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AG Finanzmathematik

Reflected Anticipated Backward Stochastic Differential Equations with Default Risk, Numerical Algorithms and Applications

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Abstrakt

AG Finanzmathematik
Fachbereich Mathematik

Promotion

Reflected Anticipated Backward Stochastic Differential Equations with Default risk, Numerical Algorithms and Applications

von Jingnan Wang

Die Hauptthemen dieser Doktorarbeit sind die theoretischen Eigenschaften numerischer Algorithmen und zugehöriger Anwendungen reflektierter antizipativer stochastischer Rückwärtsdifferentialgleichungen (RABSDE), die von einer Brownschen Bewegung und einem von ihr unabhängigen Martingal in einer standardmäßigen Umgebung generiert werden. Der Generator einer RABSDE enthält die gegenwärtigen und zukünftigen Werte der Lösung. RABSDES finden Anwendung in der Finanzmodellierung (z.B. optimales Stoppen mit Ausfallrisiko oder amerikanische Spieloptionen) oder auch in biologischen Modellen (z.B. Populationsmodelle), wenn die Dynamik der zugrunde liegenden Prozesse nicht nur von ihrem Barwert, sondern auch von einigen zukünftigen Informationen abhängt.

Die vorliegende Arbeit besteht aus zwei Teilen, dem theoretischen Hintergrund von (R) ABSDE, einschließlich grundlegender Theoreme, theoretischer Beweise und Eigenschaften (Kapitel 2-4) sowie numerischer Algorithmen und Simulationen für (R) ABSDES (Kapitel 5). Für den theoretischen Teil untersuchen wir ABSDEs (Kapitel 2), RABSDEs mit einem Hindernis (Kapitel 3) und RABSDEs mit zwei Hindernissen (Kapitel 4) unter Standardvoraussetzungen, einschließlich der zugehörigen Existenz- und Eindeutigkeitssätze, Anwendungen und des Vergleichssatzes für ABSDEs, sowie ihre Beziehungen zu PDEs und stochastischer Delay-Gleichungen (SDDE). Im Kapitel über numerischen Algorithmen (Kapitel 5) führen wir zwei Hauptalgorithmen ein, ein diskretes Penaltyverfahren und ein diskretes reflektiertes Schema, das auf einer diskreten Approximation der Brownschen Bewegung sowie einer diskreten Approximation des Standard-Martingals basiert. Wir geben die Konvergenzergebnisse der Algorithmen an, geben ein numerisches Beispiel und eine Anwendung bei amerikanischen Spieloptionen, um die Funktionsweise der Algorithmen zu veranschaulichen.

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Abstract

Financial Mathematics Group
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Doctorate

Reflected Anticipated Backward Stochastic Differential Equations with Default risk, Numerical Algorithms and Applications

by Jingnan Wang

The main subjects of this thesis are the theoretical properties, numerical algorithms and related applications of reflected anticipated backward stochastic differential equations (RABSDE) driven by a Brownian motion and a mutually independent martingale in a defaultable setting. The generator of a RABSDE includes the present and future values of the solution. RABSDES have applications in financial modeling (such as optimal stopping with default risk or American game options) and also in biological models (e.g. population growth) when the dynamics of the underlying processes are not only depending on their present value but also on some future information.

This thesis consists of two parts, i.e. the theoretical background of (R)ABSDE including basic theorems, theoretical proofs and properties (Chapter 2-4), as well as numerical algorithms and simulations for (R)ABSDES (Chapter 5). For the theoretical part, we study ABSDEs (Chapter 2), RABSDEs with one obstacle (Chapter 3) and RABSDEs with two obstacles (Chapter 4) in the defaultable setting respectively, including the existence and uniqueness theorems, applications, the comparison theorem for ABSDEs, their relations with PDEs and stochastic differential delay equations (SDDE). The numerical algorithm part (Chapter 5) introduces two main algorithms, a discrete penalization scheme and a discrete reflected scheme based on a random walk approximation of the Brownian motion as well as a discrete approximation of the default martingale; we give the convergence results of the algorithms, provide a numerical example and an application in American game options in order to illustrate the performance of the algorithms.

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My time at Technische Universität Kaiserslautern was enjoyable, memorable and precious. Three years of foreign study experience have had a great impact on my life path, worldview and personal development. I want to thank all the friends I met here, the time I spent with them and the kind help I got from them.

Lastly, I would like to thank my family for all their love and encouragement. For my parents who raised and cultivated me with love and supported me in all my pursuits. For my lovely daughter Yutang who brings me faithful support and spiritual encouragement during my tough time in Ph.D. pursuit, without her I would not have been able to complete this doctoral degree. Thank you all.

Contents

Chapter 1

Introduction

To give the reader a sound introduction to the thesis, we collect some basic notations and previous research background in this chapter that are necessary to understand the topic of the thesis.

1.1 Basic Notations

We first introduce the following basic notations and spaces ($p \in [0, \infty)$):

- $\mathcal{L}^p(\mathcal{G}_T; \mathbb{R}) := \{\varphi \in \mathbb{R} |\ \varphi$ is a $\mathcal{G}_T$-measurable random variable and $\mathbb{E}|\varphi|^p < \infty\}$;
- $\mathcal{L}^p_{\mathcal{G}}(0, t; \mathbb{R}^d) := \{\varphi : \Omega \times [0, t] \to \mathbb{R}^d |\ \varphi_t$ is $\mathcal{G}_t$-progressively measurable and $\mathbb{E}\int_0^t |\varphi_s|^p ds < \infty\}$;
- $\mathcal{S}^p_{\mathcal{G}}(0, t; \mathbb{R}) := \{\varphi : \Omega \times [0, t] \to \mathbb{R} |\ \varphi_t$ is $\mathcal{G}_t$-progressively measurable rcll process and $\mathbb{E}\left[\sup_{0 \le s \le t} |\varphi_s|^p\right] < \infty\}$;
- $\mathcal{L}^{p,\tau}_{\mathcal{G}}(0, t; \mathbb{R}^k) := \{\varphi : \Omega \times [0, t] \to \mathbb{R}^k |\ \varphi_t$ is $\mathcal{G}_t$-progressively measurable and $\mathbb{E}\left[\int_0^t |\ \varphi_s|^p 1_{\{\tau > s\}} \gamma_s ds\right] = \mathbb{E}\left[\int_0^t \sum_{i=1}^k |\varphi_{i,s}|^p 1_{\{\tau_i > s\}} \gamma^i_s ds\right] < \infty\}$;
- $\mathcal{A}^p_{\mathcal{G}}(0, T; \mathbb{R}) := \{K : \Omega \times [0, T] \to \mathbb{R} |\ K_t$ is a $\mathcal{G}_t$-adapted rcll increasing process and $K_0 = 0, K_T \in \mathcal{L}^p(\mathcal{G}_T; \mathbb{R})\}$;
- $\mathcal{T}$ stands for the set of all stopping times with values in $[0, T]$, and $\mathcal{T}_t = \{v \in \mathcal{T};\ t \le v \le T\}$.

1.2 Backward Stochastic Differential Equations

The backward stochastic differential equation (BSDE) theory plays a significant role in financial modeling, which will be shown below. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $B := (B_t)_{t \ge 0}$ is a d-dimensional standard Brownian motion, $\mathcal{F} := (\mathcal{F}_t)_{t \ge 0}$ is the associated natural filtration of B, $\mathcal{F}_t = \sigma(B_s;\ 0 \le s \le t)$, and $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. We first consider the following form of BSDE with the generator f and the terminal value ξ under the smooth square integrability assumption for ξ and the Lipschitz condition for f:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \qquad t \in [0, T]. \tag{1.1}$$

The setting of this problem is to find a pair of $\mathcal{F}_t$-adapted processes $(Y, Z) \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ satisfying the BSDE (1.1).

BSDEs with linear generators first appeared as the adjoint processes in option pricing and the maximum principle for the stochastic control problems. It was first introduced by Bismut [1] (1973), when he studied the maximum principle in stochastic optimal control. Pardoux and Peng [2] (1990) studied the general non-linear BSDEs, they gave a probabilistic interpretation of a solution of the second order quasi-linear partial differential equation and proved the existence and uniqueness of the adapted solution under the certain assumptions. Duffie and Epstein [3] (1992) independently used a class of BSDEs to describe the stochastic differential utility function theory in uncertain economic environments. Since then, the BSDE theory has been studied in many different areas, such as mathematical finance, stochastic control, economical management, etc. More about BSDEs and related applications can be found in El Karoui et. al [4] (1997), Lepeltier and Martín [5] (1998), Peng [6] (1999), Kobylanski [7] (2000), Rozkosz [8] (2003), Jiang [9] (2004), Buckdahn and Ichihara [10] (2005), Jiang [11] (2005), Briand and Hu [12] (2006), Jiang [13] (2006), Crépey [14] (2011), etc.

Example 1.2.1. ***(Pricing of contingent claims)*** *In the complete market, the expected return of a contingent claim at the terminal time T can be replicated by a dynamic portfolio, where the solution Y and Z can be represented as its wealth process and the related hedging strategy respectively (see e.g. El Karoui and Quenez [15] (1997), El Karoui et. al [4] (1997)). For the case of the incomplete market, El Karoui and Quenez [16] (1995) studied that for a contingent claim, there exists an upper process which can be obtained as the increasing limit of a sequence of processes associated with the solutions of non-linear BSDEs. More can be found in El Karoui [17] (1997), Buckdahn and Hu [18] (1998), Kohlmann and Zhou [19] (2000), etc.*

We consider a complete market model consisting of a riskless asset and a risky asset with the following price process:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t, \qquad t \in [0, T].$$

The strategy π_t is the amount of money invested in S_t at time t. The investor can borrow or lend money at a riskless rate r. Y is the wealth process obtained from trading with a self-financing strategy π and a riskless asset in the market. Then we can obtain

$$\begin{aligned} dY_t =& \frac{\pi_t}{S_t} dS_t + r_t(Y_t - \pi_t)dt \\ =& \big(r_t Y_t + \pi_t(\mu_t - r_t)\big)dt + \pi_t \sigma_t dB_t, \qquad t \in [0, T]. \end{aligned}$$

Suppose that we want to construct a portfolio with a final payoff ξ at the terminal time T (for example European call option). We plan to get the minimal amount of Y_0 so that we can cover ξ

by a strategy π at the terminal time T, i.e. $Y_T = \xi$. Consider the BSDE (1.2) as below:

$$Y_t = \xi - \int_t^T \big(r_s Y_s + \pi_s(\mu_s - r_s)\big)ds - \int_t^T \pi_s \sigma_s dB_s, \qquad t \in [0, T], \tag{1.2}$$

where (Y, π) is the solution of the BSDE (1.2). Set $Z = \pi\sigma$, the BSDE (1.2) can be transformed into the following form:

$$Y_t = \xi - \int_t^T \left(r_s Y_s + \frac{\mu_s - r_s}{\sigma_s} Z_s\right) ds - \int_t^T Z_s dB_s, \qquad t \in [0, T]. \tag{1.3}$$

The BSDE (1.3) has an explicit solution:

$$Y_t = \mathbb{E}^{\mathbb{Q}}\left[\xi e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t\right], \qquad t \in [0, T],$$

where $\mathbb{Q}$ is the risk-neutral measure.

Example 1.2.2. ***(Stochastic control problem)*** *Bismut [1] (1973) studied the maximum principle in a stochastic optimal control problem by a linear BSDE approach. See more in Peng [20] (1993), Hamadène and Lepeltier [21] (1995), etc. Consider the following controlled function:*

$$\begin{cases} dX_t^v = b(t, X_t, v_t)dt + \sigma(t, X_t, v_t)dB_t, & t \in [0, T]; \\ X_0^v = x. \end{cases} \tag{1.4}$$

A feasible control $(v_t)_{0 \le t \le T}$ is a continuous adapted process valued in a compact set $V \in \mathbb{R}^d$. Denote by $\mathcal{V}$ the set of the feasible controls. Our aim is to maximize the objective function below:

$$J(v) = \mathbb{E}\left[g(X_T^v) + \int_0^T f(t, X_t^v, v_t)dB_t\right].$$

Define the following Hamiltonian function:

$$H(t, x, p, q) = b(t, x, v)p + \sigma(t, x, v)q + f(t, x, v).$$

We can get the associated BSDE:

$$P_t = \partial_x g(X_t^{v^*}) + \int_t^T \partial_x H(t, X_s^{v^*}, P_s, Q_s)ds - \int_t^T Q_s dB_s, \qquad t \in [0, T],$$

where the optimal control v^* is given by

$$v^* = \arg\max_{v \in \mathcal{V}} H(t, X_t^v, v, P_t, Q_t).$$

Example 1.2.3. ***(Representation of non-linear f-expectations)*** *Peng [22] (1997) introduced the notion of f-expectation. We first give the definition of the non-linear expectation:*

A nonlinear expectation $\mathcal{E} : \mathcal{L}^2 \to \mathbb{R}$ is an operator, such that

- *$X \le Y \Rightarrow \mathcal{E}[X] \le \mathcal{E}[Y]$, moreover, $X = Y \Leftrightarrow \mathcal{E}(X) = \mathcal{E}(Y)$, $\mathbb{P} - a.s.$;*
- *for any constant $c \in \mathbb{R}$, $\mathcal{E}(c) = c$.*

Apart from the property of linearity, f-expectations preserve all the other properties of classical expectation. Similarly to the classical case, we can define a related conditional f-expectation with respect to $\mathcal{F}$. For the BSDE (1.1), the generator f is Lipschitz uniformly in (Y, Z), More precisely, the f-expectation for a random variable ξ is defined as the initial value Y_0 of a classical BSDE (1.1), i.e. we denote by f-expectation the operator $\mathcal{E}^f$, where $\mathcal{E}^f(\xi) = Y_0$.

Example 1.2.4. ***(Feynman-Kac representation of PDEs)*** *There is a connection between semi-linear parabolic equations and BSDEs (see e.g. Nualart and Schoutens [23], Hu and Ma [24] (2004), Peng and Wang [25] (2016), etc.). Let X be the forward solution with the following form:*

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad & t \in [0, T]; \\ X_0 = x. \end{cases} \tag{1.5}$$

Denote by $\mathcal{L}$ the operator $\mathcal{L}v(t, x) = b(t, x)\partial_x v(t, x) + \frac{1}{2}\sigma^2(t, x)\partial^2_{xx}v(t, x)$. Let u be the solution of the PDE (1.6) below:

$$\begin{cases} \partial_t v(t, x) + \mathcal{L}v(t, x) + f(t, x, v(t, x), \sigma_x v(t, x)) = 0, \\ v(T, x) = g(x). \end{cases} \tag{1.6}$$

We can get the solution of the PDE (1.6) by the Feynman-Kac approach with the following BSDE (1.7). Set $Y_t := u(t, X_t)$, $Z_t := \partial_x u(t, X_t)$, apply the Itô's formula, we know that (Y, Z) is the solution of the BSDE (1.7) below:

$$Y_t = g(X_T) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \qquad t \in [0, T]. \tag{1.7}$$

This approach allows us to replace the numerical schemes for PDEs with BSDEs by Monte Carlo simulation (especially in higher dimensions). Similarly, we can also solve BSDEs by the viscosity solutions of PDEs.

1.2.1 Backward Stochastic Differential Equations with Jumps

In connection with optimal stochastic control, Tang and Li [26] (1994) considered the following form of BSDE (1.8) driven by a Brownian motion and an independent Poisson random measure μ:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s)ds - \int_t^T Z_s dB_s - \int_t^T \int_{\mathbb{E}} V_s(e)\tilde{\mu}(ds, de), \quad t \in [0, T], \tag{1.8}$$

where $\tilde{\mu}$ is the compensated measure associated with μ. Suppose that the filtration is generated by the following two mutually independent processes:

- a d-dimensional standard Brownian motion $B := (B_t)_{t \geq 0}$;
- a Poisson random measure μ on $\mathbb{R}^+ \times \mathbb{E}$, where $\mathbb{E} := \mathbb{R}^k \setminus \{0\}$ is equipped with its Borel fields $\mathcal{E}$, with compensator $\mu(dt, de) = dt\lambda(de)$, such that $\{\tilde{\mu}([0, t] \times A) = (\mu - \nu)([0, t] \times A)\}_{0 \leq t \leq T}$ is a martingale, for any $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. The measure λ is assumed to be σ-finite on $(\mathbb{E}, \mathcal{E})$ satisfying $\int_{\mathbb{E}} (1 \wedge |e|^2)\, \lambda(de) < \infty$.

It was first introduced by Tang and Li [26] (1994), then Barles et al. [27] (1997) proved the existence and uniqueness theorem under the smooth square integrability assumption and the Lipschitz condition, they also studied the relation with integral-partial differential equations. More can be found in Buckdahn and Pardoux [28] (1994), Situ [29] (1999), Hassani and Ouknine [30] (2002), Yin and Situ [31] (2003), Becherer [32] (2006), etc.

1.2.2 Reflected Backward Stochastic Differential Equations with One Obstacle

Reflected BSDE with one continuous lower reflecting obstacle driven by a Brownian motion was first considered by El Karoui et al. [33] (1997). A triple $(Y, Z, K) := (Y_t, Z_t, K_t)_{0\le t\le T}$ is a solution of the RBSDE (1.9) with the generator f, the terminal value ξ and the lower obstacle L:

$$\begin{cases} (i) & Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \quad t \in [0, T]; \\ (ii) & Y_t \ge L_t, \quad t \in [0, T]; \\ (iii) & \int_0^T (Y_t - L_t) dK_t = 0, \end{cases} \tag{1.9}$$

where K is a continuous increasing process to push upward the process Y above the obstacle L in a minimal way, the constraint (iii) expresses the fact that K_t only increases when $Y_t = L_t$. The obstacle L satisfies $L \in \mathcal{S}^2_{\mathcal{G}}(0, T; \mathbb{R})$ and $L_T \le \xi_T$. El Karoui et al. [33] (1997) proved the existence and uniqueness of RBSDEs under the smooth square integrability assumption and the Lipschitz condition through the two methods below:

- **Penalization method**: they considered the following classical penalized BSDE:

  $$Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s)ds + K^n_T - K^n_t - \int_t^T Z^n_s dB_s, \quad t \in [0, T],$$

  where
  $$K^n_t = n \int_0^t (Y^n_s - L_s)^- ds.$$

  The comparison theorem for BSDE (Pardoux and Peng [2]) implies the convergence of the sequence $(Y^n)_{n\ge 0}$, where $Y^n \le Y^{n+1}$. There exists an $\mathcal{F}$-adapted solution (Y, Z, K), which is the limit process of (Y^n, Z^n, K^n) in the following sense:

  $$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} |Y^n_t - Y_t|^2 + \int_0^T |Z^n_s - Z_s|^2 ds + \sup_{0\le t\le T} |K^n_t - K_t|^2\right] = 0.$$

- **Snell envelope method**: at each step of a Picard-type iterative procedure, any solution of the RBSDE with one obstacle is also the value of an optimal stopping problem.

After that, Hamadène and Ouknine [34] (2003) studied RBSDE with one obstacle driven by a Brownian motion and an independent Poisson random measure, and gave the existence and uniqueness theorem. Hamadène [35] (2008) and Essaky et al. [36] (2008) studied RBSDEs with one obstacle under different conditions. RBSDE with one obstacle can be applied in the pricing problem for American options (see El Karoui et. al [37] (1997)). See more in Kobylanski et. al [38] (2002), Peng and Xu [39] (2010), Essaky and Hassani [40] (2011), etc.

1.2.3 Reflected Backward Stochastic Differential Equations with Two Obstacles

Cvitanic and Karatzas [41] (1996) first studied reflected BSDE with a continuous continuous lower obstacle and a continuous upper obstacle under the smooth square integrability assumption and the Lipschitz condition. A quadruple $(Y, Z, K^+, K^-) := (Y_t, Z_t, K_t^+, K_t^-)_{0\le t\le T}$ is the solution of the RBSDE (1.10) with the generator f, the terminal value ξ and the obstacles L and V:

$$\begin{cases} (i) \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) \\ \qquad\qquad - \int_t^T Z_s dB_s, \qquad t \in [0, T]; \\ (ii) \quad V_t \ge Y_t \ge L_t, \qquad t \in [0, T]; \\ (iii) \quad \int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (V_t - Y_t)dK_t^- = 0. \end{cases} \tag{1.10}$$

where K^+ and K^- are continuous increasing processes, K^+ is to keep Y above the lower obstacle L, while K^- is to keep Y under the upper obstacle V. When $V \equiv \infty$ and $K^- \equiv 0$, we obtain a RBSDE with one lower obstacle (see Section 1.2.2). Cvitanic and Karatzas [41] proved that at each step of a Picard-type iterative procedure, any solution of a RBSDE with two obstacles can also be represented as the value of a Dynkin game (Dynkin and Yushkevich [42] (1969)). They further applied the penalization method and established the existence and uniqueness theorem of RBSDE (1.10) under a condition that the obstacles can be approximated by semi-martingales with absolutely continuous finite variation parts.

The existence of a solution of a RBSDE with two obstacles was obtained under one of the following two assumptions:

A.1 One of the obstacles L and V is regular (see e.g. Cvitanic and Karatzas [41] (1996), Hamadène et al. [43] (1997);

A.2 Mokobodski's condition, which requires that a process exists which is the difference between two non-negative super-martingales and lies between the obstacles L and V (see e.g. Hamadène and Lepeltier [44] (2000), Lepeltier and Xu [45] (2007).

However, both of these two conditions have disadvantages, **A.1** is somewhat restrictive, **A.2** is difficult to verify in practice. See more in Bahlali and Mezerdi [46] (2005), Essaky et. al [47] (2005), Hamadène and Hdhiri [48] (2006), Xu [49] (2007), etc.

1.2.4 Anticipated Backward Stochastic Differential Equations

A new type of BSDE, called anticipated BSDE whose generator includes the values of both the present and the future, was introduced by Peng and Yang [50] (2009), with the following form:

$$\begin{cases} (i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}), \;\; Z \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^d); \\ (ii) & Y_t = \xi_T + \int_t^T f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}\big)ds - \int_t^T Z_s dB_s, \quad t \in [0,T]; \\ (iii) & Y_t = \xi_t, \quad t \in (T, T+T^\delta]; \\ (iv) & Z_t = \alpha_t, \quad t \in (T, T+T^\delta], \end{cases} \tag{1.11}$$

where the anticipated processes ξ and α satisfy the assumption **H 2.1**, the anticipated times δ^1 and δ^2 satisfy **H 2.2**, the generator f satisfies the Lipschitz condition. Peng and Yang [50] gave the existence and uniqueness theorem and the comparison theorem for anticipated BSDE (1.11), showed the duality between anticipated BSDEs and stochastic differential delay equations. Øksendal et al. [51] (2011) extended this topic to ABSDEs driven by a Brownian motion and an independent Poisson random measure. Jeanblanc et al. [52] (2017) studied ABSDEs driven by a Brownian motion and a single jump process (with a jump at time τ). More can be seen in Wu et. al [53] (2012), Lu and Ren [54] (2013), Yang and Elliott [55] (2013), Yang and Elliott [56] (2016). etc.

1.2.5 Numerical Algorithms for Backward Stochastic Differential Equations

Chevance [57] (1997) provided the numerical methods for backward stochastic differential equations, [57] used the random time discretization scheme introduced by Bally [58] (1997) for the discrete algorithms and theoretical convergence proof. Ma and Zhang [59] (2005), Bouchard and Chassagneux [60] (2008) studied the representations and regularities of discrete RBSDE. Peng and Xu [61] (2011) studied numerical algorithms for BSDEs driven by Brownian motion. Gobet and Turkedjiev [62] (2011) used the least-squares regression method for the approximation of discrete BSDE. Xu [63] (2011) introduced a discrete penalization scheme and a discrete reflected scheme for RBSDE with two obstacles. Later Dumitrescu and Labart [64] (2016) extended to RBSDE with two obstacles driven by Brownian motion and an independent compensated Poisson process. Lin and Yang [65] (2014) studied the discrete BSDE with random terminal horizon.

1.2.6 Stochastic Differential Delay Equations

Stochastic differential delay equation (SDDE) is a new kind of SDE with coefficients containing present and past values of the solution process X. SDDEs can be applied in the area of finance, where the delay part can represent the memory or inertia in the financial system. It can also be employed in the area of biology, delays occur naturally in population dynamics models, e.g. the optimal harvesting problem of biological systems. SDDE was first introduced by Itô and Nisio [66] (1964).

$$\begin{cases} dX_s = f(s, X_s, X_{s-\delta})ds + g(s, X_s, X_{s-\delta})dB_s, \quad s \in [t, T+\delta]; \\ X_t = 1; \\ X_s = 0, \quad s \in [t-\delta, t). \end{cases} \tag{1.12}$$

where coefficients f and g satisfy local Lipschitz condition and linear growth condition. Mohammed [67] (1984) and Mao [68] (2007) gave the existence and uniqueness theorem of the following SDDE driven by a Brownian motion by the standard technique of Picard's iteration. Buckwar [69] (2000), Baker and Buckwar [70] (2000) studied the numerical algorithms for SDDEs.

1.3 Backward Stochastic Differential Equations with Default Risk

Default risk is the risk that an investor suffers a loss due to the inability of taking back the initial investment, it arises from a borrower failing to make required payments. The loss may be complete or partial (more see Kusuoka [71] (1999), Elliott et al. [72] (2000)). Blanchet-Scalliet and Jeanblanc [73] (2004) provided a concise exposition of theoretical results that appear in the defaultable model. Jeanblanc and Le Cam [74] (2009) proposed a study of the set of equivalent martingale measures in the context of credit modeling. Peng and Xu [75] (2009) studied BSDEs with default risk. Jiao and Pham [76] (2011) studied the optimal investment with counterparty risk. Song [77] (2014) studied the optional splitting formula in a progressively enlarged filtration and developed practical sufficient conditions for validity in the defaultable model. Jiao et al. [78] (2013) continued the research on the optimal investment under multiple default risk through a BSDE approach. Cordoni and Di Persio [79] (2016) studied the BSDE with delayed generator in a defaultable setting. In this paper, we focus on the study of reflected anticipated BSDE with two obstacles and default risk.

Peng and Xu [75] (2009) gave the existence and uniqueness theorem and the related comparison theorem for the following BSDE (1.13) with default risk in the enlarged filtration $\mathcal{G}$:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0, T]. \tag{1.13}$$

In the defaultable financial market, the terminal value ξ represents a contingent claim aimed to be replicated at the terminal time T, (Z, U) represents the hedging strategies.

1.3.1 Basis of the Defaultable Model

Let $\tau = \{\tau_i;\ i = 1, 2, ..., k\}$ be k non-negative random variables on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ satisfying

$$\mathbb{P}(\tau_i > 0) = 1; \qquad \mathbb{P}(\tau_i > t) > 0,\ \forall t > 0; \qquad \mathbb{P}(\tau_i = \tau_j) = 0,\ \ i \neq j.$$

For each i $(i = 1, ..., k)$, we define a right-continuous default process $H^i := (H^i_t)_{t\geq 0}$ by setting $H^i_t := 1_{\{\tau_i \leq t\}}$ and denote by $\mathcal{H}^i := (\mathcal{H}^i_t)_{t\geq 0}$ the associated filtration $\mathcal{H}^i_t := \sigma(H^i_s;\ 0 \leq s \leq t)$. We assume that $\mathcal{F}_0$ is trivial (it follows that $\mathcal{G}_0$ is trivial as well).

For the fixed terminal time $T \geq 0$, there are two kinds of information:

- one is from the assets prices, denoted by $\mathcal{F} := (\mathcal{F}_t)_{0\leq t\leq T}$;
- the other is from the default times $\{\tau_i;\ i = 1, ..., k\}$, denoted by $\{\mathcal{H}^i;\ i = 1, ..., k\}$.

Barlow [80] (1978) presented a martingale approach to work on the decomposition of a process into its past and future relative to a random time and studied the related enlarged filtration. Al-Hussaini and Elliott [81] (1987) studied the enlarged filtrations for diffusions. Song [82] (2013) provided the local solution methods for the problem of enlarged filtrations. Jeanblanc and Song [83] (2015) considered the martingale representation property in progressively enlarged filtrations.

For the defaultable model, the enlarged filtration is denoted by $\mathcal{G} := (\mathcal{G}_t)_{0\leq t\leq T}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee ... \vee \mathcal{H}_t^k$. Generally, the processes H^i $(i = 1, ..., k)$ are obviously $\mathcal{G}$-adapted, but they are not necessarily $\mathcal{F}$-adapted, i.e. a $\mathcal{G}$-stopping time is not necessarily an $\mathcal{F}$-stopping time. Let $G := G_t$, where $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$, i.e. $G_t^i = \mathbb{P}(\tau_i > t|\mathcal{F}_t)$, for each $i = 1, ..., k$. In the following, G^i is assumed to be continuous. The random default times τ_i are totally inaccessible $\mathcal{G}$-stopping times.

We introduce the following assumptions (see Kusuoka [71] (1999), Bielecki et al. [84] (2007)):

H 1.1. *There exist $\mathcal{F}$-adapted processes $\gamma^i \geq 0$ $(i = 1, ..., k)$, such that*

$$M_t^i = H_t^i - \int_0^t 1_{\{\tau_i > s\}} \gamma_s^i ds$$

are $\mathcal{G}$-martingales under $\mathbb{P}$.

