface germ, if we would consider them as subsets of  $(\mathbb{R}^2, \mathbf{0})$  are not isomorphic, since they have a different number of irreducible components. This is a counterexample to the Mather–Yau theorem in the case of the real numbers. Due to this we put our focus on the complex numbers.

### 1.5.2 Quasi-Homogeneous Isolated Hypersurface Singularities

In the underlying thesis we are going to focus on a special type of isolated hypersurface singularities, namely so-called **quasi-homogeneous isolated hypersurface singularities** (short: QHIS).

**Definition 1.85.** Let  $f \in \mathbb{C}\{\mathbf{x}\}$ . We say  $f$  is *quasi-homogeneous power series* if there exists an integer  $d \in \mathbb{N}_{\geq 1}$  and a  $w \in \mathbb{N}_{\geq 1}^n$  such that all monomials  $m \in \text{Supp}(f)$  have weighted degree  $d$  with respect to  $w$ .

Let  $(X, \mathbf{0})$  be an isolated hypersurface singularity. We call  $(X, \mathbf{0})$  a *quasi-homogeneous isolated hypersurface singularity* if there exists a quasi-homogeneous power series  $f \in \mathbb{C}\{\mathbf{x}\}$ , such that  $(X, \mathbf{0}) \cong (V(f), \mathbf{0})$ .

**Remark 1.86.** Due to the positivity of the entries of  $w$ ,  $f$  has to be a polynomial. So we will assume from now on that  $(X, \mathbf{0})$  is defined by a quasi-homogeneous polynomial  $f \in \mathbb{C}[\mathbf{x}]$ .

We start with an elementary property regarding the monomial structure of a polynomial defining the QHIS.

**Lemma 1.87.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS. Then for each  $1 \leq i \leq n$  one of the following statements holds:*

- (1)  $x_i^m \in \text{Supp}(f)$  for a certain  $m \in \mathbb{N}$ .
- (2)  $x_i^m x_j \in \text{Supp}(f)$  for a certain  $m \in \mathbb{N}$  and  $1 \leq j \leq n$ .

*Proof.* See [Sai71, Korollar 1.6]. □

Lemma 1.87 implies the following result.

**Corollary 1.88.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS. Assume that  $\text{ord}(f) \geq 3$  and that  $f$  is quasi-homogeneous with respect to the weight-vectors  $w = (w_1, \dots, w_n)$  and  $v = (v_1, \dots, v_n)$  with weighted degrees  $d_w$  respectively  $d_v$ . Then*

$$\frac{w_i}{d_w} = \frac{v_i}{d_v}$$

for all  $1 \leq i \leq n$ .

We conclude this structural part with the following uniqueness result.

**Theorem 1.89.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS. Assume that  $\text{ord}(f) \geq 3$ ,  $f$  is quasi-homogeneous with respect to the weight-vector  $w = (w_1, \dots, w_n)$  and  $f$  has weighted degree  $d$ . Then for all  $1 \leq i \leq n$  the rational numbers  $\frac{w_i}{d}$  are uniquely determined and satisfy  $0 < \frac{w_i}{d} < \frac{1}{2}$ .*

*Proof.* See [Sai71, Satz 1.3]. □

Next we present relations between the Milnor number, the weight-vector and the weighted degree. We begin with the definition of an auxiliary function.

**Definition 1.90.** *Denote by  $(p_n)_{n \in \mathbb{N}_{\geq 1}}$  the monotonously increasing sequence of prime numbers. We define the function  $l : \mathbb{N}_{\geq 1} \rightarrow \mathbb{Q}, n \mapsto \prod_{i=1}^n \frac{p_i}{p_i - 1}$ .*

Using this function we obtain the following theorem.

**Theorem 1.91.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS. Assume that  $f$  is quasi-homogeneous with respect to the weight-vector  $w = (w_1, \dots, w_n)$  and has weighted degree  $d$ . Then the following hold:*

- (1)  $\mu_f = \tau_f = \prod_{i=1}^n \left( \frac{d}{w_i} - 1 \right)$ .
- (2) If  $n \geq 2$ , then  $d \leq l(n) \cdot \mu_f$ .
- (3) If  $\text{ord}(f) \geq 3$  and  $n \geq 2$ , then  $d \leq l(n-1) \cdot \mu_f$ .

*Proof.* See [HK12, Theorem 4.3]. □

An interesting theoretical property of quasi-homogeneous hypersurface singularities is the fact that being one is a generic property.

Let  $\mathcal{P} = \{p_1, \dots, p_m\} \subseteq \langle \mathbf{x} \rangle$  be a finite set of polynomials. Define  $f_i = \sum_{k=1}^m a_{ij} p_j$  for  $1 \leq i \leq k$  and  $a_{ij} \in \mathbb{C}$ . We want to express the existence of  $a_{ij}$  such that  $\mathbb{C}[\mathbf{x}]_m / \langle f_1, \dots, f_k \rangle$  is a complete intersection ring in more algebraic terms. Consider the rings  $\mathbb{C}[\mathbf{y}]$  and  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$  with maximal ideals  $\mathfrak{n} = \langle \mathbf{y} \rangle$  and  $\mathfrak{o} = \langle \mathbf{x}, \mathbf{y} \rangle$ .

Define  $A = \mathbb{C}[\mathbf{y}]_{\mathfrak{n}}$ ,  $B = \mathbb{C}[\mathbf{x}, \mathbf{y}]_{\mathfrak{o}}$  and  $F_i = \sum_{k=1}^m (y_{ij} + a_{ij}) p_j$  for  $1 \leq i \leq k$ . Moreover, let  $C = B / \langle F_1, \dots, F_k \rangle$ . The canonical maps of  $A$ -algebras

$$A \hookrightarrow B \twoheadrightarrow C$$

induce isomorphisms

$$\mathbb{C}[\mathbf{x}]_m / \langle f_1, \dots, f_k \rangle \cong C / \bar{\mathfrak{n}}C \cong C \otimes_A A / \mathfrak{n}A.$$

In particular the  $a_{ij}$  yield a complete intersection if and only if  $C \otimes_A A / \mathfrak{n}A$  is isomorphic to a complete intersection ring. Now we have every ingredient for the proof of the genericity result. A sketch of the proof has been communicated to us by Claus Hertling.

**Theorem 1.92.** *Let  $\mathcal{M} = \{m_1, \dots, m_l\} \subseteq \mathbb{C}[\mathbf{x}]$  be a finite set and  $I \subseteq \mathbb{C}[\mathbf{x}]$  an ideal generated by  $k \leq n$  polynomials  $f_1, \dots, f_k$  which are linear combinations of elements of  $\mathcal{M}$ , that is*

$$f_i = \sum_{j=0}^l a_{ij} m_j \text{ for certain } a_{ij} \in \mathbb{C}. \text{ If } \mathbb{C}[\mathbf{x}] / I \text{ is a } n - k \text{ dimensional complete intersection,}$$

*then  $\mathbb{C}[\mathbf{x}] / \langle \sum_{j=0}^l \tilde{a}_{ij} p_j \mid 1 \leq i \leq k \rangle$  is a  $n - k$  dimensional complete intersection for generic  $\tilde{a}_{ij} \in \mathbb{C}$ .*

*Proof.* The geometric idea behind this theorem is the following:

We consider the  $a_{ij}$  as values for variables of a polynomial ring, this means that we consider the  $f_i$  as elements of the polynomial ring  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ . Define  $R = \mathbb{C}[\mathbf{x}, \mathbf{y}] / \langle f_1, \dots, f_n \rangle$  and  $S = \mathbb{C}[\mathbf{y}]$ . We obtain a morphism of schemes

$$f : \text{Spec}(R) \rightarrow \text{Spec}(S),$$

where the fibre of the maximal ideal  $\mathfrak{p}$  corresponding to the point  $P = (a_{ij}) \in \mathbb{C}^{n \cdot m}$  is isomorphic to

$$\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle} / \langle \sum_{j=0}^l a_{ij} m_j \mid 1 \leq i \leq n \rangle.$$

By assumption the fibre is isomorphic to a complete intersection ring. We apply [Gro67, Théorème 6.9.1] in order to obtain an open  $U$ , such that  $f|_U : U \rightarrow \text{Spec}(S)$  is a flat morphism. Since flat morphisms are open (see [Gro67, Théorème 2.4.6]), thus we obtain that  $f(U) \subseteq \text{Spec}(S)$  is non-empty and open, hence dense in  $\text{Spec}(S)$ , since  $S$  is a domain. In our setup Krull's Principal ideal theorem (see [BH93, Theorem A.1]) implies that each fibre is of dimension  $\geq n - k$ . By Chevalley's theorem (see [Gro67, Théorème 13.1.3]) we know that locus where the fibres have dimension  $\leq n - k$  is open. We can shrink  $U$  such that the assumption on the dimension of the fibres hold. Since  $U$  is non-empty, we obtain that generic fibres have dimension  $n - k$ .  $\square$

The fact that being an isolated hypersurface singularity is equivalent to  $J_f$  being a zero-dimensional complete intersection yields the following corollary.

**Corollary 1.93.** *Assume we are given a finite set  $\mathcal{M} = \{m_1, \dots, m_l\} \subseteq \langle \mathbf{x} \rangle \subseteq \mathbb{C}[\mathbf{x}]$  and a polynomial  $f = \sum_{i=0}^l a_i m_i$  for certain  $a_i \in \mathbb{C}$ . If  $f$  defines an isolated hypersurface singularity, then  $\tilde{f} = \sum_{i=0}^l \tilde{a}_i m_i$  defines an isolated hypersurface singularity for generic  $\tilde{a}_j \in \mathbb{C}$ .*

### 1.5.3 A General Mather–Yau Theorem

In 1985 Gaffney and Hauser generalized the Mather–Yau theorem to a larger class of singularities, which are not necessarily hypersurface singularities (see [GH85]). There are two types of singularities for which we can extend this result. The first type are singularities defined by an  $f \in \mathbb{C}\{\mathbf{x}\}$  satisfying  $f \in \mathfrak{m}J_f$ . These singularities are so-called *strongly Euler-homogeneous singularities* and we consider this type of singularity in more detail in Chapter 3. The result in this case for isolated hypersurface singularities has been proven by Shoshitaishvili in [Sho76]. The second type for which we can extend the result are so-called *harmonic singularities* (see [HM86]). In this section we state the proof of the aforementioned theorem in the hypersurface case, since it is the theoretical foundation of the underlying thesis. Before we restate the proof, we need to define basic notions.

**Definition 1.94.** *Let  $(X, \mathbf{0})$  be a hypersurface singularity.*

- (1) *We say  $(X, \mathbf{0})$  is of isolated singularity type, if there exists a representative  $U$  of  $(\text{Sing}(X), \mathbf{0})$ , such that*

$$(U, x) \not\cong (U, \mathbf{0})$$

*for all  $x \in U \setminus \{\mathbf{0}\}$ .*

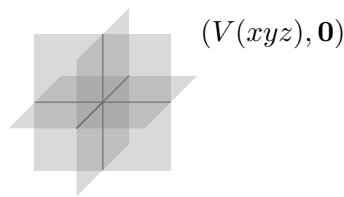
- (2) *We say  $(X, \mathbf{0})$  is a harmonic singularity if there exists a singularity  $(X', \mathbf{0})$  of isolated singularity type and a  $k \in \mathbb{N}$ , such that*

$$(X, \mathbf{0}) \cong (X' \times \mathbb{C}^k, (\mathbf{0}, \mathbf{0})).$$

**Remark 1.95.** *We have the following chain of proper inclusions for hypersurface singularities:*

$$\{\text{isolated}\} \subsetneq \{\text{isolated singularity type}\} \subsetneq \{\text{harmonic}\}.$$

*Singularities with very simple defining equations suffice to show these proper inclusions. As already seen in Example 1.68 and 1.75 the hypersurface singularity  $(V(xy), \mathbf{0}) \subseteq (\mathbb{C}^2, \mathbf{0})$  is isolated, whereas the singularity  $(V(xy), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$  is not. Nevertheless the latter is a suspension of the first, hence it is a harmonic singularity. It is not of isolated singularity type, since for any point  $p = (0, 0, t) \in \mathbb{C}^3$  with  $t \neq 0$ , we have  $\text{Sing}(V(xy), \mathbf{0}) \cong \text{Sing}(V(xy), p)$ . At last we need a singularity, which is of isolated singularity type but not isolated. Therefore we consider  $(V(xyz), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$ . A simple computation shows that  $(\text{Sing}(V(xyz)), \mathbf{0}) = (V(xy) \cup V(xz) \cup V(yz), \mathbf{0})$  and one can see that  $(\text{Sing}(V(xyz)), \mathbf{0}) \not\cong (\text{Sing}(V(xyz)), p)$  for any  $p \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ , hence  $(V(xyz), \mathbf{0})$  is of isolated singularity type, but not isolated. The real picture looks as follows:*



**Figure 1.11:** Real picture of the complex space  $(V(xyz), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$  and of its singular locus.

In the remainder of this section we want to prove a theorem similar to the Mather–Yau theorem for harmonic singularities. We adapt the proof of [JP00, Theorem 9.1.8] to our setup. In the case of hypersurface singularities it is the same as in [GH85], we just fill in more details, since this thesis relies theoretically on this result.

We start our preparations with a lemma that allows us to assume equality of Jacobian ideals.

**Lemma 1.96.** *Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $A = \mathbb{C}[[\mathbf{x}]]$  and  $\varphi = (\varphi_1, \dots, \varphi_n) \in \text{Aut}(A)$ . Then for every  $f \in A$  it holds that*

$$\varphi(J_f) = J_{\varphi(f)}.$$

*Proof.* First note that  $\varphi(\partial_{x_i} f) = (\partial_{x_i} f)(\varphi(\mathbf{x}))$ , hence

$$\varphi(J_f) = \langle \varphi(\partial_{x_1} f), \dots, \varphi(\partial_{x_n} f) \rangle = \langle (\partial_{x_1} f)(\varphi(\mathbf{x})), \dots, (\partial_{x_n} f)(\varphi(\mathbf{x})) \rangle.$$

Applying the chain rule to  $\varphi(f) = f(\varphi(\mathbf{x}))$  yields

$$\partial_{x_j} \varphi(f) = \sum_{i=1}^n (\partial_{x_i} f)(\varphi(\mathbf{x})) \cdot \partial_{x_j} \varphi_i = \sum_{i=1}^n \varphi(\partial_{x_i} f) \cdot \partial_{x_j} \varphi_i.$$

We can rewrite this as

$$\begin{pmatrix} \partial_{x_1} \varphi(f) \\ \vdots \\ \partial_{x_n} \varphi(f) \end{pmatrix} = J_\varphi \cdot \begin{pmatrix} \varphi(\partial_{x_1} f) \\ \vdots \\ \varphi(\partial_{x_n} f) \end{pmatrix}. \quad (1.1)$$

Due to the Inverse Function Theorem (Theorem 1.9) the Jacobian matrix  $J_\varphi$  is invertible. This means that we can rewrite Equation (1.1) as

$$(J_\varphi)^{-1} \cdot \begin{pmatrix} \partial_{x_1} \varphi(f) \\ \vdots \\ \partial_{x_n} \varphi(f) \end{pmatrix} = \begin{pmatrix} \varphi(\partial_{x_1} f) \\ \vdots \\ \varphi(\partial_{x_n} f) \end{pmatrix}. \quad (1.2)$$

Equation (1.1) and (1.2) now imply  $\varphi(J_f) = J_{\varphi(f)}$ .  $\square$

The next lemma is concerned with equality of a special type of ideals.

**Lemma 1.97.** *Let  $A = \mathbb{C}\{\mathbf{x}, t\}$  or  $A = \mathbb{C}[[\mathbf{x}, t]]$ . Furthermore, let  $k \in \mathbb{N}_{\geq 1}$  and  $f_1, \dots, f_k, g_1, \dots, g_k \in A$ . We define  $I := \langle f_1, \dots, f_k \rangle \subseteq A$  and  $I_t := \langle f_1 + t \cdot (g_1 - f_1), \dots, f_k + t \cdot (g_k - f_k) \rangle \subseteq A$ . If  $I = \langle g_1, \dots, g_k \rangle$ , then  $I = I_t$ .*



*Proof.* The assumption  $I = \langle g_1, \dots, g_k \rangle$  implies  $I_t \subseteq I$ . To prove the equality we consider the quotient  $\bar{I} := I/I_t \subseteq A/I_t =: B$ . Due to the definition of  $I_t$ , we obtain

$$\overline{f_i} = t \cdot \overline{(f_i - g_i)} \in \mathfrak{m}_B \bar{I} \text{ for all } i = 1, \dots, k.$$

This implies  $\bar{I} \subseteq \mathfrak{m}_B \bar{I}$ , hence  $\bar{I} = \bar{0}$  by Nakayama's lemma. Thus  $I = I_t$ .  $\square$

Our next result is a statement about path-connectedness of complex space germs.

**Lemma 1.98.** *Let  $k \in \mathbb{N}_{\geq 1}$  and  $f_1, \dots, f_k, g_1, \dots, g_k \in \mathbb{C}\{\mathbf{x}\}$ . Fix an open neighborhood  $U \subseteq \mathbb{C}^n$  of  $\mathbf{0}$  such that  $f_i, g_j$  are holomorphic functions on  $U$  for all  $1 \leq i, j \leq k$ . We define the ideal sheaf  $\mathcal{I} := f_1 \mathcal{O}_U + \dots + f_k \mathcal{O}_U$  and assume that  $\mathcal{I} = g_1 \mathcal{O}_U + \dots + g_k \mathcal{O}_U$  holds. Furthermore, we define for any  $t_0 \in \mathbb{C}$  the ideal sheaf  $\mathcal{I}_{t_0} := (f_1 + t_0 \cdot (g_1 - f_1)) \mathcal{O}_U + \dots + (f_k + t_0 \cdot (g_k - f_k)) \mathcal{O}_U$ . We define  $X := V(\mathcal{I})$  and  $X_{t_0} = V(\mathcal{I}_{t_0})$  for any  $t_0 \in \mathbb{C}$ . Then the following hold:*

(1) *There exists a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  satisfying  $\gamma(0) = 0$  and  $\gamma(1) = 1$ , such that*

$$(X, \mathbf{0}) \cong (X_{t_0}, \mathbf{0})$$

*for all  $t_0 \in \gamma([0, 1])$ .*

(2) *There exists an open neighborhood  $W \subseteq U$  of  $\mathbf{0}$  and  $V \subseteq \mathbb{C}$  of  $0$  with the property that for all  $t_0 \in V$  it holds that*

$$W \cap X = W \cap X_{t_0}.$$

*Proof.* The idea for the proof of the first claim is to connect the ideals  $I_0 := \mathcal{I}_0 = \langle f_1, \dots, f_k \rangle = \langle g_1, \dots, g_k \rangle =: I_1$  via the ideals  $I_{t_0} := (\mathcal{I}_{t_0})_{\mathbf{0}} = \langle f_1 + t_0 \cdot (g_1 - f_1), \dots, f_k + t_0 \cdot (g_k - f_k) \rangle$  by a complex line and check that there exists a path  $\gamma$  as in the claim. In order to show the existence of  $\gamma$  we use sheaf theory. We define the ideal sheaves

$$\mathcal{J} := f_1 \mathcal{O}_{U \times \mathbb{C}} + \dots + f_k \mathcal{O}_{U \times \mathbb{C}} = g_1 \mathcal{O}_{U \times \mathbb{C}} + \dots + g_k \mathcal{O}_{U \times \mathbb{C}}$$

and

$$\mathcal{J}_t := (f_1 + t \cdot (g_1 - f_1)) \mathcal{O}_{U \times \mathbb{C}} + \dots + (f_k + t \cdot (g_k - f_k)) \mathcal{O}_{U \times \mathbb{C}}.$$

By construction we have an inclusion of ideal sheaves  $\mathcal{J}_t \subseteq \mathcal{J}$  on the analytic set  $U \times \mathbb{C}$  and in particular on the analytic subset  $\{\mathbf{0}\} \times \mathbb{C}$ . We know by Proposition 1.20 that  $\text{Supp}(\mathcal{J}/\mathcal{J}_t) \cap (\{\mathbf{0}\} \times \mathbb{C})$  is an analytic subset of  $\{\mathbf{0}\} \times \mathbb{C}$ . By Example 1.22 we know that the only closed complex subspaces of  $\{\mathbf{0}\} \times \mathbb{C}$  are either unions of isolated points or the whole space. If we can show that  $(\mathbf{0}, 0), (\mathbf{0}, 1) \notin \text{Supp}(\mathcal{J}/\mathcal{J}_t) \cap (\{\mathbf{0}\} \times \mathbb{C})$ , then there exists a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$ , which avoids the isolated points and which satisfies  $\gamma(0) = 0$  and  $\gamma(1) = 1$ . So it only remains to show that the points  $(\mathbf{0}, 0)$  and  $(\mathbf{0}, 1)$  are not in  $\text{Supp}(\mathcal{J}/\mathcal{J}_t)$ . We only prove this for the stalk at  $(\mathbf{0}, 0)$ , as the stalk at  $(\mathbf{0}, 1)$  works analogously. Consider the ideal  $J_t := (\mathcal{J}_t)_{(\mathbf{0}, 0)} = \langle f_1 + t(g_1 - f_1), \dots, f_k + t(g_k - f_k) \rangle \subseteq \mathbb{C}\{\mathbf{x}, t\}$  and define  $J_0 := I_0 \cdot \mathbb{C}\{\mathbf{x}, t\} = \mathcal{J}_{(\mathbf{0}, 0)}$ . Since  $I_0 = I_1$ , we can apply Lemma 1.97 to  $J_0$  and  $J_t$  and obtain

$$J_0 = \mathcal{J}_{(\mathbf{0}, 0)} = (\mathcal{J}_t)_{(\mathbf{0}, 0)} = J_t \tag{1.3}$$

and hence  $(\mathbf{0}, 0) \notin \text{Supp}(\mathcal{J}/\mathcal{J}_t) \cap (\{\mathbf{0}\} \times \mathbb{C})$ . This proves the first part of the statement. Coherence of ideal sheaves and Equation (1.3) imply the existence of an open neighborhood  $W \subseteq U$  of  $\mathbf{0}$  and an open neighborhood  $V \subseteq \mathbb{C}$  of  $0$ , such that we obtain an equality of ideal sheaves

$$f_1 \mathcal{O}_{W \times V} + \dots + f_k \mathcal{O}_{W \times V} = (f_1 + t \cdot (g_1 - f_1)) \mathcal{O}_{W \times V} + \dots + (f_k + t \cdot (g_k - f_k)) \mathcal{O}_{W \times V}. \tag{1.4}$$



Setting the value of  $t$  in Equation (1.4) to an arbitrary but fixed  $t_0 \in V$  yields the equality of ideal sheaves

$$\mathcal{I}|_W = f_1\mathcal{O}_W + \dots + f_k\mathcal{O}_W = (f_1 + t_0 \cdot (g_1 - f_1))\mathcal{O}_W + \dots + (f_k + t_0 \cdot (g_k - f_k))\mathcal{O}_W = (\mathcal{I}_{t_0})|_W. \quad (1.5)$$

Equation (1.5) implies

$$W \cap X = W \cap X_{t_0}$$

for every  $t_0 \in V$ . □

The next lemma shows how the isomorphism class of hypersurface singularities behave if they lie on a line connecting two fixed hypersurface singularities.

**Lemma 1.99.** *Let  $f, g \in \mathbb{C}\{\mathbf{x}\}$ . Fix an open neighborhood  $U \subseteq \mathbb{C}^n$  of  $\mathbf{0}$  such that  $f, g$  are holomorphic functions on  $U$ . We define the ideal sheaves  $\mathcal{I}_f := f\mathcal{O}_U + \partial_{x_1}f\mathcal{O}_U + \dots + \partial_{x_n}f\mathcal{O}_U$  and  $\mathcal{I}_g := g\mathcal{O}_U + \partial_{x_1}g\mathcal{O}_U + \dots + \partial_{x_n}g\mathcal{O}_U$ . Furthermore, we assume  $(\mathcal{I}_f)_\mathbf{0} = \langle f, J_f \rangle = \langle g, J_g \rangle = (\mathcal{I}_g)_\mathbf{0}$ . On  $U \times \mathbb{C}$  we define the holomorphic function  $F := f + t \cdot (g - f) \in \mathbb{C}\{\mathbf{x}, t\}$  and on  $U$  we define for any  $t_0 \in \mathbb{C}$  the holomorphic function  $F_{t_0} := F(\mathbf{x}, t_0) \in \mathbb{C}\{\mathbf{x}\}$ . Then the following hold:*

- (1)  $(V(f) \times \mathbb{C}, (\mathbf{0}, 0)) \cong (V(F), (\mathbf{0}, 0))$ .
- (2) *There exist an open neighborhood  $V$  of  $0 \in \mathbb{C}$  and a continuous family of points  $(p_t)_{t \in V}$  with:*
  - (a)  $\lim_{t \rightarrow 0} p_t = \mathbf{0}$ , and
  - (b)  $(V(f), \mathbf{0}) \cong (V(F_{t_0}), p_{t_0})$ .
- (3) *If  $f \in \mathfrak{m}J_f$  and  $g \in \mathfrak{m}J_g$ , then  $p_{t_0} = \mathbf{0}$  for all  $t_0 \in V$ .*

*Proof.* By coherence of ideal sheaves the equality  $(\mathcal{I}_f)_\mathbf{0} = (\mathcal{I}_g)_\mathbf{0}$  implies the existence of an open neighborhood  $U' \subseteq U$  of  $\mathbf{0}$  such that  $\mathcal{I}_f|_{U'} = \mathcal{I}_g|_{U'}$ . After possibly shrinking  $U$  we can assume without loss of generality  $U = U'$  and hence  $\mathcal{I}_f = \mathcal{I}_g$ . We define the ideal sheaf  $\mathcal{I} := F\mathcal{O}_{U \times \mathbb{C}} + \partial_{x_1}F\mathcal{O}_{U \times \mathbb{C}} + \dots + \partial_{x_n}F\mathcal{O}_{U \times \mathbb{C}}$ . Since  $F = f + t \cdot (g - f)$ ,  $\partial_{x_i}F = \partial_{x_i}f + t \cdot (\partial_{x_i}g - \partial_{x_i}f)$  and  $\mathcal{I}_f = \mathcal{I}_g$ , we obtain the inclusion of ideal sheaves

$$\mathcal{I} \subseteq f\mathcal{O}_{U \times \mathbb{C}} + \partial_{x_1}f\mathcal{O}_{U \times \mathbb{C}} + \dots + \partial_{x_n}f\mathcal{O}_{U \times \mathbb{C}} =: \mathcal{I}'. \quad (1.6)$$

Define  $I := \mathcal{I}_{(\mathbf{0}, 0)}$  and  $I' := \mathcal{I}'_{(\mathbf{0}, 0)} = \langle f, J_f \rangle \cdot \mathbb{C}\{\mathbf{x}, t\}$ . Applying Lemma 1.97 to the ideals  $I$  and  $I'$  implies  $I = \mathcal{I}'_{(\mathbf{0}, 0)} = \langle f, J_f \rangle \cdot \mathbb{C}\{\mathbf{x}, t\} = I'$ . In this setup we obtain  $\partial_t F = g - f \in \langle f, J_f \rangle \cdot \mathbb{C}\{\mathbf{x}, t\} = I$  and Theorem 1.69 implies the existence of  $\varphi_1, \dots, \varphi_n \in \mathbb{C}\{\mathbf{x}, t\}$  and a unit  $u \in \mathbb{C}\{\mathbf{x}, t\}^*$ , such that

$$u \cdot F(\varphi_1(\mathbf{x}, t), \dots, \varphi_n(\mathbf{x}, t), 0) = F(\mathbf{x}, t). \quad (1.7)$$

The morphism  $\varphi : \mathbb{C}\{\mathbf{x}, t\} \rightarrow \mathbb{C}\{\mathbf{x}, t\}$ , defined by  $\varphi := (\varphi_1, \dots, \varphi_n, t)$  is an isomorphism due to Theorem 1.69, 2.(b) and the Inverse Function Theorem (Theorem 1.9). Then there exist  $\psi_1, \dots, \psi_n \in \mathbb{C}\{\mathbf{x}, t\}$ , such that  $\psi := (\psi_1, \dots, \psi_n, t)$  is the inverse of  $\varphi$ . Applying  $\psi$  to Equation (1.7) yields

$$f = F(\mathbf{x}, 0) = \psi(u)^{-1}F(\psi_1(\mathbf{x}, t), \dots, \psi_n(\mathbf{x}, t), t). \quad (1.8)$$

This proves the first statement. Since  $\psi$  defines an automorphism of  $\mathbb{C}\{\mathbf{x}, t\}$ , there must exist open neighborhoods  $W, W' \subseteq U$  of  $\mathbf{0}$  and  $V, V' \subseteq \mathbb{C}$  of  $0$ , such that  $\psi : W \times v \rightarrow W' \times V'$  is a biholomorphic map satisfying  $\psi(\mathbf{0}, 0) = (\mathbf{0}, 0)$ . To prove the second part of theorem we define for any  $t_0 \in V$  the maps  $\psi_{t_0} := (\psi_1(\mathbf{x}, t_0), \dots, \psi_n(\mathbf{x}, t_0)) : W \rightarrow W'$ . Due to the Inverse Function Theorem the  $\psi_{t_0}$  are isomorphisms. Define  $p_{t_0} := \psi_{t_0}(\mathbf{0})$ . By construction the family  $(p_t)_{t \in V}$  depends continuously on  $t$  and  $\psi(\mathbf{0}, 0) = (\mathbf{0}, 0)$  implies  $\lim_{t \rightarrow 0} p_t = \mathbf{0}$ . Furthermore, setting  $t := t_0$  in Equation (1.8) implies

$$(V(f), \mathbf{0}) \cong (V(F_{t_0}), p_{t_0})$$

for all  $t_0 \in V$ . This proves the second statement.

To prove the third part of the theorem we carefully analyze the proof of the first part. The assumptions  $f \in \mathfrak{m}_{J_f}$  and  $g \in \mathfrak{m}_{J_g}$  imply  $(\mathcal{I}_f)_0 = J_f$  and  $(\mathcal{I}_g)_0 = J_g$ . By assumption we know that  $\mathcal{I}_f = \mathcal{I}_g = \partial_{x_1} f \mathcal{O}_U + \dots + \partial_{x_n} f \mathcal{O}_U$ . Applying Lemma 1.97 to the ideals  $I$  and  $J_f \cdot \mathbb{C}\{\mathbf{x}, t\}$  we obtain  $I = J_f \cdot \mathbb{C}\{\mathbf{x}, t\}$ . We get  $\partial_t F = g - f \in \langle x_1, \dots, x_n \rangle J_f \cdot \mathbb{C}\{\mathbf{x}, t\} = \langle x_1, \dots, x_n \rangle I$ . In this case Theorem 1.69 yields  $\varphi(\mathbf{0}, t_0) = (\mathbf{0}, t_0)$ . Then also  $\psi(\mathbf{0}, t_0) = (\mathbf{0}, t_0)$  and by construction  $p_{t_0} = \mathbf{0}$ .  $\square$

Before we state the general Mather–Yau theorem, let us define the notion of (strongly) Euler-homogeneous singularities:

**Definition 1.100.** Let  $X \subseteq \mathbb{C}^n$  be a hypersurface singularity. Denote by  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  the local equation of  $X$  in  $p \in X$ . We call  $X$  **Euler-homogeneous at**  $p \in X$  if, and only if, there exists a derivation  $\chi_p \in \text{Der}(\mathcal{O}_{X,p})$ , such that  $\chi_p(f_p) = f_p$ , where  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  is the local equation of  $X$  at  $p \in X$ . A derivation  $\chi_p$  is called **Euler-derivation of  $f$  at  $p$** . We call  $X$  **strongly Euler-homogeneous at**  $p \in X$  if, and only if, there exists an Euler derivation  $\chi_p$  satisfying  $\chi_p(p) = 0$ . We call  $X$  **(strongly) Euler-homogeneous**, if  $X$  is (strongly) Euler-homogeneous at all  $p \in X$ . Let  $f \in \mathbb{C}\{\mathbf{x} - p\}$  be holomorphic on  $U \subseteq \mathbb{C}^n$ . We say  $f$  is (strongly) Euler-homogeneous, if  $X = V(f) \subseteq U$  is (strongly) Euler-homogeneous at  $p$ . We call a complex space germ  $(X, p)$  (strongly) Euler homogeneous (at  $p$ ), if there exists a representant which is (strongly) Euler-homogeneous (at  $p$ ).

Now we are able to state and prove the general Mather–Yau theorem by Gaffney and Hauser in the hypersurface case. We combine the original proof of the main theorem in [GH85] with the proof of the Mather–Yau theorem in [JP00]. As a byproduct we also obtain a proof for singularities satisfying  $f \in J_f$  generalizing the result from [Sho76].

**Theorem 1.101.** Let  $f, g \in \mathbb{C}\{\mathbf{x}\}$  define singularities  $(X, \mathbf{0})$  respectively  $(Y, \mathbf{0})$ . Assume either

- (a)  $(X, \mathbf{0})$  and  $(Y, \mathbf{0})$  are harmonic singularities, or
- (b)  $(X, \mathbf{0})$  and  $(Y, \mathbf{0})$  are strongly Euler-homogeneous at  $\mathbf{0}$ .

Then the following are equivalent:

1.  $f \sim g$ .
2.  $(X, \mathbf{0}) \cong (Y, \mathbf{0})$ .
3.  $T_f \cong T_g$  as  $\mathbb{C}$ -algebras.

4.  $(\text{Sing}(X), \mathbf{0}) \cong (\text{Sing}(Y), \mathbf{0})$ .

*Proof.* The equivalence of 1. and 2. and of 3. and 4. follow immediately from Proposition 1.39. In case  $f \sim g$  the chain rule of differentiation implies  $T_f \cong T_g$  as  $\mathbb{C}$ -algebras. So only the implication 3. to 1. has to be proven. We start with the proof in case (a). Applying Lemma 1.71 we can reduce our setup to the case that  $f$  and  $g$  define singularities of isolated singularity type.

We fix an open neighborhood  $U \subseteq \mathbb{C}^n$  of  $\mathbf{0} \in \mathbb{C}^n$ , such that  $f, g$  are holomorphic functions on  $U$ . If  $T_f \cong T_g$  as  $\mathbb{C}$ -algebras, then by Lemma 1.11 there exists an automorphism  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$  satisfying

$$\varphi(\langle f, J_f \rangle) = \langle \varphi(f), J_{\varphi(f)} \rangle = \langle g, J_g \rangle. \quad (1.9)$$

Since  $f \sim \varphi(f)$ , we can replace by abuse of notation  $f$  with  $\varphi(f)$ . Define the ideal sheaves  $\mathcal{I}_f := f\mathcal{O}_U + \partial_{x_1}f\mathcal{O}_U + \dots + \partial_{x_n}f\mathcal{O}_U$  and  $\mathcal{I}_g := g\mathcal{O}_U + \partial_{x_1}g\mathcal{O}_U + \dots + \partial_{x_n}g\mathcal{O}_U$ . Equation (1.9) is equivalent to saying that  $(\mathcal{I}_f)_0 = (\mathcal{I}_g)_0$ . By coherence of ideal sheaves there exists an open neighborhood  $U' \subseteq \mathbb{C}^n$  of  $\mathbf{0} \in \mathbb{C}^n$ , such that  $\mathcal{I}_f|_{U'} = \mathcal{I}_g|_{U'}$ . After possibly shrinking  $U$  we can assume without loss of generality  $U = U'$  and hence  $\mathcal{I}_f = \mathcal{I}_g$ . In the same way as in Lemma 1.99 we define  $F := f + t \cdot (g - f) \in \mathbb{C}\{\mathbf{x}, t\}$  and  $F_{t_0} := F(\mathbf{x}, t_0) \in \mathbb{C}\{\mathbf{x}\}$  for  $t_0 \in \mathbb{C}$ . Since  $F_{t_0}$  is a holomorphic function on  $U$  for every  $t_0 \in \mathbb{C}$ , we define for any  $t_0 \in \mathbb{C}$  the ideal sheaf  $\mathcal{I}_{F_{t_0}} := F_{t_0}\mathcal{O}_U + \partial_{x_1}F_{t_0}\mathcal{O}_U + \dots + \partial_{x_n}F_{t_0}\mathcal{O}_U$ . Using this notation we have  $\text{Sing}(V(f)) = V(\mathcal{I}_f)$  and  $\text{Sing}(V(F_{t_0})) = V(\mathcal{I}_{F_{t_0}})$ . Lemma 1.98 yields the existence of a path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  such that  $(\text{Sing}(V(f)), \mathbf{0}) \cong (\text{Sing}(V(F_{t_0})), \mathbf{0})$  for  $t_0 \in \gamma([0, 1])$ . Furthermore, we know that there exist open neighborhoods  $W \subseteq U$  of  $\mathbf{0}$  and  $V \subseteq \mathbb{C}$  of 0, such that

$$W \cap \text{Sing}(V(f)) = W \cap \text{Sing}(V(F_{t_0})) \quad (1.10)$$

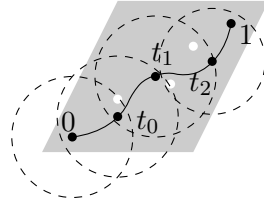
for any  $t_0 \in V$ . Since  $(V(f), \mathbf{0})$  is of isolated singularity type, we can pick a representative  $U'$  of  $(\text{Sing}(V(f)), \mathbf{0})$ , such that  $(U', \mathbf{0}) \not\cong (U', x)$  for all  $x \in U' \setminus \{\mathbf{0}\}$ . Due to the fact that  $W \cap U'$  is an open neighborhood of  $\mathbf{0}$ , we assume without loss of generality that  $W = U'$ . Lemma 1.99 implies the existence of a family of points  $(p_{t_0})_{t_0 \in V} \subseteq \mathbb{C}^n$ .  $(V(f), \mathbf{0}) \cong (V(F_{t_0}), p_{t_0})$ . Then it holds that  $(\text{Sing}(V(f)), \mathbf{0}) \cong (\text{Sing}(V(F_{t_0})), p_{t_0})$ . Due to  $\lim_{t_0 \rightarrow 0} p_{t_0} = \mathbf{0}$  we obtain  $p_{t_0} \in W$  for all  $t_0$  in an open neighborhood  $V'$  of 0. Again we can assume without loss of generality that  $V = V'$ . Using Equation (1.10), we obtain

$$(\text{Sing}(V(F_{t_0})), p_{t_0}) \cong (\text{Sing}(V(f), \mathbf{0})) \cong (\text{Sing}(V(F_{t_0})), \mathbf{0}) \quad (1.11)$$

for any  $t_0 \in V$ . Combining the Equations (1.10) and (1.11) with the fact that  $f$  defines a singularity of isolated singularity type yields  $p_{t_0} = \mathbf{0}$  for any  $t_0 \in V$ . This results in

$$(V(f), \mathbf{0}) = (V(F_{t_0}), \mathbf{0}).$$

So far we have shown that the isomorphism of the singular loci implies the existence of a contact equivalent hypersurface for small values of  $t_0$ . Iterating this process we construct a sequence of open sets  $(V_k)_{k \in \mathbb{N}}$  with  $V_0 = V$  and a sequence  $(t_k)_{k \in \mathbb{N}}$  with  $t_k \in \gamma([0, 1]) \cap V_k$  of points converging to 1, such that  $(V(F_{t_k}), \mathbf{0}) \cong (V(F_{t_{k+1}}), \mathbf{0})$ . The final step of the proof is to show that this process stops after finitely many steps. Since the  $V_k$  cover the compact set  $\gamma([0, 1])$ , we can pick a finite number of them, such that they still cover the whole path as sketched in Figure 1.12.



**Figure 1.12:** Sketch of the open covering of  $\gamma([0, 1])$ .

Then we have a finite subsequence  $(t_{k_j})_{0 \leq j \leq r}$ , satisfying

$$(V(f), \mathbf{0}) \cong (V(F_{t_{k_0}}), \mathbf{0}) \cong \dots \cong (V(F_{t_{k_r}}), \mathbf{0}) \cong (V(g), \mathbf{0}).$$

This finishes the proof in case (a).

To prove case (b) we have to assume  $f \in \mathfrak{m}J_f$  and  $g \in \mathfrak{m}J_g$ . In this case Lemma 1.99 yields  $p_t = \mathbf{0}$ , so that we can proceed from this point on as in case (a).  $\square$

**Remark 1.102.** *The proof of Theorem 1.101 shows that, if we do not assume  $f \in \mathfrak{m}J_f$ , the property of being of isolated singularity type arises as a natural condition in order to show that the isomorphism class of the singular locus determines the isomorphism class of the hypersurface singularity. We show in Example 1.103 that this condition is necessary. The proof we presented does not work if the defining ideal of the singularity has multiple generators, since the Local analytic triviality theorem cannot be extended in such a way, that we can work with the usual singular locus. Therefore Gaffney and Hauser defined a different singular locus in order to keep the analogy with the hypersurface case.*

Next we give an example of a family of hypersurface singularities, such that the singular loci are all isomorphic, but the singularities themselves are not. The example is a specialized version of [GH85, 4. Example].

**Example 1.103.** *Consider the polynomial  $f = x_1^3 x_2 + x_2^5 x_3 + x_3^5 x_1 + x_1 x_2 x_3 \in \mathbb{C}\{x_1, x_2, x_3\}$ . A SINGULAR computation (see [Dec+19]) shows  $f \notin J_f$ . Define  $F := f(\mathbf{x}) + (1 + z + t) \cdot f(\mathbf{y}) \in \mathbb{C}\{\mathbf{x}, \mathbf{y}, z, t\}$ . Furthermore, we set  $F_{t_0} = F(\mathbf{x}, \mathbf{y}, z, t_0) \in \mathbb{C}\{\mathbf{x}, \mathbf{y}, z\}$ ,  $X := V(F) \subseteq \mathbb{C}^8$  and  $X_{t_0} := V(F_{t_0}) \subseteq \mathbb{C}^7$  for fixed  $t_0 \in \mathbb{C}$ . By definition we have  $(X_0, \mathbf{0}) = (V(f), \mathbf{0})$ . We want to show that there exists an open neighborhood  $V \subseteq \mathbb{C}$  of  $0 \in \mathbb{C}$ , such that:*

- (1) *We have  $(\text{Sing}(V(F), \mathbf{0}) \cong (\text{Sing}(V(F_0)) \times V, \mathbf{0})$ ,*
- (2) *but  $(X, \mathbf{0}) \not\cong (X_0 \times V, \mathbf{0})$ .*

*We start with the first claim. Computing the partial derivatives of  $F$  yields*

$$\partial_{x_i} F = \partial_{x_i} f, \partial_{y_j} F = (1 + z + t) \cdot \partial_{y_j} f \text{ and } \partial_z F = \partial_t F = f(\mathbf{y})$$

*for  $1 \leq i, j \leq 3$ . Using that  $(1 + z + t)$  is a unit in  $\mathbb{C}\{\mathbf{x}, \mathbf{y}, z, t\}$  we obtain*

$$\langle F, \partial_{x_i} F, \partial_{y_j} F, \partial_z F, \partial_t F | 1 \leq i, j \leq 3 \rangle = \langle f(\mathbf{x}), f(\mathbf{y}), \partial_{x_i} f, \partial_{y_j} f | 1 \leq i, j \leq 3 \rangle.$$

*Thus the defining equations of the singular locus  $(\text{Sing}(V(F), \mathbf{0}))$  do not depend on  $t$  and we have shown the first claim. Next we prove the second claim. Assume the converse. Then Theorem 1.69 yields*

$$\partial_t F = f(\mathbf{y}) \in \langle \mathbf{x}, \mathbf{y}, z \rangle \langle \partial_{x_i} F, \partial_{y_j} F | 1 \leq i, j \leq 3 \rangle + \langle F \rangle. \quad (1.12)$$

If we plug  $\mathbf{x} = \mathbf{0}$  and  $z = 0$  into Equation (1.12), then there exist  $g_1, \dots, g_3, g \in \mathbb{C}\{\mathbf{y}\}$  such that

$$f(\mathbf{y}) = \sum_{i=1}^3 g_i \partial_{y_i} f(\mathbf{y}) + g \cdot (1+t) \cdot f(\mathbf{y}). \quad (1.13)$$

The partial derivatives of  $f(\mathbf{x})$  vanish, since  $f \in \mathfrak{m}^2$ . If  $\text{ord}(g) \geq 1$ , we can solve for  $f(\mathbf{y})$  in Equation (1.13) and obtain either  $f(\mathbf{y}) = (1 - g \cdot (1+t))^{-1} \cdot \sum_{i=1}^3 g_i \partial_{y_i} f(\mathbf{y})$ , hence  $f \in J_f$ , or, if  $\text{ord}(g) = 0$ , we obtain

$$f(\mathbf{y}) \in \langle \mathbf{x}, \mathbf{y}, z \rangle \langle \partial_{x_i} F, \partial_{y_j} F \mid 1 \leq i, j \leq 3 \rangle + (1 + \langle \mathbf{x}, \mathbf{y}, z \rangle) \cdot \langle F \rangle. \quad (1.14)$$

In the second case we plug  $\mathbf{y} = \mathbf{0}$  and  $z = 0$  into Equation (1.14) and a similar argument as for  $f(\mathbf{y})$  yields  $f(\mathbf{x}) \in \langle \partial_{x_1} f(\mathbf{x}), \dots, \partial_{x_n} f(\mathbf{x}) \rangle$ , hence  $f \in J_f$ . Since both cases yield the contradiction  $f \in J_f$ , the second claim is proven.

We conclude this chapter by pointing out that Theorem 1.101 combined with a result by Hauser and Müller classify weighted-homogeneous hypersurface singularities, which are determined by their singular locus. The result by Hauser and Müller is the following.

**Theorem 1.104.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  be a weighted-homogeneous hypersurface singularity with respect to the weight vector  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{C}^n$  and with weighted degree  $d := \deg_{\mathbf{w}}(f)$ . Assume that  $d - w_i \neq 0$  for  $1 \leq i \leq n$ , or, equivalently, that  $\deg_{\mathbf{w}}(\partial_{x_i} f) \neq 0$  for  $1 \leq i \leq n$ . Then  $(X, \mathbf{0}) := (V(f), \mathbf{0})$  is a harmonic hypersurface singularity.*

*Proof.* See [HM86, Theorem 4]. □

Theorem 1.104 states that any generic weighted-homogeneous hypersurface singularity is determined by its singular locus. With Theorem 1.101 we obtain that any weighted-homogeneous singularity  $f$  with weighted degree  $\deg_{\mathbf{w}}(f) \neq 0$  is determined by its singular locus, since these singularities satisfy  $f \in \mathfrak{m}J_f$ .

**Corollary 1.105.** *Let  $f, g \in \mathbb{C}\{\mathbf{x}\}$  define weighted-homogeneous hypersurface singularities with weight-vector  $\mathbf{0} \neq \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n$  and with weighted degree  $d := \deg_{\mathbf{w}}(f) = \deg_{\mathbf{w}}(g)$ . Denote the singularities defined by  $f$  and  $g$  by  $(X, \mathbf{0})$ , respectively  $(Y, \mathbf{0})$ . Assume that either*

- (i)  $d \neq 0$ , or
- (ii)  $d - w_i \neq 0$  for  $1 \leq i \leq n$ .

*Then the following are equivalent:*

- (1)  $(X, \mathbf{0}) \cong (Y, \mathbf{0})$ .
- (2)  $T_f \cong T_g$  as  $\mathbb{C}$ -algebras.

First we state an example of a singularity, which is weighted-homogeneous, but not covered by Theorem 1.104, but by Corollary 1.105.

**Example 1.106.** Consider the polynomial  $f = x_2x_3x_5 + x_1^6x_5^6 + x_4^6x_5^6 + x_5^{12} + x_2x_4^{13} \in \mathbb{C}\{x_1, x_2, x_3, x_4, x_5\}$ . It is easy to see that  $f$  is weighted-homogeneous with respect to the vector  $\mathbf{w} = (1, -1, 12, 1, 1)$  with weighted degree  $d = 12$  and that  $\deg_{\mathbf{w}}(\partial_{x_3}f) = d - 12 = 0$ . Using Theorem 1.70 one can show that  $(V(f), \mathbf{0})$  is a harmonic singularity.

The next example shows that in case  $\deg_{\mathbf{w}}(f) = 0$  and at least one of the weights being 0, the isomorphism class of a singularity is not determined the isomorphism class of its singular locus.

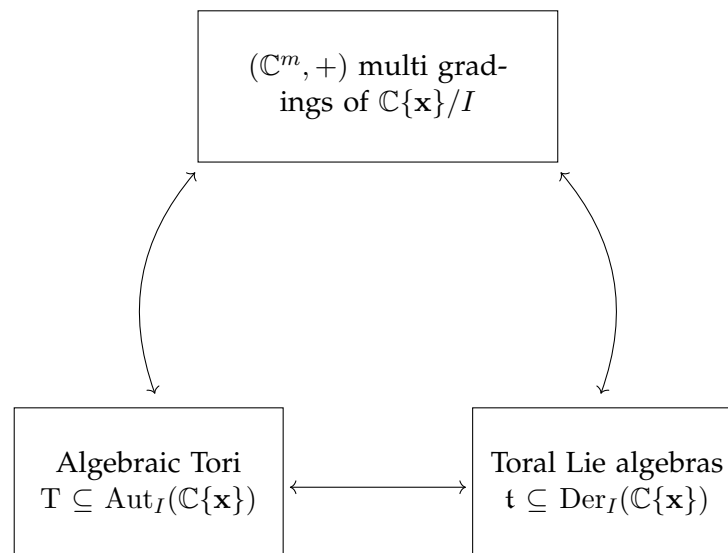
**Example 1.107.** Consider the polynomial  $g = x_1^3x_2 + x_2^5x_3 + x_3^5x_1 + x_1x_2x_3 \in \mathbb{C}\{x_1, x_2, x_3\}$  as in Example 1.103. We already know that  $g \notin \mathfrak{m}J_g$ . Then the polynomial  $f = g + x_4x_5 \in \mathbb{C}\{x_1, \dots, x_5\}$  is weighted-homogeneous with respect to weight vector  $\mathbf{w} = (0, 0, 0, 1, -1) \in \mathbb{Z}^5$  and has weighted degree  $\deg_{\mathbf{w}}(f) = 0$ . It also holds that  $f \notin \mathfrak{m}J_f$ . Next we define the polynomial  $F = f(\mathbf{x}) + (1 + z + t) \cdot f(\mathbf{y}) \in \mathbb{C}\{\mathbf{x}, \mathbf{y}, z, t\}$  as in Example 1.103. Then  $F$  is weighted homogeneous with respect to the weight vector  $\mathbf{v} = (\mathbf{w}, \mathbf{w}, 0, 0) \in \mathbb{Z}^{12}$  and satisfies  $\deg_{\mathbf{v}}(F) = 0$ . Define  $F_{t_0} = F(\mathbf{x}, \mathbf{y}, z, t_0)$ ,  $X := V(F) \subseteq \mathbb{C}^{12}$  and  $X_{t_0} = (V(F_{t_0})) \subseteq \mathbb{C}^{11}$ . As seen in Example 1.103 we conclude that  $F \notin J_F$  and that  $(\text{Sing}(V(F)), \mathbf{0}) \cong (\text{Sing}(V(F_0)) \times V, \mathbf{0})$ , but  $(X, \mathbf{0}) \not\cong (X_0 \times V, \mathbf{0})$ , where  $V$  is an open neighborhood of  $0 \in \mathbb{C}$ .

**Remark 1.108.** The full relationship between harmonic singularities and strongly Euler-homogeneous singularities is unclear to us. By considering a non-quasi homogeneous isolated hypersurface singularity it is easy to see that not every harmonic singularity is strongly Euler-homogeneous in  $\mathbf{0}$ . We do not know if every singularity, which is strongly Euler-homogeneous in  $\mathbf{0}$ , is harmonic.

## Chapter 2

# Gradings of Analytic Algebras and Modules

In the upcoming chapter we are introducing the theory of grading of Zariski rings and modules over Zariski rings by abelian groups as introduced by Scheja and Wiebe (see [SW73]). Due to the fact that every analytic algebra is a Zariski ring we obtain a grading theory for analytic algebras and analytic modules. The main purpose of this thesis is to understand singularities that admit multigradings, that is they are  $(\mathbb{C}^m, +)$ -graded. To understand this grading we first need to understand the interplay between  $(\mathbb{C}, +)$ -gradings, diagonalizable logarithmic derivations and algebraic tori.



**Figure 2.1:** Visualization of the aim of this chapter.

We are going to make use of results regarding derivations presented in Section 1.4.1. Parts of the upcoming chapter, in particular Section 2.1 and Section 2.2 have already been presented in the author's master thesis (see [Epu15]). For this thesis to be self contained we restate them.



## 2.1 Gradings of Rings and Modules

In the following section we state the classical definition of group gradings of rings and modules. Building up on these we present the definition of grading for analytic algebras and analytic modules due to Scheja and Wiebe. For the classical definition of grading in the context of rings or modules, we refer the reader to [GP08, Chapter 2.2]. We start with the basic definition of finitely graded rings and modules:

**Definition 2.1.** Let  $(G, +)$  be an abelian group,  $R$  a ring and  $M$  an  $R$ -module.  $R$  is a **finitely graded ring**, if we have a system of group homomorphisms  $\pi_g^R : R \rightarrow R$  for  $g \in G$  with the property  $\pi_g^R(R)\pi_h^R(R) \subseteq \pi_{g+h}^R(R)$  for all  $g, h \in G$ , such that  $R$  can be written as a direct sum of the subgroups  $\pi_g^R(R)$ , that is  $R = \bigoplus_{g \in G} \pi_g^R(R)$ . Furthermore,  $M$  is a **finitely graded module**, if  $R$  is graded with respect to a system of group homomorphisms  $\pi_g^R, g \in G$  as before, which is compatible with group homomorphisms  $\pi_g^M : M \rightarrow M$ , that is  $\pi_g^R(R)\pi_h^M(M) \subseteq \pi_{g+h}^M(M)$  for all  $g, h \in G$ , such that  $M$  can be written as a direct sum of the subgroups  $\pi_g^M(M)$ , that is  $M = \bigoplus_{g \in G} \pi_g^M(M)$ .

**Remark 2.2.** Definition 2.1 basically extends the well known idea of grading rings in the multivariate polynomial case. Consider for example the polynomial ring  $R := \mathbb{Q}[x_1, \dots, x_n]$ . Using multi-indices  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , we can write any  $f \in R$  as  $f = \sum_{|\alpha|=0}^m c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ , where  $m$  is the total degree of  $f$ . To keep notation short, we write  $f = \sum_\alpha f_\alpha$ , where  $f_\alpha$  denotes the homogeneous degree  $|\alpha|$  part of  $f$ . For more details on the grading of multivariate polynomial rings see [GP08]. Now  $R$  can be written as  $R = \bigoplus_{|\alpha| \geq 0} \mathbb{Q}x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ . If we consider the group  $(G, +) := (\mathbb{Z}, +)$  and the group homomorphisms

$$\begin{aligned} \pi_g : R &\rightarrow R \\ f &\mapsto \begin{cases} 0, & \text{if } g < 0 \\ f_\alpha, & \text{with } |\alpha| = g \end{cases} \end{aligned}$$

We directly get the desired properties of  $(\pi_g)_{g \in G}$  as in Definition 2.1.

The next interesting aspect is the general, not necessarily finite, grading of rings and modules. We start with the definition of Zariski rings (see for example [AM69, Chapter 10, Exercise 6]), as this is the setup in which we are able to define general gradings.

**Definition 2.3.** Let  $R$  be a ring. We say  $R$  is a **Zariski ring**, if  $R$  is a commutative unitary Noetherian topological ring whose topology is defined by an ideal  $\mathfrak{m}$  contained in the Jacobson ideal of  $R$ .

Now we can define gradings for Zariski rings.

**Definition 2.4.** Let  $(G, +)$  be an abelian group,  $R$  a Zariski ring and  $M$  a finitely generated  $R$ -module.  $R$  is a **graded ring** if we have a system of group homomorphisms  $\pi_g^R : R \rightarrow R$  for  $g \in G$ , which induce group homomorphisms  $\overline{\pi}_g^R : R/\mathfrak{m}^n \rightarrow R/\mathfrak{m}^n$  that define a finite grading on  $R/\mathfrak{m}^n$  for all  $n \in \mathbb{N}$ .  $M$  is a **graded module**, if  $R$  is graded with respect to a system of group homomorphisms  $\pi_g^R, g \in G$  as before, which is compatible with group homomorphisms  $\pi_g^M : M \rightarrow M$  which induce group homomorphism  $\overline{\pi}_g^M : M/\mathfrak{m}^n M \rightarrow M/\mathfrak{m}^n M$  that define a finite grading on  $M/\mathfrak{m}^n M$  as an  $R/\mathfrak{m}^n$ -module for all  $n \in \mathbb{N}$ .

**Remark 2.5.** The grading in the sense of Definition 2.4 is basically a grading of  $\mathfrak{m}$ -adic completions, as we reduce the grading of a ring  $R$  to gradings on all  $R/\mathfrak{m}^k$ . The same holds also for modules. We extend this idea to the grading of projective limits in Section 2.2.



**Example 2.6.** Let us consider the ring  $R := \mathbb{Q}[[x_1, \dots, x_n]]$ ,  $\mathfrak{m} := \langle x_1, \dots, x_n \rangle$  and  $(G, +) := (\mathbb{Z}, +)$ . Define  $\pi_g$  as in Remark 2.2, just extended to power series. We get that the  $\pi_g$  induce a finite grading on  $R/\mathfrak{m}^k$  for all  $k \in \mathbb{N}$ , as  $R/\mathfrak{m}^k = \mathbb{Q}[x_1, \dots, x_n]/\langle x_1, \dots, x_n \rangle^k$ . Thus  $R$  is graded in the sense of Definition 2.4.

The following statements generalize basic results of graded modules, as stated for example in [GP08].

**Theorem 2.7.** Let  $R$  be a graded Zariski ring and  $M$  be a graded  $R$ -module with system of group homomorphisms  $(\pi_g^M)_{g \in G}$ . Every  $m \in M$  can be written as  $m = \sum_{g \in G} \pi_g^M(m)$ . If  $m = \sum_{g \in G} m_g$  with  $m_g \in \pi_g^M(M)$ , then we already have  $m_g = \pi_g^M(m)$  for all  $g \in G$ .  $m_g$  is called the  $g$ -th homogeneous component of  $m$ .

*Proof.* See [SW73, (1.1)]. □

**Proposition 2.8.** Let  $R$  be a graded Zariski ring and  $M$  a graded  $R$ -module with systems of group homomorphisms  $(\pi_g^R)_{g \in G}$  respectively  $(\pi_g^M)_{g \in G}$ . Then for all  $g, h \in G$  it holds that:  $\pi_g^2 = \pi_g$ ,  $\pi_g \circ \pi_h = 0$ , if  $g \neq h$ , and  $\pi_g^R(R)\pi_h^M(M) \subseteq \pi_{g+h}^M(M)$ .

*Proof.* See [SW73, (1.2)]. □

The next natural step is to consider submodules of graded modules.

**Definition 2.9.** Let  $R$  be a graded Zariski ring,  $M$  be a graded  $R$ -module and  $N$  a subgroup of  $M$ .  $N$  is called **homogeneous submodule** of  $M$ , if  $\pi_g^M(N) \subseteq N$  for all  $g \in G$ .

The following three theorems characterize homogeneous submodules, resulting quotient modules and their grading.

**Theorem 2.10.** Let  $R$  be a graded Zariski ring,  $M$  be a graded  $R$ -module and  $N$  a submodule of  $M$ .  $N$  is homogeneous if and only if  $N$  can be generated by homogeneous elements.

*Proof.* See [SW73, (1.3)]. □

**Theorem 2.11.** Let  $R$  be a graded Zariski ring and  $M$  be a graded  $R$ -module with system of group homomorphisms  $(\pi_g^M)_{g \in G}$  and  $N$  a homogeneous submodule of  $M$ . Then the group homomorphisms  $\pi_g^M|_N : N \rightarrow N$ ,  $g \in G$ , induce a grading of  $N$  as an  $R$ -module.

*Proof.* See [SW73, (1.4)]. □

**Theorem 2.12.** Let  $R$  be a graded Zariski ring and  $M$  be a graded  $R$ -module with system of group homomorphisms  $(\pi_g^M)_{g \in G}$  and  $N$  a homogeneous submodule of  $M$ . Then the group homomorphisms  $\overline{\pi_g^M} : M/N \rightarrow M/N$ ,  $g \in G$ , induce a grading of  $M/N$  as an  $R$ -module.

*Proof.* See [SW73, (1.5)]. □

The next natural setup we can consider is the product of abelian groups, which yields so-called multigradings.

**Definition 2.13.** Let  $R$  be a Zariski ring and  $M$  an  $R$ -module. We say  $R$  is a **multigraded ring** if there exists an  $m \in \mathbb{N}_{\geq 1}$  and abelian groups  $G_1, \dots, G_m$ , such that  $R$  is graded with respect to  $G := G_1 \times \dots \times G_m$  in the sense of Definition 2.1. We say  $M$  is a **multigraded module** if there exists an  $m \in \mathbb{N}_{\geq 1}$  and abelian groups  $G_1, \dots, G_m$ , such that  $R$  and  $M$  are graded with respect to  $G := G_1 \times \dots \times G_m$  in the sense of Definition 2.1.

Our next results characterize the property of a Zariski ring  $R$  being multigraded.

**Proposition 2.14.** Let  $(G_1, +), \dots, (G_m, +)$  be abelian groups,  $R$  a Zariski ring and  $m \in \mathbb{N}_{\geq 1}$ . Furthermore, we denote for any  $k \in \mathbb{N}$  the natural projection  $R \twoheadrightarrow R/\mathfrak{m}_R^k$  by  $\pi_k$ . Then  $R$  is graded by  $(G_1 \times \dots \times G_m, +)$  with group homomorphism  $\Psi_{(g_1, \dots, g_m)}, (g_1, \dots, g_m) \in G_1 \times \dots \times G_m$ , if and only if there exist pairwise commuting group homomorphisms  $\pi_{g_1}, \dots, \pi_{g_m}$ ,  $g_i \in G_i$ , grading  $R$ , satisfying

$$R/\mathfrak{m}_R^k = \bigoplus_{g_1 \in G_1} \dots \bigoplus_{g_m \in G_m} \overline{\pi_{g_1}^R}(R/\mathfrak{m}_R^k) \cap \dots \cap \overline{\pi_{g_m}^R}(R/\mathfrak{m}_R^k)$$

for all  $k \in \mathbb{N}$ . In particular,  $\Psi_{(g_1, \dots, g_m)} = \pi_{g_m} \circ \dots \circ \pi_{g_1}$  for all  $(g_1, \dots, g_m) \in G_1 \times \dots \times G_m$ .

*Proof.* See [Epu15, Corollary 4.27]. □

The next proposition is the analogous result for multigraded modules.

**Proposition 2.15.** Let  $(G_1, +), \dots, (G_m, +)$  be abelian groups,  $m \in \mathbb{N}_{\geq 1}$ ,  $R$  a graded Zariski ring and  $M$  an  $R$ -module. Then  $M$  is graded by  $(G_1 \times \dots \times G_m, +)$  with group homomorphism  $\Psi_{(g_1, \dots, g_m)}^M, (g_1, \dots, g_m) \in G_1 \times \dots \times G_m$ , if and only if there exist pairwise commuting group homomorphisms  $\pi_{g_1}^M, \dots, \pi_{g_m}^M$ ,  $g_i \in G_i$  grading  $M$  and  $\psi_{g_1}^R, \dots, \psi_{g_m}^R$  the corresponding gradings of  $R$  satisfying

$$R/\mathfrak{m}_R^k = \bigoplus_{g_1 \in G_1} \dots \bigoplus_{g_m \in G_m} \overline{\psi_{g_1}^R}(R/\mathfrak{m}_R^k) \cap \dots \cap \overline{\psi_{g_m}^R}(R/\mathfrak{m}_R^k)$$

and

$$M/\mathfrak{m}^k M = \bigoplus_{g_1 \in G_1} \dots \bigoplus_{g_m \in G_m} \overline{\pi_{g_1}^M}(M/\mathfrak{m}^k M) \cap \dots \cap \overline{\pi_{g_m}^M}(M/\mathfrak{m}^k M)$$

for all  $k \in \mathbb{N}$ . In particular,  $\Psi_{(g_1, \dots, g_m)}^M = \pi_{g_m}^M \circ \dots \circ \pi_{g_1}^M$  and  $\Psi_{(g_1, \dots, g_m)}^R = \psi_{g_m}^R \circ \dots \circ \psi_{g_1}^R$  for all  $(g_1, \dots, g_m) \in G_1 \times \dots \times G_m$ , where the latter is the corresponding grading of  $R$ .

## 2.2 $(\mathbb{C}^m, +)$ Gradings of Analytic Algebras and Derivations

In this section we present the connection of  $(\mathbb{C}^m, +)$  gradings of analytic algebras and derivations. We start with the classical results by Scheja and Wiebe connecting derivations to  $(\mathbb{C}, +)$  gradings of analytic algebras and then state the analogous results for  $(\mathbb{C}^m, +)$  gradings. Before we start with the result we need to define the notion of lifted, respectively projected, derivations.

**Definition 2.16.** Let  $A$  and  $B$  be analytic algebras and assume there exists a surjective map  $\pi : A \twoheadrightarrow B$ . Let  $\delta \in \text{Der}(A)$  and  $\tilde{\delta} \in \text{Der}(B)$ . We call  $\delta$  a **lifting of  $\tilde{\delta}$** , respectively  $\tilde{\delta}$  a **projection of  $\delta$** , if there exists  $x_1, \dots, x_n \in A$  with  $\mathfrak{m}_A = \langle x_1, \dots, x_n \rangle$  and  $\mathfrak{m}_B = \langle \pi(x_1), \dots, \pi(x_n) \rangle$ , such that  $\tilde{\delta} \circ \pi(x_i) = \pi \circ \delta(x_i)$  for  $i = 1, \dots, n$ .

**Definition 2.17.** Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $A = \mathbb{C}[[\mathbf{x}]]$  and let  $\delta = \sum_{i=1}^n a_i \partial_{x_i} \in \text{Der}'(A)$ . Denote by  $a_{ij} \in \mathbb{C}, 1 \leq i, j \leq n$  the uniquely determined complex numbers such that

$$a_i - \sum_{j=1}^n a_{ij} x_j \in \mathfrak{m}_A^2 \text{ for } 1 \leq i \leq n.$$

We call the matrix  $A = (a_{ij}) \in \mathbb{C}^{n \times n}$  the **representation matrix** of  $\delta$  and the derivation  $\delta_0 := \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_j \partial_{x_i}$  the **linear part** of  $\delta$ .

In case we have two derivations which equal their linear part, we can compute their Lie bracket by computing the Lie bracket of their respective representation matrices.

