A Kinetic Model for Vehicular Traffic: Existence of stationary solutions

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Abstract

In this paper the kinetic model for vehicular traffic developed in [3, 4] is considered and theoretical results for the space homogeneous kinetic equation are presented. Existence and uniqueness results for the time dependent equation are stated. An investigation of the stationary equation leads to a boundary value problem for an ordinary differential equation. Existence of the solution and some properties are proven. A numerical investigation of the stationary equation is included.

1 Introduction

Kinetic or Boltzmann-like models for vehicular traffic have been developed for example in [8], [6, 7], [5, 2] In [3, 4] a new kinetic multilane model is described and numerically investigated. Here we give a theoretical investigation of the space homogeneous model presented in the above cited papers. We are especially concerned with the solution of the stationary equation, i.e. the counterparts of the Maxwellian distributions in the kinetic theory of gases. These stationary solutions are used in [3, 4] to derive the coefficients of macroscopic equations based on the kinetic model.

The paper is organized in the following way: In Section 2 we describe the equation under consideration and prove some basic properties. Section 3 contains an investigation of the stationary equation. We prove existence of solutions. Finally in the last section we present some numerical results.

2 The Space Homogeneous Kinetic Equation

Let $f = f(v,t), v \in [0,w], t \in [0,\infty)$ be the time dependent velocity distribution of vehicles in a space homogeneous situation. w denotes the maximal velocity of the cars. We set $w = 1$. The density ρ is defined as $\rho(t) = \int f(v,t)dv$. ρ may vary between 0 and a maximal density ρ_m, which we set equal to 1 as well. We consider, compare [3, 4], the equation

$$\partial_t f = C(f) \tag{1}$$

with the interaction operator $C(f)$ defined by

$$C(f) = G_B(f) - L_B(f) + G_A(f) - L_A(f),$$

where G_B, L_B, G_A, L_A denote the kinetic gain and loss terms due to braking and acceleration. The gain term due to braking is given by

$$G_B(f) = \frac{1}{\rho}\int\int_{v_1>v_2} P_B(v_1,v_2,f)|v_1-v_2|\sigma_B(v,v_1)q_B(v_1,f)f(v_1)f(v_2)dv_1dv_2$$

with

$$\sigma_B(v,v_1) = \frac{1}{v_1(1-\beta)}\chi_{[\beta v_1, v_1]}(v), 0 \le \beta \le 1.$$

σ_B expresses the fact that a driver with velocity v_1 approaching its leading car with velocity $v_2 < v_1$ is braking into a range of velocities below v_1. The loss term due to braking is

$$L_B(f) = \frac{1}{\rho} \int_{v>v_2} P_B(v, v_2, f)|v - v_2| q_B(v, f) f(v) f(v_2) dv_2.$$

The gain term due to acceleration is

$$G_A(f) = \frac{1}{\rho} \int \int_{v_1 < v_2} |v_1 - v_2| \sigma_A(v, v_1) q_A(v_1, f) f(v_1) f(v_2) dv_1 dv_2$$

with

$$\sigma_A(v, v_1) = \frac{1}{\min(1, \alpha v_1) - v_1} \chi_{[v_1, \min(1, \alpha v_1)]}(v), 1 \leq \alpha \leq \infty.$$

This expresses the fact that a driver with velocity v_1 having a leading car with velocity $v_2 > v_1$ is accelerating into a range of velocities above v_1. The loss term due to acceleration is

$$L_A(f) = \frac{1}{\rho} \int_{v<v_2} |v - v_2| q_A(v, f) f(v) f(v_2) dv_2.$$

$q_B(v, f), q_A(v, f)$ denote correlation functions. They fulfill $\int_0^1 q_B(v, f) dv = 1$ and $\int_0^1 q_A(v, f) dv = 1$. For an explicit example we refer to [3]. $P_B(v_1, v_2, f)$ denotes the probability of braking. It depends in general on the velocities of the cars involved in the braking interaction and on the distribution function itself.

One observes immediately that, due to the fact that integrating σ_B and σ_A with respect to v gives 1, the density $\rho = \int f(v) dv$ is a conserved quantity during the evolution.

For simplicity we restrict ourselves in this paper to quantities q_B, q_A, P_B of the following form, for more complicated situations, see [3]: The correlation functions q_B, q_A are assumed to depend only on ρ. For high densities, i.e. ρ near 1, the number of possible braking interactions must be much higher than the number of acceleration interactions; in contrast, for small densities, i.e. ρ near 0, the number of possible braking and acceleration interactions is approximately equal. This means the relation $q_B(\rho)/q_A(\rho)$ ranges from 1 to ∞ for ρ ranging from 0 to 1. The probability of braking P_B is also assumed to depend only on ρ. P_B ranges from $P_B(\rho = 0) = 0$ to $P_B(\rho = 1) = 1$ corresponding to the fact that for low densities the probability of braking is small and for high densities the probability of braking is 1. Thus, we introduce the parameter $k = k(\rho)$ defined by $k = \frac{P_B(\rho)q_B(\rho)}{q_A(\rho)}$. Due to the above k ranges from 0 to ∞ as ρ tends from 0 to 1. Moreover, we set $\beta = 0$ and $\alpha = \infty$.

The kinetic equation is now rewritten in terms of the normalized distribution function $F = F(v, t)$ with $f = \rho F$, $\int_0^1 F(v)dv = 1$. One obtains

$$\partial_t F = C_k(F, F), \tag{2}$$

where the collision operator $C_k(F, G)$ is defined by

$$C_k(F, G) = kG_B(F, G) + G_A(F, G) - kL_B(F, G) - L_A(F, G)$$

with

$$\begin{aligned}
G_B(F,G) &= \int_0^1 \int_0^{v_1} (v_1 - v_2)F(v_1)G(v_2)\frac{1}{v_1}\chi_{[0,v_1]}(v)dv_2dv_1 \\
G_A(F,G) &= \int_0^1 \int_{v_1}^1 (v_2 - v_1)F(v_1)G(v_2)\frac{1}{1-v_1}\chi_{[v_1,1]}(v)dv_2dv_1 \\
L_B(F,G) &= \int_0^v (v - v_2)F(v)G(v_2)dv_2 \\
L_A(F,G) &= \int_v^1 (v - v_2)F(v)G(v_2)dv_2
\end{aligned}$$

k is a parameter ranging from 0 to ∞, corresponding to a range of interactions which are purely dominated by acceleration to interactions dominated by braking.