H 1.2. *Every $\mathcal{F}$-local martingale is a $\mathcal{G}$-local martingale.*

For the sake of simplicity, we denote

$$H_t := \left(H_t^1, H_t^2, ..., H_t^k\right)^T; \qquad M_t := \left(M_t^1, M_t^2, ..., M_t^k\right)^T;$$
$$1_{\{\tau_i > t\}}\gamma_t := \left(1_{\{\tau_1 > t\}}\gamma_t^1, 1_{\{\tau_2 > t\}}\gamma_t^2, ..., 1_{\{\tau_k > t\}}\gamma_t^k\right)^T.$$

where $(\,\cdot\,)^T$ is the transpose.

Remark 1.3.1. ***(Interpretation of intensity)*** *Bielecki et al. [84] (2007) studied the $\mathcal{G}$-intensity of the default times τ_i $(i = 1, ..., k)$ and gave the explicit formula to compute it. From the submartingale property of G and the Doob-Meyer decomposition theorem, we know that $G^i = Z^i + A^i$, where Z^i is an $\mathcal{F}$-martingale and A^i an $\mathcal{F}$-predictable increasing process. Since the process G^i is assumed to be continuous, from Proposition 3.1.2 in Bielecki et al. [84], it follows that the process M^i is a $\mathcal{G}$-martingale independent of the Brownian motion B,*

$$M_t^i = H_t^i - \int_0^t 1_{\{\tau_i > s\}} \frac{dA_s^i}{1 - G_s^i} ds, \qquad t \in [0, T].$$

Suppose that the increasing process A^i is absolutely continuous with respect to the Lebesgue measure, then there exists an $\mathcal{F}$-adapted process γ^i which is called as the intensity process, such that

$$\gamma_t^i := \frac{dA_t^i}{1 - G_t^i} dt.$$

From Lemma 3.1.5 in Bielecki et al. [84], it follows that the intensity process γ has the following form

$$\gamma_t^i = \lim_{\Delta \to 0} \frac{\mathbb{P}(t < \tau_i \leq t + \Delta | \mathcal{F}_t)}{\Delta\, \mathbb{P}(\tau_i > t | \mathcal{F}_t)}, \qquad t \in [0, T].$$

1.3.2 Some Results for BSDEs with Default Risk

Peng and Xu [75] (2009) introduced the following assumptions for the terminal value and the generator:

H 1.3. *The terminal value* $\xi \in \mathcal{L}^2(\mathcal{G}_T; \mathbb{R})$.

H 1.4. *The generator* $f(w, t, y, z, u) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$, *satisfies:*

- ***(a)*** $f(\cdot, 0, 0, 0) \in \mathcal{L}^2_{\mathcal{G}}(0, T; \mathbb{R})$;
- ***(b) Lipschitz condition****: for any* $(t, y, z, u), (t, y', z', u') \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k$, *there exists a constant* $L \geq 0$ *such that*

  $$|f(t, y, z, u) - f(t, y', z', u')| \leq L \left(|y - y'| + |z - z'| + |u - u'| 1_{\{\tau > t\}} \sqrt{\gamma_t} \right);$$

- ***(c)*** *for any* $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $(u^i - \bar{u}^i) 1_{\{\tau^i > t\}} \gamma_t^i \neq 0$, *the following holds:*

  $$\frac{f(t, y, z, \tilde{u}^{i-1}) - f(t, y, z, \tilde{u}^i)}{(u^i - \bar{u}^i) 1_{\{\tau^i > t\}} \gamma_t^i} > -1,$$

  where $\tilde{u}^i = (\bar{u}^1, \bar{u}^2, ..., \bar{u}^i, u^{i+1}, ..., u^k)$, u^i *is the i-th element of* u.

Then we introduce the existence and uniqueness theorem (Theorem 3.1 in Peng and Xu [75]).

Theorem 1.3.1. *(**Existence and uniqueness theorem for BSDEs with default risk**) Suppose that the terminal value* ξ *and the generator* f *satisfy the assumptions* ***H 1.3*** *and* ***H 1.4 (a)***, ***H 1.4 (b)***. *Then the BSDE (1.13) has the unique triple solution* $(Y, Z, U) \in \mathcal{S}^2_{\mathcal{G}}(0, T; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T; \mathbb{R}^k)$.

Peng and Xu [75] also gave the comparison theorem for 1-dimensional BSDEs with default risk (Theorem 3.2 in [75]), where it needs the assumption **H 1.4 (c)** for u.

Theorem 1.3.2. *(**Comparison theorem for BSDEs with default risk**) Suppose that the terminal value* ξ *and the generator* f *satisfy the assumptions* ***H 1.3*** *and* ***H 1.4***. $\bar{f} \in \mathcal{L}^2_{\mathcal{G}}(0, T; \mathbb{R})$. (Y, Z, U) *is the unique solution of the BSDE (1.13), and* $(\bar{Y}, \bar{Z}, \bar{U})$ *is the unique solution of the following equation:*

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}_s ds - \int_t^T \bar{Z}_s dB_s - \int_t^T \bar{U}_s dM_s, \quad t \in [0, T]. \tag{1.14}$$

If $\xi_t \geq \bar{\xi}_t$ *and* $f(t, \bar{Y}_t, \bar{Z}_t, \bar{U}_t) \geq \bar{f}_t$, *a.e., a.s, then* $Y_t \geq \bar{Y}_t$, *a.e., a.s.*
Besides, the strict comparison theorem holds true, i.e. $Y_0 = \bar{Y}_0 \Leftrightarrow \xi = \bar{\xi}, f(t, \bar{Y}_t, \bar{Z}_t, \bar{U}_t) \equiv \bar{f}_t$.

From Kusuoka [71] (1999), there exists the following martingale representation theorem.

Theorem 1.3.3. *(**Kusuoka's martingale representation theorem**) Suppose that the assumptions **H** 1.1 and **H** 1.2 hold, for any $\mathcal{G}$-square integrable martingale $(\varphi_t)_{0\leq t\leq T}$, there exist the $\mathcal{G}$-adapted processes $\eta : \Omega \times [0,T] \to \mathbb{R}^d$ and $\mu := (\mu^1, \mu^2, ..., \mu^k)^T : \Omega \times [0,T] \to \mathbb{R}^k$, such that the martingale φ has the following unique representation:*

$$\varphi_t = \mathbb{E}[\varphi_0] + \int_0^t \eta_s dB_s + \int_0^t \mu_s dM_s, \qquad t \in [0,T],$$

where

$$\int_0^t \mu_s dM_s := \sum_{i=1}^k \int_0^t \mu_s^i dM_s, \qquad t \in [0,T],$$

and satisfying

$$\mathbb{E}\left[\int_0^T \left(|\eta_s|^2 + |\mu_s^i|^2 1_{\{\tau^i > s\}} \gamma_s^i\right) ds\right] < \infty, \quad i = 1, ..., k.$$

For completeness, we also give we the Itô's formula for rcll semi-martingale (Protter [85]).

Theorem 1.3.4. *(**Itô's formula for rcll semi-martingale**) Let $X := (X_t)_{0\leq t\leq T}$ be a rcll semi-martingale, g be a real value function in $\mathcal{C}^2$. Therefore, $g(X)$ is also a semi-martingale, such that*

$$\begin{aligned} g(X_t) =& g(X_0) + \int_0^t g'(X_{s-})dX_s + \frac{1}{2}\int_0^t g''(X_s)d[X]_s^c \\ &+ \sum_{0<s\leq t} \left[g(X_s) - g(X_{s-}) - g'(X_{s-})\Delta X_s\right], \end{aligned}$$

where $[X]$ is the second variation of X, $[X]^c$ is the continuous part of $[X]$, $\Delta X_s = \Delta X_s - \Delta X_{s-}$.

1.4 Dissertation Structure

The dissertation is organized as follows:

Chapter 2 Anticipated BSDEs with Default Risk

In Chapter 2, we study ABSDEs with default risk and its applications. A triple $(Y, Z, U) := (Y_t, Z_t, U_t)_{0\leq t\leq T+T^\delta}$ is a solution of the ABSDE with the generator f, the terminal value

ξ_T, the anticipated processes ξ, α and β, and the anticipated times δ^1, δ^2, δ^3, such that

$$\begin{cases} (i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}), \ \ Z \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^d), \\ & U \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^k); \\ (ii) & Y_t = \xi_T + \int_t^T f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big)ds \\ & \qquad - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0, T]; \\ (iii) & Y_t = \xi_t, \qquad t \in (T, T+T^\delta]; \\ (iv) & Z_t = \alpha_t, \qquad t \in (T, T+T^\delta]; \\ (v) & U_t = \beta_t, \qquad t \in (T, T+T^\delta]. \end{cases} \tag{1.15}$$

In Section 2.2, we prove the existence and uniqueness theorem of the ABSDE (1.15). Section 2.4 illustrates the duality between anticipated BSDEs and the stochastic differential delay equations (SDDE). Section 2.5 represents an application in stochastic control problem in the default setting. We study the relation between ABSDEs and obstacle problems for non-linear parabolic PDEs in a defaultable setting in Section 2.6.

Chapter 3 Reflected Anticipated BSDEs with One Obstacle and Default Risk

In Chapter 3, we study RABSDEs with one obstacle and default risk and the relevant applications. A quadruple $(Y, Z, U, K) := (Y_t, Z_t, U_t, K_t)_{0 \le t \le T+T^\delta}$ is a solution for the following RABSDE (1.16) with the generator f, the terminal value ξ_T, the anticipated processes ξ, α and β, the anticipated times δ^1, δ^2, δ^3, and the obstacle L, such that

$$\begin{cases} (i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}), \ \ Z \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^d), \\ & U \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^k), \ \ K \in \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R}); \\ (ii) & Y_t = \xi_T + \int_t^T f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big)ds \\ & \qquad + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0, T]; \\ (iii) & Y_t \ge L_t, \qquad t \in [0, T]; \\ (iv) & Y_t = \xi_t, \qquad t \in (T, T+T^\delta]; \\ (v) & Z_t = \alpha_t, \qquad t \in (T, T+T^\delta]; \\ (vi) & U_t = \beta_t, \qquad t \in (T, T+T^\delta]; \\ (vii) & \int_0^T (Y_t - L_t) dK_t = 0. \end{cases} \tag{1.16}$$

In Section 3.2, we use two methods, i.e. penalization method and Snell envelope method to prove the existence and uniqueness theorem of the RABSDE (1.16). Section 3.3 represents an application in optimal stopping-control problem in the default setting. We illustrate the relation between linear RABSDEs with one obstacle and stochastic differential delay equations in a defaultable setting in Section 3.4.

Chapter 4 Reflected Anticipated BSDEs with Two Obstacles and Default Risk

In Chapter 4, we study RABSDEs with two obstacles and default risk and the relevant applications. $(Y, Z, U, K^+, K^-) := (Y_t, Z_t, U_t, K_t^+, K_t^-)_{0 \le t \le T+T^\delta}$ is a solution for RABSDE with the generator f, the terminal value ξ_T, the anticipated processes ξ, α, β, the anticipated times δ^1, δ^2, δ^3, and the obstacles L and V, such that

$$\begin{cases} (i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R}), \ \ Z \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R}^d), \\ & U \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R}^k); \ \ K^{\pm} \in \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R}); \\ (ii) & Y_t = \xi_T + \int_t^T f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big) ds \\ & \qquad +(K^+_T - K^+_t) - (K^-_T - K^-_t) - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0, T]; \\ (iii) & V_t \geq Y_t \geq L_t, \qquad t \in [0, T]; \\ (iv) & Y_t = \xi_t, \qquad t \in (T, T+T^{\delta}]; \\ (v) & Z_t = \alpha_t, \qquad t \in (T, T+T^{\delta}]; \\ (vi) & U_t = \beta_t, \qquad t \in (T, T+T^{\delta}]; \\ (vii) & \int_0^T (Y_t - L_t) dK^+_t = \int_0^T (V_t - Y_t) dK^-_t = 0. \end{cases} \tag{1.17}$$

In Section 4.2, we combine the penalization method and the fixed point method to prove the existence and uniqueness theorem of the RABSDE (1.17). We represent the relation between linear RABSDEs with two obstacles and stochastic differential delay equations in a defaultable setting in Section 4.3. Section 4.4 illustrates the relation between RABSDEs and obstacle problem for non-linear parabolic PDEs in a defaultable setting.

Chapter 5 Numerical Algorithms for RABSDEs with Two Obstacles and Default Risk

In Chapter 5, we study numerical algorithms for RABSDEs with two obstacles and default risk. We introduce the implicit and the explicit versions of two discrete schemes, i.e. the discrete penalization scheme in Section 5.2 and the discrete reflected scheme in Section 5.3. Section 5.4 completes the convergence results of the numerical algorithms which were provided in the previous sections. In Section 5.5 and Section 5.6, we illustrate the performance of the algorithms by a simulation example and an application in American game options in the defaultable setting.

5.2.1 Implicit Discrete Penalization Scheme

$$\begin{cases} y_i^{p,n} = y_{i+1}^{p,n} + f^n(t_i, y_i^{p,n}, \bar{y}_i^{p,n}, z_i^{p,n}, u_i^{p,n}) \Delta^n + k_i^{+p,n} - k_i^{-p,n} \\ \qquad\qquad -z_i^{p,n} \Delta B_{i+1}^n - u_i^{p,n} \Delta M_{i+1}^n, \qquad i \in [0, n-1]; \\ k_i^{+p,n} = p\Delta^n (y_i^{p,n} - L_i^n)^-, \qquad i \in [0, n-1]; \\ k_i^{-p,n} = p\Delta^n (y_i^{p,n} - V_i^n)^+, \qquad i \in [0, n-1]; \\ y_i^{p,n} = \xi_i^n, \qquad i \in [n, n^{\delta}], \end{cases}$$

5.2.2 Explicit Discrete Penalization Scheme

$$\begin{cases} \tilde{y}_i^{p,n} = \tilde{y}_{i+1}^{p,n} + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}], \bar{\tilde{y}}_i^{p,n}, \tilde{z}_i^{p,n}, \tilde{u}_i^{p,n})\Delta^n + \tilde{k}_i^{+p,n} - \tilde{k}_i^{-p,n} \\ \qquad -\tilde{z}_i^{p,n}\Delta B_{i+1}^n - \tilde{u}_i^{p,n}\Delta M_{i+1}^n, \quad i \in [0, n-1]; \\ \tilde{k}_i^{+p,n} = p\Delta^n(\tilde{y}_i^{p,n} - L_i^n)^-, \quad i \in [0, n-1]; \\ \tilde{k}_i^{-p,n} = p\Delta^n(\tilde{y}_i^{p,n} - V_i^n)^+, \quad i \in [0, n-1]; \\ \tilde{y}_i^{p,n} = \xi_i^n, \quad i \in [n, n^\delta], \end{cases}$$

5.3.1 Implicit Discrete Reflected Scheme

$$\begin{cases} y_i^n = y_{i+1}^n + f^n(t_i, y_i^n, \bar{y}_i^n, z_i^n, u_i^n)\Delta^n + k_i^{+n} - k_i^{-n} \\ \qquad -z_i^n\Delta B_{i+1}^n - u_i^n\Delta M_{i+1}^n, \quad i \in [0, n-1]; \\ V_i^n \geq y_i^n \geq L_i^n, \quad i \in [0, n-1]; \\ k_i^{+n} \geq 0,\ k_i^{-n} \geq 0,\ k_i^{+n}k_i^{-n} = 0, \quad i \in [0, n-1]; \\ (y_i^n - L_i^n)k_i^{+n} = (y_i^n - V_i^n)k_i^{-n} = 0, \quad i \in [0, n-1]; \\ y_i^n = \xi_i^n, \quad i \in [n, n^\delta]. \end{cases}$$

5.3.2 Explicit Discrete Reflected Scheme

$$\begin{cases} \tilde{y}_i^n = \tilde{y}_{i+1}^n + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n], \bar{\tilde{y}}_i^n, \tilde{z}_i^n, \tilde{u}_i^n)\Delta^n + \tilde{k}_i^{+n} - \tilde{k}_i^{-n} \\ \qquad -\tilde{z}_i^n\Delta B_{i+1}^n - \tilde{u}_i^n\Delta M_{i+1}^n, \quad i \in [0, n-1]; \\ V_i^n \geq \tilde{y}_i^n \geq L_i^n, \quad i \in [0, n-1]; \\ \tilde{k}_i^{+n} \geq 0,\ \tilde{k}_i^{-n} \geq 0,\ \tilde{k}_i^{+n}\tilde{k}_i^{-n} = 0, \quad i \in [0, n-1]; \\ (\tilde{y}_i^n - L_i^n)\tilde{k}_i^{+n} = (\tilde{y}_i^n - V_i^n)\tilde{k}_i^{-n} = 0, \quad i \in [0, n-1]; \\ \tilde{y}_i^n = \xi_i^n, \quad i \in [n, n^\delta]. \end{cases}$$

where $\bar{y}_i^{p,n} = \mathbb{E}^{\mathcal{G}_i^n}[y_{\bar{i}}^{p,n}]$, $\bar{i} = i + [\frac{\delta}{\Delta^n}]$.

Chapter 2

Anticipated BSDEs with Default Risk

Peng and Yang [50] (2009) introduced anticipated BSDEs whose generator includes the values of both the present and the future. Øksendal et al. [51] (2011) extended this topic to ABSDEs driven by a Brownian motion and an independent Poisson random measure. Jeanblanc et al. [52] (2017) studied ABSDEs driven by a Brownian motion and a single jump process (with a jump at time τ). More previous research can be seen in Section 1.2.4.

In this chapter, we study anticipated backward stochastic differential equations driven by a Brownian motion and a mutually independent martingale in a defaultable setting. It can be used in financial and other natural models (e.g. population growth), where people's memory plays a role in the dynamics system.

This chapter is organized as follows, Section 2.1 states the basic assumptions for ABSDEs with default risk. In Section 2.2, we prove the existence and uniqueness theorem of the ABSDE (2.1). Section 2.4 illustrates the duality between anticipated BSDEs and the stochastic differential delay equations (SDDE). Section 2.5 represents an application in stochastic control problem in the default setting. We study the relation between ABSDEs and obstacle problems for non-linear parabolic PDEs in a defaultable setting in Section 2.6.

2.1 Basic Assumptions

In this Chapter, we consider the following ABSDE with default risk and the coefficient $(f, \xi, \alpha, \beta, \delta^1, \delta^2, \delta^3)$. A triple $(Y, Z, U) := (Y_t, Z_t, U_t)_{0 \le t \le T+T^\delta}$ is a solution of the ABSDE with the generator f, the terminal value ξ_T, the anticipated processes ξ, α and β, and the anticipated times δ^1, δ^2, δ^3, such that

$$\begin{cases} (i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}), \;\; Z \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^d), \\ & U \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^k); \\ (ii) & Y_t = \xi_T + \int_t^T f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big) ds \\ & \qquad - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0, T]; \\ (iii) & Y_t = \xi_t, \qquad t \in (T, T+T^\delta]; \\ (iv) & Z_t = \alpha_t, \qquad t \in (T, T+T^\delta]; \\ (v) & U_t = \beta_t, \qquad t \in (T, T+T^\delta]. \end{cases} \tag{2.1}$$

We first introduce the following assumptions for the ABSDE (2.1) (Peng and Yang [50]):

H 2.1. *The anticipated processes* $\xi \in \mathcal{L}^2_{\mathcal{G}}(T, T+T^\delta; \mathbb{R}^d)$, $\alpha \in \mathcal{L}^2_{\mathcal{G}}(T, T+T^\delta; \mathbb{R}^d)$, $\beta \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(T, T+T^\delta; \mathbb{R}^k)$, *here* ξ, α *and* β *are the given processes.*

H 2.2. $\delta^1(\cdot)$, $\delta^2(\cdot)$ *and* $\delta^3(\cdot) : [0,T] \to \mathbb{R}^+$ *are continuous functions satisfying:*

(a) *there exists a constant* $T^\delta \geq 0$*, such that for any* $t \in [0,T]$*:*

$$t+\delta^1(t) \leq T+T^\delta, \quad t+\delta^2(t) \leq T+T^\delta, \quad t+\delta^3(t) \leq T+T^\delta;$$

(b) *there exists a constant* $L^\delta \geq 0$*, such that for any* $t \in [0,T]$ *and any non-negative integrable function* $g(\cdot)$*:*

$$\int_t^T g(s+\delta^1(s))ds \leq L^\delta \int_t^{T+T^\delta} g(s)ds, \quad \int_t^T g(s+\delta^2(s))ds \leq L^\delta \int_t^{T+T^\delta} g(s)ds,$$
$$\int_t^T g(s+\delta^3(s))ds \leq L^\delta \int_t^{T+T^\delta} g(s)ds.$$

H 2.3. *The generator* $f(w,t,y,\bar{y}_r,z,\bar{z}_r,u,\bar{u}_r) : \Omega \times [0,T+T^\delta] \times \mathbb{R} \times \mathcal{S}^2_{\mathcal{G}}(t,T+T^\delta;\mathbb{R}) \times \mathbb{R}^d \times \mathcal{L}^2_{\mathcal{G}}(t,T+T^\delta;\mathbb{R}^d) \times \mathbb{R}^k \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(t,T+T^\delta;\mathbb{R}^k) \to \mathbb{R}$ *satisfies:*

(a) $f(\cdot,0,0,0,0,0,0) \in \mathcal{L}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R})$;

(b) ***Lipschitz condition****: for any* $t \in [0,T]$, $r \in [t,T+T^\delta]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u, u' \in \mathbb{R}^k$, $\bar{y}, \bar{y}' \in \mathcal{S}^2_{\mathcal{G}}(t,T+T^\delta;\mathbb{R})$, $\bar{z}, \bar{z}' \in \mathcal{L}^2_{\mathcal{G}}(t,T+T^\delta;\mathbb{R}^d)$, $\bar{u}, \bar{u}' \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t,T+T^\delta;\mathbb{R}^k)$, *there exists a constant* $L \geq 0$*, such that*

$$\begin{aligned}
&\big|f(t,y,\bar{y}_r,z,\bar{z}_r,u,\bar{u}_r) - f(t,y',\bar{y}'_r,z',\bar{z}'_r,u',\bar{u}'_r)\big| \\
\leq &L\Big(|y-y'| + \mathbb{E}^{\mathcal{G}_t}|\bar{y}_r-\bar{y}'_r| + |z-z'| + \mathbb{E}^{\mathcal{G}_t}|\bar{z}_r-\bar{z}'_r| \\
&+ |u-u'|1_{\{\tau>t\}}\sqrt{\gamma_t} + \mathbb{E}^{\mathcal{G}_t}|\bar{u}_r-\bar{u}'_r|1_{\{\tau>t\}}\sqrt{\gamma_t}\Big);
\end{aligned}$$

(c) *for any* $t \in [0,T]$, $r \in [t,T+T^\delta]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u, u' \in \mathbb{R}^k$, $\bar{y}, \bar{y}' \in \mathcal{L}^2_{\mathcal{G}}(t,T+T^\delta;\mathbb{R})$, $\bar{z}, \bar{z}' \in \mathcal{L}^2_{\mathcal{G}}(t,T+T^\delta;\mathbb{R}^d)$, $\bar{u}, \bar{u}' \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t,T+T^\delta;\mathbb{R}^k)$, *the following holds:*

$$\frac{f(t,y,\bar{y}_r,z,\bar{z}_r,\tilde{u}^{i-1},\bar{u}_r) - f(t,y,\bar{y}_r,z,\bar{z}_r,\tilde{u}^i,\bar{u}_r)}{(u^i-\tilde{u}^i)1_{\{\tau^i>t\}}\gamma^i_t} > -1,$$

where $\tilde{u}^i = (\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^i, u^{i+1}, ..., u^k)$, u^i *is the i-th element of u.*

Remark 2.1.1. $U := (U_t)_{0\leq t\leq T}$ *is well defined only on* $[0,\tau\wedge T] \cap \{t;\ \gamma_t \neq 0\}$*, i.e.* $dM_t \equiv 0$ *on* $[\tau\wedge T, T] \cup \{t;\ \gamma_t = 0\}$.

Remark 2.1.2. *For any* $t \in [\tau\wedge T, T]$*, the generator is independent of u, i.e. we have* $f(t,y,\bar{y}_r,z,\bar{z}_r,u,\bar{u}_r) \equiv f(t,y,\bar{y}_r,z,\bar{z}_r)$*. The financial explanation is that contingent claim is no longer influenced by default risk after the default has taken place.*

2.2 Existence and Uniqueness Theorem for ABSDEs with Default Risk

We first introduce the following approximation lemma (Lemma 3.1 in Peng and Xu [75] (2009)).

Lemma 2.2.1. *For a fixed $\xi \in \mathcal{L}^2(\mathcal{G}_T;\mathbb{R})$, let $g(t)$ be a $\mathcal{G}_t$-adapted process satisfying*

$$\mathbb{E}\left(\int_0^T |g(t)|dt\right)^2 < \infty,$$

there exists a unique triple of processes $(y,z,u) \in \mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R}^{1\times d})$, satisfying the BSDE below:

$$y_t = \xi + \int_t^T g(s)ds - \int_t^T z_s dB_s - \int_t^T u_s dM_s, \qquad t \in [0,T]. \tag{2.2}$$

If $g \in \mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R})$, then $(y_\cdot, z_\cdot, u_\cdot) \in \mathcal{S}^2_{\mathcal{G}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^k)$. We have the following basic estimation:

$$\begin{aligned}&|y_t|^2 + \mathbb{E}^{\mathcal{G}_t}\left[\int_t^T e^{c(s-t)}\left(\frac{c}{2}|y_s|^2 + |z_s|^2 + |u_s|^2 1_{\{\tau>s\}}\gamma_s\right)ds\right]\\ \leq&\mathbb{E}^{\mathcal{G}_t}\left[e^{c(T-t)}|\xi|^2\right] + \frac{2}{c}\left[\mathbb{E}^{\mathcal{G}_t}\int_t^T e^{c(s-t)}|g(s)|^2 ds\right].\end{aligned} \tag{2.3}$$

In particular,

$$\begin{aligned}&|y_0|^2 + \mathbb{E}\left[\int_0^T e^{cs}\left(\frac{1}{2}|y_s|^2 + |z_s|^2 + |u_s|^2 1_{\{\tau>s\}}\gamma_s\right)ds\right]\\ \leq&\mathbb{E}\left[e^{cT}|\xi|^2\right] + \frac{2}{c}\mathbb{E}\left[\int_0^T e^{cs}|g(s)|^2 ds\right],\end{aligned} \tag{2.4}$$

where $c > 0$ is an arbitrary constant. Moreover,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y_t|^2\right] \leq K\mathbb{E}\left[|\xi|^2 + \int_0^T |g(s)|^2 ds\right], \tag{2.5}$$

where K is a constant depending only on T.

Now we use the fixed point method to obtain the existence and uniqueness theorem for anticipated BSDEs with default risk in the general frame.