**Lemma 2.18.** Let  $A$  be an analytic algebra and  $\delta, \epsilon \in \text{Der}'(A)$ . Assume  $\mathfrak{m}_A$  has a minimal set of generators  $x_1, \dots, x_n$  for some  $n \in \mathbb{N}$ ,  $\delta = \sum_{i=1}^n \lambda_i x_i \partial_{x_i}$  and  $\epsilon = \epsilon_0$ . Then  $[\delta, \epsilon] = \mathbf{x}[M_\delta, M_\epsilon] \partial^T$ , where  $M_\delta, M_\epsilon \in \mathbb{C}^{n \times n}$  are the representation matrices of the linear parts of  $\delta$  respectively  $\epsilon$ .

*Proof.* See [GS06, Lemma 2.2]. □

The first two theorems are very important, as they state that every grading of an analytic algebra arises from a derivation and vice versa.

**Theorem 2.19.** Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$ , such that  $\mathfrak{m}_A$  has a system of generators containing only eigenvectors of  $\delta$ . Then there exists a unique  $(\mathbb{C}, +)$  grading  $\pi_g$  of  $A$ ,  $g \in \mathbb{C}$ , such that each  $\pi_g^A(A)$  contains only  $g$ -eigenvectors of  $\delta$ .

*Proof.* See [SW73, (2.2)]. □

**Theorem 2.20.** Let  $A$  be an analytic algebra and let  $\pi_g^A, g \in \mathbb{C}$ , be a  $(\mathbb{C}, +)$  grading of  $A$ . Then there exists a unique diagonalizable derivation  $\delta \in \text{Der}'(A)$ , such that each  $\pi_g(A)$  contains only  $g$ -eigenvectors of  $\delta$ .

*Proof.* See [SW73, (2.3)]. □

**Remark 2.21.** By Theorem 2.19 and 2.20 the diagonalizable derivations are in one-to-one correspondence with the  $(\mathbb{C}, +)$  gradings of analytic algebras.

The next theorems are crucial in an application of the Formal Structure Theorem, which we are going to state in Section 2.4.2.

**Theorem 2.22.** Let  $A$  be a  $(\mathbb{C}, +)$  graded analytic algebra. Furthermore, let  $I$  be an ideal of  $A$  and  $\delta \in \text{Der}'(A)$  be the derivation corresponding to the grading. Then  $I$  is homogeneous, if and only if  $I$  is  $\delta$ -invariant.

*Proof.* See [SW73, (2.4)]. □

**Theorem 2.23.** Let  $A$  be an analytic algebra,  $I$  be an ideal of  $A$  and  $\delta \in \text{Der}'(A)$ . If  $I$  is  $\delta$ -invariant, then every associated prime ideal  $P$  of  $I$  is  $\delta$ -invariant.

*Proof.* See [SW73, (2.5)]. □

The next theorem in this section is a surprising result, which states that we can write every diagonalizable derivation as a finite sum of diagonalizable derivations with rational eigenvalues.

**Theorem 2.24.** *Let  $A$  be an analytic algebra and let  $\delta \in \text{Der}'(A)$  be diagonalizable. Then there exist diagonalizable  $\delta_j \in \text{Der}'(A) \setminus \{0\}$  and  $a_j \in \mathbb{C}, j = 1, \dots, m$  for some  $m \in \mathbb{N}$ , such that  $\delta = \sum_{j=1}^m a_j \delta_j$ , every  $\delta_j$  has the same eigenvectors as  $\delta$  and the  $\delta_j$  have only rational eigenvalues.*

*Proof.* See [SW73, (3.2)]. □

The last lemma in this section characterizes diagonalizable and nilpotent derivations by their linear part.

**Lemma 2.25.** *Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$ . Then  $\delta$  is diagonalizable if and only if there exists a set of coordinates such that  $\delta = \delta_0$  and the representation matrix is diagonalizable.  $\delta$  is nilpotent if and only if  $\delta_0$  is nilpotent.*

*Proof.* We prove the theorem for analytic algebras of type  $A = \mathbb{C}\{\mathbf{x}\}/I$  for some ideal  $I \subseteq \mathbb{C}\{\mathbf{x}\}$ . The complete case works analogously. We start with the statement regarding diagonalizability. First assume  $\delta$  is diagonalizable. Then Theorem 2.22 implies that there exists a lift  $\tilde{\delta}$  of  $\delta$  in  $\text{Der}'(\mathbb{C}\{\mathbf{x}\})$  satisfying  $\tilde{\delta}(I) \subseteq I$ . Then there exists a set of coordinates, say  $x_1, \dots, x_n$ , for some  $n \in \mathbb{N}$ , such that  $\mathfrak{m}_{\mathbb{C}\{\mathbf{x}\}} = \langle x_1, \dots, x_n \rangle$  with the property that there exist  $\lambda_i \in \mathbb{C}$ , such that  $\tilde{\delta}(x_i) = \lambda_i x_i$ . By Theorem 1.43 we get that  $\tilde{\delta} = \sum_{i=1}^n \lambda_i x_i \partial_{x_i}$ , hence  $\tilde{\delta} = \tilde{\delta}_0$  and the representation matrix is obviously diagonalizable. Passing to  $A$  we obtain the same result for  $\delta$ . Now if  $\delta = \delta_0$  and the representation matrix is diagonalizable, then there exists a linear coordinate change, such that  $\delta$  is of type  $\sum_{i=1}^n \lambda_i x_i \partial_{x_i}$  for a set of coordinates  $x_1, \dots, x_n$ , some  $\lambda_i \in \mathbb{C}$  and some  $n \in \mathbb{N}$ . Then  $\delta$  is obviously diagonalizable. The statement for nilpotency follows immediately from Lemma 1.51. □

In the last part of we want to exhibit the connection between multigradings and pairwise commuting diagonalizable derivations. The following theorems are the multi-graded analogues to Theorem 2.19 and Theorem 2.20.

**Theorem 2.26.** *Let  $A$  be an analytic algebra and  $\delta_1, \dots, \delta_m \in \text{Der}'(A)$  pairwise commuting diagonalizable derivations. Then there exists a grading of  $A$  with group homomorphisms  $\pi_g^A, g \in \mathbb{C}^m$ , such that each  $\pi_g^A(A)$  contains only common eigenvectors of  $\delta_1, \dots, \delta_m$ .*

*Proof.* [Epu15, Theorem 4.35] □

**Theorem 2.27.** *Let  $A$  be a  $(\mathbb{C}^m, +)$  multigraded analytic algebra, where the grading is induced by group homomorphisms  $\pi_g^A, g \in \mathbb{C}^m$ . Then there exist pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_m \in \text{Der}'(A)$ , such that each  $\pi_g^A(A)$  contains only common eigenvectors of  $\delta_1, \dots, \delta_m$ .*

*Proof.* It suffices to consider the case  $m = 2$ . Proposition 2.14 translates to the fact that  $A/\mathfrak{m}_A^k$  decomposes as a direct sum of common eigenvectors of the induced derivations. In particular, it holds that  $\delta_1 \circ \delta_2(x) = \delta_2 \circ \delta_1(x) \pmod{\mathfrak{m}_A^k}$  for all  $k \in \mathbb{N}$ . This implies that  $\delta_1 \circ \delta_2 = \delta_2 \circ \delta_1$ . □

Using these results we can define the following notions of graded objects using derivations.

**Definition 2.28.** Let  $A$  be an analytic algebra and  $\delta \in \text{Der}'(A)$  diagonalizable. We call an element  $f \in A$   $\delta$ -**homogeneous of degree**  $\lambda$  or **quasi-homogeneous**, if  $\delta(f) = \lambda \cdot f$  for some  $\lambda \in \mathbb{C}$ . If we have a set of diagonal and commuting derivations, say  $\delta_1, \dots, \delta_m$ , for some  $s \in \mathbb{N}$ , we call  $f$   $\Lambda$ -**multihomogeneous**, if for all  $1 \leq j \leq m$  there exist  $\lambda_j \in \mathbb{C}$  with  $\delta_j(f) = \lambda_j \cdot f$ , where  $\Lambda := (\lambda_1, \dots, \lambda_m)$ . We say  $A$  is  $\delta$ -**graded** or just **graded**, if  $\delta \in \text{Der}'(A)$  and  $\delta$  is diagonalizable. We say  $A \cong \mathbb{C}\{\mathbf{x}\}/I$  is **multigraded** with respect to  $\delta_1, \dots, \delta_m$  if  $\delta_j \in \text{Der}'(A)$  are diagonalizable for  $j = 1, \dots, s$  and commute pairwise. We call a complex space germ  $(X, \mathbf{0})$  **graded**, respectively **multigraded**, if the corresponding analytic algebra  $\mathcal{O}_{X, \mathbf{0}}$  is isomorphic to a graded, respectively multigraded analytic algebra.

We finish this section by applying the theory of gradings through derivations to investigate the compatibility of gradings with suspensions.

We start with the following lemma.

**Lemma 2.29.** Let  $A := \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  and  $A' := \mathbb{C}\{\mathbf{x}\}$  and let  $I \subseteq A$  and  $I' \subseteq A'$  be ideals. Assume there exists an isomorphism  $\varphi \in \text{Aut}(A)$  with  $\varphi(I) = AI'$ . Then the map  $\Phi : \text{Der}'_I \rightarrow \text{Der}'_{AI'}(A)$ ,  $\delta \mapsto \varphi \circ \delta \circ \varphi^{-1}$  is a bijection. In particular,  $\Phi$  maps diagonalizable derivations to diagonalizable derivations.

*Proof.* Consider the map  $\Psi : \text{Der}'_{AI'}(A) \rightarrow \text{Der}'_I$ ,  $\delta \mapsto \varphi^{-1} \circ \delta \circ \varphi$ . In case  $\Phi$  and  $\Psi$  are well-defined, they are obviously inverse maps for each other. Thus we just have to check well-definedness. It suffices to show the well-definedness in case of  $\Phi$ , since  $\Psi$  works analogously. The map  $\tilde{\delta} = \varphi \circ \delta \circ \varphi^{-1}$  is a derivation, as one can easily compute. It only remains to show that  $\delta(I) \subseteq I$  implies  $\tilde{\delta}(AI') \subseteq AI'$ . To show this, let  $f \in AI'$  be arbitrary. Then  $\varphi^{-1}(f) \in I$ , hence  $\delta \circ \varphi^{-1}(f) \in I$  and thus  $\tilde{\delta}(f) = \varphi \circ \delta \circ \varphi^{-1}(f) \in AI'$ .  $\square$

Lemma 2.29 allows us to relate derivations in  $\text{Der}_{AI'}(A)$  and  $\text{Der}_{I'}(A')$ . We capture this in the following definition.

**Definition 2.30.** Let  $A := \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  and  $A' := \mathbb{C}\{\mathbf{x}\}$  and let  $I \subseteq A$  and  $I' \subseteq A'$  be ideals. Let  $\delta \in \text{Der}'_I(A)$  and  $\tilde{\delta} = \sum_{i=1}^n a_i(\mathbf{x}, \mathbf{y}) \partial_{x_i} + \sum_{j=1}^m b_j(\mathbf{x}, \mathbf{y}) \partial_{y_j} \in \text{Der}_{AI'}(A)$  be the image under the map  $\Phi$  from Lemma 2.29. We call the derivation  $\bar{\delta} = \sum_{i=1}^n a_i(\mathbf{x}, \mathbf{0}) \partial_{x_i} \in \text{Der}_{I'}(A')$  the **projection of**  $\tilde{\delta}$ . Given a derivation  $\delta = \sum_{i=1}^n a_i(\mathbf{x}) \partial_{x_i} \in \text{Der}_{I'}(A')$ , we call the derivation  $\tilde{\delta} = \sum_{i=1}^n a_i(\mathbf{x}) \partial_{x_i} \in \text{Der}_{AI'}(A)$  the **suspension of**  $\delta$ .

**Proposition 2.31.** Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^{n+m}, \mathbf{0})$  be a complex space germ. Assume that  $(X, \mathbf{0}) \cong (X', \mathbf{0}) \times (\mathbb{C}^m, \mathbf{0})$ . If  $(X', \mathbf{0})$  is graded by  $\tilde{\delta} \in \text{Der}'(\mathcal{O}_{X', \mathbf{0}})$ , then there exists a suspension  $\delta \in \text{Der}'(\mathcal{O}_{X, \mathbf{0}})$  of  $\tilde{\delta}$ , such that  $(X, \mathbf{0})$  is graded by  $\delta$ .

*Proof.* Define  $A := \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  and  $A' := \mathbb{C}\{\mathbf{x}\}$ . Assume  $\mathcal{O}_{X, \mathbf{0}} \cong A/I$ , where  $I := \langle f_1, \dots, f_k \rangle \subseteq A$  and  $\mathcal{O}_{X', \mathbf{0}} \cong A'/I'$ , where  $I' := \langle g_1, \dots, g_l \rangle \subseteq A'$ . We consider the lifted situation. Due to the fact that  $A/I \cong A'/I' \hat{\otimes} \mathbb{C}\{\mathbf{y}\}$ , we know by Lemma 1.11 that there exists an isomorphism  $\varphi : A \rightarrow A$  satisfying  $\varphi(I'A) = I$ .

Since  $(X', \mathbf{0})$  is graded by  $\tilde{\delta} \in \text{Der}'(\mathcal{O}_{X', \mathbf{0}})$  there exists a diagonalizable  $\tilde{\delta}' \in \text{Der}'_{I'}(A')$ . Denote by  $x'_1, \dots, x'_n$  a minimal generating system of  $\mathfrak{m}_{A'}$  satisfying  $\tilde{\delta}'(x'_i) = \lambda_i x'_i$  for certain  $\lambda_i \in \mathbb{C}$ . Define  $\tilde{\delta}'' = \sum_{i=1}^n \lambda_i x'_i \partial_{x'_i} \in \text{Der}'(A)$ . By Lemma 2.29 there exists a diagonalizable derivation  $\delta \in \text{Der}'(A)$  with  $\delta(I) \subseteq I$ , hence  $(X, \mathbf{0})$  is graded.  $\square$

The proof of Proposition 2.31 also works in the multigraded case, if we assume that the pairwise commuting diagonalizable derivations can be simultaneously diagonalized.

**Proposition 2.32.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^{n+m}, \mathbf{0})$  be a complex space germ. Assume that  $(X, \mathbf{0}) \cong (X', \mathbf{0}) \times (\mathbb{C}^m, \mathbf{0})$ . If  $(X', \mathbf{0})$  is multigraded by the diagonal derivations  $\tilde{\delta}_1, \dots, \tilde{\delta}_m \in \text{Der}'(\mathcal{O}_{X', \mathbf{0}})$ , then there exists liftings  $\delta_1, \dots, \delta_m \in \text{Der}'(\mathcal{O}_{X, \mathbf{0}})$  of the  $\tilde{\delta}_i$ , such that  $(X, \mathbf{0})$  is multigraded.*

*Proof.* The proof for the multigraded case works in the same as the proof of Proposition 2.31 by using a multihomogeneous generating systems.  $\square$

The converse of the previous propositions does not hold in general.

**Example 2.33.** *Let  $f \in \mathbb{C}\{x, y\}$  be any power series, which is not homogeneous and not a unit. In this case  $(V(f), \mathbf{0})$  is not graded. In case we consider  $f$  extended to  $\mathbb{C}\{x, y, z\}$ , then  $f$  is homogeneous with respect to the derivation  $\delta = z\partial_z$ , which implies that  $(V(f) \times \mathbb{C}, \mathbf{0})$  is graded. This shows that Proposition 2.31 cannot hold in general.*

**Definition 2.34.** *Let  $A := \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  and  $A' := \mathbb{C}\{\mathbf{x}\}$  and let  $I \subseteq A$  and  $I' \subseteq A'$  be ideals. Let  $\delta \in \text{Der}'_I(A)$ . We call  $\delta$  **suspension compatible**, if  $\bar{\delta}$  is not the zero derivation.*

Using suspension compatible derivations we can state converse statements to Proposition 2.31 and 2.32.

**Proposition 2.35.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^{n+m}, \mathbf{0})$  be a complex space germ. Assume that  $(X, \mathbf{0}) \cong (X', \mathbf{0}) \times (\mathbb{C}^m, \mathbf{0})$ . If  $(X, \mathbf{0})$  is graded by a suspension compatible  $\delta \in \text{Der}'(\mathcal{O}_{X, \mathbf{0}})$ , then  $(X', \mathbf{0})$  is graded by the projection  $\bar{\delta} \in \text{Der}'(\mathcal{O}_{X', \mathbf{0}})$ .*

*Proof.* Define  $A := \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  and  $A' := \mathbb{C}\{\mathbf{x}\}$ . Assume  $\mathcal{O}_{X, \mathbf{0}} \cong A/I$ , where  $I := \langle f_1, \dots, f_k \rangle \subseteq \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  and  $\mathcal{O}_{X', \mathbf{0}} \cong A'/I'$ , where  $I' := \langle g_1, \dots, g_l \rangle \subseteq \mathbb{C}\{\mathbf{x}\}$ . We consider the lifted situation. Due to our assumption,  $(X, \mathbf{0})$  is graded with grading induced by  $\delta \in \text{Der}'(A)$  with  $\delta(I) \subseteq I$ . By Lemma 2.29, there exists a diagonalizable derivation  $\tilde{\delta} \in \text{Der}_{A'}(A)$ . The fact that  $\bar{\delta}$  is not the zero derivation implies that  $\tilde{\delta}$  is not the zero-derivation, hence  $(X', \mathbf{0})$  is graded.  $\square$

**Proposition 2.36.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^{n+m}, \mathbf{0})$  be a complex space germ. Assume that  $(X, \mathbf{0}) \cong (X', \mathbf{0}) \times (\mathbb{C}^m, \mathbf{0})$ . If  $(X, \mathbf{0})$  is multigraded by the suspension compatible diagonal derivations  $\delta_1, \dots, \delta_m \in \text{Der}'(\mathcal{O}_{X, \mathbf{0}})$ , then  $(X', \mathbf{0})$  is multigraded by the respective projections  $\bar{\delta}_1, \dots, \bar{\delta}_m$ .*

*Proof.* The proof for the multigraded case works in the same way as the proof of Proposition 2.35 by using a multihomogeneous generating systems.  $\square$

## 2.3 Linear Algebraic Subgroups of $\text{Aut}(A)$

In the following section we present the connection between algebraic tori and the automorphism group of an analytic algebra  $A = \mathbb{C}\{\mathbf{x}\}$  or  $A = \mathbb{C}[[\mathbf{x}]]$ , as well as the connection to derivations. Therefore we are going to use the theory of linear algebraic groups and Lie algebras. A good reference for the theory of linear algebraic groups is [Hum75] or for a more modern approach [Mil17]. In the theory of linear



algebraic groups, it is a well known result that we can associate to each linear algebraic group a Lie algebra (see for example [Hum75]). In the following section we state results about the subgroups  $\text{Aut}_I(A) := \{\varphi \in \text{Aut}(A) \mid \varphi(I) = I\}$  of  $\text{Aut}(A)$ , where  $I$  is an ideal of  $A$ . We show that we can associate, in a formal sense, the Lie algebra  $\text{Der}'_I(A) := \text{Der}'(A) \cap \text{Der}_I(A)$  to these algebraic groups. We also show that the canonical projection to  $A \rightarrow A/\mathfrak{m}_A^2$  induces a bijection of reductive linear algebraic groups. We use this bijection to prove that the dimension of maximal algebraic tori is an invariant of an analytic algebra  $A/I$  and that it is connected to the maximal possible  $(\mathbb{C}^k, +)$ -grading of  $A/I$ . Before we start with our results, we fix the notation for this section.

**Notation 2.37.** From now on  $A$  either denotes  $\mathbb{C}\{\mathbf{x}\}$  or  $\mathbb{C}[[\mathbf{x}]]$ . The maximal ideal of  $A$  will always be denoted by  $\mathfrak{m}_A$ . We denote by  $\pi_k$  the canonical projection  $A \rightarrow A/\mathfrak{m}_A^{k+1}$ . By abuse of notation the induced projection  $\text{Der}'(A) \rightarrow \text{Der}'(A/\mathfrak{m}_A^{k+1})$  and the induced projection  $\text{Aut}(A) \rightarrow \text{Aut}(A/\mathfrak{m}_A^{k+1})$  are also denoted by  $\pi_k$ .

### 2.3.1 Linear Reductive Subgroups of $\text{Aut}_I(A)$

The aim of this subsection is to understand linear algebraic subgroups of  $\text{Aut}(A)$ . After stating general result, we focus on algebraic tori. In order to have precise notion of the latter we adapt [Mül86, Definition] to our setup.

**Definition 2.38.**

- (1) The group  $\text{Aut}_I(A) := \{\varphi \in \text{Aut}(A) \mid \varphi(I) = I\}$  is called the **group of  $I$ -invariant automorphisms**.
- (2) Let  $G$  be an linear algebraic group. A homomorphism  $\alpha : G \rightarrow \text{Aut}_I(A)$  is called **rational action**, if all  $\pi_k \circ \alpha$  are morphisms of algebraic groups.
- (3) A subgroup  $G \subseteq \text{Aut}_I(A)$  isomorphic to a linear algebraic group is called **linear algebraic subgroup of  $\text{Aut}_I(A)$** , if the injection  $G \hookrightarrow \text{Aut}_I(A)$  is a rational action.

**Remark 2.39.** From now on we drop the term linear, since all algebraic groups that will appear from now on are linear.

The first important result we need, is the fact that we can always find a coordinate system for  $A$ , such that reductive algebraic groups act linearly. To prove this result we need the following lemma.

**Lemma 2.40.** Let  $I \subseteq A$  be an ideal and let  $G \subseteq \text{Aut}_I(A)$  be a reductive algebraic subgroup. Then there exists a minimal generating system  $f_1, \dots, f_m$  of  $I$ , such that the  $\mathbb{C}$ -vector space  $\langle f_1, \dots, f_m \rangle_{\mathbb{C}}$  is  $G$ -invariant.

*Proof.* See [Mül86, Hilfssatz 2]. □

**Proposition 2.41.** Let  $I \subseteq A$  be an ideal and let  $G \subseteq \text{Aut}_I(A)$  be a reductive algebraic subgroup. Then there exists a coordinate system  $x_1, \dots, x_n$  of  $A$  such that  $G$  acts linearly.

*Proof.* We apply Lemma 2.40 to the ideal  $\mathfrak{m}_A$ . Then we have a coordinate system  $x_1, \dots, x_n$ , such that the  $\mathbb{C}$ -vector space  $V := \langle x_1, \dots, x_n \rangle_{\mathbb{C}}$  is  $G$ -invariant. Thus for every  $g \in G$  there exists a matrix  $A_g$ , such that

$$g \cdot \mathbf{x} = A_g \mathbf{x}.$$

This defines a representation  $\rho : G \rightarrow \mathrm{GL}(V)$ ,  $g \mapsto A_{g^{-1}}$ . By construction this induces a rational action  $\alpha : G \rightarrow \mathrm{Aut}_I(A)$ ,  $g \mapsto \rho(g)$ .  $\square$

Since every representation of a reductive group is semi-simple (see [Mil17, Corollary 22.43]), we can use [Kau67, Satz] to obtain that  $\pi_1$  restricted to a linear reductive subgroup of  $\mathrm{Aut}(A)$  is injective.

**Theorem 2.42.** *Let  $G \subseteq \mathrm{Aut}(A)$  be an algebraic reductive subgroup. Then the map  $\pi_1 : G \rightarrow \mathrm{Aut}(A/\mathfrak{m}_A^2)$  is an injection.*

This result generalizes to  $\pi_k$  for all  $k \in \mathbb{N}$ .

**Corollary 2.43.** *Let  $G \subseteq \mathrm{Aut}(A)$  be a reductive algebraic subgroup. Then the map  $\pi_k : \mathrm{Aut}(A) \rightarrow \mathrm{Aut}(A/\mathfrak{m}_A^{k+1})$  restricted to  $G$  is injective.*

*Proof.* It follows from Theorem 2.42 that  $\pi_1$  restricted to  $G$  is injective. Due to the fact that we are working with a projective system, we have  $\pi_1 = \pi_1^k \circ \pi_k$ . Then  $\pi_k$  restricted to  $G$  is injective for any  $k \in \mathbb{N}$ .  $\square$

The next important result is the lifting of conjugacy of reductive algebraic subgroups of  $\mathrm{Aut}_I(A)$ .

**Theorem 2.44.** *Let  $I \subseteq A$  be an ideal and  $G, H \subseteq \mathrm{Aut}_I(A)$  be reductive algebraic subgroups. Assume there exists a  $\varphi_1 \in \pi_1(\mathrm{Aut}_I(A))$ , such that  $\pi_1(G) = \varphi_1 \pi_1(H) \varphi_1^{-1}$ . Then there exists a  $\varphi \in \mathrm{Aut}_I(A)$ , such that  $G = \varphi H \varphi^{-1}$ .*

*Proof.* See [Mül86, Satz 2].  $\square$

From now on we will focus on algebraic tori. By [Mil17, Proposition 12.54] algebraic tori are reductive. It is a classical result that so-called maximal algebraic tori are conjugated, see for example [Hum75, Corollary 21.3, A]. We want to prove the analogous result in our setup.

**Definition 2.45.** *Let  $I \subseteq A$  be an ideal and  $T \subseteq \mathrm{Aut}_I(A)$  an algebraic torus. We call  $T$  a **maximal algebraic torus** if for any algebraic torus  $T' \subseteq \mathrm{Aut}_I(A)$  containing  $T$  we have  $T = T'$ .*

Theorem 2.44 implies the following result.

**Corollary 2.46.** *Let  $I \subseteq A$  be an ideal and let  $T, T' \subseteq \mathrm{Aut}_I(A)$  be maximal algebraic tori. Then there exists a  $\varphi \in \mathrm{Aut}_I(A)$ , such that  $\varphi T' \varphi^{-1} = T$ .*

In the case of algebraic tori we can make Proposition 2.41 even more explicit.

**Proposition 2.47.** *Let  $I \subseteq A$  be an ideal. Furthermore, let  $T \subseteq \mathrm{Aut}_I(A)$  be an algebraic torus. Then there exists a coordinate system  $x_1, \dots, x_n$ , such that  $T$  acts via characters on  $A$ .*



*Proof.* Assume that we have  $T$  is a  $k$ -dimensional algebraic torus. To keep the notation simple, we assume that  $T = (\mathbb{C}^*)^k$ . For any  $t \in T$  we denote by  $t_i$  the  $i$ -th component of  $t$ . Applying Lemma 2.40 to the maximal ideal  $\mathfrak{m}_A$ , we know that there exists a minimal generating set  $y_1, \dots, y_n$  of  $\mathfrak{m}_A$ , such that the vector space  $V := \langle y_1, \dots, y_n \rangle_{\mathbb{C}}$  is  $T$ -invariant. This implies the existence of a group representation  $\rho : T \rightarrow \mathrm{GL}(V)$ . By [Mil17, Theorem 12.12] we can write

$$V = \bigoplus_{\chi \in X(T)} V_{\chi}, \quad (2.1)$$

where  $X(T)$  denotes the character group of  $T$  and  $V_{\chi}$  the eigenspace for  $T$  with character  $\chi \in X(T)$ , see for example [Mil17, Chapter 4, g]. It is known that  $X(T) = \mathbb{Z}^k$ , see for example [Mil17, Chapter 12, e]. Thus we can rewrite Equation 2.1 as

$$V = \bigoplus_{\Lambda \in \mathbb{Z}^k} V_{\Lambda}. \quad (2.2)$$

Let  $\Lambda = (\lambda_1, \dots, \lambda_k)$ . Then  $t = (t_1, \dots, t_k)$  acts via  $\rho(t)(v) = t_1^{-\lambda_1} \cdot \dots \cdot t_k^{-\lambda_k} v$  on  $V_{\Lambda}$ . Thus there exists a basis consisting of eigenvectors for  $V$  with respect to this action. Let  $x_1, \dots, x_n$  be the basis vectors. Then  $\mathfrak{m}_A = \langle x_1, \dots, x_n \rangle$  and we obtain that  $T$  acts via characters in these coordinates.  $\square$

We finish this section by showing that algebraic tori induce  $(\mathbb{Z}^k, +)$  gradings of  $A/I$ .

**Theorem 2.48.** *Let  $I \subseteq A$  be an ideal and let  $T \subseteq \mathrm{Aut}_I(A)$  be a  $k$ -dimensional algebraic torus. Then there exists a  $(\mathbb{Z}^k, +)$  grading of  $A/I$ . Equivalently, there exist  $k$  pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_k \in \mathrm{Der}_I(A)$  with integer eigenvalues.*

*Proof.* By Proposition 2.47 we can choose coordinates for  $A$  such that  $T \cong (\mathbb{C}^*)^k$  acts via characters on them. This means that there exist  $\lambda_{ij} \in \mathbb{Z}$  with

$$t \cdot x_i = \prod_{j=1}^k t_j^{-\lambda_{ij}} x_i, \quad (2.3)$$

for any  $t = (t_1, \dots, t_k) \in T$ . By Lemma 2.40 and [Mil17, Theorem 12.12] we obtain a minimal generating system  $f_1, \dots, f_m$  of  $I$  such that there exist  $d_{lj} \in \mathbb{Z}$  with

$$t \cdot f_l = \prod_{j=1}^k t_j^{-d_{lj}} f_l, \quad (2.4)$$

for any  $t = (t_1, \dots, t_k) \in T$ . Combining Equation 2.3 and Equation 2.4 we obtain

$$t_j \cdot f_l = f(t_j \cdot x_1, \dots, t_j \cdot x_n) = f(t_j^{-\lambda_{1j}} x_1, \dots, t_j^{-\lambda_{nj}} x_n) = t^{-d_{lj}} f_l(x_1, \dots, x_n). \quad (2.5)$$

Equation 2.5 is equivalent so saying that the  $f_l$  are multi-homogeneous with respect to weights  $\lambda_i := (\lambda_{i1}, \dots, \lambda_{in})$  for  $1 \leq i \leq k$ . This again implies that  $I$  is  $(\mathbb{Z}^k, +)$  graded.  $\square$

### 2.3.2 $(\mathbb{Z}^m, +)$ Gradings, Algebraic Tori and Derivations

In this section we want to see an explicit correspondence between the Lie algebra  $\mathfrak{g}$  generated by  $\delta_1, \dots, \delta_m$  pairwise commuting diagonalizable derivations with integer eigenvalues, that is  $(\mathbb{Z}^m, +)$  gradings, and algebraic tori contained in  $\text{Aut}_I(A)$ . The first step is to see that  $\pi_k(\text{Aut}_I(A))$  as an algebraic group with Lie algebra  $\pi_k(\text{Der}'_I(A))$ . This result extends the classical result for Artinian algebras, which are finite dimensional vector spaces. See for example [Hum75, Corollary 13.2].

**Lemma 2.49.** *Let  $A$  be an analytic algebra and  $I \subseteq A$  an ideal. Then  $\pi_k(\text{Aut}_I(A))$  is an algebraic group with Lie algebra  $\pi_k(\text{Der}'_I(A))$ .*

*Proof.* See [Kra78, (4.32)]. □

We need some preparations before we can state the relation between algebraic tori and pairwise commuting diagonalizable derivations with integer eigenvalues. This section relies on the fact that we can find a coordinate system such that a set of pairwise commuting diagonalizable derivations are in diagonal shape.

**Remark 2.50.** *For the moment, we restrict ourselves to the case  $A = \mathbb{C}[[\mathbf{x}]]$ .*

*To keep our computations as simple as possible we assume that our coordinate system is always chosen in such a way that  $\delta_i = \sum_{j=1}^n \lambda_{ij} x_j \partial_{x_j}$  with  $\lambda_{ij} \in \mathbb{Z}$ .*

We can consider any element of  $\pi_1(\text{Der}'_I(A))$  as an element of  $\mathfrak{gl}(n, \mathbb{C})$  by considering the linear part of the derivations. In particular, we have a representation  $\pi_1 : \mathfrak{g} \rightarrow \mathfrak{gl}(n, \mathbb{C}), \delta \mapsto \delta_0$ . Since we deal with pairwise commuting diagonalizable derivations, we need the following definition.

**Definition 2.51.** *Let  $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{C})$  be a finite dimensional Lie subalgebra. We call  $\mathfrak{t} \subseteq \mathfrak{g}$  a **toral subalgebra** of  $\mathfrak{g}$  if all elements of  $\mathfrak{t}$  are diagonalizable. We say  $\mathfrak{t}$  is a **maximal toral subalgebra** of  $\mathfrak{g}$ , if for every toral subalgebra  $\mathfrak{t}'$  of  $\mathfrak{g}$  with  $\mathfrak{t} \subseteq \mathfrak{t}'$  it holds that  $\mathfrak{t} = \mathfrak{t}'$ . If  $\mathfrak{g}$  is clear from the context, we call  $\mathfrak{t}$  a **(maximal) toral Lie algebra**.*

**Remark 2.52.** *It is easy to see that being a maximal toral Lie algebra is equivalent to the following condition:*

*For every diagonalizable matrix  $D \in \mathfrak{g}$  with the property that  $[D, L] = 0$  for all  $L \in \mathfrak{t}$ , implies  $D \in \mathfrak{t}$ .*

**Example 2.53.** *Let  $I \subseteq A$  be an ideal. Consider the Lie algebra  $\mathfrak{h}$  generated by the pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_m \in \text{Der}'_I(A)$ . The  $\delta_i$  can be represented by diagonal matrices  $D_i = \text{diag}(\lambda_{i1}, \dots, \lambda_{in})$ . Let  $\mathfrak{t} = \langle D_1, \dots, D_m \rangle_{\mathbb{C}}$ . By definition  $\mathfrak{t}$  is a toral subalgebra of  $\mathfrak{g} = \pi_1(\text{Der}'_I(A))$ . In this setup we have  $\mathfrak{t} = \pi_1(\mathfrak{h})$ .*

We want to show that, in the formal case, every set of pairwise commuting diagonalizable derivations induces an algebraic torus  $T \subseteq \text{Aut}_I(A)$ . Any toral Lie algebra  $\mathfrak{t}$  generated by diagonal matrices over  $\mathbb{C}$  gives rise to an algebraic torus  $T = e^{\mathfrak{t}}$ , since for any two matrices  $A, B \in \mathfrak{gl}(n, \mathbb{C})$  with  $[A, B] = 0$  it holds that  $e^{A+B} = e^A \cdot e^B$ .

To prove  $T \leftrightarrow \text{Aut}_I(A)$ , we need some preparations.

**Definition 2.54.** *Let  $v_1, \dots, v_m \in \mathbb{R}^n$  with  $m \leq n$  be  $\mathbb{R}$ -linearly independent vectors. We call the  $\mathbb{Z}$ -module  $L = \langle v_1, \dots, v_m \rangle_{\mathbb{Z}}$  a **lattice** of rank  $m$ . In case  $L = \langle v_1, \dots, v_m \rangle_{\mathbb{R}} \cap \mathbb{Z}^n$ , we say  $L$  is **saturated**.*

The toral Lie algebra  $\mathfrak{t}$  from Example 2.53 restricted to the real span of its generators satisfies the condition of being a saturated lattice. We use this to show that  $T$  is isomorphic to  $(\mathbb{C}^*)^n$  as an algebraic group.

**Proposition 2.55.** *Let  $\mathfrak{t} \subseteq \mathfrak{gl}(n, \mathbb{C})$  be a toral Lie algebra. Assume  $\mathfrak{t}$  admits a  $\mathbb{C}$ -vector space basis  $D_1, \dots, D_m$ , such that all eigenvalues of the  $D_i$  are integers. Denote the  $j$ -th diagonal entry of  $D_i$  by  $\lambda_{ij}$ . Define  $v_i := (\lambda_{ij})_{1 \leq j \leq n} \in \mathbb{R}^n$ . Let  $L = \langle v_1, \dots, v_m \rangle_{\mathbb{Z}}$  be the lattice spanned by the  $v_i$ . If  $L$  is saturated, then  $T := e^{\mathfrak{t}}$  is isomorphic to  $(\mathbb{C}^*)^m$  as an algebraic group.*

*Proof.* Denote by  $D(n, \mathbb{C})$  the algebraic group of diagonal matrices (see for example [Hum75, Section 7.1]). We define the map  $\varphi : (\mathbb{C}^*)^m \rightarrow D(n, \mathbb{C})$  via

$$(t_1, \dots, t_m) \mapsto \text{diag}(t_1^{\lambda_{11}} \cdot \dots \cdot t_m^{\lambda_{m1}}, \dots, t_1^{\lambda_{1n}} \cdot \dots \cdot t_m^{\lambda_{mn}}).$$

It is obvious that  $\varphi$  satisfies  $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$  for any  $a, b \in (\mathbb{C}^*)^m$ . Since  $\varphi$  is defined by rational functions, it also defines a morphism of algebraic varieties. So it is a group homomorphism of linear algebraic groups. Our goal now is to show that  $\varphi$  maps injectively onto  $T$ .

First we show  $\text{im}(\varphi) = T$ . We know that for every  $t_i \in \mathbb{C}^*$  there exists a  $z_i \in \mathbb{C}$  with  $e^{z_i} = t_i$ . The surjectivity of the complex exponential map implies that we can write  $\text{diag}(t_i^{\lambda_{i1}}, \dots, t_i^{\lambda_{in}})$  as  $e^{z_i D_i}$ . By definition,  $\varphi$  maps  $(1, \dots, 1, t_i, 1, \dots, 1)$  to this element. Since the  $e^{z_i D_i}$  form a generating set of  $T$  and since  $\varphi$  is multiplicative, we obtain  $\text{im}(\varphi) = T$ .

It remains to show that  $\varphi$  is injective. Denote the real part of  $z_i$  by  $x_i \in \mathbb{R}$  and the imaginary part by  $y_i \in [0, 2\pi)$ . Then we can write  $\varphi(t_1, \dots, t_m) = e^{z_1 D_1 + \dots + z_m D_m}$ . Since we are dealing with diagonal matrices  $\varphi(t_1, \dots, t_m) = \text{id}$  implies  $z_1 D_1 + \dots + z_m D_m = 2\pi i \text{diag}(k_1, \dots, k_n)$  for integers  $k_1, \dots, k_n$ . Thus we obtain

$$x_1 D_1 + \dots + x_m D_m = 0 \text{ and } \frac{y_1}{2\pi} D_1 + \dots + \frac{y_m}{2\pi} D_m = \text{diag}(k_1, \dots, k_n).$$

The fact that the  $D_i$  are linearly independent implies  $x_i = 0$ . The  $v_i$  being a basis of a saturated lattice translates to the fact that the  $\frac{y_i}{2\pi}$  are integers. This implies  $y_i = 0$ . So we have  $z_i = 0$  and in particular  $t_i = 1$  for  $1 \leq i \leq m$ . Thus  $\varphi$  is injective and we obtain  $T \cong (\mathbb{C}^*)^m$ .  $\square$

**Corollary 2.56.** *Let  $I \subseteq A$  be an ideal and let  $\mathfrak{g}$  be the Lie algebra generated by the pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_m \in \text{Der}_I(A)$  with integer eigenvalues. Define  $\mathfrak{t} := \pi_1(\mathfrak{g}) \hookrightarrow \mathfrak{gl}(n, \mathbb{C})$  and  $T := e^{\mathfrak{t}}$ . For any diagonal matrix  $D = \text{diag}(d_1, \dots, d_n)$  with  $d_i \in \mathbb{C}$  we define the map  $\varphi_D : A \rightarrow A, x_i \mapsto d_i x_i$ . Then the map*

$$\psi : T \hookrightarrow \text{Aut}_I(A), D \mapsto \varphi_D$$

*defines a rational action.*

*Proof.* We keep the notation from Proposition 2.55. The map  $\varphi_D$  defines an automorphism, since  $\det(D) \neq 0$  for any  $D \in T$ . Let  $f \in A$  be multihomogeneous, that is  $\delta_i(f) = w_i f$  for some  $w_i \in \mathbb{Z}$ . Define  $D_i := \text{diag}(t_i^{-\lambda_{i1}}, \dots, t_i^{-\lambda_{in}})$  for  $i = 1, \dots, m$ . Then  $\varphi_{D_i}(f) = f(t_i^{-\lambda_{i1}} x_1, \dots, t_i^{-\lambda_{in}} x_n) = t_i^{-w_i} f$ , since we assume that we are in a multihomogeneous coordinate system. This implies  $\varphi_{D_i}(I) \subseteq I$ , since  $I$  can be generated by multihomogeneous elements. Using that  $\varphi_{D_i D_j} = \varphi_{D_i} \circ \varphi_{D_j}$  we obtain

$\varphi_D \in \text{Aut}_I(A)$  for all  $D \in T$ . Since any automorphism of  $A$  is uniquely determined by its action on the  $x_i$  we obtain an injection of groups  $\alpha : T \hookrightarrow \text{Aut}_I(A), D \mapsto \varphi_D$ . It remains to show that the maps  $\pi_k \circ \alpha$  are homomorphisms of algebraic groups. For any  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$  we have that  $(\pi_k \circ \varphi_D)(\overline{x_i}) = \lambda_i \overline{x_i}$ . This implies that the  $\pi_k \circ \varphi_D$ , considered as elements of  $\text{End}_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^{k+1})$ , are diagonal matrices, where the diagonal entries are polynomial expressions in the diagonal entries of  $D$ . This implies that  $\pi_k \circ \alpha$  is a morphism of algebraic groups and thus  $T \hookrightarrow \text{Aut}_I(A)$  defines a rational action.  $\square$

Corollary 2.56, combined with Theorem 2.48, yields a correspondence between  $(\mathbb{Z}^m, +)$  gradings, algebraic tori and toral Lie algebras of derivations with integer eigenvalues in the complete case. To our knowledge, these results cannot be proven for ideals of the convergent power series ring in general. Nevertheless, it can be proven for ideals  $I \subseteq \mathbb{C}\{\mathbf{x}\}$ , which are generated by algebraic power series. Let us make this notion more precise.

**Definition 2.57.** Let  $R = \mathbb{C}[x]$  and  $f \in \mathbb{C}\{\mathbf{x}\}$ . We call  $f$  an **algebraic power series**, if there exists a polynomial  $p \in R[t]$ , such that  $p(f) = 0$ . We say an ideal  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  is **algebraic**, if it can be generated by algebraic power series.

The following theorem is a crucial tool in passing to algebraic ideals.

**Theorem 2.58.** Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an algebraic ideal and let  $G \subseteq \text{GL}(n, \mathbb{C})$  be a reductive algebraic group. Define  $\hat{I} := IC[[\mathbf{x}]]$ . Then  $G \subseteq \text{Aut}_I(\mathbb{C}\{\mathbf{x}\})$  if and only if  $G \subseteq \text{Aut}_{\hat{I}}(IC[[\mathbf{x}]])$ .

*Proof.* See [HM89, Theorem 2].  $\square$

Theorem 2.58 combined with Corollary 2.56 implies:

**Corollary 2.59.** Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an algebraic ideal and let  $\mathfrak{g}$  be the Lie algebra generated by the pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_m \in \text{Der}_I(A)$  with integer eigenvalues. Define  $\mathfrak{t} := \pi_1(\mathfrak{g}) \hookrightarrow \mathfrak{gl}(n, \mathbb{C})$  and  $T := e^{\mathfrak{t}}$ . For any diagonal matrix  $D = \text{diag}(d_1, \dots, d_n)$  with  $d_i \in \mathbb{C}$  we define the map  $\varphi_D : A \rightarrow A, x_i \mapsto d_i x_i$ . Then the map

$$\psi : T \hookrightarrow \text{Aut}_I(A), D \mapsto \varphi_D$$

defines a rational action.

### 2.3.3 Maximal Multihomogeneity

Our next objective is to see that there is a one-to-one correspondence between maximal  $(\mathbb{C}^m, +)$  gradings, maximal toral Lie algebras and maximal algebraic tori. So far we have dealt with  $(\mathbb{Z}^m, +)$  gradings. Due to Theorem 2.24, we are able to reduce to the complex case to the integer case.

**Definition 2.60.** Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $A = \mathbb{C}[[\mathbf{x}]]$  and  $I \subseteq A$ . Assume that  $A/I$  is  $(\mathbb{C}^s, +)$  multigraded. We say  $A/I$  is **maximal multihomogeneous of rank  $s$** , if for all  $(\mathbb{C}^k, +)$  gradings of  $A/I$  it holds that  $k \leq s$ . This can be equivalently reformulated using derivations. There exist pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_s \in \text{Der}_I(A)$  with the following properties

- (a) For all diagonalizable derivations  $\delta \in \text{Der}_I(A)$  with  $[\delta, \delta_i] = 0$  for  $i = 1, \dots, s$  it holds that  $\delta \in \langle \delta_1, \dots, \delta_s \rangle_{\mathbb{C}}$ .
- (b)  $s$  is maximal with respect to all sets of pairwise commuting derivations satisfying property (a).

In order to prove that the maximal multihomogeneity is an invariant of the algebra, we need some preparations.

**Lemma 2.61.** *Let  $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{C})$  be an abelian Lie algebra. Define  $G := e^{\mathfrak{g}} \subseteq \text{GL}(n, \mathbb{C})$ . If  $G$  is connected and  $\dim(G) = \dim_{\mathbb{C}}(\mathfrak{g})$ , then  $\text{Lie}(G) = \mathfrak{g}$ .*

*Proof.* Any linear algebraic group  $G \subseteq \text{GL}(n, \mathbb{C})$  can be considered as a Lie group (see [OV90, Chapter 3, §1, Theorem 2]). We use [Hal15, Definition 3.18] and compute the corresponding Lie algebra as

$$\text{Lie}(G) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid e^{tX} \in G, \text{ for all } t \in \mathbb{R}\}.$$

The inclusion  $\mathfrak{g} \subseteq \text{Lie}(G)$  is obvious. Our assumptions now imply:

$$\dim_{\mathbb{C}}(\text{Lie}(G)) = \dim(G) = \dim_{\mathbb{C}}(\mathfrak{g}),$$

hence  $\mathfrak{g} = \text{Lie}(G)$ . □

Next we show that the Lie-functor commutes with the canonical projections  $\pi_k$  in the complete case.

**Corollary 2.62.** *Let either*

- (i)  $A = \mathbb{C}[[\mathbf{x}]]$  and  $I \subseteq A$ , or
- (ii)  $A = \mathbb{C}\{\mathbf{x}\}$  and  $I \subseteq A$  be algebraic.

*Furthermore, let  $\mathfrak{g}$  be the Lie algebra generated by the pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_m \in \text{Der}_I(A)$  with integer eigenvalues. Define  $\mathfrak{t} := \pi_1(\mathfrak{g}) \hookrightarrow \mathfrak{gl}(n, \mathbb{C})$  and  $T := e^{\mathfrak{t}} \subseteq \text{Aut}_I(A)$ . Then  $\text{Lie}(\pi_k(T)) = \pi_k(\mathfrak{t})$  for all  $k \in \mathbb{N}$ .*

*Proof.* It suffices to show the result in the complete case. By Corollary 2.43 we know that  $\pi_k : T \rightarrow \pi_k(T)$  is an isomorphism of algebraic groups for all  $k \in \mathbb{N}$ . Since we can assume that our coordinates are chosen such that  $\delta_1, \dots, \delta_m$  are diagonal, we know that  $\pi_1 : \mathfrak{g} \rightarrow \pi_1(\mathfrak{g})$  is an isomorphism of Lie algebras. Using that  $\pi_1 = \pi_1^k \circ \pi_k$ , we obtain that  $\pi_k : \mathfrak{g} \rightarrow \pi_k(\mathfrak{g})$  is an isomorphism of Lie algebras. The same result holds for  $\pi_k(T)$ . Combining these results we also obtain that the maps  $\pi_k^l : \pi_l(T) \rightarrow \pi_k(T)$  and  $\pi_k^l : \pi_l(\mathfrak{g}) \rightarrow \pi_k(\mathfrak{g})$  are isomorphisms of algebraic groups, respectively Lie algebras. Since we are working over characteristic zero, we know by [Hum75, Theorem 12.5] that the Lie functor is exact. Then the isomorphism  $\pi_k^l : \pi_l(T) \rightarrow \pi_k(T)$  induces the isomorphism  $d\pi_k^l : \text{Lie}(\pi_l(T)) \rightarrow \text{Lie}(\pi_k(T))$ . For  $k = 1$  Lemma 2.61 implies  $\text{Lie}(\pi_1(T)) = \pi_1(\mathfrak{g})$ . The fact that the differential  $d\pi_k^l$  coincides with a restriction of the map  $\pi_k^l : \pi_l(\text{Der}'_I(A)) \rightarrow \pi_k(\text{Der}'_I(A))$ , implies  $\pi_k(\mathfrak{g}) = \text{Lie}(\pi_k(T))$  for any  $k \in \mathbb{N}$ . □

The following theorem shows us that  $T = e^{\mathfrak{g}}$  is a maximal torus in  $\text{Aut}_I(A)$ , if the grading induced by  $\mathfrak{g}$  is maximal.

**Theorem 2.63.** *Let either*

- (i)  $A = \mathbb{C}[[\mathbf{x}]]$  and  $I \subseteq A$ , or
- (ii)  $A = \mathbb{C}\{\mathbf{x}\}$  and  $I \subseteq A$  be algebraic.

Furthermore, let  $\mathfrak{g}$  be the Lie algebra generated by the pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_s \in \text{Der}_I(A)$ . Define  $\mathfrak{t} := \pi_1(\mathfrak{g}) \hookrightarrow \mathfrak{gl}(n, \mathbb{C})$  and  $T := e^{\mathfrak{t}} \subseteq \text{Aut}_I(A)$ . If  $\delta_1, \dots, \delta_s$  induce a maximal multigrading on  $A/I$ , then  $T$  is a maximal torus in  $\text{Aut}_I(A)$ .

*Proof.* We split the proof in two steps. First we show statement for the complete case, then we reduce the algebraic case to the complete case.

**Step 1:** Let  $A = \mathbb{C}[[\mathbf{x}]]$ . Denote by  $T' \subseteq \text{Aut}_I(A)$  an algebraic torus with  $T \subseteq T'$ . Since tori are reductive, Corollary 2.43 implies  $\pi_k(T) \subseteq \pi_k(T')$  for every  $k \in \mathbb{N}$ . We have isomorphisms of Lie algebras  $\pi_k^m : \text{Lie}(\pi_m(T')) \rightarrow \text{Lie}(\pi_k(T'))$ . Define  $\mathfrak{g}' := \varprojlim_{k \in \mathbb{N}} \text{Lie}(\pi_k(T'))$ . Lemma 2.49 combined with [Mil17, (10.14)] and [Hum75, Theorem 13.1] implies

$$\text{Lie}(\pi_k(T)) \subseteq \text{Lie}(\pi_k(T')) \subseteq \text{Lie}(\pi_k(\text{Aut}_I(A))) = \pi_k(\text{Der}'_I(A))$$

for any  $k \in \mathbb{N}$ . Passing to the projective limit this yields  $\mathfrak{g} \subseteq \mathfrak{g}' \subseteq \text{Der}'_I(A)$ . By [Hum75, Theorem 13.4(b)] we know that the Lie algebra of a connected commutative algebraic group is abelian. Considering  $\pi_k(T)$  as a subset of  $\text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$  and using [Hum75, Theorem 15.5], we obtain that  $\text{Lie}(\pi_k(T))$  is a toral Lie subalgebra of  $\mathfrak{gl}(\mathfrak{m}/\mathfrak{m}^{k+1})$ . Then  $\mathfrak{g}'$  consists of pairwise commuting diagonalizable elements. Since  $\mathfrak{g} \subseteq \mathfrak{g}'$ , we also have  $[\delta_i, \mathfrak{g}'] = 0$  for all  $i = 1, \dots, m$ . By assumption this implies  $\mathfrak{g} = \mathfrak{g}'$  and in particular  $\pi_1(\mathfrak{g}') = \pi_1(\mathfrak{g})$ . Then [Hum75, Theorem 13.1] yields  $\pi_1(T) = \pi_1(T')$ . From Corollary 2.46 we obtain that  $T$  is a maximal algebraic torus.

**Step 2:** Let  $A = \mathbb{C}\{\mathbf{x}\}$  and  $I \subseteq A$ , be an algebraic ideal. Passing to the completion  $\hat{A}$ , we can assume that the  $\delta_i$  are in diagonal shape. By Corollary 2.56 they induce an algebraic torus  $T \subseteq \text{Aut}_I(\hat{A})$  with  $\dim(T) = s$ . If  $T$  is not maximal, there exists a maximal algebraic torus  $T' \subseteq \text{Aut}_I(\hat{A})$  with  $s' := \dim(T') > s$ . By Theorem 2.58 we have  $T, T' \subseteq \text{Aut}_I(A)$ . Then Theorem 2.48 implies that  $A/I$  is  $(\mathbb{C}^{s'}, +)$  graded. This contradicts the maximality of  $s$  and thus  $T$  is a maximal algebraic torus.  $\square$

Combining Theorem 2.48 and Theorem 2.63 we obtain the following.

**Remark 2.64.** *Let  $I \subseteq A$  be an ideal. Then there exists a correspondence between maximal algebraic tori and maximal multigradings of  $A/I$ .*

**Remark 2.65.** *At the moment we cannot drop the assumption that  $I$  is an algebraic ideal, since the proof of Theorem 2.58 makes use of an approximation theorem proven by Popescu in [Pop86], which works only in the algebraic setup. For the convergent case of this approximation theorem, a counterexample was given by Gabrielov in [Gab71].*

**Remark 2.66.** *Analyzing the proofs of Theorem 2.63 and Theorem 2.48 combined with the fact that all maximal algebraic tori are conjugated and that all algebraic tori are contained in a maximal torus, implies that the weights of all maximal multigradings are the same. Thus the information regarding the weights of an ideal can be recovered from the eigenvalues of a toral Lie algebra  $\mathfrak{t} \subseteq \pi_1(\text{Der}_I(A))$  or from the characters of  $T \subseteq \text{Aut}_I(A)$ . From a computational point of view it turns out, that it easier to obtain the eigenvalues. Corollary 2.59 implies that*



if  $A/I$  is  $(\mathbb{C}^m, +)$  graded, then  $m \leq n$ . In particular the maximal value of  $k$  is given by the dimension of the maximal tori. Thus the maximal multihomogeneity is an invariant which can be computed from algebraic objects. This completes the picture presented in the beginning of this chapter.

In Chapter 3 we are going to investigate the case when  $s = n$ , that is the maximal multihomogeneity obtains its greatest possible value.

## 2.4 The Structure Theorem for Analytic Algebras

In this section, we extend the abstract definition of grading from the previous sections to so-called Lie–Rinehart algebras and generalize the Formal Structure Theorem from [GS06], [Sch07] and [Epu15] to certain Lie–Rinehart subalgebras of  $\text{Der}'(A)$ , where  $A \cong \mathbb{C}\{\mathbf{x}\}/I$  is an analytic algebra and  $I$  is generated by algebraic elements or  $I$  is positively graded. Subsection 2.4.1 is an adapted version of [Epu15, Section 4.2], which we included in order to keep the material self contained.

### 2.4.1 Lie–Rinehart Algebras

In the following section, we introduce the notion of Lie–Rinehart algebras, which combine the structure of modules with the structure of Lie algebras and relate both to derivation modules. We also define a notion of (multi-)grading for Lie–Rinehart algebras.

Let us start with the definition of a Lie–Rinehart algebra. The definition is taken from [Hue98] and is slightly modified to fit in our context.

**Definition 2.67.** *Let  $A$  be an algebra over  $\mathbb{C}$ ,  $\mathfrak{g}$  be a Lie algebra over  $\mathbb{C}$  and  $\rho : \mathfrak{g} \rightarrow \text{Der}(A)$  a morphism of Lie algebras. Define  $\alpha(f) := \rho(\alpha)(f)$  for all  $\alpha \in \mathfrak{g}$  and  $f \in A$ . We call the pair  $(A, \mathfrak{g}, \rho)$  a **Lie–Rinehart algebra**, if the following conditions are satisfied:*

- (i)  $\mathfrak{g}$  is an  $A$ -module,
- (ii)  $[\alpha, f\beta] = \alpha(f)\beta + f[\alpha, \beta]$  for all  $f \in A, \alpha, \beta \in \mathfrak{g}$ , and
- (iii)  $(f\alpha)(g) = f(\alpha(g))$  for all  $f, g \in A, \alpha \in \mathfrak{g}$ .

**Remark 2.68.** *Condition ii) in the previous definition implies that the Lie algebra morphism  $\rho$  is also  $A$ -linear.*

The next topic we need to talk about is morphisms of Lie–Rinehart algebras. The following definition is taken from [Hue90, Chapter 1].

**Definition 2.69.** *Let  $(A, \mathfrak{g}, \rho)$  and  $(B, \mathfrak{h}, \sigma)$  be Lie–Rinehart algebras, where  $A, B$  are algebras over  $\mathbb{C}$ . Then  $(\varphi, \psi)$  is a **morphism of Lie–Rinehart algebras**, if:*

- i)  $\varphi : A \rightarrow B$  is a morphism of  $\mathbb{C}$ -algebras,
- ii)  $\psi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a morphism of Lie algebras, which in the same time is a morphism of  $A$ -modules, where  $A$  acts on  $B$  via  $\varphi$ , and

iii) for all  $f \in A, \alpha \in \mathfrak{g}$  it holds that

$$\varphi \circ \alpha(f) = \psi(\alpha)(\varphi(f)).$$

Our standard example for a Lie–Rinehart algebra is the module of derivations of an analytic algebra.

**Example 2.70.** *Let  $A$  be an analytic algebra and  $\mathfrak{g} = \text{Der}(A)$ . Then  $(A, \mathfrak{g}, \text{id})$  is a Lie–Rinehart algebra, since all properties are basic properties of the module of derivations.*

Let us now define a notion of grading for a special type of Lie–Rinehart algebras.

**Definition 2.71.** *Let  $(G, +)$  be an abelian group,  $A$  be an algebra over  $\mathbb{C}$  and  $(A, \mathfrak{g}, \rho)$  a Lie–Rinehart algebra, where  $\mathfrak{g} \subseteq \text{Der}(A)$  and  $\rho : \mathfrak{g} \hookrightarrow \text{Der}'(A)$  the canonical inclusion map. We say  $(A, \mathfrak{g}, \rho)$  is **finitely graded**, if the following conditions hold:*

- i)  $A$  is finitely graded in the sense of Definition 2.1,
- ii)  $\mathfrak{g}$  is finitely graded as an  $A$ -module in the sense of Definition 2.1, and
- iii) the group homomorphisms  $\pi_g, g \in G$ , arising from Definition 2.1, have to satisfy  $[\pi_g(\mathfrak{g}), \pi_h(\mathfrak{g})] \subseteq \pi_{g+h}(\mathfrak{g})$  for all  $g, h \in G$ .

Next, we take a look at general gradings of Lie–Rinehart algebras. We restrict ourselves to the case where  $A$  is an analytic algebra. We denote the natural projection  $A \rightarrow A/\mathfrak{m}_A^k$  by  $p_k^A$  for  $k \in \mathbb{N}$ . To keep notation short, we write  $A_k := A/\mathfrak{m}_A^k$ . Let  $\mathfrak{g} \subseteq \text{Der}(A)$  and  $\rho : \mathfrak{g} \hookrightarrow \text{Der}'(A)$ . The map  $p_k^A$  induces natural morphisms  $p_k : \text{Der}(A) \rightarrow \text{Der}(A_k)$ , which induce a natural projection of Lie algebras  $p_k^{\mathfrak{g}} : \mathfrak{g} \rightarrow \pi_k(\mathfrak{g})$ . Define  $\mathfrak{g}_k := p_k^{\mathfrak{g}}(\mathfrak{g})$ . This again induces morphisms  $(p_k^A, p_k^{\mathfrak{g}})$  of Lie–Rinehart algebras.

**Definition 2.72.** *Let  $(G, +)$  be an abelian group,  $A$  an analytic algebra and  $(A, \mathfrak{g}, \rho)$  a Lie–Rinehart algebra, where  $\mathfrak{g} \subseteq \text{Der}(A)$  and  $\rho : \mathfrak{g} \hookrightarrow \text{Der}'(A)$  is the canonical inclusion map. We say  $(A, \mathfrak{g}, \rho)$  is a **graded Lie–Rinehart algebra** with respect to  $G$ , if the following hold:*

- i) For all  $g \in G$  there are group homomorphisms  $\pi_g^A : (A, +) \rightarrow (A, +)$  grading  $A$  in the sense of Definition 2.4, and
- ii) for all  $g \in G$  there are group homomorphisms  $\pi_g^{\mathfrak{g}} : (\mathfrak{g}, +) \rightarrow (\mathfrak{g}, +)$ , such that the induced morphisms  $\pi_g^{\mathfrak{g}_k} : \mathfrak{g}_k \rightarrow \mathfrak{g}_k$  grade  $(A_k, \mathfrak{g}_k, \rho_k)$  in the sense of Definition 2.71 for all  $k \in \mathbb{N}$ .

Our definition of a graded Lie–Rinehart algebra allows us to use our results regarding graded modules. We can also switch the perspective from which we are looking at our Lie–Rinehart algebra, as it is useful to consider it sometimes as a module, sometimes as a Lie algebra. Before we go on with examples and the most important theorem of this section, we have the following remark regarding the usual notion of grading of finite Lie-algebras.

**Remark 2.73.** *The usual grading of a finite Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  is a special case of Definition 2.71. If we let  $\mathfrak{g}$  operate trivially on  $\mathbb{C}$ , that is,  $\alpha(f) = 0$  for all  $f \in \mathbb{C}$  and  $\alpha \in \mathfrak{g}$ , we can satisfy all conditions from Definition 2.67, hence  $(\mathbb{C}, \mathfrak{g}, \rho)$  is a Lie–Rinehart algebra, with  $\rho : \mathfrak{g} \rightarrow \text{Der}(K)$  being the trivial morphism. Now we can simply take  $A = \mathbb{C}$  and grade it trivially. Then condition i) in Definition 2.71 is superfluous and conditions ii) and iii) state basically, that our Lie algebra can be written as a direct sum of graded components, which are compatible with the Lie brackets, which is the usual definition of a graded Lie algebra.*



The following theorem shows, that gradings of analytic algebras induce gradings of the corresponding Lie algebra of derivations.

**Theorem 2.74.** *Let  $A$  be an analytic algebra and  $(A, \text{Der}'(A), \text{id})$  a Lie–Rinehart algebra. Denote the projections  $\text{Der}'(A) \rightarrow \text{Der}(A/\mathfrak{m}_A^k)$  by  $p_k$ , where  $\mathfrak{g}_k := p_k(\text{Der}'(A))$  for  $k \in \mathbb{N}$ . Assume that  $A$  is  $(\mathbb{C}, +)$  graded, where the grading is induced by  $\delta \in \text{Der}'(A)$ . Then  $\delta$  induces a grading on  $(A, \text{Der}'(A), \text{id})$  in the sense of Definition 2.72. In particular, every homogeneous  $\epsilon \in \text{Der}'(A)$  satisfies  $\text{ad}_\delta(\epsilon) = \lambda\epsilon$ , for some  $\lambda \in \mathbb{C}$ .*

*Proof.* Define  $\mathfrak{g} = \text{Der}'(A)$ . In the following proof, we use that if  $\delta \in \text{Der}'(A)$  is diagonalizable, also  $\text{ad}_\delta$  is diagonalizable on the finite-dimensional Lie algebras  $\mathfrak{g}_k$ , where  $\bar{\delta}$  denotes the image of  $\delta$  under  $p_k$ . Next we show that this property on the finite-dimensional Lie algebras induces our grading on  $\mathfrak{g}$ . The first property of Definition 2.72 is satisfied automatically, as we assume that  $A$  is graded. To show the second property, we use that  $\mathfrak{g}_k = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_{k,\lambda}$ , where  $\mathfrak{g}_{k,\lambda}$  denotes the eigenspace with respect to the eigenvalue  $\lambda$ . Define  $\pi_\lambda^{\mathfrak{g}_k} : (\mathfrak{g}_k, +) \rightarrow (\mathfrak{g}_{k,\lambda}, +)$  as the projection to  $\mathfrak{g}_{k,\lambda}$ , for any  $\lambda \in \mathbb{C}$ . Next we show that the  $\mathfrak{g}_k$  are finitely graded as  $A_k$ -modules. Consider any  $k \in \mathbb{N}$ , and  $\lambda, \mu \in \mathbb{C}$ , then we have for any homogeneous elements  $f_\mu \in A_k$  and  $\tau_\lambda \in \mathfrak{g}_{k,\lambda}$ :

$$\text{ad}_{\bar{\delta}}(f_\mu \tau_\lambda) = \mu f_\mu \tau_\lambda + \lambda f_\mu \tau_\lambda = (\mu + \lambda) f_\mu \tau_\lambda \in \mathfrak{g}_{k,\mu+\lambda},$$

hence  $\mathfrak{g}_k$  is a graded  $A_k$ -module. Next we need to prove the property of Definition 2.71, namely the finite grading as a Lie algebra, that is  $[\mathfrak{g}_{k,\lambda}, \mathfrak{g}_{k,\mu}] \subseteq \mathfrak{g}_{k,\lambda+\mu}$ . Consider any  $\tau_\mu \in \mathfrak{g}_{k,\mu}$  and  $\tau_\lambda \in \mathfrak{g}_{k,\lambda}$ , then

$$\text{ad}_{\bar{\delta}}([\tau_\mu, \tau_\lambda]) = -[\tau_\mu, [\tau_\lambda, \bar{\delta}]] - [\tau_\lambda, [\bar{\delta}, \tau_\mu]] = \lambda[\tau_\mu, \tau_\lambda] - \mu[\tau_\lambda, \tau_\mu] = (\mu + \lambda)[\tau_\mu, \tau_\lambda],$$

hence  $[\mathfrak{g}_{k,\mu}, \mathfrak{g}_{k,\lambda}] \subseteq \mathfrak{g}_{k,\mu+\lambda}$ .

Let  $\epsilon \in \mathfrak{g}$  be homogeneous. Then  $\text{ad}_\delta(\epsilon) = \lambda\epsilon$  follows by Lemma 1.5, since the equality holds modulo every power of the maximal ideal.  $\square$

The next corollary is an immediate consequence of Theorem 2.74.

**Corollary 2.75.** *Let  $A$  be a graded analytic algebra with grading induced by a diagonalizable derivation  $\delta \in \text{Der}'(A)$  and  $(A, \mathfrak{g}, \rho)$  a Lie–Rinehart algebra, where  $\mathfrak{g} \subseteq \text{Der}(A)$  and  $\rho : \mathfrak{g} \hookrightarrow \text{Der}'(A)$  is the canonical inclusion map. If  $[\delta, \mathfrak{g}] \subseteq \mathfrak{g}$ , then  $\mathfrak{g}$  is a graded Lie–Rinehart subalgebra of  $\text{Der}'(A)$  with respect to  $\delta$ .*

Now we are able to extend the notion of multigradings to Lie–Rinehart subalgebras of  $\text{Der}(A)$ .