We start the investigation of equation (2) by determining the collision invariants of the interaction operator C_k, i.e. we have to find the functions $\varphi = \varphi(v)$ such that

$$\int_0^1 \varphi(v)C_k(F, G)(v)dv = 0$$

for any functions F and G, i.e. the functions φ that are not changed during an interaction process.

As already stated the above equality is true for all constant functions $\varphi =$ constant. This expresses the conservation of the number of cars during an interaction. The constants are the only conserved quantities due to

Proposition 1
Let $\varphi \in \mathcal{C}^1[0,1]$. $\int_0^1 \varphi(v) C_k(F,G)(v)dv = 0$ for any function F and G in $\mathcal{C}^0[0,1]$. Then $\varphi =$ constant.

Proof:

Multiplying the interaction operator $C_k(F,G)$ with φ, integrating with respect to v and changing variables in the gain terms gives

$$k\int_0^1\int_0^1 (v-v_2)F(v)G(v_2)\chi_{[0,v]}(v_2)[\varphi(v) - \frac{1}{v}\int_0^v \varphi(\tilde{v})d\tilde{v}]dv_2 dv$$
$$+\int_0^1\int_0^1 (v_2-v)F(v)G(v_2)\chi_{[v,1]}(v_2)[\varphi(v) - \frac{1}{1-v}\int_v^1 \varphi(\tilde{v})d\tilde{v}]dv_2 dv.$$

This is assumed to be equal to 0 for all $F, G \in \mathcal{C}^0[0,1]$. We obtain

$$k(v-v_2)\chi_{[0,v]}(v_2)[\varphi(v) - \frac{1}{v}\int_0^v \varphi(\tilde{v})$$
$$= (v-v_2)\chi_{[v,1]}(v_2)[\varphi(v) - \frac{1}{1-v}\int_v^1 \varphi(\tilde{v})d\tilde{v}]$$

for all $v, v_2 \in [0,1]$. Considering $0 = v_2 < v$ we get

$$v\varphi(v) = \int_0^v \varphi(\tilde{v}d\tilde{v}$$

or

$$\varphi(v) + v\varphi'(v) = \varphi(v)$$

This gives immediately $\varphi =$ constant, since φ is continuous. ■

Remark:
In comparison to the gas dynamic case, where density, mean velocity and energy is conserved in a collisison, we have in the present case only conservation of the density, i.e. conservation of the number of cars.

One can easily prove existence and uniqueness of the solutions of the non-stationary equation:

Proposition 2
There exists a unique solution $F(\cdot,t) \in \mathcal{L}^1[0,1]$, $\int_0^1 F(v,t)dv = 1, F \geq 0$ of the time dependent kinetic equation

$$\partial_t F = C_k(F,F)$$

with inital value $F(v,0) = F_0(v) \in \mathcal{L}^1[0,1]$, $\int_0^1 F_0(v)dv = 1, F_0 \geq 0$.

Proof:

The following estimates are obtained straightforwardly: Defining for $f \in \mathcal{L}^1[0,1], f \geq 0$

$$R(f) = k \int_0^v (v - v_2) f(v_2) dv_2 + \int_v^1 (v_2 - v) f(v_2) dv_2$$

one obtains

$$\|R(f)\|_1 \leq M \int_0^1 f(v) dv$$

with a positive constant M. Moreover, for $f, g \in \mathcal{L}^1[0,1]$ one has

$$\|C_k(f,g)\|_1 \leq M \|f\|_1 \|g\|_1,$$

since for example the gain term due to braking can be estimated by

$$\begin{aligned} \int_0^1 [\int_0^1 \int_0^{v_1} (v_1 - v_2) f(v_1) g(v_2) \frac{1}{v_1} \chi_{[0,v_1]}(v) dv_2 dv_1] dv \\ \leq \int_0^1 \int_0^1 \int_0^1 f(v_1) g(v_2) \frac{1}{v_1} \chi_{[0,v_1]}(v) dv dv_2 dv_1 \\ = \int_0^1 f(v) dv \int_0^1 g(v) dv \end{aligned}$$

One can proceed with the proof exactly as in the case of the homogeneous Boltzmann equation, see, e.g. [1]. ■

A more challenging problem is to find stationary solutions of the kinetic equation, i.e. solutions of the equation $C_k(F,F) = 0, k \in [0,\infty]$. This will be treated in the next section.

Considering the stationary equation a first simplification arises from the fact that due to the following proposition it is sufficient to consider the case $k \in [0,1]$. The other cases are found by symmetry arguments:

Proposition 3
Let the functions $F^*(v)$ be defined by $F^*(v) = F(1-v)$, then

$$C_k(F,F) = 0, \text{ iff } C_{\frac{1}{k}}(F^*,F^*) = 0, \; k \in [0,1].$$

That means a solution $F_k(v)$ of the stationary equation for $k \in [1,\infty]$ can be found from a solution G_k for $k \in [0,1]$ by defining $F_k(v) = G_{\frac{1}{k}}(1-v), k \in [1,\infty]$.