Theorem 2.2.1. ***(Existence and uniqueness theorem for ABSDEs with default risk)*** *Suppose that the anticipated processes ξ, α and β satisfy the assumption **H** 2.1, the generator f satisfies **H** 2.3 **(a)** and **H** 2.3 **(b)**. δ^1, δ^2 and δ^3 satisfy **H** 2.2. Then ABSDE (2.1) has the unique triple solution $(Y,Z,U) \in \mathcal{S}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}^k)$.*

Proof. Define $\mathcal{D} := \mathcal{S}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}^k)$. Define the following mapping:

$$\Phi:\ \mathcal{D} \to \mathcal{D};$$

$$(y,z,u) \to \Phi(y,z,u) := (Y,Z,U).$$

First, we prove that Φ is a contraction mapping. Set

$$\begin{cases} Y_t = & \xi_T + \int_t^T f\left(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\right) ds \\ & - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0,T]; \\ Y_t = & \xi_t, \qquad t \in (T, T+T^\delta]; \\ Z_t = & \alpha_t, \qquad t \in (T, T+T^\delta]; \\ U_t = & \beta_t, \qquad t \in (T, T+T^\delta]. \end{cases} \tag{2.6}$$

For any (y,z,u), $(y',z',u') \in \mathcal{D}$, denote:

$$\hat{y} = y - y'; \qquad \hat{z} = z - z'; \qquad \hat{u} = u - u';$$
$$\hat{Y} = Y - Y'; \qquad \hat{Z} = Z - Z'; \qquad \hat{U} = U - U'.$$

Applying the Itô's formula for rcll semi-martingale (Theorem 1.3.4), we can obtain

$$\begin{aligned} & |\hat{Y}_0|^2 + \int_0^T e^{cs} \left(c|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau > s\}} \gamma_s \right) ds \\ =& 2 \int_0^T e^{cs} \hat{Y}_s \big| f\left(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\right) \\ & - f\left(s, y'_s, y'_{s+\delta^1(s)}, z'_s, z'_{s+\delta^2(s)}, u'_s, u'_{s+\delta^3(s)}\right) \big| ds \\ & - 2\int_0^T e^{cs} \hat{Y}_s \hat{Z}_s dB_s - \int_0^T e^{cs} \left(2\hat{Y}_{s-}\hat{U}_s + |\hat{U}_s|^2 \right) dM_s, \end{aligned}$$

where $c > 0$ is a constant. Since $\int_0^T e^{cs} \hat{Y}_s^{(n)} \hat{Z}_s^{(n)} dB_s$ and $\int_0^T e^{cs} \left(2\hat{Y}_{s-}^{(n)} \hat{U}_s^{(n)} + |\hat{U}_s^{(n)}|^2 \right) dM_s$ are $\mathcal{G}$-martingales, $|\hat{Y}_0^{(n)}|^2 \geq 0$, by the Fubini's Theorem, **H** 2.2 and the Lipschitz condition for the generator f, it follows

$$\begin{aligned} & \mathbb{E}\left[\int_0^T e^{cs} \left(\frac{c}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau > s\}} \gamma_s \right) ds \right] \\ \leq & \frac{2}{c} \mathbb{E}\Big[\int_0^T e^{cs} \hat{Y}_s \Big| f\left(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\right) \\ & \qquad - f\left(s, y'_s, y'_{s+\delta^1(s)}, z'_s, z'_{s+\delta^2(s)}, u'_s, u'_{s+\delta^3(s)}\right) \Big|^2 ds \Big] \\ \leq & \frac{2L^2}{c} \mathbb{E}\Big[\int_0^T e^{cs} |\hat{Y}_s| \Big(|\hat{y}_s| + \mathbb{E}^{\mathcal{G}_s} |\hat{y}_{s+\delta^1(s)}| + |\hat{z}_s| + \mathbb{E}^{\mathcal{G}_s} |\hat{z}_{s+\delta^2(s)}| \\ & \qquad + |\hat{u}_s| 1_{\{\tau > s\}} \sqrt{\gamma_s} + \mathbb{E}^{\mathcal{G}_s} |\hat{u}_{s+\delta^3(s)}| 1_{\{\tau > s\}} \sqrt{\gamma_s} \Big)^2 ds \Big] \\ \leq & \frac{12L^2}{c} \mathbb{E}\left[\int_0^T e^{cs} \left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau > s\}} \gamma_s \right) ds \right] \\ & + \frac{12L^2 L^\delta}{c} \mathbb{E}\Big[\int_0^{T+T^\delta} e^{cs} \left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau > s\}} \gamma_s \right) ds \Big] \\ \leq & \frac{12L^2 + 12L^2 L^\delta}{c} \mathbb{E}\Big[\int_0^{T+T^\delta} e^{cs} \left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau > s\}} \gamma_s \right) ds \Big]. \end{aligned}$$

where $L \geq 0$, $L^\delta \geq 0$ are constants. Set $c = 12L^2 + 12L^2L^\delta + 2$, consequently,

$$\mathbb{E}\left[\int_0^{T+T^\delta} e^{cs}\left(\frac{c}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right]$$
$$\leq \frac{1}{2}\mathbb{E}\left[\int_0^{T+T^\delta} e^{cs}\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right].$$

Therefore, Φ is a contraction mapping on $\mathcal{D}$ equipped with the norm defined as below:

$$\|(Y, Z, U)\|_c := \left\{\mathbb{E}\int_0^{T+T^\delta} e^{cs}\left(|Y_s|^2 + |Z_s|^2 + |U_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right\}^{\frac{1}{2}}.$$

From the Banach fixed point theorem, there exists a unique fixed point $(Y, Z, U) \in \mathcal{L}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}^k)$, which is the solution of the ABSDE (2.1). By **H 2.3** and **H 2.2**, it follows that $f(t, Y_t, Y_{t+\delta^1(t)}, Z_t, Z_{t+\delta^2(t)}, U_t, U_{t+\delta^3(t)}) \in \mathcal{L}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R})$, by Lemma 2.2.1, we can obtain that $Y \in \mathcal{S}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R})$. □

2.3 Comparison Theorem for 1-dimensional ABSDEs with Default Risk

Now we give the comparison theorem for 1-dimensional ABSDEs with default risk (Theorem 2.3.1), which can be used to compare the solutions of two ABSDEs, in order to get the upper price of a contingent claim in the evaluation or hedging problem. Here we need the third assumption **H 2.3 (c)** for u, which is stronger than the conditions for the existence and uniqueness theorem.

Theorem 2.3.1. ***(Comparison theorem for ABSDEs with default risk)*** *Suppose that the anticipated processes $\xi^{(1)}$, $\xi^{(2)}$ satisfy the assumption **H 2.1**, the generator f_1, f_2 satisfy **H 2.3 (a)**, **H 2.3 (b)** and **H 2.3 (c)**, δ satisfies **H 2.2**. $(Y^{(1)}, Z^{(1)}, U^{(1)})$ and $(Y^{(2)}, Z^{(2)}, U^{(2)})$ are the unique solutions of the two 1-dimensional anticipated BSDEs below:*

$$\begin{cases} Y_t^{(i)} = \xi_T^{(i)} + \int_t^T f_i(s, Y_s^{(i)}, Y_{s+\delta(s)}^{(i)}, Z_s^{(i)}, U_s^{(i)}) ds - \int_t^T Z_s^{(i)} dB_s \\ \qquad\quad - \int_t^T U_s^{(i)} dM_s, \quad t \in [0, T]; \\ Y_t^{(i)} = \xi_t^{(i)}, \quad t \in (T, T + T^\delta], \end{cases} \tag{2.7}$$

where $i = 1, 2$. For all $t \in [0, T]$, $r \in [t, T + T^\delta]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $u \in \mathbb{R}^k$, $\bar{y}, \bar{y}' \in \mathcal{L}^2_{\mathcal{G}}(t, T + T^\delta; \mathbb{R})$, $f_2(t, y, \bar{y}, z, u)$ is increasing in $\bar{y}$, i.e. if $\bar{y} \geq \bar{y}'$, then $f_2(t, y, \bar{y}, z, u) \geq f_2(t, y, \bar{y}', z, u)$. The following result holds:

If

$$\begin{cases} (1) \quad \xi_t^{(1)} \geq \xi_t^{(2)}, \qquad t \in [T, T + T^\delta]; \\ (2) \quad f_1(t, y, \bar{y}_r, z, \bar{z}_r, u, \bar{u}_r) \geq f_2(t, y, \bar{y}_r, z, \bar{z}_r, u, \bar{u}_r), \qquad r \in [T, T + T^\delta], \end{cases} \tag{2.8}$$

then, the following holds,

$$Y_t^{(1)} \geq Y_t^{(2)}, \qquad t \in [0, T+T^\delta], \quad a.e., \ a.s.$$

Besides, suppose that $f_2(t,y,\bar{y},z,u)$ is strictly increasing in $\bar{y}$ and $[T, T+T^\delta] \subset \{t+\delta(t), t \in [0,T]\}$, then the following holds true (strict comparison theorem):

$$Y_0 = \bar{Y}_0 \iff \begin{cases} (1) \quad \xi_t^{(1)} = \xi_t^{(2)}, & t \in [T, T+T^\delta] \\ (2) \quad f_1\big(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_t^{(1)}, , U_t^{(1)}\big) & \\ \qquad = f_2\big(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_t^{(1)}, , U_t^{(1)}\big), & t \in [0,T]. \end{cases}$$

Proof. Define the following BSDE (2.9):

$$\begin{cases} Y_t^{(3)} = & \xi_T^{(2)} + \int_t^T f_2\big(s, Y_s^{(3)}, Y_{s+\delta(s)}^{(1)}, Z_s^{(3)}, U_s^{(3)}\big) ds - \int_t^T Z_s^{(3)} dB_s \\ & - \int_t^T U_s^{(3)} dM_s, \qquad t \in [0,T]; \\ Y_t^{(3)} = & \xi_t^{(2)}, \qquad t \in (T, T+T^\delta]. \end{cases} \tag{2.9}$$

From Theorem 2.2.1, there exists a unique $\mathcal{G}$-adapted solution $(Y^{(3)}, Z^{(3)}, U^{(3)}) \in \mathcal{S}_\mathcal{G}^2(0, T+T^\delta; \mathbb{R}) \times \mathcal{L}_\mathcal{G}^2(0,T;\mathbb{R}^d) \times \mathcal{L}_\mathcal{G}^{2,\tau}(0,T;\mathbb{R}^k)$ of the BSDE (2.9). Then we set

$$\begin{aligned} \bar{f}_t =& f_1\big(t, Y_t^{(1)}, Y_{t+\delta(s)}^{(1)}, Z_t^{(1)}, , U_t^{(1)}\big) - f_2\big(t, Y_t^{(1)}, Y_{t+\delta(s)}^{(1)}, Z_t^{(1)}, U_t^{(1)}\big); \\ \bar{Y}_t =& Y_t^{(1)} - Y_t^{(3)}; \qquad \bar{Z}_t = Z_t^{(1)} - Z_t^{(3)}; \\ \bar{U}_t =& U_t^{(1)} - U_t^{(3)}; \qquad \bar{\xi}_t = \xi_t^{(1)} - \xi_t^{(2)}. \end{aligned}$$

Consequently, $(\bar{Y}, \bar{Z}, \bar{U})$ can be regarded as the solution of the linear BSDE (2.10):

$$\begin{cases} \bar{Y}_t = & \bar{\xi}_T + \int_t^T \big(a_s \bar{Y}_s + b_s \bar{Z}_s + c_s \bar{U}_s + \bar{f}_s\big) ds - \int_t^T \bar{Z}_s dB_s \\ & - \int_t^T \bar{U}_s dM_s, \qquad t \in [0,T]; \\ \bar{Y}_t = & \bar{\xi}_t, \qquad t \in (T, T+T^\delta], \end{cases} \tag{2.10}$$

where

$$a_t := \begin{cases} \dfrac{f_2\left(t, Y_t^{(1)}, Y_{t+\delta(s)}^{(1)}, Z_t^{(1)}, U_t^{(1)}\right) - f_2\left(t, Y_t^{(3)}, Y_{t+\delta^1(s)}^{(1)}, Z_t^{(1)}, U_t^{(1)}\right)}{Y_t^{(1)} - Y_t^{(3)}}, & Y_t^{(1)} \neq Y_t^{(3)}; \\ 0, & Y_t^{(1)} = Y_t^{(3)}; \end{cases}$$

$$b_t := \begin{cases} \dfrac{f_2\left(t, Y_t^{(3)}, Y_{t+\delta(s)}^{(1)}, Z_t^{(1)}, U_t^{(1)}\right) - f_2\left(t, Y_t^{(3)}, Y_{t+\delta^1(s)}^{(1)}, Z_t^{(3)}, U_t^{(1)}\right)}{Z_t^{(1)} - Z_t^{(3)}}, & Z_t^{(1)} \neq Z_t^{(3)}; \\ 0, & Z_t^{(1)} = Z_t^{(3)}; \end{cases}$$

$$c_t^i := \begin{cases} \dfrac{f_2\left(t,Y_t^{(3)},Y_{t+\delta(s)}^{(1)},Z_t^{(3)},\tilde{U}_t^{i-1}\right)-f_2\left(t,Y_t^{(3)},Y_{t+\delta 1(s)}^{(1)},Z_t^{(3)},\tilde{U}_t^{i}\right)}{(U_t^{(1)i}-U_t^{(3)i})1_{\{\tau^i>t\}}\gamma_t^i}, & \left(U_t^{(1)i}-U_t^{(3)i}\right)1_{\{\tau^i>t\}}\gamma_t^i\neq 0; \\ 0, & \left(U_t^{(1)i}-U_t^{(3)i}\right)1_{\{\tau^i>t\}}\gamma_t^i=0, \end{cases}$$

here $\tilde{U}^i = (U^{(3)1}, U^{(3)2}, ..., U^{(3)i}, U^{(1)i+1}, ..., U^{(1)k})$, $U^{(1)i}$ is the i-th element of $U^{(1)}$.

Since f_2 satisfies the Lipschitz condition, thus, $|a_t| \leq L$, $|b_t| \leq L$ and $c_t^i \geq -1$. Set

$$Q_t := \exp\left\{\int_0^t a_s ds + \int_0^t b_s dB_s - \frac{1}{2}\int_0^t b_s^2 ds + \int_0^t ln(1+c_s)dH_s - \int_0^t c_s 1_{\{\tau>s\}}\gamma_s ds\right\}.$$

Applying the Itô formula for rcll semi-martingale (Theorem 1.3.4) to $Q_s\bar{Y}_s$, it follows

$$\begin{aligned} dQ_s\bar{Y}_s =& -Q_s\left(c_s\bar{U}_s 1_{\{\tau>s\}}\gamma_s + \bar{f}_s\right)ds + Q_s\left(\bar{Z}_s + b_s\bar{Y}_s\right)dB_s \\ &+ Q_{s-}\left(\bar{U}_s + c_s\bar{Y}_{s-}\right)dM_s + Q_{s-}c_s\bar{Y}_{s-}dH_s \\ =& -Q_s\bar{f}_s ds + Q_s\left(\bar{Z}_s + b_s\bar{Y}_s\right)dB_s + Q_{s-}\left(\bar{U}_s + c_s\bar{Y}_{s-} + c_s\bar{U}_s\right)dM_s. \end{aligned}$$

Integrate from t to T, and taking conditional expectation on both sides, since $Q_t\bar{Y}_T = \bar{\xi}_T \geq 0$, $\bar{f}_s \geq 0$, we can obtain

$$Q_t\bar{Y}_t = \mathbb{E}^{\mathcal{G}_t}\left[Q_T\bar{Y}_T + \int_t^T Q_s\bar{f}_s ds\right] \geq 0, \qquad a.e.,\ a.s. \tag{2.11}$$

Hence,

$$Y_t^{(1)} \geq Y_t^{(3)}, \qquad a.e., a.s. \tag{2.12}$$

Then, define the following BSDE (2.13):

$$\begin{cases} Y_t^{(4)} = & \xi_T^{(2)} + \int_t^T f_2\big(s, Y_s^{(4)}, Y_{s+\delta(s)}^{(3)}, Z_s^{(4)}, U_s^{(4)}\big)ds - \int_t^T Z_s^{(4)} dB_s \\ & - \int_t^T U_s^{(4)} dM_s, \qquad t \in [0,T]; \\ Y_t^{(4)} = & \xi_t^{(2)}, \qquad t \in (T, T+T^\delta]. \end{cases} \tag{2.13}$$

Similarly to the proof of (2.12), we can deduce

$$Y_t^{(3)} \geq Y_t^{(4)}, \qquad a.e., a.s. \tag{2.14}$$

For $n = 5, 6, ...$, we consider the following BSDE:

$$\begin{cases} Y_t^{(n)} = & \xi_T^{(2)} + \int_t^T f_2\big(s, Y_s^{(n)}, Y_{s+\delta(s)}^{(n-1)}, Z_s^{(n)}, U_s^{(n)}\big)ds - \int_t^T Z_s^{(n)} dB_s \\ & - \int_t^T U_s^{(n)} dM_s, \qquad t \in [0,T]; \\ Y_t^{(n)} = & \xi_t^{(2)}, \qquad t \in (T, T+T^\delta]. \end{cases}$$

By induction on $n \geq 5$, similarly, we can obtain

$$Y_t^{(4)} \geq Y_t^{(5)} \geq Y_t^{(6)} \geq ... \geq Y_t^{(n)} \geq ..., \qquad a.e., a.s. \tag{2.15}$$

For any $n \geq 4$, denote

$$\hat{Y}^{(n)} = Y^{(n)} - Y^{(n-1)}; \quad \hat{Z}^{(n)} = Z^{(n)} - Z^{(n-1)}; \quad \hat{U}^{(n)} = U^{(n)} - U^{(n-1)}.$$

By Lemma 2.2.1, we can obtain

$$\begin{aligned}
&\mathbb{E}|\hat{Y}_0^{(n)}|^2 + \mathbb{E}\Big[\int_0^T e^{cs}\left(\frac{c}{2}|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2 + |\hat{U}_s^{(n)}|^2 1_{\{\tau>s\}}\gamma_s\right) ds\Big] \\
\leq& \frac{2}{c}\mathbb{E}\Big[\int_0^T e^{cs}\Big|f_2\big(s, Y_s^{(n)}, Y_{s+\delta(s)}^{(n-1)}, Z_s^{(n)}, U_s^{(n)}\big) \\
&\qquad - f_2\big(s, Y_s^{(n-1)}, Y_{s+\delta^1(s)}^{(n-2)}, Z_s^{(n-1)}, U_s^{(n-1)}\big)\Big|^2 ds\Big] \\
\leq& \frac{2L^2}{c}\mathbb{E}\left[\int_0^T e^{cs}\left(|\hat{Y}_s^{(n)}| + \mathbb{E}^{\mathcal{G}_s}|\hat{Y}_{s+\delta(s)}^{(n-1)}| + |\hat{Z}_s^{(n)}| + |\hat{U}_s^{(n)}| 1_{\{\tau>s\}}\sqrt{\gamma_s}\right)^2 ds\right] \\
\leq& \frac{8L^2}{c}\mathbb{E}\Big[\int_0^T e^{cs}\left(|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2 + |\hat{U}_s^{(n)}|^2 1_{\{\tau>s\}}\gamma_s\right) ds\Big] \\
&+ \frac{8L^2L^\delta}{c}\mathbb{E}\int_0^{T+T^\delta} e^{cs}|\hat{Y}_s^{(n-1)}|^2 ds \\
=& \frac{8L^2}{c}\mathbb{E}\int_0^T e^{cs}\left(|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2 + |\hat{U}_s^{(n)}|^2 1_{\{\tau>s\}}\gamma_s\right) ds \\
&+ \frac{8L^2L^\delta}{c}\mathbb{E}\int_0^T e^{cs}|\hat{Y}_s^{(n-1)}|^2 ds.
\end{aligned}$$

Set $c = 24L^2 + 24L^2L^\delta + 3$, it follows

$$\begin{aligned}
&\frac{2}{3}\mathbb{E}\int_0^T e^{cs}\left(|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2 + |\hat{U}_s^{(n)}|^2 1_{\{\tau>s\}}\gamma_s\right) ds \\
\leq& \frac{1}{3}\mathbb{E}\int_0^T e^{cs}|\hat{Y}_s^{(n-1)}|^2 ds \\
\leq& \frac{1}{3}\mathbb{E}\int_0^T e^{cs}\left(|\hat{Y}_s^{(n-1)}|^2 + |\hat{Z}_s^{(n-1)}|^2 + |\hat{U}_s^{(n-1)}|^2 1_{\{\tau>s\}}\gamma_s\right) ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{cs}\left(|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2 + |\hat{U}_s^{(n)}|^2 1_{\{\tau>s\}}\gamma_s\right) ds \\
\leq& \left(\frac{1}{2}\right)^{n-4}\mathbb{E}\int_0^T e^{cs}\left(|\hat{Y}_s^4|^2 + |\hat{Z}_s^4|^2 + |\hat{U}_s^4|^2 1_{\{\tau>s\}}\gamma_s\right) ds.
\end{aligned}$$

Then we know that $(Y_\cdot^{(n)})_{n\geq 4}$, $(Z_\cdot^{(n)})_{n\geq 4}$ and $(U_\cdot^{(n)})_{n\geq 4}$ are Cauchy sequences on the Banach spaces $\mathcal{L}_\mathcal{G}^2(0, T+T^\delta; \mathbb{R})$, $\mathcal{L}_\mathcal{G}^2(0, T; \mathbb{R}^d)$ and $\mathcal{L}_\mathcal{G}^{2,\tau}(0, T; \mathbb{R}^k)$ respectively. Let Y, Z and U be their limits respectively. Since the generator f satisfies the Lipschitz condition, when $n \to \infty$, we can obtain

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{cs}\Big|f_2\big(s,Y_s^{(n)},Y_{s+\delta(s)}^{(n-1)},Z_s^{(n)},U_s^{(n)}\big)-f_2\big(s,Y_s,Y_{s+\delta(s)},Z_s,U_s\big)\Big|^2ds\\
\leq&4L^2\mathbb{E}\int_0^T e^{cs}\left(|Y_s^{(n)}-Y_s|^2+|Z_s^{(n)}-Z_s|^2+|U_s^{(n)}-U_s|^2 1_{\{\tau>s\}}\gamma_s\right)ds\\
&+4L^2L^\delta\mathbb{E}\int_0^T e^{cs}|Y_s^{(n-1)}-Y_s|^2ds\to 0.
\end{aligned}
$$

Therefore, (Y,Z,U) satisfies the following anticipated BSDE:

$$
\begin{cases}
Y_t= & \xi_T^{(2)}+\int_t^T f_2\left(s,Y_s,Y_{s+\delta(s)},Z_s,U_s\right)ds-\int_t^T Z_s dB_s\\
 & -\int_t^T U_s dM_s,\qquad t\in[0,T];\\
Y_t= & \xi_t^{(2)},\qquad t\in(T,T+T^\delta].
\end{cases}
$$

By Theorem 2.2.1, we know that (Y,Z,U) is the unique solution. Hence,

$$
Y_t=Y_t^{(2)},\qquad a.e., a.s.
$$

Combining (2.12), (2.14) and (2.15), it follows that $Y_t^{(1)}\geq Y_t^{(3)}\geq Y_t^{(4)}\geq Y_t=Y_t^{(2)}$, consequently,

$$
Y_t^{(1)}\geq Y_t^{(2)},\qquad a.e., a.s.
$$

Now we continue to prove the strict comparison theorem.

Step 1. ($\Longrightarrow$) Suppose that $Y_0^{(1)}=Y_0^{(2)}$. By the comparison theorem for BSDE with default risk (Theorem 1.3.2), for any $t\in[0,T]$, we can get

$$
f_1\big(t,Y_t^{(1)},Y_{t+\delta(t)}^{(1)},Z_t^{(1)},U_t^{(1)}\big)=f_2\big(t,Y_t^{(1)},Y_{t+\delta(t)}^{(2)},Z_t^{(1)},U_t^{(1)}\big).
$$

Since $Y_0^{(1)}\geq Y_0^{(3)}\geq Y_0^{(2)}$, it follows that $Y_0^{(1)}=Y_0^{(3)}$. Again by the comparison theorem for BSDE with default risk (Theorem 1.3.2), we know

$$
f_1\big(t,Y_t^{(1)},Y_{t+\delta(t)}^{(1)},Z_t^{(1)},U_t^{(1)}\big)=f_2\big(t,Y_t^{(1)},Y_{t+\delta(t)}^{(1)},Z_t^{(1)},U_t^{(1)}\big).
$$

Therefore, it follows

$$
f_2\big(t,Y_t^{(1)},Y_{t+\delta(t)}^{(1)},Z_t^{(1)},U_t^{(1)}\big)=f_2\big(t,Y_t^{(1)},Y_{t+\delta(t)}^{(2)},Z_t^{(1)},U_t^{(1)}\big).
$$

For any $(t,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^k$, $f_2(t,y,\bar{y},z,u)$ is strictly increasing in $\bar{y}$, hence, for any $t\in[0,T]$, $Y_{t+\delta(t)}^{(1)}=Y_{t+\delta(t)}^{(2)}$. In particular, when $t\in[t,t+\delta]$, we can get that $\xi_t^{(1)}=\xi_t^{(2)}$.

Step 2. ($\Longleftarrow$) Suppose

$$Y_0 = \bar{Y}_0 \iff \begin{cases} (1) \quad \xi_t^{(1)} = \xi_t^{(2)}, & t \in [T, T+T^\delta] \\ (2) \quad f_1(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_t^{(1)}, U_t^{(1)}) & \\ \qquad = f_2(t, Y_t^{(1)}, Y_{t+\delta(t)}^{(1)}, Z_t^{(1)}, U_t^{(1)}), & t \in [0, T]. \end{cases}$$

Then, by (2.11), we get

$$\begin{aligned} \bar{Y}_t = Y_t^{(1)} - Y_t^{(3)} &= \mathbb{E}^{\mathcal{G}_t}\left[Q_T \bar{Y}_T + \int_t^T Q_s \bar{f}_s ds\right] \\ &= \mathbb{E}^{\mathcal{G}_t}\left[Q_T \bar{\xi}_T + \int_t^T Q_s \bar{f}_s ds\right] \equiv 0. \end{aligned}$$

Hence,

$$\begin{cases} Y_t^{(1)} = & \xi_T^{(2)} + \int_t^T f_2(s, Y_s^{(1)}, Y_{s+\delta(s)}^{(1)}, Z_s^{(3)}, U_s^{(3)}) ds - \int_t^T Z_s^{(3)} dB_s \\ & - \int_t^T U_s^{(3)} dM_s, \quad t \in [0, T]; \\ Y_t^{(1)} = & \xi_t^{(2)}, \quad t \in (T, T+T^\delta]. \end{cases}$$

By Theorem 2.2.1, we know that $Y_t^{(1)} = Y_t^{(2)}$, *a.e., a.s.* In particular, $Y_0^{(1)} = Y_0^{(2)}$. □

Remark 2.3.1. *The comparison theorem for ABSDEs with default risk (Theorem 2.3.1) needs the third assumption **H** 2.3 **(c)** for u and requires that f_2 is increasing in the anticipated term of Y. If f_2 contains anticipated terms of Z and U, this theorem can not hold (e.g. Example 5.3 in Peng and Yang [50]).*

Corollary 2.3.1. *Suppose that the anticipated processes ξ, α and β, generator f and $\delta^{1(i)}$ satisfy the assumptions **H** 2.1, **H** 2.3 and **H** 2.2. $f(t, y, \bar{y}, z, u)$ is increasing in $\bar{y}$. Let $(Y^{(1)}, Z^{(1)}, U^{(1)})$ and $(Y^{(2)}, Z^{(2)}, U^{(2)})$ be the solutions for the following anticipated BSDEs:*

$$\begin{cases} Y_t^{(i)} = & \xi_T + \int_t^T f(s, Y_s^{(i)}, Y_{s+\delta^{(i)}(s)}^{(i)}, Z_s^{(i)}, U_s^{(i)}) ds - \int_t^T Z_s^{(i)} dB_s \\ & - \int_t^T U_s^{(i)} dM_s, \quad t \in [0, T]; \\ Y_t^{(i)} = & \xi_t, \quad t \in (T, T+T^\delta], \end{cases} \tag{2.16}$$

where $i = 1, 2$. If

$$Y_{t+\delta^{(1)}(t)}^{(1)} \geq Y_{t+\delta^{(2)}(t)}^{(1)}, \qquad a.e.,\ a.s.,$$

then the following holds

$$Y_t^{(1)} \geq Y_t^{(2)}, \qquad a.e.,\ a.s.$$

Proof. Define the following BSDE (2.17):

$$\begin{cases} Y_t^{(3)} = & \xi_T + \int_t^T f(s, Y_s^{(3)}, Y_{s+\delta^{(2)}}^{(1)}, Z_s^{(3)}, U_s^{(3)}) ds - \int_t^T Z_s^{(3)} dB_s \\ & - \int_t^T U_s^{(3)} dM_s, \quad t \in [0, T]; \\ Y_t^{(3)} = & \xi_t, \quad t \in (T, T+T^\delta]. \end{cases} \tag{2.17}$$

By Theorem 2.2.1, there exists a unique $\mathcal{G}$-adapted solution $(Y^{(3)}, Z^{(3)}, U^{(3)}) \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T; \mathbb{R}^k)$ for (2.16). Since $f(t, y, \bar{y}, z, u)$ is increasing in $\bar{y}$, then we can get

$$f(t, y, Y^{(1)}_{t+\delta^{(1)}(t)}, z, u) \geq f(t, y, Y^{(1)}_{t+\delta^{(2)}(t)}, z, u).$$

From the comparison theorem for ABSDEs with default risk (Theorem 2.3.1), it follows

$$Y^{(1)}_t \geq Y^{(3)}_t, \qquad a.e.,\ a.s.$$

Similarly to the proof of Theorem 2.3.1, we can obtain

$$Y^{(1)}_t \geq Y^{(2)}_t, \qquad a.e.,\ a.s.$$

□

2.4 Duality Between Linear Anticipated BSDEs and the SDDEs

El Karoui et. al [4] (1997) studied the duality relation between BSDEs and SDEs. In this section, we consider the duality between anticipated BSDEs and the stochastic differential delay equations (SDDE). We can use this duality to solve the stochastic control problem in Section 2.5. Consider the following anticipated BSDE (δ is a given constant, t_0 is the initial time, B is a d-dimensional standard Brownian motion):

$$\begin{cases} -dY_t = \Big(\sigma_t Y_t + \hat{\sigma}_t \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}] + Z_t\theta_t + \mathbb{E}^{\mathcal{G}_t}[Z_{t+\delta}]\hat{\theta}_t + U_t\mu_t 1_{\{\tau>t\}} \\ \qquad\qquad + \mathbb{E}^{\mathcal{G}_t}[U_{t+\delta}]\hat{\mu}_t 1_{\{\tau>t\}} + l_t\Big)dt - Z_t dB_t - U_t dM_t, \qquad t \in [t_0, T]; \\ Y_t = \xi_t, \qquad t \in (T, T+\delta]; \\ Z_t = \alpha_t, \qquad t \in (T, T+\delta]; \\ U_t = \beta_t, \qquad t \in (T, T+\delta], \end{cases} \tag{2.18}$$

and the following stochastic differential delay equation with default risk (SDDE):

$$\begin{cases} dX_s = \big(\sigma_s X_s + \hat{\sigma}^T_{s-\delta} X_{s-\delta}\big)\,ds + \big(X_s\theta^T_s + X_{s-\delta}\hat{\theta}^T_{s-\delta}\big)\,dB_s \\ \qquad\quad + \Big(X_{s-}\mu^T_s 1_{\{\tau>s\}} + X_{s-\delta}\hat{\mu}^T_{(s-\delta)-} 1_{\{\tau>s\}}\Big)\,dM_s, \qquad s \in [t, T+\delta]; \\ X_s = 0, \qquad s \in [t-\delta, t), \end{cases} \tag{2.19}$$

where σ, $\hat{\sigma}$, θ, $\hat{\theta}$, μ and $\hat{\mu}$ are uniformly bounded. $\sigma, \hat{\sigma} \in \mathcal{L}^2_{\mathcal{G}}(t_0-\delta, T+\delta; \mathbb{R})$, $\theta, \hat{\theta} \in \mathcal{L}^2_{\mathcal{G}}(t_0-\delta, T+\delta; \mathbb{R}^d)$, $\mu, \hat{\mu} \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t_0-\delta, T+\delta; \mathbb{R}^k)$. $\xi \in \mathcal{S}^2_{\mathcal{G}}(T, T+\delta; \mathbb{R})$, $\alpha \in \mathcal{L}^2_{\mathcal{G}}(T, T+\delta; \mathbb{R}^d)$, $\beta \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(T, T+\delta; \mathbb{R}^k)$.