**Definition 2.76.** *Let  $A$  be an analytic algebra and  $(A, \mathfrak{g}, \rho)$  a Lie–Rinehart algebra, where  $\mathfrak{g} \subseteq \text{Der}(A)$  and  $\rho : \mathfrak{g} \hookrightarrow \text{Der}'(A)$  is the canonical inclusion map. We say  $(A, \mathfrak{g}, \rho)$  is a **multi-graded Lie–Rinehart algebra**, if there exists an  $m \in \mathbb{N}_{\geq 1}$  and abelian groups  $G_1, \dots, G_m$ , such that  $(A, \mathfrak{g}, \rho)$  is graded with respect to  $G := G_1 \times \dots \times G_m$  in the sense of Definition 2.72.*

Next we state the analogous results to Proposition 2.14 and 2.15. We skip the proof for the statement, since it works analogously to the proof of [Epu15, Lemma 4.26].

**Proposition 2.77.** *Let  $(G_1, +), \dots, (G_m, +)$  be abelian groups,  $A$  a graded analytic algebra and  $(A, \mathfrak{g}, \rho)$  a graded Lie–Rinehart algebra as in Definition 2.72, where  $m \in \mathbb{N}_{\geq 1}$ ,  $\mathfrak{g} \subseteq \text{Der}(A)$  and  $\rho : \mathfrak{g} \hookrightarrow \text{Der}'(A)$  is the canonical inclusion map. Keeping the notation and conditions of Definition 2.72, we say  $(A, \mathfrak{g}, \rho)$  is graded by  $(G_1 \times \dots \times G_m, +)$  with group homomorphism  $\Psi_{(g_1, \dots, g_m)}^{\mathfrak{g}_k}, (g_1, \dots, g_m) \in G_1 \times \dots \times G_m$ , if and only if there exist pairwise commuting group homomorphisms  $\pi_{g_1}^{\mathfrak{g}_k}, \dots, \pi_{g_m}^{\mathfrak{g}_k}, g_i \in G_i$  grading  $\mathfrak{g}_k$  and  $\pi_{g_1}^{A_k}, \dots, \pi_{g_j}^{A_k}$  the corresponding gradings of  $A_k$  satisfying*

$$A_k = \bigoplus_{g_1 \in G_1} \dots \bigoplus_{g_m \in G_m} \psi_{g_1}^{A_k}(A_k) \cap \dots \cap \psi_{g_m}^{A_k}(A_k)$$

and

$$\mathfrak{g}_k = \bigoplus_{g_1 \in G_1} \dots \bigoplus_{g_m \in G_m} \pi_{g_1}^{\mathfrak{g}_k}(\mathfrak{g}_k) \cap \dots \cap \pi_{g_m}^{\mathfrak{g}_k}(\mathfrak{g}_k)$$

for all  $k \in \mathbb{N}$ . Furthermore,  $\Psi_{(g_1, \dots, g_m)}^{\mathfrak{g}} = \pi_{g_m}^{\mathfrak{g}} \circ \dots \circ \pi_{g_1}^{\mathfrak{g}}$  and  $\Psi_{(g_1, \dots, g_m)}^A = \pi_{g_m}^A \circ \dots \circ \pi_{g_1}^A$  for all  $(g_1, \dots, g_m) \in G_1 \times \dots \times G_m$ , where the latter is the corresponding grading of  $A$ .

Due to the fact that a set of pairwise commuting derivations induces a multigrading on an analytic algebra  $A$ , we obtain, analogously to Theorem 2.74, that they induce a multigrading on  $\text{Der}(A)$ .

**Theorem 2.78.** *Let  $A$  be an analytic algebra and  $(A, \text{Der}'(A), \text{id})$  a Lie–Rinehart algebra. Denote the projections  $\text{Der}'(A) \rightarrow \text{Der}(A/\mathfrak{m}_A^k)$  by  $p_k$  and define  $\mathfrak{g}_k := p_k(\text{Der}'(A))$  for  $k \in \mathbb{N}$ . Assume that  $A$  is  $(\mathbb{C}^s, +)$  graded, where the grading is induced by the pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_s \in \text{Der}'(A)$ . Then  $\delta_1, \dots, \delta_s$  induce a multigrading on  $(A, \text{Der}'(A), \text{id})$  in the sense of Definition 2.76. In particular, every multihomogeneous  $\epsilon \in \text{Der}'(A)$  satisfies  $\text{ad}_{\delta_i}(\epsilon) = \lambda_i \epsilon$ , for  $1 \leq i \leq s$  some  $\lambda_i \in \mathbb{C}$ .*

*Proof.* The proof works analogously to the proof of Theorem 2.74. Due to Proposition 2.77 it only remains to show that for all  $1 \leq i, j \leq s$  it holds that  $\text{ad}_{\delta_i} \circ \text{ad}_{\delta_j} = \text{ad}_{\delta_j} \circ \text{ad}_{\delta_i}$ . This holds, since for any  $\tau \in \text{Der}'(A)$

$$\text{ad}_{\delta_i} \circ \text{ad}_{\delta_j}(\tau) = [\delta_i, [\delta_j, \tau]] = -[\delta_j, [\tau, \delta_i]] - [\tau, [\delta_i, \delta_j]] = [\delta_j, [\tau, \delta_i]] = \text{ad}_{\delta_j} \circ \text{ad}_{\delta_i}(\tau).$$

Now the group homomorphisms which grade  $\text{Der}(A_k)$  are the projections to the common eigenspaces of the  $\text{ad}_{\delta_i}^{\pm}$ , so the proof of Theorem 2.74 also works in this case.  $\square$

The following corollary follows from Theorem 2.78.

**Corollary 2.79.** *Let  $A$  be an analytic algebra and  $(A, \mathfrak{g}, \rho)$  a Lie–Rinehart algebra, where  $\mathfrak{g} \subseteq \text{Der}(A)$  and  $\rho : \mathfrak{g} \hookrightarrow \text{Der}'(A)$  is the canonical inclusion map. Assume that  $A$  is  $(\mathbb{C}^s, +)$  graded, where the grading is induced by the pairwise commuting diagonalizable derivations  $\delta_1, \dots, \delta_s \in \text{Der}'(A)$ . If  $[\delta_i, \mathfrak{g}] \subseteq \mathfrak{g}$ , then the  $\delta_1, \dots, \delta_s$  induce a multigrading on  $(A, \mathfrak{g}, \rho)$  in the sense of Definition 2.76. In particular, every multihomogeneous  $\epsilon \in \mathfrak{g}$  satisfies  $\text{ad}_{\delta_i}(\epsilon) = \lambda_i \epsilon$ , for  $1 \leq i \leq s$  some  $\lambda_i \in \mathbb{C}$ .*

## 2.4.2 The Structure Theorem

In this section we prove the structure theorem for logarithmic derivation modules. We make use of the fact that  $\text{Der}'_I(A)$  is a Lie–Rinehart algebra, thus we can use the notion of grading introduced in Section 2.4.1 in order to prove a statement about the structure of  $\text{Der}'_I(A)$ .

We present a new proof compared to [Epu15, Theorem 4.44] for the formal case:

**Theorem 2.80** (Structure Theorem). *Let either*

- (i)  $A = \mathbb{C}[[\mathbf{x}]]$  and  $I \subseteq A$ , or
- (ii)  $A = \mathbb{C}\{\mathbf{x}\}$  and  $I \subseteq A$  be an algebraic ideal.

Define  $\mathfrak{g} := \text{Der}'_I(A) \subseteq \text{Der}'(A)$  and let  $s \in \mathbb{N}$  be the rank of maximal multihomogeneity. Then there exist  $\delta_1, \dots, \delta_s, \nu_1, \dots, \nu_r \in \mathfrak{g}$ , such that

- (1)  $\delta_1, \dots, \delta_s, \nu_1, \dots, \nu_r$  is a minimal set of generators of  $\mathfrak{g}$  as an  $A$ -module,
- (2) if  $\sigma \in \mathfrak{g}$  with  $[\delta_i, \sigma] = 0$  for all  $i$ , then  $\sigma_S \in \langle \delta_1, \dots, \delta_s \rangle_{\mathbb{C}}$ ,
- (3)  $\delta_i$  is diagonal with eigenvalues in  $\mathbb{Q}$ ,
- (4)  $\nu_i$  is nilpotent, and
- (5)  $[\delta_i, \nu_j] \in \mathbb{Q} \cdot \nu_j$

*Proof.* By assumption there exists an algebraic torus  $T \subseteq \text{Aut}_I(A)$ . Proposition 2.41 allows us to choose the coordinates, such that  $T$  acts linearly. Due to Theorem 2.48 there exist  $s$  pairwise commuting diagonal derivations  $\delta_1, \dots, \delta_s$ . By Corollary 2.79 they induce a multigrading on  $\mathfrak{g}$ . In particular, we can extend  $\{\delta_1, \dots, \delta_s\}$  to a minimal multihomogeneous generating system of  $\mathfrak{g}$  by adding multihomogeneous elements  $\nu_1, \dots, \nu_r$ , where  $r \in \mathbb{N}$ . This proves the statements i), iii) and v). Denote by  $\underline{\lambda}_i = (\lambda_{i1}, \dots, \lambda_{is})$  the multidegree of  $\nu_i$ . If  $\lambda_{ij} \neq 0$  for a  $1 \leq j \leq s$ , then Lemma 1.59 implies that  $\nu_i$  is nilpotent. Thus we assume  $\lambda_{ij} = 0$  for all  $1 \leq j \leq s$ . Passing to the completion  $\hat{A}$ , we decompose  $\nu_i$  into its diagonal part  $\nu_{i,S}$  and into its nilpotent part  $\nu_{i,N}$ . Due to Proposition 1.57, we obtain  $[\delta_j, \nu_{i,S}] = 0$  for all  $1 \leq j \leq s$ . Since  $s$  is maximal we obtain that  $\nu_{i,S} \in \langle \delta_1, \dots, \delta_s \rangle_{\mathbb{C}}$ . This implies that  $\nu_{i,S} \in \text{Der}'_I(A)$ . Thus we can replace  $\nu_i$  with  $\nu_{i,N}$  and obtain iv). Statement ii) follows directly from the maximality of  $s$ .  $\square$

Next we present a small application of Theorem 2.80. In [Fab15] Faber gave a characterization of being normal crossing via the logarithmic derivation module of the local defining equation.

**Proposition 2.81.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and  $U \subseteq \mathbb{C}^n$  be an open neighborhood of  $\mathbf{0}$ , such that  $f$  is holomorphic on  $U$ . Define  $X := V(f) \subseteq U$ . Let  $p \in X$  be arbitrary. We define  $h_p \in \mathbb{C}\{\mathbf{x} - p\}$  to be the Taylor expansion of  $f$  around  $p$ . Then the following are equivalent:*

- (1)  $X$  has normal crossings at  $p \in X$ .
- (2)  $h_p$  is squarefree,  $\text{Der}(\mathcal{O}_{X,p})$  is free of rank  $n$  with basis  $\delta_1, \dots, \delta_n$  and  $[\delta_i, \delta_j] = 0$  for all  $1 \leq i, j \leq n$ .

*Proof.* See [Fab15, Proposition 6].  $\square$

Before we can prove our version of Proposition 2.81, we need the following lemma.

**Lemma 2.82.** *Let  $\delta = \sum_{i=1}^n a_i \partial_{x_i} \in \text{Der}(\mathbb{C}\{\mathbf{x}\})$  be a derivation with  $[\delta, \partial_{x_j}] = 0$  for  $1 \leq j \leq n$ . Then for all  $1 \leq i \leq n$  it holds that  $\partial_{x_j} a_i = 0$ .*

*Proof.* We assume, without loss of generality, that  $j = 1$ . Let  $\delta = \sum_{i=1}^n a_i \partial_{x_i}$ , where  $a_i \in \mathbb{C}\{\mathbf{x}\}$ . Then we can write:

$$[\partial_{x_1}, \delta] = \sum_{i=1}^n [\partial_{x_1}, a_i \partial_{x_i}] = \sum_{i=1}^n \partial_{x_1}(a_i) \cdot \partial_{x_i} = 0.$$

Comparing coefficients yields  $\partial_{x_1}(a_i) = 0$  for all  $1 \leq i \leq n$ .  $\square$

We can now prove the following:

**Theorem 2.83.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and  $U \subseteq \mathbb{C}^n$  be an open neighborhood of  $\mathbf{0}$ , such that  $f$  is holomorphic on  $U$ . Define  $X := V(f) \subseteq U$ . Let  $p \in X$  be arbitrary. We define  $h_p \in \mathbb{C}\{\mathbf{x} - p\}$  to be the Taylor expansion of  $f$  around  $p$ ,  $k_p := \dim_{\mathbb{C}} \text{Der}(\mathcal{O}_{X,p})(p)$  and  $o_p := \text{ord}(h_p)$ . Then the following are equivalent:*

- (1)  $X$  has normal crossings at  $p \in X$ .
- (2)  $o_p = n - k_p$ ,  $h_p$  is squarefree and  $\text{Der}(\mathcal{O}_{X,p})$  is free of rank  $n$ , such that there exists a basis consisting of  $n - k_p$  diagonalizable derivations  $\delta_1, \dots, \delta_{n-k_p}$  and  $k_p$  derivations  $\delta_{n-k_p+1}, \dots, \delta_n$  with non-constant term satisfying  $[\delta_i, \delta_j] = 0$  for all  $1 \leq i, j \leq n$ .

*Proof.* If  $X$  has normal crossings at  $p \in X$ , then the second statement is obvious. Let us assume that the second statement holds. Without loss of generality we can assume that  $p = \mathbf{0}$ . Therefore we write  $h, k$  and  $o$  instead of  $h_p, k_p$  and  $o_p$ . We prove an even stronger statement: we show that if the module  $\text{Der}_I(\mathbb{C}\{\mathbf{x}\})$ , where  $I = \langle h \rangle$ , has the desired structure, then  $h$  defines a normal crossing divisor after a suitable change of coordinates. Due to Theorem 1.70 we can assume that our coordinates are chosen in such a way, that  $\delta_{n-k+i} = \partial_{x_{n-k+i}}$  for  $1 \leq i \leq k$ . Due to order reasons, we know that  $\partial_{x_i} h = 0$  for all  $n - k + 1 \leq i \leq n$ . Lemma 2.82 implies that the derivations  $\delta_1, \dots, \delta_{n-k}$  only depend on the variables  $x_1, \dots, x_{n-k}$ , hence we can assume without loss of generality that  $k = 0$ . In this case we know that we have as many simultaneously diagonalizable derivations as variables and  $h$  has to be a monomial. By assumption,  $h$  is squarefree, so  $h = \lambda \cdot x_1 \cdot \dots \cdot x_n$  for a certain  $\lambda \in \mathbb{C} \setminus \{0\}$ . This implies that  $h$  defines a normal crossing divisor.  $\square$

**Remark 2.84.** *If we drop the assumption that  $h_p$  is squarefree, Theorem 2.83 states that  $h_p$  is a non-reduced normal crossing divisor.*

## Chapter 3

# Positive Gradings and Monomial Ideals

In this chapter we first want to show that the main results of Chapter 2, Section 2.3 hold in the presence of a positively graded analytic algebra  $\mathbb{C}\{\mathbf{x}\}/I$ . Furthermore, we show that hypersurface singularities with positively graded Tjurina algebra are strongly Euler-homogeneous at  $\mathbf{0}$  and thus are determined by their singular locus due to Theorem 1.101. Hypersurface singularities with monomial singular locus have a positively graded Tjurina algebra. Using combinatorial results on monomial ideals we are going to classify all hypersurface singularities with radical monomial Jacobian ideal.

### 3.1 Positively Graded Analytic Algebras

We know from Remark 2.66 that once our analytic algebra  $A = \mathbb{C}\{\mathbf{x}\}/I$  admits a positive grading induced by  $\delta \in \text{Der}_I(A)$ , every multigrading of  $A$  contains a positive grading induced by another derivation  $\delta' \in \text{Der}_I(A)$  which has the same eigenvalues as  $\delta$ . If the maximal multigrading of  $A$  is induced by say  $\delta_1, \dots, \delta_s$ , with  $\delta_1 = \delta$ , then the main ingredient to prove Theorem 2.63 was to use that we were able to simultaneously diagonalize the  $\delta_i$ . In the presence of a positive grading we show that we are still able to do this. The tool we need is the following approximation theorem due to Artin.

**Theorem 3.1** (Artin's Approximation Theorem). *Let  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_N)$  and  $f_1(\mathbf{x}, \mathbf{y}), \dots, f_m(\mathbf{x}, \mathbf{y}) \in \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  for certain  $m, n, N \in \mathbb{N}_{\geq 1}$ . Fix an integer  $c \in \mathbb{N}$ . If there exists a formal solution  $\hat{\mathbf{y}} \in \langle \mathbf{x} \rangle \mathbb{C}[[\mathbf{x}]]^N$  of the system of equations*

$$f_i(\mathbf{x}, \hat{\mathbf{y}}) = 0, \quad i = 1, \dots, m, \quad (3.1)$$

*then there exists also a convergent solution  $\tilde{\mathbf{y}} \in \mathbb{C}\{\mathbf{x}\}^N$  of (3.1) such that*

$$\hat{\mathbf{y}} = \tilde{\mathbf{y}} \quad \text{mod } \langle \mathbf{x} \rangle^c.$$

*Proof.* See [Art68, Theorem 1.2]. □

Using Artin's Approximation Theorem we can prove the following result regarding ideal containment.

**Corollary 3.2.** *Let  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_N), I_1, \dots, I_p \subseteq \mathbb{C}\{\mathbf{x}\}$  be ideals and  $f_{1k}(\mathbf{x}, \mathbf{y}), \dots, f_{m_k k}(\mathbf{x}, \mathbf{y}) \in \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$  for certain  $p, m_k, n, N \in \mathbb{N}_{\geq 1}$  and  $1 \leq k \leq p$ . Fix an integer  $c \in \mathbb{N}_{\geq 1}$ . If there exists a  $\hat{\mathbf{y}} \in \langle \mathbf{x} \rangle \mathbb{C}[[\mathbf{x}]]^N$ , such that*

$$f_{jk}(\mathbf{x}, \hat{\mathbf{y}}) \in I_k \mathbb{C}[[\mathbf{x}]] \text{ for all } j = 1, \dots, m_k, k = 1, \dots, p$$

*then there exists also a  $\tilde{\mathbf{y}} \in \mathbb{C}\{\mathbf{x}\}^N$  such that*

$$f_{jk}(\mathbf{x}, \tilde{\mathbf{y}}) \in I_k \text{ for all } j = 1, \dots, m_k, k = 1, \dots, p$$

*and such that*

$$\hat{\mathbf{y}} = \tilde{\mathbf{y}} \pmod{\langle \mathbf{x} \rangle^c}.$$

*Proof.* We can assume that the  $I_j$  have the same number of generators, since we can add 0 functions to the generating sets. Denote this number of generators by  $l \in \mathbb{N}$ . Assume  $I_j = \langle g_{1j}, \dots, g_{lj} \rangle$  for certain  $g_{ij} \in \mathbb{C}\{\mathbf{x}\}$ . Again by adding 0 functions we can assume that the  $m_j$  have all the same value  $m \in \mathbb{N}$ . Adding formal variables  $z_{ij}^{(k)}$  we consider the system of equations:

$$f_{jk}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^l z_{ij}^{(k)} g_{ik} \text{ for all } i = 1, \dots, m, j = 1, \dots, l, k = 1, \dots, p. \quad (3.2)$$

Then the ideal containment condition is equivalent to the existence of constants  $c_{ij}^{(k)} \in \mathbb{C}$  and formal power series  $\hat{z}_{ij}^{(k)} \in \langle \mathbf{x} \rangle \mathbb{C}[[\mathbf{x}]]$ , such that

$$f_{jk}(\mathbf{x}, \hat{\mathbf{y}}) - \sum_{i=1}^l (\hat{z}_{ij}^{(k)} + c_{ij}^{(k)}) g_{ik} = 0 \text{ for all } i = 1, \dots, m, j = 1, \dots, l, k = 1, \dots, p. \quad (3.3)$$

Applying Artin’s Approximation Theorem to Equation 3.3 yields the existence of  $\tilde{\mathbf{y}}, \tilde{\mathbf{z}} \in \mathbb{C}\{\mathbf{x}\}$ , such that Equation 3.3 holds. This is equivalent to  $f_{jk}(\mathbf{x}, \tilde{\mathbf{y}}) \in I_k$  for all  $j = 1, \dots, m_k, k = 1, \dots, p$ . The equality of the formal and convergent solution up to degree  $c$  follows also immediately from Artin’s Approximation Theorem.  $\square$

Now we are able to prove the following lemma.

**Lemma 3.3.** *Let  $I_1, \dots, I_p \subseteq \mathbb{C}\{\mathbf{x}\}$  be ideals with  $I_k = \langle f_{1k}, \dots, f_{l_k k} \rangle$  for certain  $p, l_k, n \in \mathbb{N}$ . Let  $g_{jk} \in \mathbb{C}\{\mathbf{x}\}$  for  $1 \leq j \leq l_k, 1 \leq k \leq p$ . If there exists an automorphism  $\hat{\varphi} \in \text{Aut}(\mathbb{C}[[\mathbf{x}]])$ , such that  $\hat{\varphi}(\hat{I}_k) = \langle g_{1k}, \dots, g_{m_k k} \rangle \mathbb{C}[[\mathbf{x}]]$  for  $1 \leq k \leq p$ , then there exists an automorphism  $\tilde{\varphi} \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$ , such that  $\tilde{\varphi}(I_k) = \langle g_{1k}, \dots, g_{m_k k} \rangle$  for  $1 \leq k \leq p$ .*

*Proof.* Define  $\hat{y}_i := \hat{\varphi}(x_i) \in \langle \mathbf{x} \rangle \mathbb{C}[[\mathbf{x}]]$  for  $i = 1, \dots, n$ . By adding 0 functions, we can assume that for all  $1 \leq k \leq p$  it holds that  $m_k = m$ , where  $m \in \mathbb{N}$  is a constant. In the same manner, we can assume that for all  $1 \leq k \leq p$  it holds that  $l_k = l$ , where  $l \in \mathbb{N}$  is a constant. Then  $\hat{\varphi}(\hat{I}_k) = \langle g_{1k}, \dots, g_{m_k k} \rangle \mathbb{C}[[\mathbf{x}]]$  is equivalent to the fact that  $\hat{\mathbf{y}}$  solves the ideal containment

$$f_{jk}(\hat{\mathbf{y}}) \in \langle g_{1k}, \dots, g_{m_k k} \rangle \mathbb{C}[[\mathbf{x}]] \text{ for all } j = 1, \dots, l, k = 1, \dots, p.$$

By Corollary 3.2 we obtain the existence of  $\tilde{\mathbf{y}} \in \mathbb{C}\{\mathbf{x}\}^n$  satisfying

$$f_{jk}(\tilde{\mathbf{y}}) \in \langle g_{1k}, \dots, g_{m_k k} \rangle \text{ for all } j = 1, \dots, l, k = 1, \dots, p$$

and

$$\hat{\mathbf{y}} = \tilde{\mathbf{y}} \pmod{\langle \mathbf{x} \rangle^2}.$$

Define  $\tilde{\varphi}(x_i) := \tilde{y}_i$  for  $i = 1, \dots, n$ . Due to the fact that  $\tilde{\mathbf{y}}$  equals  $\hat{\mathbf{y}}$  up to degree 1, we know that  $\tilde{\varphi} \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$ . Then this is equivalent to  $\tilde{\varphi}(I_k) \subseteq \langle g_{1k}, \dots, g_{mk} \rangle$  for all  $1 \leq k \leq p$ . By faithful flatness of  $\mathbb{C}[[\mathbf{x}]]$  over  $\mathbb{C}\{\mathbf{x}\}$  we obtain equality, since  $\tilde{\varphi}(I_k)\mathbb{C}[[\mathbf{x}]] = \langle g_{mk}, \dots, g_{mk} \rangle\mathbb{C}[[\mathbf{x}]]$ .  $\square$

Now we can prove that every positively graded ideal is algebraic.

**Proposition 3.4.** *Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal. If there exists a diagonalizable derivation  $\delta \in \text{Der}_I$  with positive eigenvalues, then there exists an automorphism  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$ , such that  $\varphi(I)$  is generated by ( $\delta$ -homogeneous) polynomials  $g_1, \dots, g_m \in \mathbb{C}[\mathbf{x}]$ . In particular,  $I$  is algebraic after a suitable change of coordinates.*

*Proof.* Applying Theorem 2.10 to the ideal  $I$ , we know that there exists a generating set consisting of multihomogeneous elements. Applying Nakayama’s Lemma allows us to reduce this to a minimal multihomogeneous generating set. There exists a formal coordinate change  $\hat{\varphi} \in \text{Aut}(\mathbb{C}[[\mathbf{x}]])$ , such that  $\delta$  is diagonal and the generators are common eigenfunctions of  $\delta$ . This implies that these generators have to be polynomials. Denote them by  $g_1, \dots, g_m \in \mathbb{C}[\mathbf{x}]$ . Then  $\hat{\varphi}(I) = \langle g_1, \dots, g_m \rangle\mathbb{C}[[\mathbf{x}]]$  and the result follows from Lemma 3.3.  $\square$

Theorem 2.63 implies the following:

**Corollary 3.5.** *Let  $A = \mathbb{C}\{\mathbf{x}\}$  and  $I \subseteq A$  be an ideal. Assume that  $A/I$  is maximal multihomogeneous of rank  $s$  and that  $A/I$  admits a positive grading. Denote by  $\delta_1, \dots, \delta_s \in \text{Der}_I(A)$  the pairwise commuting diagonalizable derivations inducing this grading. Then there exists a maximal algebraic torus  $T \subseteq \text{Aut}_I(A)$  with  $\dim(T) = s$ .*

Using positivity we obtain two sequences we can associate to an ideal. The first one depends on the choice of coordinates, whereas the second one is independent of the coordinates. Let us define the first sequence.

**Definition 3.6.** *Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal and  $\delta_1, \dots, \delta_m \in \text{Der}_I(\mathbb{C}\{\mathbf{x}\})$  be pairwise commuting diagonalizable derivations with integer eigenvalues. Assume that the eigenvalues of  $\delta_1$  are positive integers and that the coordinates are chosen in such a way, that the  $\delta_i$  are diagonal. Denote by  $\mathbf{w}_i \in \mathbb{Z}^s$  the weight-vectors arising from the  $\delta_i$ . Let  $f_1, \dots, f_k$  be a minimal multihomogeneous polynomial generating set of  $I$  and define  $d_{ij} := \deg_{\mathbf{w}_j}(f_i)$  as well as  $\mathbf{d}_i := (d_{i1}, \dots, d_{is}) \in \mathbb{Z}^s$ . Assume that  $\mathbf{d}_1, \dots, \mathbf{d}_k$  are already ordered increasingly with respect to the lexicographical ordering. Then we call the sequence*

$$(\mathbf{d}_1, \dots, \mathbf{d}_k)$$

*the weight-sequence of  $I$ .*

It remains to show that once a coordinate system is fixed, the weight-sequence does not depend on the minimal generators of  $I$ .

**Proposition 3.7.** *Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal and  $\delta_1, \dots, \delta_m \in \text{Der}_I(\mathbb{C}\{\mathbf{x}\})$  be pairwise commuting diagonalizable derivations with integer eigenvalues. Assume that the eigenvalues of  $\delta_1$  are positive integers and that the coordinates are chosen in such a way, that the  $\delta_i$  are diagonal. Let  $(\mathbf{d}_1, \dots, \mathbf{d}_k)$  as well as  $(\mathbf{d}'_1, \dots, \mathbf{d}'_k)$  be weight-sequences of  $I$ . Then it holds that*

$$(\mathbf{d}_1, \dots, \mathbf{d}_k) = (\mathbf{d}'_1, \dots, \mathbf{d}'_k).$$



*Proof.* Due to the fact that all multihomogeneous minimal generating systems consist of polynomials the result follows from the statement in the polynomial case. For details, see for example [KR05, Proposition 4.7.8].  $\square$

The next sequence is independent of the chosen coordinate system.

**Definition 3.8.** Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal with minimal generating system  $f_1, \dots, f_k$ . Define  $d_i := \text{ord}(f_i)$ . Assume that the  $d_i$  are ordered increasingly. Then we call the sequence

$$(d_1, \dots, d_k)$$

the *order-sequence* of  $I$ .

We finish this section with the proof that the order sequence is an invariant of the ideal  $I$ .

**Proposition 3.9.** Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal with  $k$  minimal generators and let  $(d_1, \dots, d_k)$  as well as  $(d'_1, \dots, d'_k)$  be order-sequences of  $I$ . Then it holds that

$$(d_1, \dots, d_k) = (d'_1, \dots, d'_k).$$

*Proof.* Being in a fixed coordinate system, the equality of the sequences follows from Nakayama's Lemma and the fact that  $\text{ord}(x_i) = 1 > 0$ . The coordinate invariance follows from the fact that every  $\varphi \in \text{Aut}_I(\mathbb{C}\{\mathbf{x}\})$  satisfies  $\varphi(\mathfrak{m}) \subseteq \mathfrak{m}$ .  $\square$

## 3.2 Positively Graded Tjurina Algebra

In this section we want to prove that every hypersurface singularity  $f$  with positively graded Tjurina algebra is strongly Euler-homogeneous. From a computational point of view, it is harder to check if the Tjurina algebra is positively graded, than to check whether  $f \in \mathfrak{m}J_f$  or not. The main importance for this result is the fact that we can use it to classify a special type of singularities, so-called **normal crossing divisors**, using only the Jacobian ideal  $J_f$ . Moreover, the class of hypersurface singularities with monomial singular locus have a positively graded Tjurina algebra, hence they are strongly Euler-homogeneous and thus their isomorphy class is determined by the isomorphy class of their singular locus (see Theorem 1.101). The main property of strongly Euler homogeneous singularities we are going to use is the following:

**Lemma 3.10.** Let  $X \subseteq \mathbb{C}^n$  be a hypersurface singularity. Assume that  $X \cong X' \times \mathbb{C}^k$  for  $1 \leq k < n$ . Then  $X$  is strongly Euler-homogeneous if and only if  $X'$  is strongly Euler homogeneous.

*Proof.* This follows from [GS06, Lemma 3.2].  $\square$

We start with the main result of this section. The statement and proof are generalized versions of [XY96, Theorem 1.2], where we use the analytic grading instead of the classical grading.

**Theorem 3.11.** Let  $f \in \mathfrak{m} \subseteq \mathbb{C}\{\mathbf{x}\}$  and assume that the Tjurina algebra  $\mathbb{T}_f$  admits a positive grading. Then  $f \in \mathfrak{m}J_f$ . Equivalently, the germ  $(V(f), \mathbf{0})$  is strongly Euler-homogeneous at  $\mathbf{0}$ .



*Proof.* We assume that  $f$  is not analytically trivial, that is for every derivation  $\epsilon \in \text{Der}_{\langle f \rangle}(\mathbb{C}\{\mathbf{x}\})$  it holds that  $\epsilon(\mathbf{0}) = \mathbf{0}$ . Using Proposition 2.31 this does not change the assumption on the positive grading of  $\mathbb{T}_f$ .

In case  $f \in \mathfrak{m} \setminus \mathfrak{m}^2$  we know that we can find a  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$ , such that  $\varphi(f) = x_1$ . In this case  $f$  is obviously strongly Euler-homogeneous. Due to this we assume from now on that  $f \in \mathfrak{m}^2$ . Define  $I := \langle f, f_{x_1}, \dots, f_{x_n} \rangle$ . By assumption, we know that there exists a diagonalizable derivation  $\delta \in \text{Der}_I(\mathbb{C}\{\mathbf{x}\})$ , such that  $\delta(I) \subseteq I$  and  $\delta$  has positive integer eigenvalues. Due to Proposition 3.4 we can assume that  $\delta = \sum_{i=1}^n w_i x_i \partial_{x_i}$  with  $w_i \in \mathbb{Z}_{>0}$ . In this case we can write  $I = \sum_{d \in \mathbb{Z}_{\geq 0}} I_d$ , where the  $I_d$  are the homogeneous

components of  $I$  with weighted degree  $d$ . The positivity of  $\delta$  implies the finiteness of  $\dim_{\mathbb{C}} I_d$ . Since  $\delta$  acts by multiplication with  $d$  on  $I_d$  we can consider it as a bijective linear operator on all  $I_d$  with  $d > 0$ . By assumption  $f$  is a non-unit, hence we know that  $I_0 = \{0\}$ . This implies that  $\delta$  acts bijectively on  $I$ . The inclusion  $\delta(I) \subseteq I$  yields the existence of  $a_i \in \mathbb{C}\{\mathbf{x}\}$  and  $a_{i1}, \dots, a_{in} \in \mathbb{C}\{\mathbf{x}\}$  such that

$$\delta(f_{x_i}) = a_i f + \sum_{j=1}^n a_{ij} f_{x_j} \quad (3.4)$$

for all  $1 \leq i \leq n$ . Due to the surjectivity of  $\delta$  there exist  $d_i \in \mathbb{C}\{\mathbf{x}\}$  and  $d_{i1}, \dots, d_{in} \in \mathbb{C}\{\mathbf{x}\}$  satisfying

$$f_{x_i} = \delta \left( d_i f + \sum_{j=1}^n d_{ij} f_{x_j} \right) = \left( \delta(d_i) + \sum_{j=1}^n d_{ij} a_j \right) f + \sum_{j=1}^n \left[ d_i w_j x_j + \delta(d_{ij}) + \sum_{l=1}^n d_{il} a_{lj} \right] f_{x_j}. \quad (3.5)$$

Define the derivations

$$\tau_i = \sum_{j=1}^n \left[ d_i w_j x_j + \delta(d_{ij}) + \sum_{l=1}^n d_{il} a_{lj} - e_{ij} \right] \frac{\partial}{\partial x_j} \in \text{Der}(\mathbb{C}\{\mathbf{x}\}),$$

where  $(e_{ij})_{ij} \in \mathbb{R}^{n \times n}$  is the unit matrix. Then Equation 3.5 implies  $\tau_i(f) \in \langle f \rangle$ . Write  $\tau_i = \sum_{j=1}^n \tau_{ij} \frac{\partial}{\partial x_j}$ . By assumption  $f$  is not analytically trivial, hence  $\tau_i(\mathbf{0}) = \mathbf{0}$ , or equivalently  $\tau_{ij}(\mathbf{0}) = 0$  for  $1 \leq i, j \leq n$ . This implies

$$\left( \sum_{l=1}^n d_{il} a_{lj} - e_{ij} \right) (\mathbf{0}) = 0 \quad (3.6)$$

for all  $1 \leq i \leq n$ . Equation 3.6 is equivalent to the matrix  $(a_{ij}(\mathbf{0}))_{ij}$  being invertible. Using the bijectivity of  $\delta$  on  $I$ , there exist  $c, c_1, \dots, c_n \in \mathbb{C}\{\mathbf{x}\}$  such that:

$$f = \delta \left( c f + \sum_{j=1}^n c_j f_{x_j} \right) = \left( \delta(c) + \sum_{j=1}^n a_j c_j \right) f + \sum_{j=1}^n \left[ c w_j x_j + \delta(c_j) + \sum_{i=1}^n c_i a_{ij} \right] f_{x_j} \quad (3.7)$$

Define the derivation

$$\eta := \sum_{j=1}^n \left[ c w_j x_j + \delta(c_j) + \sum_{i=1}^n c_i a_{ij} \right] \frac{\partial}{\partial x_j} \in \text{Der}(\mathbb{C}\{\mathbf{x}\}).$$

Then  $\eta(f) = \left[ 1 - \delta(c) - \sum_{i=1}^n a_i c_i \right] f$ . By construction  $\eta(f) \in \langle f \rangle$ . Since  $f$  is not analytically trivial, we obtain

$$\left( \sum_{i=1}^n c_i a_{ij} \right) (\mathbf{0}) = 0 \quad (3.8)$$

for  $1 \leq i \leq n$ . Equation 3.8 implies that the vector  $(c_i(\mathbf{0}))_i$  is in the kernel of the matrix  $(a_{ij}(\mathbf{0}))_{ij}$ . Since the latter is invertible, we obtain  $c_i(\mathbf{0}) = 0$  for  $1 \leq i \leq n$ . This implies that  $u := 1 - \delta(c) - \sum_{i=1}^n a_i c_i$  is a unit in  $\mathbb{C}\{\mathbf{x}\}$ . Now  $u^{-1}\eta(f) = f$ , hence  $f$  is strongly Euler-homogeneous.  $\square$

Theorem 3.11 implies the following two corollaries.

**Corollary 3.12.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and  $(X, \mathbf{0}) := (V(f), \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$ . If  $(\text{Sing}(X), \mathbf{0})$  is positively graded, then the isomorphism class of  $(X, \mathbf{0})$  is determined by the isomorphism class of  $\mathbb{T}_f$ .*

**Corollary 3.13.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and  $(X, \mathbf{0}) := (V(f), \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$ . If  $(\text{Sing}(X), \mathbf{0})$  is defined by a monomial ideal, then the isomorphism class of  $(X, \mathbf{0})$  is determined by the isomorphism class of  $\mathbb{T}_f$ .*

The following lemma shows that the Tjurina ideal  $\langle f, J_f \rangle$  and the Jacobian ideal  $J_f$  coincide in case they are positively graded and radical.

**Lemma 3.14.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$ . If either  $\langle f, J_f \rangle$  is positively graded, or  $J_f$  is radical, then  $\langle f, J_f \rangle = J_f$ .*

*Proof.* In case  $\langle f, J_f \rangle$  is positively graded and radical, Theorem 3.11 implies  $f \in \mathfrak{m}J_f$ , hence  $\langle f, J_f \rangle = J_f$ . Now assume that  $J_f$  is radical. By [Fab15, Lemma 1]  $J_f$  being radical implies  $f \in J_f$  and the statement follows.  $\square$

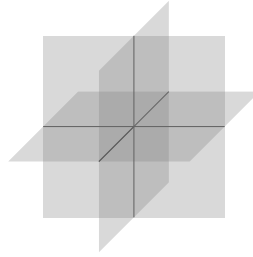
The next aim of this section is to deduce a numerical characterization for normal crossing divisors. First we need to define them.

**Definition 3.15.** *Let  $X \subseteq \mathbb{C}^n$ . We say  $X$  is **normal crossing at**  $p \in X$ , if there exists a coordinate system  $x_1, \dots, x_n$  at  $p$  and an integer  $k \in \mathbb{N}_{\geq 1}$  with  $k \leq n$ , such that*

$$(X, p) \cong (V(x_1 \cdots x_k), p).$$

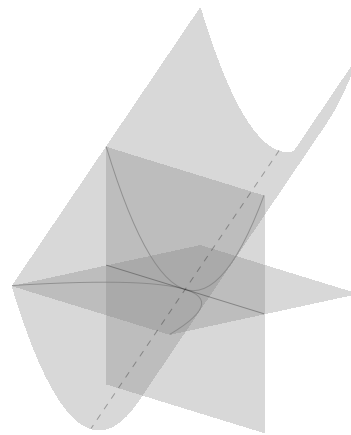
*If  $X$  is normal crossing at  $p$  we also call  $(X, p)$  a **normal crossing divisor**. If  $f \in \mathbb{C}\{\mathbf{x}\}$  defines a normal crossing singularity in  $\mathbf{0}$  we also say that  $f$  is a normal crossing divisor.*

**Example 3.16.** *We have already encountered an example for a normal crossing divisor in Remark 1.95. We consider  $f = xyz \in \mathbb{C}\{x, y, z\}$ . The real picture of  $(V(f), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$  looks as follows:*



**Figure 3.1:** Real picture of the normal crossing divisor  $(V(xyz), \mathbf{0}) \subseteq (\mathbb{C}^3, \mathbf{0})$ .

To see an example of a non-normal crossing divisor at  $\mathbf{0}$ , we consider the equation  $g = xz(x + z - y^2) \in \mathbb{C}\{x, y, z\}$ . The name of corresponding surface is **Tülle**. Tülle looks as follows:



**Figure 3.2:** Real picture of Tülle.

Outside of  $\mathbf{0}$  Tülle has normal crossings, but in  $\mathbf{0}$  it does not.

In [Fab15] Faber gave an algebraic characterization of being normal crossing. The characterization given there needs the property of a hypersurface being free and it uses the normalization. For the notion of normalization see [GLS07, Chapter 1.9]. The notion of freeness is due to Saito (see [Sai80]) and looks as follows.

**Definition 3.17.** Let  $X \subseteq \mathbb{C}^n$  be a divisor. We call  $X$  a **free divisor at a point**  $p \in X$  if the module of logarithmic derivations  $\text{Der}(\mathcal{O}_{X,p})$  is a free module. We say  $X$  is a **free divisor** if  $X$  is free at any point  $p \in X$ . We say the complex space germ  $(X, \mathbf{0})$  is a free divisor, if there exists a representant which is a free divisor.

Faber states the following criterion to decide whether a hypersurface singularity is normal crossing at a point  $p \in X$ .

**Theorem 3.18.** Let  $X \subseteq \mathbb{C}^n$ . For every  $p \in X$  we denote by  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  the local equation of  $X$  at  $p$ . Denote by  $\pi : \tilde{X} \rightarrow X$  the normalization of  $X$ . Then the following are equivalent:

- (1)  $X$  has normal crossings at  $p \in X$ .
- (2)  $X$  is free at  $p$ ,  $J_{f,p}$  is radical and  $(\tilde{X}, \pi^{-1}(p))$  is smooth.

*Proof.* See [Fab15, Theorem 1].  $\square$

We want to give other criteria to check whether a hypersurface has normal crossings at a point  $p$  or not, which do not use the normalization. The first one is the following.

**Theorem 3.19.** *Let  $X \subseteq \mathbb{C}^n$ . For every  $p \in X$  we denote by  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  the local equation of  $X$  at  $p$ . Furthermore, we define  $k_p := \dim_{\mathbb{C}} \text{Der}(\mathcal{O}_{X,p})(p)$  and  $o_p := \text{ord}(h_p)$ . Then the following are equivalent:*

- (1)  $X$  has normal crossings at  $p \in X$ .
- (2)  $o_p = n - k_p$ ,  $J_{f,p}$  is radical,  $\langle f_p, J_{f,p} \rangle$  is, after a suitable change of coordinates, minimally generated by  $n - k_p$  monomials of order  $o_p - 1$ .

*Proof.* The second statement follows by a simple computation in case  $X$  has normal crossings at  $p$ , thus we only need to prove the converse. For simplicity we assume  $p = \mathbf{0}$  and we consider the complex space germ  $(X, \mathbf{0})$ . Corollary 3.13 implies that  $(X, \mathbf{0})$  is determined by  $(\text{Sing}(X), \mathbf{0})$ , since the monomiality of  $\langle f, J_f \rangle$  implies that  $T_f$  is positively graded. Before we prove the result, we reduce to the case  $k_{\mathbf{0}} = 0$ . By Theorem 1.70 it holds that  $(X, \mathbf{0}) \cong (X', \mathbf{0}) \times (\mathbb{C}^{k_{\mathbf{0}}}, \mathbf{0})$ . This implies the existence of a  $f' \in \mathbb{C}\{\mathbf{x}'\}$  with  $(X', \mathbf{0}) = (V(f'), \mathbf{0})$ . Then it also holds that  $(\text{Sing}(X), \mathbf{0}) \cong (\text{Sing}(X'), \mathbf{0}) \times (\mathbb{C}^{k_{\mathbf{0}}}, \mathbf{0})$ . In our case  $(\text{Sing}(X), \mathbf{0})$  corresponds to the algebra  $T_f$ , which is generated by monomials. In particular, it has maximal multihomogeneity  $n$ . Thus, by Proposition 2.32,  $(\text{Sing}(X'), \mathbf{0})$  has maximal multihomogeneity  $n - k_{\mathbf{0}}$  and  $T_{f'}$  is also generated by monomials. Theorem 3.11 yields  $f' \in J_{f'}$ . Using [GR71, §5 Satz 17] we know that reducedness is preserved under analytic tensor products and since  $T_f = T_{f'} \hat{\otimes}_{\mathcal{O}_{\mathbb{C}^{k_{\mathbf{0}}}, \mathbf{0}}}$ , the ideal  $\langle f', J_{f'} \rangle = J_{f'}$  is radical. These considerations show that we can replace  $f$  by  $f'$  or, equivalently, assume  $k_{\mathbf{0}} = 0$ . Let  $g = x_1 \cdot \dots \cdot x_n$ . The assumption on the monomial generators yields that

$$J_f = \left\langle \frac{g}{x_1}, \dots, \frac{g}{x_n} \right\rangle = J_g. \quad (3.9)$$

Equation 3.9 is equivalent to saying that  $T_f = T_g$ , which by Theorem 1.101 is equivalent to

$$(X, \mathbf{0}) \cong (V(x_1 \cdot \dots \cdot x_n), \mathbf{0}).$$

By definition  $X$  has a normal crossing at  $\mathbf{0}$ .  $\square$

**Remark 3.20.** *The property of an ideal being generated by monomials is equivalent to having maximal multihomogeneity  $n$ . Due to this all quantities respectively properties appearing in Theorem 3.19 are invariants. Theorem 1.70 implies that  $k_{\mathbf{0}}$  is an invariant, being radical is an algebraic property, which does not depend on the chosen coordinate system, and the order of an element respectively the order-sequence of a minimal generating system are invariants due to Proposition 3.9. We are going to see in Chapter 5 how to compute these invariants.*

Let us have a look at two examples.

**Example 3.21.** *Let us have a look at a simple example to verify our criterion. Consider the polynomial  $f = (x+y^2)(y+xy) \in \mathbb{C}\{x, y, z\}$ . We want to see whether  $X := V(f)$  has normal crossings in  $\mathbf{0}$  or not. First we notice that  $k_{\mathbf{0}} = 1$  and that  $o_{\mathbf{0}} = 2 = n - k_{\mathbf{0}}$ . In this simple case both invariants can be read off from the defining equation. By SINGULAR computation we*

obtain  $\langle f, J_f \rangle = \langle x, y \rangle$ . Thus  $\langle f, J_f \rangle$  is a monomial ideal, which immediately implies that  $T_f$  is positively graded. Due to this we know  $f \in \mathfrak{m}J_f$ , hence  $J_f$  is also a radical ideal. The order sequence of the Jacobian ideal is  $(1, 1) = (n - k_{\mathbf{0}} - 1, n - k_{\mathbf{0}} - 1)$ . Then Theorem 3.19 yields  $(X, \mathbf{0}) \cong (V(x \cdot y), \mathbf{0})$  and  $X$  has normal crossings at  $\mathbf{0}$ .

Next we want to show that Tülle from Example 3.16 has no normal crossings at  $\mathbf{0}$ . Let  $g = xz(x + z - y^2) \in \mathbb{C}\{x, y, z\}$  and  $Y := V(g)$ . A SINGULAR computation shows that

$$\langle g, J_g \rangle = \langle x^2 + 2xz - xy^2, 2xz + z^2 - y^2z, yz^2 - y^3z \rangle.$$

This generating system is minimal, hence we have the order-sequence  $(2, 2, 3)$ . By Theorem 3.19  $Y$  cannot have normal crossings at  $\mathbf{0}$ .

**Remark 3.22.** Theorem 3.19 has the advantage that, if the order sequence has different entries, we are already done in disproving that the given hypersurface has normal crossings, whereas all checks in Theorem 3.18 can be time consuming.

### 3.3 Stanley–Reisner Monomial Singular Loci

In the previous section we have seen, that we can characterize normal crossing divisors almost only by numerical properties of the Jacobian ideal. Motivated by the second problem of Hauser and Schicho in [HS11] to classify all hypersurface singularities  $f$  where the ideal  $\langle f, J_f \rangle$  is monomial, we investigate this setup for the special class of ideals of so-called Stanley–Reisner type. We give a complete characterization of hypersurfaces, where the ideal  $\langle f, J_f \rangle$  is of Stanley–Reisner type. The motivation for question arises from the fact that monomial ideals in general admit combinatorial descriptions, see for example [HH11, Part III]. In order to state the main result of this section we need the following definitions:

**Definition 3.23.** Let  $A = \mathbb{C}\{\mathbf{x}\}$  or  $\mathbb{C}[[\mathbf{x}]]$ . Let  $I \subseteq A$  be an ideal. We say  $I$  is an ideal of **monomial type**, if there exists an automorphism  $\varphi \in \text{Aut}(A)$ , such that  $\varphi(I)$  is a monomial ideal. We say  $I$  is an ideal of **Stanley–Reisner type**, if  $I$  is a radical ideal of monomial type.

**Definition 3.24.** Let  $f \in \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ . We say  $f$  is of **Sebastiani–Thom type**, if there exist  $g \in \mathbb{C}\{\mathbf{x}\}$  and  $h \in \mathbb{C}\{\mathbf{y}\}$ , such that  $f = g + h$ . We say that a hypersurface singularity  $X \subseteq \mathbb{C}^{n+m}$  is of Sebastiani–Thom type at  $p = (p_1, p_2) \in X$  if there exists an isomorphism such that  $(X, p) \cong (V(f), p)$ , where  $f \in \mathbb{C}\{\mathbf{x} - p_1, \mathbf{y} - p_2\}$  is of Sebastiani–Thom type. We call  $X$  a **Sebastiani–Thom type hypersurface singularity**, if it is of Sebastiani–Thom type for all  $p \in X$ . We say a complex space germ  $(X, \mathbf{0})$  is of Sebastiani–Thom type, if there exists a representant which is of Sebastiani–Thom type. Consider the germ  $(X, \mathbf{0}) \cong (V(f), \mathbf{0})$  with  $f = g + h$  and  $g \in \mathbb{C}\{\mathbf{x}\}$ ,  $h \in \mathbb{C}\{\mathbf{y}\}$ . We call the germs  $(X_1, \mathbf{0}) = (V(g), \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$  and  $(X_2, \mathbf{0}) = (V(h), \mathbf{0}) \subseteq (\mathbb{C}^m, \mathbf{0})$  the **Sebastiani–Thom components** of  $(X, \mathbf{0})$ .

**Theorem 3.25.** Let  $f \in \mathbb{C}\{\mathbf{x}\}$ . Then  $\langle f, J_f \rangle$  being an ideal of Stanley–Reisner type is equivalent to the existence of an automorphism  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$  and a partition of the  $\mathbf{x}$  variables, denoted by  $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l+1)}$ , such that

$$\varphi(f) = \sum_{j=1}^{r_1} \left(x_j^{(0)}\right)^2 + \sum_{i=1}^l g_i,$$

where  $g_i \in \mathbb{C}[\mathbf{x}^{(i)}]$  is a normal crossing divisor for  $1 \leq i \leq l$ . This means that all singularities with Stanley–Reisner singular locus are of Sebastiani–Thom type where the summands are

$A_1$ -singularities or normal crossing divisors. In particular, the Sebastiani–Thom components are unique up to isomorphism and permutation.

Theorem 3.25 is similar to the Splitting Lemma (see Lemma 1.80). It states that every singularity with Stanley–Reisner singular locus is the sum of an  $A_1$ -singularity and a Sebastiani–Thom singularity whose summands are all normal crossing divisors.

**Definition 3.26.** Let  $f \in \mathbb{C}\{\mathbf{x}\}$ . We say  $f$  defines a **generalized normal crossing divisor**, if there exists a coordinate change  $\varphi \in \mathbb{C}\{\mathbf{x}\}$  and a partition of  $\mathbf{x}$  denoted by  $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l+1)}$ , such that

$$\varphi(f) = \sum_{i=0}^l g_i,$$

where  $g_0 \in \mathbb{C}[\mathbf{x}]$  is an  $A_1$ -singularity and  $g_i \in \mathbb{C}[\mathbf{x}^{(i)}]$  is a normal crossing divisor for  $1 \leq i \leq l$ . We call a hypersurface singularity  $X \subseteq \mathbb{C}^n$  a **generalized normal crossing singularity at  $p$** , if the local equation  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  of  $X$  at  $p$  defines a generalized normal crossing divisor. We call a complex space germ  $(X, p)$  **generalized normal crossing**, if there exists a representant which is generalized normal crossing.

### 3.4 Proof of Theorem 3.25

This section is dedicated to the proof of Theorem 3.25. We start by fixing the notation and certain standing assumptions.

In our setup, Theorem 3.11 implies  $f \in \mathfrak{m}J_f$ , hence we can directly assume that the Jacobian ideal of  $f$  is a Stanley–Reisner ideal. We fix a minimal generating system of  $J_f$  consisting of partial derivatives, and order them increasingly with respect to their order.

**Notation 3.27.** Denote by  $\mathbf{k}$  the order sequence of  $J_f$ . We write  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_l)$  where the  $\mathbf{k}_i$  are constant sequences with entry  $k_i \in \mathbb{N}_{\geq 1}$  for  $i = 1, \dots, l$ .

**Notation 3.28.** We partition the  $\mathbf{x}$  variables into  $l + 1$  blocks  $\mathbf{x}^{(j)}$ , such that for  $1 \leq j \leq l$  the power series  $\partial_{x_i^{(j)}} f$  appear as minimal generators of  $J_f$  and are of order  $k_j$ . The  $\mathbf{x}^{(0)}$  variables correspond to power series  $\partial_{x_j^{(0)}} f$ , which do not appear as a minimal generators of  $J_f$  in our fixed minimal generating system.

**Notation 3.29.** By  $r_i$  we denote the number of variables in  $\mathbf{k}_i$ .

Let  $j \in \mathbb{N}$  with  $1 \leq j \leq l$ . We can further partition the  $\mathbf{x}$  variables. We write  $u_i^{(j)}$  for the minimal monomial generators of  $J_f$  of order  $k_j$ .

The first result we need is a statement from linear algebra.

**Lemma 3.30.** Let  $n \in \mathbb{N}_{\geq 1}$  and  $M \in \mathbb{C}^{n \times n}$  be an invertible matrix. Then we can permute the rows of  $M$  such that the resulting matrix  $B = (b_{ij}) \in \mathbb{C}^{n \times n}$  satisfies  $b_{ii} \neq 0$  for  $1 \leq i \leq n$ .

*Proof.* We do the proof by induction on  $n$ . For  $n = 1$  the statement is trivial. In case the first column of  $M$  does not contain any non-zero value,  $M$  does not have full rank, which is not possible. Denote by  $M_{i,j}$  the  $(n - 1) \times (n - 1)$  matrix which is obtained if we delete the  $i$ -th row and  $j$ -th column of  $M$ . We have to show that  $M_{i,1}$  has full rank

for some  $i$  with  $m_{i1} \neq 0$ . Assume this is not the case, this means for all  $i$  with  $m_{i1} \neq 0$  we have  $\det(M_{i,1}) = 0$ . Then  $\det(M) = 0$  by Laplace expansion. Thus there exists an index  $k$  with  $m_{k1} \neq 0$  and  $\det(M_{k,1}) \neq 0$ . We swap the first row with the  $k$ -th row and we obtain a new matrix  $B = (b_{ij})$ . By construction  $B$  has full rank and we can argue by induction.  $\square$

Now we can prove the auxiliary lemma we are going to use in every upcoming proof.

**Lemma 3.31.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . Then, after possibly renumbering the variables  $x_1^{(j)}, \dots, x_{r_j}^{(j)}$ , we can assume*

$$u_i^{(j)} \in \text{Supp} \left( \partial_{x_i^{(j)}} f \right)$$

for all  $1 \leq j \leq l$  and  $1 \leq i \leq r_j$ .

*Proof.* Due to the fact that we have two minimal generating systems, there exists an invertible matrix  $M \in \mathbb{C}\{\mathbf{x}\}^{n \times n}$ , such that

$$\begin{pmatrix} \partial_{x_1^{(1)}} f \\ \vdots \\ \partial_{x_{r_l}^{(l)}} f \end{pmatrix} = M \begin{pmatrix} u_1^{(1)} \\ \vdots \\ u_{r_l}^{(l)} \end{pmatrix}.$$

In order to see which monomial is contained in the support of the partial derivatives we consider the non-zero entries of the matrix  $M(\mathbf{0})$ , which is also invertible, since  $A$  is invertible. By our assumption on the order of the  $u_i^{(j)}$  and  $\partial_{x_i^{(j)}} f$ , we know that  $u_i^{(j)} \notin \text{Supp} \left( \partial_{x_i^{(j')}} f \right)$  for any  $j' > j$ . This means that the shape of  $M(\mathbf{0})$  is as follows:

$$M(\mathbf{0}) = \begin{pmatrix} M_1 & * & * & \dots & * \\ 0 & M_2 & * & \dots & * \\ 0 & 0 & M_3 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & M_l \end{pmatrix},$$

where  $M_j \in \mathbb{C}^{r_j \times r_j}$  are invertible matrices. By Lemma 3.30 we can reorder the rows of the  $M_j$  independently, such that the diagonal entries of the resulting matrices are non-zero. This is equivalent to renumbering the variables  $x_1^{(j)}, \dots, x_{r_j}^{(j)}$ , such that

$$u_i^{(j)} \in \text{Supp} \left( \partial_{x_i^{(j)}} f \right)$$

for all  $1 \leq j \leq l$  and  $1 \leq i \leq r_j$ .  $\square$

Next we need an auxiliary lemma, which implies that a monomial of a certain type is a minimal monomial generator of  $J_f$ .



**Lemma 3.32.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . Suppose that for all indices  $j$  with  $1 \leq j \leq l$  and for all  $1 \leq i \leq r_j$*

$$u_i^{(j)} \in \mathbb{C}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j)}].$$

*Then, for fixed index  $j$ , the following holds:*

*If, after renumbering the minimal monomial generators,  $u_1^{(j)} = m \cdot x_i^{(j')} \in \text{Supp}(\partial_{x_1^{(j)}} f)$ , where  $m \in \mathbb{C}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j)}]$  is a monomial,  $0 \leq j' \leq j$ ,  $1 \leq i \leq r_{j'}$  and  $i \neq 1$  in case  $j = j'$ , then the monomial  $u = m \cdot x_1^{(j)}$  is a minimal monomial generator of  $J_f$ .*

*Proof.* After renumbering the minimal monomial generators and the  $\mathbf{x}^{(j')}$ ,  $\mathbf{x}^{(j)}$  variables, we can consider  $u_1^{(j)} = m \cdot x_1^{(j')} \in \text{Supp}(\partial_{x_1^{(j)}} f)$  or  $u_1^{(j)} = m \cdot x_2^{(j)} \in \text{Supp}(\partial_{x_1^{(j)}} f)$ . Let  $u_1^{(j)} = m \cdot x_1^{(j')} \in \text{Supp}(\partial_{x_1^{(j)}} f)$ . Then the monomial  $m \cdot x_1^{(j')} \cdot x_1^{(j)} \in \text{Supp}(f)$  and hence  $m \cdot x_1^{(j)} \in \text{Supp}(\partial_{x_1^{(j')}} f) \subseteq J_f$ . In case  $u_1^{(j)} = m \cdot x_2^{(j)}$  it follows analogously that  $m \cdot x_1^{(j)} \in \text{Supp}(\partial_{x_2^{(j)}} f) \subseteq J_f$ . We consider two cases.

- (1) In case  $x_1^{(j)}$  divides  $m$ , we know that  $m \in J_f$ , since  $J_f$  is radical. Due to the fact that  $\text{ord}(m) \leq k_j - 1$ , we know that there exists a minimal monomial generator  $u_{i'}^{(j')}$ , where  $1 \leq j' < j$  and  $1 \leq i' \leq r_{j'}$ , which divides  $m$ . Then  $u_1^{(j)}$  is divisible by  $u_{i'}^{(j')}$ . In this case  $u_1^{(j)}$  is not a minimal monomial generator, which yields a contradiction to our assumption.
- (2) In case  $x_1^{(j)}$  does not divide  $m$ , we obtain that  $m \cdot x_1^{(j)}$  is another minimal monomial generator of order  $k_j$ , since no  $u_{i'}^{(j')}$  for  $1 \leq j' < j$  and  $1 \leq i' \leq r_{j'}$  is divisible by  $x_1^{(j)}$  and since no  $u_{i'}^{(j')}$  can divide  $m$ , since  $u_1^{(j)}$  is a minimal generator.

□

The first application of Lemma 3.31 are the following two results.

**Lemma 3.33.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . Suppose that for all for all indices  $j$  with  $1 \leq j \leq l$  and for all  $1 \leq i \leq r_j$*

$$u_i^{(j)} \in \mathbb{C}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j)}].$$

*Then, for fixed index  $j$ , the following holds:*

*If, after renumbering the minimal monomial generators,  $u_1^{(j)} = m \cdot x_i^{(j')}$ , where  $m \in \mathbb{C}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j)}]$  is a monomial,  $0 \leq j' < j$  and  $1 \leq i \leq r_{j'}$ , then the monomials  $u_1 = m \cdot x_1^{(j)}, \dots, u_{r_j} = m \cdot x_{r_j}^{(j)}$  are minimal monomial generators of  $J_f$ .*

*Proof.* Assume that, after renumbering the minimal monomial generators,  $u_1^{(j)} = m \cdot x_1^{(j')}$  for some  $0 \leq j' < j \leq l$  and  $m \in \mathbb{C}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j)}]$  a monomial of order

$k_j - 1$ . By Lemma 3.31 we can assume that  $u_1^{(j)} \in \text{Supp}(\partial_{x_1^{(j)}} f)$  after renumbering the  $\mathbf{x}^{(j)}$  variables. At this point we apply Lemma 3.32 and obtain that  $u_1 = m \cdot x_1^{(j)}$  is a minimal monomial generator of order  $k_j$ . We denote this monomial by  $u_2^{(j)}$ . Assume that we have constructed the minimal monomial generators  $u_1^{(j)} = m \cdot x_1^{(j)}$ ,  $u_2^{(j)} = m \cdot x_1^{(j)}$ ,  $\dots$ ,  $u_k^{(j)} = m \cdot x_{k-1}^{(j)}$ , where  $k \leq r_j$ . Lemma 3.31 implies that, after possibly renumbering the variables  $\mathbf{x}^{(j)}$ , we can assume that  $u_i^{(j)} \in \text{Supp}(\partial_{x_i^{(j)}} f)$  for  $1 \leq i \leq k$ .

Denote the permutation on the indices of the  $\mathbf{x}^{(j)}$  variables by  $\sigma$ . Then  $u_1^{(j)} = m \cdot x_1^{(j)}$ ,  $u_2^{(j)} = m \cdot x_{\sigma(1)}^{(j)}$ ,  $\dots$ ,  $u_k^{(j)} = m \cdot x_{\sigma(k-1)}^{(j)}$ . Denote by  $i$  the index, such that  $1 \leq i \leq k$  and  $i \notin \sigma(\{1, \dots, k-1\})$ .

In the case  $i = 1$  we use  $u_1^{(j)} = m \cdot x_1^{(j)} \in \text{Supp}(\partial_{x_1^{(j)}} f)$ . In case  $i > 1$  we use  $u_i^{(j)} = m \cdot x_{\sigma(i-1)}^{(j)} \in \text{Supp}(\partial_{x_i^{(j)}} f)$ . After applying Lemma 3.32, we obtain in both cases that  $m \cdot x_i^{(j)}$  is another minimal monomial generator of order  $k_j$ . We denote this monomial by  $u_{k+1}^{(j)}$ . This shows that we can construct the  $r_j$  minimal monomial generators as claimed.  $\square$

Lemma 3.32 and Lemma 3.33 allow us to show that the  $u_i^{(j)}$  can only be divisible by the variables, where the partial derivative of  $f$  with respect to them has the same order as  $u_i^{(j)}$ .

**Lemma 3.34.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . Then*

$$u_i^{(j)} \in \mathbb{C}[\mathbf{x}^{(j)}]$$

for all  $1 \leq j \leq l$  and  $1 \leq i \leq r_j$ .

*Proof.* In order to prove the lemma we have to be able to apply Lemma 3.32 and Lemma 3.33. We do so in two steps.

**Step 1:**

The first step is to show that the  $u_i^{(j)}$  are not divisible by the  $\mathbf{x}^{(j')}$  variables with  $j' > j$ . Assume the contrary, that is, after renumbering, we can assume  $u_1^{(j)} = m \cdot x_1^{(j')}$  for some  $1 \leq j < j' \leq l$  and  $m \in \mathbb{C}[\mathbf{x}]$  a monomial of order  $k_j - 1$ . By Lemma 3.31 we can assume that  $u_1^{(j)} \in \text{Supp}(\partial_{x_1^{(j)}} f)$ . Then  $m \cdot x_1^{(j)} \cdot x_1^{(j')} \in \text{Supp}(f)$  and hence  $m \cdot x_1^{(j)} \in \text{Supp}(\partial_{x_1^{(j')}} f)$ . This contradicts the fact that  $\text{ord}(\partial_{x_1^{(j')}} f) = k_{j'} > k_j$ .

**Step 2:**

Let  $1 \leq j \leq l$ . Due to Step 1 we already know that  $u_i^{(j)} \in \mathbb{C}[\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(j)}]$  for all  $1 \leq i \leq r_j$ . Assume that, after renumbering the minimal monomial generators,  $u_1^{(j)} \notin \mathbb{C}[\mathbf{x}^{(j)}]$ . This is equivalent to saying that there exists a monomial  $m \in \mathbb{C}[\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(j)}]$  of order  $k_j - 1$  and an index  $0 \leq j' < j$ , such that after renumbering the  $\mathbf{x}^{(j)}$  variables, we can assume  $u_1^{(j)} = m \cdot x_1^{(j')}$ . Lemma 3.33 implies that we have  $r_j + 1$  minimal monomial generators of order  $k_j$ , which contradicts our assumption on the maximal number  $r_j$  of minimal monomial generators of order  $k_j$ . Thus  $u_i^{(j)} \in \mathbb{C}[\mathbf{x}^{(j)}]$  for all  $1 \leq j \leq l$  and  $1 \leq i \leq r_j$ .  $\square$

So far we know that  $u_i^{(j)} \in \mathbb{C}[\mathbf{x}^{(j)}]$ . The next step is to show that the minimal monomial generators arise as derivatives of normal crossing divisors. We need the following auxiliary lemma.

**Lemma 3.35.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . We fix an index  $j$ , such that  $k_j \geq 2$ . Let  $N \subseteq \{1, \dots, r_j\}$  with  $|N| = k_j$  and assume that, after renumbering the minimal monomial generators,  $u_1^{(j)} = \prod_{i \in N} x_i^{(j)}$ . Then  $u_1^{(j)} \notin \text{Supp} \left( \partial_{x_i^{(j)}} f \right)$  for all  $i \in N$ .*

*Proof.* Assume the contrary, that is, there exists an  $i' \in N$ , such that  $u_1^{(j)} \in \text{Supp} \left( \partial_{x_{i'}^{(j)}} f \right)$ .