Proof:

Looking at the gain terms we get

$$
\begin{aligned}
G_A(F,F)(v) &= \int_0^v \int_{v_1}^1 (v_2 - v_1) F(v_1) F(v_2) \frac{1}{1-v_1} dv_2 dv_1 \\
&= \int_0^v \int_{v_1}^1 (1 - \frac{1-v_2}{1-v_1}) F(v_1) F(v_2) dv_2 dv_1.
\end{aligned}
$$

Substituting now $v_2' = 1 - v_2$ and $v_1' = 1 - v_1$ we get

$$
G_A(F,F)(v) = \int_{1-v}^1 \int_0^{v_1'} (1 - \frac{1-v_2'}{1-v_1'}) F(1-v_1') F(1-v_2') dv_2' dv_1'
$$
$$
= \int_{1-v}^1 \int_0^{v_1} (1 - \frac{v_2}{v_1}) F^*(v_1) F^*(v_2) dv_2 dv_1 = G_B(F^*, F^*)(1-v).
$$

In the same way one obtains

$$
L_A(F,F)(v) = L_B(F^*, F^*)(1-v).
$$

This yields for all $v \in [0,1]$

$$
\begin{aligned}
& \frac{1}{k} C_k(F,F)(v) \\
= \;& G_B(F,F)(v) + \frac{1}{k} G_A(F,F)(v) - L_B(F,F)(v) - \frac{1}{k} L_A(F,F)(v) \\
= \;& G_A(F^*,F^*)(1-v) + \frac{1}{k} G_B(F^*,F^*)(1-v) \\
& - L_A(F^*,F^*)(1-v) - \frac{1}{k} L_B(F^*,F^*)(1-v) \\
= \;& C_{\frac{1}{k}}(F^*,F^*)(1-v).
\end{aligned}
$$

The statement follows immediately. ■

3 Existence of Solutions for the Stationary Equation

In this section we consider the equation $C_k(F,F) = 0, k \in [0, \infty]$. In order to investigate the equation we distinguish between two sets of values for the parameter k, namely $k = 0, \infty$ and $k \in (0, \infty)$.

The first case $k = 0$ corresponds to a situation where only acceleration interactions are present. One expects the stationary solutions to be the δ distribution at 1 for $k = 0$ (all cars are driving with maximal speed). Using Proposition 3 the corresponding solution for $k = \infty$ is a δ distribution at 0 (all cars have velocity zero).

However, as one observes immediately, the other δ-functions $\delta_{v_0}, v_0 \in [0,1]$ are solutions of the stationary equation as well for $k = 0, \infty$. This corresponds to the unstable case of cars having the same velocity and moving, in spite of a high density, with high speed very close to each other. We prove

Proposition 4
Let $k = 0, \infty$. Then, the support of the solutions of the stationary equation is concentrated in one point v_0 with $v_0 \in [0,1]$; the only solutions of the stationary equation are the δ-distributions $\delta_{v_0}(v)$.

Proof:

We show that the support of F is concentrated in a single point. Due to Proposition 3 we restrict to the case $k = 0$.

Multiplication of $0 = C_k(F,F)$ with a test function φ and integration with respect to v gives for $k = 0$:

$$\int_0^1 \int_v^1 (v_2 - v) F(v) F(v_2) [\varphi(v) - \frac{1}{1-v} \int_v^1 \varphi(\tilde{v}) d\tilde{v}] dv_2 dv = 0.$$

Considering this equation the following Lemma 1 shows that

$$\int_0^1 \int_v^1 (v_2 - v) F(v) F(v_2) \psi(v) dv_2 dv = 0$$

for all $\psi \in \mathcal{C}^1[0,1]$, $\frac{\psi}{1-v} \in \mathcal{L}^1[0,1]$. This gives

$$\int_0^1 [\int_0^1 \chi_{[}v, 1](v_2)(v_2 - v) F(v) F(v_2) dv_2] \psi(v) dv = 0$$

or for all v

$$\int_0^1 \chi_{[}v, 1](v_2)(v_2 - v) F(v) F(v_2) dv_2 = 0.$$

Since $\chi_[v,1](v_2)(v_2 - v)F(v)F(v_2) \geq 0$ for all v, v_2 we get

$$\chi_[v,1](v_2)(v_2 - v)F(v)F(v_2) = 0$$

or $F(v)F(v_2) = 0$ for $v_2 > v$. In other words for $F(v_0) \neq 0$ we have $F(v) = 0, v \neq v_0$. This shows that the support of F is concentrated in one point and F is a δ-distribution. ■

Lemma 1
Let $\psi \in \mathcal{C}^1[0,1]$, $\frac{\psi}{1-v} \in \mathcal{L}^1[0,1]$. Then there exist $\varphi \in \mathcal{C}^0[0,1]$ with

$$\psi = \varphi(v) - \frac{1}{1-v}\int_v^1 \varphi(\tilde{v})d\tilde{v}.$$

Proof:

Define

$$\varphi(v) = \varphi(0) + \psi(v) - \int_v^1 \frac{\psi(\tilde{v})}{1-\tilde{v}}d\tilde{v}.$$

Then

$$\varphi'(v) = \psi'(v) + \frac{\psi(v)}{1-v}$$

or

$$((1-v)\varphi)' = ((1-v)\psi)' - \varphi.$$

Integrating gives

$$(1-v)\psi = (1-v)\varphi - \int_v^1 \varphi(\tilde{v})d\tilde{v}.$$

■

A much more difficult problem is to find the solutions of the stationary equation with parameters $k \in (0,\infty)$. Again the δ-functions $\delta_{v_0}, v_0 \in [0,1]$ are solutions of the stationary equation associated to the above mentioned unstable traffic flow situations. However, additionally smooth solutions exist. One expects a unique stable smooth solution.

In the following we want to prove the existence of a smooth solution of the stationary equation. We consider again the weak formulation

$$
\begin{aligned}
k\int_0^1\int_0^v (v-v_2)F(v)F(v_2)[\varphi(v)-\frac{1}{v}\int_0^v\varphi(\tilde{v})d\tilde{v}]dv_2dv \\
+\int_0^1\int_v^1 (v_2-v)F(v)F(v_2)[\varphi(v)-\frac{1}{1-v}\int_v^1\varphi(\tilde{v})d\tilde{v}]dv_2dv = 0
\end{aligned}
$$