We give the main result of the duality between the ABSDE (2.18) and the SDDE (2.19) as below.

Theorem 2.4.1. *The solution Y of the above anticipated BSDE (2.18) can be given by the closed formula:*

$$\begin{aligned} Y_t = \mathbb{E}^{\mathcal{G}_t}\Bigg[& X_T\xi_T + \int_T^{T+\delta} \left(\xi_s\hat{\sigma}_{s-\delta} + \alpha_s\hat{\theta}_{s-\delta} + \beta_s\hat{\mu}_{s-\delta}1_{\{\tau>s\}}\right)X_{s-\delta}ds \\ & + \int_t^T X_s l_s ds\Bigg], \qquad a.e.,\ a.s., \end{aligned} \tag{2.20}$$

where X is the solution of the SDDE (2.19).

Proof. **Step 1**. First, we prove that the SDDE (2.19) has a unique solution.

When $s \in [t, t+\delta]$, SDDE (2.19) can be transformed into the SDE (2.21) with the following form:

$$\begin{cases} dX_s = \sigma_s X_s ds + X_s\theta_s^T dB_s + \mu_s X_{s-} dM_s, & s \in [t, t+\delta]; \\ X_t = 1, \end{cases} \tag{2.21}$$

obviously, there exists a unique solution $X^{(1)}$ for the SDE (2.21) above.

When $s \in [t+\delta, t+2\delta]$, there exists a unique solution $X^{(2)}$ for the following SDE (2.22):

$$\begin{aligned} dX_s = & \left(\sigma_s X_s + \hat{\sigma}_{s-\delta}^T X_s^{(1)}\right) ds + \left(X_s\theta_s^T + X_s^{(1)}\hat{\theta}_{s-\delta}^T\right) dB_s \\ & + \left(X_{s-}\mu_s^T + X_s^{(1)}\hat{\mu}_{(s-\delta)-}\right) dM_s. \end{aligned} \tag{2.22}$$

When $s \in [t+2\delta, t+3\delta]$, there exists a unique solution $X^{(3)}$ for the following SDE (2.23):

$$\begin{aligned} dX_s = & \left(\sigma_s X_s + \hat{\sigma}_{s-\delta}^T X_s^{(2)}\right) ds + \left(X_s\theta_s^T + X_s^{(2)}\hat{\theta}_{s-\delta}^T\right) dB_s \\ & + \left(X_{s-}\mu_s^T + X_s^{(2)}\hat{\mu}_{(s-\delta)-}\right) dM_s. \end{aligned} \tag{2.23}$$

By the induction on $[t+3\delta, t+4\delta]$, $[t+4\delta, t+5\delta]$,..., we can prove that there exists a unique solution for the SDDE (2.19).

Step 2. Applying the Itô formula to X_sY_s on $[t, T]$, and taking conditional expectation under $\mathcal{G}_t$, we can obtain

$$\begin{aligned} & \mathbb{E}^{\mathcal{G}_t}\left[X_T Y_T\right] - X_t Y_t \\ = & \mathbb{E}^{\mathcal{G}_t}\Big[\int_t^T \Big(Y_s\hat{\sigma}_{s-\delta}X_{s-\delta} - \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}]\hat{\sigma}_s X_s + Z_s\hat{\theta}_{s-\delta}X_{s-\delta} - \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}]\hat{\theta}_s X_s \\ & + U_s\hat{\mu}_{s-\delta}X_{s-\delta}1_{\{\tau>s\}} - \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\hat{\mu}_s X_s 1_{\{\tau>s\}} - X_s l_s\Big)ds\Big]. \end{aligned}$$

When $s \in [t-\delta, t)$, $X_s = 0$, and $X_t = 1$, it follows

$$
\begin{aligned}
Y_t =& \mathbb{E}^{\mathcal{G}_t}\Big[X_T Y_T + \int_t^T X_s l_s ds - \int_t^T Y_s \hat{\sigma}_{s-\delta} X_{s-\delta} ds + \int_{t+\delta}^{T+\delta} Y_s \hat{\sigma}_{s-\delta} X_{s-\delta} ds \\
&- \int_t^T Z_s \hat{\theta}_{s-\delta} X_{s-\delta} ds + \int_{t+\delta}^{T+\delta} Z_s \hat{\theta}_{s-\delta} X_{s-\delta} ds \\
&- \int_t^T U_s \hat{\mu}_{s-\delta} X_{s-\delta} 1_{\{\tau > s\}} ds + \int_{t+\delta}^{T+\delta} U_s \hat{\mu}_{s-\delta} X_{s-\delta} 1_{\{\tau > s\}} ds\Big] \\
=& \mathbb{E}^{\mathcal{G}_t}\left[X_T \xi_T + \int_t^T X_s l_s ds + \int_T^{T+\delta} \left(\xi_s \hat{\sigma}_{s-\delta} + \alpha_s \hat{\theta}_{s-\delta} + \beta_s \hat{\mu}_{s-\delta} 1_{\{\tau > s\}}\right) X_{s-\delta} ds\right].
\end{aligned}
$$

Consequently, we prove (2.20). □

2.5 Application in Stochastic Control Problem

El Karoui et. al [4] (1997) applied the duality between BSDEs and SDEs to stochastic control problems. In this section, we use the duality between ABSDEs and SDDEs studied in Section 2.4 to solve the stochastic control problem in the defaultable setting. We consider the following controlled function ($\delta > 0$ is a given constant):

$$
\begin{cases}
dX_s^v = \left(\sigma(s, v_s) X_s^v + \hat{\sigma}(s-\delta, v_{s-\delta}) X_{s-\delta}^v\right) ds + X_s^v \theta^T(s, v_s) dB_s \\
\qquad\qquad + X_{s-}^v \mu^T(s-, v_{s-}) dM_s, \qquad s \in [t, T+\delta]; \\
X_t^v = 1; \\
X_s^v = 0, \qquad s \in [t-\delta, t),
\end{cases}
\tag{2.24}
$$

where $\sigma(t,v) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $\hat{\sigma}(t,v) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $\theta(t,v) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $\mu(t,v) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^k$ are adapted processes uniformly continuous with respect to (t, v), and uniformly bounded. A feasible control $(v_t)_{-\delta \le t \le T+\delta}$ is a continuous adapted process valued in a compact set $V \in \mathbb{R}^d$. Denote by $\mathcal{V}$ the set of feasible controls. Our aim is to maximize the following objective function:

$$
J(v) = \mathbb{E}\left[X_T^v \xi_T + \int_0^T X_s^v l(s, v_s) ds + \int_T^{T+\delta} \xi_s \hat{\sigma}(s-\delta, v_{s-\delta}) X_{s-\delta}^v ds\right],
$$

where $\xi \in \mathcal{S}_{\mathcal{G}}^2(T, T+\delta; \mathbb{R})$ is the anticipated process, $l(t,v)$ is an adapted process uniformly continuous with respect to (t,v) and uniformly bounded, $l(w,t,v_t)_{0 \le t \le T}$ is the running cost associated with the control process v. Consider the following linear anticipated BSDE:

$$
\begin{cases}
-dY_t^v = \left(\sigma(t, v_t) Y_t^v + \hat{\sigma}(t, v_t) \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^v] + Z_t^v \theta(t, v_t) \right. \\
\qquad\qquad \left. + U_t^v \mu(t, v_t) 1_{\{\tau > t\}} + l(t, v_t)\right) dt - Z_t^v dB_t - U_t^v dM_t, \quad t \in [0, T]; \\
Y_t^v = \xi_t, \qquad t \in [T, T+\delta]; \\
Z_t^v = \alpha_t, \qquad t \in (T, T+\delta]; \\
U_t^v = \beta_t, \qquad t \in (T, T+\delta].
\end{cases}
\tag{2.25}
$$

From Theorem 2.4.1, we know that $J(u) = Y_0^v$, where (Y^v, Z^v, U^v) is the solution of the ABSDE (2.25), Y_t^v has the following form:

$$Y_t^v = \mathbb{E}^{\mathcal{G}_t}\left[X_T^v \xi_T + \int_t^T X_s^v l(s, v_s)ds + \int_T^{T+\delta} \xi_s \hat{\sigma}(s-\delta, v_{s-\delta}) X_{s-\delta}^v ds\right], \quad a.e., \ a.s. \quad (2.26)$$

For the sake of simplicity, for all $t \in [0,T]$, $r \in [t, T+\delta]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $u \in \mathbb{R}^k$, $\bar{y} \in \mathcal{S}_{\mathcal{G}}^2(t, T+\delta; \mathbb{R})$, we denote:

$$\begin{aligned} f^v(t, y, \bar{y}_r, z, , u) =& \sigma(t, v_t)y + \hat{\sigma}(t, v_t)\mathbb{E}^{\mathcal{G}_t}[\bar{y}_r] + z\theta(t, v_t) + u\mu(t, v_t)1_{\{\tau > t\}} + l(t, v_t); \\ f^*(t, y, \bar{y}_r, z, u) =& ess \sup_{v \in \mathcal{V}} f^v(t, y, \bar{y}_r, z, \bar{z}_r, u, \bar{u}_r). \end{aligned} \quad (2.27)$$

Consider the following ABSDE (2.28):

$$\begin{cases} -dY_t^* = \Big(\sigma(t, v_t)Y_t^* + \hat{\sigma}(t, v_t)\mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^*] + Z_t^*\theta(t, v_t) + U_t^*\mu(t, v_t)1_{\{\tau > t\}} \\ \qquad\qquad + l(t, v_t)\Big)dt - Z_t^* dB_t - U_t^* dM_t, \quad t \in [0, T]; \\ Y_t^* = \xi_t, \qquad t \in [T, T+\delta]; \\ Z_t^* = \alpha_t, \qquad t \in (T, T+\delta]; \\ U_t^* = \beta_t, \qquad t \in (T, T+\delta]. \end{cases} \quad (2.28)$$

We suppose that $\sigma(t, v)$, $\hat{\sigma}(t, v)$, $\theta(t, v)$ and $\mu(t, v)$ are uniformly bounded by a constant $M \geq 0$, for all $t \in [0, T]$, $s \in [T, T+\delta]$, $r \in [t, T+\delta]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u, u' \in \mathbb{R}^k$, $\bar{y}$, $\bar{y}' \in \mathcal{S}_{\mathcal{G}}^2(t, T+\delta; \mathbb{R})$. It follows

$$\begin{aligned} & f^*(t, y, \bar{y}_r, z, u) - f^*(t, y', \bar{y}_r', z', u') \\ \leq & ess \sup_{v \in \mathcal{V}} \Big[\sigma(t, v_t)(y - y') + \hat{\sigma}(t, v_t)(\bar{y}_r - \bar{y}_r') + (z - z')\theta(t, v_t) \\ & \qquad + (u - u')\mu(t, v_t)1_{\{\tau > t\}}\Big] \\ \leq & M\Big[|y - y'| + \mathbb{E}^{\mathcal{G}_t}|\bar{y}_r - \bar{y}_r'| + |z - z'| + |u - u'|1_{\{\tau > t\}}\Big]. \end{aligned}$$

Since $\mathbb{E}\int_0^T |f(t, 0, 0, 0, 0)|^2 dt \leq M^2 T$, we know that the ABSDE (2.28) has a unique solution (Y^*, Z^*, U^*) (Theorem 2.2.1).

Theorem 2.5.1. *The solution Y^* of the ABSDE (2.28) is the value function of the stochastic control problem above, i.e.*

$$Y_t^* = ess \sup_{v \in \mathcal{V}} Y_t^v, \qquad t \in [0, T].$$

Proof. Since for all $v \in \mathcal{V}$, $f^v(t, y, \bar{y}_r, z, u)$ is increasing in $\bar{y}$, and $f^v(t, y, \bar{y}_r, z, u) \leq f^*(t, y, \bar{y}_r, z, u)$, by the comparison theorem for ABSDEs with default risk (Theorem 2.3.1), we can get that $Y_t \geq Y_t^v$, therefore, $Y_t \geq Y_t^*$, *a.e.*, *a.s.*

By the definition of f^* (2.27), we know for any $\epsilon > 0$ and any $(w,t) \in \Omega \times [0,T)$, the following set is not empty:

$$\Big\{ \quad f^*(w,t,Y_t^*(w),Y_{t+\delta}^*(w),Z_t^*(w),U_t^*(w)) \\ \le \sigma(t,v)Y_t^*(w) + \hat{\sigma}(t,v)\mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^*(w)] + Z_t^*(w)\theta(t,v) \\ + U_t^*(w)\mu(t,v)1_{\{\tau>t\}} + l(w,t,v) + \epsilon \quad \Big\} \neq \varnothing.$$

By Beneš' selection theorem (Beneš [86] (1971)), there exists a $v^\epsilon \in \mathcal{V}$, such that

$$f^*\left(t,Y_t^*,Y_{t+\delta}^*,Z_t^*,U_t^*\right) \le f^{v^\epsilon}\left(t,Y_t^*,Y_{t+\delta}^*,Z_t^*,U_t^*\right) + \epsilon, \quad a.e., a.s. \tag{2.29}$$

Denote by $(Y^{v^\epsilon},Z^{v^\epsilon},U^{v^\epsilon})$ the solution of the ABSDE with the coefficient (f^{v^ϵ},ξ).

Step 1. First, when $t \in [T-\delta,T]$, it follows that $Y_{t+\delta}^* = Y_{t+\delta}^{v^\epsilon}$. By (2.29), it follows

$$\begin{aligned} & f^{v^\epsilon}\left(t,Y_t^{v^\epsilon},Y_{t+\delta}^{v^\epsilon},Z_t^{v^\epsilon},U_t^{v^\epsilon}\right) - f^*\left(t,Y_t^*,Y_{t+\delta}^*,Z_t^*,U_t^*\right) \\ \ge & f^{v^\epsilon}\left(t,Y_t^{v^\epsilon},Y_{t+\delta}^{v^\epsilon},Z_t^{v^\epsilon},U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t,Y_t^*,Y_{t+\delta}^{v^\epsilon},Z_t^*,U_t^*\right) - \epsilon \\ = & a_t(Y_t^{v^\epsilon} - Y_t^*) + b_t(Z_t^{v^\epsilon} - Z_t^*) + c_t(U_t^{v^\epsilon} - U_t^*) - \epsilon. \end{aligned}$$

When $t \in [T-\delta,T]$, denote

$$a_t := \begin{cases} \dfrac{f^{v^\epsilon}\left(t,Y_t^{v^\epsilon},Y_{t+\delta}^{v^\epsilon},Z_t^{v^\epsilon},U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t,Y_t^*,Y_{t+\delta}^{v^\epsilon},Z_t^{v^\epsilon},U_t^{v^\epsilon}\right)}{Y_t^{v^\epsilon}-Y_t^*}, & Y_t^{v^\epsilon} \neq Y_t^*; \\ 0, & Y_t^{v^\epsilon} = Y_t^*; \end{cases}$$

$$b_t := \begin{cases} \dfrac{f^{v^\epsilon}\left(t,Y_t^*,Y_{t+\delta}^{v^\epsilon},Z_t^{v^\epsilon},U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t,Y_t^*,Y_{t+\delta}^{v^\epsilon},Z_t^*,U_t^{v^\epsilon}\right)}{Z_t^{v^\epsilon}-Z_t^*}, & Z_t^{v^\epsilon} \neq Z_t^*; \\ 0, & Z_t^{v^\epsilon} = Z_t^*; \end{cases}$$

$$c_t^i := \begin{cases} \dfrac{f^{v^\epsilon}\left(t,Y_t^*,Y_{t+\delta}^{v^\epsilon},Z_t^*,\tilde{U}_t^{i-1}\right) - f^{v^\epsilon}\left(t,Y_t^*,Y_{t+\delta}^{v^\epsilon},Z_t^*,\tilde{U}_t^i\right)}{(U_t^{v^\epsilon i}-U_t^{*i})1_{\{\tau^i>t\}}\gamma_t^i}, & (U_t^{v^\epsilon}-U_t^*)1_{\{\tau^i>t\}}\gamma_t^i \neq 0; \\ 0, & (U_t^{v^\epsilon}-U_t^*)1_{\{\tau^i>t\}}\gamma_t^i = 0, \end{cases}$$

where $\tilde{U}^i = (U^{*1},U^{*2},...,U^{*i},U^{v^\epsilon i+1},...,U^{v^\epsilon k})$, $U^{v^\epsilon i}$ is the i-th element of U^{v^ϵ}, $i = 1,2,...k$.

Therefore, for all $t \in [T-\delta,T]$, we can obtain

$$\begin{aligned} Y_t^{v^\epsilon} - Y_t^* \ge & \int_t^T \left[a_s(Y_s^{v^\epsilon} - Y_s^*) + b_s(Z_s^{v^\epsilon} - Z_s^*) + c_s(U_s^{v^\epsilon} - U_s^*) - \epsilon\right] ds \\ & - \int_t^T (Z_s^{v^\epsilon} - Z_s^*)dB_s - \int_t^T (U_s^{v^\epsilon} - U_s^*)dM_s. \end{aligned}$$

Denote by $\tilde{Y}_t^{(1)}$ the solution of the BSDE below:

$$\tilde{Y}_t^{(1)} = \int_t^T \left(a_s\tilde{Y}_t^{(1)} + b_s\tilde{Z}_t^{(1)} + c_s\tilde{U}_t^{(1)} - \epsilon\right) ds - \int_t^T \tilde{Z}_t^{(1)} dB_s \\ - \int_t^T \tilde{U}_t^{(1)} dM_s, \qquad t \in [T-\delta, T].$$

By the comparison theorem for ABSDEs with default risk (Theorem 2.3.1), we can obtain

$$Y_t^{v^\epsilon} - Y_t^* \geq \tilde{Y}_t^{(1)}.$$

Denote

$$Q_s^{(1)} = exp\Big\{ \int_0^s a_r dr + \int_0^s b_r dB_r + \int_0^s ln(1+c_r) dH_r - \frac{1}{2}\int_0^s |b_r|^2 dr \\ - \int_0^s c_r 1_{\{\tau > r\}} \gamma_r dr \Big\},$$

since $|a_t| \leq M$, $|b_t| \leq M$, $|c_t| \leq M$ and $c_t \geq -1$, applying the Itô formula for rcll semimartingale (Theorem 1.3.4) to $Q_s^{(1)}\tilde{Y}_s^{(1)}$ on $[t, T]$ and taking conditional expectation on both sides, we can get

$$\tilde{Y}_t^{(1)} = -\epsilon \mathbb{E}^{\mathcal{G}_t}\left[\int_t^T Q_s^{(1)} ds\right], \qquad t \in [T-\delta, T].$$

Hence, there exists a constant $\rho^1 > 0$, such that

$$Y_t^{v^\epsilon} - Y_t^* \geq \tilde{Y}_t^{(1)} \geq -\rho^1\epsilon, \qquad t \in [T-\delta, T]. \tag{2.30}$$

where ρ depending only on M, T and δ.

Step 2. Second, when $t \in [T-2\delta, T-\delta]$, then $t+\delta \in [T-\delta, T]$, we have that $Y_{t+\delta}^* \leq Y_{t+\delta}^{v^\epsilon} + \rho^1\epsilon$. By (2.29) and (2.30), it follows

$$\begin{aligned} & f^{v^\epsilon}\left(t, Y_t^{v^\epsilon}, Y_{t+\delta}^{v^\epsilon}, Z_t^{v^\epsilon}, U_t^{v^\epsilon}\right) - f^*\left(t, Y_t^*, Y_{t+\delta}^*, Z_t^*, U_t^*\right) \\ \geq & f^{v^\epsilon}\left(t, Y_t^{v^\epsilon}, Y_{t+\delta}^{v^\epsilon}, Z_t^{v^\epsilon}, U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^*, Z_t^*, U_t^*\right) - \epsilon \\ \geq & f^{v^\epsilon}\left(t, Y_t^{v^\epsilon}, Y_{t+\delta}^{v^\epsilon}, Z_t^{v^\epsilon}, U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^*, U_t^*\right) \\ & + f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^*, U_t^*\right) - f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^*, Z_t^*, U_t^*\right) - \epsilon \\ \geq & f^{v^\epsilon}\left(t, Y_t^{v^\epsilon}, Y_{t+\delta}^{v^\epsilon}, Z_t^{v^\epsilon}, U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^*, U_t^*\right) - M\rho^1\epsilon - \epsilon \\ = & a_t(Y_t^{v^\epsilon} - Y_t^*) + b_t(Z_t^{v^\epsilon} - Z_t^*) + c_t(U_t^{v^\epsilon} - U_t^*) - \epsilon(M\rho^1 + 1). \end{aligned}$$

When $t \in [T-2\delta, T-\delta]$, similarly to the proof of **Step 1**, denote

$$a_t := \begin{cases} \dfrac{f^{v^\epsilon}\left(t, Y_t^{v^\epsilon}, Y_{t+\delta}^{v^\epsilon}, Z_t^{v^\epsilon}, U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^{v^\epsilon}, U_t^{v^\epsilon}\right)}{Y_t^{v^\epsilon} - Y_t^*}, & Y_t^{v^\epsilon} \neq Y_t^*; \\ 0, & Y_t^{v^\epsilon} = Y_t^*; \end{cases}$$

$$b_t := \begin{cases} \frac{f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^{v^\epsilon}, U_t^{v^\epsilon}\right) - f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^*, U_t^{v^\epsilon}\right)}{Z_t^{v^\epsilon} - Z_t^*}, & Z_t^{v^\epsilon} \neq Z_t^*; \\ 0, & Z_t^{v^\epsilon} = Z_t^*; \end{cases}$$

$$c_t^i := \begin{cases} \frac{f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^*, \tilde{U}_t^{i-1}\right) - f^{v^\epsilon}\left(t, Y_t^*, Y_{t+\delta}^{v^\epsilon}, Z_t^*, \tilde{U}_t^{i}\right)}{(U_t^{v^\epsilon i} - U_t^{*i})1_{\{\tau^i > t\}}\gamma_t^i}, & (U_t^{v^\epsilon} - U_t^*)1_{\{\tau^i > t\}}\gamma_t^i \neq 0; \\ 0, & (U_t^{v^\epsilon} - U_t^*)1_{\{\tau^i > t\}}\gamma_t^i = 0, \end{cases}$$

where $\tilde{U}^i = (U^{*1}, U^{*2}, ..., U^{*i}, U^{v^\epsilon i+1}, ..., U^{v^\epsilon k})$, $U^{v^\epsilon i}$ is the i-th element of U^{v^ϵ}, $i = 1, 2, ...k$.

Therefore, for all $t \in [T - 2\delta, T - \delta]$,

$$\begin{aligned} Y_t^{v^\epsilon} - Y_t^* \geq & Y_{T-\delta}^{v^\epsilon} - Y_{T-\delta}^* + \int_t^{T-\delta} \Big[a_s(Y_s^{v^\epsilon} - Y_s^*) + b_s(Z_s^{v^\epsilon} - Z_s^*) \\ & + c_s(U_s^{v^\epsilon} - U_s^*) - \epsilon(M\rho^1 + 1)\Big]ds - \int_t^{T-\delta} (Z_s^{v^\epsilon} - Z_s^*)dB_s \\ & - \int_t^{T-\delta} (U_s^{v^\epsilon} - U_s^*)dM_s. \end{aligned}$$

Consequently, $Y_t^{v^\epsilon} - Y_t^* \geq \tilde{Y}_t^{(2)}$, where $\tilde{Y}_t^{(2)}$ is the solution of the following BSDE:

$$\begin{aligned} \tilde{Y}_t^{(2)} = & Y_{T-\delta}^{v^\epsilon} - Y_{T-\delta}^* + \int_t^{T-\delta} \Big[a_s\tilde{Y}_s^{(2)} + b_s\tilde{Z}_s^{(2)} + c_s\tilde{U}_s^{(2)} - \epsilon(M\rho^1 + 1)\Big]ds \\ & - \int_t^{T-\delta} \tilde{Z}_s^{(2)}dB_s - \int_t^{T-\delta} \tilde{U}_s^{(2)}dM_s. \end{aligned}$$

Since $|a_t| \leq M$, $|b_t| \leq M$, $|c_t| \leq M$ and $c_t \geq -1$, we can get

$$\tilde{Y}_t^{(2)} = \mathbb{E}^{\mathcal{G}_t}\left[\left(Y_{T-\delta}^{v^\epsilon} - Y_{T-\delta}^*\right)Q_{T-\delta}^{(2)} + \int_t^{T-\delta} \epsilon(M\rho^1 + 1)Q_s^{(2)}ds\right],$$

here

$$\begin{aligned} Q_s^{(2)} = \exp\Big\{ & \int_0^s a_r dr + \int_0^s b_r dB_r + \int_0^s ln(1 + c_r)dH_r - \frac{1}{2}\int_0^s |b_r|^2 dr \\ & - \int_0^s c_r 1_{\{\tau > r\}}\gamma_r dr\Big\}. \end{aligned}$$

Since $Y_{T-\delta}^{v^\epsilon} - Y_{T-\delta}^* \geq -\rho^1$, therefore, there exists a constant $\rho^2 > 0$, such that

$$Y_t^{v^\epsilon} - Y_t^* \geq \tilde{Y}_t^{(2)} \geq -\rho^2\epsilon, \qquad t \in [T - 2\delta, T - \delta].$$

Similarly, there exist constants ρ^2, $\rho^4 ..., \rho^{[\frac{T}{\delta}]+1} > 0$, such that

$$\begin{aligned} & Y_t^{v^\epsilon} - Y_t^* \geq -\rho^n\epsilon, && t \in [T - n\delta, T - (n-1)\delta], \quad n = 3, 4, ..., \Big[\frac{T}{\delta}\Big]; \\ & Y_t^{v^\epsilon} - Y_t^* \geq -\rho^{[\frac{T}{\delta}]+1}\epsilon, && t \in \Big[0, T - \Big[\frac{T}{\delta}\Big]\delta\Big]. \end{aligned}$$

Set $\rho := \{\rho^1,\ \rho^2...,\rho^{[\frac{T}{\delta}]+1}\}$, we can obtain

$$Y_t^{v^\epsilon} - Y_t^* \geq -\rho\epsilon, \qquad t \in [0,T].$$

Since $Y_t^{v^\epsilon} \leq Y_t^*$, as $\epsilon \to 0$, we can get

$$Y_t^{v^\epsilon} \to Y_t^*, \qquad a.e.,\ a.s.$$

So we can prove

$$Y_t = Y_t^*, \qquad a.e.,\ a.s.$$

□

2.6 Relation with the Obstacle Problems for Non-linear Parabolic PDEs

Example 1.2.4 in Section 1.2 illustrates the connection between semi-linear parabolic equations and BSDEs. In this section, we will show that the ABSDE studied in the previous sections allows us to give a probabilistic representation of the solution of some obstacle problems for PDEs. For that purpose, we will put the ABSDE in a Markovian framework. Consider the following PDE (2.31). We want to get the solution of ABSDE with the following PDE:

$$\begin{cases} \partial_t v(t,x,h) + \mathcal{L}^{t,x}v(t,x,h) + f^{t,x}(t,x,h) = 0; \\ v(T,x,h) = \varphi(x,h), \end{cases} \tag{2.31}$$

where $h := \{0,1\}$, $v : [0,T] \times \mathbb{R} \times \{0,1\} \to \mathbb{R}$,

$$\begin{aligned} \mathcal{L}^{t,x}v(t,x,h) &:= \sigma(t,x)\partial_x v(t,x,h) + \frac{1}{2}\theta^2(t,x)\partial_{xx}^2 v(t,x,h) \\ &\quad + \big(\Delta v(t,x) - \mu(t,x)\partial_x v(t,x,h)\big)(1-h)\gamma_t; \\ f^{t,x}(t,x,h) &:= f\big(t,x,h,v(t,x,h),\mathbb{E}^{\mathcal{G}_t}[v(t+\delta,\bar{x},\bar{h})],\theta(t,x)\partial_x v(t,x,h), \\ &\qquad \theta(t,x)\mathbb{E}^{\mathcal{G}_t}[\partial_x v(t+\delta,\bar{x},\bar{h})],\Delta v(t,x),\mathbb{E}^{\mathcal{G}_t}[\Delta v(t+\delta,\bar{x})]\big); \\ \Delta v(t,x) &:= v(t,x+\mu(t,x),1) - v(t,x,0), \end{aligned}$$

where $\bar{x} = x_{t+\delta}$, $\bar{h} = h_{t+\delta}$.

Then we consider a state process X, for each initial time $t \in [0,T]$ and each initial condition $x \in \mathbb{R}$, let $X^{t,x}$ be the solution of the following SDE (2.32):

$$\begin{cases} dX_s^{t,x} = \sigma(s,X_s^{t,x})ds + \theta(s,X_s^{t,x})dB_s + \mu(s,X_{s-}^{t,x})dM_s, \qquad s \in [t,T]; \\ X_t^{t,x} = x. \end{cases} \tag{2.32}$$

and the ABSDE (2.33) below:

$$\begin{cases} -dY_s^{t,x} = f\big(s, X_s^{t,x}, H_s, Y_s^{t,x}, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}^{t,x}], Z_s^{t,x}, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}^{t,x}], U_s^{t,x}, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}^{t,x}]\big)ds \\ \qquad\qquad -Z_s^{t,x}dB_s - U_s^{t,x}dM_s, \qquad s \in [t,T]; \\ Y_s^{t,x} = \varphi(X_s^{t,x}, H_s), \qquad s \in [T, T+\delta]; \\ Z_s^{t,x} = \alpha_s, \qquad s \in (T, T+\delta]; \\ U_s^{t,x} = \beta_s, \qquad s \in (T, T+\delta]; \\ \mathbb{E}\int_0^T \Big(|Y_s^{t,x}|^2 + |Z_s^{t,x}|^2 + |U_s^{t,x}|^2 1_{\{\tau>s\}}\Big)\,ds < \infty, \end{cases} \tag{2.33}$$

where $\sigma : [0,T]\times\mathbb{R} \to \mathbb{R}$, $\theta : [0,T]\times\mathbb{R} \to \mathbb{R}$, $\mu : [0,T]\times\mathbb{R} \to \mathbb{R}$, $\varphi : \mathbb{R}\times\{0,1\} \to \mathbb{R}$, $f : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$.