Then  $g := \left( x_{i'}^{(j)} \right)^2 \cdot \prod_{i \in N \setminus \{i'\}} x_i^{(j)} \in \text{Supp}(f)$ . Since  $k_j \geq 2$ , there exists an index  $i'' \in N \setminus \{i'\}$ . Then  $\partial_{x_{i''}^{(j)}} g = \left( x_{i'}^{(j)} \right)^2 \cdot \prod_{i \in N \setminus \{i', i''\}} x_i^{(j)} \in J_f$ . This monomial is not squarefree so it must be divisible by some  $u_{i''}^{(j)}$  with  $1 \leq i'' < j$  and  $1 \leq i'' \leq r_{j'}$ . Since, due to Lemma 3.34,  $u_{i''}^{(j)} \notin \mathbb{C}[\mathbf{x}^{(j)}]$ , this yields a contradiction.  $\square$

**Lemma 3.36.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . We fix an index  $j$  with  $k_j \geq 2$ . Let  $N \subseteq \{1, \dots, r_j\}$  with  $|N| = k_j$  and assume that, after renumbering the minimal monomial generators,  $u_1^{(j)} = \prod_{i \in N} x_i^{(j)}$ . Then there exists an index  $i' \in \{1, \dots, r_j\} \setminus N$ , such that:*

- (1)  $g := x_{i'}^{(j)} \cdot \prod_{i \in N} x_i^{(j)} \in \text{Supp}(f)$ , and
- (2) for all  $s \in N \cup \{i'\}$ , we obtain that  $\partial_{x_s^{(j)}} g \in J_f$  are minimal monomial generators of order  $k_j$ .

In particular, if  $u_1^{(j)} \in \text{Supp} \left( \partial_{x_k^{(j)}} f \right)$  for a  $k \in \{1, \dots, r_j\} \setminus N$ , then one can choose  $i' = k$ .

*Proof.* Due to Lemma 3.31 we know that  $u_1^{(j)} \in \text{Supp} \left( \partial_{x_{i'}^{(j)}} f \right)$  for a certain  $1 \leq i' \leq r_j$ . By Lemma 3.35 we know that  $i' \notin N$ . Thus  $g := x_{i'}^{(j)} \cdot \prod_{i \in N} x_i^{(j)} \in \text{Supp}(f)$ . Due to the fact that, by Lemma 3.34,  $u_{i''}^{(j)} \notin \mathbb{C}[\mathbf{x}^{(j)}]$  for all  $1 \leq i'' < j$  and  $1 \leq i'' \leq r_{j'}$ , the monomials  $\partial_{x_s^{(j)}} g \in \mathbb{C}[\mathbf{x}^{(j)}]$  are minimal monomial generators for all  $s \in N \cup \{i'\}$ .  $\square$

**Lemma 3.37.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . We fix an index  $j$  with  $k_j \geq 2$ . Then for all  $1 \leq i \leq r_j$  there exists a monomial  $m \in \mathbb{C}[\mathbf{x}^{(j)}]$ , such that  $u = x_i^{(j)} \cdot m$  is a minimal monomial generator of  $J_f$  of order  $k_j$ .*

*Proof.* Lemma 3.31 implies that, after renumbering the  $\mathbf{x}^{(j)}$  variables,  $u_i^{(j)} \in \text{Supp}\left(\partial_{x_i^{(j)}} f\right)$  for all  $1 \leq i \leq r_j$ . By Lemma 3.36 we obtain that all partial derivatives of  $g = x_i^{(j)} \cdot u_i^{(j)}$  are minimal monomial generators of  $J_f$  of order  $k_j$ , hence the claim follows.  $\square$

**Lemma 3.38.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . We fix an index  $j$  with  $k_j \geq 2$ . Assume that, after renumbering the  $\mathbf{x}^{(j)}$  variables, the parts of partition  $\mathbf{x}^{(j,1)}, \dots, \mathbf{x}^{(j,t)}$  have been constructed for some  $t \geq 1$  and that these parts of partition are containing precisely the variables  $x_1^{(j)}, \dots, x_a^{(j)}$ , where  $1 \leq a < r_j$ . Furthermore, we assume that, after renumbering the minimal monomial generators, it holds that  $u_1^{(j)}, \dots, u_b^{(j)} \in \mathbb{C}[\mathbf{x}^{(j,1)}, \dots, \mathbf{x}^{(j,t)}]$  for  $1 \leq b < r_j$ . Assume that, after renumbering minimal monomial generators with index greater than  $b$  and after renumbering the  $\mathbf{x}^{(j)}$  variables with index greater than  $a$ , there exist an integer  $w \in \mathbb{N}$  with  $1 \leq w \leq k_j$ , and two sets  $S \subseteq \{1, \dots, a\}$  and  $T = \{a+1, \dots, a+w\}$  with  $|S| = k_j - w$  and  $|T| = w$ , such that*

$$u_{b+1}^{(j)} = \prod_{i_1 \in S} x_{i_1}^{(j)} \cdot \prod_{i_2 \in T} x_{i_2}^{(j)}.$$

Assume that  $u_{b+1}^{(j)} \in \text{Supp}\left(\partial_{x_s^{(j)}} f\right)$  for some  $1 \leq s \leq r_j$ . Then one the following holds:

- (1) If  $s \geq a+1$ , then the partial derivatives of  $g := x_s^{(j)} \cdot u_{b+1}^{(j)}$  are minimal monomial generators of  $J_f$  of order  $k_j$  distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ .
- (2) If  $w = k_j$  and  $s \leq a$ , then there exists a minimal monomial generator  $u = x_s^{(j)} \cdot m$  of order  $k_j$  distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ , where  $m \in \mathbb{C}[x_{a+1}^{(j)}, \dots, x_{a+k_j}^{(j)}]$ .
- (3) If  $w < k_j$  and  $s \leq a$ , then at least  $k_j$  of the partial derivatives of  $g := x_s^{(j)} \cdot u_{b+1}^{(j)}$  are minimal monomial generators of  $J_f$  distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ .

*Proof.*

- (1) Let  $s \geq a+1$ . By Lemma 3.35 we can assume, after renumbering the  $\mathbf{x}^{(j)}$  variables with index not contained in  $\{1, \dots, a\}$ , that  $s = a+w+1$ . Then Lemma 3.36 implies that  $g = x_s^{(j)} \cdot u_{b+1}^{(j)} \in \text{Supp}(f)$  and all partial derivatives of  $g$  are minimal monomial generators of  $J_f$ , which are by construction distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ .
- (2) Let  $w = k_j$  and  $s \leq a$ . We can assume, after renumbering the  $\mathbf{x}^{(j)}$  variables with index contained in  $\{1, \dots, a\}$  and the so far constructed parts of partition, that  $s = 1$ . Then Lemma 3.36 implies that  $g = x_1^{(j)} \cdot \prod_{i \in T} x_i^{(j)} \in \text{Supp}(f)$  and all partial derivatives of  $g$  with respect to the variables  $x_i^{(j)}$  for  $i \in T$  are minimal monomial generators of  $J_f$ , which are distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ . In particular, since  $k_j \geq 2$ , we obtain a minimal monomial generator  $u = x_1^{(j)} \cdot \prod_{i \in T \setminus \{a+1\}} x_i^{(j)}$ .
- (3) Let  $w < k_j$  and  $s \leq a$ . We can assume, after renumbering the  $\mathbf{x}^{(j)}$  variables with index contained in  $\{1, \dots, a\}$  and the so far constructed parts of partition, that  $s = 1$ . Then Lemma 3.36 implies that  $g = x_1^{(j)} \cdot u_{b+1}^{(j)} \in \text{Supp}(f)$  and all partial

derivatives of  $g$  are minimal monomial generators of  $J_f$ . By construction at least  $k_j$  are distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ .

□

**Lemma 3.39.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  and assume that  $J_f$  is a Stanley–Reisner ideal with order-sequence  $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_r) \in (\mathbb{N}_{\geq 1})^n$  and minimal monomial generating system  $u_1^{(j)}, \dots, u_{r_j}^{(j)}$ , where  $1 \leq j \leq l$ . Then for each  $1 \leq j \leq l$  with  $k_j \geq 2$  the following hold:*

(1) *There exists a partition of the  $\mathbf{x}^{(j)}$  variables denoted by  $\mathbf{x}^{(j,1)}, \dots, \mathbf{x}^{(j,l_j)}$  for some  $l_j \in \mathbb{N}$  with  $1 \leq l_j \leq r_j$ , where each part of the partition contains  $k_j + 1$  variables.*

(2) *We define  $g^{(j,l')} := \prod_{i=1}^{k_j+1} x_i^{(j,l')} \in \mathbb{C}[\mathbf{x}^{(j,l')}]$  for any  $1 \leq l' \leq l_j$ . Then for all  $1 \leq l' \leq l_j$  and  $1 \leq s \leq k_j + 1$  the monomials  $\partial_{x_s^{(j,l')}} g^{(j,l')}$  are minimal monomial generators of  $J_f$ .*

*Proof.* Fix a  $j \in \{1, \dots, l\}$  with  $k_j \geq 2$ . Due to Lemma 3.34 we can assume  $u_i^{(j)} \in \mathbb{C}[\mathbf{x}^{(j)}]$  for all  $1 \leq i \leq r_j$ . The upcoming proof is going to be constructive. Due to Lemma 3.31 we can assume that  $u_1^{(j)} \in \text{Supp}(\partial_{x_1^{(j)}} f)$ . By Lemma 3.35 we can assume that, after renumbering the  $\mathbf{x}^{(j)}$  variables,  $u_1^{(j)} = \prod_{i=2}^{k_j+1} x_i^{(j)}$ . Define  $\mathbf{x}^{(j,1)} = (x_1^{(j)}, \dots, x_{k_j+1}^{(j)})$  and  $g^{(j,1)} = \prod_{i=1}^{k_j+1} x_i^{(j)}$ . The statement for  $g^{(j,1)}$  follows from Lemma 3.36. If  $\mathbf{x}^{(j)}$  contains only  $k_j + 1$  variables the statement is proved. Assume  $\mathbf{x}^{(j)}$  contains more than  $k_j + 1$  variables and assume that, after renumbering the  $\mathbf{x}^{(j)}$  variables, the parts of partition  $\mathbf{x}^{(j,1)}, \dots, \mathbf{x}^{(j,t)}$  have been constructed for some  $t \geq 1$  and that these parts of partition are containing precisely the variables  $x_1^{(j)}, \dots, x_a^{(j)}$ , where  $1 \leq a < r_j$ . Furthermore, we assume that, after renumbering the minimal monomial generators, it holds that  $u_1^{(j)}, \dots, u_b^{(j)} \in \mathbb{C}[\mathbf{x}^{(j,1)}, \dots, \mathbf{x}^{(j,t)}]$  for a  $1 \leq b \leq r_j$ . By construction the ideal  $J_f$  has at most  $n$  minimal monomial generators. Due to our convention, there are exactly  $r_j$  minimal monomial generators of order  $k_j$  of  $J_f$ . By Lemma 3.37 there exists a minimal monomial generator  $u_{b+1}^{(j)} \notin \mathbb{C}[\mathbf{x}^{(j,1)}, \dots, \mathbf{x}^{(j,t)}]$ . In particular,  $b < r_j$ . After renumbering the  $\mathbf{x}^{(j)}$  variables with index greater than  $a$ , there exist an integer  $w \in \mathbb{N}$  with  $1 \leq w \leq k_j$ , and two sets  $S \subseteq \{1, \dots, a\}$  and  $T = \{a+1, \dots, a+w\}$  with  $|S| = k_j - w$  and  $|T| = w$ , such that

$$u_{b+1}^{(j)} = \prod_{i_1 \in S} x_{i_1}^{(j)} \cdot \prod_{i_2 \in T} x_{i_2}^{(j)}.$$

Due to Lemma 3.31 we know that  $u_{b+1}^{(j)} \in \text{Supp}(\partial_{x_s^{(j)}} f)$  for some  $1 \leq s \leq r_j$ . We are now in the setup of Lemma 3.38. We construct the next part of partition  $\mathbf{x}^{(j,t+1)}$  and minimal monomial generators  $u_{b+2}^{(j)}, \dots, u_{b+k_j}^{(j)}$ , respectively  $u_{b+2}^{(j)}, \dots, u_{b+k_j+1}^{(j)}$  distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$  iteratively.

**Case 1:** Let  $w = k_j$  and  $s \geq a + 1$ . By Lemma 3.35 we can assume, after renumbering the  $\mathbf{x}^{(j)}$  variables with index not contained in  $\{1, \dots, a\}$ , that  $s = a + k_j + 1$ . We define  $\mathbf{x}^{(j,t+1)} := (x_{a+1}^{(j)}, \dots, x_{a+k_j+1}^{(j)})$  and  $g^{(j,t+1)} := x_s^{(j)} \cdot u_{b+1}^{(j)}$ . Lemma 3.38 implies that all

$k_j + 1$  partial derivatives of  $g^{(j,t+1)}$  are minimal monomial generators of  $J_f$  of order  $k_j$ , which are distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ .

**Case 2:** Let  $w = k_j$  and  $s \leq a$ . We can assume, after renumbering the  $\mathbf{x}^{(j)}$  variables with index contained in  $\{1, \dots, a\}$  and the so far constructed partitions, that  $s = 1$ . Then Lemma 3.38 implies that  $u = x_1^{(j)} \cdot \prod_{i \in T \setminus \{a+1\}} x_i^{(j)}$  is a minimal monomial generator of  $J_f$ . Redefine  $u_{b+1}^{(j)} := u$ .

**Case 3:** Let  $w < k_j$  and  $s \geq a + 1$ . By Lemma 3.35 we can assume, after renumbering the  $\mathbf{x}^{(j)}$  variables with index not contained in  $\{1, \dots, a\}$ , that  $s = a + w + 1$ . In this case we define  $\mathbf{x}^{(j,t+1)} := (x_{a+1}^{(j)}, \dots, x_{a+w+1}^{(j)})$  and  $g^{(j,t+1)} := x_s^{(j)} \cdot u_{b+1}^{(j)}$ . Lemma 3.38 implies that all  $k_j + 1$  partial derivatives of  $g^{(j,t+1)}$  are minimal monomial generators of  $J_f$  of order  $k_j$ , which are distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ .

**Case 4:** Let  $w < k_j$  and  $s \leq a$ . We can assume, after renumbering the  $\mathbf{x}^{(j)}$  variables with index contained in  $\{1, \dots, a\}$  and the so far constructed partitions, that  $s = 1$ . Define  $\mathbf{x}^{(j,t+1)} := (x_{a+1}^{(j)}, \dots, x_{a+w}^{(j)})$  and  $g^{(j,t+1)} := x_1^{(j)} u_{b+1}^{(j)}$ . Lemma 3.38 implies that at least  $k_j$  partial derivatives of  $g^{(j,t+1)}$  are minimal monomial generators of  $J_f$  of order  $k_j$ , which are distinct from  $u_1^{(j)}, \dots, u_b^{(j)}$ .

Note that Case 2 changes the input of our iteration so that we end up in either Case 3 or Case 4. In Case 3 and Case 4 we partition strictly less variables than minimal monomial generators that are constructed. This implies that we can continue the iteration after a new part of the partition has been constructed in Case 3 or Case 4. Since we deal with only finitely many variables this yields a contradiction. So every part of the partition has been constructed in Case 1. The statement then follows by construction.  $\square$

Now we have all tools to prove Theorem 3.25.

*Proof of Theorem 3.25.* Due to Corollary 3.12, it suffices to find a partition of the variables  $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(t)}$  and a  $g \in \mathbb{C}\{\mathbf{x}\}$ , such that  $T_f = T_g$  and  $g = \sum_{j=1}^{r_1} (x_j^{(1)})^2 + \sum_{i=2}^t g_i$ ,

where  $g_i \in \mathbb{C}[\mathbf{x}^{(i)}]$  is a normal crossing divisor for  $2 \leq i \leq t$ . We define  $\mathbf{x}^{(0)}$ , as before, to be variables, whose partial derivatives do not appear in a fixed minimal generating system of  $J_f$ . First assume that  $\text{ord}(f) = 2$ . Define  $\mathbf{x}^{(1)}$  to be the variables which are minimal monomial generators of  $J_f$  of order 1. We can partition the remaining variables according to Lemma 3.39 into  $\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(t)}$ , where the minimal monomial generators with respect to each set of variables arise as the partial derivatives of a normal crossing divisor  $g_i \in \mathbb{C}[\mathbf{x}^{(i)}]$ . We define  $g := \sum_{j=1}^{r_1} (x_j^{(1)})^2 + \sum_{i=2}^t g_i$  and obtain

$T_f = T_g$ . In case  $\text{ord}(f) \geq 3$ , the statement follows analogously, since we can drop the sum of squares in the definition of  $g$ . The uniqueness statement follows immediately from the uniqueness of the order-sequence.  $\square$





## Chapter 4

# Free Singularities and Generalized Normal Crossing Divisor

This chapter is joint work with D. Pol (see [EP20]). Section 4.1 is common work with D. Pol. Section 4.2 has been contributed by D. Pol, whereas the sections 4.3 and 4.4 have been contributed by the author of this thesis. The purpose of this chapter is to give new families of free singularities and to investigate properties of generalized normal crossing divisors, which are related to the module of (multi-)logarithmic derivations. We first show that a generic equidimensional subspace arrangement of codimension  $k$  in  $\mathbb{C}^n$  is free if the number of subspaces is lower than or equal to  $\binom{n}{k}$  (see Theorem 4.18). Afterwards we show that a product of two Cohen–Macaulay subspaces is free if and only if the two subspaces are free (see Theorem 4.30). In the particular case of divisors, it follows that the product of two divisors is a free complete intersection of codimension 2 if and only if both divisors are free. All computations have been performed using the computer algebra system SINGULAR ([Dec+19]). In order to compute all mentioned algebraic objects we provide the SINGULAR-library `logmodules.lib` which can be downloaded under <https://www.math.univ-angers.fr/~pol/logmodules.lib>.

We conclude this chapter by showing that singular loci of generalized normal crossing divisors are free, as well as that these singularities are Saito holonomic.

### 4.1 Preliminaries

Let  $n \in \mathbb{N}_{\geq 1}$ . Throughout this section, if not stated otherwise, let  $A$  be either  $\mathbb{C}[\mathbf{x}]$  or  $\mathbb{C}\{\mathbf{x}\}$ . For the rest of this section, we will also write  $\mathbb{C}^n$  in the local case instead of  $(\mathbb{C}^n, \mathbf{0})$ .

We denote by  $\text{Der}(A)$  the  $A$ -module of vector fields on  $\mathbb{C}^n$ , which is a free  $A$ -module of rank  $n$ , generated by the vector fields  $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ .

For  $q \in \mathbb{N}$  we denote by  $\Omega_{\mathbb{C}^n}^q$  the module of differential  $q$ -forms on  $\mathbb{C}^n$  and we consider the usual pairing  $\langle \cdot, \cdot \rangle : \bigwedge^q \text{Der}(A) \times \Omega_{\mathbb{C}^n}^q \rightarrow A$ .

A generalization of the module of logarithmic vector fields along singular hypersurfaces (see [Sai80]) is introduced in [GS12] for complete intersections and in [Pol20] for general equidimensional subspaces. We give here the equivalent definition as stated in [ST18, Definition 3.19]:

**Definition 4.1.** Let  $X \subseteq \mathbb{C}^n$  be a Cohen–Macaulay subspace of codimension  $k$  defined as the vanishing set of the radical ideal  $I_X \subseteq A$ . The module of multi-logarithmic  $k$ -vector fields along  $X$  is defined by

$$\mathrm{Der}^k(-\log X) = \left\{ \delta \in \bigwedge^k \mathrm{Der}(A) \mid \forall (f_1, \dots, f_k) \in I_X^k, \langle \delta, df_1 \wedge \dots \wedge df_k \rangle \in I_X \right\}.$$

**Remark 4.2.** Let  $\{h_1, \dots, h_r\}$  be a generating set of  $I_X$ . Let  $\delta \in \bigwedge^k \mathrm{Der}(A)$ . Then  $\delta \in \mathrm{Der}^k(-\log X)$  if and only if for all  $(i_1 < \dots < i_k) \subseteq \{1, \dots, r\}$ ,  $\langle \delta, dh_{i_1} \wedge \dots \wedge dh_{i_k} \rangle \in I_X$ .

A hypersurface  $D$  is called free if and only if  $\mathrm{Der}(-\log D) := \mathrm{Der}^1(-\log D)$  is a free  $A$ -module (see [Sai80]). A generalization of this notion to higher codimensional subspaces is the following (see [Pol20, Definition 4.3]):

**Definition 4.3.** An equidimensional reduced subspace  $X \subseteq \mathbb{C}^n$  of codimension  $k$  is called free if and only if

$$\mathrm{projdim} \left( \mathrm{Der}^k(-\log X) \right) = k - 1.$$

In the case of hypersurfaces, the criterion of Terao and Aleksandrov ([Ter80], [Ale88]) gives a characterization of freeness in terms of a property of the singular locus. It is shown in [Pol20] that this property can be extended to Cohen–Macaulay spaces.

Let  $X \subseteq \mathbb{C}^n$  be a reduced equidimensional subspace. One can prove that there exists a regular sequence  $(f_1, \dots, f_k) \subseteq I_X$  such that the ideal  $I_C$  generated by  $f_1, \dots, f_k$  is radical (see [AT08, Remark 4.3] or [Pol16, Proposition 4.2.1] for a detailed proof of this result). We fix such a sequence  $(f_1, \dots, f_k)$  and denote by  $C$  the complete intersection defined by the ideal  $I_C = \langle f_1, \dots, f_k \rangle$ .

**Notation 4.4.** Let  $X$  be a reduced equidimensional subspace of codimension  $k$  in  $\mathbb{C}^n$  and  $C$  be a reduced complete intersection of codimension  $k$  in  $\mathbb{C}^n$  containing  $X$ . Let  $J_{X/C} = J_C + I_X$ , where  $J_C$  is the Jacobian ideal of  $C$ , that is to say, the ideal of  $A$  generated by the  $k \times k$  minors of the Jacobian matrix of  $(f_1, \dots, f_k)$ .

**Remark 4.5.** The vanishing set of the ideal  $J_{X/C}$  is the restriction of the singular locus of  $C$  to  $X$ . If  $X$  is not a complete intersection, it does not describe the singular locus of  $X$ .

The following proposition generalizes [GS12, Definition 5.1]:

**Proposition 4.6.** Let  $X \subseteq \mathbb{C}^n$  be a reduced Cohen–Macaulay subspace of codimension  $k$  in  $\mathbb{C}^n$  and  $C$  be a reduced complete intersection of codimension  $k$  containing  $X$ . Then  $X$  is free if and only if  $A/J_{X/C} = 0$  or  $A/J_{X/C}$  is Cohen–Macaulay of dimension  $n - k - 1$ .

*Proof.* See [Pol20, Proposition 4.2]. □

**Remark 4.7.** If  $C'$  is another reduced complete intersection of codimension  $k$  containing  $X$ , the modules  $A/J_{X/C}$  and  $A/J_{X/C'}$  are isomorphic as  $A/I_X$ -modules (see [Pol20, Remark 3.8]).

The module of multi-logarithmic  $k$ -vector fields of a union of reduced equidimensional subspaces of the same codimension satisfies the following property:

**Proposition 4.8.** *Let  $X$  be a reduced equidimensional subspace of codimension  $k$ , with irreducible components  $X_1, \dots, X_s$ . Then:*

$$\mathrm{Der}^k(-\log X) = \bigcap_{i=1}^s \mathrm{Der}^k(-\log X_i).$$

*Proof.* See [Pol20, Proposition 5.1]. □

Before giving some basic motivating examples of free singularities, let us introduce the following notation:

**Notation 4.9.** *We denote by  $K(\underline{f})$  the Koszul complex of a sequence  $(f_1, \dots, f_k)$  in  $A$ :*

$$K(\underline{f}) : 0 \rightarrow \bigwedge^k A^k \xrightarrow{d_k} \dots \xrightarrow{d_2} \bigwedge^1 A^k \xrightarrow{d_1} A \rightarrow 0. \quad (4.1)$$

The maps  $d_p$  are given by

$$d_p(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{j=1}^p (-1)^{j+1} f_j e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_p}.$$

We also set  $\widetilde{K}(\underline{f})$  the complex obtained from  $K(\underline{f})$  by removing the last  $A$ .

The complex  $0 \rightarrow A \rightarrow 0$  is denoted by  $\mathcal{C}$ .

**Example 4.10.** *Let  $E_0 = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$  and let  $X$  be the vector subspace of  $\mathbb{C}^n$  defined by the regular sequence  $(x_{i_1}, \dots, x_{i_k})$ . Then a generating set of  $\mathrm{Der}^k(-\log X)$  is*

$$\{x_j \wedge_{i \in E_0} \partial_{x_i} \mid j \in E_0\} \cup \{\wedge_{i \in E} \partial_{x_i} \mid E \neq E_0\}.$$

A minimal free resolution of  $\mathrm{Der}^k(-\log X)$  is then given by

$$\widetilde{K}((x_i)_{i \in E_0}) \oplus \bigoplus_{1 \leq i \leq \binom{n}{k}-1} \mathcal{C}.$$

In particular,  $\mathrm{projdim}(\mathrm{Der}^k(-\log X)) = k - 1$  so that  $X$  is free.

More generally, the following holds:

**Proposition 4.11.** *Let  $X$  be an equidimensional union of coordinate subspaces. Then  $X$  is free.*

*Proof.* See [Pol20, Corollary 5.5] □

## 4.2 Generic Subspace Arrangements and Freeness

In this section we assume  $A = \mathbb{C}[\mathbf{x}]$ .

**Definition 4.12.** *An equidimensional subspace arrangement of codimension  $k$  in  $\mathbb{C}^n$  is a finite union of pairwise distinct vector subspaces of codimension  $k$  in  $\mathbb{C}^n$ . We denote by  $I_X \subseteq S$  the ideal of vanishing polynomials on  $X$ .*

**Definition 4.13.** *Let  $\delta \in \bigwedge^k \text{Der}(A)$ . We say that  $\delta$  is homogeneous of degree  $p$  if there exist homogeneous polynomials  $(a_E)_{|E|=k, E \subseteq \{1, \dots, n\}}$  of degree  $p$  such that*

$$\delta = \sum_{\substack{E \subseteq \{1, \dots, n\} \\ |E|=k}} \left( a_E \bigwedge_{i \in E} \partial_{x_i} \right).$$

**Notation 4.14.** *Let  $M$  be a graded  $A$ -module. For  $p \in \mathbb{N}$  we denote by  $M_p$  the submodule of  $M$  composed of the homogeneous elements of  $M$  of degree  $p$  and  $0 \in M$ .*

**Definition 4.15.** *Let  $\Lambda$  be a finite index set and let  $X = \bigcup_{i \in \Lambda} X_i$  be an equidimensional subspace arrangement of codimension  $k$ . We say that  $X$  is generic if for  $j = \min \{|\Lambda|, \binom{n}{k}\}$  and for all  $I \subseteq \Lambda$  with  $|I| = j$ , it holds that*

$$\dim_{\mathbb{C}} \left( \bigcap_{i \in I} \text{Der}^k(-\log X_i)_0 \right) = \binom{n}{k} - j.$$

**Remark 4.16.** *The condition given in Definition 4.15 generalizes the usual definition of generic hyperplane arrangement (see [OT92, Definition 5.22]), since for a hyperplane  $H$ ,  $\text{Der}^1(-\log H)_0$  is equal to the vector fields tangent to the hyperplane.*

**Remark 4.17.** *If the coefficients of the defining linear equations of the irreducible components are chosen randomly, the condition of Definition 4.15 is satisfied. This remark can be used to create examples in a computer algebra system such as SINGULAR.*

Up to a change of coordinates, it is easy to see that a generic hyperplane arrangement in  $\mathbb{C}^n$  with at most  $n$  hyperplanes is isomorphic to a normal crossing divisor, and thus is free. The purpose of this section is to prove the following generalization of this result:

**Theorem 4.18.** *Let  $X = X_1 \cup \dots \cup X_s$  be an equidimensional subspace arrangement of codimension  $k$  in  $\mathbb{C}^n$  such that for all  $i \in \{1, \dots, s\}$ ,  $X_i$  is a vector subspace defined by the regular sequence  $(h_{i,1}, \dots, h_{i,k})$ .*

*If  $s \leq \binom{n}{k}$  and  $X$  is a generic subspace arrangement, then there exists a basis  $(\delta_1, \dots, \delta_{\binom{n}{k}})$  of  $\bigwedge^k \text{Der}(A)$  such that a minimal generating set of  $\text{Der}^k(-\log X)$  is given by*

$$\{h_{i,j} \delta_i \mid i \in \{1, \dots, s\}, j \in \{1, \dots, k\}\} \cup \{\delta_i \mid i \geq s+1\}. \quad (4.2)$$

**Corollary 4.19.** *Let  $X = X_1 \cup \dots \cup X_s$  be an equidimensional subspace arrangement of codimension  $k$  in  $\mathbb{C}^n$  satisfying the hypothesis of Theorem 4.18. Then  $X$  is free.*

In order to prove Theorem 4.18, we need the following auxiliary lemmas.

**Lemma 4.20.** *Let  $h_1, \dots, h_k$  be  $k$  linear polynomials defining a vector subspace  $X$  of codimension  $k$ . Let  $\{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$ . We assume that the  $k \times k$  minor of  $J_h$  relative to the columns indexed by  $i_1, \dots, i_k$  is non-zero. Then a minimal generating set of  $\text{Der}^k(-\log X)$  is of the form:*

$$\left\{ h_i \partial_{x_{i_1}} \wedge \dots \wedge \partial_{x_{i_k}} \mid i \in \{1, \dots, k\} \right\} \cup \left\{ \delta_2, \dots, \delta_{\binom{n}{k}-1} \right\}, \quad (4.3)$$

where for  $i \in \{2, \dots, \binom{n}{k}-1\}$ ,  $\delta_i$  is homogeneous of degree 0 and such that  $\left\{ \partial_{x_{i_1}} \wedge \dots \wedge \partial_{x_{i_k}}, \delta_2, \dots, \delta_{\binom{n}{k}-1} \right\}$  is a basis of  $\bigwedge^k \text{Der}(A)$ .

*Proof.* Let us consider new coordinates  $(y_1, \dots, y_n)$  such that for  $j \in \{1, \dots, k\}$ ,  $y_{i_j} = h_j$  and for all  $j \notin \{i_1, \dots, i_k\}$ ,  $y_j = x_j$ . The condition on the minor ensures that it is indeed a change of coordinates.

Let  $S \in \text{GL}(\mathbb{C}, n)$  be the matrix such that  $(y_1, \dots, y_n)^T = S(x_1, \dots, x_n)^T$ .

In the new system of coordinates, the subspace  $X$  is defined by  $y_{i_1}, \dots, y_{i_k}$  so that a minimal generating set of  $\text{Der}^k(-\log X)$  is given by Example 4.10.

It holds that  $(\partial_{y_1}, \dots, \partial_{y_n}) = (\partial_{x_1}, \dots, \partial_{x_n}) S^{-1}$ .

Let  $B = (b_{ij})_{1 \leq i, j \leq n} = S^{-1}$ .

Since for all  $j \notin \{i_1, \dots, i_k\}$ ,  $y_j = x_j$ , we have that for all  $(i, j)$  such that  $i \notin \{i_1, \dots, i_k\}$  and  $j \neq i$ ,  $b_{ij} = 0$ . Therefore, for  $j \in \{1, \dots, k\}$ ,  $\partial_{y_{i_j}}$  is a linear combination of  $\partial_{x_{i_1}}, \dots, \partial_{x_{i_k}}$ . Thus,  $\partial_{y_{i_1}} \wedge \dots \wedge \partial_{y_{i_k}}$  can be expressed as a non-zero multiple of  $\partial_{x_{i_1}} \wedge \dots \wedge \partial_{x_{i_k}}$ .  $\square$

**Remark 4.21.** *With the same assumptions as for Lemma 4.20, for any  $0 \neq \delta \in \left( \bigwedge^k \text{Der}(A) \right)_0 \setminus \text{Der}^k(-\log X)_0$  and  $\mathcal{B}$  a basis of  $\text{Der}^k(-\log X)_0$ , one can see that  $\mathcal{B} \cup \{h_i \delta \mid i \in \{1, \dots, k\}\}$  is a minimal generating set of  $\text{Der}^k(-\log X)$ .*

**Lemma 4.22.** *Let  $A$  be a graded ring and  $F$  be a free graded  $A$ -module of rank  $n \in \mathbb{N}_{>0}$  with bases  $\mathcal{B} = \{b_1, \dots, b_n\}$  and  $\mathcal{C} = \{c_1, \dots, c_n\}$ . For  $k \in \{1, \dots, n-1\}$ , let  $I, I_1, \dots, I_k \subseteq A$  be homogeneous ideals. Define the graded modules  $V = \bigoplus_{i=1}^k I_i b_i \oplus \bigoplus_{j=k+1}^n A b_j$  and  $W = I c_1 \oplus \bigoplus_{i=2}^n A c_i$ . If  $\dim_{\mathbb{C}}(V_0 \cap W_0) = n - k - 1$ , then there exists a basis  $\mathcal{B}' = \{b'_1, \dots, b'_n\}$  of  $F$ , such that:*

$$V \cap W = \bigoplus_{i=1}^k I_i b'_i \oplus I b'_{k+1} \oplus \bigoplus_{j=k+2}^n A b'_j.$$

*Proof.* Let  $V' = \langle V_0 \rangle$  and  $W' = \langle W_0 \rangle$ . After renumbering the  $b_i$  with index  $i \geq k+1$ , we can assume  $b_{k+1} \notin V_0 \cap W_0$ . Then  $\bar{\mathcal{B}} = \{\bar{b}_{k+1}\}$  is a basis of  $F/W'$ , which yields the existence of  $a_i \in A$  and  $w_i \in W'$ , such that  $b_i = a_i b_{k+1} + w_i$  for  $i \in \{1, \dots, k, k+2, \dots, n\}$  and the existence of a unit  $a_{k+1} \in A$  and of  $w_{k+1} \in W'$  with  $c_1 = a_{k+1} b_{k+1} + w_{k+1}$ . This implies that  $\mathcal{B}' = \{w_1, \dots, w_k, b_{k+1}, w_{k+2}, \dots, w_n\}$  is a basis of  $F$ . We obtain

$$V = \bigoplus_{i=1}^k I_i w_i \oplus A b_{k+1} \oplus \bigoplus_{j=k+2}^n A w_j$$

and

$$W = \bigoplus_{i=1}^k A w_i \oplus I b_{k+1} \oplus \bigoplus_{j=k+2}^n A w_j.$$

Then

$$V \cap W = \bigoplus_{i=1}^k I_i w_i \oplus I b_{k+1} \oplus \bigoplus_{j=k+2}^n A w_j.$$

□

*Proof of Theorem 4.18.* Let us prove Theorem 4.18 by induction. The initialisation for  $s = 1$  is given by Lemma 4.20. Let  $N = \binom{n}{k}$  and  $s \in \{1, \dots, N - 1\}$ .

We assume that  $X_1, \dots, X_{s+1}$  are linear subspaces of  $\mathbb{C}^n$  of codimension  $k$  which are in generic position.

Let  $X = \bigcup_{i=1}^s X_i$ ,  $V = \text{Der}^k(-\log X)$ ,  $W = \text{Der}^k(-\log X_{s+1})$  and  $F = A^N$ . By the induction hypothesis,  $\dim_{\mathbb{C}} V_0 = N - s$  and by Lemma 4.20,  $\dim_{\mathbb{C}} W_0 = N - 1$ . Then  $\dim_{\mathbb{C}} V_0 \cap W_0 = N - s - 1$  follows from the genericity of the subspace arrangement. By Proposition 4.8 it holds that

$$\text{Der}^k \left( -\log \left( \bigcup_{i=1}^{s+1} X_i \right) \right) = V \cap W.$$

Then Lemma 4.22 yields the result. □

*Proof of Corollary 4.19.* Let  $\{\delta_1, \dots, \delta_N\}$  be a basis of  $\bigwedge^k \text{Der}_{\mathbb{C}^n}$  such that a minimal generating set of  $\text{Der}^k(-\log X)$  is given by (4.2). Since for all  $i \in \{1, \dots, s\}$ ,  $(h_{i,1}, \dots, h_{i,k})$  is a regular sequence, a minimal free resolution of the ideal  $\langle h_{i,1}, \dots, h_{i,k} \rangle$  is given by the truncated Koszul complex  $\tilde{K}_i := \tilde{K}(h_{i,1}, \dots, h_{i,k})$ . Since

$$\text{Der}^k(-\log X) = \bigoplus_{i=1}^s \langle h_{i,1}, \dots, h_{i,k} \rangle \delta_i \oplus \bigoplus_{i=s+1}^N S \delta_i,$$

we deduce that a minimal free resolution of  $\text{Der}^k(-\log X)$  is

$$\tilde{K}_1 \oplus \dots \oplus \tilde{K}_s \oplus \bigoplus_{i=s+1}^N \mathcal{C}$$

where  $\mathcal{C}$  is defined as in Notation 4.9. Thus, the projective dimension of  $\text{Der}^k(-\log X)$  is  $k - 1$  and  $X$  is free. □

The following example shows that the genericity assumption cannot be dropped in Theorem 4.18.

**Example 4.23.** Let us consider the subspace arrangement  $X$  defined by the equations  $h_1 = xy(x - y + z - t)$  and  $h_2 = zt$ . It is the union of 6 planes in  $\mathbb{C}^4$ . Computations using SINGULAR show that  $X$  is not free, since a minimal free resolution is given by:

$$0 \rightarrow A \rightarrow A^5 \rightarrow A^{10} \rightarrow \text{Der}^k(-\log X) \rightarrow 0.$$

**Remark 4.24.** The condition on the number of subspaces in Theorem 4.18 cannot be dropped, as we observed by considering randomly generated examples with more than  $\binom{n}{k}$  subspaces with SINGULAR.

### 4.3 Constructing Free Singularities via Products

In this section we describe two ways of constructing new free singularities from known free singularities via two kinds of products: scheme-theoretic products and a generalization of the product in the sense of hyperplane arrangements.

**Notation 4.25.** Let  $A_1 = \mathbb{C}\{\mathbf{x}\}$  and  $A_2 = \mathbb{C}\{\mathbf{y}\}$ .

We set  $A = A_1 \hat{\otimes} A_2 \simeq \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ .

**Notation 4.26.** The following notations are fixed in this section.

For  $i \in \{1, 2\}$  let  $(X_i, \mathbf{0}) \subseteq (\mathbb{C}^{n_i}, \mathbf{0})$  be a reduced Cohen–Macaulay subspace of codimension  $k_i$  and  $(f_{i,1}, \dots, f_{i,k_i}) \subseteq A_i$  be the equations of a reduced complete intersection  $(C_i, \mathbf{0})$  of codimension  $k_i$  containing  $(X_i, \mathbf{0})$ .

The next lemma recalls basic properties of analytic tensor products which will be used after.

**Theorem 4.27.** Let  $R_1$  and  $R_2$  be two analytic  $\mathbb{C}$ -algebras and  $R = R_1 \hat{\otimes} R_2$ . Let  $M_i$  be an  $R_i$ -module for  $i \in \{1, 2\}$  and define  $M_{iR} = M_i \otimes_{R_i} R$ . Then

- (1)  $\text{depth}_R(M_{1R} \otimes_R M_{2R}) = \text{depth}_{R_1}(M_1) + \text{depth}_{R_2}(M_2)$ ,
- (2)  $\dim_R(M_{1R} \otimes_R M_{2R}) = \dim_{R_1}(M_1) + \dim_{R_2}(M_2)$ .
- (3)  $R_1$  and  $R_2$  are reduced if and only if  $R$  is reduced.

*Proof.* See [GR71, Kapitel III §5 Satz 10], [GR71, Kapitel III §5 Satz 17] and [GR71, Kapitel III §5 Satz 19].  $\square$

It follows that:

**Corollary 4.28.** With the hypothesis of Notations 4.26, the product  $(X_1, \mathbf{0}) \times (X_2, \mathbf{0}) \subseteq (\mathbb{C}^{n_1}, \mathbf{0}) \times (\mathbb{C}^{n_2}, \mathbf{0})$  is a reduced Cohen–Macaulay subspace.

**Remark 4.29.** We define  $(X, \mathbf{0}) := (X_1, \mathbf{0}) \times (X_2, \mathbf{0})$ . A reduced complete intersection  $(C, \mathbf{0})$  containing  $(X, \mathbf{0})$  is defined by the regular sequence  $(f_{1,1}, \dots, f_{1,k_1}, f_{2,1}, \dots, f_{2,k_2}) \subseteq A$ . In particular,  $\text{codim}(X) = \text{codim}(C) = k_1 + k_2$  and  $J_C = AJ_{C_1} \cdot AJ_{C_2}$ .

The main result of this section is:

**Theorem 4.30.** Let  $(X_1, \mathbf{0}) \subseteq (\mathbb{C}^{n_1}, \mathbf{0})$  and  $(X_2, \mathbf{0}) \subseteq (\mathbb{C}^{n_2}, \mathbf{0})$  be reduced Cohen–Macaulay subspaces and  $(X, \mathbf{0}) = (X_1, \mathbf{0}) \times (X_2, \mathbf{0}) \subseteq (\mathbb{C}^{n_1}, \mathbf{0}) \times (\mathbb{C}^{n_2}, \mathbf{0})$ . Then  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0})$  are free if and only if  $(X, \mathbf{0})$  is free.

**Remark 4.31.** In particular, if  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0})$  are hypersurfaces, then  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0})$  are free divisors if and only if  $(X_1, \mathbf{0}) \times (X_2, \mathbf{0})$  is a free complete intersection of codimension 2.

We will need the following results.



**Lemma 4.32** (Depth Lemma). *Let  $R$  be a local Noetherian ring and consider a short exact sequence of  $R$ -modules :*

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0.$$

Then

$$\text{depth}(M_2) \geq \min(\text{depth}(M_1), \text{depth}(M_3)).$$

In case this inequality is strict, we have  $\text{depth}(M_1) = \text{depth}(M_3) + 1$ .

*Proof.* See [JP00, Lemma 6.5.18]. □

**Lemma 4.33.** *Let  $R_1$  and  $R_2$  be two analytic  $\mathbb{C}$ -algebras and  $R = R_1 \hat{\otimes} R_2$ . Let  $I \subseteq R_1$  and  $J \subseteq R_2$ . We assume that  $\text{depth}(R_1/I) < \text{depth}(R_1)$  and  $\text{depth}(R_2/J) < \text{depth}(R_2)$ . Then:*

- (1)  $\text{depth}(R/(RI + RJ)) = \text{depth}(R_1/I) + \text{depth}(R_2/J)$ ,
- (2)  $\text{depth}(R/(RI \cap RJ)) = \text{depth}(R_1/I) + \text{depth}(R_2/J) + 1$ .

*Proof.*

- (1) The statement follows from Lemma 4.27 noticing that  $R/(RI + RJ) \simeq (R_1/I) \hat{\otimes} (R_2/J)$ .
- (2) Let us consider the exact sequence

$$0 \rightarrow R/(RI \cap RJ) \rightarrow (R/RI) \oplus (R/RJ) \rightarrow R/(RI + RJ) \rightarrow 0. \quad (4.4)$$

Applying Lemma 4.27 to  $R/RI = (R_1/I) \hat{\otimes} R_2$  yields

$$\text{depth}(R/RI) = \text{depth}(R_1/I) + \text{depth}(R_2).$$

By assumption  $\text{depth}(R_2) > \text{depth}(R_2/J)$ , hence statement 1. and Lemma 4.27 imply

$$\text{depth}(R/RI) > \text{depth}(R/(RI + RJ)).$$

Analogously we obtain

$$\text{depth}(R/RJ) > \text{depth}(R/(RI + RJ)).$$

Since  $\text{depth}((R/RI) \oplus (R/RJ)) = \min(\text{depth}(R/RI), \text{depth}(R/RJ))$ , we get

$$\text{depth}((R/RI) \oplus (R/RJ)) > \text{depth}(R/(RI + RJ)).$$

In this case the inequality in Lemma 4.32 is strict, hence

$$\text{depth}(R/(RI \cap RJ)) = \text{depth}(R/(RI + RJ)) + 1. \quad \square$$

**Proposition 4.34.** *Let  $R_1$  and  $R_2$  be two analytic  $\mathbb{C}$ -algebras and  $R = R_1 \hat{\otimes} R_2$ . Let  $I \subseteq R_1$  and  $J \subseteq R_2$ . We assume that  $\text{depth}(R_1/I) < \text{depth}(R_1)$  and  $\text{depth}(R_2/J) < \text{depth}(R_2)$ . Then the following are equivalent:*

- (1)  $R/(RI \cap RJ)$  is Cohen–Macaulay,

- (2)  $R_1, R_2, R_1/I$  and  $R_2/J$  are Cohen–Macaulay,  $\dim(R_1/I) = \dim(R_1) - 1$  and  $\dim(R_2/J) = \dim(R_2) - 1$ .

*Proof.* By Lemma 4.33, we have:

$$\text{depth}(R/(RI \cap RJ)) = \text{depth}(R_1/I) + \text{depth}(R_2/J) + 1. \quad (4.5)$$

Furthermore, Lemma 4.27 and our assumptions imply the following inequality:

$$\begin{aligned} \dim(R/(RI \cap RJ)) &= \max(\dim(R/RI), \dim(R/RJ)) \\ &= \max(\dim(R_1/I) + \dim(R_2), \dim(R_1) + \dim(R_2/J)) \\ &\geq \min(\dim(R_1/I) + \dim(R_2), \dim(R_1) + \dim(R_2/J)) \\ &\geq \min(\text{depth}(R_1/I) + \text{depth}(R_2), \text{depth}(R_1) + \text{depth}(R_2/J)) \\ &\geq \text{depth}(R_1/I) + \text{depth}(R_2/J) + 1. \end{aligned} \quad (4.6)$$

Assume first that the hypothesis of the second statement is satisfied. In this case Inequality (4.6) becomes an equality. Then the first statement follows by using Equation (4.5).

Next we assume that  $R/(RI \cap RJ)$  is Cohen–Macaulay. Due to Equation (4.5) and Inequality (4.6) we obtain:

$$\begin{aligned} \text{depth}(R/(RI \cap RJ)) &= \text{depth}(R_1/I) + \text{depth}(R_2/J) + 1 \\ &\leq \dim(R/(RI \cap RJ)) \end{aligned}$$

Since  $R/(RI \cap RJ)$  is Cohen–Macaulay, equality holds everywhere, which yields that  $R_1, R_2, R_1/I$  and  $R_2/J$  are Cohen–Macaulay and  $\dim(R_2/J) = \dim(R_2) - 1$  and  $\dim(R_1/I) = \dim(R_1) - 1$ .  $\square$

**Lemma 4.35.** *Let  $R_1$  and  $R_2$  be two analytic  $\mathbb{C}$ -algebras and  $R = R_1 \hat{\otimes} R_2$ . Let  $I \subseteq R_1$  and  $J \subseteq R_2$  be ideals. Then the following equality holds in the ring  $R$ :*

$$RI \cdot RJ = RI \cap RJ.$$

*Proof.* [GR71, Kapitel III, §5 Korollar zu Satz 5]  $\square$

*Proof of Theorem 4.30.* We set for  $i \in \{1, 2\}$ ,  $R_i = A_i/I_{X_i}$  and  $R = A/I_X = A_1/I_{X_1} \hat{\otimes} A_2/I_{X_2}$ .

For  $i \in \{1, 2\}$ , let  $J_{X_i/C_i} \subseteq A_i$  and  $J_{X/C} \subseteq A$  be defined as in Notation 4.4. We denote by  $\pi : A \rightarrow R$ , respectively  $\pi_i : A_i \rightarrow R_i$  the canonical surjections. Then, by Remark 4.29,  $J_C = AJ_{C_1} \cdot AJ_{C_2} \subseteq A$ , hence Lemma 4.35 implies

$$\begin{aligned} \pi(J_{X/C}) &= \pi(J_C) \\ &= R\pi_1(J_{C_1}) \cdot R\pi_2(J_{C_2}) \\ &= R\pi_1(J_{X_1/C_1}) \cdot R\pi_2(J_{X_2/C_2}) \\ &= R\pi_1(J_{X_1/C_1}) \cap R\pi_2(J_{X_2/C_2}) \end{aligned} \quad (4.7)$$

First we assume  $J_{X_i/C_i} \neq A_i$  for  $i \in \{1, 2\}$ . Then, by Proposition 4.6,  $X$  is free if and only if  $R/\pi(J_{X/C})$  is Cohen–Macaulay of  $R$ -codimension 1. By Equation (4.7) and Proposition 4.34 we obtain that  $R/\pi(J_{X/C})$  is Cohen–Macaulay if and only if for

$i \in \{1, 2\}$  it holds that  $R_i$  and  $R_i/\pi_i(J_{X_i/C_i})$  are Cohen–Macaulay and  $\dim(R_i) = \dim(R_i/\pi_i(J_{X_i/C_i})) + 1$ . This is, again by Proposition 4.6, equivalent to the fact that  $X_1$  and  $X_2$  are free. Next we consider the case  $J_{X_i/C_i} = A_i$  for at least one  $i \in \{1, 2\}$ . In case  $J_{X/C} = A$  the statement is obvious, hence we assume without loss of generality  $J_{X/C} = AJ_{X_1/C_1}$ . Then  $R/\pi(J_{X/C}) \cong R_1/\pi_1(J_{X_1/C_1}) \hat{\otimes} R_2$ . In this setup the statement follows from Theorem 4.27.  $\square$

**Remark 4.36.** *As a consequence, if  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0})$  are free Cohen–Macaulay subspaces, we have*

$$\text{projdim} \left( \text{Der}^{k_1+k_2}(-\log X_1 \times X_2) \right) = \\ \text{projdim} \left( \text{Der}^{k_1}(-\log X_1) \right) + \text{projdim} \left( \text{Der}^{k_2}(-\log X_2) \right) + 1$$

A different notion of product for hyperplane arrangements is considered in [OT92, Definition 2.13]. We use the following notation.

**Notation 4.37.** *Let  $X_1 \subseteq \mathbb{C}^{n_1}$  and  $X_2 \subseteq \mathbb{C}^{n_2}$  be two reduced equidimensional subspaces, both of the same codimension  $k$ . Let  $X'_1 = X_1 \times \mathbb{C}^{n_2}$  and  $X'_2 = \mathbb{C}^{n_1} \times X_2$ .*

*For  $i \in \{1, 2\}$  let  $\iota_i : \bigwedge^k \text{Der}_{\mathbb{C}^{n_i}} \rightarrow \bigwedge^k \text{Der}_{\mathbb{C}^{n_1+n_2}}$  be the canonical maps. By abuse of notation we identify  $\text{Der}^k(-\log X_i)$  with the submodule of  $\bigwedge^k \text{Der}_{\mathbb{C}^{n_1+n_2}}$  generated by  $\iota_i(\text{Der}^k(-\log X_i))$ .*

*Consider the decomposition:*

$$\bigwedge^k \text{Der}_{\mathbb{C}^{n_1+n_2}} = D_1 \oplus D_2 \oplus D_{1,2}$$

*where  $D_i$  is the submodule generated by the image of  $\bigwedge^k \text{Der}_{\mathbb{C}^{n_i}}$  in  $\bigwedge^k \text{Der}_{\mathbb{C}^{n_1+n_2}}$  and  $D_{1,2}$  is the free submodule of  $\bigwedge^k \text{Der}_{\mathbb{C}^{n_1+n_2}}$  generated by the elements of the form  $\partial_{x_{i_1}} \wedge \cdots \wedge \partial_{x_{i_p}} \wedge \partial_{y_{j_1}} \wedge \cdots \wedge \partial_{y_{j_{k-p}}}$  where  $p \in \{1, \dots, k-1\}$ .*

It can be generalized to subspaces of higher codimension as follows:

**Definition 4.38.** *Let  $X_1 \subseteq \mathbb{C}^{n_1}$  and  $X_2 \subseteq \mathbb{C}^{n_2}$  be two equidimensional subspaces, both of the same codimension  $k$ . We set  $X_1 * X_2 = X'_1 \cup X'_2$ .*

A similar result as Theorem 4.30 is satisfied, which generalizes [OT92, Proposition 4.28]:

**Proposition 4.39.** *Let  $X_1 \subseteq \mathbb{C}^{n_1}$  and  $X_2 \subseteq \mathbb{C}^{n_2}$  be two reduced equidimensional subspaces, both of the same codimension  $k$ . Then, with Notation 4.37:*

$$\text{Der}^k(-\log X_1 * X_2) = \text{Der}^k(-\log X_1) \oplus \text{Der}^k(-\log X_2) \oplus D_{1,2}.$$

*In particular,  $X_1 * X_2$  is free if and only if both  $X_1$  and  $X_2$  are free.*

*Proof.* We have:

$$\text{Der}^k(-\log X'_1) = \text{Der}^k(-\log X_1) \oplus D_2 \oplus D_{1,2},$$

$$\text{Der}^k(-\log X'_2) = D_1 \oplus \text{Der}^k(-\log X_2) \oplus D_{1,2}.$$

By Proposition 4.8,  $\text{Der}^k(-\log X_1 * X_2) = \text{Der}^k(-\log X_1) \cap \text{Der}^k(-\log X_2)$ . Thus  $\text{Der}^k(-\log X_i) \subseteq D_i$  implies:

$$\text{Der}^k(-\log X_1 * X_2) = \text{Der}^k(-\log X_1) \oplus \text{Der}^k(-\log X_2) \oplus D_{1,2}$$

A minimal free resolution of  $\text{Der}^k(-\log X_1 * X_2)$  is thus given as the direct sum of minimal free resolutions of  $\text{Der}^k(-\log X_1)$ ,  $\text{Der}^k(-\log X_2)$  and  $D_{1,2}$ . Since  $D_{1,2}$  is free, the projective dimension of  $\text{Der}^k(-\log X_1 * X_2)$  is

$$\max \left\{ \text{projdim} \left( \text{Der}^k(-\log X_1) \right), \text{projdim} \left( \text{Der}^k(-\log X_2) \right) \right\}.$$

Since by [Pol20, Proposition 4.2],  $\text{projdim}(\text{Der}^k(-\log X_i)) \geq k - 1$ , we have  $\text{projdim}(\text{Der}^k(-\log X_1 * X_2)) = k - 1$  if and only if

$$\text{projdim}(\text{Der}^k(-\log X_1)) = \text{projdim}(\text{Der}^k(-\log X_2)) = k - 1. \quad \square$$

## 4.4 Properties of Generalized Normal Crossing Divisors

### 4.4.1 Freeness and Generalized Normal Crossing Divisors

In this section we investigate freeness of generalized normal crossing divisors respectively their singular loci.

Theorem 4.30 implies directly that generalized normal crossing divisors have free singular loci:

**Proposition 4.40.** *Let  $(X, \mathbf{0}) \subseteq \mathbb{C}^n$  be a generalized normal crossing divisor. Then  $(\text{Sing}(X), \mathbf{0})$  is a free singularity.*

*Proof.* By Theorem 3.25 we can assume that  $f = \sum_{i=0}^l g_i$ , where  $g_0$  is an  $A_1$ -singularity and the  $g_i$  are normal crossing divisors for  $1 \leq i \leq l$ . Due to Proposition 4.11, we obtain that  $J_{g_i}$  defines a free singularity. The result follows from Theorem 4.30.  $\square$

Next we want to investigate when generalized normal crossing divisors are free. In order to do so, we need the following result from Aleksandrov-Terao, which is a special version of Proposition 4.6.

**Lemma 4.41.** *Let  $(X, \mathbf{0}) \subseteq \mathbb{C}^n$  be the germ of a hypersurface singularity. Denote by  $f \in \mathbb{C}\{\mathbf{x}\}$  a local equation of  $(X, \mathbf{0})$ . Then  $(X, \mathbf{0})$  is free if and only if the  $\mathcal{O}_{X, \mathbf{0}}/J_f$  is Cohen–Macaulay of dimension  $n - 2$ .*

*Proof.* See [Ter80, Proposition 2.4] and [Ale88, §2, Theorem].  $\square$

Now we are able to show the following:

**Proposition 4.42.** *Let  $(X, \mathbf{0}) \subseteq \mathbb{C}^n$  be the germ of a hypersurface singularity. Denote by  $f \in \mathbb{C}\{\mathbf{x}\}$  a local equation of  $(X, \mathbf{0})$ . Then the following are equivalent:*

- (1)  $(X, \mathbf{0})$  is a normal crossing divisor.

(2)  $(X, \mathbf{0})$  is free and  $J_f$  is of Stanley–Reisner type.

*Proof.* In case  $(X, \mathbf{0})$  has normal crossings the freeness of  $(X, \mathbf{0})$  and  $J_f$  being a Stanley–Reisner ideal follow immediately. Therefore only the converse has to be shown. Due to Theorem 3.25 we may assume that  $f = \sum_{i=1}^r \left(x_i^{(1)}\right)^2 + \sum_{i=1}^l g_i \in \mathbb{C}\{\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(l+1)}\}$  where  $r, l \in \mathbb{N}$ , and  $g_i \in \mathbb{C}\{\mathbf{x}^{(i)}\}$  are normal crossing. For simplicity define  $g_0 := \sum_{i=1}^r \left(x_i^{(0)}\right)^2 \in \mathbb{C}\{\mathbf{x}^{(0)}\}$ . Then

$$\mathbb{C}\{\mathbf{x}\}/J_f \cong \mathbb{C}\{\mathbf{x}^{(1)}\}/J_{g_1} \hat{\otimes} \dots \hat{\otimes} \mathbb{C}\{\mathbf{x}^{(l)}\}/J_{g_l}. \quad (4.8)$$

Applying Theorem 4.27 to  $\mathbb{C}\{\mathbf{x}\}/J_f$  and Proposition 4.6 to  $\mathbb{C}\{\mathbf{x}\}/J_{g_i}$ , we obtain  $\dim \mathbb{C}\{\mathbf{x}\}/J_f = n - 2 \cdot l - r$ . In order for  $f$  to be free, we must have either  $l = 1$  and  $r = 0$  or  $l = 0$  and  $r = 2$ . In the first this case we have that  $f$  defines a normal crossing divisor, and the claim is shown. In the second case we have that  $f = \left(x_1^{(1)}\right)^2 + \left(x_2^{(1)}\right)^2$ . Define  $x := x_1^{(1)} + i \cdot x_2^{(1)}$  and  $y := x_1^{(1)} - i \cdot x_2^{(1)}$  then the automorphism  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$  defined by  $\varphi\left(x_i^{(0)}\right) = x_i^{(0)}$ ,  $\varphi\left(x_1^{(1)}\right) = \frac{x+y}{2}$  and  $\varphi\left(x_2^{(1)}\right) = \frac{x-y}{2i}$  satisfies  $\varphi(f) = x \cdot y$ . Thus  $f$  defines a normal crossing divisor.  $\square$

#### 4.4.2 Saito Holonomicity and Generalized Normal Crossing Divisors

In this subsection we investigate the holonomicity of strongly Euler homogeneous divisors of Sebastiani–Thom type. The notion of holonomicity as introduced by K. Saito in [Sai80] is closely related to the module of logarithmic derivations of a hypersurface singularity. We show the following theorem:

**Theorem 4.43.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^{n+m}, \mathbf{0})$  be a strongly Euler-homogeneous singularity of Sebastiani–Thom type. We denote the Sebastiani–Thom components of  $(X, \mathbf{0})$  by  $(X_1, \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$  and  $(X_2, \mathbf{0}) \subseteq (\mathbb{C}^m, \mathbf{0})$ . Then the following hold:*

- (1)  $(Y, \mathbf{0}) \subseteq (\text{Sing}(X), \mathbf{0})$  is a logarithmic stratum if and only if there exists a logarithmic stratum  $(X_{1,\alpha}, \mathbf{0}) \subseteq (\text{Sing}(X_1), \mathbf{0})$  and a logarithmic stratum  $(X_{2,\beta}, \mathbf{0}) \subseteq (\text{Sing}(X_2), \mathbf{0})$ , such that

$$(Y, \mathbf{0}) = (X_{1,\alpha}, \mathbf{0}) \times (X_{2,\beta}, \mathbf{0}) =: (X_{(\alpha,\beta)}, \mathbf{0}).$$

- (2)  $(X, \mathbf{0})$  is holonomic if and only if  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0})$  are holonomic.

We start with the basic notions in order to define the term **holonomic divisor**, which is based on the definition in [Sai80, (3.8)].

**Notation 4.44.** *From now on let  $S \subseteq \mathbb{C}^n$  be an  $n$ -dimensional complex manifold and  $X \subseteq S$  a hypersurface singularity. Any given index set will be denoted by  $\mathcal{I}$ .*

**Definition 4.45.** *Let  $\text{Der}_S(-\log X)$  be the sheaf of logarithmic vector fields along  $X$ . For any point  $p \in S$  we denote by  $\text{Der}_S(\log X)(p)$  the subspace of the tangent space  $T_{S,p}$  at  $p$ , which consists of the vectors  $\delta(p)$  of the values of the vector field  $\delta \in \text{Der}_S(-\log \mathcal{O}_{X,p})$  at  $p$ .*

We obtain the following lemma:

**Lemma 4.46.** *There exists a unique stratification  $\{X_\alpha \mid \alpha \in \mathcal{I}\}$  of  $S$  with the following properties:*

- (1) *Each stratum  $X_\alpha, \alpha \in \mathcal{I}$  is a smooth connected immersed submanifold of  $S$  and  $S$  is a disjoint union  $\bigcup_{\alpha \in \mathcal{I}} X_\alpha$  of the strata.*
- (2) *Let  $p \in S$  belong to a stratum  $X_\alpha$ . Then the tangent space  $T_{X_\alpha, p}$  of  $X_\alpha$  at  $p$  equals  $\text{Der}_S(-\log(X))(p)$ .*
- (3) *If  $X_\alpha \cap \overline{X_\beta} \neq \emptyset$  for some  $\alpha, \beta \in \mathcal{I}$  and  $\alpha \neq \beta$ , then  $X_\alpha \subseteq \partial X_\beta$ .*

*Proof.* See [Sai80, (3.2)]. □

Now we are able to define the notion of logarithmic stratification.

**Definition 4.47.** *The stratification of Lemma 4.46 is called **logarithmic stratification of  $S$** . The strata  $X_\alpha$  are called **logarithmic strata**.*

Using the notion of logarithmic strata we can define the notion of holonomic points, holonomic strata and holonomic divisors.

**Definition 4.48.** *A point  $p \in S$  is called **holonomic**, if there exists an open neighborhood  $V$  of  $p$ , such that  $V$  intersects only finitely many logarithmic strata. A stratum  $X_\alpha$  is called **holonomic stratum**, if there exists an open neighborhood  $V$  of  $X_\alpha$ , such that  $V$  intersects only finitely many logarithmic strata. A divisor  $X$  is called **holonomic divisor**, if every  $p \in S$  is holonomic. We say a complex space germ  $(X, \mathbf{0})$  is holonomic, if there exists a holonomic representant.*

In order to state criteria to determine the holonomicity of divisor, we need the following notation.

**Notation 4.49.** *We define  $\mathcal{I}_p = \{\beta \in \mathcal{I} \mid p \in \overline{X_\beta}\}$  for a point  $p \in S$  and  $\mathcal{I}_\alpha = \{\beta \in \mathcal{I} \mid X_\alpha \subseteq \overline{X_\beta}\}$  for a stratum  $X_\alpha$ . For any  $r \geq 0$  we define  $V_r = \{p \in S \mid \text{rank}_{\mathbb{C}} \text{Der}_S(-\log X)(p)\}$ .*

**Remark 4.50.** *By definition the sets  $V_r$  are closed analytic subsets and it holds that  $V_r = \bigcup_{\dim X_\alpha \leq r} X_\alpha$ .*

The following lemma gives us an algebraic tool to decide whether a divisor is holonomic or not.

**Lemma 4.51.**

- (1) *Let  $p$  be a point of a stratum  $X_\alpha$ . Then  $p$  is holonomic if and only if*

$$\dim_p V_r \leq r \text{ for } \dim X_\alpha \leq r.$$

- (2) *Let  $S' \subseteq S$  be an open subset and let  $X' = X \cap S'$ . Then a point  $p \in S'$  is holonomic with respect to the logarithmic stratification by  $X'$  if and only if  $p \in S$  is holonomic with respect to the logarithmic stratification by  $X$ .*
- (3) *Let  $X = X' \times \mathbb{C}^k \subseteq S' \times \mathbb{C}^k = S$  for some  $1 \leq k \leq n$ . Then  $p = (p', p'') \in S$  is holonomic if and only if  $p' \in S'$  is holonomic.*

*Proof.* See [Sai80, (3.13)] and [Sai80, (3.14)].  $\square$

Lemma 4.51 allows us to verify algorithmically if a divisor is holonomic by computing the dimension of the varieties  $V_r$ , which are just radicals of minor ideals. Furthermore, Lemma 4.51 allows us to reduce to the case of unsuspended divisors. To make the computational aspect more explicit, we consider two examples of divisors, one holonomic the other one not holonomic.

**Example 4.52.**

- (1) Let  $X = V(xyz) \subseteq \mathbb{C}^3$ . Then a SINGULAR computation yields that  $\text{Der}_{\mathbb{C}^3}(\log X)$  is generated by the derivations

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \underbrace{\begin{pmatrix} x & y & z \\ x & 0 & -z \\ 0 & y & -z \end{pmatrix}}_{=A} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$

From the radicals of minor ideals of the matrix  $A$  we obtain

$$\begin{aligned} V_0 &= V(x, y, z) \\ V_1 &= V(xy, xz, yz) \\ V_2 &= V(xyz) \\ V_r &= \mathbb{C}^3 \text{ for all } r \geq 3. \end{aligned}$$

In particular,  $\dim(V_r) \leq r$  for all  $r \geq 0$ , hence  $X$  is a holonomic divisor by Lemma 4.51.

- (2) Let  $X = V(xy(x+y)(x-y)(y-xz)) \subseteq \mathbb{C}^3$ . Then a SINGULAR computation yields that  $\text{Der}_{\mathbb{C}^3}(\log X)$  is generated by the derivations

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \underbrace{\begin{pmatrix} x & y & 0 \\ 0 & x^2y - y^3 & 2xy - x^2z - 3y^2z + 2xyz^2 \\ 0 & 0 & y - xz \end{pmatrix}}_{=A} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$

In this particular case, we obtain  $V_0 = V(x, y)$ . This implies that  $X$  is not holonomic, since  $\dim(V_0) = 1 > 0$ .

As we can see in the previous example, we need to compute the module of logarithmic derivations of a given divisor to determine whether it is holonomic or not.

To keep the notation as simple as possible, we reduce to the unsuspended case.

**Remark 4.53.** By Lemma 3.10, we assume from now on that all divisors are unsuspended.

We introduce the following definition.