Since adding a constant to the test function φ will not change the equation we can restrict to test functions $\varphi\in\mathcal{C}^0[0,1]$ fulfilling

$$
\int_0^1\varphi(v)dv = 0.
$$

We will derive in the following a boundary value problem for an ordinary differential equation which is equivalent to the above integral equation. Defining

$$
\psi(v) = \int_0^v\varphi(\tilde{v})d\tilde{v}
$$

we have

$$
\psi'(v) = \varphi(v).
$$

Since $\int_0^1\varphi(v)dv = 1$ we have as well

$$
\psi(0) = 0 = \psi(1).
$$

Moreover,

$$
\begin{aligned}
\varphi(v)-\frac{1}{v}\int_0^v\varphi(\tilde{v})d\tilde{v} &= \psi'(v)-\frac{1}{v}\psi(v) \\
\varphi(v)-\frac{1}{1-v}\int_v^1\varphi(\tilde{v})d\tilde{v} &= \psi'(v)+\frac{1}{1-v}\psi(v).
\end{aligned}
$$

This yields

$$
\begin{aligned}
k\int_0^1\int_0^v (v-v_2)F(v)F(v_2)[\psi'(v)-\frac{1}{v}\psi(v)]dv_2dv \\
+\int_0^1\int_v^1 (v_2-v)F(v)F(v_2)[\psi'(v)+\frac{1}{1-v}\psi(v)]dv_2dv = 0.
\end{aligned}
$$

Using partial integration and $\psi(0) = 0 = \psi(1)$ one obtains

$$
\begin{aligned}
&\int_0^1\int_0^v (v-v_2)F(v)F(v_2)\psi'(v)dv_2dv \\
= -&\int_0^1\int_0^v [F(v)+(v-v_2)F'(v)]F(v_2)\psi(v)dv_2dv
\end{aligned}
$$

and

$$\int_0^1 \int_v^1 (v_2 - v) F(v) F(v_2) \psi'(v) dv_2 dv$$
$$= \int_0^1 \int_v^1 [F(v) - (v_2 - v) F'(v)] F(v_2) \psi(v) dv_2 dv.$$

Alltogether we get

$$k \int_0^1 \int_0^v [-F(v) - (v - v_2) F'(v) - (v - v_2) F(v) \frac{1}{v}] F(v_2) \psi(v) dv_2 dv$$
$$+ \int_0^1 \int_v^1 [F(v) - (v_2 - v) F'(v) + (v_2 - v) F(v) \frac{1}{1-v}] F(v_2) \psi(v) dv_2 dv = 0$$

for all $\psi \in \mathcal{C}_0^1[0,1]$. That gives

$$k \int_0^v [F(v) + (v - v_2) F'(v) + (v - v_2) F(v) \frac{1}{v}] F(v_2) dv_2$$
$$= \int_v^1 [F(v) - (v_2 - v) F'(v) + (v_2 - v) F(v) \frac{1}{1-v}] F(v_2) dv_2$$

or

$$\begin{aligned} & k \int_0^v F(v_2) dv_2 [2F(v) + vF'(v)] \\ & + k \int_0^v v_2 F(v_2) dv_2 [-F'(v) - F(v) \frac{1}{v}] \\ = & \int_v^1 F(v_2) dv_2 [F(v)(1 - \frac{v}{1-v}) + vF'(v)] \\ & + \int_v^1 v_2 F(v_2) dv_2 [-F'(v) + F(v) \frac{1}{1-v}]. \end{aligned} \tag{3}$$

We define the functions

$$K(v) = \int_0^v \int_0^{v_1} F(v_2) dv_2 dv_1$$

and

$$G(v) = \int_0^v v_2 F(v_2) dv_2.$$

Then

$$\begin{aligned} K'(v) &= \int_0^v F(v_2) dv_2 \\ K''(v) &= F(v). \end{aligned}$$

Moreover

$$K(0) = 0, K'(0) = 0, K'(1) = 1,$$

where the last equality is due to $\int_0^1 F(v)dv = 1$. Since $G'(v) = vF(v) = vK''(v) = (vK'(v))' - K'(v)$ and $K(0) = 0$, we get

$$\begin{aligned} G(v) &= vK'(v) - K(v) \\ G(1) &= 1 - K(1) \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int_v^1 f(v_2)dv_2 &= 1 - K'(v) \\ \int_v^1 v_2 f(v_2)dv_2 &= 1 - K(1) - vK'(v) + K(v). \end{aligned}$$

Using these expressions in (3) we get

$$\begin{aligned} & kK'(v)[2K''(v) + vK'''(v)] + k(vK'(v) - K(v))(-K'''(v) - \frac{1}{v}K''(v)) \\ = \; & (1 - K'(v))[K''(v)(1 - \frac{v}{1-v}) + vK'''(v)] \\ + \; & (1 - K(1) - vK'(v) + K(v))[-K'''(v) + \frac{1}{1-v}K''(v)]. \end{aligned}$$

Reordering gives

$$\begin{aligned} & K'''(v)[(k+1)K(v) - v + 1 - K(1)] \\ +K''(v)[(1+k)K'(v) - 2 + k\frac{K(v)}{v} + & \frac{K(1) - K(v)}{1-v}] = 0 \end{aligned}$$

with $K(0) = 0, K'(0) = 0, K'(1) = 1$. We mention that K, K' and K'' must fulfill $0 \leq K \leq 1$, $0 \leq K' \leq 1$ and $0 \leq K''$ due to the definition of K and the properties of F.

We define

$$\begin{aligned} a(K(v), v) &= (k+1)K(v) - v + 1 - K(1), \\ b(K(v), K'(v), v) &= -(1+k)K'(v) + 2 - k\frac{K(v)}{v} - \frac{K(1) - K(v)}{1-v}. \end{aligned}$$

The dependence on $K(1)$ is neglected in the notation. One obtains the equation

$$K'''(v)a(K(v), v) = K''(v)b(K(v), K'(v), v). \tag{4}$$

with $K(0) = 0, K'(0) = 0, K'(1) = 1$. In the following we want to prove existence of smooth solutions K of the above boundary value problem.