We introduce the following assumptions to make sure the existence of the SDE (2.32) and the ABSDE (2.33) above.

(a) $\sigma(t,x)$ and $\theta(t,x)$ are continuous and invertible mappings, $\mu(t,x)$ is progressively measurable and invertible, $\varphi(x,0)$ and $\varphi(x,1)$ are continuous in x. $\sigma^{-1}(t,x)$, $\theta^{-1}(t,x)$ and $\mu^{-1}(t,x)$ are bounded;

(b) $\sigma(t,x)$, $\theta(t,x)$, $\mu(t,x)$, $\varphi(x,0)$ and $\varphi(x,1)$ are uniformly with respect to t and Lipschitz with respect to x, i.e. $\forall(t,x),\ (t,x') \in [0,T]\times\mathbb{R}$, there exists a constant $C_1 \geq 0$, such that

$$|\sigma(t,x) - \sigma(t,x')| + |\theta(t,x) - \theta(t,x')| \leq C_1|x-x'|;$$
$$|\varphi(x,0) - \varphi(x',0)| + |\varphi(x,1) - \varphi(x',1)| \leq C_1|x-x'|;$$
$$|\mu(t,x) - \mu(t,x')| \leq C_1|x-x'|1_{\{\tau>t\}};$$

(c) f is continuous in t uniformly with respect to x, y, z and u, and continuous in x uniformly with respect to y, z and u. There exists a constant $C_3 > 0$, for any $t \in [0,T]$, $r \in [t, T+T^\delta]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}$, $u, u' \in \mathbb{R}$, $\bar{y}, \bar{y}' \in \mathcal{S}^2_{\mathcal{G}}(t, T+\delta; \mathbb{R})$, $\bar{z}$, $\bar{z}' \in \mathcal{L}^2_{\mathcal{G}}(t, T+\delta; \mathbb{R})$, $\bar{u}, \bar{u}' \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t, T+\delta; \mathbb{R})$, there exists a constant $L \geq 0$, such that

$$\begin{aligned} &|f(t,x,y,\bar{y},z,\bar{z},u,\bar{u}) - f(t,x,y',\bar{y}',z',\bar{z}',u',\bar{u}')| \\ \leq &L\Big(|y-y'| + \mathbb{E}^{\mathcal{G}_t}|\bar{y}-\bar{y}'| + |z-z'| + \mathbb{E}^{\mathcal{G}_t}|\bar{z}-\bar{z}'| \\ &\quad + |u-u'|1_{\{\tau>t\}}\sqrt{\gamma_t} + \mathbb{E}^{\mathcal{G}_t}|\bar{u}-\bar{u}'|1_{\{\tau>t\}}\sqrt{\gamma_t}\Big). \end{aligned}$$

Remark 2.6.1. *$v(t,x,0)$ is called the pre-default pricing function, while $v(t,x,1)$ is called the post-default pricing function. $X^{t,x}$ represents the dynamics of wealth process, $\varphi(X_T^{t,x}, H_T)$ is the contingent claim which we want to replicate. $Y_s^{t,x}$ also relies on H_s. As for the case of multiple assets, we can set $\mu_i = 0$ for the assets without default risk, $\mu_i = -1$ for the assets with total default risk, $0 \neq \mu_i > -1$ for the assets with a non-zero recovery.*

Theorem 2.6.1. *Suppose that $v(t,x,0)$, $v(t,x,1) \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}, \mathbb{R})$, then we have*

$$Y_t^{t,x} = v(t, X^{t,x}, H_t).$$

and

$$Y_s^{t,x} = v(s, X_s^{t,x}, H_s); \qquad Z_s^{t,x} = \theta(s, X_s^{t,x})\partial_x v(s, X_s^{t,x}, H_s);$$
$$U_s^{t,x} = \Delta v(s, X_{s-}^{t,x}).$$

Proof. Denote $A_s := v(s, X_s^{t,x}, H_s)$. We know that A_s only has jumps at the default times, $\Delta A_s := A_s - A_{s-}$,

$$\Delta A_s = 1_{\{\tau=s\}}\left[v(s, X_s^{t,x}, 1) - v(s, X_{s-}^{t,x}, 0)\right] = 1_{\{\tau=s\}}\Delta v(s, X_{s-}^{t,x}).$$

Applying the Itô's formula for rcll semi-martingale (Theorem 1.3.4) to A_s,

$$\begin{aligned} dA_s =& \partial_t v(s, X_{s-}^{t,x}, H_{s-})ds + \partial_x v(s, X_{s-}^{t,x}, H_{s-})dX_s^{t,x} \\ &+ \frac{1}{2}\theta^2(s, X_{s-}^{t,x})\partial_{xx}^2 v(s, X_{s-}^{t,x}, H_{s-})ds \\ &+ \left[\Delta v(s, X_{s-}^{t,x}) - \mu(s, X_{s-}^{t,x})\partial_x v(s, X_{s-}^{t,x}, H_{s-})\right] dH_s \\ =& \left[\partial_t v(s, X_{s-}^{t,x}, H_{s-}) + \mathcal{L}^{t,x} v(s, X_{s-}^{t,x}, H_{s-})\right] ds \\ &+ \theta(s, X_s^{t,x})\partial_x v(s, X_{s-}^{t,x}, H_{s-})dB_s + \Delta v(s, X_{s-}^{t,x})dM_s, \end{aligned} \tag{2.34}$$

where $A_T = \varphi(X_T^{t,x}, H_T)$. By (2.31), it follows

$$\begin{aligned} &\int_0^t \partial_t v(s, X_{s-}^{t,x}, H_{s-}) + \mathcal{L}^{t,x} v(s, X_{s-}^{t,x}, H_{s-})ds \\ =& -\int_0^t f^{t,x}(s, X_{s-}^{t,x}, H_{s-})ds \\ =& -\int_0^t f\Big(s, X_{s-}^{t,x}, H_{s-}, v(s, X_{s-}^{t,x}, H_{s-}), \mathbb{E}^{\mathcal{G}_s}[v(s+\delta, X_{(s+\delta)-}^{t,x}, H_{(s+\delta)-})], \\ &\qquad \theta(s, X_{s-}^{t,x})\partial_x v(s, X_{s-}^{t,x}, H_{s-}), \theta(s, X_{s-}^{t,x})\mathbb{E}^{\mathcal{G}_s}[\partial_x v(s+\delta, X_{(s+\delta)-}^{t,x}, H_{(s+\delta)-})], \\ &\qquad \Delta v(s, X_{s-}^{t,x}), \mathbb{E}^{\mathcal{G}_s}[\Delta v(s+\delta, X_{(s+\delta)-}^{t,x})]\Big)ds \\ =& -\int_0^t f\Big(s, X_s^{t,x}, H_s, v(s, X_s^{t,x}, H_s), \mathbb{E}^{\mathcal{G}_s}[v(s+\delta, X_{s+\delta}^{t,x}, H_{s+\delta})], \\ &\qquad \theta(s, X_s^{t,x})\partial_x v(s, X_s^{t,x}, H_s), \theta(s, X_s^{t,x})\mathbb{E}^{\mathcal{G}_s}[\partial_x v(s+\delta, X_{s+\delta}^{t,x}, H_{s+\delta})], \\ &\qquad \Delta v(s, X_{s-}^{t,x}), \mathbb{E}^{\mathcal{G}_s}[\Delta v(s+\delta, X_{(s+\delta)-}^{t,x})]\Big)ds, \end{aligned}$$

and

$$\int_0^t \theta(s, X_s^{t,x})\partial_x v(s, X_{s-}^{t,x}, H_{s-})dB_s = \int_0^t \theta(s, X_s^{t,x})\partial_x v(s, X_s^{t,x}, H_s)dB_s.$$

In (2.34), we set

$$Y_s^{t,x} = v(s, X_s^{t,x}, H_s); \qquad Z_s^{t,x} = \theta(s, X_s^{t,x})\partial_x v(s, X_s^{t,x}, H_s);$$
$$U_s^{t,x} = \Delta v(s, X_{s-}^{t,x}).$$

Obviously, $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x})$ is the unique solution for ABSDE (2.33). When $s = t$, we get that $Y_t^{t,x} = v(t, X_t^{t,x}, H_t)$. $\square$

Chapter 3

Reflected Anticipated BSDEs with One Obstacle and Default Risk

Reflected BSDEs with one continuous lower reflecting obstacle driven by a Brownian motion was first considered by El Karoui et al. [33] (1997). Guo and Xu [87] (2013) studied RBSDE with one obstacle and default risk, they provided the existence and uniqueness theorem and an application in optimal stopping-control problem. More previous research about reflected BSDEs can be seen in Section 1.2.2.

In this chapter, we study reflected anticipated backward stochastic differential equations (RABSDE) with one obstacle driven by a Brownian motion and a mutually independent martingale in a defaultable setting. The generator of a RABSDE includes the present and future values of the solution. The proof of the existence theorem for RABSDEs with one obstacle and default risk is the foundation of the proof for RABSDEs with two obstacles and default risk in Chapter 4. We study the theoretical existence and uniqueness result and provide the related applications of RABSDE with one obstacles and default risk.

This chapter is organized as follows, Section 3.1 states the basic assumptions for RABSDEs with one obstacle and default risk. In Section 3.2, we use two methods, i.e. penalization method and the Snell envelope method to prove the existence and uniqueness theorem of the RABSDE (3.1). Section 3.3 represents an application in optimal stopping-control problem in the default setting. We illustrate the relation between linear RABSDEs with one obstacle and stochastic differential delay equations in a defaultable setting in Section 3.4.

3.1 Basic Assumptions

In this Chapter, we consider the following RABSDE (3.1) with one obstacle and default risk with the coefficient $(f, \xi, \alpha, \beta, \delta^1, \delta^2, \delta^3, L)$.

A quadruple $(Y, Z, U, K) := (Y_t, Z_t, U_t, K_t)_{0 \le t \le T+T^\delta}$ is a solution for the RABSDE with the generator f, the terminal value ξ_T, the anticipated processes ξ, α and β, the anticipated times δ^1, δ^2, δ^3, and the obstacle L. K is a continuous increasing process to keep Y above obstacle L, therefore the jumps of Y are only from default part. The anticipated time δ^1, δ^2 and δ^3 satisfy **H 2.2**.

$$
\begin{cases}
(i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R}),\ \ Z \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R}^d), \\
& U \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R}^k),\ \ K \in \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R}); \\
(ii) & Y_t = \xi_T + \int_t^T f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big) ds \\
& \qquad + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0, T]; \\
(iii) & Y_t \geq L_t, \qquad t \in [0, T]; \\
(iv) & Y_t = \xi_t, \qquad t \in (T, T+T^{\delta}]; \\
(v) & Z_t = \alpha_t, \qquad t \in (T, T+T^{\delta}]; \\
(vi) & U_t = \beta_t, \qquad t \in (T, T+T^{\delta}]; \\
(vii) & \int_0^T (Y_t - L_t) dK_t = 0,
\end{cases}
\tag{3.1}
$$

Now we introduce the following assumptions for RABSDE with one obstacle and default risk (3.1):

H 3.1. *The anticipated processes* $\xi \in \mathcal{L}^2_{\mathcal{G}}(T, T+T^{\delta}; \mathbb{R}^d)$, $\alpha \in \mathcal{L}^2_{\mathcal{G}}(T, T+T^{\delta}; \mathbb{R}^d)$, $\beta \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(T, T+T^{\delta}; \mathbb{R}^k)$, *here* ξ, α *and* β *are given processes;*

H 3.2. *The generator* $f(w, t, y, \bar{y}_r, z, \bar{z}_r, u, \bar{u}_r) : \Omega \times [0, T+T^{\delta}] \times \mathbb{R} \times \mathcal{S}^2_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R}) \times \mathbb{R}^d \times \mathcal{L}^2_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R}^d) \times \mathbb{R}^k \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R}^k) \to \mathbb{R}$ *satisfies:*

- ***(a)*** $f(\cdot, 0, 0, 0, 0, 0, 0) \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R})$;

- ***(b)*** ***Lipschitz condition****: for any* $t \in [0, T]$, $r \in [t, T+T^{\delta}]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u, u' \in \mathbb{R}^k$, $\bar{y}, \bar{y}' \in \mathcal{L}^2_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R})$, $\bar{z}, \bar{z}' \in \mathcal{L}^2_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R}^d)$, $\bar{u}, \bar{u}' \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R}^k)$, *there exists a constant* $L \geq 0$ *such that*

$$
\begin{aligned}
&|f(t, y, \bar{y}_r, z, \bar{z}_r, u, \bar{u}_r) - f(t, y', \bar{y}'_r, z', \bar{z}'_r, u', \bar{u}'_r)| \\
\leq & L\Big(|y - y'| + \mathbb{E}^{\mathcal{G}_t}|\bar{y}_r - \bar{y}'_r| + |z - z'| + \mathbb{E}^{\mathcal{G}_t}|\bar{z}_r - \bar{z}'_r| \\
&\quad + |u - u'| 1_{\{\tau > t\}} \sqrt{\gamma_t} + \mathbb{E}^{\mathcal{G}_t}|\bar{u}_r - \bar{u}'_r| 1_{\{\tau > t\}} \sqrt{\gamma_t}\Big);
\end{aligned}
$$

- ***(c)*** *for any* $t \in [0, T]$, $r \in [t, T+T^{\delta}]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u, u' \in \mathbb{R}^k$, $\bar{y}, \bar{y}' \in \mathcal{L}^2_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R})$, $\bar{z}, \bar{z}' \in \mathcal{L}^2_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R}^d)$, $\bar{u}, \bar{u}' \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t, T+T^{\delta}; \mathbb{R}^k)$, *the following holds:*

$$
\frac{f(t, y, \bar{y}_r, z, \bar{z}_r, \tilde{u}^{i-1}, \bar{u}_r) - f(t, y, \bar{y}_r, z, \bar{z}_r, \tilde{u}^{i}, \bar{u}_r)}{(u^i - \tilde{u}^i) 1_{\{\tau^i > t\}} \gamma^i_t} > -1,
$$

  where $\tilde{u}^i = (\tilde{u}^1, \tilde{u}^2, ..., \tilde{u}^i, u^{i+1}, ..., u^k)$, u^i *is the i-th element of* u.

Then we introduce the assumption for the obstacle process $L \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^{\delta}; \mathbb{R})$:

H 3.3. *The obstacle L is rcll (right continuous with left limits), and its jumping times are totally inaccessible, such that:*

$$
L_T \leq \xi_T, \quad \mathbb{P} - a.s.
$$

3.2 Existence and Uniqueness Theorem for RABSDEs with One Obstacle and Default Risk

3.2.1 Uniqueness Theorem for RABSDEs with One Obstacle and Default Risk

Theorem 3.2.1. ***(Uniqueness theorem for RABSDEs with one obstacle and default risk)*** *Suppose that the anticipated processes ξ, α and β, the generator f and the obstacle process L satisfy assumptions **H 3.1**, **H 3.2** and **H 3.3**. δ^1, δ^2 and δ^3 satisfy **H 2.2**. Then RABSDE (3.1) with the coefficient $(f, \xi, \alpha, \beta, \delta^1, \delta^2, \delta^3, L)$ has no more than one solution $(Y, Z, U, K) \in \mathcal{S}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}^k) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R})$.*

Proof. Suppose that (Y, Z, U, K) and (Y', Z', U', K') are two solutions of RABSDE (3.1). Denote:

$$\bar{Y} = Y - Y', \qquad \bar{Z} = Z - Z', \qquad \bar{U} = U - U', \qquad \bar{K} = K - K'.$$

Applying the Itô's formula for rcll semi-martingale (Theorem 1.3.4) to $|\bar{Y}_t|^2$ on $[t, T]$, we can obtain

$$\begin{aligned}
&|\bar{Y}_t|^2 + \int_t^T |\bar{Z}_s|^2 ds + \int_t^T |\bar{U}_s|^2 1_{\{\tau > s\}} \gamma_s ds \\
=&2 \int_t^T |\bar{Y}_s| \Big[f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big) \\
&- f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big) \Big] ds \\
&+ 2 \int_t^T \bar{Y}_s d\bar{K}_s - 2 \int_t^T \bar{Y}_s \bar{Z}_s dB_s - 2 \int_t^T \left(2\bar{Y}_{s-} \bar{U}_s + |\bar{U}_s|^2 \right) dM_s.
\end{aligned}$$

Since the generator f satisfies the Lipschitz condition, $\int_t^T \bar{Y}_s \bar{Z}_s dB_s$ and $\int_t^T \left[2\bar{Y}_{s-} \bar{U}_s + |\bar{U}_s|^2 \right] dM_s$ are $\mathcal{G}_t$-martingales, and

$$\begin{aligned}
\int_t^T \bar{Y}_s d\bar{K}_s = &\int_t^T (Y_{s-} - L_{s-}) dK_s + \int_t^T (L_{s-} - Y'_{s-}) dK_s \\
&+ \int_t^T (Y'_{s-} - L_{s-}) dK'_s + \int_t^T (L_{s-} - Y_{s-}) dK'_s \le 0.
\end{aligned}$$

Therefore, from the Fubini's Theorem and assumption **H 2.2**. Taking expectation in both sides, it follows

$$\begin{aligned}
&\mathbb{E} \left[|\bar{Y}_t|^2 + \int_t^T |\bar{Z}_s|^2 ds + \int_t^T |\bar{U}_s|^2 1_{\{\tau > s\}} \gamma_s ds \right] \\
\le &2L\mathbb{E} \Big[\int_t^T |\bar{Y}_s| \Big(|\bar{Y}_s| + \mathbb{E}^{\mathcal{G}_s} |\bar{Y}_{s+\delta^1(s)}| + |\bar{Z}_s| + \mathbb{E}^{\mathcal{G}_s} |\bar{Z}_{s+\delta^2(s)}| \\
&+ |\bar{U}_s| 1_{\{\tau > s\}} \sqrt{\gamma_s} + \mathbb{E}^{\mathcal{G}_s} |\bar{U}_{s+\delta^3(s)}| 1_{\{\tau > s\}} \sqrt{\gamma_s} \Big)^2 ds \Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq 2L\mathbb{E}\left[\int_t^T |\bar{Y}_s|\Big(|\bar{Y}_s| + |\bar{Z}_s| + |\bar{U}_s|1_{\{\tau>s\}}\sqrt{\gamma_s}\Big)ds\right] \\
&\quad + 2LL^\delta\mathbb{E}\left[\int_t^{T+T^\delta} |\bar{Y}_s|\Big(|\bar{Y}_s| + |\bar{Z}_s| + |\bar{U}_s|1_{\{\tau>s\}}\sqrt{\gamma_s}\Big)ds\right] \\
&\leq \left(2L + \frac{2L^2}{\lambda}\right)\mathbb{E}\left[\int_t^T |\bar{Y}_s|^2 ds\right] + \lambda\mathbb{E}\left[\int_t^T \left(|\bar{Z}_s| + |\bar{U}_s|1_{\{\tau>s\}}\gamma_s\right)ds\right] \\
&\quad + \left(2L + \frac{2L^2L^{\delta 2}}{\lambda}\right)\mathbb{E}\left[\int_t^{T+T^\delta} |\bar{Y}_s|^2 ds\right] \\
&\quad + \lambda\mathbb{E}\left[\int_t^{T+T^\delta} \left(|\bar{Z}_s| + |\bar{U}_s|1_{\{\tau>s\}}\gamma_s\right)ds\right] \\
&\leq \left(4L + \frac{2L^2 + 2L^2L^{\delta 2}}{\lambda}\right)\mathbb{E}\left[\int_t^{T+T^\delta} |\bar{Y}_s|^2 ds\right] \\
&\quad + 2\lambda\mathbb{E}\left[\int_t^{T+T^\delta} \left(|\bar{Z}_s| + |\bar{U}_s|1_{\{\tau>s\}}\gamma_s\right)ds\right],
\end{aligned}
$$

where $L \geq 0$, $\lambda > 0$ are constants. Set $\lambda = \frac{1}{4}$, it follows

$$
\begin{aligned}
&\mathbb{E}\left[|\bar{Y}_t|^2 + \int_t^{T+T^\delta} |\bar{Z}_s|^2 ds + \int_t^{T+T^\delta} |\bar{U}_s|^2 1_{\{\tau>s\}}\gamma_s ds\right] \\
&\leq \left(4L + 8L^2 + 8L^2L^{\delta 2}\right)\mathbb{E}\left[\int_t^{T+T^\delta} |\bar{Y}_s|^2 ds\right].
\end{aligned}
$$

Since the process $\bar{Y}$ is right continuous, by Gronwall's inequality, we can obtain that $Y = Y'$. Consequently, we can get

$$(Y, Z, U, K) = (Y', Z', U', K').$$

□

3.2.2 Existence Theorem for RABSDEs with One Obstacle and Default Risk

Similarly to the methodology used in El Karoui et. al [33] (1997), we prove the existence theorem for the RABSDE (3.1) through two methods, i.e. the penalization method in Section 3.2.2.1 and the Snell envelope method in Section 3.2.2.2. The comparison theorem for ABSDEs with default risk (Theorem 2.3.1) requires that the generator f is increasing in the anticipated term of Y and can not contain the anticipated terms of Z and U. Thus, for both of the methods, we suppose that the generator is independent on (y, z, u), then use the fixed point method to obtain the result in the general frame.

3.2.2.1 Existence Theorem – Penalization Method

In this section, we use the penalization method to prove the existence of the RABSDE (3.1). We first introduce the following penalized BSDE associated to RABSDE (3.1):

$$\begin{cases} Y_t^n = \xi_T + \int_t^T g(s)ds + K_T^n - K_t^n - \int_t^T Z_s^n dB_s - \int_t^T U_s^n dM_s, \quad t \in [0,T]; \\ K_t^n = n \int_0^t (Y_s^n - L_s)^- ds, \qquad t \in [0,T]. \end{cases} \tag{3.2}$$

where the terminal value ξ_T satisfies the assumption **H 3.1**, the generator g is independent on (y,z,u), i.e. for all $(w,t) \in \Omega \times [0,T]$, $f(w,t,y,z,u) \equiv g(w,t)$. For any $n \in \mathbb{N}$, the triple $(Y^n, Z^n, U^n) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k$ is a $\mathcal{G}$-adapted solution of the penalized BSDE (3.2).

In order to prove the existence theorem for the RABSDE (3.1), we first use penalization method to prove the existence of the penalized BSDE (3.2). For the proof of the existence theorem for the penalized BSDE (3.2) (Theorem 3.2.2), we introduce Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.3. Lemma 3.2.1 represents the approximation of the penalized BSDE (3.2). Lemma 3.2.2 illustrates the existence of the limiting process Y of Y^n in the sense of (3.9). Lemma 3.2.3 completes the existence of the limiting processes (Z,U,K) of (Z^n,U^n,K^n).

Lemma 3.2.1 represents the approximation of the penalized BSDE (3.2).

Lemma 3.2.1. *Suppose that the triple $(Y^n, Z^n, U^n) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k$ is the unique $\mathcal{G}$-adapted solution of penalized BSDE (3.2), then there exists a constant $C \geq 0$ independent of n, for all $t \in [0,T]$, such that*

$$\mathbb{E}\left[\sup_{0 \le t \le T} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds + \int_0^T |U_s^n|^2 1_{\{\tau > s\}} \gamma_s ds + |K_T^n|^2\right] \le C, \quad n \in \mathbb{N}. \tag{3.3}$$

Proof. **Step 1**. First we prove

$$\mathbb{E}\left[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + \int_t^T |U_s^n|^2 1_{\{\tau > s\}} \gamma_s ds + |K_T^n|^2\right] \le C_1, \tag{3.4}$$

where $C_1 \geq 0$ is a constant. Since K_t^n is continuous, then Y_t^n only has jumps from the random default times, applying Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|Y_s|^2$ on $[t,T]$, we can obtain

$$\begin{aligned} &|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + \int_t^T |U_s^n|^2 dH_s \\ =&|\xi_T|^2 + 2\int_t^T Y_s^n g(s) ds + 2\int_t^T Y_s^n dK_s^n - 2\int_t^T Y_s^n Z_s^n dB_s - 2\int_t^T Y_{s-}^n U_s^n dM_s, \end{aligned} \tag{3.5}$$

therefore,

$$\begin{aligned} &|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + \int_t^T |U_s^n|^2 1_{\{\tau > s\}} \gamma_s ds \\ =&|\xi_T|^2 + 2\int_t^T Y_s^n g(s) ds + 2\int_t^T Y_s^n dK_s^n - 2\int_t^T Y_s^n Z_s^n dB_s \\ &- 2\int_t^T \left[Y_{s-}^n U_s^n + |U_s^n|^2\right] dM_s. \end{aligned} \tag{3.6}$$

Since $\int_t^T Y_s^n Z_s^n dB_s$ and $\int_t^T \left[Y_{s-}^n U_s^n + |U_s^n|^2\right] dM_s$ are $\mathcal{G}_t$-martingales, δ^1, δ^2, and δ^3 satisfy **H 2.2**. Taking expectation in both sides, it follows

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + \int_t^T |U_s^n|^2 1_{\{\tau>s\}}\gamma_s ds\right] \\
=&\mathbb{E}|\xi_T|^2 + 2\mathbb{E}\left[\int_t^T Y_s^n g(s)ds\right] + 2\mathbb{E}\left[\int_t^T Y_s^n dK_s^n\right] \\
\leq&\mathbb{E}|\xi_T|^2 + \mathbb{E}\left[\int_t^T (|Y_s^n|^2 + |g(s)|^2)ds\right] + 2\mathbb{E}\Big[\int_t^T L_s^n dK_s^n\Big] \\
\leq&\mathbb{E}|\xi_T|^2 + \mathbb{E}\left[\int_t^T |Y_s^n|^2 ds\right] + \mathbb{E}\left[\int_t^T |g(s)|^2 ds\right] \\
&+ \frac{1}{\lambda_1}\mathbb{E}\left[\sup_{0\leq t\leq T} |L_t|^2\right] + \lambda_1 \mathbb{E}\left|K_T^n - K_t^n\right|^2,
\end{aligned}$$

where $\lambda_1 > 0$ are constants. Since

$$K_T^n - K_t^n = Y_t^n - \xi_T - \int_t^T g(s)ds + \int_t^T Z_s^n dB_s + \int_t^T U_s^n dM_s,$$

hence,

$$\begin{aligned}
\mathbb{E}\left|K_T^n - K_t^n\right|^2 \leq&\lambda_2\mathbb{E}\Big[|Y_t^n|^2 + |\xi_T|^2 + \int_t^T |g(s)|^2 ds \\
&+ \int_t^T \left(|Z_s^n|^2 + |U_s^n|^2 1_{\{\tau>s\}}\gamma_s\right)ds\Big].
\end{aligned} \tag{3.7}$$

where $\lambda_2 \geq 0$ is a constant. Hence,

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + \int_t^T |U_s^n|^2 1_{\{\tau>s\}}\gamma_s ds\right] \\
\leq& (1+\lambda_1\lambda_2)\,\mathbb{E}|\xi_T|^2 + \lambda_1\lambda_2\mathbb{E}|Y_t^n|^2 + \mathbb{E}\left[\int_t^T |Y_s^n|^2 ds\right] \\
&+ (1+\lambda_2)\mathbb{E}\left[\int_t^T |g(s)|^2 ds\right] + \frac{1}{\lambda_1}\mathbb{E}\left[\sup_{0\leq t\leq T}|L_t|^2\right] \\
&+ \lambda_1\lambda_2\mathbb{E}\left[\int_t^T \left(|Z_s^n|^2 + |U_s^n|^2 1_{\{\tau>s\}}\gamma_s\right)ds\right].
\end{aligned}$$

Set $\lambda_1\lambda_2 = \frac{1}{2}$, by the assumption **H 3.1**, we can obtain

$$\mathbb{E}\left[|Y_t^n|^2 + \frac{1}{2}\int_t^T |Z_s^n|^2 ds + \frac{1}{2}\int_t^T |U_s^n|^2 1_{\{\tau>s\}}\gamma_s ds\right] \leq C_2\left[1 + \mathbb{E}\int_t^T |Y_s^n|^2 ds\right],$$

where $C_2 \geq 0$ is a constant. By Gronwall's inequality, it follows consequently that $\mathbb{E}|Y_t^n|^2$ is bounded, therefore, $\mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right]$ and $\mathbb{E}\left[\int_t^T |U_s^n|^2 1_{\{\tau>s\}}\gamma_s ds\right]$ are bounded. By (3.7), we know $|K_T^n|^2$ is bounded, then we can get (3.6).