**Definition 4.54.** Let  $X \subseteq \mathbb{C}^n$  define a hypersurface singularity. Let  $p \in X$ . Denote by  $f_p \in \mathbb{C}\{\mathbf{x}-p\}$  the local equation of  $X$  at  $p$ . The module  $\text{Der}_{X,p}(-\log f_p) = \{\delta \in \text{Der}_{\mathbb{C}^n}(-\log \mathcal{O}_{X,p}) \mid \delta(f_p) = 0\}$  is called the **module of annihilating derivations of  $f$  at  $p$** .

In our particular case, we obtain the module of logarithmic derivations very easily from the following lemma:



**Lemma 4.55.** *Let  $X \in \mathbb{C}^n$  define an Euler-homogeneous hypersurface singularity. Denote by  $f_p \in \mathbb{C}\{\mathbf{x} - p\}$  the local equation of  $X$  at  $p \in X$  and by  $\chi_p$  an Euler-derivation at the point  $p \in X$ . Then*

$$\mathrm{Der}_{\mathbb{C}^n}(-\log \mathcal{O}_{X,p}) = \langle \chi_p \rangle + \mathrm{Der}_{X,p}(-\log f_p).$$

*Proof.* For every  $\delta \in \mathrm{Der}_{\mathbb{C}^n}(-\log \mathcal{O}_{X,p})$  it holds that  $\delta(f_p) = g \cdot f_p$  for some  $g \in \mathcal{O}_{X,p}$ . Then we can write  $\delta$  as

$$\delta = \underbrace{\delta - g \cdot \chi_p}_{\in \mathrm{Der}_{X,p}(-\log f_p)} + g \cdot \chi_p.$$

Thus the claim follows.  $\square$

In other words, Lemma 4.55 states that every logarithmic derivation of  $f$  can be written as a sum of an Euler-derivation and an annihilating derivation.

**Remark 4.56.** *Note that  $\mathrm{Der}_{X,p}(-\log f_p)$  is isomorphic to the module  $\mathrm{syz}_{\mathcal{O}_{X,p}}(J_{f_p})$ . Hence computing the module of annihilating derivations is reduced to computing the syzygies of the Jacobian ideal.*

The next proposition is crucial for the computation of the annihilating derivations. The proof of Proposition 4.57 involves some basic computer algebra, in particular the theory of standard bases over power series rings. We refer the reader for more details on this topic to [JP00, Chapter 7] and [GP08, Chapter 6].

**Proposition 4.57.** *Let  $A = \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ ,  $A_1 = \mathbb{C}\{\mathbf{x}\}$  and  $A_2 = \mathbb{C}\{\mathbf{y}\}$ . Let  $f \in A$  with  $f = g_1 + g_2$ , where  $g_1 \in A_1$  and  $g_2 \in A_2$ . Denote by  $e_1, \dots, e_{n+m}$  the canonical basis vectors of  $A^{n+m}$ . Then*

$$\mathrm{syz}_A(J_f) = R_1 + R_2 + J \subseteq A^{n+m},$$

where  $R_1 = \mathrm{syz}_{A_1}(J_{g_1}) \otimes_{A_1} A \subseteq \langle e_1, \dots, e_n \rangle$ ,  $R_2 = \mathrm{syz}_{A_2}(J_{g_2}) \otimes_{A_2} A \subseteq \langle e_{n+1}, \dots, e_{n+m} \rangle$  and  $J$  is generated by vectors of type  $\sum_{i=1}^n a_i e_i + \sum_{j=1}^m b_j e_{n+j}$  with  $a_i \in J_{g_2}$ ,  $b_j \in J_{g_1}$ .

We need the following theorem to show Proposition 4.57.

**Theorem 4.58.** *Let  $F = (f_1, \dots, f_s)^T \in \mathbb{C}\{\mathbf{x}\}^s$  and  $G = (g_1, \dots, g_m)^T \in \mathbb{C}\{\mathbf{x}\}^m$ . Assume that  $S = \{g_1, \dots, g_m\}$  is a standard basis of the ideal  $I = \langle f_1, \dots, f_s \rangle$ . Denote by  $E_s$  the  $s \times s$  unit matrix, by  $R$  the matrix whose rows form a generating set of  $\mathrm{syz}_{\mathbb{C}\{\mathbf{x}\}}(\langle S \rangle)$ , by  $U$  the matrix satisfying  $G = UF$  and by  $V$  the matrix satisfying  $F = VG$ . Then the rows of  $Q$  form a basis of  $\mathrm{syz}_{\mathbb{C}\{\mathbf{x}\}}(I)$ , where*

$$Q = \begin{pmatrix} E_s - V \cdot U \\ R \cdot U \end{pmatrix}.$$

*Proof.* The result is shown in [Win96, Theorem 8.4.8] for the polynomial case and works verbatim for the power series case.  $\square$

*Proof of Proposition 4.57.* We fix an arbitrary local ordering  $>_1$  on  $A_1$  and an arbitrary local ordering  $>_2$  on  $A_2$ . On  $A$  we consider the local ordering  $> = (>_1, >_2)$ . We prove the result in two steps.

**Step 1:** We assume that  $S' = \{\partial_{x_1} g_1, \dots, \partial_{x_n} g_1, \partial_{y_1} g_2, \dots, \partial_{y_m} g_2\}$  is a standard basis for  $J_f$ . Using the Product Criterion (see [JP00, Exercise 7.2.19]) and [GP08, Theorem 2.5.9] the result follows. In particular,  $J = \langle \partial_{y_j} g_2 e_i - \partial_{x_i} g_1 e_j \mid 1 \leq i \leq n, 1 \leq j \leq m \rangle$ .

**Step 2:** We assume that  $S' = \{\partial_{x_1}g_1, \dots, \partial_{x_n}g_1, \partial_{y_1}g_2, \dots, \partial_{y_m}g_2\}$  is not a standard basis for  $J_f$ . By applying the Product Criterion it follows that, if  $S_1$  is a standard basis for  $J_{g_1} \in A_1$  and if  $S_2$  is a standard basis for  $J_{g_2} \in A_2$ ,  $S = S_1 \cup S_2$  is a standard basis for  $J_f$ . Let  $s_1 = |S_1|$ ,  $s_2 = |S_2|$  and  $s = s_1 + s_2$ . Define  $F_1 = (\partial_{x_1}g_1, \dots, \partial_{x_n}g_1)^T \in A_1^n$ ,  $F_2 = (\partial_{y_1}g_2, \dots, \partial_{y_m}g_2)^T \in A_2^m$ ,  $G_1 \in A_1^{s_1}$  be the vector containing the elements of  $S_1$  and  $G_2 \in A_2^{s_2}$  be the vector containing the elements of  $S_2$ . Furthermore, we denote by  $U_1 \in A_1^{n \times s_1}$  the matrix satisfying  $G_1 = U_1 F_1$ , by  $U_2 \in A_2^{m \times s_2}$  the matrix satisfying  $G_2 = U_2 F_2$ , by  $V_1 \in A_1^{n \times s_1}$  the matrix satisfying  $F_1 = V_1 G_1$  and by  $V_2 \in A_2^{m \times s_2}$  the matrix satisfying  $F_2 = V_2 G_2$ . Denote by  $R'_1$  the matrix whose rows form a generating set of  $\text{syz}_{A_1}(\langle S_1 \rangle)$ , by  $R'_2$  the matrix whose rows form a generating set of  $\text{syz}_{A_2}(\langle S_2 \rangle)$ , and by  $J'$  the matrix whose rows are the vectors  $(0, \dots, \partial_{y_j}g_2, 0, \dots, 0, \partial_{x_i}g_1, 0, \dots, 0)$  for  $1 \leq i \leq n, 1 \leq j \leq m$ . This means that  $J = \begin{pmatrix} J_1 & J_2 \end{pmatrix}$ , where the entries of  $J_1$  are in  $J_{g_2}$  and the entries of  $J_2$  are in  $J_{g_1}$ . Then, by Step 1,  $\text{syz}_A(\langle S \rangle)$  is generated by the rows of the matrix

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \\ J_1 & J_2 \end{pmatrix}.$$

Let  $F$  be the concatenation of  $F_1$  and  $F_2$ ,  $G$  the concatenation of  $G_1$  and  $G_2$ ,  $U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$  and  $V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$ . Then  $G = UF$  and  $F = VG$ . The result follows from Theorem 4.58 and the block structure of the involved matrices.  $\square$

**Remark 4.59.** Proposition 4.57 shows that, in the setup of Theorem 4.43, every derivation  $\delta \in \text{Der}_{\mathbb{C}^{n+m}}(-\log \mathcal{O}_{X,p})$  as

$$\delta = \chi_p + \underbrace{\delta_1}_{\in R_1} + \underbrace{\delta_2}_{\in R_2} + \underbrace{\delta_3}_{\in J},$$

where the derivations  $\delta_1, \delta_2$  and  $\delta_3$  satisfy  $\delta_1(g_1) = 0, \delta_2(g_2) = 0$  and  $\delta_3(p) = 0$  for  $p \in \text{Sing}(X)$ . This means that  $\delta_1$  and  $\delta_2$  are annihilating derivations of  $g_1$  respectively  $g_2$ .

In our setup we obtain a block-matrix structure for the generating set of  $\text{Der}_{\mathbb{C}^{n+m}}(-\log \mathcal{O}_{X,p})$ , as we can see in the following example:

**Example 4.60.** Let  $f = xyz + abcd \in \mathbb{C}[x, y, z, a, b, c, d]$  and  $X = V(f) \subseteq \mathbb{C}^7$ . Then  $\text{Der}_{\mathbb{C}^7}(-\log X)$  is generated by the columns of the following matrix, where we drop the partial derivatives:

$$\begin{pmatrix} 4x & x & 0 & 0 & 0 & 0 & -bcd & -acd & -abd & -abc & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4y & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -bcd & -acd & -abd & -abc & 0 & 0 & 0 & 0 \\ 4z & -z & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -bcd & -acd & -abd & -abc \\ 3a & 0 & 0 & a & 0 & 0 & yz & 0 & 0 & 0 & xz & 0 & 0 & 0 & xy & 0 & 0 & 0 \\ 3b & 0 & 0 & 0 & b & 0 & 0 & yz & 0 & 0 & 0 & xz & 0 & 0 & 0 & xy & 0 & 0 \\ 3c & 0 & 0 & 0 & 0 & c & 0 & 0 & yz & 0 & 0 & 0 & xz & 0 & 0 & 0 & xy & 0 \\ 3d & 0 & 0 & -d & -d & -d & 0 & 0 & 0 & yz & 0 & 0 & 0 & xz & 0 & 0 & 0 & xy \end{pmatrix}.$$

Next we want to show, that it suffices to consider only the annihilating derivations of  $f$  evaluated at every point  $p \in \text{Sing}(X)$ .

**Lemma 4.61.** Let  $X \subseteq \mathbb{C}^n$  be an Euler-homogeneous hypersurface singularity. Then for every point  $p \in X$  and for every Euler-derivation  $\chi_p \in \text{Der}_{\mathbb{C}^n}(-\log \mathcal{O}_{X,p})$  it holds that  $\chi_p(p) \in \text{Der}_{X,p}(-\log f_p)(p)$  if, and only if  $X$  is strongly Euler-homogeneous.

*Proof.* Denote the local equation of  $X$  at  $p$  by  $f_p$ .

Let  $p \in X$  be arbitrary and  $\chi_p \in \text{Der}_{\mathbb{C}^n}(-\log \mathcal{O}_{X,p})$  any Euler-derivation. The equation  $\chi_p(p) \in \text{Der}_{X,p}(-\log f_p)(p)$  implies the existence of  $\delta \in \text{Der}_{X,p}(-\log f_p)$  with  $\chi_p(p) = \delta(p)$ . Then the derivation  $\chi'_p = \chi_p - \delta$  is an Euler-derivation with  $\chi'_p(p) = 0$ , hence  $X$  is strongly Euler-homogeneous at  $p$ . Now we assume that  $X$  is strongly Euler-homogeneous at  $p \in X$ . Denote by  $\chi_p$  the Euler-derivation satisfying  $\chi_p(p) = 0$ . By Lemma 4.55 every Euler-derivation  $\chi'_p$  of  $f_p$  can be written as

$$\chi'_p = \chi_p + \delta,$$

where  $\delta \in \text{Der}_{X,p}(-\log f_p)$ . This implies  $\chi'_p(p) = \chi_p(p) + \delta(p) = \delta(p)$ , hence  $\chi'_p(p) \in \text{Der}_{X,p}(-\log f_p)(p)$ .  $\square$

The final ingredient we need is the following lemma.

**Lemma 4.62.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^{n+m}, \mathbf{0})$  be a hypersurface singularity of Sebastiani–Thom type. Then  $(X, \mathbf{0})$  is strongly Euler-homogeneous if and only if the Sebastiani–Thom components of  $(X, \mathbf{0})$  are strongly Euler-homogeneous.*

*Proof.* Let  $A = \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ ,  $A_1 = \mathbb{C}\{\mathbf{x}\}$ ,  $A_2 = \mathbb{C}\{\mathbf{y}\}$ ,  $(X, \mathbf{0}) \cong (V(f), \mathbf{0}) \subseteq \mathbb{C}^{n+m}$ , with  $f = g + h \in A$ , where  $g \in A_1$  and  $h \in A_2$ . If  $g$  and  $h$  are strongly Euler-homogeneous with respect to the Euler-derivations  $\chi_g = \sum_{i=1}^n a_i \partial_{x_i}$  and  $\chi_h = \sum_{j=1}^m b_j \partial_{y_j}$ , then  $f$  is strongly Euler-homogeneous with respect to the Euler-derivation  $\chi = \sum_{i=1}^n a_i \partial_{x_i} + \sum_{j=1}^m b_j \partial_{y_j}$ . Now we prove the converse. Assume  $(X, \mathbf{0})$  is strongly Euler-homogeneous. If  $g = 0$  or  $h = 0$ , we are in the case of Lemma 3.10, hence we can assume  $g \neq 0$  and  $h \neq 0$ . The fact that  $f$  is strongly Euler-homogeneous is equivalent to  $f \in \mathfrak{m}_A J_f$ . This is equivalent to the existence of  $a_i, b_j \in \mathfrak{m}_A$  for  $1 \leq i \leq n, 1 \leq j \leq m$ , such that

$$f = \sum_{i=1}^n a_i \partial_{x_i} g + \sum_{j=1}^m b_j \partial_{y_j} h. \quad (4.9)$$

Since  $f$  defines a singularity, we know that  $f \in \mathfrak{m}^2$ , hence  $g \in \mathfrak{m}_{A_1}^2$  and  $h \in \mathfrak{m}_{A_2}^2$ . This implies that  $\partial_{x_i} g(\mathbf{0}) = 0$  and  $\partial_{y_j} h(\mathbf{0}) = 0$  for  $1 \leq i \leq n$  and  $1 \leq j \leq m$ . Equation 4.9 implies

$$g(\mathbf{x}) = f(\mathbf{x}, \mathbf{0}) = \sum_{i=1}^n a_i(\mathbf{x}, \mathbf{0}) \partial_{x_i} g(\mathbf{x}).$$

By assumption  $g \neq 0$ , so for all  $1 \leq i \leq n$  it holds that  $a_i(\mathbf{x}, \mathbf{0}) \neq 0$ , hence  $g \in \mathfrak{m}_{A_2} J_g$ . The result for  $h$  follows analogously.  $\square$

**Remark 4.63.** *The converse statement of Lemma 4.62 does not hold in general, if we replace the property of being strongly Euler-homogeneous with the property of being Euler-homogeneous. To see this, consider any  $g \in \mathbb{C}[\mathbf{x}]$ , such that  $(X_1, \mathbf{0}) = (V(g), \mathbf{0})$  is not Euler-homogeneous. Define  $f = g \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ . Then  $X = V(f) \subseteq \mathbb{C}^{n+1}$  has Sebastiani–Thom components  $(X_1, \mathbf{0})$  and  $(X_2, \mathbf{0}) = (\mathbb{C}, \mathbf{0})$  at  $p = \mathbf{0}$ . Additionally,  $(X, \mathbf{0}) \cong (V(e^y \cdot g), \mathbf{0})$ , so  $(X, \mathbf{0})$  is Euler-homogeneous with Euler-derivation  $\partial_z$ .*

Now we are able to prove Theorem 4.43.

*Proof of Theorem 4.43.* Let  $A = \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ ,  $A_1 = \mathbb{C}\{\mathbf{x}\}$ ,  $A_2 = \mathbb{C}\{\mathbf{y}\}$ ,  $(X, \mathbf{0}) \cong (V(f), \mathbf{0}) \subseteq \mathbb{C}^{n+m}$ , with  $f = g + h \in A$ , where  $g \in A_1$  and  $h \in A_2$ . Define  $(X_1, \mathbf{0}) = (V(g), \mathbf{0}) \subseteq \mathbb{C}^n$  and  $(X_2, \mathbf{0}) = (V(h), \mathbf{0}) \subseteq \mathbb{C}^m$ . It holds that  $(\text{Sing}(X), \mathbf{0}) = (\text{Sing}(X_1), \mathbf{0}) \times (\text{Sing}(X_2), \mathbf{0})$ .

We fix a basic open neighbourhood  $U = U_1 \times U_2 \in \mathbb{C}^{n+m}$  of  $\mathbf{0}$  and consider the representants on  $U$ , respectively on  $U_1$  and  $U_2$ . We start by proving the first statement. Due to Lemma 4.62 we know that  $g$  and  $h$  are strongly Euler-homogeneous. Proposition 4.57 and Lemma 4.61 imply that for every  $p = (p_1, p_2) \in U$  it holds that:

$$\mathrm{Der}_{\mathbb{C}^{n+m}}(-\log V(f))(p) \cong \mathrm{Der}_{\mathbb{C}^n}(-\log V(g))(p_1) \oplus \mathrm{Der}_{\mathbb{C}^m}(-\log V(h))(p_2). \quad (4.10)$$

Let  $X_{1,\alpha}$  be a logarithmic stratum of  $X_1$  and  $X_{2,\beta}$  be a logarithmic stratum of  $X_2$ . Define  $X_{(\alpha,\beta)} = X_{1,\alpha} \times X_{2,\beta}$ . By construction the  $X_{(\alpha,\beta)}$  are smooth, connected, immersed submanifolds of  $U$  with  $\mathrm{Sing}(X) = \bigcup_{(\alpha,\beta)} X_{(\alpha,\beta)}$  and

$$T_{X_{(\alpha,\beta)},p} \cong T_{X_{1,\alpha},p_1} \oplus T_{X_{2,\beta},p_2}. \quad (4.11)$$

Lemma 4.46, Equation (4.10) and Equation (4.11) imply that the  $X_{(\alpha,\beta)}$  are the unique logarithmic strata of  $\mathrm{Sing}(X)$ , hence the statement follows. The second statement follows directly from the first statement, since by Lemma 4.51 only the logarithmic strata contained in the singular locus have to satisfy the finiteness property.  $\square$

Since generalized normal crossing divisors satisfy the assumptions of Theorem 4.43, we obtain the following corollary:

**Corollary 4.64.** *Let  $(X, \mathbf{0}) \subseteq (\mathbb{C}^n, \mathbf{0})$  be a generalized normal crossing divisor. Then  $(X, \mathbf{0})$  is a holonomic divisor.*

## Chapter 5

# Algorithms and Examples

In this chapter we present a Las Vegas algorithm, which can reconstruct the defining equation  $f$  of a quasi-homogeneous isolated hypersurface singularity from a zero-dimensional  $\mathbb{C}$ -algebra isomorphic to  $\mathbb{C}\{\mathbf{x}\}/J_f$ . The algorithm can also be used to check if a zero-dimensional  $\mathbb{C}$ -algebra is isomorphic to  $\mathbb{C}\{\mathbf{x}\}/J_f$ , where  $f$  defines a quasi-homogeneous isolated hypersurface singularity.

We focus on quasi-homogeneous isolated hypersurface singularities, because they admit enough structure which makes it possible to obtain the needed information. A similar algorithm for the homogeneous case has been presented in [IK14]. The main obstacle we have to overcome is the fact that in the quasi-homogeneous case the weights of  $f$ , which is uniquely determined if  $\text{ord}(f) \geq 3$ , are not known. We are going to use computational methods to not only recover the weights, but also to show that maximal toral Lie algebras contained in  $\text{Der}_{J_f}(\mathbb{C}\{\mathbf{x}\})$  contain the information needed to find a suitable coordinate system in which our defining polynomial is quasi-homogeneous.

For basic computational aspects we refer the reader to [GP08] for an introduction to Computer algebra and to [Gra00] for an algorithmic treatment of Lie algebras. We start by presenting methods from linear algebra for vector fields and continue with basic results regarding quasi-homogeneous isolated hypersurface singularities. An implementation, in particular of Algorithm 10, can be found at <https://github.com/raulepure/reconstruction.jl>. Algorithm 10 has been announced in [ERS17].

### 5.1 Linear Algebra for Vector Fields

In this section we want to extend methods from linear algebra, such as Jordan decomposition and simultaneous diagonalization of matrices, to vector fields. These methods allow us to obtain information about a coordinate change into a coordinate system in which a maximal multihomogeneous system of generators exists as well as to give us the corresponding weights. The methods we present work theoretically for any analytic algebra, but they do not necessarily terminate in case we do not deal with a zero-dimensional analytic algebra.

Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal generated by  $f_1, \dots, f_k \in \mathbb{C}\{\mathbf{x}\}$ . Then we can use syzygies to compute  $\text{Der}_I(\mathbb{C}\{\mathbf{x}\})$ .

**Lemma 5.1.** *Let  $I = \langle f_1, \dots, f_k \rangle \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal. Then  $\text{Der}_I(\mathbb{C}\{\mathbf{x}\})$  is isomorphic to the projection of the first  $n$  components of  $\text{syz}(A)$ , where the matrix  $A$  is defined as follows:*

$$A = \begin{pmatrix} \partial_{x_1} f_1 & \dots & \partial_{x_n} f_1 & f_1 & \dots & f_k & 0 & \dots & 0 \\ \vdots & & \vdots & \ddots & & \ddots & & & \\ \partial_{x_1} f_k & \dots & \partial_{x_n} f_k & 0 & \dots & 0 & f_1 & \dots & f_k \end{pmatrix} \in \text{Mat}(\mathbb{C}\{\mathbf{x}\}, k \times (n+1) \cdot k).$$

Using SINGULAR we can compute the aforementioned syzygies and we obtain vectors in  $\mathbb{C}\{\mathbf{x}\}^n$  which represent our vector fields. The next operation we need to apply coordinate changes to these vector fields. We denote different systems of coordinates by  $\mathbf{x}, \mathbf{y}$  and  $\mathbf{z}$ . Let  $\varphi : \mathbb{C}\{\mathbf{z}\} \rightarrow \mathbb{C}\{\mathbf{y}\}$  and  $\psi : \mathbb{C}\{\mathbf{y}\} \rightarrow \mathbb{C}\{\mathbf{x}\}$  be automorphisms and denote by  $J_\varphi$  and  $J_\psi$  their respective Jacobian matrices. Using the chain rule we can write

$$\partial_{x_i} = \sum_{j=1}^n \partial_{x_i} \varphi_j(\mathbf{x}) \cdot \partial_{y_j}. \quad (5.1)$$

Equation (5.1) yields

$$\nabla_{\mathbf{x}} = J_\varphi \cdot \nabla_{\mathbf{y}}. \quad (5.2)$$

Equation (5.2) implies the following lemma.

**Lemma 5.2.** *A be an analytic algebra isomorphic to  $\mathbb{C}\{\mathbf{x}\}$ . Denote by  $\mathbf{x}, \mathbf{y}$  and  $\mathbf{z}$  three systems of coordinates and let  $\varphi : \mathbb{C}\{\mathbf{z}\} \rightarrow \mathbb{C}\{\mathbf{y}\}$  and  $\psi : \mathbb{C}\{\mathbf{y}\} \rightarrow \mathbb{C}\{\mathbf{x}\}$  be automorphisms. Then the following hold:*

- (1)  $J_{\psi \circ \varphi} = \psi(J_\varphi) \cdot J_\psi$ , and
- (2)  $J_{\varphi^{-1}} = (\varphi^{-1}(J_\varphi))^{-1}$ .

Now we are able to transform vector fields which are represented by vectors.

**Lemma 5.3.** *Let  $\delta \in \text{Der}'_I(\mathbb{C}\{\mathbf{x}\})$  be represented by the vector  $V = V(\mathbf{x}) \in \mathbb{C}\{\mathbf{x}\}^n$  and let  $\varphi : \mathbb{C}\{\mathbf{y}\} \rightarrow \mathbb{C}\{\mathbf{x}\}$  be an automorphism. Then*

$$V(\mathbf{y}) = (\varphi^{-1}(V(\mathbf{x})))^T \cdot J_{\varphi^{-1}} \in \mathbb{C}\{\mathbf{y}\}^n.$$

Lemma 5.3 is formulated in this way, since the usual coordinate transformation is of type  $y_i = \varphi_i(\mathbf{x})$ , but we want to express our vector field in terms of the  $\mathbf{y}$  variables. This show us that it is necessary to compute the inverse of an algebra morphism. By the inverse function theorem we obtain that our morphism are invertible and the inverse is a power series expression, even if the input is polynomial. At this point it is important to work with a zero-dimensional algebra. Here, using the SINGULAR to compute the highest corner of  $I$ , we obtain a bound  $k$  such that  $\mathfrak{m}^k \subseteq I$ . The following result gives us an algorithm to compute such inverse up to to a given bound.

**Lemma 5.4.** *Let  $\varphi \in \mathbb{C}[\mathbf{x}]^n$  and let  $I := \langle y_1 - \varphi_1(\mathbf{x}), \dots, y_n - \varphi_n(\mathbf{x}) \rangle + \mathfrak{m}^{k+1} \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$ , where  $k \in \mathbb{N}$  is a given bound. Denote by  $G$  a reduced Gröbner basis of  $I$  with respect to an elimination ordering for  $\mathbf{x}$ . Then  $\varphi$  induces an automorphism of  $\mathbb{C}[\mathbf{x}]/\mathfrak{m}^{k+1}$  if and only if there exist  $\psi \in \mathbb{C}[\mathbf{y}]^n$  and  $q_1, \dots, q_m \in \mathbb{C}[\mathbf{y}]$  such that*

$$G = \{x_1 - \psi_1(\mathbf{y}), \dots, x_n - \psi_n(\mathbf{y}), q_1, \dots, q_m\}.$$

*In particular, if  $\varphi$  induces an automorphism of  $\mathbb{C}[\mathbf{x}]/\mathfrak{m}^{k+1}$ , then  $\psi$  induces the inverse map.*

*Proof.* See [Ess00, Theorem 3.2.1]. □

The algorithm looks as follows:

---

**Algorithm 1** Inversion of Algebra Morphism in  $\mathbb{C}[x]/\mathfrak{m}^{k+1}$

---

**INPUT:** A  $\mathbb{C}$ -algebra morphism  $\varphi(\mathbf{x})$  defined by  $\varphi_i = \varphi(x_i)$  and a bound  $k \in \mathbb{N}_{\geq 1}$ .

**OUTPUT:** An automorphism  $\psi(\mathbf{y})$  which is an inverse of  $\varphi$  modulo  $\mathfrak{m}^{k+1}$ .

- 1: Compute generating set  $Q = \{q_1, \dots, q_m\}$  of  $\mathfrak{m}^{k+1}$ .
  - 2: Compute a reduced Gröbner Basis  $G$  for the ideal  $I = \langle y_1 - \varphi_1(\mathbf{x}), \dots, y_n - \varphi_n(\mathbf{x}), q_1, \dots, q_m \rangle \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$  with respect to an elimination ordering for  $\mathbf{x}$ .
  - 3: **return**  $\psi_1(\mathbf{y}) := \text{NF}(x_1 \mid G), \dots, \psi_n(\mathbf{y}) := \text{NF}(x_n \mid G)$ .
- 

Lemma 5.4 implies the following proposition.

**Proposition 5.5.** *Algorithm 1 terminates and works correctly.*

Now we can describe how to compute the Chevalley decomposition of a vector field and how to simultaneously diagonalize pairwise commuting diagonalizable vector fields. We start with the simultaneous diagonalization. The theory for the Chevalley decomposition will only be sketched, since it can be found in detail in [Sai71].

**Remark 5.6.** *We assume from now on that all vector fields appearing in this section are contained in  $\text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$ , where  $I \subseteq \mathbb{C}[[\mathbf{x}]])$  is an ideal.*

We want to show how to algorithmically diagonalize a given finite set of pairwise commuting and diagonalizable vector fields simultaneously. Define  $V_k := \mathbb{C}[[\mathbf{x}]]/\mathfrak{m}^k$  and denote the image of  $\delta$  under the projection  $\text{Der}'_I(\mathbb{C}[[\mathbf{x}]]) \rightarrow \text{End}_{\mathbb{C}}(V_k)$  by  $\bar{\delta}_k$ . In case  $\delta_1, \dots, \delta_m$  is a set of pairwise commuting and diagonalizable derivations, also their linear parts are pairwise commuting and diagonalizable. It is well known from linear algebra, that we can find a linear coordinate change, such that the linear parts of the  $\delta_i$  are diagonal. We assume this setup from now on. Our goal is to show, that we can find iterative coordinate changes of type  $y_i = x_i + h_i$ , where  $h_i$  is homogeneous of degree  $l$ , such that we can write  $\delta_j$  in the new coordinate system as  $\delta_j = \delta_{j,0} + \sum_{i=1}^n a_{j,i} \partial_{y_i}$  with  $\text{ord}(a_i) \geq l + 1$ , where  $l$  denotes the number of iterations. In terms of linear algebra, this is equivalent to saying that a common eigenvector of  $\bar{\delta}_{1,k}, \dots, \bar{\delta}_{m,k}$  lifts under the canonical projection  $\pi_k : \text{End}_{\mathbb{C}}(V_{k+1}) \rightarrow \text{End}_{\mathbb{C}}(V_k)$  to a common eigenvector of  $\bar{\delta}_{1,k+1}, \dots, \bar{\delta}_{m,k+1}$ . We use the linear algebraic characterization to prove the result.

**Lemma 5.7.** *Let  $I \subseteq \mathbb{C}[[\mathbf{x}]]$  be an ideal and  $\delta_1, \dots, \delta_m \in \text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$  be pairwise commuting and diagonalizable derivations with diagonal linear parts  $\delta_{1,0}, \dots, \delta_{m,0}$ . Write  $\delta_j = \delta_{j,0} + \sum_{i=1}^n a_{j,i} \partial_{x_i}$  and assume  $\text{ord}(a_{j,i}) \geq l$  for some  $l \in \mathbb{N}_{\geq 1}$ . Then there exists a coordinate change  $y_i = x_i + h_i$  with  $h_i$  homogeneous of degree  $l$ , such that  $\delta_j = \delta_{j,0} + \sum_{i=1}^n \tilde{a}_{j,i} \partial_{y_i}$  and  $\text{ord}(\tilde{a}_{j,i}) \geq l + 1$ .*

*Proof.* By assumption  $\bar{\delta}_{1,l}, \dots, \bar{\delta}_{m,l} \in \text{End}_{\mathbb{C}}(V_l)$  are diagonalizable. So it suffices to show that the common eigenspaces of  $\bar{\delta}_{1,l+1}, \dots, \bar{\delta}_{m,l+1}$  map surjectively to the common eigenspaces of  $\bar{\delta}_{1,l}, \dots, \bar{\delta}_{m,l}$ . To see this we consider the following commutative diagram:



$$\begin{array}{ccc} V_{l+1} & \xrightarrow{\bar{\delta}_{i,l+1}} & V_{l+1} \\ \downarrow \pi_l & & \downarrow \pi_l \\ V_l & \xrightarrow{\bar{\delta}_{i,l}} & V_l. \end{array}$$

Let  $w$  be a common eigenvector of  $\bar{\delta}_{1,l+1}, \dots, \bar{\delta}_{m,l+1}$  with  $\bar{\delta}_{i,l+1}(w) = \lambda_i w$ . Then

$$\bar{\delta}_{i,l} \circ \pi_l(w) = \pi_l \circ \bar{\delta}_{i,l+1}(w) = \lambda_i \pi_l(w).$$

This computation shows that, if  $\pi_l(w) \neq 0$ , then  $\pi_l(w)$  is a common eigenvector of  $\bar{\delta}_{1,l}, \dots, \bar{\delta}_{m,l}$  with  $\bar{\delta}_{i,l}(\pi_l(w)) = \lambda_i \pi_l(w)$ . Since  $\pi_l$  is a surjection, the claim follows  $\square$

Lemma 5.7 yields the existence of a Cauchy sequence of coordinate systems  $(\mathbf{x}^{(n)})_{n \in \mathbb{N}}$  (in the  $\mathfrak{m}$ -adic topology). Denote the limit by  $\mathbf{z}$ . By construction we obtain the following result:

**Theorem 5.8.** *Let  $I \subseteq \mathbb{C}[[\mathbf{x}]]$  be an ideal and  $\delta_1, \dots, \delta_m \in \text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$  be pairwise commuting and diagonalizable derivations with diagonal linear parts  $\delta_{1,0} = \mathbf{x}D_1\partial^T, \dots, \delta_{m,0} = \mathbf{x}D_m\partial^T$ . Then there exists a coordinate system  $\mathbf{z}$ , such that  $\delta_i = \mathbf{z}D_i\partial^T \in \text{Der}'_I(\mathbb{C}[[\mathbf{z}]])$  for  $i = 1, \dots, m$ .*

It remains to state the algorithm on how to explicitly diagonalize a set of given vector fields simultaneously up to a given degree bound  $k$ . In general the algorithm does not terminate in a finite number of steps, because we have to consider our derivations modulo all possible powers of the maximal ideal  $\mathfrak{m}$ . For the investigation in case of zero-dimensional ideals a bound is sufficient, since we know that  $\mathfrak{m}^k \subseteq I$  for some  $k \in \mathbb{N}_{\geq 1}$ .

We keep the notation from Theorem 5.8. Let  $y_i = x_i + h_i$ , where  $h_i$  is homogeneous of degree  $l \leq k$ . Then

$$\delta_j(y_i) = \delta_j(x_i + h_i) = \delta_{j,0}(x_i) + g_j^{(i)}(\mathbf{x}) + \delta_{j,0}(h_i) \pmod{\mathfrak{m}^{l+1}}, \quad (5.3)$$

where  $g_j^{(i)}$  is homogeneous of degree  $l$ . Lemma 5.7 now tells us that we can find  $h_i$  in such a way that all  $g_j + \delta_{j,0}(h_i) = 0$  for  $j = 1, \dots, s$ . Write  $W_l$  for the vector space generated by all monomials of degree  $l$  and denote by  $A_j^{(l)}$  the representation matrix of  $\delta_{j,0}$  on  $W_l$ . Let  $r = \dim_{\mathbb{C}}(W_l)$ . From now on we consider the  $g_j^{(i)}$  and  $h_i$  as elements of  $W_l$ . Define

$$\mathbf{A}^{(l)} = \begin{pmatrix} A_1^{(l)} \\ \vdots \\ A_s^{(l)} \end{pmatrix} \in \mathbb{C}^{sr \times r}$$

and

$$\mathbf{g}^{(i)} = \begin{pmatrix} g_1^{(i)} \\ \vdots \\ g_s^{(i)} \end{pmatrix} \in \mathbb{C}^{sr}.$$

Equation 5.3 now implies that finding a coordinate change such that the  $\delta_j$  are diagonal modulo  $\mathfrak{m}^{l+1}$  is equivalent to finding solutions  $h_i$  of the linear systems of equations

$\mathbf{A}^{(l)}h_i + \mathbf{g}^{(i)} = 0$ . The pseudo code of the algorithm to compute a simultaneous diagonalization of vector fields looks as follows:

---

**Algorithm 2** Simultaneous Diagonalization of Vector Fields

---

**INPUT:**  $\delta_1, \dots, \delta_s \in \text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$  pairwise commuting and diagonalizable, a degree bound  $k \in \mathbb{N}_{\geq 1}$ .

**OUTPUT:** An automorphism  $\varphi \in \text{Aut}_I(\mathbb{C}[[\mathbf{x}]])$ , such that  $\delta_1, \dots, \delta_s$  are diagonal in the coordinate system  $\mathbf{y} = \varphi(\mathbf{x})$ , and the  $\delta_1, \dots, \delta_s$  transformed with respect to  $\varphi$ .

- 1: Compute a transformation matrix  $S \in \mathbb{C}^{n \times n}$ , such that  $\delta_{1,0}, \dots, \delta_{s,0}$  are diagonal.
  - 2: Set  $\varphi(\mathbf{x}) = S\mathbf{x}$  and transform  $\delta_1, \dots, \delta_s$  into the coordinate system  $\mathbf{y} = \varphi(\mathbf{x})$ .
  - 3: **for**  $l = 2$  to  $k$  **do**
  - 4:     Compute the matrix  $\mathbf{A}^{(l)}$ .
  - 5:     **for**  $i = 1$  to  $n$  **do**
  - 6:         Compute the vectors  $\mathbf{g}^{(i)}$ .
  - 7:         Compute solutions  $h_i$  of  $\mathbf{A}^{(l)}h_i + \mathbf{g}^{(i)} = 0$ .
  - 8:     **end for**
  - 9:     Define the map  $\psi$  via  $\psi(x_i) = x_i + h_i$ .
  - 10:      $\varphi = \psi \circ \varphi$ .
  - 11:     Transform  $\delta_1, \dots, \delta_s$  into the coordinate system  $\mathbf{y} = \varphi(\mathbf{x})$ .
  - 12: **end for**
  - 13: **return**  $\varphi$  and  $\delta_1, \dots, \delta_s$ .
- 

The previous discussion as well as Theorem 5.8 imply the following proposition.

**Proposition 5.9.** *Algorithm 2 terminates and works correctly.*

The idea for the Chevalley decomposition of a vector field  $\delta$  is to write  $\delta = \sum_{i=1}^n a_i \partial_{x_i}$  with  $a_i \in \mathbb{C}[[\mathbf{x}]]$  and to find coordinate changes such that  $a_i$  is weighted homogeneous with respect with respect to the weight vector  $\lambda = (\lambda_1, \dots, \lambda_n)$  defined by the eigenvalues of the matrix  $M$  defining the linear part  $\delta_0 = \mathbf{x}M\partial^T$ . We obtain this iteratively by first applying a linear coordinate change to bring  $M$  into Jordan normal form. Saito has shown the following:

**Lemma 5.10.** *Let  $I \subseteq \mathbb{C}[[\mathbf{x}]]$  be an ideal and  $\delta \in \text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$  be a vector field with linear part  $\delta_0$ , which is in Jordan normal form. Denote the weight vector defined by the eigenvalues of  $\delta_0$  by  $\lambda$ . Let  $l \in \mathbb{N}_{\geq 2}$ . Write  $\delta = \sum_{i=1}^n (a_i + b_i)\partial_{x_i}$  and assume  $\text{ord}(a_{j,i}) \geq l$  for some  $l \in \mathbb{N}_{\geq 1}$ .*

- (1)  $a_i \in \mathbb{C}[\mathbf{x}]$  is weighted homogeneous with respect to  $\lambda$  and of degree  $\leq l - 1$ , and
- (2)  $b_i \in \mathbb{C}[[\mathbf{x}]]$  is of order  $\geq l$ .

Then there exists a coordinate change  $y_i = x_i + h_i$  with  $h_i \in W_l$ , such that, after the coordinate transformation  $y_i = x_i + h_i$ , we can write  $\delta = \sum_{i=1}^n (a'_i + b'_i)\partial_{y_i}$  where

- (1)  $a'_i \in \mathbb{C}[\mathbf{y}]$  is weighted homogeneous with respect to  $\lambda$  and of degree  $\leq l$ , and
- (2)  $b'_i \in \mathbb{C}[[\mathbf{y}]]$  is of order  $\geq l + 1$ .

*Proof.* This follows from [Sai71, Lemma 2.4]. □

Lemma 5.10 yields a Cauchy sequence of coordinate systems in the  $\mathfrak{m}$ -adic topology. Denote the limit by  $\mathbf{z}$  and write  $\delta = \sum_{i=1}^n a_i \partial_{z_i}$ . Let  $\delta_0 = \mathbf{z}J\partial^T$ , where  $J \in \mathbb{C}^{n \times n}$  is in Jordan normal form. Let  $J = D + N$ , where  $D$  is a diagonal matrix and  $N$  is a nilpotent matrix. Keeping this notation we obtain the following theorem.

**Theorem 5.11.** *Let  $I \subseteq \mathbb{C}[[\mathbf{x}]]$  be an ideal and  $\delta \in \text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$ . Then there exists a coordinate system  $\mathbf{z}$ , such that:*

- (1)  $\delta_0 = \mathbf{z}J\partial^T$ ,
- (2)  $\delta_S = \mathbf{z}D\partial^T$ ,
- (3)  $\delta_N = \delta - \delta_S$ , and
- (4)  $[\delta_S, \delta_N] = 0$ .

*Proof.* See [Sai71, Satz 3.1]. □

Now we present the algorithm. As in the case of the simultaneous diagonalization, it suffices to use methods from linear algebra. We keep the notation of Lemma 5.10 and denote by  $W_{l,\lambda}$  the vector space of all monomials of degree  $l$  which are weighted homogeneous with respect to  $\lambda$ . Fix an integer  $l \in \mathbb{N}_{\geq 1}$ . Due to the structure of the Jordan normal form we have to distinguish the two cases

- (1)  $\delta_0(x_i) = \lambda_i x_i$ , and
- (2)  $\delta_0(x_i) = x_{i-1} + \lambda_i x_i$ .

In the first case we write

$$\delta(y_i) = \delta_j(x_i + h_i) = a_i + g^{(i)}(\mathbf{x}) + \delta_0(h_i) \pmod{\mathfrak{m}^{l+1}}, \quad (5.4)$$

and in the second case we write

$$\delta(y_i) = \delta_j(x_i + h_i) = a_i + g^{(i)}(\mathbf{x}) + \delta_0(h_i) + h_{i-1} \pmod{\mathfrak{m}^{l+1}}. \quad (5.5)$$

Our goal is to find  $h_i$ , such that  $\chi_i := \delta(y_i) - a_i - \delta_0(h_i) \in W_{l,\lambda}$ . Write  $U$  for a vector space complement of  $W_{l,\lambda}$  in  $W_l$  and write  $\pi_U$  for the canonical projection  $\pi_U : W_l \rightarrow U$ . Denote the representation matrix of  $\delta_0$  on  $U$  by  $\mathbf{A}_l$  and the vector representing  $\chi_i$  on  $U$  by  $\mathbf{g}^{(i)}$ . Note that the fact that diagonal and nilpotent part commute implies  $A^{(l)}W_{l,\lambda} \subseteq W_{l,\lambda}$ . Then  $\chi_i$  is weighted homogeneous with respect to  $\lambda$ , if and only if  $\pi_U(\chi_i) = 0$ . This yields the following algorithm:

**Algorithm 3** Chevalley Decomposition of Vector Fields**INPUT:**  $\delta \in \text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$ , a degree bound  $k \in \mathbb{N}_{\geq 1}$ .**OUTPUT:** An automorphism  $\varphi \in \text{Aut}_I(\mathbb{C}[[\mathbf{x}]])$ , such that  $\delta_S = \delta_{0,S}$  and  $\delta_N = \delta - \delta_S$  modulo  $\mathfrak{m}^{k+1}$  in the coordinate system  $\mathbf{y} = \varphi(\mathbf{x})$ .

- 1: Compute a transformation matrix  $S \in \mathbb{C}^{n \times n}$ , such that  $\delta_0$  is in Jordan normal form.
- 2: Set  $\varphi(\mathbf{x}) = S\mathbf{x}$  and transform  $\delta$  into the coordinate system  $\mathbf{y} = \varphi(\mathbf{x})$ .
- 3: **for**  $l = 2$  to  $k$  **do**
- 4:     Compute the matrix  $\mathbf{A}^{(l)}$ .
- 5:     **for**  $i = 1$  to  $n$  **do**
- 6:         Compute the vectors  $\mathbf{g}^{(i)}$ .
- 7:         Compute solutions  $h_i$  of  $\mathbf{A}^{(l)}\pi_U(h_i) + \mathbf{g}^{(i)} = 0$ .
- 8:     **end for**
- 9:     Define the map  $\psi$  via  $\psi(x_i) = x_i + h_i$ .
- 10:     $\varphi = \psi \circ \varphi$ .
- 11:    Transform  $\delta$  into the coordinate system  $\mathbf{y} = \varphi(\mathbf{x})$ .
- 12: **end for**
- 13: **return**  $\varphi, \delta, \delta_{0,S}$  and  $\delta_N$ .

The previous discussion and Theorem 5.11 imply the following proposition.

**Proposition 5.12.** *Algorithm 3 terminates and works correctly.*

**Remark 5.13.** *Algorithm 2 and 3 have been implemented by Adrian Rettich under our supervision in the SINGULAR library VECFIELD.LIB. We can use this library to compute explicit examples.*

**Example 5.14.** *We consider the ideal  $I = \langle x^2 + 2xy + y^2, y^2 - 2x^2y + x^4 \rangle \subseteq \mathbb{C}[[x, y]]$ . A SINGULAR computation shows that  $D = \text{Der}_I(\mathbb{C}[[x, y]])$  is generated by the derivations*

$$\begin{aligned} \delta_1 &= (y - y^2)\partial_x + (-y + y^2 + 2xy^2 + 2y^3)\partial_y \\ \delta_2 &= -(x^2 + 2xy + y^2)\partial_y \\ \delta_3 &= (-y^2 + y^3)\partial_x + (2x^2y + 2xy^2 + y^3 - y^4)\partial_y \\ \delta_4 &= -(x + y)\partial_x + (2xy + 2y^2)\partial_y \end{aligned}$$

Using the SINGULAR library VECFIELD.LIB we obtain that  $\delta = 1$  is diagonalizable,  $\delta_2$  and  $\delta_3$  are nilpotent and that  $\delta_4$  decomposes into a diagonalizable and nilpotent part. The explicit computation of does not terminate, but since  $\mathfrak{m}^3 \subseteq I$ , we obtain by truncation that the derivation  $\delta$  with the following Jordan–Chevalley decomposition:

$$\begin{aligned} \delta_S &= (-x - y - 2xy - 2y^2 - 4x^2y - 12xy^2 - 8y^3)\partial_x + (2xy + 2y^2 + 4xy^2 + 4y^3)\partial_y \\ \delta_N &= (2xy + 2y^2 + 4x^2y + 12xy^2 + 8y^3)\partial_x + (-4xy^2 - 4y^3)\partial_y \end{aligned}$$

is contained in  $D$ . Diagonalizing  $\delta_1$ , up to degree 3, yields

$$\tilde{\delta}_1 = -x\partial_x$$

with respect to the coordinate transformation  $\varphi : \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ ,  $x \mapsto y + y^2 + 2xy^2 + 3y^3$ ,  $y \mapsto x + y + xy^2 + y^3$ . We obtain  $\varphi^{-1}(I) = \langle x^2, y^2 \rangle$ . This means that  $I$  is monomial, hence has maximal multihomogeneity 2. It holds that  $[\delta_1, \delta_S] \neq 0$ , so we are not able to simultaneously diagonalize both derivations. This shows that, in view of Theorem 2.80, more work has to be put into finding the two pairwise commuting diagonalizable derivations in  $D$ . We cover more details regarding this topic in the next section.

## 5.2 Computing Weights and Monomiality

This subsection serves two purposes. On the one hand side we want to present an algorithm, which is able to check, whether a given hypersurface singularity  $(V(f), \mathbf{0})$  is isomorphic to a non-reduced normal crossing divisor, that is isomorphic to  $(V(x_1^{a_1} \cdots x_n^{a_n}), \mathbf{0})$  for certain  $a_i \in \mathbb{N}$ . The second purpose is to give a simple method to obtain the maximal multihomogeneity of any ideal  $I \subseteq \mathbb{C}[[\mathbf{x}]]$  which is defined by polynomials. As a byproduct we also obtain a check if a given ideal is of monomial type or not. We use the results from Chapter 2 and basic results regarding Lie algebras.

**Remark 5.15.** Consider the canonical projection  $\pi_1 : \text{Der}(\mathbb{C}[[\mathbf{x}]]) \rightarrow \text{Der}(\mathbb{C}[[\mathbf{x}]]/\mathfrak{m}^2)$ . Throughout this section we define  $\mathfrak{g} := \pi_1(\text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$ .

Due to Theorem 2.48 and Theorem 2.63 we obtain all possible weights of  $\mathbb{C}[[\mathbf{x}]]/I$  as the eigenvalues of any maximal toral Lie subalgebra contained in  $\mathfrak{g}$ . Using one of the corner stones of OSCAR, the computer algebra system GAP (see [19]), we are able to perform computations with Lie algebras. The computation of a maximal toral Lie subalgebra is based on [Gra00, Algorithm ToralSubalgebra]:

---

### Algorithm 4 Maximal Toral Subalgebra

---

**INPUT:** A basis of a finite-dimensional Lie algebra  $\mathcal{L}$ .

**OUTPUT:** A basis of a maximal toral subalgebra  $\mathfrak{t} \subseteq \mathcal{L}$ .

- 1: Compute a Cartan subalgebra  $\mathfrak{h} \subseteq \mathcal{L}$ .
  - 2: Compute a basis  $\mathcal{B}'$  of the center of  $\mathfrak{h}$ .
  - 3: Define  $L := \emptyset$ .
  - 4: **for**  $x \in \mathcal{B}'$  **do**
  - 5:     Compute the semi-simple part  $x_s$  of  $x$ .
  - 6:      $L = L \cup \{x_s\}$ .
  - 7: **end for**
  - 8: Compute a basis  $\mathcal{B}$  of the Lie algebra generated by  $L$ .
  - 9: **return**  $\mathcal{B}$ .
- 

**Proposition 5.16.** Algorithm 4 terminates and works correctly.

*Proof.* The correctness follows from [Hum67, Proposition 15.2]. The termination follows from the fact that all algorithms which are used, terminate, see [Gra00].  $\square$

We obtain the following algorithm to decide whether an ideal is of monomial type or not.

---

### Algorithm 5 Is of monomial type

---

**INPUT:** An ideal  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  generated by polynomials.

**OUTPUT:** 1 if  $I$  is of monomial type, 0 else.

- 1: Compute a vector space basis of  $\mathfrak{g} = \pi_1(\text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$ .
  - 2: Use Algorithm 4 to compute a basis  $\mathcal{B}$  of a maximal toral subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$ .
  - 3: **if**  $|\mathcal{B}| = n$  **then**
  - 4:     **return** 1.
  - 5: **else**
  - 6:     **return** 0.
  - 7: **end if**
-

**Proposition 5.17.** *Algorithm 5 terminates and works correctly.*

*Proof.* The correctness follows from the fact that an ideal is monomial if and only if it is invariant under a  $(\mathbb{C}^*)^n$  action, which is equivalent to  $\dim(\mathfrak{t}) = n$ . The termination follows from the fact that all algorithms, which are used, terminate.  $\square$

Let us consider an example.

**Example 5.18.** *We continue Example 5.14. The Lie algebra  $\mathfrak{g}$  is generated by the matrices*

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

*It holds that  $[A_1, A_2] = 0$  and that both matrices are diagonalizable, so we obtain that the ideal  $I$  is of monomial type, as we have already seen in Example 5.14.*

Algorithm 4 also allows us to compute a maximal set of weight vectors for a given ideal  $I$ .

---

#### Algorithm 6 Maximal set of Weight Vectors

---

**INPUT:** An ideal  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  generated by polynomials.

**OUTPUT:** A matrix  $M$  whose rows contain a maximal set of weight vectors for  $I$ .

- 1: Compute a vector space basis of  $\mathfrak{g} = \pi_1(\text{Der}'_I(\mathbb{C}[[\mathbf{x}]])$ .
  - 2: Use Algorithm 4 to compute a basis  $\mathcal{B}$  of a maximal toral subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$ .
  - 3: Simultaneously diagonalize the matrices of  $\mathcal{B}$  and store them in a list  $L$ .
  - 4: Denote by  $A \in \mathbb{C}^{|\mathcal{B}| \times n}$  the matrix containing the diagonals of the matrices in  $L$  as rows.
  - 5: Compute  $M \in \mathbb{C}^{|\mathcal{B}| \times n}$ , the reduced row echelon form of  $A$ .
  - 6: **return**  $M$ .
- 

**Proposition 5.19.** *Algorithm 6 terminates and works correctly.*

*Proof.* The termination follows from the fact that all algorithms, which are used, terminate. The correctness up to Step 4 follows from the fact that we are considering the eigenvalues of diagonalizable derivations. In order to obtain weights, we need rational numbers. So it remains to show that  $M \in \mathbb{Q}^{|\mathcal{B}| \times n}$ . Due to Theorem 2.24, we know that the vector space spanned by the rows of  $A$  has a basis consisting only of vectors with rational entries. This is equivalent to saying that we can find a matrix  $U \in \text{GL}(|\mathcal{B}|, \mathbb{C})$ , such that  $Q := U \cdot A \in \mathbb{Q}^{|\mathcal{B}| \times n}$ . The matrices  $A$  and  $Q$  have the same reduced row echelon form, which is given by the matrix  $M$ . Since computing the reduced row echelon form for  $Q$  involves only operations over  $\mathbb{Q}$ , we obtain  $M \in \mathbb{Q}^{|\mathcal{B}| \times n}$ .  $\square$

Let us consider an example.

**Example 5.20.** *Consider the ideal  $I = \langle x^2, 3y^2 + z^3, yz^2, z^5 \rangle \subseteq \mathbb{C}[[x, y, z]]$ . Using OSCAR we obtain*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

*This shows us additionally that the ideal  $I$  cannot be of monomial type.*

### 5.3 Classifying Non-Reduced and Generalized Normal Crossing Divisors

In the upcoming section we are going to state algorithms which classify non-reduced normal crossing divisors and generalized normal crossing divisors. We start with the algorithm for the classification of non-reduced normal crossing divisors, since it also serves as a toy example to show how to perform computations over a finite algebraic extension  $K$  of  $\mathbb{Q}$  instead of  $\mathbb{C}$ , since symbolic computations on a computer cannot be performed over  $\mathbb{C}$ . For this algorithm, we assume that our singularity is defined by a polynomial  $f \in K[\mathbf{x}]$ .

---

#### Algorithm 7 Classify non-reduced normal crossing

---

**INPUT:** A polynomial  $f \in K[\mathbf{x}]$ .

**OUTPUT:** 0, if  $f$  does not define a non-reduced normal crossing divisor, or a polynomial  $g = x_1^{a_1} \cdot \dots \cdot x_n^{a_n} \in L[\mathbf{x}]$ , where  $L$  is a finite extension of  $K$ .

- 1: Compute a vector space basis of  $\mathfrak{g} = \pi_1(\text{Der}'_L(\mathbb{C}[[\mathbf{x}])))$ .
  - 2: Use Algorithm 6 to compute a basis  $\mathcal{B}$  of a maximal toral subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$ .
  - 3: **if**  $|\mathcal{B}| < n$  **then**
  - 4:     **return** 0.
  - 5: **end if**
  - 6: Simultaneously diagonalize the matrices of  $\mathcal{B}$  and denote the splitting field by  $L$ .
  - 7: Denote by  $o_f$  the order of  $f$ .
  - 8: Denote by  $h$  the  $o_f$ -jet of  $f$ .
  - 9: Factor the polynomial  $h$  over  $L[\mathbf{x}]$  and denote the exponents of the irreducible factors by  $a_i$ . If we have fewer exponents than variables we define the remaining exponents to be 0.
  - 10: **return**  $g = x_1^{a_1} \cdot \dots \cdot x_n^{a_n}$ .
- 

**Proposition 5.21.** *Algorithm 7 terminates and works correctly.*

*Proof.* The termination follows from the fact that all algorithms, which are used, terminate. For the correctness we note that  $f$  defines a non-reduced normal crossing divisor if and only if  $|\mathcal{B}| = n$ . It remains to show why the computations over the splitting field  $L$  of  $K$  are sufficient to obtain the exponents  $a_1, \dots, a_n$ . Note that Step 6 is the same as the first step in Algorithm 2. From a theoretical point of view, we can use Algorithm 2 to compute a coordinate change  $\varphi \in \text{Aut}_f(\mathbb{C}\{\mathbf{x}\})$ , such that

$$\langle \varphi(f) \rangle = \langle x_1^{a_1} \cdot \dots \cdot x_n^{a_n} \rangle. \quad (5.6)$$

All computations in Algorithm 2 take place over the field  $L$ , in particular all coefficients of the defining equations of  $\varphi$  are in  $L$ . The equality of ideals in Equation 5.6 implies the existence of a unit  $u = u_0 + u_1 \in \mathbb{C}\{\mathbf{x}\}$ , where  $u_0 \in \mathbb{C} \setminus \{0\}$  and  $u_1 \in \mathfrak{m}$ , such that

$$\varphi(f) = u \cdot x_1^{a_1} \cdot \dots \cdot x_n^{a_n}. \quad (5.7)$$

In order to pass to a statement about  $f$ , we have to apply the inverse of  $\varphi$  to Equation 5.7. Since Algorithm 1 also performs all its computations over the field  $L$ , we obtain that the defining equations of  $\psi := \varphi^{-1}$  are contained in  $L$ . Denote by  $o_f$  the order



of  $f$  and by  $h$  the  $o_f$ -jet of  $f$ . Furthermore write  $\psi_1$  for the linear part of  $\psi$ , that is  $\psi(x_i) = \psi_1(x_i) + g_i$ , where  $g_i \in \mathfrak{m}^2$ . Applying  $\psi_1$  and comparing components of equal degree yields

$$h = u_0 \cdot \psi_1(x_1)^{a_1} \cdot \dots \cdot \psi_1(x_n)^{a_n}. \quad (5.8)$$

From Equation 5.8 we can read off, that the exponents of the factors of  $h$  are the exponents  $a_i$ .  $\square$

**Example 5.22.** Consider the polynomial  $f = x^2z^2 + y^2z^2 + 2x^5z + 2x^3y^2z + x^8 + x^6y^2 \in \mathbb{Q}\{x, y, z\}$ . Algorithm 6 implies that, after a coordinate change,  $f$  is defined by a monomial over  $\mathbb{Q}(i)\{x, y, z\}$ . The order of  $f$  is 4, so it cannot define a normal crossing divisor. We obtain

$$h = z^2x^2 + z^2y^2 = (x + iy) \cdot (x - iy) \cdot z^2,$$

hence  $f$  defines a non-reduced normal crossing divisor.

Now we can state an algorithm to classify generalized normal crossing divisors.

---

**Algorithm 8** Classify generalized normal crossing

---

**INPUT:** A polynomial  $f \in \mathbb{C}[\mathbf{x}]$ .

**OUTPUT:** 0, if  $f$  does not define a generalized normal crossing divisor, or a polynomial  $g$  defining a generalized normal crossing divisor.

- 1: Compute  $J_f$ , apply Algorithm 5 to  $J_f$  and store the result in  $m$ .
  - 2: **if**  $m = 0$  **then**
  - 3:     **return** 0.
  - 4: **end if**
  - 5: Compute a minimal generating set  $M$  of  $J_f$ .
  - 6: Define  $\nu : \mathbb{N} \rightarrow \mathbb{N}, x \mapsto |\{m \in M \mid \text{ord}(m) = x\}|$
  - 7: Create a list of tuples  $L = \{(a, \nu(a)) \mid \text{ord}(m) = a \text{ for some } m \in M, \}$ .
  - 8: Define  $g := 0$ .
  - 9: **for**  $(a, \nu(a)) \in L$  **do**
  - 10:     Generate  $n_a := \frac{\nu(a)}{a+1}$  normal crossing divisors  $g_{a,1} \dots g_{a,n_a}$  of order  $a + 1$  in distinct sets of variables.
  - 11:      $g = g + \sum_{i=1}^{n_a} g_{a,i}$ .
  - 12: **end for**
  - 13: **return**  $g$ .
- 

**Proposition 5.23.** Algorithm 8 terminates and works correctly.

*Proof.* The termination follows from the fact that all algorithms, which are used, terminate. Correctness follows from Theorem 3.25.  $\square$

**Example 5.24.** Consider the polynomial  $f = -x^2 + xy - xy^2 + y^3 + abc + b^2c + a^3b + a^2b^2 \in \mathbb{C}\{x, y, a, b, c\}$ . Using OSCAR, we obtain that  $J_f$  is indeed of monomial type and using SINGULAR we obtain that  $J_f$  is radical. A minimal generating set of  $J_f$  is given by

$$\{x, y, bc + 3a2b + 2ab2, ac + 2bc + a3 + 2a2b, ab + b2\}.$$

Considering the order of the elements, we obtain  $g = xy + abc$  as a normal form.

## 5.4 Reconstructing QHIS from their Milnor Algebra

In this section we present a Las Vegas algorithm, which can reconstruct the defining equation  $f$  of a quasi-homogeneous isolated hypersurface singularity from a zero-dimensional  $\mathbb{C}$ -algebra isomorphic to  $\mathbb{C}\{\mathbf{x}\}/J_f$ . The algorithm can also be used to check if a zero-dimensional  $\mathbb{C}$ -algebra is isomorphic to  $\mathbb{C}\{\mathbf{x}\}/J_f$ , where  $f$  defines a quasi-homogeneous isolated hypersurface singularity.

We focus on quasi-homogeneous isolated hypersurface singularities, because they admit enough structure which makes it possible to obtain the needed information. A similar algorithm for the homogeneous case has been presented in [IK14]. The main obstacle we have to overcome, is the fact that in the quasi-homogeneous case the weights of  $f$ , which are uniquely determined if  $\text{ord}(f) \geq 3$ , are not known. From a theoretical point of view, we can compute this information from a maximal toral subgroup of  $\text{Aut}_f(\mathbb{C}\{\mathbf{x}\})$ , as presented in Chapter 2.

### 5.4.1 More about QHIS

Basic theoretical properties have been presented in Section 1.5.2. Here we continue with more specific theoretical results.

Due to Theorem 1.91 we have a relation between the weight vector, the weighted degree and the Milnor number of a QHIS. In our algorithm the main task is to find possible weight vectors for  $f$ . Knowing a weight vector and the corresponding weighted degree we obtain the Milnor number. The Milnor number on the other hand side gives us a bound on the weighted degree. This implies that if we are given any weight vector, knowing the corresponding weighted degree would help us to decide if this weight vector can be a possible one for  $f$ . The key ingredient for this is to use the socle of  $M_f$ .

**Definition 5.25.** *Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k$ . Then we define the socle of  $R$  as*

$$\text{Soc}(R) = 0 : \mathfrak{m}.$$

**Remark 5.26.** *Note that  $\text{Soc}(R) \cong \text{Hom}(k, R)$ .*

In our setup we consider  $R = M_f$ , which is a zero-dimensional complete intersection ring. Due to [BH93, Proposition 3.1.20]  $R$  is a Gorenstein ring, so we obtain the following result.

**Proposition 5.27.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define an isolated hypersurface singularity. Then  $\dim_{\mathbb{C}} \text{Soc}(M_f) = 1$ .*

*Proof.* This follows from [BH93, Theorem 3.2.10]. □

Denote by  $w$  the weight-vector of  $f$ . Then  $M_f$  is quasi-homogeneous with respect to the weight-vector  $w$ . Next we want to show that the socle of  $M_f$  can be generated by a quasi-homogeneous element and that we can even compute its degree. Therefore we need the following lemma.

**Lemma 5.28.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS. Denote the Hessian matrix of  $f$  by  $H_f$ . Then the following hold:*

- (1)  $\det(H_f) \notin J_f$ .
- (2) For every  $g \in \mathbb{C}\{\mathbf{x}\} \setminus J_f$  there exists an  $h \in \mathbb{C}\{\mathbf{x}\}$ , such that

$$h \cdot g - \det(H_f) \in J_f.$$

*Proof.* See [Sai74, Lemma 3.3]. □

In the upcoming statements we denote by  $\bar{g}$  the image of  $g$  under the canonical projection  $\mathbb{C}\{\mathbf{x}\} \rightarrow M_f$ . Now we can show the following.

**Proposition 5.29.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS. Assume that  $f$  is quasi-homogeneous with respect to the weight-vector  $w = (w_1, \dots, w_n)$  and has weighted degree  $d$ . Denote the Hessian matrix of  $f$  by  $H_f$ . Then  $\overline{\det(H_f)}$  generates  $\text{Soc}(M_f)$ . In particular,  $\text{Soc}(M_f)$  is generated by an element of weighted degree  $n \cdot d - 2 \cdot \sum_{i=1}^n w_i$ .*

*Proof.* Let  $g \in \mathbb{C}\{\mathbf{x}\} \setminus J_f$  be an element, such that  $\text{Soc}(M_f) = \langle \bar{g} \rangle_{\mathbb{C}}$ . By Lemma 5.28 there exists an element  $h \in \mathbb{C}\{\mathbf{x}\}$ , such that  $\overline{h \cdot g} = \overline{\det(H_f)}$ . In case  $h \in \mathfrak{m}$  we obtain  $\overline{h \cdot g} = \bar{0}$ , which contradicts  $\det(H_f) \notin J_f$ . Thus  $h$  is a unit and we can assume  $h \in \mathbb{C}^*$ . This implies that  $\overline{\det(H_f)}$  generates  $\text{Soc}(M_f)$ .

The formula for the degree follows from the Leibniz-formula to compute the determinant. □

The next object we introduce is the so-called highest corner of an ideal. This object allows us to compute a bound for the determinacy of our isolated hypersurface singularity and it yields a monomial representative for the socle. We state the definition only in the local case.

**Definition 5.30.** *Let  $>$  be a local monomial ordering on the set of monomials  $\text{Mon}(\mathbf{x})$  and let  $I \subseteq \mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}$  be an ideal. A monomial  $m \in \text{Mon}(\mathbf{x})$  is called **highest corner of  $I$** , if*

- (1)  $m \notin L(I)$ , and
- (2)  $m' \in \text{Mon}(\mathbf{x})$  with  $m' < m$  implies  $m' \in L(I)$ .

We obtain the following two results

**Theorem 5.31.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS and fix a local ordering on  $\text{Mon}(\mathbf{x})$ . Denote by  $m_i \in \text{Mon}(\mathbf{x})$  the highest corner of  $\mathfrak{m}^i J_f$ ,  $i = 0, 1, 2$ . Then  $f$  is  $\min(\deg(m_i) + 2 - i | i = 0, 1, 2)$  determined.*

*Proof.* See [GP08, Corollary A.9.7]. □

**Proposition 5.32.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a QHIS, fix a local ordering on  $\text{Mon}(\mathbf{x})$  and denote by  $m \in \text{Mon}(\mathbf{x})$  the highest corner of  $J_f$ . Then  $\bar{m}$  generates  $\text{Soc}(M_f)$ .*

*Proof.* First we note that  $m$  being a monomial means that  $m \in L(J_f)$  if and only if  $m \in J_f$ , hence by definition  $\bar{m} \neq 0$ . Let  $m' \in \mathfrak{m}$  be an arbitrary monomial. Since we work over a local ordering we obtain  $m < 1$ , thus  $m' \cdot m < m'$ , hence by definition  $m' \cdot m \in J_f$ . This implies  $\mathfrak{m} \cdot m \subseteq J_f$  and thus  $\bar{m}$  generates  $\text{Soc}(M_f)$ . □

The standing assumption of this section, so far, was that we have our coordinates chosen in such a way that  $f$  is quasi-homogeneous. Our final theoretical result is to show that we can always find such a coordinate system. In fact it turns out that any coordinate system in which a maximal Torus is linear, is sufficient.