By straightforward formal manipulations we can transform equation (4) in the following way: We integrate

$$\frac{(K'')'(v)}{K''(v)} = \frac{b(K(v), K'(v), v)}{a(K(v), v)}$$

and obtain

$$\ln(K''(v)) = \int_0^v \frac{b(K(\tilde{v}), K'(\tilde{v}), \tilde{v})}{a(K(\tilde{v}), \tilde{v})} d\tilde{v} + \text{constant}.$$

This yields

$$K''(v) = K''(0) exp(\int_0^v \frac{b(K(\tilde{v}), K'(\tilde{v}), \tilde{v})}{a(K(\tilde{v}), \tilde{v})} d\tilde{v}).$$

Choosing $K''(0)$ in such a way that $K'(1) = 1$ is fulfilled we get

$$1 = K'(1) = K''(0) \int_0^1 exp(\int_0^{\tilde{v}} \frac{b(K(\hat{v}), K'(\hat{v}), \hat{v})}{a(K(\hat{v}), \hat{v})} d\hat{v}) d\tilde{v}.$$

The resulting equation is

$$K''(v) = \frac{exp(\int_0^v \frac{b(K(\hat{v}), K'(\hat{v}), \hat{v})}{a(K(\hat{v}), \hat{v})} d\hat{v})}{\int_0^1 exp(\int_0^{\tilde{v}} \frac{b(K(\hat{v}), K'(\hat{v}), \hat{v})}{a(K(\hat{v}), \hat{v})} d\hat{v}) d\tilde{v}} \tag{5}$$

with $K(0) = 0$ and $K'(0) = 0$. Integrating twice one obtains

$$K(v) = \frac{\int_0^v \int_0^{v'} exp(\int_0^{\tilde{v}} \frac{b(K(\hat{v}), K'(\hat{v}), \hat{v})}{a(K(\hat{v}), \hat{v})} d\hat{v}) d\tilde{v} dv'}{\int_0^1 exp(\int_0^{\tilde{v}} \frac{b(K(\hat{v}), K'(\hat{v}), \hat{v})}{a(K(\hat{v}), \hat{v})} d\hat{v}) d\tilde{v}}. \tag{6}$$

In the following we restrict to the case $k = 1$. In this case an additional symmetry property of the solutions allows to use formulation (6) and a fixpoint argument to prove the existence of a solution of the boundary value problem (4). Our main result is

Theorem 1
Let $k = 1$. Then, there exists a solution $K \in \mathcal{C}^3[0, 1]$ such that

$$K'''(v) a(K(v), v) = K''(v) b(K(v), K'(v), v) \tag{7}$$

with

$$a(K(v), v) = 2K(v) - v + 1 - K(1)$$
$$b(K(v), K'(v), v) = -2K'(v) + 2 - \frac{K(v)}{v} - \frac{K(1) - K(v)}{1 - v}.$$

$K(0) = 0, K'(0) = 0, K'(1) = 1.$

Furthermore, the function K has the following properties:

1. For all $v \in [0, \frac{1}{2}]$:

$$K'''(\frac{1}{2} + v) = -K'''(\frac{1}{2} - v) \quad , \quad K''(\frac{1}{2} + v) = K''(\frac{1}{2} - v)$$
$$K'(\frac{1}{2} + v) = 1 - K'(\frac{1}{2} - v) \quad , \quad K(\frac{1}{2} + v) = v + K(\frac{1}{2} - v).$$

2. There exists an $\epsilon_0 \in (0, \frac{1}{4})$ such that for all $v \in [0, \frac{1}{2}]$:

$$\epsilon_0 v^2 \le K(v) \le \frac{v}{2} \quad , \quad 2\epsilon_0 \le K''(v).$$

3. $K(1) = \frac{1}{2}, K'(\frac{1}{2}) = \frac{1}{2}.$

Proof:

The key to the proof is the restriction to solutions which satisfy the symmetry condition

$$K''(\frac{1}{2} + v) \quad = \quad K''(\frac{1}{2} - v). \tag{8}$$

Indeed let us assume that there exists a solution of (7) which satisfies (8). Then it is easy to verify that $K'''(\frac{1}{2} + v) = -K'''(\frac{1}{2} - v)$ and $K'(\frac{1}{2} + v) = K'(1) + K'(0) - K'(\frac{1}{2} - v)$ for all $v \in [0, \frac{1}{2}]$. Making use of the boundary conditions $K'(0) = 0$ and $K'(1) = 1$ gives $K'(\frac{1}{2} + v) = 1 - K'(\frac{1}{2} - v)$ for all $v \in [0, \frac{1}{2}]$, especially $K'(\frac{1}{2}) = \frac{1}{2}$. We furthermore get $K(\frac{1}{2} + v) = v - \frac{1}{2} + K(1) - K(0) + K(\frac{1}{2} - v)$ for all $v \in [0, \frac{1}{2}]$. The boundary condition $K(0) = 0$ gives $K(\frac{1}{2} + v) = v - \frac{1}{2} + K(1) + K(\frac{1}{2} - v)$ for all $v \in [0, \frac{1}{2}]$. Setting $v = 0$ we deduce $K(1) = \frac{1}{2}$ and therefore $K(\frac{1}{2} + v) = v + K(\frac{1}{2} - v)$ for all $v \in [0, \frac{1}{2}]$.
We conclude: Any solution K of (7) *and* (8) satisfies 1. and 3.

The symmetry condition (8) has two important consequences. First, the value $K(1)$ which arises in the model equation is known: $K(1) = \frac{1}{2}$. Second, it sufficies to determine $K(v)$ for all $v \in [0, \frac{1}{2}]$, because due to 1. we get $K(v) = v - \frac{1}{2} + K(1-v)$ for all $v \in (\frac{1}{2}, 1]$.
Let f denote the restriction of K to the interval $[0, \frac{1}{2}]$. Then f will satisfy

$$f'''(v)a(f(v), v) = f''(v)b(f(v), f'(v), v) \tag{9}$$

with

$$\begin{aligned} a(f(v), v) &= 2f(v) - v + \frac{1}{2}, \\ b(f(v), f'(v), v) &= -2f'(v) + 2 - \frac{f(v)}{v} - \frac{\frac{1}{2} - f(v)}{1-v} \\ f(0) = 0, f'(0) &= 0, f'(\frac{1}{2}) = \frac{1}{2}. \end{aligned}$$