Step 2. Then we prove there exists a constant $C_3 \geq 0$, such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^2\right] \leq C_3. \tag{3.8}$$

By (3.5), we can obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^2\right] \leq& \mathbb{E}\sup_{0\leq t\leq T}\left[|Y_t^n|^2 + \int_t^T |Z_s^n|^2 ds + \int_t^T |U_s^n|^2 dH_s\right] \\ \leq& \mathbb{E}|\xi_T|^2 + 2\mathbb{E}\left[\int_0^T Y_s^n g(s) ds\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_s^n dK_s^n\right] \\ &+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_s^n Z_s^n dB_s\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_{s-}^n U_s^n dM_s\right] \\ \leq& \mathbb{E}|\xi_T|^2 + \mathbb{E}\left[\int_0^T |Y_s^n|^2 ds\right] + \mathbb{E}\left[\int_0^T |g(s)|^2 ds\right] \\ &+ \frac{1}{\lambda}\mathbb{E}\sup_{0\leq t\leq T}|L_t^+|^2 + \lambda\mathbb{E}|K_T^n|^2 \\ &+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_s^n Z_s^n dB_s\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_{s-}^n U_s^n dM_s\right],\end{aligned}$$

where $\lambda > 0$ is a constant. By Burkholder-Davis-Gundy inequality (Theorem A.1.1), it follows

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_s^n Z_s^n dB_s^n\right] &\leq \frac{1}{C_4}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^2\right] + C_4\mathbb{E}\left[\int_0^T |Z_s^n|^2 ds\right]; \\ \mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_{s-}^n U_s^n dM_s\right] &\leq \frac{1}{C_5}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n|^2\right] + C_5\mathbb{E}\left[\int_0^T |U_s^n|^2 1_{\{\tau>s\}}\gamma_s ds\right],\end{aligned}$$

where $C_4, C_5 > 0$ are constants. Therefore, we obtain (3.8). So we can prove (3.3). □

The following Lemma 3.2.2 illustrates the existence of the limiting process Y of Y^n in the sense of (3.9).

Lemma 3.2.2. *$(Y^n)_{n\geq 0}$ is a non-decreasing sequence. For any $t \in [0, T]$, $(Y_t^n)_{n\geq 0}$ converges to Y_t, and satisfies*

$$\begin{aligned} Y_t &= \lim_{n\to\infty} Y_t^n; \\ \lim_{n\to\infty}\mathbb{E}&\left[\sup_{0\leq t\leq T}|(Y_t^n - L_t)^-|^2\right] = 0, \qquad a.s. \end{aligned}\tag{3.9}$$

Proof. **Step 1**. By the comparison theorem for BSDEs with default risk (Theorem 1.3.2), we know that $(Y^n)_{n\geq 0}$ is non-decreasing,

$$\mathbb{E}Y_t^1 \leq \mathbb{E}Y_t = \mathbb{E}\left[\lim_{n\to\infty} Y_t^n\right] \leq \lim_{n\to\infty}\mathbb{E}Y_t^n \leq C,$$

where $C \geq 0$ is a constant. That is

$$Y_t^n \uparrow Y_t, \quad n \to \infty, \ \mathbb{P}-a.s.$$

Moreover, Lebesgue's dominated convergence theorem implies

$$\mathbb{E}\left[\int_0^T |Y_t^n - Y_t|^2 dt\right] \to 0, \ a.s., \quad n \to \infty.$$

Step 2. Then, let $(\bar{Y}, \bar{Z}, \bar{U})$ be the solution of the following BSDE with the coefficient $(g - n(y - L), \xi)$:

$$\bar{Y}_t^n = \xi_T + \int_t^T [g(s) - n(\bar{Y}_s^n - L_s)]\, ds - \int_t^T \bar{Z}_s^n dB_s - \int_t^T \bar{U}_s^n dM_s.$$

By the comparison theorem for BSDEs with default risk (Theorem 1.3.2), we know for any $t \in [0, T]$,

$$Y_t^n \geq \bar{Y}_t^n.$$

On the other hand, let $e^{-nt}\bar{Y}_t^n$ be the solution of the following BSDE with the coefficient $(e^{-nt}(g - nL), e^{-nT}\xi)$:

$$e^{-nt}\bar{Y}_t^n = e^{-nT}\xi_T + \int_t^T e^{-ns}\left(g(s) + nL_s\right) ds - \int_t^T e^{-ns}\bar{Z}_s^n dB_s - \int_t^T e^{-ns}\bar{U}_s^n dM_s.$$

Taking conditional expectation on both sides, let v be a $\mathcal{G}$-stopping time, such that

$$\bar{Y}_v^n = \mathbb{E}^{\mathcal{G}_v}\left[e^{-n(T-v)}\xi_T + \int_v^T e^{-n(s-v)}\left(g(s) + nL_s\right) ds\right].$$

Since the obstacle L is right continuous, and

$$\left|\int_v^T e^{-n(s-v)} g(s) ds\right| \leq \frac{1}{\sqrt{n}}\left(\int_v^T |g(s)|^2 ds\right)^{\frac{1}{2}},$$

as $n \to \infty$, we have the following convergences in $\mathcal{L}^2(\Omega; \mathbb{P})$:

$$\int_v^T g(s)e^{-n(s-v)} ds \to 0,, \qquad \mathbb{P}-a.s.;$$

$$e^{-n(T-v)}\xi_T + n\int_v^T e^{-n(s-v)} L_s ds \to \xi_T 1_{\{v=T\}} + L_v 1_{\{v<T\}}, \quad \mathbb{P}-a.s.,$$

Consequently,

$$\bar{Y}_v^n \to \xi_T 1_{\{v=T\}} + L_v 1_{\{v<T\}}, \quad n \to \infty, \ \mathbb{P}-a.s., \quad \text{in } \mathcal{L}^2(\Omega; \mathbb{P}).$$

Therefore,

$$Y_v \geq L_v, \quad \mathbb{P}-a.s.$$

By the section theorem (Theorem A.1.2), we deduce that for any $t \in [0, T]$,

$$Y_t \geq L_t, \qquad \mathbb{P} - a.s.,$$

consequently,

$$(Y_t^n - L_t)^- \downarrow 0, \qquad \mathbb{P} - a.s.$$

Since $Y^n \uparrow Y$, we know that ${}^pY^n \uparrow {}^pY$ and ${}^pY \geq {}^pL$ (${}^pX$ is the predictable projection of the process X). For any $n \in \mathbb{N}$, the jumps of Y^n are from the default process H, which are inaccessible (the default times are inaccessible). Therefore, for any predictable stopping time σ, we have $Y_\sigma^n = Y_{\sigma-}^n$, it follows that ${}^pY^n = Y_-^n$. Similarly, we can obtain ${}^pL = L_-$, since the obstacle process L only has the inaccessible jumps.

Consequently, we can prove

$$Y_-^n = {}^pY^n \uparrow {}^pY \geq {}^pL = L_-,$$

hence,

$$(Y_-^n - L_-) \uparrow ({}^pY - L_-) \geq 0,$$

it follows that, for any $t \in [0, T]$,

$$(Y_{t-}^n - L_{t-})^- \downarrow 0, \qquad n \to \infty, \qquad \mathbb{P} - a.s.$$

From a weak version of Dini's theorem (Theorem A.1.3), we deduce that

$$\sup_{0 \leq t \leq T} (Y_t^n - L_t)^- \downarrow 0, \qquad n \to \infty, \qquad \mathbb{P} - a.s.$$

Since $Y_t^1 - L_t^+ \leq Y_t^n - L_t$, then $(Y_t^n - L_t)^- \leq |Y_t^1| + L_t^+$, by the Lebesgue's dominated convergence theorem, it follows (3.9). □

The following Lemma 3.2.3 completes the existence of the limiting processes (Z, U, K) of (Z^n, U^n, K^n).

Lemma 3.2.3. *There exist $\mathcal{G}$-adapted processes $Z = (Z_t)_{0 \leq t \leq T}$, $U = (U_t)_{0 \leq t \leq T}$ and $K = (K_t)_{0 \leq t \leq T}$, such that*

$$\begin{aligned} \lim_{n \to \infty} \mathbb{E} \Bigg[& \sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 ds + \int_0^T |U_s^n - U_s|^2 1_{\{\tau > s\}} \gamma_s ds \\ & + \sup_{0 \leq t \leq T} |K_t^n - K_t|^2 \Bigg] = 0. \end{aligned} \tag{3.10}$$

Proof. **Step 1**. We first prove

$$\lim_{n \to \infty} \mathbb{E} \left[\int_0^T |Z_s^n - Z_s|^2 ds + \int_0^T |U_s^n - U_s|^2 1_{\{\tau > s\}} \gamma_s ds \right] = 0. \tag{3.11}$$

Applying the Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|Y_s^n - Y_s^m|^2$ on $[t,T]$, we can obtain, for any $m \geq n > 0$,

$$
\begin{aligned}
&\mathbb{E}\left[|Y_t^n - Y_t^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2 ds + \int_t^T |U_s^n - U_s^m|^2 1_{\{\tau>s\}}\gamma_s ds\right]\\
=&2\mathbb{E}\left[\int_t^T (Y_s^n - Y_s^m) d(K_s^n - K_s^m)\right]\\
\leq&2\mathbb{E}\left[\int_t^T (Y_s^n - L_s)^- dK_s^m\right] + 2\mathbb{E}\left[\int_t^T (Y_s^m - L_s)^- dK_s^n\right]\\
\leq&\left(\mathbb{E}\left[\sup_{0\leq t\leq T} |(Y_t^n - L_t)^-|^2\right]\right)^{\frac{1}{2}} \cdot K_T^m + \left(\mathbb{E}\left[\sup_{0\leq t\leq T} |(Y_t^m - L_t)^-|^2\right]\right)^{\frac{1}{2}} \cdot K_T^n.
\end{aligned}
$$

By Lemma 3.2.1 and Lemma 3.2.2, when $n \to \infty$, it follows

$$
\mathbb{E}\left[\int_0^T |Z_s^n - Z_s^m|^2 ds + \int_0^T |U_s^n - U_s^m|^2 1_{\{\tau>s\}}\gamma_s ds\right] \to 0.
$$

Therefore, $(Z^n)_{n\geq 0}$ and $(U^n)_{n\geq 0}$ are Cauchy sequences in complete spaces, then there exist $\mathcal{G}$-progressively measurable processes Z and U, such that sequences $(Z^n)_{n\geq 0}$ and $(U^n)_{n\geq 0}$ converge to Z and U in $\mathcal{L}_{\mathcal{G}}^2(0,T;\mathbb{R}^d)$ and $\in \mathcal{L}_{\mathcal{G}}^{2,\tau}(0,T;\mathbb{R}^k)$ respectively. Then we can obtain (3.11).

Step 2. Then, we prove

$$
\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t^n - Y_t|^2 + \sup_{0\leq t\leq T} |K_t^n - K_t|^2\right] = 0. \tag{3.12}
$$

Similarly to the proof of Lemma 3.2.1, by the Burkholder-Davis-Gundy inequality

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t^n - Y_t^m|^2\right]\\
\leq&\mathbb{E}\sup_{0\leq t\leq T}\left[|Y_t^n - Y_t^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2 ds + \int_t^T |U_s^n - U_s^m|^2 dH_s\right]\\
\leq&2\mathbb{E}\Big[\sup_{0\leq t\leq T} \int_t^T (Y_s^n - Y_s^m) d(K_s^n - K_s^m)\Big]\\
&+2\mathbb{E}\Big[\sup_{0\leq t\leq T} \int_t^T Y_s^n (Z_s^n - Z_s^m) dB_s\Big] + 2\mathbb{E}\Big[\sup_{0\leq t\leq T} \int_t^T Y_{s-}^n (U_s^n - U_s^m) dM_s\Big]\\
\leq&\Big(\mathbb{E}\Big[\sup_{0\leq t\leq T} |(Y_t^n - L_t)^-|^2\Big]\Big)^{\frac{1}{2}} \cdot K_T^m + \Big(\mathbb{E}\Big[\sup_{0\leq t\leq T} |(Y_t^m - L_t)^-|^2\Big]\Big)^{\frac{1}{2}} \cdot K_T^n\\
&+\lambda\mathbb{E}\left[\int_0^T |Z_s^n - Z_s^m|^2 ds\right] + \lambda\mathbb{E}\left[\int_0^T |U_s^n - U_s^m|^2 1_{\{\tau>s\}}\gamma_s ds\right]\\
&+\frac{2}{\lambda}\mathbb{E}\Big[\sup_{0\leq t\leq T} |Y_t^n - Y_t^m|^2\Big],
\end{aligned}
$$

where $\lambda > 0$ is a constant. When $n,\ m \to \infty$, we get

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2\right] \to 0, \qquad n,\ m \to \infty.$$

Thus,

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n - Y_t|^2\right] \to 0, \qquad n \to \infty,$$

here $Y = (Y_t)_{0\le t\le T} \in \mathcal{S}^2_{\mathcal{G}}(0,T;\mathbb{R})$.
Since

$$K_t^n = Y_0^n - Y_t^n - \int_0^t g(s)ds + \int_0^t Z_s^n dB_s + \int_0^t U_s^n dM_s,$$

it follows

$$\mathbb{E}\left[\sup_{0\le t\le T}|K_t^n - K_t^m|^2\right] \to 0, \qquad n,\ m \to \infty.$$

Hence there exists a $\mathcal{G}_t$-adapted increasing process $K = (K_t)_{0\le t\le T}$, $K_0 = 0$, such that

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|K_t^n - K_t|^2\right] = 0.$$

So we have proved (3.12), it follows (3.10). □

Therefore, we have the following existence theorem for the penalized BSDE (3.2).

Theorem 3.2.2. *(Existence theorem for the penalized BSDE (3.2))* $(Y^n, Z^n, U^n, K^n)_{n\ge 0}$ *has a limit process* (Y, Z, U, K)*, which is the solution of the following RBSDE with one obstacle associated with the coefficient* (g, ξ, L)*:*

$$\begin{cases} (i) \quad Y_t = \xi_T + \int_t^T g(s)ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \quad t\in[0,T];\\ (ii) \quad Y_t \ge L_t, \quad t\in[0,T];\\ (iii) \quad \int_0^T (Y_t - L_t)dK_t = 0, \quad \mathbb{P}-a.s. \end{cases} \tag{3.13}$$

Proof. From Lemma 3.2.2 and Lemma 3.2.3, $\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|(Y_t^n - L_t)^-|^2\right] = 0$, we can obtain

$$Y_t \ge L_t, \qquad t\in[0,T], \qquad \mathbb{P}-a.s.$$

Then, we give the proof of the condition (iii) of (3.13).

By Lemma 3.2.3, there exists a subsequence of $(K^n)_{n\ge 0}$ (we still denote as $(K^n)_{n\ge 0}$), such that

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|K_t^n - K_t|\right] = 0.$$

For any $w\in\Omega$, since the function $Y(w) - L(w) : t\in[0,T] \to Y_t(w) - L_t(w)$ is rcll, then there exists a sequence of staircase functions $(l^n(w))_{n\ge 0}$, which converges uniformly to $Y(w) - L(w)$ on $[0,T]$. From Lemma 3.2.3, it follows that, for any $\epsilon^1 > 0$, there exists a constant N^ϵ, such that for all $n \geq N^\epsilon$,

$$\begin{aligned} Y_t(w) - L_t(w) \leq Y_t^n(w) - L_t(w) + \epsilon^1; \\ K_T^n(w) \leq K_T(w) + \epsilon^1. \end{aligned} \tag{3.14}$$

Since

$$\int_0^T (Y_t^n - L_t) dK_t^n = -n \int_0^T |(Y_t^n - L_t)^-|^2 dt \leq 0,$$

hence, by (3.14), for all $n \geq N^\epsilon$, we get

$$\int_0^T (Y_t - L_t) dK_t^n \leq \epsilon^1 K_T(w) + \epsilon^2. \tag{3.15}$$

On the other hand, there exists a constant M^ϵ, such that for all $n \geq M^\epsilon$,

$$|Y_t(w) - L_t(w) - l_t^n(w)| < \epsilon^1,$$

since $l^n(w)$ is a staircase function, then $\int_0^T h_t^m(w) d(K_t - K_t^n) \to 0$, as $n \to \infty$. Therefore, it follows

$$\begin{aligned} &\int_0^T (Y_t - L_t) d(K_t - K_t^n) \\ =& \int_0^T (Y_t - L_t - l_t^m(w)) d(K_t - K_t^n) + \int_0^T h_t^m(w) d(K_t - K_t^n) \\ \leq& \epsilon^1 (K_T(w) - K_T^n(w)) + \int_0^T l_t^m(w) d(K_t - K_t^n) \to 2\epsilon^1 K_T(w). \end{aligned}$$

Thus, we can obtain

$$\lim_{n \to \infty} \left[\sup_{0 \leq t \leq T} \int_0^T (Y_t - L_t) d(K_t - K_t^n) \right] \leq 2\epsilon^1 K_T(w). \tag{3.16}$$

Finally, by (3.15) and (3.16), we can get

$$\begin{aligned} \int_0^T (Y_t - L_t) dK_t =& \int_0^T (Y_t - L_t) d(K_t - K_t^n) + \int_0^T (Y_t^n - L_t) dK_t^n \\ \leq& 3\epsilon^1 K_T(w) + \epsilon^2. \end{aligned}$$

As ϵ^1 and ϵ^2 are arbitrary, and $Y \leq L$, consequently, we can prove *(iii)* $\int_0^T (Y_t - L_t) dK_t = 0$. □

3.2.2.2 Existence Theorem – Snell Envelope Method

In this Section, we use the Snell envelope method to prove the existence of the RABSDE (3.1). Some definitions, properties and theorems of Snell envelope theory for the proof of the existence theorem for RABSDE can be found in the Appendix. More previous research about the Snell envelope theory can be seen in Meyer [88] (1966), El Karoui [89] (1981), Hamadène [35] (2008), etc.

Similarly to the penalization method in Section 3.2.2.1, we introduce the following RBSDE whose generator is independent on (y, z, u), as below:

$$\begin{cases} Y_t = \xi_T + \int_t^T g(s)ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, & t \in [0, T]; \\ Y_t = \xi_t, \quad t \in (T, T + T^\delta], \end{cases} \tag{3.17}$$

where the terminal value ξ_T and the obstacle process L satisfy **H 3.1** and **H 3.3**. δ^1, δ^2 and δ^3 satisfy **H 2.2**. K is a continuous increasing process.

Lemma 3.2.4. *There exists a solution $(Y_t, Z_t, U_t, K_t)_{0 \le t \le T}$ for RBSDE (3.17) with the coefficient (g, ξ, L).*

Proof. **Step 1**. Define $\theta := (\theta_t)_{0 \le t \le T}$ as below:

$$\theta_t = \xi_T 1_{\{t=T\}} + L_t 1_{\{t<T\}} + \int_0^t g(s)ds, \qquad t \in [0, T], \tag{3.18}$$

therefore, θ is rcll and has the same inaccessible jumping times $[0, T)$ as the obstacle L (may have a positive jump at the terminal time T). Moreover,

$$\sup_{0 \le t \le T} |\theta_t| \in \mathcal{L}^2(\Omega).$$

By the Definition A.1.2, its Snell envelop $(\mathcal{S}_t(\theta))_{0 \le t \le T}$ is of class $\mathcal{D}$ (Definition A.1.1) and satisfies

$$\mathbb{E}\left[\sup_{0 \le t \le T} |\mathcal{S}_t(\theta)|^2\right] < \infty. \tag{3.19}$$

Hence, from Doob-Meyer decomposition theorem for Snell envelope (Theorem A.1.4), the Snell envelop $\mathcal{S}_t(\theta)$ has the following decomposition:

$$\mathcal{S}_t(\theta) = \mathbb{E}^{\mathcal{G}_t}\left[\xi_T + \int_0^t g(s) + K_T\right] - K_t,$$

where $(K_t)_{0 \le t \le T}$ is a $\mathcal{G}_t$-adapted rcll non-decreasing process, $K_0 = 0$. By (3.19) and Theorem A.1.4, consequently, $\mathbb{E}|K_T|^2 < \infty$. Therefore,

$$\mathbb{E}\left[\sup_{0 \le t \le T} \left|\mathbb{E}^{\mathcal{G}_t}[\xi_T + K_T]\right|^2\right] < \infty.$$

By Kusuoka martingale representation theorem (Theorem 1.3.3), there exist two processes $(Z_t)_{0 \le t \le T}$ and $(U_t)_{0 \le t \le T}$, such that

$$\begin{aligned} N_t := & \mathbb{E}^{\mathcal{G}_t}\left[\xi_T + \int_0^t g(s) + K_T\right] \\ = & \mathbb{E}[\xi_T + K_T] + \int_0^t Z_s dB_s + \int_0^t U_s dM_s, \qquad t \in [0, T], \end{aligned} \tag{3.20}$$

where

$$\mathbb{E}\left[\int_0^T \left(|Z_s|^2 + |U_s|^2 1_{\{\tau > s\}} \gamma_s\right) ds\right] < \infty.$$

Step 2. Set

$$Y_t = ess \sup_{\sigma \in \mathcal{T}_t} \mathbb{E}^{\mathcal{G}_t}\left[\xi_T 1_{\{v=T\}} + L_t 1_{\{v<T\}} + \int_t^v g(s)ds\right],$$

then it follows

$$Y_t + \int_0^t g(s)ds = \mathcal{S}_t(\theta) = M_t - K_t,$$

from the definition of θ (3.18), we can obtain

$$Y_t \geq L_t.$$

Combining (3.20), it follows

$$Y_t + \int_0^t g(s)ds = \mathbb{E}[\xi_T + K_T] + \int_0^t Z_s dB_s + \int_0^t U_s dM_s - K_t, \qquad t \in [0,T].$$

Hence, for any $t \in [0,T]$,

$$Y_t = \xi_T + \int_t^T g(s)ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t \in [0,T].$$

Step 3. Then we prove that K is continuous.

By Theorem A.1.4, we know that $\{\Delta K^d > 0\} \subset \{\mathcal{S}_-(\theta) = \theta_-\}$, i.e. the jumping times of K are included in the set $\{\mathcal{S}_-(\theta) = \theta_-\}$, where $\limsup_{s\uparrow t} \theta_s = \theta_{t-}$.

Let v be a predictable stopping time on $[0,T]$. Since the process θ only has the inaccessible jumps on $[0,T)$ (may have a positive jump at $t = T$). Hence, when K has a jump at time v, it follows

$$\begin{aligned} \mathbb{E}\left[\mathcal{S}_{v-}(\theta)1_{\{\Delta K_v>0\}}\right] =& \mathbb{E}\left[\theta_{v-}1_{\{\Delta K_v>0\}}\right] \\ \leq& \mathbb{E}\left[\theta_v 1_{\{\Delta K_v>0\}}\right] \\ \leq& \mathbb{E}\left[\mathcal{S}_v(\theta)1_{\{\Delta K_v>0\}}\right]. \end{aligned} \tag{3.21}$$

If there is no jump of K at v, since the jumping times of M are from default part which are inaccessible, then $M_{v-} = M_v$, it follows

$$\begin{aligned} \mathbb{E}\left[\mathcal{S}_{v-}(\theta)1_{\{\Delta K_v=0\}}\right] =& \mathbb{E}\left[(M_{v-} + K_v)1_{\{\Delta K_v=0\}}\right] \\ \leq& \mathbb{E}\left[(M_v + K_v)1_{\{\Delta K_v=0\}}\right] \\ \leq& \mathbb{E}\left[\mathcal{S}_v(\theta)1_{\{\Delta K_v=0\}}\right]. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22), we can get

$$\mathbb{E}\left[\mathcal{S}_{v-}(\theta)\right] \leq \mathbb{E}\left[\mathcal{S}_v(\theta)\right],$$

since $\mathcal{S}(\theta)$ is a super-martingale, it follows that, for any predictable stopping time v,

$$\mathbb{E}\left[\mathcal{S}_{v-}(\theta)\right] = \mathbb{E}\left[\mathcal{S}_{v}(\theta)\right].$$

Thus, from Definition A.1.3, we know that ${}^{p}\mathcal{S}(\theta) = \mathcal{S}_{-}(\theta)$, i.e. $\mathcal{S}(\theta)$ is regular, consequently, the process K is continues.

Step 4. Finally we prove $\int_0^T (Y_t - L_t)dK_t = 0$.

Define

$$v_t := \inf\{s \geq t;\ K_s \geq K_t\} \wedge T,$$

Since $\mathcal{S}(\theta)$ is regular, by Theorem A.1.5, it follows that v_t is optimal on $[t, T]$, consequently,

$$\mathcal{S}_{v_t}(\theta) = \theta_{v_t}.$$

Therefore, for any $s \in [t, v_t]$,

$$\begin{aligned}(\mathcal{S}_t(\theta) - \theta_t)dK_t &= 0\\ &= (Y_t - L_t)dK_t.\end{aligned}$$

It follows that $\int_0^T (Y_t - L_t)dK_t = 0$. □

Then we use Theorem 3.2.1 and Theorem 3.2.3 to prove the existence of RABSDE (3.1).

3.2.2.3 Existence Theorem for the RABSDE (3.1) in the general frame

We have proved the existence of the penalized BSDE (3.2) through the penalization method in Section 3.2.2.1, and the existence of RBSDE (3.17) through the Snell envelope method in Section 3.2.2.2. By the uniqueness theorem (Theorem 3.2.1) in Section 3.2.1, we can use the fixed point method (Banach fixed-point theorem) to prove the following existence and uniqueness theorem of RABSDE (3.1) in the general frame.

Theorem 3.2.3. ***(Existence and uniqueness theorem for RABSDEs with one obstacle and default risk)*** *Suppose that the anticipated processes ξ, α and β, the generator f and the obstacle process L satisfy the assumptions* ***H 3.1***, ***H 3.2*** *and* ***H 3.3****. δ^1, δ^2 and δ^3 satisfy* ***H 2.2****. Then RABSDE (3.1) with the coefficient $(f, \xi, \alpha, \beta, \delta^1, \delta^2, \delta^3, L)$ has a unique solution $(Y, Z, U, K) \in \mathcal{S}^2_{\mathcal{G}}(0, T + T^{\delta}; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T + T^{\delta}; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T + T^{\delta}; \mathbb{R}^k) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R})$.*

Proof. Define $\mathcal{D} := \mathcal{S}^2_{\mathcal{G}}(0, T + T^{\delta}; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T + T^{\delta}; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T + T^{\delta}; \mathbb{R}^k)$. Define the following mapping:

$$\begin{aligned}\Phi :\ &\mathcal{D} \to \mathcal{D};\\ (y, z, u) &\to \Phi(y, z, u) := (Y, Z, U).\end{aligned}$$

First, we prove that Φ is a contraction mapping of $\mathcal{D}$.

We define

$$\begin{aligned} K_t =&Y_t - Y_0 - \int_0^t f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big)ds \\ &+ \int_0^t Z_s dB_s + \int_0^t U_s dM_s, \qquad t \in [0, T], \end{aligned} \tag{3.23}$$

then (Y, Z, U, K) solves the penalized BSDE with the generator $g(s) = f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big)$, i.e. (Y, Z, U, K) is the solution of the following RABSDE:

$$\begin{cases} Y_t = \xi_T + \int_t^T f\big(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\big)ds \\ \qquad +K_T - K_t - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \quad t \in [0, T]; \\ Y_t \geq L_t, \quad t \in [0, T]; \\ Y_t = \xi_t, \quad t \in (T, T + T^\delta]; \\ Z_t = \alpha_t, \quad t \in (T, T + T^\delta]; \\ U_t = \beta_t, \quad t \in (T, T + T^\delta]; \\ \int_0^T (Y_t - L_t) dK_t = 0, \quad \mathbb{P} - a.s. \end{cases} \tag{3.24}$$

From Theorem 3.2.1, (Y, Z, U) is the unique solution of the penalized BSDE (3.2). For any (y, z, u), $(y', z', u') \in \mathcal{D}$, denote:

$$\begin{aligned} &\hat{y} = y - y', \quad \hat{z} = z - z', \quad \hat{u} = u - u'; \\ &\hat{Y} = Y - Y', \quad \hat{Z} = Z - Z', \quad \hat{U} = U - U', \quad \hat{K} = K - K'. \end{aligned}$$

Applying the Itô's formula for rcll semi-martingale (Theorem 1.3.4), we can obtain

$$\begin{aligned} &|\hat{Y}_t|^2 + \int_0^T e^{cs}\left(c|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau > s\}}\gamma_s\right)ds \\ =&2\int_t^T e^{cs}\hat{Y}_s \big| f\big(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\big) \\ &- f\big(s, y'_s, y'_{s+\delta^1(s)}, z'_s, z'_{s+\delta^2(s)}, u'_s, u'_{s+\delta^3(s)}\big)\big| ds + 2\int_t^T e^{cs}|\hat{Y}_s| d\hat{K}_s \\ &- 2\int_t^T e^{cs}\hat{Y}_s\hat{Z}_s dB_s - \int_t^T e^{cs}\left(2\hat{Y}_{s-}\hat{U}_s + |\hat{U}_s|^2\right)dM_s, \end{aligned}$$

where $c > 0$ is a constant. Since $\int_t^T e^{cs}\hat{Y}_s^{(n)}\hat{Z}_s^{(n)} dB_s$ and $\int_t^T e^{cs}\left(2\hat{Y}_{s-}^{(n)}\hat{U}_s^{(n)} + |\hat{U}_s^{(n)}|^2\right)dM_s$ are $\mathcal{G}$-martingales, $|\hat{Y}_t^{(n)}|^2 \geq 0$, $\int_t^T e^{cs}|\hat{Y}_s| d\hat{K}_s \leq 0$, by the Fubini's Theorem, **H 2.2** and the Lipschitz condition for f, it follows

$$\begin{aligned} &\mathbb{E}\left[\int_t^T e^{cs}\left(\frac{c}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau > s\}}\gamma_s\right)ds\right] \\ \leq&\frac{2}{c}\mathbb{E}\Big[\int_t^T e^{cs}\hat{Y}_s \big| f\big(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\big) \\ &\qquad - f\big(s, y'_s, y'_{s+\delta^1(s)}, z'_s, z'_{s+\delta^2(s)}, u'_s, u'_{s+\delta^3(s)}\big)\big|^2 ds\Big] \end{aligned}$$

$$
\begin{aligned}
\leq & \frac{2L^2}{c}\mathbb{E}\Big[\int_0^T e^{cs}\Big(|\hat{y}_s| + \mathbb{E}^{\mathcal{G}_s}|\hat{y}_{s+\delta^1(s)}| + |\hat{z}_s| + \mathbb{E}^{\mathcal{G}_s}|\hat{z}_{s+\delta^2(s)}| \\
& \qquad + |\hat{u}_s|1_{\{\tau>s\}}\sqrt{\gamma_s} + \mathbb{E}^{\mathcal{G}_s}|\hat{u}_{s+\delta^3(s)}|1_{\{\tau>s\}}\sqrt{\gamma_s}\Big)^2 ds\Big] \\
\leq & \frac{12L^2}{c}\mathbb{E}\left[\int_t^T e^{cs}|\hat{Y}_s|\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right] \\
& + \frac{12L^2L^\delta}{c}\mathbb{E}\left[\int_t^{T+T^\delta} e^{cs}\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right] \\
\leq & \frac{12L^2 + 12L^2L^\delta}{c}\mathbb{E}\left[\int_t^{T+T^\delta} e^{cs}\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right].
\end{aligned}
$$

where $L \geq 0$, $L^\delta \leq 0$ are constants. Set $c = 12L^2 + 12L^2L^\delta + 2$, it follows

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^{T+T^\delta} e^{cs}\left(\frac{c}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right] \\
\leq & \frac{1}{2}\mathbb{E}\left[\int_0^{T+T^\delta} e^{cs}\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right].
\end{aligned}
$$

Therefore, Φ is a contraction mapping on $\mathcal{D}$ equipped with the norm defined as below:

$$
\|(Y, Z, U)\|_c := \left\{\mathbb{E}\int_0^{T+T^\delta} e^{cs}\left(|Y_s|^2 + |Z_s|^2 + |U_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right\}^{\frac{1}{2}}.
$$

From the Banach fixed-point theorem, there exists a unique fixed point $(Y, Z, U) \in \mathcal{L}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T + T^\delta; \mathbb{R}^k)$, with K (defined as (3.23)), is the solution of the RABSDE (3.1). Combining with the assumptions **H 2.3** and **H 2.2**, it follows that $f(t, Y_t, Y_{t+\delta^1(t)}, Z_t, Z_{t+\delta^2(t)}, U_t, U_{t+\delta^3(t)}) \in \mathcal{L}^2_{\mathcal{G}}(0, T + T^\delta; \mathbb{R})$. □

3.3 Application in Optimal Stopping-Control Problem

El Karoui et al. [33] (1997) studied the relation between RBSDEs with one obstacle and optimal stopping-control problems. Guo and Xu [87] (2013) extended this topic to RBSDEs with default risk. More research on this topic can be found in Ren and Xia [90] (2006), Elliott and Siu [91] (2013), Dumitrescu [92] (2016), etc.