**Proposition 5.33.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  define a quasi-homogeneous isolated hypersurface singularity. Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal, such that  $M_f \cong \mathbb{C}\{\mathbf{x}\}/I$ . For every maximal algebraic torus  $T \subseteq \text{Aut}_I(\mathbb{C}\{\mathbf{x}\})$ , which is linear with respect to the chosen coordinates, there exists a quasi-homogeneous  $g_T \in I \subseteq \mathbb{C}\{\mathbf{x}\}$ , such that :*

- (1)  $(V(g_T), \mathbf{0}) \cong (V(f), \mathbf{0})$ , and
- (2)  $J_{g_T}$  is  $T$ -equivariant.

*Proof.* Since  $M_f \cong \mathbb{C}\{\mathbf{x}\}/I$ , we obtain the existence of an isomorphism  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$  with  $\varphi(J_f) = I$ . By Lemma 1.96 it holds that  $I = J_{\varphi(f)}$ . This means that we can replace  $f$  with  $\varphi(f)$  and we can reduce to the case  $I = J_f$ . By quasi-homogeneity of  $f$  we can assume that the coordinates  $\mathbf{x}$  are chosen, such that  $f$  is defined by a quasi-homogeneous power series. This implies the existence of an algebraic torus  $T_f$ , which is linear with respect to the chosen coordinate system. Due to the fact that the partial derivatives of  $f$  are also quasi-homogeneous, we obtain  $T_f \subseteq \text{Aut}_{J_f}(\mathbb{C}\{\mathbf{x}\})$ . Let  $T_{J_f} \subseteq \text{Aut}_{J_f}(\mathbb{C}\{\mathbf{x}\})$  be a maximal algebraic torus containing  $T_f$ . Due to Corollary 2.46 any algebraic torus  $T \subseteq \text{Aut}_{J_f}(\mathbb{C}\{\mathbf{x}\})$  is conjugated to  $T_{J_f}$ , this means there exists  $\psi \in \text{Aut}_{J_f}(\mathbb{C}\{\mathbf{x}\})$ , such that

$$T = \psi T_{J_f} \psi^{-1}.$$

Define  $g_T := \psi(f)$  and  $T_{g_T} := \psi T_f \psi^{-1}$ . Using the definitions we obtain

$$T_{g_T}(g_T) = \psi T_f \psi^{-1}(g_T) = \psi T_f(f) \subseteq \mathbb{C}^* g_T.$$

This implies that  $g_T$  is quasi-homogeneous with respect to the weights induced by the characters of  $T_{g_T}$ . In particular, it holds that  $g_T \in J_{g_T} = J_{\psi(f)}$ . Using Lemma 1.96 we obtain that  $M_{g_T} \cong M_f$ . The Mather–Yau theorem now implies  $V(g_T, \mathbf{0}) \cong V(f, \mathbf{0})$ . By construction  $J_{g_T}$  is  $T$ -equivariant.  $\square$

## 5.4.2 The Reconstruction Algorithm

After we prepared the theoretical foundation, we are able to show how an algorithmic reconstruction works. The main idea is to recover candidates for the weight-vector of the QHIS and to use Corollary 1.93. We start by presenting an algorithm to find possible weight-vectors. The key to finding them is Algorithm 6 and some elementary convex geometry, as it can be found in [Zie95].

We assume from now on, that we are in a coordinate system in which a maximal torus  $T \subseteq \text{Aut}_{J_f}(\mathbb{C}\{\mathbf{x}\})$  is linear. This can be achieved by using Algorithm 2. After changing the coordinate system we can find candidates for our weight-vectors. Let  $M$  be the output of Algorithm 6. We write  $W$  for the vector-space generated by the rows of  $M$ . The following are known about the weight-vector  $w_f = (w_1, \dots, w_n)$  and the weighted degree  $d_f$  :

- (1)  $w_f \in \mathbb{Z}_{>0}^n$  (see Theorem 1.89),

(2) If the class of the monomial  $m \in \mathbb{C}[\mathbf{x}]$  generates  $\text{Soc}(M_f)$ , then  $\deg_{w_f}(m) = n \cdot d_f - 2 \cdot \sum_{i=1}^n w_i$  (see Proposition 5.32),

(3)  $d_f \leq C \cdot \mu_f$ , where  $C$  can be computed explicitly (see Theorem 1.91) and

(4)  $\mu_f = \prod_{i=1}^n \left( \frac{d_f}{w_i} - 1 \right)$  (see Theorem 1.91).

The first property of  $w_f$  allows us to restrict our considerations to the convex cone  $C_f = W \cap \mathbb{R}_{\geq 0}^n$ . Assume  $m = \prod_{i=1}^n x_i^{a_i}$ , then the second property combined with the third yields the inequation

$$w_1 \cdot (a_1 + 2) + \dots + w_n \cdot (a_n + 2) \leq n \cdot C \cdot \mu_f. \tag{5.9}$$

Restricting  $C_f$  with inequation 5.9, we obtain a polytope  $\mathcal{P}$  containing  $w_f$  as a point. For every point  $p = (p_1, \dots, p_n) \in \mathcal{P}$  we can use Proposition 5.32 to compute a candidate for the weighted degree of  $f$ , which we denote by  $d_p$ . We know that our weight-vector can be chosen to be integral, so it suffices to consider weight-vectors, where the gcd of the weights and the weighted degree is 1. Due to our previous arguments, it suffices to consider the weights, which are in the following set:

$$\mathcal{A} := \left\{ p \in \mathcal{P} \mid p \in \mathbb{Z}^n, \gcd(p_1, \dots, p_n, d_p) = 1 \text{ and } \mu_f = \prod_{i=1}^n \left( \frac{d_p}{w_i} - 1 \right) \right\}.$$

The algorithm to compute the set  $\mathcal{A}$  looks as follows:

**Algorithm 9** Candidates for weight-vectors

**INPUT:** A zero-dimensional complete intersection ideal  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  generated by polynomials, a set  $L \subseteq \text{Der}_I(\mathbb{C}\{\mathbf{x}\})$  of simultaneously diagonal derivations, where  $|L|$  is equal to the maximal multihomogeneity of  $I$ .

**OUTPUT:** A set  $\mathcal{A}$  containing candidates for the weight-vectors of a possible polynomial  $f$  satisfying  $M_f \cong \mathbb{C}\{\mathbf{x}\}/I$ .

- 1: Let  $W$  be the vector space generated by the diagonals of the elements of  $L$ .
- 2: Compute  $\mu = \dim_{\mathbb{C}} \mathbb{C}\{\mathbf{x}\}/I$ .
- 3: Compute  $m$ , such that  $\langle m \rangle_{\mathbb{C}} = \text{Soc}(\mathbb{C}[\mathbf{x}]/I)$ .
- 4: Denote the exponents of  $m$  by  $a_1, \dots, a_n$ .
- 5: Compute a bound  $d$  for the weighted degree  $d_f$  using Theorem 1.91.
- 6: Compute the Polytope

$$\mathcal{P} := W \cap \mathbb{R}_{\geq 0}^n \cap \left\{ (p_1, \dots, p_n) \in \mathbb{R}^n \mid \sum_{i=1}^n p_i \cdot (a_i + 2) \leq n \cdot d \right\}$$

7:  $\mathcal{A} = \emptyset$ .

8: **for**  $p \in \mathcal{P} \cap \mathbb{Z}^n$  **do**

9:     Compute  $d_p = \frac{\deg_p(m) + 2 \cdot \sum_{i=1}^n p_i}{n}$ .

10:    **if**  $\gcd(p_1, \dots, p_n, d_p) = 1$  **and**  $\mu = \prod_{i=1}^n \left( \frac{d_p}{w_i} - 1 \right)$  **then**

11:        $\mathcal{A} = \mathcal{A} \cup \{(p_1, \dots, p_n, d_p)\}$ .

12:    **end if**

13: **end for**

14: **return**  $\mathcal{A}$ .

**Proposition 5.34.** *Algorithm 9 terminates and works correctly.*

*Proof.* The termination follows from the fact that all algorithms, which are used, terminate. Correctness follows from the discussion prior to the algorithm.  $\square$

**Example 5.35.** *We consider the ideal  $I = \langle x^2, 3y^2 + z^3, yz^2, z^5 \rangle \subseteq \mathbb{C}\{x, y, z\}$  as in Example 5.20. We already know that  $W$  is generated by the rows of*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

*Further OSCAR computations show that  $\mu = 14$ ,  $d = 53$  is an upper degree bound and that the polytope  $\mathcal{P} \subseteq \mathbb{R}^3$  is described by the equation*

$$3x_1 - 2x_2 = 0$$

*and the inequations*

$$\begin{aligned} 159 &\geq 6x_1 + 2x_2 + 3x_3 \\ 0 &\leq x_1 \\ 0 &\leq x_2 \\ 0 &\leq x_3 \end{aligned}$$

*Reducing the integer points of  $\mathcal{P}$  yields*

$$\mathcal{A} = \{(2, 3, 3, 9), (2, 3, 4, 10)\}.$$

Using Algorithm 9 we are able to state our reconstruction algorithm, which is based on Proposition 5.33 and Corollary 1.93. The proposition guarantees the existence of a polynomial  $g$ , which is right-equivalent to  $f$ , in case we are in a coordinate system, where a maximal torus  $T$  acts linearly on  $J_f$ . Algorithm 9 allows us the computation of candidates for our weight-vector we are looking for. For each candidate  $p \in \mathcal{A}$ , we can compute the set of monomials

$$\mathcal{M}_p = \{m \in \text{Mon}(\mathbb{C}[\mathbf{x}]) \mid \deg_p(m) = d_p\}.$$

Due to the positivity of the weight-vectors in  $\mathcal{A}$ , the set  $\mathcal{M}_p$  is finite. Denote its cardinality by  $k_p$ . Let  $\mathcal{M}_p = \{m_1, \dots, m_{k_p}\}$  and  $J_f$  be generated by  $f_1, \dots, f_k$ . Using syzygies, we can check for the existence of  $a_1, \dots, a_{k_p} \in \mathbb{C}$ , such that  $g_p := \sum_{i=1}^{k_p} a_i \cdot m_i$  is right-equivalent to  $f$ . To see this, let  $\pi : \mathbb{C}\{\mathbf{x}\}^{k_p+n \cdot k} \rightarrow \mathbb{C}\{\mathbf{x}\}^{k_p}$  be the projection to the first  $k_p$  entries. We compute

$$S_p := \pi(\text{syz}(E_p)) \cap \mathbb{C}^{k_p},$$

where

$$E_p = \begin{pmatrix} m_1 & \dots & m_{k_p} & f_1 & \dots & f_k & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \partial_{x_1} m_1 & \dots & \partial_{x_1} m_{k_p} & 0 & \dots & 0 & f_1 & \dots & f_k & \dots & 0 & \dots & 0 \\ \vdots & & & & \vdots & & \vdots & & \ddots & & \vdots & & \\ \partial_{x_n} m_1 & \dots & \partial_{x_n} m_{k_p} & 0 & \dots & 0 & 0 & \dots & 0 & \dots & f_1 & \dots & f_k \end{pmatrix}.$$

We know that for some  $p \in \mathcal{A}$ ,  $S_p$  cannot be empty, hence we can choose a generic linear combination of a basis of  $S_p$  and denote the result by  $a_p$ . By construction of  $\mathcal{A}$ , the resulting polynomial  $g_p$  satisfies  $\mu_{g_p} = \mu_f$ , and by construction of  $a_p$ , we obtain  $g_p \in J_f$  and  $J_{g_p} \subseteq J_f$ . The genericity in the choice of  $a_p$  and Corollary 1.93 imply that, if  $p$  is the correct weight-vector,  $g_p$  defines an isolated hypersurface singularity. Combining all these, we obtain an isomorphism of  $\mathbb{C}$ -algebras between the Milnor-algebras  $M_{g_p}$  and  $M_f$ . Due to the Mather – Yau theorem, this is equivalent to  $g_p$  being right-equivalent to  $f$ . This leads to the following algorithm:



**Algorithm 10** Reconstructing QHIS from their Milnor-algebra

**INPUT:** A zero-dimensional complete intersection ideal  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  generated by polynomials.

**OUTPUT:** A quasi-homogeneous polynomial  $g \in \mathbb{C}[\mathbf{x}]$ , such that  $M_g \cong \mathbb{C}\{\mathbf{x}\}/I$ , or false if none is found.

- 1: Compute  $\mathfrak{g} := \text{Der}'_{J_f}(\mathbb{C}[\mathbf{x}])$ .
- 2: Compute a toral Lie algebra  $\mathfrak{t}$  of  $\mathfrak{g}$ .
- 3: Use Algorithm 2 to simultaneously diagonalize the basis of  $\mathfrak{t}$  and store the result in a list  $L$ .
- 4: Denote the coordinate change for the simultaneous diagonalization by  $\varphi$ .
- 5: Define  $I := \varphi(I)$ .
- 6: Compute the set  $\mathcal{A}$  by applying Algorithm 9 to  $L$ .
- 7: Compute  $m$ , such that  $\langle m \rangle_{\mathbb{C}} = \text{Soc}(\mathbb{C}[\mathbf{x}]/I)$ .
- 8: **for**  $p \in \mathcal{A}$  **do**
- 9:     Compute  $d_p = \frac{\deg_p(m) + 2 \cdot \sum_{i=1}^n p_i}{n}$ .
- 10:    Compute  $\mathcal{M}_p = \{m \in \text{Mon}(\mathbb{C}[\mathbf{x}]) \mid \deg_p(m) = d_p\}$ .
- 11:    Let  $\mathcal{M}_p = \{m_1, \dots, m_{k_p}\}$ .
- 12:    Compute a basis for the vector-space  $S_p$ , which is defined as in the previous discussion.
- 13:    **if**  $S_p \neq \emptyset$  **then**
- 14:       Choose a generic  $\mathbf{0} \neq a_p \in S_p$  and construct  $g_p := \sum_{i=1}^{k_p} a_i \cdot m_i$ .
- 15:       **if**  $g_p$  defines an isolated singularity **then**
- 16:          **return**  $g_p$ .
- 17:       **end if**
- 18:    **end if**
- 19: **end for**
- 20: **return** false

**Proposition 5.36.** *Algorithm 10 terminates and works correctly.*

*Proof.* The termination follows from the fact that all algorithms, which are used, terminate. Correctness follows from the discussion prior to the algorithm.  $\square$

We consider three examples.

**Example 5.37.** *We continue Example 5.35. We have seen already that  $|\mathcal{A}| = 2$ , so only few tests have to be done. In our case Algorithm 10 returns, for example,*

$$g = -38x^3 + 62 \cdot (y^3 + yz^3).$$

*This implies  $\mathbb{C}\{\mathbf{x}\}/I \cong M_g$ .*

**Example 5.38.** *We consider the ideal  $I = \langle x^2 + y^2 + 3z^2, 3y^2 - 2xz + 9z^2, 2xz + 2yz - 9z^2, 4yz^2 - 9z^3, z^4 \rangle \mathbb{C}\{x, y, z\}$ . We obtain  $\mu = 8, d = 30$  and  $\mathcal{A} = \{(1, 1, 1, 3)\}$ . Algorithm 10 returns, for example,*

$$g = 30x^3 + 30y^3 + 30x^2z + 30y^2z + 30z^3.$$

Example 5.37 and Example 5.38 have been positive results. The next example is going to be a negative result, which shows that not all quasi-homogeneous zero-dimensional complete intersections arise as Milnor algebras of QHIS. Before we state the example, we consider the following theorem due to Yau.

**Theorem 5.39.** *Let  $I = \langle f_1, \dots, f_k \rangle \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal with homogeneous generators  $f_1, \dots, f_k$ , where  $1 \leq k \leq n$ . Assume that  $f_1, \dots, f_k$  are of degree  $d \in \mathbb{N}_{\geq 2}$ . Then there exists a  $g \in \mathbb{C}\{\mathbf{x}\}$  with  $J_g = I$  if, and only if there exist homogeneous polynomials  $F_1, \dots, F_n \in \mathbb{C}[\mathbf{x}]$  of degree  $d$  and a matrix  $B \in \mathbb{C}^{k \times n}$  of rank  $k$ , such that*

$$\begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = B \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} \tag{5.10}$$

and such that

$$\partial_{x_i} F_j = \partial_{x_j} F_i \text{ for all } 1 \leq i, j \leq n. \tag{5.11}$$

*Proof.* See [Yau87, Theorem 5]. □

**Example 5.40.** *Consider the ideal  $I = \langle f_1, f_2, f_3 \rangle \subseteq \mathbb{C}\{x, y, z\}$ , where*

$$\begin{aligned} f_1 &= 4x^3 + x^2y - 5xy^2 + 2y^3 - 3x^2z - 3xyz + 4y^2z + 2xz^2 + 3yz^2 - 2z^3 \\ f_2 &= x^3 + x^2y - 5xy^2 - 5y^3 - 4x^2z + 5xyz - 3y^2z + 5xz^2 + yz^2 + 3z^3 \\ f_3 &= -5x^3 - 5xy^2 + 3y^3 - 2xyz - 2y^2z + 4xz^2 + 2yz^2 - 2z^3 \end{aligned}$$

As invariants we obtain  $\mu = 27, d = 102$  and maximal multihomogeneity  $s = 1$ . For this example Algorithm 10 returns false. So far, this result does not mean too much, since our algorithm is a Las Vegas algorithm. This implies that we need to prove that  $\mathbb{C}\{\mathbf{x}\}/I$  cannot be the Milnor algebra of a QHIS. We apply Theorem 5.39.  $I$  is homogeneous, since it has three homogeneous generators. Equation (5.10) and Equation (5.11) in Theorem 5.39 are equivalent to solving a linear system of equations to determine, whether a matrix  $B \in \mathbb{C}^{3 \times 3}$  exists in our case. Using for example OSCAR we obtain that such a matrix  $B$  cannot exist, hence there exists no  $f \in \mathbb{C}\{\mathbf{x}\}$ , such that  $M_f \cong \mathbb{C}\{\mathbf{x}\}/I$ .



## Chapter 6

# On Singularities of Sebastiani–Thom type

In the upcoming chapter we are going to establish a connection between the property of being a Sebastiani–Thom singularity in case of quasi-homogeneous hypersurface singularities and the maximal multihomogeneity of their singular locus. To be more precise, we investigate the following conjecture:

**Conjecture 6.1.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  define a quasi-homogeneous isolated hypersurface singularity. Then  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components if, and only if, the maximal multihomogeneity of  $J_f$  is at least 2.*

We show the following:

**Theorem 6.2.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  be a quasi-homogeneous isolated hypersurface singularity with respect to the weight-vector  $\mathbf{w} \in \mathbb{N}_{>0}^n$ . Assume that  $J_f$  is multihomogeneous with respect to  $\mathbf{w}$  and  $\mathbf{v} \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$ , where  $\mathbf{w}$  and  $\mathbf{v}$  are linearly independent, and that one of the following properties holds:*

(a)  $J_f$  is of monomial type.

(b)  $\mathbf{w}$  satisfies, after possibly permuting the variables,

$$w_1 > \dots > w_n > \frac{w_1}{2}.$$

(c)  $\mathbf{w}$  satisfies, after possibly permuting the variables,

$$w_1 \geq \dots \geq w_n > \frac{w_1}{2}$$

and  $\mathbf{v} = (1, \dots, 1)$ .

(d)  $n \leq 3$ .

Then  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components.

The rest of this chapter is concerned with the proof of Theorem 6.2. We start at this point by reducing first to the case that  $\text{ord}(f) \geq 3$ .

**Proposition 6.3.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  define a quasi-homogeneous isolated hypersurface singularity. Assume  $\text{ord}(f) = 2$ . Then  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components.*

*Proof.* By the Splitting Lemma (see Lemma 1.80) we know that  $f$  is right-equivalent to  $x_1^2 + \dots + x_k^2 + g \in \mathbb{C}\{\mathbf{x}\}$ , where  $g \in \mathbb{C}\{x_{k+1}, \dots, x_n\}$  defines an isolated hypersurface singularity and  $\text{ord}(g) \geq 3$ . We obtain the isomorphism of Tjurina-algebras  $T_f \cong T_g$ . In particular,  $T_g$  is positively graded, since  $f$  is quasi-homogeneous. It follows from [XY96, Theorem 1.2] and Theorem 1.83 that  $g$  is right-equivalent to a quasi-homogeneous polynomial  $h \in \mathbb{C}\{x_{k+1}, \dots, x_n\}$ . Thus  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components.  $\square$

**Remark 6.4.** *Proposition 6.3 justifies to consider Theorem 6.2 only for the case  $\text{ord}(f) \geq 3$ .*

For the next result, we need the following definition.

**Definition 6.5.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$ . We say  $f$  defines a **Brieskorn–Pham singularity**, if there exists integers  $a_1, \dots, a_n \in \mathbb{N}_{>0}^n$ , such that  $f$  is right-equivalent to  $x_1^{a_1} + \dots + x_n^{a_n}$ .*

Theorem 6.2 (a) follows from the following proposition.

**Proposition 6.6.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  define an isolated hypersurface singularity. Assume  $J_f$  is of monomial type. Then  $f$  is a Brieskorn–Pham singularity.*

*Proof.* Since  $J_f$  is of monomial type, there exists an isomorphism  $\varphi \in \text{Aut}(\mathbb{C}\{\mathbf{x}\})$ , such that  $\varphi(J_f) = J_{\varphi(f)}$  is monomial. Since  $J_{\varphi(f)}$  defines a zero-dimensional complete intersection ideal, we obtain that it is generated by  $x_1^{a_1}, \dots, x_n^{a_n}$  for certain  $a_1, \dots, a_n \in \mathbb{N}_{>0}^n$ . Then  $g = x_1^{a_1+1} + \dots + x_n^{a_n+1}$  satisfies  $J_g = J_{\varphi(f)}$ . By Theorem 1.83  $f$  is right-equivalent to  $g$ , hence  $f$  defines a Brieskorn–Pham singularity.  $\square$

## 6.1 Proof of Theorem 6.2 (b) and (c)

Before we prove our result, we state a characterization of a zero-dimensional algebra to be the Milnor algebra of a quasi-homogeneous isolated hypersurface singularity.

To prove our result we need the following version of the Poincaré-Lemma:

**Lemma 6.7.** *Let  $F_1, \dots, F_n \in \mathbb{C}\{\mathbf{x}\}$  with*

$$\partial_{x_j} F_i = \partial_{x_i} F_j$$

*for all  $1 \leq i, j \leq n$ . Then there exists an  $f \in \mathbb{C}\{\mathbf{x}\}$ , such that  $F_i = \partial_{x_i} f$ .*

Furthermore, we need the following auxiliary lemma, which is part of the proof of [Yau87, Theorem 2]:

**Lemma 6.8.** *Let  $f \in \mathfrak{m} \subseteq \mathbb{C}\{\mathbf{x}\}$ . Assume that there exists a weight-vector  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Q}_{>0}^n$ , such that the partial derivatives of  $f$  are  $\mathbf{w}$ -homogeneous.*

*Then  $f = \sum_{i=1}^n \frac{w_i}{\deg_{\mathbf{w}}(\partial_{x_i} f) + w_i} x_i \partial_{x_i} f$ . In particular,  $f$  is weighted-homogeneous.*

*Proof.* The result follows immediately from the following computation:

$$f = \int_0^1 \frac{d}{dt} f(t^{w_1} x_1, \dots, t^{w_n} x_n) dt = \sum_{i=1}^n \frac{w_i}{\deg_{\mathbf{w}}(\partial_{x_i} f) + w_i} x_i \partial_{x_i} f.$$

□

Using the Poincaré-Lemma and Lemma 6.8, Yau (see [Yau87, Theorem 3]) proved the following theorem:

**Theorem 6.9.** *Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal and  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}_{>0}^n$ . Then  $A := \mathbb{C}\{\mathbf{x}\}/I$  is the Milnor-Algebra of a  $\mathbf{w}$ -homogeneous hypersurface singularity  $f \in \mathbb{C}\{\mathbf{x}\}$  if and only if  $I$  is generated by  $F_1, \dots, F_n \in \mathbb{C}\{\mathbf{x}\}$  with the following properties:*

- (1) *the  $F_i$  are weighted-homogeneous with respect to  $\mathbf{w}$ ,*
- (2)  *$\partial_{x_j} F_i = \partial_{x_i} F_j$  for all  $1 \leq i, j \leq n$ , and*
- (3)  *$\partial_{x_i} f = F_i$  for all  $1 \leq i \leq n$ .*

**Remark 6.10.** *Let  $f \in \mathbb{C}[\mathbf{x}]$  define a quasi-homogeneous isolated hypersurface singularity, which is weighted-homogeneous with respect to the weight-vector  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}_{>0}^n$ . Furthermore, we assume that the partial derivatives of  $f$  are weighted homogeneous with respect to  $\mathbf{w}$  and to an additional weight-vector  $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$ , where  $\mathbf{w}$  and  $\mathbf{v}$  are linearly independent. By adding a sufficient multiple of  $\mathbf{w}$  to  $\mathbf{v}$ , we can assume  $v_i \geq w_i$  for all  $1 \leq i \leq n$ . By choosing  $k$  with  $v_k - w_k \leq v_i - w_i$  for all  $i \neq k$  and scaling  $\mathbf{v}$  by  $\frac{w_k}{v_k}$ , we can additionally assume that  $v_k = w_k$ . This assumption implies, by using Lemma 6.8, that  $\deg_{\mathbf{v}} f \geq \deg_{\mathbf{w}} f$  for all  $f \in \mathbb{C}[\mathbf{x}]$ .*

Now we are able to prove a weak version of our result.

**Proposition 6.11.** *Let  $I \subseteq \mathbb{C}\{\mathbf{x}\}$  be an ideal and  $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{N}_{>0}^n$ . Then  $A := \mathbb{C}\{\mathbf{x}\}/I$  is the Milnor-Algebra of a  $\mathbf{w}$ -homogeneous isolated hypersurface singularity  $f \in \mathbb{C}\{\mathbf{x}\}$  of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components if and only if  $I$  is generated by  $F_1, \dots, F_n \in \mathfrak{m}^2$  with the following properties:*

- (1)  *$I$  is zero-dimensional and there exists a  $\mathbf{v} \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$ , where  $\mathbf{w}$  and  $\mathbf{v}$  are linearly independent, such that the  $F_i$  are multihomogeneous with respect to  $\mathbf{w}$  and  $\mathbf{v}$ , and*
- (2)  *$\partial_{x_j} F_i = \partial_{x_i} F_j$  for all  $1 \leq i, j \leq n$ .*

*Proof.* We show the "if" direction, since the other direction is trivial.

Due to Theorem 6.9 there exists a  $\mathbf{w}$ -homogeneous  $f \in \mathbb{C}\{\mathbf{x}\}$  satisfying  $\partial_{x_i} f = F_i$  for  $1 \leq i \leq n$ . In particular, it holds that  $\text{ord}(f) \geq 3$ . By assumption  $f$  defines an isolated hypersurface singularity, since  $I$  is zero-dimensional. Due to Remark 6.10 we assume that  $v_i \geq w_i$  and that there exists an index  $k$ , such that  $v_k = w_k$ . By assumption, the partial derivatives of  $f$  are  $\mathbf{w}$ -homogeneous, as well as  $\mathbf{v}$ -homogeneous. Due to Lemma 6.8 and due to the uniqueness of the weights of  $f$ , see Theorem 1.89, we obtain

$$\frac{w_i}{\deg_{\mathbf{w}}(\partial_{x_i} f) + w_i} = \frac{v_i}{\deg_{\mathbf{v}}(\partial_{x_i} f) + v_i} \tag{6.1}$$

for all  $1 \leq i \leq n$ . For any index  $k$  with  $w_k = v_k$  Equation (6.1) implies

$$\deg_{\mathbf{w}}(\partial_{x_k} f) = \deg_{\mathbf{v}}(\partial_{x_k} f). \quad (6.2)$$

Assume  $\partial_{x_i, x_k}^2 f \neq 0$ . For any  $1 \leq i \leq n$  with  $v_i > w_i$  Equation (6.2) yields

$$\deg_{\mathbf{w}}(\partial_{x_i, x_k}^2 f) = \deg_{\mathbf{w}}(\partial_{x_k} f) - w_i > \deg_{\mathbf{v}}(\partial_{x_k} f) - v_i = \deg_{\mathbf{v}}(\partial_{x_i, x_k}^2 f). \quad (6.3)$$

Inequality (6.3) and Remark 6.10 yield a contradiction, hence

$$\partial_{x_i, x_k}^2 f = 0.$$

This means that the partial derivatives with respect to variables  $x_k$  which satisfy  $w_k = v_k$  do not depend on variables  $x_i$  with  $v_i > w_i$  and vice versa. We reorder the  $\mathbf{x}$  variables together with the corresponding weights, such that  $v_i = w_i$  for  $1 \leq i \leq r$  and  $v_i > w_i$  for  $r+1 \leq i \leq n$ . Under our assumptions, Lemma 6.7 and Lemma 6.8 imply the existence of  $\mathbf{w}$ -homogeneous  $f_1(x_1, \dots, x_r)$  and  $f_2(x_{r+1}, \dots, x_n)$ , such that  $\partial_{x_i} f_1 = \partial_{x_i} f$  for  $1 \leq i \leq r$  and  $\partial_{x_i} f_2 = \partial_{x_i} f$  for  $r+1 \leq i \leq n$ . This proves the claim.  $\square$

The main problem with Proposition 6.11 is that the statement is highly coordinate dependent. We aim to prove a coordinate independent version, which holds under stronger assumptions on the weights.

**Theorem 6.12.** *Let  $f \in \mathbb{C}\{\mathbf{x}\}$  be a quasi-homogeneous isolated hypersurface singularity with respect to the weight-vector  $\mathbf{w} \in \mathbb{N}_{>0}^n$ . Assume that  $J_f$  is multihomogeneous with respect to  $\mathbf{w}$  and  $\mathbf{v} \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$ , where  $\mathbf{w}$  and  $\mathbf{v}$  are linearly independent, and that one of the following properties holds:*

- (1)  $w_1 > \dots > w_n > \frac{w_1}{2}$ .
- (2)  $w_1 \geq \dots \geq w_n > \frac{w_1}{2}$  and  $\mathbf{v} = (1, \dots, 1)$ .

*Then  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type with quasi-homogeneous Sebastiani–Thom components.*

*Proof.* Denote by  $d = \deg_{\mathbf{w}}(f)$  the  $\mathbf{w}$ -degree of  $f$ . Furthermore, we assume that  $\deg_{\mathbf{w}}(F_i) = \deg_{\mathbf{w}}(\partial_{x_i} f) = d - w_i$  for  $1 \leq i \leq n$ . Write  $\partial_{x_i} f = \sum_{j=1}^n a_{ij} F_j$ . Due to the  $\mathbf{w}$ -homogeneity, we obtain

$$\deg_{\mathbf{w}}(a_{ij}) = w_j - w_i. \quad (6.4)$$

The fact that  $w_1 \geq \dots \geq w_n$  together with Equation (6.4) imply that the matrix  $A = (a_{ij}) \in \mathbb{C}\{\mathbf{x}\}^{n \times n}$  is an invertible, lower triangular matrix. Both assumptions imply

$$w_j - w_i < 2w_n - w_n = w_n,$$

hence the matrix  $A$  has only constant entries.

In case  $w_1 > \dots > w_n$ , the matrix  $A$  is diagonal, so the partial derivatives of  $f$  are already multihomogeneous and we can apply Proposition 6.11.

In case  $w_1 \geq \dots \geq w_n$ , the matrix  $A$  is block diagonal, where the size of the blocks corresponds to the number of weights with the same value. Since  $A$  is invertible, the linear coordinate change  $\varphi(\mathbf{x}) = (A^T)^{-1} \mathbf{x}$  combined with Lemma 5.3 implies that  $\varphi(f)$  is  $\mathbf{w}$ -homogeneous with  $\mathbf{v}$ -homogeneous partial derivatives. Yet again we can apply Proposition 6.11.  $\square$



## 6.2 Proof of Theorem 6.2 (d)

**Theorem 6.13.** *Let  $f \in \mathbb{C}\{x, y, z\}$  be quasi-homogeneous and of order  $\geq 3$ . Assume  $f$  defines an isolated hypersurface singularity. Then  $f$  is of Sebastiani–Thom type if and only if the maximal multihomogeneity of  $J_f$  is at least 2.*

The proof will be split into a rigorous case by case analysis taking into account possible cases for the ordering of the weights of  $f$ . The following sections then will deal with the respective proofs of the subcases.

**Remark 6.14.** *Throughout this chapter, we will always assume that  $f$  is given in a coordinate system, in which it is quasi-homogeneous with respect to the weights  $w_1, w_2, w_3$ .*

**Notation 6.15.** *We assume that the Jacobian ideal  $J_f$  is generated by multihomogeneous polynomials  $h_1, h_2$ , and  $h_3$ . We can assume this, due to Proposition 5.33. The maximal multihomogeneity of  $J_f$  will be denoted by  $s$ .  $\text{Supp}(f)$  denotes the monomial support of  $f$ .*

The main idea of the upcoming computations is to consider the monomial diagram of the monomial support  $\text{Supp}(f)$  for the defining equation  $f \in \mathbb{C}[x, y, z]$ . We use the fact that being quasi-homogeneous implies that the exponents of all monomials in  $\text{Supp}(f)$ , considered as points in  $\mathbb{R}^3$  lie on one hyperplane  $H$ . In a next step we consider the minimal multihomogeneous generating set  $\{h_1, h_2, h_3\}$  of  $J_f$ . By assumption we have maximal multihomogeneity  $s = 2$ , hence the exponents of the monomials of the  $h_i$  lie on lines  $L_i$  in  $\mathbb{R}^3$ , with the feature, that the lines  $L_1, L_2, L_3$  are pairwise parallel. We use this fact, to show that  $f$  cannot contain a monomial of type  $x^a y^b z^c$  for  $a, b, c \in \mathbb{N}_{>0}$ , in a suitably chosen coordinate system. Since a line  $L$  in  $\mathbb{R}^3$  is determined by a point  $p \in L$  on the line and a vector  $v \in \mathbb{R}^3$ , determining the direction of the line, we introduce the following definition.

**Definition 6.16.** *Let  $h \in \mathbb{C}[x, y, z]$  be a multihomogeneous polynomial with  $|\text{Supp}(h)| \geq 2$ . Assume that  $p = (p_1, p_2, p_3)$  and  $q = (q_1, q_2, q_3)$  in  $\mathbb{R}^3$  are different exponents of monomials of  $h$ . Then we say that the vector  $v = p - q$  is a direction vector of  $h$ .*

The proof relies on a case by case analysis of the weights and of the exponents of  $f$ .

## 6.3 The case $w_1 = w_2 = w_3 = 1$

The first case we are dealing with is the homogeneous case. We start with the following auxiliary lemma.

**Lemma 6.17.** *Let  $f \in \mathbb{C}[x, y, z]$  define a homogeneous isolated hypersurface singularity with  $\deg(f) = d \geq 3$  and  $s = 2$ . Then the following hold:*

- (1) *There exists a multihomogeneous set of generators  $h_1, h_2, h_3$  of  $J_f$  with  $\deg(h_i) = d - 1$ .*
- (2) *There exists an invertible matrix  $M \in \mathbb{C}^{3 \times 3}$ , such that*

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = M \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

*Proof.* The first result follows immediately from Proposition 3.9. The second result follows from elementary linear algebra.  $\square$

In a first step we are going to reduce to the case that  $J_f$  is multihomogeneous with respect to the weight vectors  $v_1 = (1, 1, 0)$  and  $v_2 = (0, 0, 1)$ . If this were not the case, we can assume that, after renumbering the variables,  $J_f$  is weighted homogeneous with respect to the weight vectors  $v_1 = (1, 1, 1)$  and  $v_2 = (w, 1, 0)$  for a  $w \in \mathbb{Q}_{>1}$ . In this case we are able to show the following:

**Lemma 6.18.** *Let  $f \in \mathbb{C}[x, y, z]$  be a homogeneous isolated hypersurface singularity with  $\deg(f) = d \geq 3$  and assume  $J_f$  is weighted homogeneous with respect to the weight vectors  $v_1 = (1, 1, 1)$  and  $v_2 = (w, 1, 0)$  for some  $w \in \mathbb{Q}_{>1}$ . Then  $f$  is of Sebastiani–Thom type.*

*Proof.* Lemma 6.17 implies that all monomials appearing in  $\text{Supp}(\partial_x f)$ ,  $\text{Supp}(\partial_y f)$  and  $\text{Supp}(\partial_z f)$  are also appearing in  $\text{Supp}(h_1)$ ,  $\text{Supp}(h_2)$  and  $\text{Supp}(h_3)$ . Note that by Lemma 1.87  $\{x^{d-1}, y^{d-1}, z^{d-1}\} \subseteq \text{Supp}(\partial_x f) \cup \text{Supp}(\partial_y f) \cup \text{Supp}(\partial_z f)$ , hence  $\{x^{d-1}, y^{d-1}, z^{d-1}\} \subseteq \text{Supp}(h_1) \cup \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Due to this we can assume that  $\deg_{v_2}(h_1) = \deg_{v_2}(x^{d-1}) = w(d-1)$ ,  $\deg_{v_2}(h_2) = \deg_{v_2}(y^{d-1}) = d-1$  and  $\deg_{v_2}(h_3) = \deg_{v_2}(z^{d-1}) = 0$ . We show in multiple steps, that no monomial of type  $x^a y^b z^c$  with  $a, b, c \geq 1$  is contained in the support of  $f$ . This shows, that  $f$  is of Sebastiani–Thom type.

**Step 1:** We show that no monomial of type  $y^b z^c$  with  $b, c \geq 1$  is contained in the support of  $f$ . Assume the contrary.

The assumptions imply in particular  $1 \leq b \leq d-1$ . Then  $y^{b-1} z^c \in \text{Supp}(\partial_y f)$  and in particular  $y^{b-1} z^c \in \text{Supp}(h_i)$  for some  $i = 1, 2, 3$ . It holds that

$$0 \leq b-1 = \deg_{v_2}(y^{b-1} z^c) < d-1.$$

This implies  $b = 1$  and  $c = d-1$ , hence  $yz^{d-2} \in \text{Supp}(\partial_z f)$ . Next it holds that

$$0 < \deg_{v_2}(yz^{d-2}) = 1 < d-1,$$

hence  $yz^{d-1} \notin \text{Supp}(h_i)$  for  $i = 1, 2, 3$ , which is a contradiction.

**Step 2:** We show that no monomial of type  $x^a y^b$  with  $a, b \geq 1$  is contained in the support of  $f$ . Assume the contrary.

The assumptions imply in particular  $1 \leq b \leq d-1$ . Then  $x^{a-1} y^b, x^a y^{b-1} \in \text{Supp}(\partial_x f) \cup \text{Supp}(\partial_y f)$  and in particular  $x^{a-1} y^b, x^a y^{b-1}$  are contained  $\text{Supp}(h_i)$  for certain  $i = 1, 2, 3$ . It holds that

$$0 < w(a-1) + b < wa + b - 1.$$

This implies  $\deg_{v_2}(x^{a-1} y^b) = w(d-1)$ , since we obtain a contradiction otherwise. Using that  $a + b = d$ , we obtain  $a = d-1$  and  $b = 1$ . On the other hand, in order to avoid a contradiction, we obtain

$$d-1 = \deg_{v_2}(x^{a-1} y^b) = w(d-2) + 1.$$

This equation implies  $w = 1$ , which contradicts  $w > 1$ .

**Step 3:** We show that no monomial of type  $x^a y^b z^c \in \text{Supp}(f)$  with  $a, b, c \geq 1$  is contained in the support of  $f$ . Assume the contrary.

Then  $x^a y^b z^{c-1} \in \text{Supp}(\partial_z f)$  and in particular  $x^a y^b z^{c-1} \in \text{Supp}(h_i)$  for some  $i = 1, 2, 3$ . It holds that

$$\deg_{v_2}(h_1) = w(d-1) \geq wa + b = \deg_{v_2}(x^a y^b z^{c-1}) > 0.$$

We consider two subcases:

- (1) Assume that  $\deg_{v_2}(h_1) = \deg_{v_2}(x^a y^b z^{c-1})$ . Then  $a + b \leq d - 1$  implies that  $a = d - 1, b = 0$  and  $c = 1$ . This contradicts  $b \geq 1$ .
- (2) Assume that  $\deg_{v_2}(h_2) = \deg_{v_2}(x^a y^b z^{c-1}) = d - 1$ . Then  $x^{a-1} y^b z^c \in \text{Supp}(\partial_x f)$  and in particular  $x^{a-1} y^b z^c \in \text{Supp}(h_i)$  for some  $i = 1, 2, 3$ . It holds that

$$\deg_{v_2}(h_2) = \deg_{v_2}(x^a y^b z^{c-1}) = wa + b > w(a - 1) + b = \deg_{v_2}(x^{a-1} y^b z^c).$$

This implies  $w(a - 1) + b = 0$ , hence  $a = 1, b = 0$  and  $c = d - 1$ . This contradicts  $b \geq 1$ .

□

Due to Lemma 6.18 we can assume that  $J_f$  is multihomogeneous with respect to the weight vectors  $v_1 = (1, 1, 0)$  and  $v_2 = (0, 0, 1)$ .

Being weighted homogeneous with respect to  $v_2$  implies  $h_i = z^{k_i} \cdot g_i(x, y)$  with  $k_i \in \mathbb{N}$  and  $g_i \in \mathbb{C}[x, y]$ . In order to prove that  $f$  is of Sebastiani–Thom type, we need the following lemmas.

**Lemma 6.19.** *Let  $f \in \mathbb{C}[x, y, z]$  be a homogeneous isolated hypersurface singularity with  $\text{ord}(f) \geq 3, s = 2$  and assume that  $J_f$  is weighted homogeneous with respect to the weights  $v_1 = (1, 1, 0)$  and  $v_2 = (0, 0, 1)$ . Furthermore, denote by  $h_1, h_2, h_3$  a multihomogeneous system of generators of  $J_f$ . Then, after possibly renumbering the  $h_i$ ,  $h_1 = h_1(x, y)$  and  $h_2 = h_2(x, y)$ . In particular, it holds that  $k_3 \geq 2$ .*

*Proof.* If  $k_i > 0$  for more than one index  $i$ , then  $f$  does not define an isolated singularity, hence we can assume without loss of generality that  $h_1 = h_1(x, y)$  and  $h_2 = h_2(x, y)$ . The fact that  $f$  defines an isolated singularity also implies in this case that  $k_3 > 0$ . Furthermore, Lemma 1.87 implies that either  $z^{r+1}, z^r x, z^r y \in \text{Supp}(f)$  with  $r \geq 2$ . The fact that  $h_1 = h_1(x, y)$  and  $h_2 = h_2(x, y)$  implies together with Lemma 6.17 that  $k_3 \geq 2$ .

□

**Lemma 6.20.** *Let  $f \in \mathbb{C}[x, y, z]$  be a homogeneous isolated hypersurface singularity with  $\text{ord}(f) \geq 3, s = 2$  and assume that  $J_f$  is weighted homogeneous with respect to the weights  $v_1 = (1, 1, 0)$  and  $v_2 = (0, 0, 1)$ . Furthermore, denote by  $h_1, h_2, h_3$  a multihomogeneous system of generators of  $J_f$ . Then  $h_3 = z^{k_3}, z^{k_3} \notin \text{Supp}(f_x)$  and  $z^{k_3} \notin \text{Supp}(f_y)$ .*

*Proof.* By Lemma 6.19 we assume that  $h_1$  and  $h_2$  do not depend on the  $z$ -variable. Let  $x^i y^j z^{k_3} \in \text{Supp}(h_3)$  for  $i, j \in \mathbb{N}$ . We first show that  $x^i y^j z^{k_3} \notin \text{Supp}(f_x) \cup \text{Supp}(f_y)$ . We consider two cases:

**Case 1:**  $x^i y^j z^{k_3} \in \text{Supp}(f_x)$ .

Then  $x^{i+1} y^j z^{k_3-1} \in \text{Supp}(f_z)$ . In this case  $f_z$  is a sum of at least two elements with non-zero  $v_2$ -degree, since  $k_3 \geq 2$  by Lemma 6.19. This contradicts Lemma 6.17.

**Case 2:**  $x^i y^j z^{k_3} \in \text{Supp}(f_y)$ .

This case works analogously to Case 1.

Next we show that if either  $i > 0$  or  $j > 0$ , then  $x^i y^j z^{k_3} \notin \text{Supp}(f_z)$ . We can assume without loss of generality that  $i > 0$ . Assume the contrary, that is  $x^i y^j z^{k_3} \in \text{Supp}(f_z)$ . Then  $x^{i-1} y^j z^{k_3+1} \in \text{Supp}(f_x)$ . In this case  $f_x$  is a sum of at least two elements with non-zero  $v_2$ -degree. This contradicts Lemma 6.17. This shows that  $h_3 = z^{k_3}$  and that  $z^{k_3} \notin \text{Supp}(f_x) \cup \text{Supp}(f_y)$ .

□

The next step is to show that we can give more structure to the matrix  $M$  in Lemma 6.17 in case  $h_1$  and  $h_2$  do not depend on the  $z$ -variable and in case  $h_3 = z^{k_3}$ .

**Lemma 6.21.** *Let  $f \in \mathbb{C}[x, y, z]$  be a homogeneous isolated hypersurface singularity and assume  $J_f$  is weighted homogeneous with respect to the weights  $v_1 = (1, 1, 0)$  and  $v_2 = (0, 0, 1)$ . Furthermore, denote by  $h_1, h_2, h_3$  a multihomogeneous system of generators of  $J_f$ . Assume that  $h_1 = h_1(x, y)$ ,  $h_2 = h_2(x, y)$  and  $h_3 = z^{k_3}$ . Then, after a suitable linear change of coordinates,*

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

for some  $\beta_j \in \mathbb{C}$ .

*Proof.* Due to Lemma 6.20 we can assume the matrix appearing in Lemma 6.17 to be

$$M = \begin{pmatrix} m_1 & m_2 & 0 \\ m_3 & m_4 & 0 \\ m_5 & m_6 & m_7 \end{pmatrix}. \text{ The submatrix } M' := \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \text{ of } M \text{ is invertible, since } M$$

is invertible. We consider a linear coordinate change of type

$$\varphi(x, y, z) = (a_1x + a_2y, a_3x + a_4y, z)$$

with  $a_i \in \mathbb{C}$ . Note that this type of coordinate change does not affect the homogeneity of  $f$  or the multihomogeneity of  $J_f$ . We show that we can determine certain values for the  $a_i \in \mathbb{C}$ , in order to obtain the desired shape of the matrix. Define

$$A := \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \text{ and } B := \begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We use that  $\varphi$  being a coordinate change is equivalent to  $A$  and  $B$  being invertible. Applying the chain rule to  $f'(x, y, z) = f \circ \varphi = f(a_1x + a_2y, a_3x + a_4y, z)$  and using Lemma 6.17, we obtain:

$$\begin{pmatrix} f'_x \\ f'_y \\ f'_z \end{pmatrix} = B^T \begin{pmatrix} f_x \circ \varphi \\ f_y \circ \varphi \\ f_z \circ \varphi \end{pmatrix} = B^T M \begin{pmatrix} h_1 \circ \varphi \\ h_2 \circ \varphi \\ h_3 \circ \varphi \end{pmatrix} = \left( \begin{array}{cc|c} A^T M' & & 0 \\ \hline m_5 & m_6 & m_7 \end{array} \right) \begin{pmatrix} h_1 \circ \varphi \\ h_2 \circ \varphi \\ h_3 \circ \varphi \end{pmatrix}.$$

Choosing the  $a_i$  such that  $A^T = (M')^{-1}$  we can define  $\beta_1 := m_5$ ,  $\beta_2 := m_6$  and  $\beta_3 := m_7$ , and we obtain our desired result.  $\square$

Now we can show that  $f$  is of Sebastiani–Thom type.

**Proposition 6.22.** *Let  $f \in \mathbb{C}[x, y, z]$  be a homogeneous isolated hypersurface singularity with  $\text{ord}(f) \geq 3$ ,  $s = 2$  and assume that  $J_f$  is weighted homogeneous with respect to the weights  $v_1 = (1, 1, 0)$  and  $v_2 = (0, 0, 1)$ . Then  $(V(f), \mathbf{0})$  is of Sebastiani–Thom type.*

*Proof.* By Lemma 6.19 and by Lemma 6.20 we can assume that  $h_1 = h_1(x, y)$ ,  $h_2 = h_2(x, y)$  and  $h_3 = z^{k_3}$ . Due to Lemma 6.21 it holds that

$$\partial_y h_1 = \partial_x h_2.$$

Then Lemma 6.7 implies that there exists a  $g \in \mathbb{C}\{x, y\}$ , such that  $\partial_x g = h_1$  and  $\partial_y g = h_2$ . Define  $F = g(x, y) + z^{k_3+1}$ . Then  $J_F = J_f$ , hence the Mather–Yau theorem implies that  $(V(F), \mathbf{0}) \cong (V(f), \mathbf{0})$  is of Sebastiani–Thom type.  $\square$

### 6.3.1 The case $w_1 > w_2 > w_3$

**Lemma 6.23.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $w = (w_1, w_2, w_3)$ , where  $w_1 > w_2 > w_3$  and assume  $s = 2$ . Then the following hold:*

- (1)  $f_x$  is multihomogeneous with respect to the weights of  $J_f$ .
- (2) There exist  $g_1, g_2 \in \mathbb{C}[y, z]$  and  $g_3 \in \mathbb{C}[z]$ , such that

$$f_y = g_1 f_x + h_2 \text{ and } f_z = g_2 f_x + g_3 h_2 + h_3.$$

In particular we can always assume  $h_1 = f_x$ .

*Proof.* The inequality  $w_1 > w_2 > w_3$  combined with Proposition 3.7 implies

$$\deg_w(f_x) = \deg_w(h_1) < \deg_w(f_y) = \deg_w(h_2) < \deg_w(f_z) = \deg_w(h_3).$$

We also know that

$$\deg_w(f_x) + w_1 > \deg_w(f_y), \deg_w(f_z) \text{ and } \deg_w(f_y) + w_2 > \deg_w(f_z). \quad (6.5)$$

Using these inequalities we can prove the statements:

- (1) Proposition 3.7 implies that there exists an  $a \in \mathbb{C} \setminus \{0\}$ , such that  $h_1 = a f_x$ . This immediately implies that  $f_x$  is multihomogeneous with respect to weights of  $J_f$ .
- (2) Proposition 3.7 also yields the existence of  $g_1 \in \mathfrak{m}$  and  $a_1 \in \mathbb{C} \setminus \{0\}$ , such that  $f_y = g_1 f_x + a_1 h_2$ . Equation (6.5) implies  $g_1 \in \langle y, z \rangle \subseteq \mathbb{C}[y, z]$ . Since  $a_1 \neq 0$ , we can assume  $a_1 = 1$  after a coordinate change of type  $\varphi(x, y, z) = \lambda(x, y, z)$ , which does not affect the quasihomogeneity of  $f$  and the multihomogeneity of  $J_f$ . Then we obtain

$$f_y = g_1 f_x + h_2,$$

with  $g_1 \in \mathbb{C}[y, z]$ . The same reasoning implies the analogous result for  $f_z$ .

□

The first step is to show that no monomial of type  $x^a y^b z^c$  with  $a, b, c \geq 1$  occurs in  $f$ . In order to do so, we prove some lemmas assuming such a monomial exists in  $\text{Supp}(f)$ .

**Lemma 6.24.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and assume that there exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Then the following statements hold:*

- (1) If  $y^{q+1} \in \text{Supp}(f)$  for some  $q \in \mathbb{N}$ , then  $y^q \in \text{Supp}(h_2)$ .
- (2)  $y^q x \notin \text{Supp}(f)$  for any  $q \in \mathbb{N}$ .
- (3) If  $y^{q+1} z \in \text{Supp}(f)$  for some  $q \in \mathbb{N}$ , then  $y^q z \in \text{Supp}(h_2)$ .

*Proof.*

- (1) Let  $y^{q+1} \in \text{Supp}(f)$ . Assume  $y^q \notin \text{Supp}(h_2)$ , then  $f_y = g_1(y, z)f_x + h_2$  implies the existence of an  $l \in \mathbb{N}_{\geq 2}$ , such that  $y^l \in \text{Supp}(f_x)$ . Now we have to consider three cases:

- (a) If  $x^{p+1} \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c)$$

and in the same way

$$u' = (a - 1, b, c) - (0, l, 0) = (a - 1, b - l, c).$$

The vector  $u$  cannot be parallel to  $u'$ , since the last components are equal, but  $b > b - l$ .

- (b) If  $x^{p+1}y \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c)$$

and in the same way

$$u' = (a - 1, b, c) - (0, l, 0) = (a - 1, b - l, c).$$

The vector  $u$  cannot be parallel to  $u'$ , since the last components are equal, but  $b - 1 > b - l$ .

- (c) If  $x^{p+1}z \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1)$$

and in the same way

$$u' = (a - 1, b, c) - (0, l, 0) = (a - 1, b - l, c).$$

The vector  $u$  cannot be parallel to  $u'$ , since  $c > c - 1$  and  $u = \lambda u'$  imply  $1 > \lambda > 0$ , contradicting  $b > b - l$ .

All cases are impossible so  $y^q \in \text{Supp}(h_2)$ .

- (2) If  $y^q x \in \text{Supp}(f)$ , then  $y^q \in \text{Supp}(f_x)$ , which is impossible due to the proof of part (1).

- (3) Let  $y^{q+1}z \in \text{Supp}(f)$ . Assume  $y^q z \notin \text{Supp}(h_2)$ , then  $f_y = g_1(y, z)f_x + h_2$  implies the existence of an  $l \in \mathbb{N}_{\geq 1}$  with  $y^l z \in \text{Supp}(f_x)$ . We are going to proceed as in the proof of (1) by showing that direction vectors cannot be parallel. To assure that the direction vectors we deal with are not equal to  $(0, 0, 0)$ , we have to assume  $l \neq b$  or  $a > 1$  or  $c > 1$ . The case  $a = c = 1$  and  $l = b$  is treated separately. We have to consider three cases:

- (a) If  $x^{p+1} \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c)$$

and in the same way

$$u' = (a - 1, b, c) - (0, l, 1) = (a - 1, b - l, c - 1).$$

Both vectors are different from  $(0, 0, 0)$ , since  $l \neq b$  or  $a > 1$  or  $c > 1$ .  $u$  cannot be parallel to  $u'$ , since  $c > c - 1$  and  $u = \lambda u'$  imply  $\lambda > 1$ , contradicting  $a - 1 > a - 1 - p$ .

(b) If  $x^{p+1}y \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c)$$

and in the same way

$$u' = (a - 1, b, c) - (0, l, 1) = (a - 1, b - l, c - 1).$$

Both vectors are different from  $(0, 0, 0)$ , since since  $l \neq b$  or  $a > 1$  or  $c > 1$ .  $u$  cannot be parallel to  $u'$ , since  $c > c - 1$  and  $u = \lambda u'$  imply  $\lambda > 1$ , contradicting  $a - 1 > a - 1 - p$ .

(c) If  $x^{p+1}z \in \text{Supp}(f)$ , then the direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1)$$

and in the same way

$$u' = (a - 1, b, c) - (0, l, 1) = (a - 1, b - l, c - 1).$$

Both vectors are different from  $(0, 0, 0)$ , since  $l \neq b$  or  $a > 1$  or  $b > 1$ .  $u$  cannot be parallel to  $u'$ , since the last components are equal, but  $b > b - l$ .

Next we consider the case  $a = c = 1$  and  $l = b$ . We assume that  $xy^l z$  is the only monomial of  $f$  of type  $x^i y^j z^k$  for  $i, j, k \in \mathbb{N}_{\geq 1}$ , otherwise we are in the previous case. Since all monomials of  $f_x = h_1$  lie on a line, only  $y^l z$  and one of the monomials  $x^p, x^p y$  or  $x^p z$  can appear in  $\text{Supp}(f_x)$ . If another monomial were to appear, then it would be of type  $x^i y^j$  or  $x^i z^j$  for some  $i, j \in \mathbb{N}$  and a simple computation as in (a) - (c) shows that the monomials of  $f_x$  would not lie on a line.

The weighted homogeneity of  $f$  yields:

$$\deg_w(y^{q+1}z) = (q + 1)w_2 + w_3 = \deg_w(xy^l z) = w_1 + lw_2 + w_3 > (l + 1)w_2 + w_3.$$

This implies  $q > l + 1$ . Since  $f_y = g_1(y, z)f_x + h_2$  and  $y^l z$  divides  $y^q z$ , we know that  $y^{q-l} \in \text{Supp}(g_1)$ . We use this in the following three cases:

(a) If  $x^{p+1} \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (0, l, 1) - (p, 0, 0) = (-p, l, 1).$$

We have  $\{x^p, y^l z\} = \text{Supp}(f_x)$ , which implies  $y^{q-l}x^p \in \text{Supp}(g_1 f_x)$ . We consider two different subcases:

i. Assume  $y^{q-l}x^p \in \text{Supp}(h_2)$ . We know that  $xy^{l-1}z \in \text{Supp}(f_y)$ . The structure of  $f_x$  and  $g_1$  yields  $xy^{l-1}z \notin \text{Supp}(g_1 f_x)$ . This implies  $xy^{l-1}z \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is

$$u_1 = (1, l - 1, 1) - (p, q - l + 1, 0) = (1 - p, 2l - q - 2, 1).$$

$u$  cannot be parallel to  $u_1$ , since the last components are equal, but  $1 - p > -p$ .

ii. Assume  $y^{q-l}x^p \notin \text{Supp}(h_2)$ . Then  $y^{q-l}x^p \in \text{Supp}(f_y)$ , which implies  $y^{q-l+1}x^{p-1} \in \text{Supp}(f_x)$ . Thus another direction vector of  $f_x$  is given by

$$u_2 = (p, 0, 0) - (p - 1, q - l + 1, 0) = (1, l - q - 1, 0).$$

$u$  cannot be parallel to  $u_2$ , since  $1 \neq 0$  in the last component.



(b) If  $x^{p+1}y \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (0, l, 1) - (p, 1, 0) = (-p, l - 1, 1).$$

Since  $f_y = g_1(y, z)f_x + h_2$  and  $x^p \in \text{Supp}(f_y)$ , we obtain  $x^{p+1} \in \text{Supp}(h_2)$ . We know that  $y^{q-l+1}x^p \in \text{Supp}(y^{q-l}f_x)$ . We consider two different subcases:

i. Assume  $y^{q-l+1}x^p \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is

$$u_1 = (p + 1, 0, 0) - (p, q - l + 1, 0) = (1, l - q - 1, 0).$$

$u$  cannot be parallel to  $u_1$ , since  $1 \neq 0$  in the last component.

ii. Assume  $y^{q-l+1}x^p \notin \text{Supp}(h_2)$ . Then  $y^{q-l+1}x^p \in \text{Supp}(f_y)$ , which implies  $y^{q-l+2}x^{p-1} \in \text{Supp}(f_x)$ . Thus another direction vector of  $f_x$  is given by

$$u_2 = (p, 1, 0) - (p - 1, q - l + 2, 0) = (1, l - q - 1, 0).$$

$u$  cannot be parallel to  $u_2$ , since  $1 \neq 0$  in the last component.

(c) If  $x^{p+1}z \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (p, 0, 1) - (0, l, 1) = (p, -l, 0).$$

We know that  $\{x^{p+1}z, y^{q+1}z, xy^l z\} \subset \text{Supp}(f)$ . By our assumption on  $f_x$  and since  $z$  is not allowed to divide  $f$ , if  $f$  defines an isolated hypersurface singularity, we obtain  $y^{m+1} \in \text{Supp}(f)$  for some  $m \in \mathbb{N}_{\geq 2}$ . By (1) we know  $y^m \in \text{Supp}(h_2)$ . We know  $y^{q+1} \in \text{Supp}(f_z)$ . The proof of (1) excludes  $y^{q+1} \in \text{Supp}(g_2 f_x)$ , so we only have to consider two cases:

i. If  $y^{q+1} \notin \text{Supp}(h_3)$  then, there must exist a  $k \in \mathbb{N}$  with  $y^k \in \text{Supp}(h_2)$  and  $k < q + 1$ . The multihomogeneity of  $h_2$  implies  $m = k < q + 1$ . Then  $y^{q+1}z, y^{m+1} \in \text{Supp}(f)$  and the weighted homogeneity of  $f$  imply  $(q + 1)w_2 + w_3 = (m + 1)w_2$ . This implies

$$w_3 = (m - q)w_2 \leq 0,$$

which contradicts  $w_3 > 0$ .

ii. If  $y^{q+1} \in \text{Supp}(h_3)$ , then  $f_z = g_2(y, z)f_x + g_3(z)h_2 + h_3$  and  $\deg_w(f_x) < \deg_w(h_2) < \deg_w(h_3)$  imply  $x^{p+1} \in \text{Supp}(h_3)$ . A direction vector of  $h_3$  is

$$u' = (p + 1, 0, 0) - (0, q + 1, 0) = (p + 1, -q - 1, 0).$$

$u$  cannot be parallel to  $u'$ , since  $p + 1 > p$  and  $u = \lambda u'$  imply  $1 > \lambda > 0$ , contradicting  $-l > -q - 1$ .

All cases are impossible so  $y^q z \in \text{Supp}(h_2)$ .

□

Now we prove a similar statement for the monomials close to the  $z$ -axis.

**Lemma 6.25.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and assume that there exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Then the following statements hold:*

- (1) If  $z^{r+1} \in \text{Supp}(f)$  for some  $r \in \mathbb{N}$ , then  $z^r \in \text{Supp}(h_3)$ .
- (2)  $z^r x \notin \text{Supp}(f)$  for any  $r \in \mathbb{N}$ .
- (3)  $z^r y \notin \text{Supp}(f)$  for any  $r \in \mathbb{N}$ .

*Proof.*

(1) Let  $z^r \in \text{Supp}(f_z)$ . With  $f_z = g_2(y, z)f_x + g_3(z)h_2 + h_3$ , we have to exclude  $z^r \in \text{Supp}(g_2f_x)$  and  $z^r \in \text{Supp}(g_3h_2)$  in order to show  $z^r \in \text{Supp}(h_3)$ .

(a) Assume  $z^r \in \text{Supp}(g_2f_x)$ . Then there exist an  $l \in \mathbb{N}_{\geq 2}$  with  $z^l \in \text{Supp}(f_x)$ . Now we have to consider three possible cases:

i. If  $x^{p+1} \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c)$$

and in the same way

$$u' = (a - 1, b, c) - (0, 0, l) = (a - 1, b, c - l).$$

$u$  cannot be parallel to  $u'$ , since the second components are equal, but  $c > c - l$ .

ii. If  $x^{p+1}y \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c)$$

and in the same way

$$u' = (a - 1, b, c) - (0, 0, l) = (a - 1, b, c - l).$$

$u$  cannot be parallel to  $u'$ , since  $b > b - 1$  and  $u = \lambda u'$  imply  $1 > \lambda > 0$ , contradicting  $c > c - l$ .

iii. If  $x^{p+1}z \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1)$$

and in the same way

$$u' = (a - 1, b, c) - (0, 0, l) = (a - 1, b, c - l).$$

$u$  cannot be parallel to  $u'$ , since the second components are equal, but  $c - 1 > c - l$ .

This computation implies  $z^l \notin \text{Supp}(f_x)$  and thus  $z^r \notin \text{Supp}(g_2f_x)$ .

(b) Assume  $z^r \in \text{Supp}(g_3h_2)$ . Then there exists an  $l \in \mathbb{N}_{\geq 2}$  with  $z^l \in \text{Supp}(h_2)$ . Using Lemma 6.24, we know that either  $y^q \in \text{Supp}(h_2)$  or  $y^q z \in \text{Supp}(h_2)$  for some  $q \in \mathbb{N}$ . We also know that the monomials of  $h_1$  and  $h_2$  have to lie on parallel lines. We consider both cases separately:

i. Assume  $y^q \in \text{Supp}(h_2)$  for some  $q \geq 2$ . Again we have to distinguish all cases with monomials close to the x-axis.

- A. If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c)$$

and a direction vector of  $h_2$  is given by

$$u' = (0, q, 0) - (0, 0, l) = (0, q, -l).$$

Since  $a, b, c \geq 1$  and  $\deg_w(x^a y^b z^c) = \deg_w(x^{p+1})$  we obtain  $aw_1 + bw_2 + cw_3 = (p+1)w_1$ , hence  $p+1 > a$ . Then  $u$  cannot be parallel to  $u'$ , since  $a - p - 1 < 0$ .

- B. If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c)$$

and a direction vector of  $h_2$  is given by

$$u' = (0, q, 0) - (0, 0, l) = (0, q, -l).$$

Since  $a, b, c \geq 1$  and  $\deg_w(x^a y^b z^c) = \deg_w(x^{p+1}y)$  we obtain  $aw_1 + bw_2 + cw_3 = pw_1 + w_2$ , hence  $p > a$ . Then  $u$  cannot be parallel to  $u'$ , since  $a - p - 1 < 0$ .

- C. If  $x^{p+1}z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1)$$

and a direction vector of  $h_2$  is given by

$$u' = (0, q, 0) - (0, 0, l) = (0, q, -l).$$

Since  $a, b, c \geq 1$  and  $\deg_w(x^a y^b z^c) = \deg_w(x^{p+1}z)$  we obtain  $aw_1 + bw_2 + cw_3 = pw_1 + w_3$ , hence  $p > a$ . Then  $u$  cannot be parallel to  $u'$ , since  $a - p - 1 < 0$ .

- ii. Assume  $y^q z \in \text{Supp}(h_2)$ .

- A. If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c)$$

and a direction vector of  $h_2$  is given by

$$u' = (0, q, 1) - (0, 0, l) = (0, q, 1 - l).$$

Since  $a, b, c \geq 1$  and  $\deg_w(x^a y^b z^c) = \deg_w(x^{p+1})$  we obtain  $aw_1 + bw_2 + cw_3 = (p+1)w_1$ , hence  $p+1 > a$ . Then  $u$  cannot be parallel to  $u'$ , since  $a - p - 1 < 0$ .

- B. If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c)$$

and a direction vector of  $h_2$  is given by

$$u' = (0, q, 1) - (0, 0, l) = (0, q, 1 - l).$$

Since  $a, b, c \geq 1$  and  $\deg_w(x^a y^b z^c) = \deg_w(x^{p+1} y)$  we obtain  $aw_1 + bw_2 + cw_3 = pw_1 + w_2$ , hence  $p > a$ . Then  $u$  cannot be parallel to  $u'$ , since  $a - p - 1 < 0$ .