Until now we have assumed that there exists a solution K of (7) *and* (8). Now let us assume that $f \in \mathcal{C}^3[0, \frac{1}{2}]$ is a solution of (9). We extend f to a function K defined on $[0, 1]$ by

$$K(v) = \begin{cases} f(v), & \text{if } 0 \le v \le \frac{1}{2} \\ v - \frac{1}{2} + f(1-v), & \text{if } \frac{1}{2} < v \le 1 \end{cases}$$

Using the boundary condition $f'(\frac{1}{2}) = \frac{1}{2}$ the reader will have no difficulty to verify that $K \in \mathcal{C}^3[0, 1]$. It is also easy to see that K satisfies properties 1. and 3. The sensitive question is whether K is a solution of (7). This can be seen as follows. First of all we observe that due to $K(1) = \frac{1}{2}$ the equations (7) and (9) coincide on $[0, \frac{1}{2}]$. Since f is a solution of (9) and since $K = f$ on $[0, \frac{1}{2}]$ we get: K satisfies (7) on $[0, \frac{1}{2}]$. It remains to prove that K satisfies (7) for all $v \in (\frac{1}{2}, 1]$. We calculate for all $v \in (\frac{1}{2}, 1]$:

$$K'''(v) = -f'''(1-v) \quad , \quad K''(v) = f''(1-v) \quad , \quad K'(v) = 1 - f'(1-v)$$

as well as

$$a(K(v), v) = 2f(1-v) + v - \frac{1}{2} = a(f(1-v), 1-v)$$

and

$$\begin{aligned} b(K(v), K'(v), v) &= 2f'(1-v) - 2 - \frac{f(1-v) - \frac{1}{2}}{v} + \frac{f(1-v)}{1-v} \\ &= -b(f(1-v), f'(1-v), 1-v). \end{aligned}$$

Hence

$$\begin{aligned} & -K'''(v)\ a(K(v),v) + K''(v)\ b(K(v),K'(v),v) \\ = \ & f'''(1-v)\ a(f(1-v),1-v) - f''(1-v)\ b(f(1-v),f'(1-v),1-v) \\ = \ & 0, \end{aligned}$$

since $1-v \in [0,\frac{1}{2})$ and f solves (9) on $[0,\frac{1}{2}]$.
We conclude: If (9) has a solution $f \in \mathcal{C}^3[0,\frac{1}{2}]$ and if $K \in \mathcal{C}^3[0,1]$ is constructed from f as described above, then K will be a solution of (7) which satisfies 1. and 3.

It remains to prove that problem (9) possesses a solution $f \in \mathcal{C}^3[0,\frac{1}{2}]$, which has the additional property 2., i.e. there is an $\epsilon_0 \in (0,\frac{1}{4})$ such that for all $v \in [0,\frac{1}{2}]$:

$$\epsilon_0 v^2 \le f(v) \le \frac{v}{2} \quad , \quad 2\epsilon_0 \le f''(v).$$

Using the same formal manipulations as those leading to (5) and writing b as

$$b(f(v),f'(v),v) = b_1(f(v),v) + b_2(f'(v),v)$$

with

$$\begin{aligned} b_1(f(v),v) &= 1 - \frac{f(v)}{v} - \frac{\frac{1}{2}-f(v)}{1-v} \\ b_2(f'(v),v) &= -\frac{d}{dv}a(f(v),v) = -2f'(v)+1 \end{aligned}$$

one obtains

$$f''(v) = T[f](v),$$

where

$$T[f](v) = \frac{1}{2}\frac{\frac{1}{a(f(v),v)}exp(\int_0^v \frac{b_1(f(\hat{v}),\hat{v})}{a(f(\hat{v}),\hat{v})}d\hat{v})}{\int_0^{\frac{1}{2}} \frac{1}{a(f(\tilde{v}),\tilde{v})}exp(\int_0^{\tilde{v}} \frac{b_1(f(\hat{v}),\hat{v})}{a(f(\hat{v}),\hat{v})}d\hat{v})d\tilde{v}}.$$

On the other hand it is easy to verify that any $g \in \mathcal{C}^2[0,\frac{1}{2}]$ with

$$\begin{aligned} a(g(v),v) &> 0 \\ g''(v) &= T[g](v) \end{aligned}$$

belongs to $\mathcal{C}^3[0,\frac{1}{2}]$ and is a solution of problem (9).

Hence problem (9) can be reformulated as fixed point problem: Find an $f \in \mathcal{C}^2[0, \frac{1}{2}]$ such that

$$f(v) = S[f](v) = \int_0^v \int_0^{v'} T[f](\tilde{v}) d\tilde{v} dv'. \tag{10}$$

We prove by means of the Schauder theorem that (10) has a solution. We have to find a convex, closed (in an appropriate topology) set M such that S maps M continuously into a compact subset of M.

For $\epsilon \in (0, \frac{1}{4})$ we introduce the set

$$M^\epsilon = \{g \in \mathcal{C}^1[0, \frac{1}{2}], \epsilon v^2 \leq g \leq \frac{v}{2}\}.$$

We observe: M^ϵ is a closed, convex subset of $\mathcal{C}^1[0, \frac{1}{2}]$. We shall prove that there exists an $\epsilon_0 \in (0, \frac{1}{4})$ such that S maps M^{ϵ_0} $\mathcal{C}^1[0, \frac{1}{2}]$-continuously into a $\mathcal{C}^1[0, \frac{1}{2}]$-compact subset of M^{ϵ_0}.
First of all we observe that $b_1(g(v), v)$ belongs to $\mathcal{C}^0[0, \frac{1}{2}]$ for all $g \in \mathcal{C}^1[0, \frac{1}{2}]$. Furthermore we have for all $g \in M^\epsilon$

$$2\epsilon v^2 - v + \frac{1}{2} \leq a(g(v), v) \leq \frac{1}{2} \tag{11}$$

and

$$0 \leq b_1(g(v), v) \leq \frac{\frac{1}{2} - v}{1 - v} - \epsilon v^2 (\frac{1}{v} - \frac{1}{1 - v}) \tag{12}$$

Hence for all $g \in M^\epsilon$:

$$a(g(v), v) > 0$$

This settles: S is well defined on M^ϵ . Furthermore, we easily get: S maps M^ϵ $\mathcal{C}^1[0, \frac{1}{2}]$-continuosly into $\mathcal{C}^1[0, \frac{1}{2}]$.