In this section, we study the relation between RABSDEs with one obstacle and optimal stopping-control problems in the default setting. We take the conditional expectation of the anticipated terms in (3.25) to make the system adapted. Consider the following RABSDE (3.25) with one obstacle and default risk (δ is a given constant), $(Y, Z, U, K) \in \mathcal{S}^2_{\mathcal{G}}(0, T + \delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T + \delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T + \delta; \mathbb{R}^k) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R})$ is a solution of the RABSDE (3.25):

$$\begin{cases} Y_t = \xi_T + \int_t^T f\big(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\big)ds \\ \qquad +(K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \quad t \in [0,T]; \\ Y_t \geq L_t, \qquad t \in [0, T+\delta]; \\ Y_t = \xi_t, \qquad t \in [T, T+\delta]; \\ Z_t = \alpha_t, \qquad t \in (T, T+\delta]; \\ U_t = \beta_t, \qquad t \in (T, T+\delta]; \\ \int_{t_0}^T (Y_t - L_t) dK_t = 0, \end{cases} \tag{3.25}$$

The following Theorem 3.3.1 illustrates that the solution Y of the RABSDE (3.25) above corresponds to the value of a optimal stopping-control problem.

Theorem 3.3.1. *Let $(Y, Z, U, K) \in \mathcal{S}^2_{\mathcal{G}}(0, T+\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T+\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+\delta; \mathbb{R}^k) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R})$ be the unique solution of the RABSDE (3.1) with the coefficient (f, ξ, L, δ). The anticipated process ξ, the generator f and the obstacle process L satisfy the assumptions **H** 3.1, **H** 3.2 and **H** 3.3. Then for all $t \in [0, T]$:*

$$Y_t = \operatorname*{ess\,sup}_{v \in \mathcal{T}_t} \mathbb{E}^{\mathcal{G}_t}\Big[\int_t^v f\left(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\right) ds + L_v 1_{\{v<T\}} + \xi_T 1_{\{v=T\}}\Big],$$

where $\mathcal{T}_t = \{v \in \mathcal{T} | \, t \leq v \leq T\}$, $\mathcal{T}$ is the set of all the stopping times on $[0, T]$.

Proof. **Step 1**. First, we prove (3.26) as below:

$$K_T - K_t = \sup_{t \leq r \leq T} \left(\xi_T + \int_r^T f\left(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\right) ds - \int_r^T Z_s dB_s - \int_r^T U_s dM_s - L_r\right)^-. \tag{3.26}$$

Since

$$\begin{aligned} Y_{T-t} =& \xi_T + \int_{T-t}^T f\left(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\right) ds \\ &+ K_T - K_t - \int_{t-T}^T Z_s dB_s - \int_{T-t}^T U_s dM_s, \end{aligned}$$

and

$$\begin{aligned} & Y_{T-t} \geq L_{T-t}, \\ & \int_0^T (Y_{t-} - L_{t-}) dK_t = 0. \end{aligned}$$

Set

$$\begin{aligned} x_t =& \Big(\xi_T + \int_{t-T}^T f\left(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\right) ds \\ &- \int_{t-T}^T Z_s dB_s - \int_{t-T}^T U_s dM_s - L_{t-T}\Big)(w); \end{aligned}$$

$$k_t = (K_T - K_{t-T})(w);$$
$$y_t = (Y_{T-t} - L_{T-t})(w).$$

by the Skorohod lemma (Lemma A.1.1), consequently it follows that (3.26).

Step 2. Denote

$$D_t = \inf\{r \le t \le T;\ Y_r = L_r\},$$

with the convention that $D_t = T$, if $Y_r \ge L_r$ and $r \le t \le T$. By the condition $\int_0^T (Y_t - L_t) dK_t = 0$, it follows

$$K_{D_t} - K_t = 0.$$

Since

$$\begin{aligned} Y_t =& \mathbb{E}^{\mathcal{G}_t}\Big[\int_t^v f\left(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\right) ds + Y_v + K_v - K_t\Big] \\ \ge& \mathbb{E}^{\mathcal{G}_t}\Big[\int_t^v f\left(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\right) ds \\ & + L_v 1_{\{v<T\}} + \xi_T 1_{\{v=T\}}\Big], \end{aligned}$$

we can obtain

$$\begin{aligned} Y_t =& \mathbb{E}^{\mathcal{G}_t}\Big[\int_t^{D_t} f\left(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}], U_s, \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\right) ds \\ & + L_{D_t} 1_{\{D_t<T\}} + \xi_T 1_{\{D_t=T\}}\Big]. \end{aligned}$$

Hence, the result follows. □

3.4 Linear Reflected Anticipated BSDEs with One Obstacle and Stochastic Differential Delay Equations

Similarly to Section 2.4, we study the relation between linear RABSDEs with one obstacle and SDDEs. Consider the following RABSDE (3.27) with one obstacle and default risk (δ is a given constant, t_0 is the initial time, B is a d-dimensional standard Brownian motion). $(Y, Z, U, K) \in \mathcal{S}^2_{\mathcal{G}}(0, T+\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T+\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+\delta; \mathbb{R}^k) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R})$ is a solution of the RABSDE (3.27):

$$\begin{cases} -dY_t = \Big(\sigma_t Y_t + \hat{\sigma}_t \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}] + \theta_t Z_t + \hat{\theta}_t \mathbb{E}^{\mathcal{G}_t}[Z_{t+\delta}] + \mu_t U_t 1_{\{\tau>t\}} \\ \qquad\quad + \hat{\mu}_t \mathbb{E}^{\mathcal{G}_t}[U_{t+\delta}] 1_{\{\tau>t\}} + l_t\Big) dt + dK_t - Z_t dB_t - U_t dM_t, \quad t \in [0, T]; \\ Y_t \ge L_t, \qquad t \in [t_0, T+\delta]; \\ Y_t = \xi_t, \qquad t \in [T, T+\delta]; \\ Z_t = \alpha_t, \qquad t \in (T, T+\delta]; \\ U_t = \beta_t, \qquad t \in (T, T+\delta]; \\ \int_{t_0}^T (Y_t - L_t) dK_t = 0, \end{cases} \tag{3.27}$$

and the following stochastic differential delay equation with default risk (SDDE):

$$
\begin{cases}
dX_s = (\sigma_s X_s + \hat{\sigma}_{s-\delta} X_{s-\delta})\, ds + \left(X_s\theta_s + X_{s-\delta}\hat{\theta}_{s-\delta}\right) dB_s \\
\qquad\qquad + \left(X_{s-}\mu_{s-} + X_{(s-\delta)-}\hat{\mu}_{(s-\delta)-}\right) dM_s, \qquad s \in [t, T+\delta]; \\
X_t = 1; \\
X_s = 0, \qquad s \in [t-\delta, t),
\end{cases}
\tag{3.28}
$$

where σ, $\hat{\sigma}$, θ, μ and $\hat{\mu}$ are uniformly bounded. σ, $\hat{\sigma} \in \mathcal{L}^2_{\mathcal{G}}(t_0-\delta, T+\delta; \mathbb{R})$, θ, $\hat{\theta} \in \mathcal{L}^2_{\mathcal{G}}(t_0-\delta, T+\delta; \mathbb{R}^d)$, $\mu, \hat{\mu} \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t_0-\delta, T+\delta; \mathbb{R}^k)$, $l \in \mathcal{L}^2_{\mathcal{G}}(t_0, T; \mathbb{R})$, $\xi \in \mathcal{S}^2_{\mathcal{G}}(T, T+\delta; \mathbb{R})$, $\alpha \in \mathcal{L}^2_{\mathcal{G}}(T, T+\delta; \mathbb{R}^d)$, $\beta \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(T, T+\delta; \mathbb{R}^k)$.

Theorem 3.4.1. *The solution Y of the RABSDE (3.27) above can be given in the following form:*

$$
\begin{aligned}
Y_t = ess\sup_{v\in\mathcal{T}_t} \mathbb{E}^{\mathcal{G}_t}\Big[\int_t^v l_s X_s ds + L_v X_v 1_{\{v<T\}} + \xi_T X_T 1_{\{v=T\}} \\
+ 1_{\{v=T\}} \int_T^{T+\delta} \left(\xi_s \hat{\sigma}_{s-\delta} + \alpha_s \hat{\theta}_{s-\delta} + \beta_s \hat{\mu}_{s-\delta} 1_{\{\tau>s\}}\right) X_{s-\delta} ds\Big],
\end{aligned}
\tag{3.29}
$$

where X is the solution of the SDDE (3.28), $\mathcal{T}_t = \{v \in \mathcal{T} \mid t \le v \le T\}$, $\mathcal{T}$ is the set of all the stopping times on $[0, T]$.

Proof. From Theorem 2.4.1, we know that the SDDE (3.28) has a unique solution. Applying the Itô's formula for rcll semi-martingale (Theorem 1.3.4) to $X_s Y_s$ on $[t, T]$, and taking conditional expectation under $\mathcal{G}_t$, we can obtain

$$
\begin{aligned}
& X_T Y_T - X_t Y_t \\
= & \int_t^T \Big(Y_s \hat{\sigma}_{s-\delta} X_{s-\delta} - \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}]\sigma_s X_s + Z_s \hat{\theta}_{s-\delta} X_{s-\delta} - \mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}]\theta_s X_s \\
& \qquad + U_s \hat{\mu}_{s-\delta} X_{s-\delta} 1_{\{\tau>s\}} - \mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\hat{\mu}_s X_s 1_{\{\tau>s\}} - X_s l_s \Big) ds \\
& - \int_t^T X_s dK_s + \int_t^T \left[Y_s \left(X_s\theta_s + X_{s-\delta}\hat{\theta}_{s-\delta}\right) + X_s Z_s \right] dB_s \\
& + \int_t^T \left[Y_s \left(X_{s-}\mu_{s-} + X_{(s-\delta)-}\hat{\mu}_{(s-\delta)-} \right) + X_s U_s \right] dM_s.
\end{aligned}
$$

When $s \in [t-\delta, t)$, $X_s = 0$, and $X_t = 1$, it follows:

$$
\begin{aligned}
X_t Y_t = & X_T Y_T + \int_t^T X_s l_s ds + \int_t^T X_s dK_s - \int_t^T Y_s \hat{\sigma}_{s-\delta} X_{s-\delta} ds \\
& + \int_{t+\delta}^{T+\delta} Y_s \hat{\sigma}_{s-\delta} X_{s-\delta} ds - \int_t^T Z_s \hat{\theta}_{s-\delta} X_{s-\delta} ds + \int_{t+\delta}^{T+\delta} Z_s \hat{\theta}_{s-\delta} X_{s-\delta} ds \\
& - \int_t^T U_s \hat{\mu}_{s-\delta} X_{s-\delta} 1_{\{\tau>s\}} ds + \int_{t+\delta}^{T+\delta} U_s \hat{\mu}_{s-\delta} X_{s-\delta} 1_{\{\tau>s\}} ds \\
& + \int_t^T \left[Y_s \left(X_s\theta_s + X_{s-\delta}\hat{\theta}_{s-\delta}\right) + X_s Z_s \right] dB_s
\end{aligned}
$$

$$\begin{aligned}
&+\int_t^T \left[Y_s\left(X_{s-}\mu_s + X_{s-\delta}\hat{\mu}_{(s-\delta)-}\right) + X_sU_s\right]dM_s \\
=&X_T\xi_T + \int_t^T X_s l_s ds + \int_T^{T+\delta}\left(\xi_s\hat{\sigma}_{s-\delta} + \alpha_s\hat{\theta}_{s-\delta} + \beta_s\hat{\mu}_{s-\delta}1_{\{\tau>s\}}\right)X_{s-\delta}ds \\
&+\int_t^T X_s dK_s + \int_t^T \left[Y_s\left(X_s\theta_s + X_{s-\delta}\hat{\theta}_{s-\delta}\right) + X_sZ_s\right]dB_s \\
&+\int_t^T \left[Y_s\left(X_{s-}\mu_s + X_{s-\delta}\hat{\mu}_{(s-\delta)-}\right) + X_sU_s\right]dM_s.
\end{aligned} \tag{3.30}$$

Denote

$$\begin{aligned}
\hat{Y}_t &= X_tY_t;\\
\hat{Z}_t &= Y_t\left(X_t\theta_t + X_{t-\delta}\hat{\theta}_{t-\delta}\right) + X_tZ_t;\\
\hat{Y}_t &= Y_t\left(X_{t-}\mu_t + X_{t-\delta}\hat{\mu}_{(t-\delta)-}\right) + X_tU_t;\\
\hat{K}_t &= \int_0^t X_s dK_s.
\end{aligned}$$

Therefore, $(\hat{Y}, \hat{Z}, \hat{U}, \hat{K})$ is a solution of RBSDE (3.30), with the lower obstacle X_tL_t, and the terminal value $X_T\xi_T + \int_T^{T+\delta}\left(\xi_s\hat{\sigma}_{s-\delta} + \alpha_s\hat{\theta}_{s-\delta} + \beta_s\hat{\mu}_{s-\delta}1_{\{\tau>s\}}\right)X_{s-\delta}ds$, the generator X_tl_t. By Theorem 3.3.1 and the condition $X_t = 1$, we can obtain (3.29). □

Chapter 4

Reflected Anticipated BSDEs with Two Obstacles and Default Risk

Cvitanic and Karatzas [41] (1996) first introduced RBSDEs with two obstacles in the framework of Brownian filtration. The existence theorem of RBSDEs with two obstacles can be obtained under the assumptions **A.1** and **A.2** (see Section 1.2.3). Due to the disadvantages of the above assumptions, we use the assumption **H 4.1** for the obstacles. More previous research can be seen in Section 1.2.3.

In this chapter, we study reflected anticipated backward stochastic differential equations with two obstacles driven by a Brownian motion and a mutually independent martingale in a defaultable setting. The generator of a RABSDE includes the present and future values of the solution. We study the theoretical existence and uniqueness result and provide the related applications of RABSDE with two obstacles and default risk.

This chapter is organized as follows, Section 4.1 states the basic assumptions for RABSDEs with two obstacles and default risk. In Section 4.2, we combine penalization method and fixed point method to prove the existence and uniqueness theorem of the RABSDE (4.1). We represent the relation between linear RABSDEs with two obstacles and stochastic differential delay equations in a defaultable setting in Section 4.3. Section 4.4 illustrates the relation between RABSDEs and obstacle problem for non-linear parabolic PDEs in a defaultable setting.

4.1 Basic Assumptions

In this chapter, we consider the RABSDE (4.1) below with two obstacles and default risk with the coefficient $(f, \xi, \alpha, \beta, \delta^1, \delta^2, \delta^3, L, V)$.

$(Y, Z, U, K^+, K^-) := (Y_t, Z_t, U_t, K_t^+, K_t^-)_{0\le t\le T+T^\delta}$ is a solution for RABSDE with the generator f, the terminal value ξ_T, the anticipated processes ξ, α, β, the anticipated times δ^1, δ^2, δ^3, and the obstacles L and V. K^+ and K^- are continuous increasing processes, the jumps of Y only originate from the default part. K^+ is to keep Y above the lower obstacle L, while K^- is to keep Y under the upper obstacle V. If we take $V \equiv \infty$ and $K^- \equiv 0$, we can obtain a RABSDE with one obstacle and default risk in the Chapter 3.

$$\begin{cases} (i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}), \;\; Z \in \mathcal{L}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}^d), \\ & U \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T+T^\delta;\mathbb{R}^k); \;\; K^\pm \in \mathcal{A}^2_{\mathcal{G}}(0,T;\mathbb{R}); \\ (ii) & Y_t = \xi_T + \int_t^T f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big)ds \\ & \qquad +(K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \quad t \in [0,T]; \\ (iii) & V_t \geq Y_t \geq L_t, \qquad t \in [0,T]; \\ (iv) & Y_t = \xi_t, \qquad t \in (T, T+T^\delta]; \\ (v) & Z_t = \alpha_t, \qquad t \in (T, T+T^\delta]; \\ (vi) & U_t = \beta_t, \qquad t \in (T, T+T^\delta]; \\ (vii) & \int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (V_t - Y_t) dK_t^- = 0, \end{cases} \tag{4.1}$$

Suppose that the anticipated processes ξ, α and β satisfy **H 3.1**, the generator f satisfies **H 3.2**. δ^1, δ^2 and δ^3 satisfy **H 2.2**. Similarly to the design of Assumption 4 in Lepeltier and San Martín [93] (2004), we introduce the following assumptions for the obstacles L and V:

H 4.1. *The obstacle processes L and $V \in \mathcal{S}^2_{\mathcal{G}}(0,T+T^\delta;\mathbb{R})$ and satisfy:*

(a) *for any $t \in [0,T]$, $V_T \geq \xi \geq L_T$, L and V are separated, i.e. $V_t > L_t$, $\mathbb{P}-a.s.$;*

(b) *L and V are rcll, whose jumping times are totally inaccessible,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} (L_t^+)^2\right] < \infty, \qquad \mathbb{E}\left[\sup_{0\leq t\leq T} (V_t^-)^2\right] < \infty;$$

(c) *there exists a process with the following form:*

$$X_t = X_0 - \int_0^t \sigma_s^{(1)} dB_s - \int_0^t \sigma_s^{(2)} dM_s + A_t^+ - A_t^-,$$

where $X_T = \xi_T$, $\sigma^{(1)} \in \mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R}^d)$, $\sigma^{(2)} \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^k)$, A^+ and A^- are $\mathcal{G}$-adapted increasing processes, $\mathbb{E}[|A_T^+|^2 + |A_T^-|^2] < \infty$, such that

$$V_t \geq X_t \geq L_t, \qquad t \in [0,T], \qquad \mathbb{P}-a.s.$$

Remark 4.1.1. *From the assumption **H 4.1** **(a)**, for any $t \in [0,T]$, $V_t > L_t$, $\mathbb{P}-a.s.$, it follows that there is no common jump of the increasing processes K^+ and K^- on $[0,T]$, i.e. $dK_t^+ \cdot dK_t^- = 0$.*

4.2 Existence and Uniqueness Theorem for RABSDE with Two Obstacles and Default Risk

4.2.1 Uniqueness Theorem for RABSDE with Two Obstacles and Default Risk

Theorem 4.2.1. *Suppose that the anticipated processes ξ, α and β, the generator f, the obstacle processes L and V satisfy assumptions **H** 3.1, **H** 3.2 and **H** 4.1 respectively. Then RABSDE (4.1) with the coefficient $(f, \xi, L, V, \delta^1, \delta^2, \delta^3)$ has no more than one solution $(Y, Z, U, K^+, K^-) \in \mathcal{S}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^k) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R}) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R})$.*

Proof. Assume that (Y, Z, U, K^+, K^-) and (Y', Z', U', K'^+, K'^-) are two solutions of the RABSDE (4.1). Denote

$$\bar{Y} = Y - Y', \qquad \bar{Z} = Z - Z', \qquad \bar{U} = U - U',$$
$$\bar{K}^+ = K^+ - K'^+, \qquad \bar{K}^- = K^- - K'^-.$$

Applying Itô's formula for rcll semi-martingale (Theorem 1.3.4) to $|\bar{Y}_t|^2$ on $[t, T]$, we can obtain

$$\begin{aligned}
&|\bar{Y}_t|^2 + \int_t^T |\bar{Z}_s|^2 ds + \int_t^T |\bar{U}_s|^2 1_{\{\tau > s\}} \gamma_s ds \\
=&2 \int_t^T |\bar{Y}_s| \Big[f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big) \\
&\qquad - f\big(s, Y_s, Y_{s+\delta^1(s)}, Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}\big)\Big] ds \\
&+ 2 \int_t^T \bar{Y}_s \left(d\bar{K}^+_s - d\bar{K}^-_s\right) - 2 \int_t^T \bar{Y}_s \bar{Z}_s dB_s - 2 \int_t^T \left(2\bar{Y}_{s-}\bar{U}_s + |\bar{U}_s|^2\right) dM_s.
\end{aligned}$$

The generator f satisfies the Lipschitz condition, $\int_t^T \bar{Y}_s \bar{Z}_s dB_s$ and $\int_t^T \left(2\bar{Y}_{s-}\bar{U}_s + |\bar{U}_s|^2\right) dM_s$ are $\mathcal{G}_t$-martingales, and

$$\int_t^T \bar{Y}_s \left(d\bar{K}^+_s - d\bar{K}^-_s\right) = \int_t^T (Y_{s-} - Y'_{s-}) d\bar{K}^+_s - \int_t^T (Y_{s-} - Y'_{s-}) d\bar{K}^-_s \leq 0.$$

Hence, similarly to the proof of the uniqueness theorem for RABSDEs with one obstacle and default risk 3.2.1, we can obtain

$$\mathbb{E}\left[|\bar{Y}_t|^2 + \int_t^{T+T^\delta} |\bar{Z}_s|^2 ds + \int_t^{T+T^\delta} |\bar{U}_s|^2 1_{\{\tau > s\}} \gamma_s ds \right] \leq M \mathbb{E}\left[\int_t^{T+T^\delta} |\bar{Y}_s|^2 ds \right],$$

where $M \geq 0$ is a constant. Since the process $\bar{Y}$ is right continuous, by Gronwall's inequality, we can obtain that $Y = Y'$. Consequently, we get

$$(Y, Z, U, K^+, K^-) = (Y', Z', U', K'^+, K'^-).$$

□

4.2.2 Existence Theorem for RABSDE with Two Obstacles and Default Risk

4.2.2.1 Existence Theorem for the Penalized RBSDE (4.2)

We use the penalization method to prove the existence theorem for RABSDE with two obstacles. We first suppose that the generator is independent on (y,z,u) and provide the existence theorem in this special frame (4.2) as below, then use fixed-point method to obtain the existence result in the general frame.

$$\begin{cases} Y_t^n = \xi_T + \int_t^T g(s)ds + (K_T^{+n} - K_t^{+n}) - (K_T^{-n} - K_t^{-n}) \\ \qquad - \int_t^T Z_s^n dB_s - \int_t^T U_s^n dM_s, \qquad t \in [0,T]; \\ K_t^{-n} = n\int_0^t (Y_s^n - V_s)^+ ds, \qquad t \in [0,T]; \\ \int_0^T (Y_t^n - L_t) dK_t^{+n} = 0, \qquad \mathbb{P} - a.s.; \\ Y_t^n \geq L_t, \qquad t \in [0,T], \end{cases} \tag{4.2}$$

where the terminal value ξ_T satisfies the assumption **H 3.1**, the generator g is independent on (y,z,u), i.e. for all $(w,t) \in \Omega \times [0,T]$, $f(w,t,y,z,u) \equiv g(w,t)$. For any $n \in \mathbb{N}$, (Y^n, Z^n, U^n, K^{+n}) is a $\mathcal{G}$-adapted solution of the penalized RBSDE (4.2) with the coefficient $(g - n(y-V)^+, \xi, L)$. From the comparison theorem for RBSDEs with default risk (Theorem 4.2 in Agram et.al [94] (2018)), it follows that $(Y^n)_{n\geq 0}$ is decreasing.

In order to prove the existence theorem for the penalized RBSDE (4.2) (Theorem 4.2.2), we introduce Lemma 4.2.1, Lemma 4.2.2 and Lemma 4.2.3. Lemma 4.2.1 represents the approximation of the penalized RBSDE (4.2). Lemma 4.2.2 illustrates the existence of the limiting process Y of Y^n in the sense of (4.9). Lemma 4.2.3 completes the existence of the limiting processes (Z, U, K^+) of (Z^n, U^n, K^{+n}).

Through the fixed point method (Banach fixed-point theorem), the existence theorem for the RABSDE (4.1) (Theorem 4.2.3) in the general frame consequently follows.

Lemma 4.2.1. *Suppose that $(Y^n, Z^n, U^n, K^{+n}) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}$ is the unique $\mathcal{G}$-adapted solution of penalized RBSDE (4.2), then there exists a constant $C \geq 0$ independent of n, for all $t \in [0,T]$, such that*

$$\begin{aligned} \mathbb{E}\Bigg[\sup_{0\leq t\leq T} |Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds + \int_0^T |U_s^n|^2 \mathbf{1}_{\{\tau > s\}} \gamma_s ds + |K_T^{+n}|^2 \\ + \left(\int_t^T n\,(Y_s^n - V_s)^+ ds \right)^2 \Bigg] \leq C, \qquad n \in \mathbb{N}. \end{aligned} \tag{4.3}$$

Proof. Let $(Y^{n,k}, Z^{n,k}, U^{n,k})$ be the solution for the following BSDE:

$$\begin{aligned} Y_t^{n,k} =& \xi_T + \int_t^T g(s)ds + k\int_t^T \left(Y_s^{n,k} - L_s\right)^- ds - n\int_t^T \left(Y_s^{n,k} - V_s\right)^+ ds \\ & - \int_t^T Z_s^{n,k} dB_s - \int_t^T U_s^{n,k} dM_s, \qquad t \in [0,T]. \end{aligned} \tag{4.4}$$

Since $g(s) + k\left(Y_s^{n,k} - L_s\right)^- - n\left(Y_s^{n,k} - V_s\right)^+$ satisfies the Lipschitz condition, by the existence and uniqueness theorem for BSDEs with default risk (Theorem 1.3.1), there exists a unique solution $(Y^{n,k}, Z^{n,k}, U^{n,k})$ for BSDE (4.4). Applying Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|Y_s^{n,k}|^2$ on $[t, T]$, we can obtain

$$\begin{aligned}
&|Y_t^{n,k}|^2 + \int_t^T |Z_s^{n,k}|^2 ds + \int_t^T |U_s^{n,k}|^2 1_{\{\tau > s\}} \gamma_s ds \\
=&|\xi_T|^2 + 2\int_t^T Y_s^{n,k} g(s) ds + 2\int_t^T k Y_s^{n,k}\left(Y_s^{n,k} - L_s\right)^- ds \\
&+ 2\int_t^T n Y_s^{n,k}\left(Y_s^{n,k} - V_s\right)^+ ds - 2\int_t^T Y_s^{n,k} Z_s^{n,k} dB_s \\
&- 2\int_t^T \left[Y_{s-}^{n,k} U_s^{n,k} + |U_s^{n,k}|^2\right] dM_s.
\end{aligned} \tag{4.5}$$

Since $\int_t^T Y_s^{n,k} Z_s^{n,k} dB_s$ and $\int_t^T \left[Y_{s-}^{n,k} U_s^{n,k} + |U_s^{n,k}|^2\right] dM_s$ are $\mathcal{G}_t$-martingales. Taking expectation in both sides, it follows

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^{n,k}|^2 + \int_t^T |Z_s^{n,k}|^2 ds + \int_t^T |U_s^{n,k}|^2 1_{\{\tau > s\}} \gamma_s ds\right] \\
=&\mathbb{E}|\xi_T|^2 + 2\mathbb{E}\left[\int_t^T Y_s^{n,k} g(s) ds\right] + 2\mathbb{E}\left[\int_t^T k Y_s^{n,k}\left(Y_s^{n,k} - L_s\right)^- ds\right] \\
&+ 2\mathbb{E}\left[\int_t^T n Y_s^{n,k}\left(Y_s^{n,k} - V_s\right)^+ ds\right] \\
\leq&\mathbb{E}|\xi_T|^2 + \lambda_1 \mathbb{E}\left[\int_t^T |Y_s^{n,k}|^2 ds\right] + \frac{1}{\lambda_1}\mathbb{E}\left[\int_t^T |g(s)|^2 ds\right] \\
&+ \frac{1}{\lambda_1}\mathbb{E}\left[\sup_{0\leq t\leq T} (L_t^+)^2\right] + \lambda_1 \mathbb{E}\left[\int_t^T k\left(Y_s^{n,k} - L_s\right)^- ds\right]^2 \\
&+ \frac{1}{\lambda_1}\mathbb{E}\left[\sup_{0\leq t\leq T} (V_t^-)^2\right] + \lambda_1 \mathbb{E}\left[\int_t^T n\left(Y_s^{n,k} - V_s\right)^+ ds\right]^2.
\end{aligned} \tag{4.6}$$

where $\lambda_1 > 0$ is a constant.