C. If  $x^{p+1} z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1)$$

and a direction vector of  $h_2$  is given by

$$u'(0, q, 1) - (0, 0, l) = (0, q, 1 - l).$$

Since  $a, b, c \geq 1$  and  $\deg_w(x^a y^b z^c) = \deg_w(x^{p+1} z)$  we obtain  $aw_1 + bw_2 + cw_3 = pw_1 + w_3$ , hence  $p > a$ . Then  $u$  cannot be parallel to  $u'$ , since  $a - p - 1 < 0$ .

These computations imply  $z^r \notin \text{Supp}(g_2 f_x)$  and  $z^r \notin \text{Supp}(g_3 h_2)$ , so we must have  $z^r \in \text{Supp}(h_3)$ .

- (2) Let  $z^r x \in \text{Supp}(f)$  for some  $r \in \mathbb{N}$ . Then  $z^r \in \text{Supp}(f_x)$ , which is impossible due to the proof of (1), (a).
- (3) Let  $z^r y \in \text{Supp}(f)$  for some  $r \in \mathbb{N}$ . Then  $z^r \in \text{Supp}(f_y)$ , which is impossible due to the proof of (1), (a) and (1), (b), since  $f_y = g_1(y, z)f_x + h_2$  implies that either  $z^r \in \text{Supp}(h_2)$  or  $z^l \in \text{Supp}(f_x)$  for some  $l \in \mathbb{N}_{\geq 2}$ .

□

Our next goal is to prove that  $x^a y^{b-1} z^c \in \text{Supp}(h_2)$ , if  $x^a y^b z^c \in \text{Supp}(f)$ . Before we can do so, we need a lemma excluding a very special case.

**Lemma 6.26.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and assume that there exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Furthermore, we assume the following:*

- (1)  $a, b, c$  are chosen with maximal  $a$ , i.e. if  $x^{a'} y^{b'} z^{c'} \in \text{Supp}(f)$  with  $a', b', c' \in \mathbb{N}_{\geq 1}$ , then  $a \geq a'$ .
- (2)  $bw_2 + cw_3 = w_1$ .
- (3)  $x^{p+1} \in \text{Supp}(f)$  for some  $p \in \mathbb{N}_{\geq 2}$ .
- (4) No monomial of type  $x^i y^j z^k$  is contained in  $\text{Supp}(h_2)$  for any  $i, j, k \in \mathbb{N}_{\geq 1}$ .

Then  $a = p$  and

- (a)  $f$  is right-equivalent to  $x^{p+1} + y^q + z^r$  or  $x^{p+1} + y^q z + z^r$  for some  $q, r \in \mathbb{N}$ , or
- (b)  $x^p z^c \in \text{Supp}(h_2)$ , if  $b = 1$  and (a) does not hold.

*Proof.* First we show that  $a = p$ . Since  $x^{p+1}, x^a y^b z^c \in \text{Supp}(f)$  and  $bw_2 + cw_3 = w_1$  the weighted homogeneity of  $f$  implies

$$(p+1)w_1 = aw_1 + bw_2 + cw_3 = (a+1)w_1,$$

which yields  $a = p$ . Next we proof the second part of the statement for  $b > 1$ . Deriving yields  $x^p y^{b-1} z^c \in \text{Supp}(f_y)$ . Since no monomial of type  $x^i y^j z^k$  is contained in  $\text{Supp}(h_2)$  for any  $i, j, k \in \mathbb{N}_{\geq 1}$ , we know that  $x^p y^{b-1} z^c \in \text{Supp}(g_1 f_x)$  with  $g_1 = g_1(y, z)$ . The only possibility to obtain  $x^p y^{b-1} z^c$  as a multiple of a monomial of  $f_x$  is to multiply  $x^p$  with  $y^{b-1} z^c$ . Otherwise multiplication of  $x^p y^{b-1-s} z^{c-t}$  with  $y^s z^t$  for some  $s, t \in \mathbb{N}$  with  $s \leq b-1$  and  $t \leq c$  contradicts  $w_2 b + w_3 c = w_1$ . Multiplying  $f_x$  with  $y^{b-1} z^c$  implies  $x^{p-1} y^{2b-1} z^{2c} \in \text{Supp}(g_1 f_x)$ . With no monomial of type  $x^i y^j z^k$  in  $\text{Supp}(h_2)$  we have  $x^{p-1} y^{2b} z^{2c} \in \text{Supp}(f)$ . Continuing this argument we obtain

$$\{x y^{pb} z^{pc}, \dots, x^{p-1} y^{2b} z^{2c}, x^p y^b z^c\} \subseteq \text{Supp}(f).$$

We also have  $y^{(p+1)b} z^{(p+1)c} \in \text{Supp}(f)$ . Assume this were not the case, then  $y^{b-1} z^c y^{pb} z^{pc} = y^{(p+1)b-1} z^{(p+1)c} \in \text{Supp}(g_1 f_x)$  and  $y^{(p+1)b-1} z^{(p+1)c} \in \text{Supp}(h_2)$ , because the corresponding monomials need to cancel each other. In this case, using Lemma 6.24, it easy to see that the resulting direction vectors of  $f_x$  and  $h_2$  cannot be parallel, since a direction vectors of  $f_x$  is given by

$$u = (1, -b, -c)$$

having 1 as the first component and the possible direction vectors of  $h_2$  have 0 as their first component.

Next we show that the only monomial of type  $y^u z^v \in \text{Supp}(f)$  with  $u, v \geq 2$  is  $y^{(p+1)b} z^{(p+1)c}$ . Assume the contrary. Then  $y^{u-1} z^v \in \text{Supp}(f_y)$ . If  $y^{u-1} z^v \notin \text{Supp}(h_2)$ , then there exist  $i, j \in \mathbb{N}$  with  $y^i z^j \in \text{Supp}(f_x)$  and  $y^i z^j$  divides  $y^{u-1} z^v$ . Since  $f_x$  is multihomogeneous we must have  $i = pb$  and  $j = pc$ . In particular,  $u > pb$  and  $v \geq pc$ . This yields  $x^p y^{u-1-pb} z^{v-pc} \in \text{Supp}(g_1 f_x)$ . If  $x^p y^{u-1-pb} z^{v-pc} \in \text{Supp}(h_2)$ , we have to consider four cases:

(1) If  $u-1 > pb$  and  $v > pc$  we have  $p, u-1-pb, v-pc \geq 1$  and  $x^p y^{u-1-pb} z^{v-pc} \in \text{Supp}(h_2)$ , contradicting assumption (4).

(2) Assume  $u-1 = pb$  and  $v = pc$ . Using Lemma 6.24, we have to consider two subcases:

(a) Assume  $y^q \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is given by

$$u_1 = (p, -q, 0).$$

$u$  cannot be parallel to  $u_1$ , since  $-c < 0$ .

(b) Assume  $y^q z \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is given by

$$u_2 = (p, -q, -1).$$

$u$  cannot be parallel to  $u_2$ , since  $p > 1$  and  $u = \lambda u_2$  imply  $1 > \lambda > 0$ , contradicting  $-c \leq -1$ .

(3) Assume  $u-1 > pb$  and  $v = pc$ . Using Lemma 6.24, we have to consider two subcases:

(a) Assume  $y^q \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is given by

$$u_1 = (p, u - 1 - pb - q, 0).$$

$u$  cannot be parallel to  $u_1$ , since  $-c < 0$ .

(b) Assume  $y^q z \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is given by

$$u_2 = (p, u - 1 - pb - q, -1).$$

$u$  cannot be parallel to  $u_1$ , since  $p > 1$  and  $u = \lambda u_2$  imply  $1 > \lambda > 0$ , contradicting  $-c \leq -1$ .

(4) Assume  $u - 1 = pb$  and  $v > pc$ . Using Lemma 6.24, we have to consider two subcases:

(a) Assume  $y^q \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is given by

$$u_1 = (p, -q, v - pc).$$

$u$  cannot be parallel to  $u_1$ , since  $p > 1$  and  $u = \lambda u_2$  imply  $1 > \lambda > 0$ , contradicting  $v - pc > 0 > -c$

(b) Assume  $y^q z \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is given by

$$u_2 = (p, -q, v - pc - 1).$$

$u$  cannot be parallel to  $u_1$ , since  $p > 1$  and  $u = \lambda u_2$  imply  $1 > \lambda > 0$ , contradicting  $v - pc - 1 \geq 0 > -c$

All the cases contradict  $x^p y^{u-1-pb} z^{v-pc} \in \text{Supp}(h_2)$ . This implies  $x^p y^{u-1-pb} z^{v-pc} \in \text{Supp}(f_y)$ , hence  $x^p y^{u-pb} z^{v-pc} \in \text{Supp}(f)$ . The multihomogeneity of  $f_x$  implies  $u - pb = b$  and  $v - pc = c$ , hence  $u = (p + 1)b, v = (p + 1)c$ . Thus only monomials close to the  $y$ - or  $z$ -axis and  $y^{(p+1)b} z^{(p+1)c}$  are allowed to appear in  $f$ , if  $x$  were not to appear in them. Applying Lemma 6.24 and Lemma 6.25 we know, after a suitable coordinate change of type  $x \mapsto \alpha x, y \mapsto \beta y$  and  $z \mapsto \gamma z$  for certain  $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$ , that

$$f = x^{p+1} + y^q + z^r + \lambda_1 x^p y^b z^c + \dots + \lambda_p y^{(p+1)b} z^{(p+1)c}$$

or

$$f = x^{p+1} + y^q z + z^r + \lambda_1 x^p y^b z^c + \dots + \lambda_p y^{(p+1)b} z^{(p+1)c}$$

with  $\lambda_i \in \mathbb{C}$ . Considering the structure of  $f$  we see that the only way of eliminating the monomials in  $f_y$  containing an  $x$  is by multiplying  $f_x$  with  $\lambda y^{b-1} z^c$  for some  $\lambda \in \mathbb{C} \setminus \{0\}$ . Comparing coefficients in the equation

$$f_y - \lambda y^{b-1} z^c f_x = qy^{q-1}$$

or

$$f_y - \lambda y^{b-1} z^c f_x = qy^{q-1} z$$

yields  $\lambda_i = \binom{p+1}{i} \left(\frac{\lambda}{b}\right)^i$ . Using the binomial formula we get

$$f = \left(x + \frac{\lambda}{b} y^b z^c\right)^{p+1} + y^q + z^r,$$

or

$$f = \left( x + \frac{\lambda}{b} y^b z^c \right)^{p+1} + y^q z + z^r.$$

After the coordinate change  $x \mapsto x - \frac{\lambda}{b} y^b z^c$ ,  $y \mapsto y$ ,  $z \mapsto z$  we obtain the claimed form for (a). Next we assume  $b = 1$ . Either we have  $x^p z^c \notin \text{Supp}(h_2)$  and we can argue precisely as before or we obtain  $x^p z^c \in \text{Supp}(h_2)$ .

□

The next step is to prove  $x^a y^{b-1} z^c \in \text{Supp}(h_2)$

**Lemma 6.27.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and assume that there exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Then  $x^a y^{b-1} z^c \in \text{Supp}(h_2)$ .*

*Proof.* Similar to the previous proofs we assume that  $x^a y^{b-1} z^c \notin \text{Supp}(h_2)$  for any  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ . For fixed  $a, b, c$ , due to  $w_1 > w_2, w_3$ , there must exist  $k, l \in \mathbb{N}$  with  $x^a y^k z^l \in \text{Supp}(f_x)$  and  $x^a y^k z^l$  divides  $x^a y^{b-1} z^c$ . We consider different cases.

- (1) Assume  $k, l \geq 1$ . Choose  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$  and  $a$  maximal, i.e. if  $x^{a'} y^{b'} z^{c'} \in \text{Supp}(f)$  with  $a', b', c' \in \mathbb{N}_{\geq 1}$ , then  $a \geq a'$ . Then  $x^{a+1} y^k z^l \in \text{Supp}(f)$ , contradicting the maximality of  $a$ .
- (2) Assume  $k = 1$  and  $l = 0$ . This implies that  $x^{a+1} y \in \text{Supp}(f)$  and  $x^{a+1} \in \text{Supp}(f_y)$ . With  $w_1 > w_2$ , we have  $x^{a+1} \in \text{Supp}(h_2)$ . By Lemma 6.24 we have  $y^q \in \text{Supp}(h_2)$  for some  $q \geq 2$  or  $y^q z \in \text{Supp}(h_2)$  for some  $q \geq 1$ . We have to compare possible direction vectors of  $f_x$  and  $h_2$  in both cases. A direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (a, 1, 0) = (-1, b - 1, c).$$

For a direction vector of  $h_2$  we have two possibilities:

- (a) Assume  $y^q \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is

$$u_1 = (a + 1, 0, 0) - (0, q, 0) = (a + 1, -q, 0).$$

$u$  cannot be parallel to  $u_1$ , since  $c > 0$ .

- (b) Assume  $y^q z \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is

$$u_2 = (a + 1, 0, 0) - (0, q, 1) = (a + 1, -q, -1).$$

$u$  cannot be parallel to  $u_2$ , since  $a + 1 > -1$  and  $u = \lambda u_2$  imply  $0 > \lambda > -1$ , contradicting  $c > 1$ .

This contradicts  $x^a y \in \text{Supp}(f_x)$ .

- (3) Assume  $k = 0$  and  $l = 1$ . This implies that  $x^{a+1} z \in \text{Supp}(f)$  and  $x^{a+1} \in \text{Supp}(f_z)$ . With  $w_1 > w_3$ , we have  $x^{a+1} \in \text{Supp}(h_3)$ . By Lemma 6.25  $z^r \in \text{Supp}(h_3)$  for some  $r \geq 2$ . We have to compare possible direction vectors of  $f_x$  and  $h_3$ . A direction vector of  $f_x$  is

$$u = (a - 1, b, c) - (a, 0, 1) = (-1, b, c - 1)$$

and a direction vector of  $h_3$  is

$$u' = (a + 1, 0, 0) - (0, 0, q) = (a + 1, 0, -q).$$

$u$  cannot be parallel to  $u'$ , since  $b > 0$ . This contradicts  $x^a z \in \text{Supp}(f_x)$ .



(4) Assume  $k \geq 2$  and  $l = 0$ . This implies that  $x^{a+1}y^k \in \text{Supp}(f)$  and  $x^ay^k \in \text{Supp}(f_x)$ . Considering all possibilities of monomials close to the  $x$ -axis we have three different monomials of  $f_x$  and we can check if the corresponding direction vectors are parallel:

(a) Assume  $x^p \in \text{Supp}(f_x)$  for some  $p \geq 2$ . The first direction vector we obtain is

$$u_1 = (a - 1, b, c) - (a, k, 0) = (-1, b - k, c)$$

and the second one is

$$u_2 = (a - 1, b, c) - (p, 0, 0) = (a - p, b, c).$$

$u_1$  cannot be parallel to  $u_2$ , since the last components are equal, but  $b > b - k$ .

(b) Assume  $x^py \in \text{Supp}(f_x)$  for some  $p \geq 1$ . The first direction vector we obtain is

$$u_1 = (a - 1, b, c) - (a, k, 0) = (-1, b - k, c)$$

and the second one is

$$u_2 = (a - 1, b, c) - (p, 1, 0) = (a - p, b - 1, c).$$

$u_1$  cannot be parallel to  $u_2$ , since the last components are equal, but  $b - 1 > b - k$ .

(c) Assume  $x^pz \in \text{Supp}(f_x)$  for some  $p \geq 1$ . The first direction vector we obtain is

$$u_1 = (a - 1, b, c) - (a, k, 0) = (-1, b - k, c)$$

and the second one is

$$u_2 = (a - 1, b, c) - (p, 0, 1) = (a - p, b, c - 1).$$

$u_1$  cannot be parallel to  $u_2$ , since  $b > b - k$  and  $u_1 = \lambda u_2$  imply  $1 > \lambda > 0$ , contradicting  $c > c - 1$ .

This contradicts  $x^ay^k \in \text{Supp}(f_x)$ .

(5) Assume  $k = 0$  and  $l \geq 2$ . This implies  $x^{a+1}y^l \in \text{Supp}(f)$  and  $x^ay^l \in \text{Supp}(f_x)$ . Considering all possibilities of monomials close to the  $x$ -axis we have three different monomials of  $f_x$  and we can check if the corresponding direction vectors are parallel:

(a) Assume  $x^p \in \text{Supp}(f_x)$  for some  $p \geq 2$ . The first direction vector we obtain is

$$u_1 = (a - 1, b, c) - (a, 0, l) = (-1, b, c - l)$$

and the second one is

$$u_2 = (a - 1, b, c) - (p, 0, 0) = (a - p, b, c).$$

$u_1$  cannot be parallel to  $u_2$ , since the second components are equal, but  $c > c - l$ .

- (b) Assume  $x^p y \in \text{Supp}(f_x)$  for some  $p \geq 1$ . The first direction vector we obtain is

$$u_1 = (a - 1, b, c) - (a, 0, l) = (-1, b, c - l)$$

and the second one is

$$u_2 = (a - 1, b, c) - (p, 1, 0) = (a - p, b - 1, c).$$

$u_1$  cannot be parallel to  $u_2$ , since  $b - 1 < b$  and  $u_1 = \lambda u_2$  imply  $\lambda > 1$ , contradicting  $c > c - l$ .

- (c) Assume  $x^p z \in \text{Supp}(f_x)$  for some  $p \geq 1$ . The first direction vector we obtain is

$$u_1 = (a - 1, b, c) - (a, 0, l) = (-1, b, c - l)$$

and the second one is

$$u_2 = (a - 1, b, c) - (p, 0, 1) = (a - p, b, c - 1).$$

$u_1$  cannot be parallel to  $u_2$ , since the second components are equal, but  $c - 1 > c - l$ .

This contradicts  $x^a z^l \in \text{Supp}(f_x)$ .

- (6) Assume  $k = l = 0$ . This case is an application of Lemma 6.26. If we are in case (a) of Lemma 6.26, we can perform a coordinate change, which does not change the multihomogeneity of  $f$  and no monomial of type  $x^a y^b z^c$  appears in  $\text{Supp}(f)$ , contradicting the assumption. If we are in case (b) of Lemma 6.26 we have  $x^p z^c \in \text{Supp}(h_2)$  and  $x^p \in \text{Supp}(f_x)$ . This yields

$$u = (p, 0, 0) - (p - 1, 1, c) = (1, -1, -c)$$

as a direction vector of  $f_x$ . For  $h_2$  Lemma 6.24 yields two possibilities:

- (a) Assume  $y^q \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is

$$u_1 = (p, 0, c) - (0, q, 0) = (p, -q, c).$$

$u$  cannot be parallel to  $u_1$ , since the last entries and  $u = \lambda u_1$  imply  $\lambda = -1$ , contradiction  $p > 1$ .

- (b) Assume  $y^q z \in \text{Supp}(h_2)$ . Then a direction vector of  $h_2$  is

$$u_1 = (p, 0, c) - (0, q, 1) = (p, -q, c - 1).$$

$u$  cannot be parallel to  $u_1$ , since  $-q < -1$  and  $u = \lambda u_1$  imply  $1 > \lambda > 0$ , contradiction  $c - 1 < c$ .

This contradicts  $x^p z^c \in \text{Supp}(h_2)$ .

The assumption that no monomial of type  $x^a y^b z^c$  satisfies  $x^a y^{b-1} z^c \in \text{Supp}(h_2)$  leads only to contradictions, so  $h_2$  must contain such a monomial.  $\square$

Now we are in the position to prove that no monomial of type  $x^a y^b z^c$  can be contained in  $\text{Supp}(f)$ , if we know that  $J_f$  is multihomogeneous and  $w_1 > w_2 > w_3$ .

**Proposition 6.28.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and with  $s = 2$ . Then there do not exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ .*

*Proof.* Assume the contrary. Then by Lemma 6.27 there exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^{b-1} z^c \in \text{Supp}(h_2)$ . Lemma 6.24 states that  $y^q \in \text{Supp}(h_2)$  or  $y^q z \in \text{Supp}(h_2)$  for some  $q \in \mathbb{N}$ . Next to these two cases we have to consider all cases of monomials close to the x-axis.

(1) Assume  $y^q \in \text{Supp}(h_2)$  for a  $q \geq 2$ . A direction vector of  $h_2$  is given by

$$u = (a, b - 1, c) - (0, q, 0) = (a, b - 1 - q, c).$$

Next we consider all possible cases of monomials close to the x-axis:

(a) If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u' = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c).$$

$u$  cannot be parallel to  $u'$ , since the last components are equal, but  $a > a - 1 - p$ .

(b) If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u' = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c).$$

$u$  cannot be parallel to  $u'$ , since the last components are equal, but  $a > a - 1 - p$ .

(c) If  $x^{p+1}z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u' = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1).$$

$u$  cannot be parallel to  $u'$ , since  $c > c - 1$  and  $u' = \lambda u$  imply  $1 > \lambda > 0$ , contradicting  $b > b - 1 - q$ .

(2) Assume  $y^q z \in \text{Supp}(h_2)$  for a  $q \geq 1$ . A direction vector of  $h_2$  is given by

$$u = (a, b - 1, c) - (0, q, 1) = (a, b - 1 - q, c - 1).$$

Next we consider all possible cases of monomials close to the x-axis:

(a) If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u' = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c).$$

$u$  cannot be parallel to  $u'$ , since  $c > c - 1$  and  $u' = \lambda u$  imply  $\lambda > 1$ , contradicting  $a > a - 1 - p$ .

(b) If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u' = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c).$$

$u$  cannot be parallel to  $u'$ , since  $c > c - 1$  and  $u' = \lambda u$  imply  $\lambda > 1$ , contradicting  $a > a - 1 - p$ .

(c) If  $x^{p+1}z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is given by

$$u' = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1).$$

$u$  cannot be parallel to  $u'$ , since the last components are equal, but  $a > a - 1 - p$ .

□

**Remark 6.29.** Proposition 6.28 allows us to assume from now on that no monomials of type  $x^a y^b z^c$  appear in  $\text{Supp}(f)$  for  $a, b, c \in \mathbb{N}_{\geq 1}$ .

The next step is to consider all possibilities for the structure of the monomial diagram of  $f_x$ , knowing that no monomial of type  $x^a y^b z^c$  appears in  $\text{Supp}(f)$ . The fact that  $f_x$  is multihomogeneous leaves us with only four possibilities:

- (1)  $f_x$  is a monomial, or
- (2)  $f_x$  only contains monomials of type  $x^i y^j$  for  $i, j \in \mathbb{N}$ , or
- (3)  $f_x$  only contains monomials of type  $x^i z^j$  for  $i, j \in \mathbb{N}$ , or
- (4)  $f_x$  contains one monomial of type  $x^i y^j$  and one monomial of type  $x^k z^l$  for certain  $i, j, k, l \in \mathbb{N}$ .

We prove for each case that either  $f$  is of Sebastiani–Thom type or that such an  $f$  cannot exist. Let us begin with the first case.

**Lemma 6.30.** Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s \geq 2$ . Let  $f_x$  be a monomial, then the following hold:

- (1) If  $f_x = x^p$  for some  $p \in \mathbb{N}$ , then  $f$  is of Sebastiani–Thom type.
- (2) If  $f_x = x^p y$  for some  $p \in \mathbb{N}$ , then  $f$  is of Sebastiani–Thom type.
- (3) If  $f_x = x^p z$  for some  $p \in \mathbb{N}$ , then  $f$  is of Sebastiani–Thom type.

*Proof.*

- (1) By integration we obtain  $f(x, y, z) = \int x^p dx = \frac{1}{p+1} x^{p+1} + h(y, z)$  for some polynomial  $h \in \mathbb{C}[y, z]$ .
- (2) By assumption the only possible monomials close to the  $y$ -axis are  $y^{q+1}$  or  $y^{q+1} z$ . Using that  $f_x = x^p y$  we have  $x^{p+1} \in \text{Supp}(f_y)$ , which implies  $x^{p+1} \in \text{Supp}(h_2)$ , since  $w_1 > w_2$ . Now we have two possible pairs:
  - (a)  $x^{p+1}, y^q \in \text{Supp}(h_2)$  or
  - (b)  $x^{p+1}, y^q z \in \text{Supp}(h_2)$ .

With  $s = 2$  the monomials of  $h_2$  have to lie on a line. This implies that no monomial of type  $y^i z^j$  for  $i \geq 1$  and  $j \geq 2$  is contained in  $\text{Supp}(f_y)$ , as it could not be canceled by a multiple of a monomial contained in  $\text{Supp}(f_x)$  and it would contradict the fact that the monomials in  $\text{Supp}(h_2)$  lie on a line. This implies that the only monomial close to the  $z$ -axis is  $z^{r+1}$  for some  $r \in \mathbb{N}$ . Due to our results so far we have two possible cases for  $\text{Supp}(f)$  :

- (a) The first case is  $\{x^{p+1} y, y^{q+1}, z^{r+1}\} = \text{Supp}(f)$ . In this case  $f$  is of Sebastiani–Thom type.

- (b) The second case is  $\{x^{p+1}y, y^{q+1}z, z^{r+1}\} = \text{Supp}(f)$ . In this case  $z^r, y^{q+1} \in \text{Supp}(h_3)$ , since no monomial of  $f_x$  or  $h_2$  could divide any of these monomials. But in this case  $h_3$  lies in the  $y$ - $z$  plane. We obtain a contradiction to the parallelity of the direction vectors of  $h_2$  and  $h_3$ , since one monomial of  $h_2$  lies in the  $y$ - $z$  plane and the other one does not.
- (3) By assumption  $f_y = f_y(y, z)$ . Then  $f_y = g_1(y, z)x^p z + h_2$  implies that we can choose  $h_2 = f_y$ . With  $w_1 > w_3$  we obtain  $x^{p+1} \in \text{Supp}(h_3)$ . Here we have to differentiate two possible cases:
- (a) The first case is  $\text{Supp}(h_3) = \{x^{p+1}\}$ . Our assumptions allow two possibilities for monomials close to the  $y$ -axis. If  $y^{q+1}z \in \text{Supp}(f)$  for some  $q \in \mathbb{N}_{\geq 1}$ , then  $y^{q+1} \in \text{Supp}(h_3)$ , since  $f_z = g_2(y, z)f_x + g_3(z)f_y + h_3$  and  $w_2 > w_3$ . This means that the only possibility is  $y^{q+1} \in \text{Supp}(f)$  for some  $q \in \mathbb{N}_{\geq 2}$ . Next we consider monomials of type  $y^i z^j \in \text{Supp}(f)$  with  $i, j \in \mathbb{N}_{\geq 1}$ . Assume such a monomial with  $j \geq 2$  minimal is contained in  $\text{Supp}(f)$ . Then there must exist  $y^i z^a \in \text{Supp}(h_2)$  dividing  $y^i z^{j-1} \in \text{Supp}(f_z)$ . The minimality assumption on  $j$  implies  $a = 0$ , so  $z^{j-1} \in \text{Supp}(g_3)$ .  $y^q \in \text{Supp}(f_y)$  yields  $y^q z^{j-1} \in \text{Supp}(f_z)$ , hence the weighted homogeneity of  $f_z$  implies  $i = q$ . With  $y^{q-1} z^j \in \text{Supp}(f_y)$ , we obtain  $y^{q-1} z^{2j-1} \in \text{Supp}(f_z)$ . Iterating this process we obtain that, after a suitable change of coordinates, as in the proof of Lemma 6.26,

$$f = x^{p+1}z + y^{q+1} + \lambda_1 y^q z^j + \dots + \lambda_q y z^{qj} + \lambda_{q+1} z^{(q+1)j},$$

with  $\lambda_i \in \mathbb{C}$ . Since  $f_z = g_2(y, z)f_x + g_3(z)f_y + h_3$ , the previous computations imply

$$f_z = \lambda z^{j-1} \cdot f_y + x^{p+1}$$

for some  $\lambda \in \mathbb{C} \setminus \{0\}$ . Comparing coefficients yields  $\lambda_i = \binom{q+1}{i} \left(\frac{\lambda}{j}\right)^i$ . Then

$$f = \left(y + \frac{\lambda}{j} z^j\right)^{q+1} + x^p z.$$

Using the coordinate change  $x \mapsto x, y \mapsto y - \frac{\lambda}{j} z^j, z \mapsto z$  we get that  $f$  is right-equivalent to  $x^p z + y^{q+1}$ , which does not define an isolated hypersurface singularity.

- (b) The next possible case is that  $h_3$  contains more than one monomial. We know that the direction vectors  $f_y$  and  $h_3$  have to be parallel. Since  $h_3$  is not contained in the  $y$ - $z$  plane, this implies that  $f_y$  contains only one monomial. Then the only monomial close to the  $z$ -axis is  $z^r$  for some  $r \in \mathbb{N}$ . We have to consider two cases for monomials close to the  $y$ -axis:
- i. If  $\text{Supp}(f_y) = \{y^q\}$  for some  $q \in \mathbb{N}_{\geq 2}$  we immediately have that  $f$  is of Sebastiani–Thom type.
  - ii. The case  $\text{Supp}(f_y) = \{y^q z\}$  is impossible, since our assumptions on  $f$  yield  $\{x^{p+1}z, y^{q+1}z, z^{r+1}\} = \text{Supp}(f)$ . In this case  $z$  divides  $f$ , so the fact that  $f$  defines an isolated hypersurface singularity implies the existence of  $y^i \in \text{Supp}(f)$ , which contradicts the assumptions.

□

In the next steps we assume that the monomials of  $f_x$  lie on a line in the x-y or x-z plane. First we consider the case where they lie in the x-y plane.

**Lemma 6.31.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $|\text{Supp}(f_x)| \geq 2$  and the monomials of  $f_x$  lie in the x-y plane. Then  $f$  is of Sebastiani–Thom type.*

*Proof.* Due to our assumption we have either  $\{x^{p+1}, x^a y^b\} \subseteq \text{Supp}(f)$  for  $a, b \geq 1$  or  $\{x^{p+1}y, x^a y^b\} \subseteq \text{Supp}(f)$  for  $a \geq 1$  and  $b > 1$ . Assume we are in the first case. We have to consider two possible cases:

- (1) Assume  $x^a y^b \in \text{Supp}(h_2)$  with  $a, b \geq 1$ . Then  $h_2 = h_2(x, y)$  since the direction vectors of  $f_x$  and  $h_2$  have to be parallel. Assume now there exist monomials of type  $y^i z^j \in \text{Supp}(f)$  for some  $i, j \in \mathbb{N}_{\geq 1}$ . Then  $y^{i-1} z^j \in \text{Supp}(f_y)$  and by assumption  $y^{i-1} z^j \in \text{Supp}(g_1 f_x)$ , since  $f_y = g_1(y, z)f_x + h_2$ . Now  $f_x = f_x(x, y)$  implies  $y^k \in \text{Supp}(f_x)$  for some  $k \in \mathbb{N}$ . Using this we see that  $x^p y^{i-1-k} z^j \in \text{Supp}(g_1 f_x)$  and thus  $x^{p-1} y^{i-k} z^j \in \text{Supp}(f_x)$  contradicting the fact that the monomials of  $f_x$  lie in the x-y plane, since  $j \geq 1$ . So no monomial of type  $y^i z^j$  with  $i, j \in \mathbb{N}_{\geq 1}$  exists and  $f$  has to be of Sebastiani–Thom type.
- (2) Assume no monomial of type  $x^a y^b$  is contained in  $h_2$ . Consider such a monomial in  $\text{Supp}(f)$  with minimal  $b$ . Then  $x^a y^{b-1}$  is in  $\text{Supp}(g_1 f_x)$ , hence there exists a  $k \in \mathbb{N}$  with  $x^a y^k$  divides  $x^a y^{b-1}$ . The minimality of  $b$  and the fact that  $f_x$  is weighted homogeneous imply  $k = 0$  and  $a = p$ . Thus we must have  $x^p y^{2b-1} \in \text{Supp}(f_y)$ . Iterating this process, and using a suitable change of coordinates we obtain

$$f = x^{p+1} + \lambda_1 x^p y^b + \dots + \lambda_p x y^{pb} + \lambda_{p+1} y^{(p+1)b} + h(y, z)$$

with  $\lambda_i \in \mathbb{C}$  and  $h \in \mathbb{C}[y, z]$  and

$$f_y = \lambda y^{b-1} f_x + h_y(y, z)$$

for some  $\lambda \in \mathbb{C}$ . Using the same argument as in the proof of Lemma 6.26, we get that  $f$  is right-equivalent to

$$x^{p+1} + r(y, z)$$

for some polynomial  $r \in \mathbb{C}[y, z]$ . Then  $f$  is of Sebastiani–Thom type.

In the case  $\{x^{p+1}y, x^a y^b\} \subseteq \text{Supp}(f)$  for  $a \geq 1$  and  $b > 1$  we see that  $x^{p+1} \in \text{Supp}(h_2)$ . Due to this  $h_2$  lies in the x-y plane and we can argue as in (1).  $\square$

Next we consider the case where  $f_x$  lies in the x-z plane.

**Lemma 6.32.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $|\text{Supp}(f_x)| \geq 2$  and the monomials of  $f_x$  lie in the x-z plane. Then  $f$  is of Sebastiani–Thom type.*

*Proof.* Due to our assumptions we have  $f_y \in \mathbb{C}[y, z]$ . Since  $f_x \in \mathbb{C}[x, z]$  we can assume that  $f_y = h_2$ . The fact that the direction vectors of  $f_x$  and  $h_2$  have to be parallel implies  $\text{Supp}(f_y) = \{y^a\}$  or  $\text{Supp}(f_y) = \{y^a z\}$ . So we have to consider two cases:

- (1) Let  $\text{Supp}(f_y) = \{y^a\}$ . Then integration yields that  $f$  is of Sebastiani–Thom type.

- (2) Let  $\text{Supp}(f_y) = \{y^q z\}$ . Then  $y^{q+1} \in \text{Supp}(h_3)$ , since the monomial cannot be divided by any monomial of  $f_x$  or  $y^q z$ . This implies  $\text{Supp}(h_3) = \{y^{q+1}\}$ , since the direction vectors of  $f_x$  and  $h_3$  have to be parallel. This also implies  $x^{p+1} \in \text{Supp}(f)$ , since the same argument would yield  $x^{p+1} \in \text{Supp}(h_3)$ , if  $x^{p+1} z \in \text{Supp}(f)$ . Any monomial of type  $x^i z^j \in \text{Supp}(f_z)$  for  $i \geq 1$  and  $j \geq 2$  has to be a multiple of a monomial of  $f_x$ . Taking such a monomial with minimal  $j$ , which must exist since  $f_x = f_x(x, z)$  and  $|\text{Supp}(f_x)| \geq 2$ , we can argue similar to the proof of Lemma 6.30, (3), (a) and after suitable change of coordinates we obtain that  $f$  is right equivalent to

$$x^{p+1} + y^{q+1} z$$

for some  $p, q \in \mathbb{N}$ . In this case  $f$  does not define an isolated hypersurface singularity.

□

Finally we consider the case where  $f_x$  lies in the  $x$ - $y$  plane and in the  $x$ - $z$  plane.

**Lemma 6.33.** *Let  $f \in \mathbb{C}[x, y, z]$  be a polynomial with unique weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $|\text{Supp}(f_x)| \geq 2$  and the monomials of  $f_x$  lie in the  $x$ - $y$  plane and in the  $x$ - $z$  plane. Then  $f$  cannot define an isolated hypersurface singularity.*

*Proof.* Assume  $f$  defines an isolated hypersurface singularity. Then  $\text{Supp}(f_x) = \{x^p y, x^i z^j\}$  or  $\text{Supp}(f_x) = \{x^p z, x^i y^j\}$  for  $i, j \in \mathbb{N}_{\geq 1}$ .

- (1) Assume  $\text{Supp}(f_x) = \{x^p y, x^i z^j\}$ . Then  $x^{p+1} \in \text{Supp}(h_2)$ , since no monomial of  $f_x$  divides  $x^{p+1}$ . By the same argument we obtain  $y^q \in \text{Supp}(h_2)$  or  $y^q z \in \text{Supp}(h_2)$  for some  $q \in \mathbb{N}$ . A possible direction vector of  $f_x$  is given by

$$u = (p - i, 1, -j).$$

We have to consider two different cases:

- (a) Let  $y^q \in \text{Supp}(h_2)$ , then a possible direction vector of  $h_2$  is given by

$$u_1 = (p + 1, -q, 0).$$

$u$  cannot be parallel to  $u_1$ , since  $-j < 0$  in the last component.

- (b) Let  $y^q z \in \text{Supp}(h_2)$ , then a possible direction vector of  $h_2$  is given by

$$u_2 = (p + 1, -q, -1).$$

$u$  cannot be parallel to  $u_2$ , since  $-j \leq -1 < 0$  and  $u = \lambda u_2$  imply  $\lambda > 0$  contradicting  $-q < 0 < 1$  in the second component.

Both cases contradict our assumptions.

- (2) Assume  $\text{Supp}(f_x) = \{x^p z, x^i y^j\}$  with  $j \geq 2$ . Then  $x^{i+1} y^{j-1} \in \text{Supp}(h_2)$ , since this monomial cannot be divided by any monomial of  $f_x$ . As in (1) we must have  $y^q \in \text{Supp}(h_2)$  or  $y^q z \in \text{Supp}(h_2)$  for some  $q \in \mathbb{N}$ . A possible direction vector of  $f_x$  is given by

$$u = (p - i, -j, 1).$$

We have to consider two different cases:



(a) Let  $y^q \in \text{Supp}(h_2)$ , then a possible direction vector of  $h_2$  is given by

$$u_1 = (i + 1, j - 1 - q, 0).$$

$u$  cannot be parallel to  $u_1$ , since  $1 > 0$  in the last component.

(b) Let  $y^q z \in \text{Supp}(h_2)$ , then a possible direction vector of  $h_2$  is given by

$$u_2 = (i + 1, j - 1 - q, -1).$$

$u$  cannot be parallel to  $u_2$ , since  $-1 < 0 < 1$  and  $u = \lambda u_2$  imply  $\lambda = -1$  contradicting  $-q < -j$  in the second component.

Both cases contradict our assumptions and an isolated hypersurface singularity defined by  $f$  cannot exist.

□

Combining the previous lemmas we obtain the following result:

**Proposition 6.34.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ , then  $f$  is of Sebastiani–Thom type.*

## 6.4 The case $w_1 > w_2 = w_3$

The following Lemma can be proven using the same techniques that appeared in the proof of Lemma 6.23. Due to this we omit the proof.

**Lemma 6.35.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $w = (w_1, w_2, w_3)$ , where  $w_1 > w_2 = w_3$  and assume  $s = 2$ . Then the following hold:*

- (1)  $f_x$  is multihomogeneous with respect to the weights of  $J_f$ .
- (2) There exist  $g_1, g_2 \in \mathbb{C}[y, z]$  and  $a_1, b_1 \in \mathbb{C}$ , such that

$$f_y = g_1 f_x + h_2 + a_1 h_3 \text{ and } f_z = g_2 f_x + b_1 h_2 + h_3.$$

In particular we can always assume  $h_1 = f_x$ .

The first step is to show that no monomial of type  $x^a y^b z^c$  with  $a, b, c \geq 1$  occurs in  $f$ . In order to do so, we first prove results assuming such a monomial exists in  $f$ . Let us start with the first result:

**Lemma 6.36.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$ ,  $s = 2$  and assume that there exist  $a, b, c \geq 1$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Then the following statements hold:*

- (1) If  $y^{q+1} \in \text{Supp}(f)$  for some  $q \in \mathbb{N}$ , then  $y^q \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .
- (2)  $y^{q+1} x \notin \text{Supp}(f)$  for any  $q \in \mathbb{N}$ .
- (3) If  $y^{q+1} z \in \text{Supp}(f)$  for some  $q \in \mathbb{N}$ , then  $y^q z \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  or there exists a  $k \in \mathbb{N}_{\geq 2}$  such that  $y^k \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .

*Proof.*

- (1) Let  $y^{q+1} \in \text{Supp}(f)$ . Then  $y^q \in \text{Supp}(f_y)$ . If  $y^q \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then there exists some  $l \geq 2$ , such that  $y^l \in \text{Supp}(f_x)$ . Using  $x^{a-1}y^bz^c \in \text{Supp}(f_x)$  we need to consider the following three cases:

- (a) If  $x^{p+1} \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u_1 = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c)$$

and in the same way

$$u_2 = (a - 1, b, c) - (0, l, 0) = (a - 1, b - l, c).$$

These cannot be parallel, since the last components are equal, but  $a - 1 - p < a - 1$ . This gives a contradiction to  $f_x$  having monomials on a line.

- (b) If  $x^{p+1}y \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u_1 = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c)$$

and in the same way

$$u_2 = (a - 1, b, c) - (0, l, 0) = (a - 1, b - l, c).$$

These vectors cannot be parallel, since their third component is equal to  $c$ , but  $b - l < b$ . This gives a contradiction to  $f_x$  having monomials on a line.

- (c) If  $x^{p+1}z \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u_1 = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1)$$

and in the same way

$$u_2 = (a - 1, b, c) - (0, l, 0) = (a - 1, b - l, c).$$

These vectors cannot be parallel, since  $0 \leq c - 1 < c$  implies  $\lambda u_2 = u_1$  only if  $1 > \lambda > 0$ , but then  $b = \lambda(b - l) < b - l < b$ . This gives a contradiction to  $f_x$  having monomials on a line.

All cases are impossible so  $y^q \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .

- (2) If  $y^{q+1}x \in \text{Supp}(f)$ , then  $y^{q+1} \in \text{Supp}(f_x)$ , which is impossible due to the proof of (1).

- (3) If  $y^{q+1}z \in \text{Supp}(f)$ , then  $y^qz \in \text{Supp}(f_y)$ . If  $y^qz \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then, as in the proof of (1), there must exist an  $l \geq 1$  with  $y^lz \in \text{Supp}(f_x)$ . In order to use the same arguments as in (1) we need to assure that the appearing direction vectors are all different from  $(0, 0, 0)$ . To obtain this, we first assume  $a > 1$  or  $b \neq l$  or  $c > 1$ . Using  $x^{a-1}y^bz^c \in \text{Supp}(f_x)$  we need to consider the following three cases:

(a) If  $x^{p+1} \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u_1 = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c)$$

and in the same way

$$u_2 = (a - 1, b, c) - (0, l, 1) = (a - 1, b - l, c - 1).$$

These vectors cannot be parallel, since  $0 \leq c - 1 < c$  and  $\lambda u_2 = u_1$  imply  $\lambda > 1$ , contradicting  $a - 1 > a - 1 - p$ . This gives a contradiction to  $f_x$  having monomials on a line.

(b) If  $x^{p+1}y \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u_1 = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c)$$

and in the same way

$$u_2 = (a - 1, b, c) - (0, l, 1) = (a - 1, b - l, c - 1).$$

These vectors cannot be parallel, since  $0 \leq c - 1 < c$  and  $\lambda u_2 = u_1$  imply  $\lambda > 1$ , contradicting  $a - 1 > a - 1 - p$ . This gives a contradiction to  $f_x$  having monomials on a line.

(c) If  $x^{p+1}z \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u_1 = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1)$$

and in the same way

$$u_2 = (a - 1, b, c) - (0, l, 1) = (a - 1, b - l, c - 1).$$

These vectors cannot be parallel, since their last components equal  $c - 1$ , but  $a - 1 - p < a - 1$ . This gives a contradiction to  $f_x$  having monomials on a line.

Next we consider the monomial  $x^a y^b z^c \in \text{Supp}(f)$  with the property that  $a$  is maximal, i.e. for any  $x^{a'} y^{b'} z^{c'} \in \text{Supp}(f)$  with  $a', b', c' \geq 1$  we have  $a \geq a'$ . Only the case  $a = c = 1$  and  $b = l$  remains to be considered. The maximality of  $a$  implies that  $xy^l z$  is the only monomial of  $f$  of type  $x^i y^j z^k$  for  $i, j, k \in \mathbb{N}_{\geq 1}$ . Since all monomials of  $f_x$  lie on a line, only  $y^l z$  and one of the monomials  $x^p, x^p y$  or  $x^p z$  can appear in  $\text{Supp}(f_x)$ . If another monomial were to appear, then it would be of type  $x^i y^j$  or  $x^i z^j$  for some  $i, j \in \mathbb{N}$  and a simple computation as in (a) - (c) yields a contradiction on the parallelity of the direction vectors of  $f_x$ . Using the weighted homogeneity of  $f$  we obtain

$$w_2(q + 1) + w_3 = w_1 + w_2 l + w_3 > w_2(l + 1) + w_3.$$

This implies  $q \geq l + 1$ . Since  $f_y = g_1 f_x + h_2 + a_1 h_3$  and  $y^l z \in \text{Supp}(f_x)$  divides  $y^q z$ , we know that  $y^{q-l} \in \text{Supp}(g_1)$ .

(i) If  $x^{p+1} \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (0, l, 1) - (p, 0, 0) = (-p, l, 1).$$

We know  $xy^{l-1}z \in \text{Supp}(f_y)$ . If  $xy^{l-1}z \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then there exists a monomial of type  $xy^i z^j \in \text{Supp}(f_x)$  dividing  $xy^{l-1}z$ .  $\text{Supp}(f_x) = \{x^p, y^l z\}$  implies  $i = j = 0$ , thus  $x \in \text{Supp}(f_x)$  contradicting  $\text{ord}(f) \geq 2$ . Thus  $xy^{l-1}z \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . We know that  $y^{q-l}x^p \in \text{Supp}(y^{q-l}f_x)$ . If  $y^{q-l}x^p \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then  $y^{q-l}x^p \in \text{Supp}(f_y)$ . In this case  $y^{q-l+1}x^{p-1} \in \text{Supp}(f_x)$ , which is not possible. So  $y^{q-l}x^p \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Having three different monomials from  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $h_2$  or  $h_3$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$ . Then possibilities are

$$\begin{aligned} u_1 &= (1, l-1, 1) - (p, q-l+1, 0) = (1-p, 2l-q-2, 1) \\ u_2 &= (1, l-1, 1) - (0, q+1, 0) = (1, 2l-q-2, 1) \\ u_3 &= (p, q-l+1, 0) - (0, q+1, 0) = (p, -l, 0) \end{aligned}$$

$u$  cannot be parallel to  $u_1$  since  $-p < 1-p$ , but the last entries of the vectors are equal.  $u$  cannot be parallel to  $u_2$  since  $1 > -p$  but the last entries of the vectors are equal.  $u$  cannot be parallel to  $u_3$  since  $1 \neq 0$  in the last entry.

(ii) If  $x^{p+1}y \in \text{Supp}(f)$ , then a direction vector of  $f_x$  is

$$u = (0, l, 1) - (p, 1, 0) = (-p, l-1, 1).$$

We know  $x^{p+1} \in \text{Supp}(f_y)$ . If  $x^{p+1} \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then  $x^{p+1} \in \text{Supp}(f_x)$ , contradicting  $\deg_w(f_x) < \deg_w(f_y)$ . Thus  $x^{p+1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . We know that  $y^{q-l+1}x^p \in \text{Supp}(y^{q-l}f_x)$ . If  $y^{q-l+1}x^p \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then  $y^{q-l+1}x^p \in \text{Supp}(f_y)$ . In this case  $y^{q-l+2}x^{p-1} \in \text{Supp}(f_x)$ , which is not possible. So  $y^{q-l+1}x^p \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Having three different monomials from  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $h_2$  or  $h_3$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$ . Then possible ones are

$$\begin{aligned} u_1 &= (p+1, 0, 0) - (p, q-l+1, 0) = (1, l-q-1, 0) \\ u_2 &= (p+1, 0, 0) - (0, q+1, 0) = (p+1, -q-1, 0) \\ u_3 &= (p, q-l+1, 0) - (0, q+1, 0) = (p, -l, 0) \end{aligned}$$

$u$  cannot be parallel to  $u_1, u_2$  or  $u_3$  since  $1 \neq 0$  in the last entry.

(iii) If  $x^{p+1}z \in \text{Supp}(f)$ , then  $\{x^{p+1}z, y^{q+1}z, xy^l z\} \subseteq \text{Supp}(f)$ . Considering that we need a monomial close to the  $z$ -axis, we have only monomials, which are divisible by  $z$ , hence we need a monomial of type  $x^i y^j$  for some  $i, j \in \mathbb{N}$ . Due to  $\text{Supp}(f_x) = \{x^p z, y^l z\}$  this implies  $i = 0$  and we obtain  $y^m \in \text{Supp}(f)$  for some  $m \in \mathbb{N}_{\geq 3}$ . Now (1) implies  $y^{m-1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Thus we obtain our result by setting  $k = m - 1$ .

□

**Lemma 6.37.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$ ,  $s = 2$  and assume that there exist  $a, b, c \geq 1$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Then the following statements hold:*

- (1) If  $z^{r+1} \in \text{Supp}(f)$  for some  $r \in \mathbb{N}$ , then  $z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .
- (2)  $z^r x \notin \text{Supp}(f)$  for any  $r \in \mathbb{N}$ .

- (3) If  $z^{r+1}y \in \text{Supp}(f)$  for some  $r \in \mathbb{N}$ , then  $z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  or there exists a  $k \in \mathbb{N}_{\geq 2}$  such that  $z^k \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .

*Proof.* The result follows from Lemma 6.36 by applying the automorphism defined by  $x \mapsto x, y \mapsto z, z \mapsto y$ , which does not affect the multihomogeneity of  $J_f$ .  $\square$

Before we can prove that  $x^a y^{b-1} z^c \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $a, b, c \geq 1$  we need a lemma excluding a very special case.

**Lemma 6.38.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 > w_2 = w_3$ ,  $\text{ord}(f) \geq 3$ ,  $s = 2$  and assume that there exist  $a, b, c \geq 1$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Furthermore, we assume the following:*

- (1)  $a, b, c$  are chosen with maximal  $a$ , i.e. if  $x^{a'} y^{b'} z^{c'} \in \text{Supp}(f)$  with  $a', b', c' \in \mathbb{N}_{\geq 1}$ , then  $a \geq a'$ .
- (2)  $bw_2 + cw_3 = w_1$ .
- (3)  $x^{p+1} \in \text{Supp}(f)$  for some  $p \in \mathbb{N}_{\geq 2}$ .
- (4) No monomial of type  $x^i y^j z^k$  is contained in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  for any  $i, j, k \in \mathbb{N}_{\geq 1}$ .

Then  $a = p$  and

- (a)  $f$  is right-equivalent to  $x^{p+1} + y^q + z^r$  or  $x^{p+1} + y^q z + z^r$  or  $x^{p+1} + y^q + z^r y$  or  $x^{p+1} + y^q z + z^r y$  for some  $q, r \in \mathbb{N}$ , or
- (b)  $x^p z^c \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , if  $b = 1$  and (a) does not hold.

*Proof.* From now on we denote the weighted degree  $\deg_w(f)$  by  $d$ . The equality  $a = p$  follows as in the proof of Lemma 6.27. First we assume  $b > 1$ . Now we also know that  $x^{p-1} y^b z^c \in \text{Supp}(f_x)$  and  $x^p y^{b-1} z^c \in \text{Supp}(f_y)$ . Assumption (4) implies  $x^p y^{b-1} z^c \in \text{Supp}(g_1 f_x)$ . Since  $g_1 = g_1(y, z)$  we must have  $y^{b-1} z^c \in \text{Supp}(g_1)$ . Thus we obtain  $x^{p-1} y^{2b-1} z^{2c} \in \text{Supp}(g_1 f_x)$ . Assumption (4) then implies  $x^{p-1} y^{2b-1} z^{2c} \in \text{Supp}(f_y)$  and thus  $x^{p-1} y^{2b} z^{2c} \in \text{Supp}(f)$ . Continuing this argument we obtain

$$\{xy^{pb} z^{pc}, \dots, x^{p-1} y^{2b} z^{2c}, x^p y^b z^c\} \subseteq \text{Supp}(f).$$

Furthermore, we have  $y^{(p+1)b} z^{(p+1)c} \in \text{Supp}(f)$ . If this were not the case, then  $y^{pb} z^{(p+1)c} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  in order to cancel the corresponding monomial from a multiple of  $f_x$ . In this case, using Lemma 6.36 and Lemma 6.37, a simple computation shows that the direction vectors of  $f_x$  and  $h_2$  or  $h_3$  cannot be parallel.

Next we show that the only monomial of type  $y^u z^v \in \text{Supp}(f)$  with  $u, v \geq 2$  is  $y^{(p+1)b} z^{(p+1)c}$ . Assume the contrary. Then  $y^{u-1} z^v \in \text{Supp}(f_y)$ . If  $y^{u-1} z^v \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then there exist  $i, j \in \mathbb{N}$  with  $y^i z^j \in \text{Supp}(f_x)$  and  $y^i z^j$  divides  $y^{u-1} z^v$ . Since all monomials of  $f_x$  lie on a line we have  $i = pb$  and  $j = pc$ . In particular  $u > pb$  and  $v \geq pc$ . Now we have  $y^{u-1-pb} z^{v-pc} \cdot x^p$  is a multiple of a monomial of  $f_x$ . If this monomial were in  $h_2$  or  $h_3$  this contradicts our assumptions, since either  $u-1 > pb$  or  $v > pc$  as the monomial is different from  $y^{pb} z^{pc}$ . This implies  $y^{u-1-pb} z^{v-pc} \cdot x^p \in \text{Supp}(f_y)$  and hence  $y^{u-pb} z^{v-pc} \cdot x^p \in \text{Supp}(f)$ . Now the structure of  $f_x$  implies  $u - pb = b$  and  $v - pc = c$ , hence  $u = (p+1)b, v = (p+1)c$ , again a contradiction. So only monomials close to the  $y$ - or  $z$ -axis and  $y^{(p+1)b} z^{(p+1)c}$  are allowed in the support of  $f$ , if  $x$  does not

divide them.

Applying Lemma 6.36 and Lemma 6.37 we know, after a suitable coordinate change, that

$$f = x^{p+1} + y^q + z^r + \lambda_1 x^p y^b z^c + \dots + \lambda_p y^{(p+1)b} z^{(p+1)c}$$

or

$$f = x^{p+1} + y^q z + z^r + \lambda_1 x^p y^b z^c + \dots + \lambda_p y^{(p+1)b} z^{(p+1)c}$$

or

$$f = x^{p+1} + y^q + z^r y + \lambda_1 x^p y^b z^c + \dots + \lambda_p y^{(p+1)b} z^{(p+1)c}$$

or

$$f = x^{p+1} + y^q z + z^r y + \lambda_1 x^p y^b z^c + \dots + \lambda_p y^{(p+1)b} z^{(p+1)c}$$

for  $\lambda_i \in \mathbb{C}$ . Checking the proof so far we see that the only way of eliminating the monomials in  $f_y$  divisible by  $x$  is by multiplying  $f_x$  with  $\lambda y^{b-1} z^c$  for some  $\lambda \in \mathbb{C} \setminus \{0\}$ . Comparing coefficients in the equation

$$f_y - \lambda y^{b-1} z^c f_x = qy^{q-1}$$

or

$$f_y - \lambda y^{b-1} z^c f_x = qy^{q-1} z$$

yields  $\lambda_i = \binom{p+1}{i} \left(\frac{\lambda}{b}\right)^i$ . Using the binomial formula we get

$$f = \left(x + \frac{\lambda}{b} y^b z^c\right)^{p+1} + y^q + z^r,$$

or

$$f = \left(x + \frac{\lambda}{b} y^b z^c\right)^{p+1} + y^q z + z^r$$

or

$$f = \left(x + \frac{\lambda}{b} y^b z^c\right)^{p+1} + y^q + z^r y$$

or

$$f = \left(x + \frac{\lambda}{b} y^b z^c\right)^{p+1} + y^q z + z^r y.$$

After the coordinate change  $x \mapsto x - \frac{\lambda}{b} y^b z^c, y \mapsto y, z \mapsto z$  we obtain the claimed forms. Next we assume  $b = 1$ . In case  $x^p z^c \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then we can argue in the same way as above. Otherwise we obtain  $x^p z^c \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .  $\square$

The next step is to prove  $x^a y^{b-1} z^c, x^a y^b z^{c-1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .

**Lemma 6.39.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3, s = 2$  and assume that there exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Then the following hold:*

- (1)  $\{x^a y^{b-1} z^c, x^a y^b z^{c-1}\} \subseteq \text{Supp}(h_2) \cup \text{Supp}(h_3)$ ,
- (2)  $\{x^a y^{b-1} z^c, x^a y^b z^{c-1}\} \not\subseteq \text{Supp}(h_2)$  and
- (3)  $\{x^a y^{b-1} z^c, x^a y^b z^{c-1}\} \not\subseteq \text{Supp}(h_3)$ .

*Proof.* First we prove (1) for  $x^a y^{b-1} z^c$ . Similar to the previous proofs we assume that  $x^a y^{b-1} z^c \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for any  $a, b, c \geq 1$  with  $x^a y^b z^c \in \text{Supp}(f)$ . Due to  $w_1 > w_2, w_3$ , there must exist  $k, l \in \mathbb{N}$  with  $x^a y^k z^l \in \text{Supp}(f_x)$ , such that  $x^a y^k z^l$  divides  $x^a y^{b-1} z^c$ . We consider different cases for  $k, l$ :

- (1) Assume  $k, l \geq 1$ . Choose  $a$  maximal with the property that  $x^a y^b z^c \in \text{Supp}(f)$ . Then  $x^{a+1} y^k z^l \in \text{Supp}(f)$ , contradicting the maximality of  $a$ .
- (2) Assume  $k = 1$  and  $l = 0$ . This implies that  $x^{a+1} y \in \text{Supp}(f)$  and  $x^{a+1} \in \text{Supp}(f_y)$ . Now  $w_1 > w_2$  implies  $x^{a+1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . By Lemma 6.36 we have  $y^q \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $q \geq 2$  or  $y^q z \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $q \geq 1$  and by Lemma 6.37 we have  $z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $r \geq 2$  or  $z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $r \geq 1$ . Since  $f$  is quasi-homogeneous we obtain

$$w_3(r+1) = w_2 + w_3 r = w_1 a + w_2 b + w_3 c > w_2 + w_3 c,$$

which implies  $r > c$ , a fact we are going to need in the following proof. We have to compare direction vectors of  $f_x$  and  $h_2$  or  $h_3$  in these cases. A possible direction vector of  $f_x$  is

$$u = (a, 1, 0) - (a-1, b, c) = (1, 1-b, -c).$$

We consider four different cases:

- (a) Assume  $y^q, z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Having three different monomials from  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $h_2$  or  $h_3$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$ , which are not equal to  $(0, 0, 0)$ :

$$\begin{aligned} u_1 &= (0, q, 0) - (0, 0, r) = (0, q, -r) \\ u_2 &= (a+1, 0, 0) - (0, q, 0) = (a+1, -q, 0) \\ u_3 &= (a+1, 0, 0) - (0, 0, r) = (a+1, 0, -r) \end{aligned}$$

$u$  cannot be parallel to  $u_1$  since  $1 \neq 0$  in the first entry.  $u$  cannot be parallel to  $u_2$  since  $-c \neq 0$  in the last entry.  $u$  cannot be parallel to  $u_3$  if  $b > 1$  since  $1-b \neq 0$  in the second entry. If  $b = 1$ , then  $0 < 1 < a+1$ .  $u = \lambda u_3$  implies  $1 > \lambda > 0$ , contradicting  $-c > -r$ . In this case  $u$  is also not parallel to  $u_3$ .

- (b) Assume  $y^q, z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Having three different monomials from  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $h_2$  or  $h_3$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$ , which are not equal to  $(0, 0, 0)$ :

$$\begin{aligned} u_1 &= (0, q, 0) - (0, 1, r) = (0, q-1, -r) \\ u_2 &= (a+1, 0, 0) - (0, q, 0) = (a+1, -q, 0) \\ u_3 &= (a+1, 0, 0) - (0, 1, r) = (a+1, -1, -r) \end{aligned}$$

$u$  cannot be parallel to  $u_1$  since  $1 \neq 0$  in the first entry.  $u$  cannot be parallel to  $u_2$  since  $-c \neq 0$  in the last entry.  $u$  cannot be parallel to  $u_3$  if  $b \neq 2$ , since  $0 < 1 < a+1$  and  $u = \lambda u_3$  implies  $1 > \lambda > 0$  contradicting  $-c > -r$ . If  $b = 2$ , then  $1 \neq a+1$  contradicts parallelity.

- (c) Assume  $y^q z, z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Having three different monomials from  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $h_2$  or  $h_3$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$ , which are not equal to  $(0, 0, 0)$  :

$$\begin{aligned} u_1 &= (0, q, 1) - (0, 0, r) = (0, q, -r + 1) \\ u_2 &= (a + 1, 0, 0) - (0, q, 1) = (a + 1, -q, -1) \\ u_3 &= (a + 1, 0, 0) - (0, 0, r) = (a + 1, 0, -r) \end{aligned}$$

$u$  cannot be parallel to  $u_1$  since  $1 \neq 0$  in the first entry.  $u$  cannot be parallel to  $u_2$  since  $0 < 1 < a + 1$  and  $u = \lambda u_2$  implies  $1 > \lambda > 0$ , contradicting  $-1 \geq -c$ .  $u$  cannot be parallel to  $u_3$  if  $b \neq 1$  since  $b - 1 \neq 0$  in the second entry. If  $b = 1$ , then  $0 < 1 < a + 1$  and  $u = \lambda u_3$  implies  $1 > \lambda > 0$ , contradicting  $-c > -r$ . In this case  $u$  is also not parallel to  $u_3$ .

- (d) Assume  $y^q z, z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Having three different monomials from  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $h_2$  or  $h_3$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$ , which are not equal to  $(0, 0, 0)$  :

$$\begin{aligned} u_1 &= (0, q, 1) - (0, 1, r) = (0, q - 1, -r + 1) \\ u_2 &= (a + 1, 0, 0) - (0, q, 1) = (a + 1, -q, -1) \\ u_3 &= (a + 1, 0, 0) - (0, 1, r) = (a + 1, -1, -r) \end{aligned}$$

The vector  $u$  cannot be parallel to  $u_1$  since  $1 \neq 0$  in the first entry.  $u$  cannot be parallel to  $u_2$  since  $0 < 1 < a + 1$  and  $u = \lambda u_2$  implies  $1 > \lambda > 0$ , contradicting  $-1 \geq -c$ .  $u$  cannot be parallel to  $u_3$  if  $b > 2$  since  $0 < 1 < a + 1$  and  $u = \lambda u_3$  implies  $1 > \lambda > 0$ , contradicting  $-1 > b - 1$ . If  $b = 2$ , then  $0 < 1 < a + 1$  but the second entries are equal. In case  $b = 1$  the we have  $-1 \neq 0$  in the second entry. In all these cases  $u$  is also not parallel to  $u_3$ .

This implies that  $x^a y \notin \text{Supp}(f_x)$ .

- (3) Assume  $k = 0$  and  $l = 1$ . We perform the coordinate change  $x \mapsto x, y \mapsto z, z \mapsto y$  and use (2). Afterwards we use the same coordinate change to obtain  $x^a z \notin \text{Supp}(f_x)$ .
- (4) This proof works in the same way as in the proof of Lemma 6.27 (4).
- (5) This proof works in the same way as in the proof of Lemma 6.27 (5).
- (6) Assume  $k = l = 0$ . This implies  $w_2 b + w_3 c = w_1$  and  $x^{a+1} \in \text{Supp}(f)$ . Now Lemma 6.38 yields that after a suitable coordinate change no monomial of type  $x^a y^b z^c$  for  $a, b, c \in \mathbb{N}_{\geq 1}$  appears in  $\text{Supp}(f)$  or  $x^a z^c \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Let us consider the latter. By Lemma 6.36 we have  $y^q \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $q \geq 2$  or  $y^q z \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $q \geq 1$  and by Lemma 6.37 we have  $z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $r \geq 2$  or  $z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for some  $r \geq 1$ . We obtain four possibilities for direction vectors of  $h_2$  :

- (a) Assume  $y^q, z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Then we obtain

$$u_1 = (a, 0, c) - (0, q, 0) = (a, -q, c)$$



and

$$u_2 = (a, 0, c) - (0, 0, r) = (a, 0, c - r)$$

as direction vectors of  $h_2$ . The vector  $u_1$  is not parallel to  $u_2$ , since  $-q \neq 0$  in the second components.

(b) Assume  $y^q z, z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Then we obtain

$$u_1 = (a, 0, c) - (0, q, 1) = (a, -q, c - 1)$$

and

$$u_2 = (a, 0, c) - (0, 0, r) = (a, 0, c - r)$$

as direction vectors of  $h_2$ . The vector  $u_1$  is not parallel to  $u_2$ , since  $-q \neq 0$  in the second components.

(c) Assume  $y^q z, z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Then we obtain

$$u_1 = (a, 0, c) - (0, q, 0) = (a, -q, c)$$

and

$$u_2 = (a, 0, c) - (0, 1, r) = (a, -1, c - r)$$

as direction vectors of  $h_2$ . The vector  $u_1$  is not parallel to  $u_2$ , since the first components are equal, but  $-1 > -q$  in the second components.

(d) Assume  $y^q z, z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Then we obtain

$$u_1 = (a, 0, c) - (0, q, 1) = (a, -q, c - 1)$$

and

$$u_2 = (a, 0, c) - (0, 1, r) = (a, -1, c - r)$$

as direction vectors of  $h_2$ . The vector  $u_1$  is not parallel to  $u_2$ , since the first components are equal, but  $-1 > -q$  in the second components.

Since all cases are impossible, we obtain  $x^a z^c \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ .

Combining all cases, there must exist a monomial of type  $x^i y^j z^k$  in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  with  $i, j, k \in \mathbb{N}_{\geq 1}$ . Having this result, we can obtain  $x^a y^b z^{c-1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  very easy. We apply the coordinate change  $x \mapsto x, y \mapsto z, z \mapsto y$ . This does not change our setup. Applying the previous result we obtain  $x^a y^{c-1} z^b \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Applying the same coordinate change again, we obtain  $x^a y^b z^{c-1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . In order to prove (2) we assume  $x^a y^{b-1} z^c \in \text{Supp}(h_2)$ . If  $x^a y^b z^{c-1} \in \text{Supp}(h_2)$ , then a direction vector of  $h_2$  is  $(0, 1, -1)$ . In all cases of monomials close to the x-axis, we obtain that the first entry of the direction vector is not equal to 0, so they cannot be parallel. This implies  $x^a y^b z^{c-1} \in \text{Supp}(h_3)$ . (3) follows analogously.  $\square$

Now we are in the position to prove that no monomial of type  $x^a y^b z^c$  can be contained in  $\text{Supp}(f)$ , if we know that  $J_f$  is multihomogeneous and  $w_1 > w_2 = w_3$ .