We also observe that for all $g \in M^\epsilon$ the following is true: $S[g] \in \mathcal{C}^2[0, \frac{1}{2}]$ and

$$\begin{aligned} 0 \leq S[g](v) &\leq \frac{1}{2} \\ 0 \leq (S[g])'(v) &\leq \frac{1}{2} \\ S[g](0) = 0, (S[g])'(0) &= 0. \end{aligned}$$

Hence for all $g \in M^\epsilon$:

$$0 \leq S[g](v) \leq \frac{v}{2}.$$

Now let us consider $(S[g])''(v)$. If we can find an $\epsilon_0 \in (0, \frac{1}{4})$ such that

$$(S[g])''(v) \geq 2\epsilon_0,$$

then S will map M^{ϵ_0} into M^{ϵ_0}, because then for all $v \in [0, \frac{1}{2}]$ there is a $\xi \in [0, \frac{1}{2}]$ such that

$$S[g](v) = S[g](0) + S[g])'(0)v + \frac{1}{2}(S[g])''(\xi v)v^2 \geq \epsilon_0 v^2.$$

To get a lower estimate for $(S[g])''$ we make use of the previous estimates (11) and (12):

$$-v + \frac{1}{2} \leq 2\epsilon v^2 - v + \frac{1}{2} \leq a(g(v), v) \leq \frac{1}{2}$$

and $b_1(g(v), v) \geq 0$ give

$$\frac{1}{a(g(v), v)} exp(\int_0^v \frac{b_1(g(\hat{v}), \hat{v})}{a(g(\hat{v}), \hat{v})} d\hat{v}) \geq 2.$$

Moreover, from

$$\frac{b_1(g(v), v)}{a(g(v), v)} \leq \frac{\frac{\frac{1}{2}-v}{1-v} - \epsilon v^2(\frac{1}{v} - \frac{1}{1-v})}{2\epsilon v^2 - v + \frac{1}{2}} \leq \frac{\frac{\frac{1}{2}-v}{1-v}}{\frac{1}{2} - v} = \frac{1}{1-v}$$

we get

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{1}{a(g(\hat{v}), \hat{v})} exp(\int_0^{\hat{v}} \frac{b_1(g(\tilde{v})), \tilde{v}}{a(g(\tilde{v}), \tilde{v})} d\tilde{v}) d\hat{v} &\leq \int_0^{\frac{1}{2}} \frac{1}{2\epsilon\hat{v}^2 - \hat{v} + \frac{1}{2}} \frac{1}{1-\hat{v}} d\hat{v} \\ &\leq \int_0^{\frac{1}{2}} \frac{d\hat{v}}{2\epsilon\hat{v}^2 - \hat{v} + \frac{1}{2}}. \end{aligned}$$

The last integral can be calculated explicitly (where we make use of the fact that $\epsilon < \frac{1}{4}$):

$$\int_0^{\frac{1}{2}} \frac{d\hat{v}}{2\epsilon\hat{v}^2 - \hat{v} + \frac{1}{2}} = \frac{1}{\sqrt{1-4\epsilon}} \ln\left(\frac{1 + \sqrt{1-4\epsilon} - 2\epsilon}{1 + \sqrt{1-4\epsilon}} \cdot \frac{1 - \sqrt{1-4\epsilon}}{1 - \sqrt{1-4\epsilon} - 2\epsilon}\right).$$

We observe that there exists an $\epsilon^* \in (0, \frac{1}{4})$ and a constant $C \in (0, \infty)$ such that for all $\epsilon \in (0, \epsilon^*]$:

$$\int_0^{\frac{1}{2}} \frac{d\hat{v}}{2\epsilon\hat{v}^2 - \hat{v} + \frac{1}{2}} \leq C \ln(\frac{1}{\epsilon}).$$

Hence, for all $\epsilon \in (0, \epsilon^*]$ and all $g \in M^\epsilon$:

$$T[g] \geq \frac{2}{C \ln(\frac{1}{\epsilon})}.$$

We observe that $\epsilon \ln(\frac{1}{\epsilon})$ tends to 0 as ϵ tends to 0. Hence there exists an $\epsilon_0 \in (0, \frac{1}{4})$ such that

$$T[g] \geq 2\epsilon_0$$

for all $v \in [0, \frac{1}{2}]$. We conclude: S maps M^{ϵ_0} into M^{ϵ_0}.

It is not difficult to obtain an upper estimate for $T[g](v)$. The reader can easily verify that there exist $C_1, C_2 \in (0, \infty)$ such that for all $v \in [0, \frac{1}{2}]$and all $g \in M^{\epsilon_0}$:

$$C_1 \leq \frac{1}{a(g(v), v)}, \frac{b_1(g(v), v)}{a(g(v), v)} \leq C_2.$$

Hence,

$$T[g] \leq \frac{1}{2} \frac{C_2 exp C_2}{C_1 exp C_1}.$$

We conclude: $S(M^{\epsilon_0})$ is a bounded subset of $\mathcal{C}^2[0, \frac{1}{2}]$. Due to the Arzela Ascoli Theorem we have: $S(M^{\epsilon_0})$ is a $\mathcal{C}^1[0, \frac{1}{2}]$-precompact subset of $\mathcal{C}^1[0, \frac{1}{2}]$, i.e. S maps M^{ϵ_0} $\mathcal{C}^1[0, \frac{1}{2}]$-compactly into M^{ϵ_0}. Now we can apply the Schauder theorem to get: There exists an $f \in M^{\epsilon_0}$ such that

$$f = S[f].$$

This finishes the proof of the theorem.

■

Remark:
From Theorem 1 we conclude the existence of a solution $F = K'' \in \mathcal{C}^1[0, 1]$ of the original integral equation $0 = C_1(F, F)$. Since $K'' \geq 2\epsilon_0$ one obtains $F \geq 2\epsilon_0$. Moreover, F is symmetric with respect to $v = \frac{1}{2}$.