**Proposition 6.40.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Then there do not exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^b z^c \in \text{Supp}(f)$ .*

*Proof.* Assume the contrary. Then by Lemma 6.39 there exist  $a, b, c \in \mathbb{N}_{\geq 1}$  with  $x^a y^{b-1} z^c \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Lemma 6.36 states that  $y^q \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  or  $y^q z \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for a certain  $q \in \mathbb{N}$  and Lemma 6.37 states that  $z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  or  $z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for certain  $r \in \mathbb{N}$ . In all cases we have  $q > b$  and  $r > c$ , since  $f$  is quasihomogeneous. Next to these four cases we have to consider all cases of monomials close to the  $x$ -axis. In the following we always have  $p > a$ , since  $w_1 > w_2, w_3$ .

- (1) Assume  $y^q, z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for  $r, q \geq 2$ . Having three different monomials in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $\text{Supp}(h_2)$  or  $\text{Supp}(h_3)$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$  :

$$\begin{aligned} u_1 &= (0, q, 0) - (0, 0, r) = (0, q, -r) \\ u_2 &= (a, b-1, c) - (0, q, 0) = (a, b-q-1, c) \\ u_3 &= (a, b-1, c) - (0, 0, r) = (a, b-1, c-r) \end{aligned}$$

- (a) If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a-1, b, c) - (p, 0, 0) = (a-1-p, b, c).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a-1-p < 0$  in the first entry.
- ii.  $u$  cannot be parallel to  $u_2$ , since  $a > a-1-p$  but the last entries are equal.
- iii.  $u$  cannot be parallel to  $u_3$ , since  $b > b-1 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a-1-p$ .

- (b) If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a-1, b, c) - (p, 1, 0) = (a-1-p, b-1, c).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a-1-p < 0$  in the first entry.
- ii.  $u$  cannot be parallel to  $u_2$ , since  $a > a-1-p$  but the last entries are equal.
- iii.  $u$  cannot be parallel to  $u_3$ , since  $a > 0 > a-1-p$ , but the second entries are equal.

- (c) If  $x^{p+1}z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a-1, b, c) - (p, 0, 1) = (a-1-p, b, c-1).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a-1-p < 0$  in the first entry.
- ii.  $u$  cannot be parallel to  $u_2$ , since  $c > c-1 \geq 0$  and  $u = \lambda u_2$  imply  $1 > \lambda > 0$  contradicting  $a > 0 > a-1-p$ .
- iii.  $u$  cannot be parallel to  $u_3$ , since  $b \geq b-1 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a-1-p$ .

- (2) Assume  $y^q z, z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for  $q \geq 1$  and  $r \geq 2$ . Having three different monomials in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $\text{Supp}(h_2)$  or  $\text{Supp}(h_3)$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$  :

$$\begin{aligned} u_1 &= (0, q, 1) - (0, 0, r) = (0, q, 1-r) \\ u_2 &= (a, b-1, c) - (0, q, 1) = (a, b-q-1, c-1) \\ u_3 &= (a, b-1, c) - (0, 0, r) = (a, b-1, c-r) \end{aligned}$$

(a) If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$  in the first entry.
- ii.  $u$  cannot be parallel to  $u_2$ , since  $c > c - 1 \geq 0$  and  $u = \lambda u_2$  imply  $\lambda > 1$  contradicting  $b > 0 > b - q - 1$ .
- iii.  $u$  cannot be parallel to  $u_3$ , since  $b > b - 1 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

(b) If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$  in the first entry.
- ii.  $u$  cannot be parallel to  $u_2$ , since  $c > c - 1 \geq 0$  and  $u = \lambda u_2$  imply  $\lambda > 1$  contradicting  $b > 0 > b - q - 1$ .
- iii.  $u$  cannot be parallel to  $u_3$ , since  $a > 0 > a - 1 - p$ , but the second entries are equal.

(c) If  $x^{p+1}z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$ .
- ii.  $u$  cannot be parallel to  $u_2$ , since  $a > 0 > a - 1 - p$ , but the last entries are equal.
- iii.  $u$  cannot be parallel to  $u_3$ , since  $b > b - 1 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

(3) Assume  $y^q, z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for  $q \geq 2$  and  $r \geq 1$ . Having three different monomials in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $\text{Supp}(h_2)$  or  $\text{Supp}(h_3)$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$  :

$$\begin{aligned} u_1 &= (0, q, 0) - (0, 1, r) = (0, q - 1, -r) \\ u_2 &= (a, b - 1, c) - (0, q, 0) = (a, b - q - 1, c) \\ u_3 &= (a, b - 1, c) - (0, 1, r) = (a, b - 2, c - r) \end{aligned}$$

$u_3$  has to be replaced in the following proofs if  $b = 1$ . We need to take the monomial  $x^a y z^{c-1}$  into account. Assuming that  $x^a z^c, y z^r \in \text{Supp}(h_2)$  we can assume by Lemma 6.39  $x^a y z^{c-1}, y^q \in \text{Supp}(h_3)$ . The corresponding direction vector is

$$u_4 = (a, 1 - q, c - 1).$$

(a) If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$ .
- ii.  $u$  cannot be parallel to  $u_2$ , since  $b > b - q - 1$  but the last entries are equal.
- iii.  $u$  cannot be parallel to  $u_3$ , if  $b > 1$ , since  $b > b - 2 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

iv.  $u$  cannot be parallel to  $u_4$ , since  $c > c - 1 \geq 0$  and  $u = \lambda u_4$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

(b) If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c).$$

i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$ .

ii.  $u$  cannot be parallel to  $u_2$ , since  $b - 1 > b - q - 1$  but the last entries are equal.

iii.  $u$  cannot be parallel to  $u_3$ , if  $b > 1$ , since  $b - 1 > b - 2 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

iv.  $u$  cannot be parallel to  $u_4$  if  $b = 1$ , since  $c > c - 1 \geq 0$  and  $u = \lambda u_4$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

(c) If  $x^{p+1}z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1).$$

i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$ .

ii.  $u$  cannot be parallel to  $u_2$ , since  $c > c - 1 \geq 0$  and  $u = \lambda u_2$  imply  $1 > \lambda > 0$  contradicting  $a > 0 > a - 1 - p$ .

iii.  $u$  cannot be parallel to  $u_3$  if  $b > 1$ , since  $b > b - 2 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

iv.  $u$  cannot be parallel to  $u_4$  if  $b = 1$ , since  $a > a - 1 - p$  and the last entries are equal.

(4) Assume  $y^q z, z^r y \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$  for  $q \geq 1$  and  $r \geq 1$ . Having three different monomials in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$  we know that two of them must be in  $\text{Supp}(h_2)$  or  $\text{Supp}(h_3)$ , so we obtain three possible direction vectors of  $h_2$  or  $h_3$ :

$$\begin{aligned} u_1 &= (0, q, 1) - (0, 1, r) = (0, q - 1, 1 - r) \\ u_2 &= (a, b - 1, c) - (0, q, 1) = (a, b - q - 1, c - 1) \\ u_3 &= (a, b - 1, c) - (0, 1, r) = (a, b - 2, c - r) \end{aligned}$$

$u_3$  has to be replaced in the following proofs if  $b = 1$ . We need to take the monomial  $x^a y z^{c-1}$  into account. Assuming that  $x^a z^c, y z^r \in \text{Supp}(h_2)$  we can assume by Lemma 6.39  $x^a y z^{c-1}, y^q z \in \text{Supp}(h_3)$ . The corresponding direction vector is

$$u_4 = (a, 1 - q, c - 2).$$

(a) If  $x^{p+1} \in \text{Supp}(f)$ , then  $x^p \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 0, 0) = (a - 1 - p, b, c).$$

i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$ .

ii.  $u$  cannot be parallel to  $u_2$ , since  $c > c - 1 \geq 0$  and  $u = \lambda u_2$  imply  $\lambda > 1$  contradicting  $b > 0 > b - q - 1$ .

iii.  $u$  cannot be parallel to  $u_3$  if  $b > 1$ , since  $b > b - 2 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

- iv.  $u$  cannot be parallel to  $u_4$  if  $b = 1$  and  $c > 1$ , since  $c > c - 2 \geq 0$  and  $u = \lambda u_4$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .
- v.  $u$  cannot be parallel to  $u_4$  if  $b = c = 1$ , since  $u = \lambda u_4$  implies  $\lambda = -1$ . Thus  $p = 2a - 1 \geq 2$  and  $q = 2$ . Then  $a \geq 2$  and

$$3w_1 \leq 4w_1 \leq 2aw_1 = 2w_2 + w_3 < 2w_1 + w_2$$

implies  $w_2 > w_1$ , which is a contradiction.

- (b) If  $x^{p+1}y \in \text{Supp}(f)$ , then  $x^p y \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 1, 0) = (a - 1 - p, b - 1, c).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$ .
- ii.  $u$  cannot be parallel to  $u_2$ , since  $c > c - 1 \geq 0$  and  $u = \lambda u_2$  imply  $\lambda > 1$  contradicting  $b - 1 \geq 0 > b - q - 1$ .
- iii.  $u$  cannot be parallel to  $u_3$  if  $b > 1$ , since  $b - 1 > b - 2 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .
- iv.  $u$  and  $u_4$  cannot be parallel if  $b = 1$  and  $q > 1$ , since  $1 - q < 0$ .
- v.  $u$  and  $u_4$  cannot be parallel if  $q = 1$ , since  $y^2 z, x^a y z^c \in \text{Supp}(f)$  implies

$$2w_2 + w_3 = aw_1 + w_2 + cw_3 \geq w_1 + w_2 + w_3 > 2w_2 + w_3,$$

which is a contradiction.

- (c) If  $x^{p+1}z \in \text{Supp}(f)$ , then  $x^p z \in \text{Supp}(f_x)$ . A direction vector of  $f_x$  is parallel to

$$u = (a - 1, b, c) - (p, 0, 1) = (a - 1 - p, b, c - 1).$$

- i.  $u$  cannot be parallel to  $u_1$ , since  $a - 1 - p < 0$ .
- ii.  $u$  cannot be parallel to  $u_2$ , since  $a > a - 1 - p$  but the last entries are equal.
- iii.  $u$  cannot be parallel to  $u_3$  if  $b > 1$ , since  $b > b - 2 \geq 0$  and  $u = \lambda u_3$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .
- iv.  $u$  cannot be parallel to  $u_4$  if  $b = 1$  and  $q = 1$ , since  $1 \neq 0$  in the second entry.
- v.  $u$  cannot be parallel to  $u_4$  if  $b = c = 1$  and  $q > 1$ , since  $-1 \neq 0$  in the third entry.
- vi.  $u$  cannot be parallel to  $u_4$  if  $c > 1 = b$ , since  $c - 1 > c - 2 \geq 0$  and  $u = \lambda u_4$  imply  $\lambda > 1$  contradicting  $a > 0 > a - 1 - p$ .

□

From now on we can assume that  $f$  has no monomials of type  $x^a y^b z^c$  for any  $a, b, c \in \mathbb{N}_{\geq 1}$ . In the next steps we have to consider all possibilities for  $f_x$ . The first possibility is that  $f_x$  is a monomial.

In the following proof we will make explicit use of the additional weights of the Jacobian ideal  $J_f$ . We denote them by  $v = (v_1, v_2, v_3)$ . The main technique we are going to use is the fact that we can compute linear combinations of  $v$  and  $w$  to obtain new weight vectors, which by abuse of notation will be denoted again by  $v$ . Since switching the  $y$  and  $z$  variable does not affect the multihomogeneity of  $J_f$ , we can always assume that  $v_2 \geq v_3$ .

This technique is being used in the next proof.

**Lemma 6.41.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s \geq 2$ . Let  $f_x$  be a monomial, then the following hold:*

- (1) *If  $f_x = x^p$  for some  $p \in \mathbb{N}$ , then  $f$  is of Sebastiani–Thom type.*
- (2) *If  $f_x = x^p y$  for some  $p \in \mathbb{N}$ , then  $f$  is of Sebastiani–Thom type.*
- (3) *If  $f_x = x^p z$  for some  $p \in \mathbb{N}$ , then  $f$  is of Sebastiani–Thom type.*

*Proof.*

- (1) By integration we obtain  $f(x, y, z) = \int x^p dx = \frac{1}{p+1} x^{p+1} + h(y, z)$  for some polynomial  $h$ .
- (2) With  $\text{ord}(f) \geq 3$  the only possible monomials close to the  $y$ -axis are  $y^{r+1}$  or  $y^r z$  for some  $r \in \mathbb{N}$  and the only possible monomials close to the  $z$ -axis are  $z^{r+1}$  or  $z^r y$ . If  $\{x^p y, y^{r+1}, z^r y\} \subseteq \text{Supp}(f)$  or  $\{x^p y, y^r z, z^r y\} \subseteq \text{Supp}(f)$ , then we know that there exists an  $i \in \mathbb{N}_{\geq 1}$  with  $z^i \in \text{Supp}(f)$ , otherwise  $f$  would not define an isolated hypersurface singularity, since  $y$  would divide all monomials of  $f$ . With  $f$  being quasihomogeneous we know that  $i = r + 1$ . So we can always assume
  - (a)  $\{x^p y, y^{r+1}, z^{r+1}\} \subseteq \text{Supp}(f)$  or
  - (b)  $\{x^p y, y^r z, z^{r+1}\} \subseteq \text{Supp}(f)$ .

The next step is to show that no monomial of type  $y^i z^j$  is contained in  $\text{Supp}(f)$  for any  $i, j \in \mathbb{N}_{\geq 1}$ . Assume the contrary. Using that  $\text{Supp}(f_x) = \{x^p y\}$ , we obtain  $f_y = h_2 + a_1 h_3, f_z = a_2 h_2 + h_3$  for  $a_1, a_2 \in \mathbb{C}$  and  $\{x^p, y^{i-1} z^j, y^i z^{j-1}, z^r\} \subseteq \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . The idea is to use the second weight vector  $v$  of  $J_f$ . Since we can perform linear operations on  $v$  using  $w$  and since switching  $y$  with  $z$  does not change the multihomogeneity, we can assume that  $v = (0, v_2, v_3)$  with  $v_2 \geq v_3$ . In case  $v_2 = v_3$ , we modify  $v$  again and obtain  $v = (1, 0, 0)$ , since  $w_1 > w_2 = w_3$ . First we assume  $v_2 > v_3$ . We consider both cases (a) and (b) assuming that a monomial of type  $y^i z^j$  with  $i, j \geq 1$  is contained in  $\text{Supp}(f)$  :

- (a) Assume  $\{x^p, y^r, y^{i-1} z^j\} \subseteq \text{Supp}(f_y)$ . Obviously

$$0 = \deg_v(x^p) < \deg_v(y^r).$$

The equality  $w_2 = w_3$  implies  $r = i - 1 + j$ . This yields

$$v_2 r = v_2(i - 1) + v_2 j > v_2(i - 1) + v_3 j.$$

In case  $\deg_v(y^{i-1} z^j) \neq 0$  we obtain

$$\deg_v(x^p) < \deg_v(y^{i-1} z^j) < \deg_v(y^r),$$

which is a sequence of three elements appearing in  $\text{Supp}(f_y)$  with different  $v$ -degrees. This is not possible, since  $f_y$  is a linear combination of at most two weighted homogeneous elements. For the case  $\deg_v(y^{i-1} z^j) = 0$  we have to consider two subcases:

i. If  $i \geq 2$  we have

$$0 = (i - 1)v_2 + v_3j > (i - 1 + j)v_3 = rv_3 = \deg_v(z^r).$$

In particular,  $v_3 < 0$ . Since  $\{x^p, y^r, z^r\} \subseteq \text{Supp}(h_2) \cup \text{Supp}(h_3)$  we have three elements with

$$\deg_v(z^r) < 0 = \deg_v(x^p) < \deg_v(y^r)$$

which is impossible. Hence a monomial of type  $y^i z^j$  with  $i \geq 2, j \geq 1$  cannot be contained in  $\text{Supp}(f)$ .

ii. If  $i = 1$ , then  $v_3 = 0$ , since  $j \geq 1$ . In this case we can assume without loss of generality  $\{x^p, z^r\} \subseteq \text{Supp}(h_2)$ , since  $\deg_v(x^p) = \deg_v(z^r) = 0 \neq \deg_v(y^r)$ . This means that  $y^r \in \text{Supp}(h_3)$ . Since  $z^r \in \text{Supp}(f_z)$  and  $x^p \notin \text{Supp}(f_z)$ , we must have  $x^p \in \text{Supp}(h_3)$  as well, since otherwise it could not be canceled from  $f_z$ . In this case  $h_3$  is not homogeneous with respect to  $v$ , which contradicts the assumption that  $h_3$  is multihomogeneous.

(b) Assume  $\{x^p, y^{r-1}z, y^r, z^r\} \subseteq \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Obviously

$$0 = \deg_v(x^p) < \deg_v(y^r).$$

We have to consider two subcases:

i. If  $v_3 \neq 0$  we have

$$0 \neq \deg_v(z^r) < \deg_v(y^r).$$

We obtain a sequence of three elements appearing in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$ , with different  $v$ -degrees. This contradicts the multihomogeneity of  $h_2$  and  $h_3$ .

ii. If  $v_3 = 0$ , then we can assume without loss of generality  $\{x^p, z^r\} \subseteq \text{Supp}(h_2)$ , since  $\deg_v(x^p) = \deg_v(z^r) = 0 \neq \deg_v(y^r)$ . This implies  $y^{r-1}z, y^r \in \text{Supp}(h_3)$ . With  $r \geq 2$  we obtain

$$0 < \deg_v(y^{r-1}z) < \deg_v(y^r).$$

This yields a contradiction to  $h_3$  being weighted homogeneous with respect to  $v$ .

From both cases we see that in both the cases no monomial of type  $y^i z^j$  with  $i, j \geq 1$  can exist. Thus  $\{x^p y, y^{r+1}, z^{r+1}\} = \text{Supp}(f)$  and  $f$  is of Sebastiani–Thom type, if  $v_2 > v_3$ . To finish the proof we have to consider the case  $v = (1, 0, 0)$  in the cases (a) or (b):

(a) Assume  $\{x^p, y^r, y^{i-1}z^j\} \subseteq \text{Supp}(f_y)$ . With out loss of generality we can assume  $\{y^r, y^{i-1}z^j\} \subseteq \text{Supp}(h_2)$ , since  $\deg_v(y^r) = \deg_v(y^{i-1}z^j) = 0 \neq \deg_v(x^p)$ . This implies  $x^p \in \text{Supp}(h_3)$ . Since  $x^p$  does not appear in  $\text{Supp}(f_z)$ , we must have  $x^p \in \text{Supp}(h_2)$ , which contradicts the multihomogeneity of  $h_2$ .

(b) Assume  $\{x^p, y^{r-1}z, y^r, z^r\} \subseteq \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . With out loss of generality we can assume  $\{y^r, z^r\} \subseteq \text{Supp}(h_2)$ , since  $\deg_v(y^r) = \deg_v(z^r) = 0 \neq \deg_v(x^p)$ . This implies  $x^p \in \text{Supp}(h_3)$ . Since  $x^p$  does not appear in  $\text{Supp}(f_z)$ , we must have  $x^p \in \text{Supp}(h_2)$ , which contradicts the multihomogeneity of  $h_2$ .

This shows that the case  $v = (1, 0, 0)$  is not possible.

- (3) The result follows immediately from (2) if we apply the coordinate change  $x \mapsto x, y \mapsto z$  and  $z \mapsto y$ .

□

In the next steps we assume that the monomials of  $f_x$  lie on a line in the  $x$ - $y$  or  $x$ - $z$  plane. First we consider the case where  $f_x$  lies in the  $x$ - $y$  plane.

**Lemma 6.42.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $|\text{Supp}(f_x)| \geq 2$  and the monomials of  $f_x$  lie in the  $x$ - $y$  plane. Then  $f$  is of Sebastiani–Thom type.*

*Proof.* Due to our assumption we have either  $\{x^{p+1}, x^a y^b\} \subseteq \text{Supp}(f)$  for  $a, b \geq 1$  or  $\{x^{p+1} y, x^a y^b\} \subseteq \text{Supp}(f)$  for  $a \geq 1$  and  $b > 1$ . Assume we are in the first case, then a possible direction vector of  $f_x$  is

$$u = (p - a + 1, -b, 0).$$

We have to consider two cases:

- (1) Assume that a monomial of type  $x^l y^m$  with  $l, m \in \mathbb{N}_{\geq 1}$  is contained in  $h_2$ , then  $h_2 = h_2(x, y)$  since its direction vector has to be parallel to a direction vector of  $f_x$ . We assume this without loss of generality, since the case of  $h_3 = h_3(x, y)$  works analogously. Next we assume that there exists a monomial of type  $y^i z^j \in \text{Supp}(f)$  with  $i, j \in \mathbb{N}_{\geq 1}$  and that  $j$  is chosen maximal, i.e. if  $y^{i'} z^{j'} \in \text{Supp}(f)$  with  $i', j' \in \mathbb{N}_{\geq 1}$ , then  $j \geq j'$ . This setup yields  $y^{i-1} z^j \in \text{Supp}(f_y)$ . We have to consider three cases.

- (a) Assume  $y^{i-1} z^j \in \text{Supp}(h_3)$ . We know that  $y^i z^{j-1} \in \text{Supp}(f_z)$ .

- i. Assume  $y^i z^{j-1} \in \text{Supp}(h_3)$ , then a possible direction vector of  $h_3$  is

$$u_1 = (0, 1, -1),$$

which is obviously not parallel to  $u$ .

- ii. Assume  $y^i z^{j-1} \in \text{Supp}(h_2)$ . Then  $h_2 = h_2(x, y)$  implies  $j = 1$ . Since  $j$  is maximal and  $f_x = f_x(x, y)$ , and we know immediately that the only possible monomial close to the  $z$ -axis is  $z^{r+1}$  for some  $r \in \mathbb{N}_{\geq 2}$ . The quasihomogeneity of  $f_x$  implies  $r + 1 = i + 1$ , thus  $r = i$ . The structure of  $f_x$  and  $h_2$  also imply  $z^r \in \text{Supp}(h_3)$ . In this case a possible direction vector of  $h_3$  is

$$u_2 = (0, 1, -1),$$

which is not parallel to  $u$ .

- iii. Assume  $y^i z^{j-1} \in \text{Supp}(g_2 f_x)$ . Then there exists  $y^k \in \text{Supp}(f_x)$  for some  $k \in \mathbb{N}_{\geq 2}$ . This implies  $x^p y^{i-k} z^{j-1} \in \text{Supp}(g_2 f_x)$ . Now we have to consider three cases for this monomial:

- A. If  $x^p y^{i-k} z^{j-1} \in \text{Supp}(h_2)$ , then  $j = 1$  and we know immediately that the only possible monomial close to the  $z$ -axis is  $z^{r+1}$  for some  $r \in \mathbb{N}_{\geq 2}$ . Hence we are in case (ii).



B. If  $x^p y^{i-k} z^{j-1} \in \text{Supp}(h_3)$ , then a possible direction vector of  $h_3$  is

$$u_3 = (p, 1 - k, -1),$$

which is obviously not parallel to  $u$ .

C. If  $x^p y^{i-k} z^{j-1} \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then  $x^p y^{i-k} z^{j-1} \in \text{Supp}(f_z)$  and in particular  $x^p y^{i-k} z^j \in \text{Supp}(f)$ , which is impossible, since  $p \geq 2$  implies  $x^{p-1} y^{i-k} z^j \in f_x$  and  $j \geq 1$  implies  $f_x = f_x(x, y, z)$ .

(b) If  $y^{i-1} z^j \notin \text{Supp}(h_3)$ , then  $h_2 = h_2(x, y)$  implies the existence of a  $k \in \mathbb{N}_{\geq 2}$  with  $y^k \in \text{Supp}(f_x)$  and this implies  $x^p y^{i-1-k} z^j \in \text{Supp}(g_1 f_x)$ . If  $x^p y^{i-1-k} z^j \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then  $x^p y^{i-1-k} z^j \in \text{Supp}(f_y)$  and in particular  $x^p y^{i-k} z^j \in \text{Supp}(f)$ , which contradicts  $f_x = f_x(x, y)$ . So the only remaining possibility is  $x^p y^{i-1-k} z^j \in \text{Supp}(h_3)$ , since  $h_2 = h_2(x, y)$ . Let us assume this from now on. As before we know that  $y^i z^{j-1} \in \text{Supp}(f_z)$ . We have to consider three different cases:

i. Assume  $y^i z^{j-1} \in \text{Supp}(h_3)$ . Then a possible direction vector of  $h_3$  is given by

$$u_1 = (p, -1 - k, -1),$$

which is not parallel to  $u$ .

ii. Assume  $y^i z^{j-1} \in \text{Supp}(h_2)$ . Then  $j = 1$  and, since  $j$  is maximal, we have  $z^{r+1} \in \text{Supp}(f)$  for some  $r \in \mathbb{N}_{\geq 2}$ . The structure of  $f_x$  and  $h_2$  implies  $z^r \in \text{Supp}(h_3)$ . In this case a possible direction vector of  $h_3$  is

$$u_2 = (p, i - 1 - k, 1 - r),$$

which is not parallel to  $u$ , since  $r \geq 2$ . So this case is impossible.

iii. Assume  $y^i z^{j-1} \in \text{Supp}(g_2 f_x)$ . This implies  $x^p y^{i-k} z^{j-1} \in \text{Supp}(g_2 f_x)$ . Now we have to consider three cases for this monomial:

A. If  $x^p y^{i-k} z^{j-1} \in \text{Supp}(h_2)$ , then  $j = 1$  and we know immediately that the only possible monomial close to the  $z$ -axis is  $z^{r+1}$  for some  $r \in \mathbb{N}_{\geq 2}$ . The structure of  $f_x$  and  $h_2$  then implies  $z^r \in \text{Supp}(h_3)$ . In this case a possible direction vector of  $h_3$  is

$$u_3 = (p, i - 1 - k, 1 - r).$$

This vector is not parallel to  $u$ , since  $r \geq 2$  implies  $1 - r \neq 0$ .

B. If  $x^p y^{i-k} z^{j-1} \in \text{Supp}(h_3)$ , then a possible direction vector of  $h_3$  is

$$u_4 = (0, 1, -1),$$

which is obviously not parallel to  $u$ .

C. If  $x^p y^{i-k} z^{j-1} \notin \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , then  $x^p y^{i-k} z^{j-1} \in \text{Supp}(f_z)$  and in particular  $x^p y^{i-k} z^j \in \text{Supp}(f)$ , which is impossible, since  $p \geq 2$  implies  $x^{p-1} y^{i-k} z^j \in f_x$  and  $j \geq 1$  implies  $f_x = f_x(x, y, z)$ .

All these cases imply that no monomial of type  $y^i z^j$  for  $i, j \in \mathbb{N}_{\geq 1}$  can be contained in  $\text{Supp}(f)$ . Thus  $f$  is of Sebastiani–Thom type.

(2) Assume no monomial of type  $x^l y^m$ ,  $l, m \in \mathbb{N}_{\geq 1}$ , is contained in  $\text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Consider a monomial of this type in  $\text{Supp}(f)$  with minimal  $m$ . If  $m \geq$

2, then  $x^l y^{m-1}$  is in  $\text{Supp}(g_1 f_x)$ , hence we must have  $l = p$ , since only  $x^p$  can divide  $x^l y^{m-1}$  due to the minimality of  $m$ . Due to our assumption we must have  $x^{p-1} y^{2m-1} \in \text{Supp}(f_y)$ , hence  $x^{p-1} y^{2m} \in \text{Supp}(f)$ . Iterating this process we see that  $f_y = \lambda y^{m-1} f_x + h(y, z)$  for some  $\lambda \in \mathbb{C} \setminus \{0\}$  and  $h \in \mathbb{C}[y, z]$  a polynomial. Using the same argument as in the proof of Lemma 6.26, we get that  $f$  is right-equivalent to  $x^{p+1} + y^{(p+1)m} + r(y, z)$  for some polynomial  $r \in \mathbb{C}[y, z]$  and this is of Sebastiani–Thom type, if  $m \geq 2$ . The case  $m = 1$  remains. In this case we have  $x^l \in \text{Supp}(f_y)$ . The structure of  $f_y$  and  $w_1 > w_2 = w_3$  implies  $x^l \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Without loss of generality we can assume  $x^l \in \text{Supp}(h_2)$ , which implies  $h_2 = h_2(x, y)$ , hence we can argue as in (1).

In the case  $\{x^{p+1}y, x^a y^b\} \subseteq \text{Supp}(f)$  for  $a \geq 1$  and  $b > 1$  we see that  $x^{p+1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Due to this we can assume that the monomials of  $h_2$  lie in the x-y plane and we can argue as in (1).  $\square$

Next we consider the case where  $f_x$  lies in the x-z plane.

**Lemma 6.43.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $|\text{Supp}(f_x)| \geq 2$  and the monomials of  $f_x$  lie in the x-z plane. Then  $f$  is of Sebastiani–Thom type.*

*Proof.* This result follows from Lemma 6.42 by applying the coordinate change  $x \mapsto x, y \mapsto z$  and  $z \mapsto y$ .  $\square$

Finally we consider the case where the monomials of  $f_x$  lie in the x-y and in the x-z plane. In the following proof we will again make explicit use of the additional weights of the Jacobian ideal  $J_f$ . We denote them by  $v = (v_1, v_2, v_3)$ . Since we can perform linear combinations of weights to obtain new ones, we assume from now on that  $v_1 = 0$  and  $v_2 \geq v_3$ . Switching the  $y$  and  $z$  coordinate does not affect the quasihomogeneity of  $f$  or the multihomogeneity of  $J_f$ .

**Lemma 6.44.** *Let  $f \in \mathbb{C}[x, y, z]$  be a polynomial with unique weights  $(w_1, w_2, w_3)$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $|\text{Supp}(f_x)| \geq 2$  and that the monomials of  $f_x$  lie in the x-y and x-z plane. Then  $f$  is of Sebastiani–Thom type.*

*Proof.* In the case where the monomials of  $f$  lie in the x-y plane as well as in the x-z plane we know that  $\text{Supp}(f_x) = \{x^p y, x^i z^j\}$  or  $\text{Supp}(f_x) = \{x^p z, x^i y^j\}$  for  $i, j \in \mathbb{N}$  with  $i + j \geq 2$ .

- (1) Assume  $\text{Supp}(f_x) = \{x^p y, x^i z^j\}$ . Then  $x^{p+1} \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ , since no monomial of  $f_x$  divides  $x^{p+1}$ . Using that either  $y^{q+1}$  or  $y^q z$  are in  $\text{Supp}(f)$  for some  $q \in \mathbb{N}$  and using that either  $z^{r+1}$  or  $z^r y$  are in  $\text{Supp}(f)$  for some  $r \in \mathbb{N}$ , we can apply the previous argument to see that  $y^q, z^r \in \text{Supp}(h_2) \cup \text{Supp}(h_3)$ . Assuming that the monomials of  $f_x$  do not lie completely in the x-y plane, we can assume  $j \in \mathbb{N}_{\geq 1}$ . In the case  $p \neq i$  checking possible direction vectors shows that  $h_2$  or  $h_3$  cannot be parallel to  $f_x$ . So let us consider the case  $p = i$  in more detail, since this case is non-trivial. In this case a direction vector of  $f_x$  is given by  $u = (0, 1, -1)$ . The weights of  $f$  imply that  $\{x^p y, x^p z\} = \text{Supp}(f_x)$ . Checking the possible direction vectors we see that the only non-trivial case is  $q = r, x^p \in \text{Supp}(h_2)$  and  $y^r, z^r \in \text{Supp}(h_3)$ , but  $y^r, z^r \notin \text{Supp}(h_2)$ . If  $v_2 > v_3$ , then  $y^r$  and  $z^r$

do not have the same multidegree and thus cannot be monomials of  $h_3$ , hence  $v_2 = v_3$ . Using our information so far we know that  $f = \lambda x^p y + \mu x^p z + h(y, z)$  with  $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ . Then the coordinate change  $\varphi$  defined by  $x \mapsto x, y \mapsto \frac{y - \mu z}{\lambda}, z \mapsto z$  is multihomogeneous and keeps the quasihomogeneity of  $f$  and the multihomogeneity of  $J_f$ . We obtain  $\varphi(f) = x^p y + \varphi(h(y, z))$  and we can apply Lemma 6.41 to obtain that  $f$  is of Thom Sebastiani type.

- (2) Assume  $\text{Supp}(f_x) = \{x^p z, x^i y^j\}$ . Apply the coordinate change  $x \mapsto x, y \mapsto z$  and  $z \mapsto y$  and we can argue using (1).

□

Combining the previous lemmas we obtain the following result:

**Proposition 6.45.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$  satisfying  $w_1 > w_2 = w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Then  $f$  is of Sebastiani–Thom type.*

## 6.5 The case $w_1 = w_2 > w_3$

In this section we will make explicit use of the additional weight of the Jacobian ideal. We denote it by  $v = (v_1, v_2, v_3)$ . Since we can perform linear combinations of weight vectors  $w$  and  $v$  to obtain new ones, we assume from now on that  $v_3 = 0$ . In the previous sections we made use of the fact that  $f_x = h_1$ . In this case we do not get this result immediately. Proposition 3.7 yields only the following result:

**Lemma 6.46.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $w = (w_1, w_2, w_3)$ , where  $w_1 = w_2 > w_3$  and assume  $s = 2$ . Then there exist  $g_1, g_2 \in \mathbb{C}[z]$  and  $\alpha, \beta \in \mathbb{C}$  such that:*

- (1)  $f_x = h_1 + \alpha h_2$  and  $f_y = \beta h_1 + h_2$ , and
- (2)  $f_z = g_1 h_1 + g_2 h_2 + h_3$ .

In our setup we can prove now that we can assume  $h_1 = f_x$  and  $h_2 = f_y$ .

**Lemma 6.47.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . If there exists a second weight vector  $v$  for  $J_f$  with  $v_1 \neq v_2$ , then  $f_x$  and  $f_y$  are multihomogeneous with respect to the weights  $w$  and  $v$ . In particular, we can choose  $h_1 = f_x$  and  $h_2 = f_y$ .*

*Proof.* We keep the notation of Lemma 6.46. The case  $\alpha = \beta = 0$  is trivial. So we assume  $\alpha \neq 0$  or  $\beta \neq 0$ . In case  $\text{Supp}(h_1) \cap \text{Supp}(h_2) \neq \emptyset$ , we know  $\deg_v(h_1) = \deg_v(h_2)$ . Then  $f_x$  and  $f_y$  form part of a system of multihomogeneous minimal generators of  $J_f$  and we can assume  $f_x = h_1, f_y = h_2$ . Thus we can assume  $\text{Supp}(h_1) \cap \text{Supp}(h_2) = \emptyset$ . We bring this to a contradiction by considering three different cases for the values of  $\alpha$  and  $\beta$ . We can assume without loss of generality  $v_1 > v_2$ , since switching the  $x$  and  $y$  variable does not affect the multihomogeneity of  $J_f$ .

- (1) First we consider the case  $\alpha \neq 0 \neq \beta$ . Then  $f_x$  and  $f_y$  contain the same monomials, as the monomials cannot cancel each other. We need to consider two cases to show that no monomial close to the  $x$ -axis can be contained in  $\text{Supp}(f)$  in order to construct a contradiction.

- (a) Let us assume  $x^i y^j \in \text{Supp}(f)$  for  $i, j \in \mathbb{N}$  and  $i + j \geq 3$ . Deriving yields  $x^{i-1} y^j \in \text{Supp}(f_x) = \text{Supp}(f_y)$ . Integrating yields  $x^{i-1} y^{j+1} \in \text{Supp}(f)$ . We can iterate this process and obtain

$$x^{i+j-1}, x^{i+j-2}y, \dots, xy^{i+j-2} \in \text{Supp}(f_x)$$

Now we have

$$\deg_v(x^{i+j-1}) > \deg_v(x^{i+j-2}y) > \dots > \deg_v(xy^{i+j-2}),$$

which is impossible since this implies that  $f_x$  has to contain at least 3 monomials of different multi-degrees.

- (b) Let us assume  $x^p z \in \text{Supp}(f)$  for some  $p \in \mathbb{N}$ . In case  $p \geq 3$  deriving and integrating yields  $x^{p-1}z, x^{p-2}yz, x^{p-3}y^2z \in \text{Supp}(f_x)$ , since  $f_x$  and  $f_y$  share the same monomials. With  $\deg_v(x^{p-1}z) > \deg_v(x^{p-2}yz) > \deg_v(x^{p-3}y^2z)$ , we see that  $f_x$  has to contain at least 3 monomials of different multi-degrees, which is impossible, since  $f_x$  can contain at most 2. If  $p = 2$  we see that  $z$  is contained in all monomials that we can obtain by deriving and integrating so far. We know, since  $f$  defines an isolated singularity, that this implies the existence of a monomial of type  $x^i y^j$  with  $i + j \geq 3$ . This case is covered by (a).

- (2) Now we consider the case  $\alpha = 0 \neq \beta$ . In this case  $f_x = h_1$  and  $f_y = \beta h_1 + h_2$ . Then all monomials of  $f_x$  also appear in  $f_y$  and they cannot cancel each other. Consider any monomial close to the x-axis. This monomial is of type  $x^p y^i z^j$  with  $p, i, j \in \mathbb{N}, p \geq 2$  and  $i, j \in \{0, 1\}$ . Deriving and integrating as in (1) yields  $x^{p-1} y^{i+1} z^j \in \text{Supp}(f)$ . This implies  $\{x^{p-1} y^i z^j, x^{p-2} y^{i+1} z^j\} \subseteq \text{Supp}(f_x)$ . Since  $\deg_v(x^{p-1} y^i z^j) > \deg_v(x^{p-2} y^{i+1} z^j)$  we obtain that  $f_x = h_1$  cannot be multihomogeneous, which is a contradiction.
- (3) The case  $\alpha \neq 0 = \beta$  works in the same way as case (2) considering the monomials close to the y-axis.

□

**Lemma 6.48.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3), w_1 = w_2 > w_3, \text{ord}(f) \geq 3$  and  $s = 2$ . If  $v_1 \neq v_2$ , then no monomial of type  $x^a y^b z^c$  with  $a, b, c \in \mathbb{N}_{\geq 1}$  is contained in  $\text{Supp}(f)$ .*

*Proof.* The idea is to assume the existence of a monomial of type  $x^a y^b z^c$  with  $a, b, c \in \mathbb{N}_{\geq 1}$  in  $\text{Supp}(f)$  and bring this to a contradiction. For the first part of the proof we assume without loss of generality  $v_1 > v_2$  and  $v_1 > 0$ , since  $v$  can be multiplied by a non-zero constant and since the coordinate change  $x \mapsto y, y \mapsto x$  and  $z \mapsto z$  does not affect the quasihomogeneity of  $f$  and the multihomogeneity of  $J_f$ . We show that the only monomial close to the x-axis can be  $x^p z$  for some  $p \in \mathbb{N}$  and the only monomial close to the y-axis can be  $y^p z$ . For the monomials close to the x-axis we consider two different cases:

- (1) If  $x^{p+1} \in \text{Supp}(f)$  for some  $p \in \mathbb{N}_{\geq 2}$ , then  $w_1 = w_2 > w_3$  and  $c \geq 1$  imply  $p + 1 > a + b$ , which is equivalent to  $p > a - 1 + b$ . Using  $v_1 > 0$  and  $v_1 > v_2$  we obtain

$$v_1 p > v_1(a - 1) + v_1 b > v_1(a - 1) + v_2 b.$$

This implies  $\deg_v(x^p) > \deg_v(x^{a-1}y^bz^c)$  and contradicts the multihomogeneity of  $f_x$ .

- (2) If  $x^py \in \text{Supp}(f)$  for some  $p \in \mathbb{N}_{\geq 1}$ , then  $w_1 = w_2 > w_3$  and  $c \geq 1$  imply  $p+1 > a+b$ , which is equivalent to  $p > a+b-1$ . Using  $v_1 > 0$  and  $v_1 > v_2$  we obtain

$$v_1p > v_1a + v_1(b-1) > v_1a + v_2(b-1).$$

This implies  $\deg_v(x^p) > \deg_v(x^ay^{b-1}z^c)$  and contradicts the multihomogeneity of  $f_y$ .

For the statement about the monomials close to the y-axis we first consider the case  $v_2 \neq 0$ . Since multiplying by a non-zero constant does not change the multihomogeneity of the  $h_i$  we get another weight vector  $v' = (v'_1, v'_2, v'_3)$  with  $v'_2 > 0$ .

- (1) If  $y^{q+1} \in \text{Supp}(f)$  for some  $q \in \mathbb{N}_{\geq 2}$ , then  $w_1 = w_2 > w_3$  and  $c \geq 1$  imply  $q+1 > a+b$ , which is equivalent to  $q > a+b-1$ . We have to consider two cases:

- (a) If  $v'_2 > v'_1$  we obtain

$$v'_2q = v'_2a + v'_2(b-1) > v'_1a + v_2(b-1),$$

which implies  $\deg_{v'}(y^q) > \deg_{v'}(x^ay^{b-1}z^c)$ .

- (b) If  $v'_2 < v'_1$  we obtain

$$v'_2q = v'_2a + v'_2(b-1) < v'_1a + v'_2(b-1),$$

which implies  $\deg_{v'}(y^q) < \deg_{v'}(x^ay^{b-1}z^c)$ .

Both possibilities contradict the multihomogeneity of  $f_y$ .

- (2) If  $y^qx \in \text{Supp}(f)$  for some  $q \in \mathbb{N}_{\geq 1}$ , then  $w_1 = w_2 > w_3$  and  $c \geq 1$  imply  $q+1 > a+b$ , which is equivalent to  $q > a-1+b$ . We have to consider two cases:

- (a) If  $v'_2 > v'_1$  we obtain

$$v'_2q > v'_2(a-1) + v'_2b > v'_1(a-1) + v'_2b,$$

which implies  $\deg_{v'}(y^q) > \deg_{v'}(x^{a-1}y^bz^c)$ .

- (b) If  $v'_2 < v'_1$  we obtain

$$v'_2q > v'_2(a-1) + v'_2b > v'_1(a-1) + v'_2b,$$

which implies  $\deg_{v'}(y^q) < \deg_{v'}(x^{a-1}y^bz^c)$ .

Both possibilities contradict the multihomogeneity of  $f_x$ .

So far we have shown that  $v_1 \neq v_2$  and  $v_2 > 0$  implies that only  $x^py$  respectively  $y^qz$  are close to the x-axis respectively y-axis. The remaining case is  $v_2 = v_3 = 0$ . In this case we can assume  $v_1 = 1$ . Then  $f_x$  being multihomogeneous implies  $f_x = x^k g(y, z)$  for some  $k \in \mathbb{N}$  and  $g \in \mathbb{C}[y, z]$ . In this setup, no matter which one of the monomials  $x^{p+1}$ ,  $x^py$  or  $x^pz$  are close to the x-axis,  $w_3, b, c \geq 1$  imply  $p > a$ . Using  $\{x^py^iz^j, x^{a-1}y^bz^c\} \subseteq \text{Supp}(f_x)$ , we see that  $f_x$  cannot be multihomogeneous, contradicting our assumptions.

The proof so far implies  $\{x^p z, y^q z\} \subseteq \text{Supp}(f)$ . Now  $w_1 = w_2$  and the quasihomogeneity of  $f$  imply  $p = q$ . In particular we obtain  $\{x^p, y^p\} \subseteq \text{Supp}(f_z)$ . With  $\deg_w(f_x) = \deg_w(f_y) < \deg_w(f_z)$  and  $g_1, g_2 \in \mathbb{C}[z]$ , we obtain  $\{x^p, y^p\} \subseteq \text{Supp}(h_3)$ . This contradicts the multihomogeneity of  $h_3$ , since these monomials have different weighted degrees with respect to  $v$  due to  $v_1 \neq v_2$ . All these results combined yield a contradiction, if we assume that a monomial of type  $x^a y^b z^c$  for any  $a, b, c \in \mathbb{N}_{\geq 1}$  is contained in  $\text{Supp}(f)$ . □

Lemma 6.48 implies that the monomials of  $f$  can lie only on the  $x$ - $y$ ,  $x$ - $z$  and the  $y$ - $z$  plane, thus  $f_x$  can only lie on the  $x$ - $y$  and/or the  $x$ - $z$  plane. We consider all possible cases of the position of  $f_x$ , starting with  $f_x$  being a monomial.

**Lemma 6.49.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $v_1 \neq v_2$  and let  $f_x$  be a monomial, then the following hold:*

- (1) *If  $f_x = x^p$  for some  $p \in \mathbb{N}_{\geq 3}$ , then  $f$  is of Sebastiani–Thom type.*
- (2) *If  $f_x = x^p y$  for some  $p \in \mathbb{N}_{\geq 2}$ , then  $f$  is of Sebastiani–Thom type.*
- (3) *If  $f_x = x^p z$  for some  $p \in \mathbb{N}_{\geq 2}$ , then  $f$  is of Sebastiani–Thom type.*

*Proof.*

- (1) By integration we obtain  $f(x, y, z) = \int x^p dx = \frac{1}{p+1} x^{p+1} + h(y, z)$  for some polynomial  $h$ .
- (2) With  $\text{ord}(f) \geq 3$  the only possible monomials close to the  $y$ -axis are  $y^{q+1}$  or  $y^{q+1} z$  for some  $q \in \mathbb{N}$  and the only possible monomials close to the  $z$ -axis are  $z^{r+1}$  or  $z^r y$  for some  $r \in \mathbb{N}$ . If  $\{x^{p+1} y, y^{q+1}, z^r y\} \subseteq \text{Supp}(f)$  or  $\{x^{p+1} y, y^q z, z^r y\} \subseteq \text{Supp}(f)$ , then there exists an  $i \in \mathbb{N}_{\geq 1}$  with  $z^i \in \text{Supp}(f)$ , otherwise  $f$  would not define an isolated hypersurface singularity, since  $y$  would divide  $f$ . Thus we have
  - (a)  $\{x^{p+1} y, y^{q+1}, z^{r+1}\} \subseteq \text{Supp}(f)$  or
  - (b)  $\{x^{p+1} y, y^{q+1} z, z^{r+1}\} \subseteq \text{Supp}(f)$ .

The next step is to show that no monomial of type  $y^i z^j$  is contained in  $\text{Supp}(f)$  for any  $i, j \in \mathbb{N}_{\geq 1}$ . Assume the contrary. We need to consider two different cases:

- (a) If  $\{x^{p+1} y, y^{q+1}, z^{r+1}\} \subseteq \text{Supp}(f)$ , then  $\{x^{p+1}, y^q, y^{i-1} z^j\} \subseteq \text{Supp}(f_y)$ . Two possible direction vectors of  $f_y$  are

$$u_1 = (p + 1, -q, 0)$$

and

$$u_2 = (0, q + 1 - i, -j).$$

$u_1$  and  $u_2$  are not parallel since  $p + 1 \neq 0$ . So no monomial of type  $y^i z^j$  for  $i, j \in \mathbb{N}_{\geq 1}$  can exist in  $\text{Supp}(f)$  and  $f$  is of Sebastiani–Thom type.

- (b) If  $\{x^{p+1} y, y^{q+1} z, z^{r+1}\} \subseteq \text{Supp}(f)$ , then  $\{x^{p+1}, y^q z\} \subseteq \text{Supp}(f_y)$  and  $\{y^q, z^r\} \subseteq \text{Supp}(f_z)$ .  $w_1 = w_2 > w_3$  implies  $y^q \in \text{Supp}(h_3)$ . For  $z^r$  we have two possibilities. Either there exists an  $l \in \mathbb{N}_{\geq 2}$  with  $z^l \in \text{Supp}(f_y)$  and  $z^{r-l} \in \text{Supp}(g_2)$  or  $z^r \in \text{Supp}(h_3)$ .

- i. If  $z^l \in \text{Supp}(f_y)$  for some  $l \in \mathbb{N}_{\geq 2}$ , then two possible direction vectors of  $f_y$  are

$$u_1 = (p + 1, 0, -l)$$

and

$$u_2 = (0, q, 1 - l).$$

$u_1$  and  $u_2$  are not parallel since  $p + 1 \neq 0$ .

- ii. If  $z^r \in \text{Supp}(h_3)$ , then a possible direction vector of  $h_3$  is

$$u_1 = (0, q + 1, -r)$$

and a possible direction vector of  $f_y$  is

$$u_2 = (p + 1, -q, -1).$$

$u_1$  and  $u_2$  are not parallel since  $p + 1 \neq 0$ .

- (3) With  $\text{ord}(f) \geq 3$  the only possible monomials close to the y-axis are  $y^{q+1}$  or  $y^{q+1}z$  for some  $q \in \mathbb{N}$  and the only possible monomials close to the z-axis are  $z^{r+1}$  or  $z^r y$  for some  $r \in \mathbb{N}$ . If  $\{x^{p+1}z, y^{q+1}z, z^{r+1}\} \subseteq \text{Supp}(f)$  or  $\{x^{p+1}z, y^q z, z^r y\} \subseteq \text{Supp}(f)$ , then there exists an  $i \in \mathbb{N}_{\geq 1}$  with  $y^i \in \text{Supp}(f)$ , otherwise  $f$  would not define an isolated hypersurface singularity, since  $z$  would divide  $f$ . Thus we have

(a)  $\{x^{p+1}z, y^{q+1}, z^{r+1}\} \subseteq \text{Supp}(f)$  or

(b)  $\{x^{p+1}z, y^{q+1}, z^r y\} \subseteq \text{Supp}(f)$ .

The next step is to show that no monomial of type  $y^i z^j$  is contained in  $\text{Supp}(f)$  for any  $i, j \in \mathbb{N}_{\geq 1}$ . Assume the contrary. We need to consider two different cases:

- (a) Assume  $\{x^{p+1}z, y^{q+1}, z^{r+1}\} \subseteq \text{Supp}(f)$ . Then  $\{y^q, y^{i-1}z^j\} \subseteq \text{Supp}(f_y)$  and  $\{x^{p+1}, z^r\} \subseteq \text{Supp}(f_z)$ .  $w_1 = w_2 > w_3$  implies  $x^p \in \text{Supp}(h_3)$ . For  $z^r$  we have two possibilities. Either there exists an  $l \in \mathbb{N}_{\geq 2}$  with  $z^l \in \text{Supp}(f_y)$  and  $z^{r-l} \in \text{Supp}(g_2)$  or  $z^r \in \text{Supp}(h_3)$ .

- i. If  $z^l \in \text{Supp}(f_y)$  and  $v_2 \neq 0$  then  $\deg_v(y^q) = qv_2 \neq 0 = \deg_v(z^l)$ . This implies that  $f_y$  contains two monomials with different  $v$ -degree, contradicting the multihomogeneity of  $f_y$ .

If  $v_2 = 0$ , then we can assume  $v_1 = 1$  and the multihomogeneity of  $J_f$  implies  $\text{Supp}(f_x) = \{x^p z\}$  and  $\text{Supp}(h_3) = \{x^{p+1}\}$ . Now we consider a monomial of type  $y^a z^b \in \text{Supp}(f)$  with  $a, b \in \mathbb{N}_{\geq 1}$ , with  $a$  being maximal. Then  $y^a z^{b-1} \in \text{Supp}(f_z)$  implies  $y^a z^{b-1} \in \text{Supp}(g_2 f_y)$ . This yields the existence of a  $k \in \mathbb{N}$  with  $y^a z^k \in \text{Supp}(f_y)$ . The maximality of  $a$  implies  $k = 0$  and thus  $a = q$ , since  $f_y$  is quasihomogeneous. Due to  $y^q \notin \text{Supp}(h_3)$  and  $w_2 > w_3$  we must have  $b \geq 2$ . The fact that  $f_x$  and  $h_3$  are monomials divisible by  $x$  imply that  $\{z^{b-1}\} = \text{Supp}(g_2)$ . In order to see this assume  $z^t \in \text{Supp}(g_2)$  with  $t \neq b - 1$ . In this setup  $\text{Supp}(f_z)$  contains  $y^q z^{b-1}$  and  $y^q z^t$ , which contradicts the quasihomogeneity of  $f_z$  with respect to  $w$ . This implies  $\{y^q z^b, y^{q-1} z^{2b}, \dots, z^{(q+1)b}\} \subseteq \text{Supp}(f)$ . Applying the same argument as in Lemma 6.26, we obtain, after a suitable change of coordinates,  $f = x^{p+1}z + y^{q+1}$ , thus  $f$  does not define an isolated hypersurface singularity. In this case no monomial of type  $y^i z^j$  for any  $i, j \in \mathbb{N}_{\geq 1}$  can be contained in  $\text{Supp}(f)$  and  $f$  is of Sebastiani–Thom type.

ii. If  $z^r \in \text{Supp}(h_3)$ , then a possible direction vector of  $h_3$  is

$$u_1 = (p + 1, 0, -r)$$

and a possible direction vector of  $f_y$  is

$$u_2 = (0, q + 1 - i, -j),$$

which are obviously not parallel. Thus no monomial of type  $y^i z^j$  for  $i, j \in \mathbb{N}_{\geq 1}$  can exist in  $\text{Supp}(f)$  and  $f$  is of Sebastiani–Thom type.

- (b) Assume  $\{x^{p+1}z, y^{q+1}, z^r y\} \subseteq \text{Supp}(f)$ . Then  $\{y^q, z^r\} \subseteq \text{Supp}(f_y)$  and  $\{x^{p+1}, y^q z\} \subseteq \text{Supp}(f_z)$ . If  $v_2 \neq 0$ , then  $\deg_v(y^q) = qv_2 \neq 0 = \deg_v(z^r)$ . This implies that  $f_y$  contains two monomials with different  $v$ -degree, contradicting the multihomogeneity of  $f_y$ .

Now we can assume  $v_1 = 1$  and  $v_2 = v_3 = 0$ . In this setup we know that  $\text{Supp}(f_x) = \{x^p z\}$  and  $\text{Supp}(h_3) = \{x^{p+1}\}$ . Consider a monomial of type  $y^a z^b \in \text{Supp}(f)$  with  $a, b \in \mathbb{N}_{\geq 1}$  with  $a$  being maximal. Then  $y^a z^{b-1} \in \text{Supp}(g_2 f_y)$ , hence there exists a  $k \in \mathbb{N}$  with  $y^a z^k \in \text{Supp}(f_y)$ . The maximality of  $a$  implies  $k = 0$  and thus  $a = q$ , since  $f_y$  is quasihomogeneous. Due to  $y^q \notin \text{Supp}(h_3)$  and  $w_2 > w_3$  we must have  $b \geq 2$ . The fact that  $f_x$  and  $h_3$  are monomials divisible by  $x$  imply that  $\{z^{b-1}\} = \text{Supp}(g_2)$ . In order to see this assume  $z^t \in \text{Supp}(g_2)$  with  $t \neq b - 1$ . In this setup  $\text{Supp}(f_z)$  contains  $y^q z^{b-1}$  and  $y^q z^t$ , which contradicts the quasihomogeneity of  $f_z$  with respect to  $w$ . This implies  $\{y^q z^b, y^{q-1} z^{2b}, \dots, z^{(q+1)b}\} \subseteq \text{Supp}(f)$  and we are in the setup of (a).

□

Next we consider the cases where  $f_x$  lies in the  $x$ - $y$  plane or the  $x$ - $z$  plane.

**Lemma 6.50.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $v_1 \neq v_2$  and assume that the monomials of  $f_x$  lie in the  $x$ - $y$  plane. Then  $f$  is of Sebastiani–Thom type.*

*Proof.* By assertion the monomials of  $f_x$  are of type  $x^i y^j$  for  $i, j \in \mathbb{N}$  with  $i + j \geq 2$ . Define  $d_w := \deg_w(f_x)$ ,  $d_v := \deg_v(f_x)$  and

$$A := \begin{pmatrix} w_1 & w_1 \\ v_1 & v_2 \end{pmatrix}.$$

$A$  has full rank, since  $v_1 \neq v_2$ . Then  $(i, j)$  has to satisfy  $(i, j)A^T = (d_w, d_v)^T$ . Since  $A$  has full rank, there exists precisely one solution for  $(i, j)$ . Thus  $f_x$  is a monomial and we can apply Lemma 6.49. □

Next we consider the case where  $f_x$  lies in the  $x$ - $z$  plane.

**Lemma 6.51.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $v_1 \neq v_2$ ,  $|\text{Supp}(f_x)| \geq 2$  and assume that the monomials of  $f_x$  lie in the  $x$ - $z$  plane. Then  $f$  is of Sebastiani–Thom type.*



*Proof.* If  $v_1 \neq 0 = v_3$ , then we can argue as in the proof of Lemma 6.50 and we obtain that  $f_x$  is a monomial. Then Lemma 6.49, (c) yields that  $f$  is of Sebastiani–Thom type. Now we assume  $v_1 = v_3 = 0$  and  $v_2 = 1$ . This implies that  $\text{Supp}(f_y) = \{y^q\}$  or  $\text{Supp}(f_y) = \{y^{q-1}z\}$  for some  $q \in \mathbb{N}_{\geq 2}$ . Applying the coordinate change  $x \mapsto y, y \mapsto x, z \mapsto z$  does not change the facts that  $f$  is quasi homogeneous and that  $J_f$  is multi-homogeneous. After the coordinate change we are in the case of Lemma 6.49 and the result follows.  $\square$

**Lemma 6.52.** *Let  $f \in \mathbb{C}[x, y, z]$  be a polynomial with unique weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Assume  $v_1 \neq v_2$ ,  $|\text{Supp}(f_x)| \geq 2$  and assume that the monomials of  $f_x$  lie in the  $x$ - $y$  plane and in the  $x$ - $z$  plane. Then  $f$  is of Sebastiani–Thom type.*

*Proof.* In the case where  $f$  lies in the  $x$ - $y$  plane as well as in the  $x$ - $z$  plane we know that  $\text{Supp}(f_x) = \{x^p y, x^{i-1} z^j\}$  or  $\text{Supp}(f_x) = \{x^p z, x^{i-1} y^j\}$  for certain  $i, j \in \mathbb{N}$  with  $i+j \geq 3$  and  $i \geq 1$ . We have to consider these two cases:

- (1) Assume  $\text{Supp}(f_x) = \{x^p y, x^{i-1} z^j\}$ . In this setup only  $y^{q+1}$  or  $y^{q+1}z$  can be the monomials in  $\text{Supp}(f)$  close to the  $y$ -axis. If  $j = 0$   $f_x$  lies in the  $x$ - $y$  plane and this case has already been covered in Lemma 6.50. From now on we assume  $j \geq 1$ . A possible direction vector of  $f_x$  is

$$u_1 = (p+1-i, 1, -j).$$

Using  $x^p \in \text{Supp}(f_y)$  we have to consider two cases:

- (a) Assume  $y^{q+1} \in \text{Supp}(f)$ . Then a direction vector of  $f_y$  is given by

$$u_2 = (p, -q, 0).$$

$u_1$  is not parallel to  $u_2$ , since  $-j \neq 0$  in the last component.

- (b) Assume  $y^{q+1}z \in \text{Supp}(f)$ . Then a direction vector of  $f_y$  is given by

$$u_3 = (p, -q, -1).$$

$u_1$  is not parallel to  $u_3$ , since  $j \geq 1$  and  $u_1 = \lambda u_3$  imply  $\lambda \geq 1$ , which contradicts  $-q < 1$ .

- (2) Assume  $\text{Supp}(f_x) = \{x^p z, x^{i-1} y^j\}$ . In this setup all monomials  $y^{q+1}, y^{q+1}x$  or  $y^{q+1}z$  can be monomials in  $\text{Supp}(f)$  close to the  $y$ -axis. If  $j = 0$   $f_x$  lies in the  $x$ - $z$  plane and this case has already been covered in Lemma 6.50. From now on we assume  $j \geq 1$ . A possible direction vector of  $f_x$  is

$$u_1 = (p+1-i, -j, 1).$$

Using  $x^i y^{j-1} \in \text{Supp}(f_y)$  we have to consider three cases:

- (a) Assume  $y^{q+1} \in \text{Supp}(f)$ . Then a direction vector of  $f_y$  is given by

$$u_2 = (-i, q+1-j, 0)$$

$u_1$  is not parallel to  $u_2$ , since  $1 \neq 0$  in the last component.

(b) Assume  $y^{q+1}x \in \text{Supp}(f)$ . Then a direction vector of  $f_y$  is given by

$$u_3 = (i - 1, j - 1 - q, 0).$$

In case  $i > 1$  or  $j \neq q + 1$  we have  $u_3 \neq (0, 0, 0)$ . In this case we see that  $u_1$  is not parallel to  $u_3$ , since  $1 \neq 0$  in the last component. We consider the case  $i = 1$  and  $j = q + 1$  separately. In this case we have  $\text{Supp}(f_x) = \{x^p z, y^{q+1}\}$ . Due to the fact that the monomials of  $f_y$  have to lie on a line only one monomial of type  $y^a z^{r+1}$  can be contained in  $\text{Supp}(f)$  for certain  $a, r \in \mathbb{N}$ . Since we need a monomial close to the  $z$ -axis, we obtain  $a \in \{0, 1\}$ . This yields

$$\{x^{p+1}z, xy^{q+1}, y^a z^{r+1}\} = \text{Supp}(f).$$

Then  $\text{Supp}(f_z) = \{x^{p+1}, y^a z^r\}$ . This implies  $f_z = h_3$ , since no monomial of  $f_x$  or  $f_y$  divides any monomial of  $f_z$ . We have to consider some possibilities:

- i. Assume  $a = 0$ . Then the multihomogeneity of  $f_z$  implies  $v_1(p + 1) = 0$ , hence  $v_1 = 0$ . Thus we can assume  $v_2 = 1$ . This contradicts the multihomogeneity of  $f_x$  since  $\deg_v(x^p z) = 0 \neq q + 1 = \deg_v(y^{q+1})$ .
- ii. Assume  $a = 1$ . Then the multihomogeneity of  $f_x$  implies  $v_1 + v_2 q = 0$  and the multihomogeneity of  $f_z$  implies  $v_1(p + 1) = v_2$ . Combining both results yields  $v_1 = 0$ , hence we obtain  $v_2 \neq 0$ . In this case we obtain  $q = 0$ , which contradicts  $\text{ord}(f) \geq 3$ .

(c) Assume  $y^{q+1}z \in \text{Supp}(f)$ . Then a direction vector of  $f_y$  is given by

$$u_4 = (-i, q + 1 - j, 1)$$

$u_1$  is not parallel to  $u_4$ , since the last components are equal, but  $-i \neq p - i$ .

□

This concludes the case  $v_1 \neq v_2$ . Next we consider the case  $v_1 = v_2$ .

**Lemma 6.53.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . If  $v_1 = v_2$ , then, after a suitable change of coordinates, we can assume  $f_x = h_1$  and  $f_y = h_2$ .*

*Proof.* Given a system of coordinates  $(x, y, z)$  our goal is to find a coordinate change  $\varphi(x, y, z)$ , such that we can assume  $f_x = h_1$  and  $f_y = h_2$ . With  $w_1 = w_2 > w_3$  and  $v_1 = v_2 \neq 0 = v_3$  we obtain  $v' = (1, 1, 0)$  as well as  $v' = (0, 0, 1)$  as possible weight vectors for  $J_f$ . This implies that the coordinate change

$$\varphi(x, y, z) = (ax + by, cx + dy, z)$$

for certain  $a, b, c, d \in \mathbb{C}$  is multihomogeneous. We know that  $f_x = h_1 + \alpha h_2$  and  $f_y = \beta h_1 + h_2$  for certain  $\alpha, \beta \in \mathbb{C}$ . Define

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B := \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}.$$

The fact that  $\varphi$  is a coordinate change yields that  $A$  is invertible. The fact that  $f_x$  and  $f_y$ , as well as  $h_1$  and  $h_2$ , form part of a minimal system of generators of  $J_f$  and have minimal weighted degree with respect to  $w$  implies that  $B$  is invertible. Define

$f' = f \circ \varphi$ , i.e.  $f'(x, y, z) = f(ax + yb, cx + dy, z)$ . In this setup we only need to consider  $f'_x$  and  $f'_y$ . Applying the chain rule yields:

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = A^T \begin{pmatrix} f_x \circ \varphi \\ f_y \circ \varphi \end{pmatrix} = A^T B \begin{pmatrix} h_1 \circ \varphi \\ h_2 \circ \varphi \end{pmatrix}.$$

We can choose  $a, b, c, d$  in such a way that  $A^T := B^{-1}$ . Then we obtain  $f'_x = h_1 \circ \varphi$  and  $f'_y = h_2 \circ \varphi$ . This means we can find a linear coordinate change such that we can assume  $f_x = h_1$  and  $f_y = h_2$ .  $\square$

With these preparations we can prove the following.

**Proposition 6.54.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . If  $v_1 = v_2$ , then  $f$  is of Sebastiani–Thom type.*

*Proof.* Using Lemma 6.53, we can assume  $f_x = h_1$  and  $f_y = h_2$ . We know that  $J_f$  is homogeneous with respect to  $v' = (0, 0, 1)$ , which implies  $h_i = z^{k_i} h'_i(x, y)$  for some  $k_i \in \mathbb{N}$  and  $h'_i \in \mathbb{C}[x, y]$ . We obtain  $f_x = z^{k_1} h'_1(x, y)$  and  $f_y = z^{k_2} h'_2(x, y)$ . If  $k_1 \neq 0 \neq k_2$ , then  $z$  divides  $f$ , which is not allowed for isolated hypersurface singularities. Hence we can assume without loss of generality that  $k_1 = 0$ , which implies  $f_x = f_x(x, y)$ . We have to consider two cases:

- (1) Assume  $k_2 = 0$ . Then  $f_x$  and  $f_y$  do not depend on  $z$ , so the only monomial containing  $z$  is  $z^r$  for some  $r \in \mathbb{N}$ . In this case  $f = g(x, y) + z^r$  for some  $g \in \mathbb{C}[x, y]$  and  $f$  is of Sebastiani–Thom type.
- (2) Assume  $k_2 \geq 1$ . Assume that  $x^i y^j z^{k_2} \in \text{Supp}(f_y)$  for some  $i \geq 1$ . Then  $x^{i-1} y^{j+1} z^{k_2} \in \text{Supp}(f_x)$ , which contradicts  $f_x = f_x(x, y)$ . This means  $f_y$  does not depend on  $x$ . Now any monomial of type  $x^i y^j \in \text{Supp}(f_x)$  for  $j \geq 1$  leads to a similar contradiction, so  $f_x$  can only depend on  $x$ , hence  $\text{Supp}(f_x) = \{x^p\}$  for some  $p \in \mathbb{N}_{\geq 3}$ . By integration we obtain that  $f = x^{p+1} + h(y, z)$  for some  $h \in \mathbb{C}[y, z]$  and  $f$  is of Sebastiani–Thom type.  $\square$

Combining the previous Lemmas and Proposition 6.54 we obtain:

**Proposition 6.55.** *Let  $f \in \mathbb{C}[x, y, z]$  be a QHIS with weights  $(w_1, w_2, w_3)$ ,  $w_1 = w_2 > w_3$ ,  $\text{ord}(f) \geq 3$  and  $s = 2$ . Then  $f$  is of Sebastiani–Thom type.*

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