Remark:
1) The solution K of Theorem 1 belongs to $\mathcal{C}^3[0, 1]$ and satisfies (8). Aside

from this function K there is another *distributional* solution $K_\circ \in \mathcal{C}^0[0,1]$ of (7) which satisfies (8), namely

$$\begin{array}{rcl} K_\circ \; : \; [0,1] & \to & [0,\infty[\\ v & \mapsto & \begin{cases} 0 & , \quad 0 \le v \le \frac{1}{2} \\ v - \frac{1}{2} & , \quad \frac{1}{2} < v \le 1 \end{cases} \end{array}$$

2) The reader may wonder why the fixed point argument is settled in a space with a lower estimate for the second derivative. At first glance a much more convenient way to prove the existence of a solution of (9) would be to replace the operator T by an operator T^η, $\eta \in]0,\infty[$ defined as

$$T^\eta[f](v) = \frac{1}{2} \frac{\frac{1}{a^\eta(f(v),v)} exp(\int_0^v \frac{b_1(f(\hat{v}),\hat{v})}{a^\eta(f(\hat{v}),\hat{v})} d\hat{v})}{\int_0^{\frac{1}{2}} \frac{1}{a^\eta(f(\tilde{v}),\tilde{v})} exp(\int_0^{\tilde{v}} \frac{b_1(f(\hat{v},\hat{v})}{a^\eta(f(\hat{v}),\hat{v})} d\hat{v}) d\tilde{v}}$$

where $a^\eta(f(v),v) := \max\{a(f(v),v),\eta\}$ and then let η tend to zero. Indeed it causes no problems to prove (again by Schauder's fixed point theorem) the existence of an $f^\eta \in \mathcal{C}^2[0,\frac{1}{2}]$ which satisfies

$$f^\eta(v) = \int_0^v \int_0^{v'} T^\eta[f^\eta](\tilde{v}) d\tilde{v} dv'.$$

The difficulty of this approach however lies in the fact that there is no mechanism in the equation which allows to find bounds for $T^\eta[f^\eta]$ which are independent of η. This is not surprising, because - as mentioned above - there *is* a (distributional) solution of (7) which satisfies (8) and whose second derivative has a δ-singularity at $v = \frac{1}{2}$.
3) For $k \in (0,1)$ the additional symmetry properties of the solution used in the proof are not fulfilled, compare the numerical results in the next section. Additional arguments would be needed in that case to prove the existence of solutions of the stationary equation.

4 Numerical Investigations

In this section a numerical investigation of the equations is presented. In particular, the time dependent equation $\partial_t F = C_k(F,F)$ is simulated for different values of the parameter k. To obtain the numerical results shown here we did use a numerical scheme as described in detail in [4].

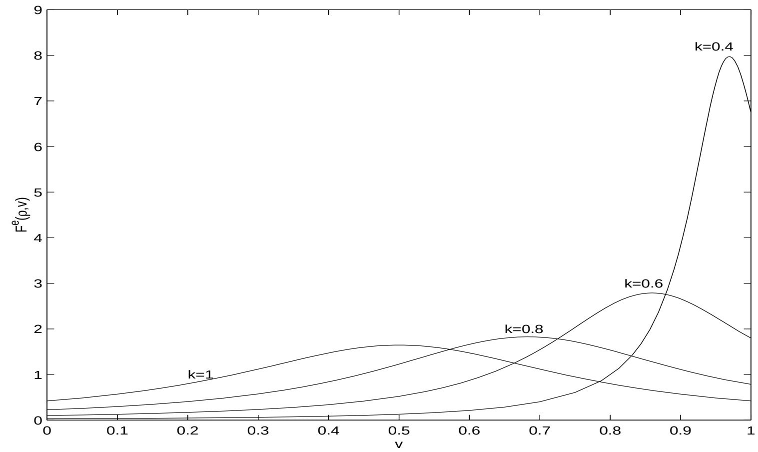

Figure 1: Equilibrium Distributions for $k = 0.4, 0.6, 0.8, 1.0$

The stationary distribution functions are computed and shown in Figure 1. One observes the symmetry of the numerical solution for $k = 1$ with respect to $v = \frac{1}{2}$. The stationary distributions are then used to determine the mean velocities for different values of k. This gives a fundamental diagram as in Figure 2. In the figures we plot only the results for values $k \in [0, 1]$.

Despite of the simplicity of this caricature of a traffic flow equation the obtained results are qualitatively reasonable.

5 Conclusions:

- In this paper a simplification of the kinetic model for vehicular traffic presented in [3] has been investigated.
- In particular, we did prove the existence of stationary solutions of the homogeneous equation for the case $k = 1$.
- The problem of existence for $k \in (0, 1)$ and of uniqueness of a smooth solution of the stationary equation in the general case is still open. In particular, we have not been able to find a counterpart of the Boltzmann H-functional for the above equation.

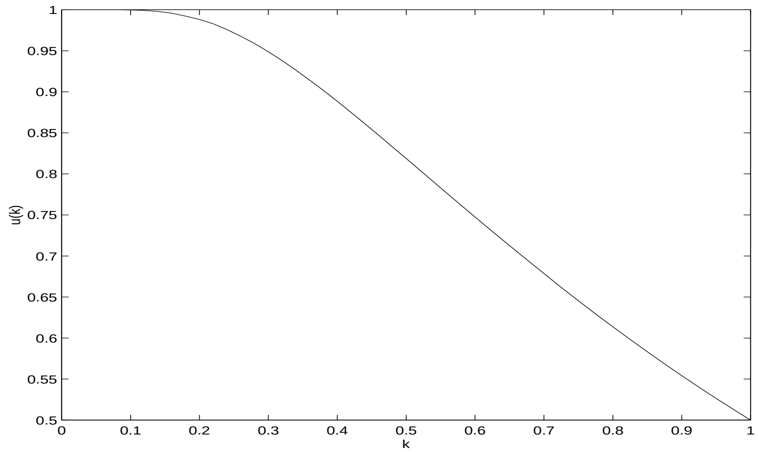

Figure 2: Mean velocities

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