Step 1. First, we prove that for any $t \in [0, T]$, there exists a constant $C_1 \geq 0$, such that

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T k\left(Y_s^{n,k} - L_s\right)^- ds\right]^2 + \mathbb{E}\left[\int_t^T n\left(Y_s^{n,k} - V_s\right)^+ ds\right]^2 \\
\leq&C_1\left[1 + \mathbb{E}\int_t^T \left(|Z_s^{n,k}|^2 + |U_s^{n,k}|^2 1_{\{\tau > s\}} \gamma_s\right) ds\right].
\end{aligned} \tag{4.7}$$

Similarly to the methodology of Lemma 2 in Lepeltier and San Martín [93] (2004), we consider the sequences of stopping times $(s_i)_{i=1,2,...}$ and $(l_i)_{i=1,2,...}$ with the following forms:

$$\begin{aligned}
s_1 &= \inf\{t \le r \le T;\ Y_r^{n,k} \ge V_r\} \wedge T;\\
l_i &= \inf\{s_i < r \le T;\ Y_r^{n,k} = L_r\} \wedge T, \quad i = 1, 2, ...;\\
s_i &= \inf\{l_{i-1} < r \le T;\ Y_r^{n,k} = V_r\} \wedge T, \quad i = 2, 3, ...
\end{aligned}$$

Since for any $t \in [0, T)$, $V_t > L_t$, $\mathbb{P} - a.s.$ (**H 4.1 (a)**), we can obtain that $s_i \to T$, $l_i \to T$, as $i \to \infty$. Since for any $r \in [s_i, l_i]$, $Y_r^{n,k} \ge L_r$, it follows

$$Y_{s_i}^{n,k} = Y_{l_i}^{n,k} + \int_{s_i}^{l_i} g(r)dr - n\int_{s_i}^{l_i}\left(Y_r^{n,k} - V_r\right)^+ dr - \int_{s_i}^{l_i} Z_r^{n,k}dB_r - \int_{s_i}^{l_i} U_r^{n,k}dM_r.$$

Moreover, by **H 4.1** (**c**), it follows

$$\begin{cases}
Y_{s_i}^{n,k} &= X_{s_i} = \xi_T, \qquad s_i = T;\\
Y_{s_i}^{n,k} &\ge X_{s_i}, \qquad s_i < T;\\
Y_{l_i}^{n,k} &= X_{l_i} = \xi_T, \qquad l_i = T;\\
Y_{l_i}^{n,k} &\ge X_{l_i}, \qquad l_i < T.
\end{cases}$$

Hence, for any $i = 1, 2, ...,$

$$\begin{aligned}
&n\int_{s_i}^{l_i}\left(Y_r^{n,k} - V_r\right)^+ ds\\
\le& X_{l_i} - X_{s_i} + \int_{s_i}^{l_i} g(r)dr - \int_{s_i}^{l_i} Z_r^{n,k}dB_r - \int_{s_i}^{l_i} U_r^{n,k}dM_r\\
\le& \int_{s_i}^{l_i}|g(r)|dr - \int_{s_i}^{l_i}(Z_r^{n,k} + \sigma_r^{(1)})dB_r - \int_{s_i}^{l_i}(U_r^{n,k} + \sigma_r^{(2)})dM_r\\
&+ A_{l_i}^+ - A_{s_i}^+ - A_{l_i}^- + A_{s_i}^-\\
\le& \int_{s_i}^{l_i}|g(r)|dr - \int_{s_i}^{l_i}(Z_r^{n,k} + \sigma_r^{(1)})dB_r - \int_{s_i}^{l_i}(U_r^{n,k} + \sigma_r^{(2)})dM_r\\
&+ A_{l_i}^+ + A_{s_i}^-.
\end{aligned}$$

Since for any $r \in [l_i, s_{i+1}]$, $Y_r^{n,k} \le V_r$,

$$\begin{aligned}
&n\int_t^T\left(Y_r^{n,k} - V_r\right)^+ ds\\
=& \sum_{i=1}^{\infty} n\int_{s_i}^{l_i}\left(Y_r^{n,k} - V_r\right)^+ ds\\
\le& \int_t^T|g(r)|dr - \int_t^T(Z_r^{n,k} + \sigma_r^{(1)})\Big(\sum_{i=1}^{\infty} 1_{[s_i,l_i)}(r)\Big)dB_r\\
&- \int_t^T(U_r^{n,k} + \sigma_r^{(2)})\Big(\sum_{i=1}^{\infty} 1_{[s_i,l_i)}(r)\Big)dM_r + A_T^+ + A_T^-,
\end{aligned}$$

it follows

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T n\left(Y_s^{n,k}-V_s\right)^+ds\right]^2\\
\leq&\lambda_1\mathbb{E}\Big[\int_t^T|g(r)|^2dr+\int_t^T|Z_r^{n,k}+\sigma_r^{(1)}|^2dr+\int_t^T|U_r^{n,k}+\sigma_r^{(2)}|^2 1_{\{\tau>s\}}\gamma_s\\
&\quad+(A_T^+)^2+(A_T^-)^2\Big]\\
\leq&\lambda_2\left[1+\mathbb{E}\int_t^T\left(|Z_s^{n,k}|^2+|U_s^{n,k}|^2 1_{\{\tau>s\}}\gamma_s\right)ds\right],
\end{aligned}$$

here λ_1, $\lambda_2 \geq 0$ are constants. Similarly, we can prove there exists a constant $\lambda_3 \geq 0$, such that

$$\mathbb{E}\left[\int_t^T k\left(Y_s^{n,k}-L_s\right)^-ds\right]^2\leq\lambda_3\left[1+\mathbb{E}\int_t^T\left(|Z_s^{n,k}|^2+|U_s^{n,k}|^2 1_{\{\tau>s\}}\gamma_s\right)ds\right],$$

consequently, we prove (4.7).

Step 2. Combining (4.6) and (4.7), we can get

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^{n,k}|^2+\int_t^T|Z_s^{n,k}|^2ds+\int_t^T|U_s^{n,k}|^2 1_{\{\tau>s\}}\gamma_s ds\right]\\
\leq&\mathbb{E}|\xi_T|^2+\lambda_1\mathbb{E}\left[\int_t^T|Y_s^{n,k}|^2ds\right]+\frac{1}{\lambda_1}\mathbb{E}\left[\int_t^T|g(s)|^2ds\right]+\frac{1}{\lambda_1}\mathbb{E}\left[\sup_{0\leq t\leq T}(L_t^+)^2\right]\\
&+\frac{1}{\lambda_1}\mathbb{E}\left[\sup_{0\leq t\leq T}(V_t^-)^2\right]+\lambda_1C_1\mathbb{E}\left[1+\int_t^T\left(|Z_s^{n,k}|^2+|U_s^{n,k}|^2 1_{\{\tau>s\}}\gamma_s\right)ds\right].
\end{aligned}\tag{4.8}$$

Set $\lambda_1 C_1 = \frac{1}{2}$, it follows

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^{n,k}|^2+\frac{1}{2}\int_t^T|Z_s^{n,k}|^2ds+\frac{1}{2}\int_t^T|U_s^{n,k}|^2 1_{\{\tau>s\}}\gamma_s ds\right]\\
\leq&C_2\left[1+\mathbb{E}\int_t^T\left(|Z_s^{n,k}|^2+|U_s^{n,k}|^2 1_{\{\tau>s\}}\gamma_s\right)ds\right],
\end{aligned}$$

where $C_2 \geq 0$ is a constant. By the Gronwall's inequality, we can obtain that $\mathbb{E}|Y_t^{n,k}|^2$ is bounded, therefore, $\mathbb{E}\left[\int_t^T|Z_s^{n,k}|^2ds\right]$ and $\mathbb{E}\left[\int_t^T|U_s^{n,k}|^2 1_{\{\tau>s\}}\gamma_s ds\right]$ are bounded. Consequently, there exists a constant $C_3 \geq 0$, such that

$$\mathbb{E}\left[\int_t^T k\left(Y_s^{n,k}-L_s\right)^-ds\right]^2+\mathbb{E}\left[\int_t^T n\left(Y_s^{n,k}-V_s\right)^+ds\right]^2\leq C_3.$$

Let $k \to \infty$, we can obtain $Y^{n,k} \to Y^n$ in $\mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R})$, $Z^{n,k} \to Z^n$ in $\mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R}^d)$, $U^{n,k} \to U^n$ in $\mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^k)$, $\int_0^T k\left(Y_s^{n,k}-L_s\right)^- ds \to K_T^{+n}$ in $\mathcal{L}^2(\mathcal{G}_T;\mathbb{R})$.

On the other hand,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{n,k}|^2\right]\\
\le&\mathbb{E}\sup_{0\le t\le T}\left[|Y_t^{n,k}|^2+\int_t^T|Z_s^{n,k}|^2ds+\int_t^T|U_s^{n,k}|^2dH_s\right]\\
\le&\mathbb{E}|\xi_T|^2+2\mathbb{E}\left[\int_0^T Y_s^{n,k}g(s)ds\right]+2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T Y_s^{n,k}dK_s^{n,k}\right]\\
&+2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T Y_s^{n,k}Z_s^{n,k}dB_s\right]+2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T Y_{s-}^{n,k}U_s^{n,k}dM_s\right]\\
\le&\mathbb{E}|\xi_T|^2+\lambda_5\mathbb{E}\left[\int_0^T|Y_s^{n,k}|^2ds\right]+\frac{1}{\lambda_5}\mathbb{E}\left[\int_0^T|g(s)|^2ds\right]\\
&+\frac{1}{\lambda_5}\mathbb{E}\left[\sup_{0\le t\le T}(L_t^+)^2\right]+\frac{1}{\lambda_5}\mathbb{E}\left[\sup_{0\le t\le T}(V_t^-)^2\right]\\
&+\lambda_5C_1\mathbb{E}\left[1+\int_t^T\left(|Z_s^{n,k}|^2+|U_s^{n,k}|^2\mathbf{1}_{\{\tau>s\}}\gamma_s\right)ds\right]\\
&+2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T Y_s^nZ_s^ndB_s\right]+2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T Y_{s-}^nU_s^ndM_s\right],
\end{aligned}$$

Similarly to the proof of Lemma 3.2.1, by the Burkholder-Davis-Gundy inequality, there exists a constant $C_4\ge 0$, such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{n,k}|^2\right]\le C_4.$$

Let $k\to\infty$, it follows (4.3). □

Lemma 4.2.2 below illustrates the existence of the limiting process Y of Y^n in the sense of (4.9).

Lemma 4.2.2. *$(Y^n)_{n\ge0}$ is a non-increasing sequence. For any $t\in[0,T]$, $(Y_t^n)_{n\ge0}$ converges to $Y_t=\underline{\lim}_{n\to\infty}Y_t^n$ and satisfy*

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|(Y_t^n-V_t)^+|^2\right]=0,\qquad a.s. \tag{4.9}$$

Proof. By the comparison theorem for RBSDEs with default risk (Theorem 4.2 in Agram et.al [94]), we know that $(Y^n)_{n\ge0}$ is non-increasing. Similarly to the proof of Lemma 3.2.2, first let $(\bar{Y},\bar{Z},\bar{U})$ be the solution of the following RBSDE with the coefficient $(g-n(y-V),\xi,L)$:

$$\bar{Y}_t^n=\xi_T+\int_t^T[g(s)-n(\bar{Y}_s^n-V_s)]\,ds+\bar{K}_T-\bar{K}_t-\int_t^T\bar{Z}_s^ndB_s-\int_t^T\bar{U}_s^ndM_s.$$

By the comparison theorem for RBSDEs with default risk (Theorem 4.2 in Agram et.al [94]), we know for any $t \in [0, T]$,

$$Y_t^n \leq \bar{Y}_t^n, \qquad \mathbb{P}-a.s.$$

On the other hand, let $e^{-nt}\bar{Y}_t^n$ be the solution of the following RBSDE with the coefficient $(e^{-nt}(g - nL), e^{-nT}\xi, e^{-nt}L)$:

$$\begin{aligned} e^{-nt}\bar{Y}_t^n =& e^{-nT}\xi_T + \int_t^T e^{-ns}\left(g(s) + nL_s\right)ds + e^{-nT}\bar{K}_T - e^{-nt}\bar{K}_t \\ & - \int_t^T e^{-ns}\bar{Z}_s^n dB_s - \int_t^T e^{-ns}\bar{U}_s^n dM_s. \end{aligned}$$

By Proposition 2.3 in El Karoui et. al [33] and the definition of X (**H 4.1** (**c**)), let v be a $\mathcal{G}_t$-stopping time, such that

$$\begin{aligned} \bar{Y}_v^n =& ess\sup_{\sigma\geq v} \mathbb{E}^{\mathcal{G}_v}\Big[\int_v^\sigma e^{-n(s-v)}\left(g(s) + nV_s\right)ds + e^{-n(\sigma-v)}\xi_T 1_{\{\sigma<T\}} \\ & \qquad + e^{-n(\sigma-v)}L_\sigma 1_{\{\sigma=T\}}\Big] \\ \leq& ess\sup_{\sigma\geq v} \mathbb{E}^{\mathcal{G}_v}\left[e^{-n(\sigma-v)}\xi_T 1_{\{\sigma=T\}} + e^{-n(\sigma-v)}X_\sigma 1_{\{\sigma<T\}} + \int_v^\sigma ne^{-n(s-v)}X_s ds\right] \\ & + \mathbb{E}^{\mathcal{G}_v}\left[\int_v^T e^{-n(s-v)}\Big(g(s) + n(V_s - X_s)\Big)ds\right] \\ =& ess\sup_{\sigma\geq v} \mathbb{E}^{\mathcal{G}_v}\left[X_v + \int_v^\sigma e^{-n(s-v)}dX_s\right] \\ & + \mathbb{E}^{\mathcal{G}_v}\Big[\int_v^T e^{-n(s-v)}\Big(g(s) + n(V_s - X_s)\Big)ds\Big] \\ \leq& \xi_T 1_{\{v=T\}} + X_v 1_{\{v<T\}} + \mathbb{E}^{\mathcal{G}_v}\Big[\int_v^T e^{-n(s-v)}d(A_s^+ + A_s^-)\Big] \\ & + \mathbb{E}^{\mathcal{G}_v}\left[\int_v^T e^{-n(s-v)}\Big(g(s) + n(V_s - X_s)\Big)ds\right]. \end{aligned}$$

It follows

$$\int_v^T e^{-n(s-v)}d(A_s^+ + A_s^-) \to 0, \qquad n \to \infty, \quad \mathbb{P}-a.s..$$

in $\mathcal{L}^2(\Omega; \mathbb{P})$. Since

$$\left|\int_v^T e^{-n(s-v)}g(s)ds\right| \leq \frac{1}{\sqrt{n}}\left(\int_v^T |g(s)|^2 ds\right)^{\frac{1}{2}},$$

and the definition of X (**H 4.1** (**c**)), we can obtain

$$\begin{aligned} & \int_v^T e^{-n(s-v)}g(s)ds \to 0; \\ & \int_v^T e^{-n(s-v)}n(V_s - X_s)ds \to (V_v - X_v)1_{\{v<T\}}, \qquad n \to \infty, \quad \mathbb{P}-a.s. \end{aligned}$$

in $\mathcal{L}^2(\Omega;\mathbb{P})$. Consequently,

$$\bar{Y}_v^n \to \xi_T 1_{\{v=T\}} + V_v 1_{\{v<T\}}, \qquad n \to \infty, \quad \mathbb{P}-a.s.$$

in $\mathcal{L}^2(\Omega;\mathbb{P})$. Therefore,

$$Y_v \le \varliminf_{n\to\infty} \bar{Y}_v^n \le \xi_T 1_{\{v=T\}} + V_v 1_{\{v<T\}} \le V_v, \qquad \mathbb{P}-a.s.$$

By the section theorem of Meyer [88] (p.220), we deduce that for any $t \in [0,T]$,

$$Y_t \le V_t, \qquad \mathbb{P}-a.s.,$$

therefore,

$$(Y_t^n - V_t)^+ \downarrow 0, \qquad \mathbb{P}-a.s.$$

Since $Y^n \downarrow Y$, we know that ${}^pY^n \downarrow {}^pY$ and ${}^pY \le {}^pV$. For any $n \in \mathbb{R}^+$, the jumps of Y^n are from the default process H, which are inaccessible (the default times are inaccessible). Then for any predictable stopping time σ, we have $Y_\sigma^n = Y_{\sigma-}^n$, therefore ${}^pY^n = Y_-^n$. So we can prove that

$$Y_-^n = {}^pY^n \downarrow {}^pY \le V,$$

hence,

$$(Y_-^n - V) \uparrow ({}^pY - V) \le 0,$$

it follows that for any $t \in [0,T]$,

$$(Y_{t-}^n - V_t)^+ \downarrow 0, \qquad n \to \infty, \quad \mathbb{P}-a.s.$$

From a weak version of Dini's theorem (Meyer [88], p.202), we deduce that

$$\sup_{0\le t\le T} (Y_t^n - V_t)^+ \downarrow 0, \qquad n \to \infty, \quad \mathbb{P}-a.s.$$

Since $(Y_t^n - V_t)^+ \le |Y_t^1| + V_t^+$, from the Lebesgue's dominated convergence theorem, we can obtain (4.9). □

From Lemma 4.2.1 and Lemma 4.2.2, we can prove that there exist the limiting process $\mathcal{G}$-adapted processes (Z, U, K^+) of (Z^n, U^n, K^{+n}).

Lemma 4.2.3. *There exist $\mathcal{G}$-adapted processes $Z = (Z_t)_{0\le t\le T}$, $U = (U_t)_{0\le t\le T}$ and $K^+ = (K_t^+)_{0\le t\le T}$, such that*

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{0\le t\le T} |Y_t^n - Y_t|^2 + \int_0^T |Z_s^n - Z_s|^2 ds + \int_0^T |U_s^n - U_s|^2 1_{\{\tau>s\}} \gamma_s ds \\ + \sup_{0\le t\le T} |K_t^{+n} - K_t^+|^2\Big] = 0. \tag{4.10}$$

Proof. **Step 1**. First, we prove

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |Z_s^n - Z_s|^2 ds + \int_0^T |U_s^n - U_s|^2 1_{\{\tau>s\}} \gamma_s ds\right] = 0. \tag{4.11}$$

Applying the Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|Y_s^n - Y_s^m|^2$ on $[t,T]$, since $\int_t^T (Y_s^n - Y_s^m) d(K_s^{+n} - K_s^{+m}) \le 0$, for any $m \ge n > 0$, it follows

$$
\begin{aligned}
&\mathbb{E}\left[|Y_t^n - Y_t^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2 ds + \int_t^T |U_s^n - U_s^m|^2 1_{\{\tau > s\}} \gamma_s ds\right] \\
=&2\mathbb{E}\left[\int_t^T (Y_s^n - Y_s^m) d(K_s^{+n} - K_s^{+m})\right] \\
&+ 2\mathbb{E}\left[\int_t^T (Y_s^n - Y_s^m)\left(n(Y_s^n - V_s)^+ - m(Y_s^m - V_s)^+\right) ds\right] \\
\le&2\mathbb{E}\left[\int_t^T (Y_s^n - V_s)^+ m(Y_s^m - V_s)^+ ds\right] + 2\mathbb{E}\left[\int_t^T (Y_s^m - V_s)^+ n(Y_s^n - V_s)^+ ds\right] \\
\le&\left(\mathbb{E}\left[\sup_{0\le t\le T} |(Y_t^n - V_t)^-|^2\right]\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\left[\int_t^T m(Y_s^m - V_s)^+ ds\right]^2\right)^{\frac{1}{2}} \\
&+ \left(\mathbb{E}\left[\sup_{0\le t\le T} |(Y_t^m - V_t)^+|^2\right]\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\left[\int_t^T n(Y_s^n - V_s)^+ ds\right]^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

From Lemma 4.2.1 and Lemma 4.2.2, when $n \to \infty$, it follows

$$
\mathbb{E}\left[\int_0^T |Z_s^n - Z_s^m|^2 ds + \int_0^T |U_s^n - U_s^m|^2 1_{\{\tau > s\}} \gamma_s ds\right] \to 0.
$$

Therefore, $(Z^n)_{n\ge 0}$ and $(U^n)_{n\ge 0}$ are Cauchy sequences in complete spaces, then there exist $\mathcal{G}$-progressively measurable processes Z and U, such that sequences $(Z^n)_{n\ge 0}$ and $(U^n)_{n\ge 0}$ converge to Z and U in $\mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R}^d)$ and $\in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^k)$ respectively. Then we can obtain (3.11).

Step 2. Then, we prove

$$
\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} |Y_t^n - Y_t|^2 + \sup_{0\le t\le T} |K_t^{+n} - K_t^+|^2\right] = 0. \tag{4.12}
$$

Similarly to Lemma 3.2.3, by Burkholder-Davis-Gundy inequality, it follows

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} |Y_t^n - Y_t^m|^2\right] \\
\le&\mathbb{E}\sup_{0\le t\le T}\left[|Y_t^n - Y_t^m|^2 + \int_t^T |Z_s^n - Z_s^m|^2 ds + \int_t^T |U_s^n - U_s^m|^2 dH_s\right] \\
\le&2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T (Y_s^n - Y_s^m)\left(n(Y_s^n - V_s)^+ - m(Y_s^m - V_s)^+\right) ds\right] \\
&+ 2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T Y_s^n (Z_s^n - Z_s^m) dB_s\right] + 2\mathbb{E}\left[\sup_{0\le t\le T}\int_t^T Y_{s-}^n (U_s^n - U_s^m) dM_s\right]
\end{aligned}
$$

$$\leq \left(\mathbb{E}\left[\sup_{0\leq t\leq T}|(Y_t^n - V_t)^-|^2\right]\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\left[\int_t^T m(Y_s^m - V_s)^+ ds\right]^2\right)^{\frac{1}{2}}$$
$$+ \left(\mathbb{E}\left[\sup_{0\leq t\leq T}|(Y_t^m - V_t)^+|^2\right]\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\left[\int_t^T n(Y_s^n - V_s)^+ ds\right]^2\right)^{\frac{1}{2}}$$
$$+ 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_s^n(Z_s^n - Z_s^m)dB_s\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T}\int_t^T Y_{s-}^n(U_s^n - U_s^m)dM_s\right],$$

we can get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n - Y_t^m|^2\right] \to 0, \qquad n,\ m \to \infty.$$

Thus,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n - Y_t|^2\right] \to 0, \qquad n \to \infty,$$

here $Y = (Y_t)_{0\leq t\leq T} \in \mathcal{S}^2_{\mathcal{G}}(0,T;\mathbb{R})$.
Since for any $t \in [0,T]$,

$$K_t^{+n} = Y_0^n - Y_t^n - \int_0^t g(s)ds - \int_0^t n(Y_s^n - V_s)^+ ds + \int_0^t Z_s^n dB_s + \int_0^t U_s^n dM_s,$$

it follows

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|K_t^{+n} - K_t^{+m}|^2\right] \to 0, \qquad n,\ m \to \infty.$$

Hence there exists a $\mathcal{G}_t$-adapted non-decreasing process $K^+ = (K_t^+)_{0\leq t\leq T}$, $K_0^+ = 0$, such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|K_t^{+n} - K_t^{+}|^2\right] \to 0, \qquad n \to \infty.$$

So we have proved (4.12), it follows (4.10). □

Therefore, we have the following existence theorem for the penalized RBSDE (4.2).

Theorem 4.2.2. ***(Existence theorem for the penalized RBSDE)*** $(Y^n, Z^n, U^n, K^{+n}, K^{-n})_{n\geq 0}$ *has a limit process (Y, Z, U, K^+, K^-), which is the solution of the following RBSDE with two obstacles associated with the coefficient (g, ξ, L, V):*

$$\begin{cases} (i) & Y_t = \xi_T + \int_t^T g(s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) \\ & \qquad - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \quad t \in [0,T]; \\ (ii) & V_t \geq Y_t \geq L_t, \quad t \in [0,T]; \\ (iii) & \int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (V_t - Y_t)dK_t^- = 0, \quad \mathbb{P}-a.s. \end{cases} \tag{4.13}$$

Proof. From Lemma 4.2.2 and Lemma 4.2.3, $\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\leq t\leq T}|(Y_t^n - V_t)^+|^2\right] = 0$, since $Y_t^n \geq L_t$, for any $t \in [0,T]$, as $n \to \infty$, we can obtain

$$V_t \geq Y_t \geq L_t, \qquad t \in [0,T], \qquad \mathbb{P}-a.s.$$

Taking limit of the penalized RBSDE (4.2), we can obtain

$$\begin{aligned} Y_t =&\xi_T + \int_t^T g(s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s \\ &- \int_t^T U_s dM_s, \qquad t \in [0,T], \qquad \mathbb{P}-a.s. \end{aligned} \tag{4.14}$$

Then, we give the proof of the condition (*iii*) of (4.13).

Step 1. Firstly, we prove $\int_0^T (Y_t - L_t) dK_t^+ = 0$.

Since

$$\int_0^T (Y_t^n - L_t) dK_t^{n+} = -n \int_0^T |(Y_t^n - L_t)^-|^2 dt \leq 0,$$

Moreover, Y^n converges to Y, $Y^n \geq Y \geq L$, hence, for all $n \geq N^\epsilon$, we get

$$0 \leq \int_0^T (Y_t - L_t) dK_t^{n+} = \int_0^T (Y_t - Y_t^n) dK_t^{n+} + \int_0^T (Y_t^n - L_t) dK_t^{n+} \leq 0,$$

it follows

$$0 = \int_0^T (Y_t - L_t) dK_t^{n+} \to \int_0^T (Y_t - L_t) dK_t^+, \qquad n \to \infty.$$

Step 2. Then, we prove that $\int_0^T (V_t - Y_t) dK_t^- = 0$.

Lemma 4.2.3 implies the convergence of $\int_0^t n(Y_s^n - V_s)^+ ds$ in $\mathcal{L}^2(\Omega;\mathbb{P})$ to a non-decreasing process that we denote by K^-, here $K^- \in \mathcal{A}^2_{\mathcal{G}}(0,T;\mathbb{R})$. There exists a subsequence of $(K^{n-})_{n\geq 0}$ (we still denote as $(K^{n-})_{n\geq 0}$), such that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |K_t^{n-} - K_t^-|\right] = 0.$$

For any $w \in \Omega$, since function $V(w) - Y(w) : t \in [0,T] \to V_t(w) - Y_t(w)$ is rcll, then there exists a sequence of staircase functions $(v^n(w))_{n\geq 0}$, which converges uniformly to $V_t(w) - Y_t(w)$ on $[0,T]$. Since Y^n converges to Y and $Y^n \geq Y$, $V \geq Y$, by Lemma 4.2.3, it follows, for any $\epsilon^1 > 0$, there exists N^ϵ, such that for all $n \geq N^\epsilon$,

$$\begin{aligned} V_t(w) - Y_t(w) &\leq V_t(w) - Y_t^n(w) + \epsilon^1; \\ K_T^{n-}(w) &\leq K_T^-(w) + \epsilon^1. \end{aligned} \tag{4.15}$$

Since

$$\int_0^T (V_t - Y_t^n) dK_t^{n-} = -n \int_0^T |(Y_t^n - V_t)^+|^2 dt \leq 0,$$

hence, by (4.15), there exists $\epsilon_2 > 0$, for all $n \geq N^\epsilon$, we get

$$\int_0^T (V_t - Y_t) dK_t^{n-} \leq \epsilon^1 K_T^-(w) + \epsilon^2. \tag{4.16}$$

On the other hand, there exists M^{ϵ}, such that for all $n \geq M^{\epsilon}$,

$$|V_t(w) - Y_t(w) - v_t^m(w)| < \epsilon^1,$$

since $v^n(w)$ is a staircase function, then $\int_0^T v_t^n(w)d(K_t^- - K_t^{n-}) \to 0$, as $n \to \infty$. Therefore, it follows

$$\begin{aligned}
&\int_0^T (V_t - Y_t)d(K_t^- - K_t^{n-}) \\
=&\int_0^T (V_t - Y_t - v_t^n(w))(dK_t^- - K_t^{n-}) + \int_0^T v_t^n(w)d(K_t^- - K_t^{n-}) \\
\leq&\epsilon^1(K_T^-(w) - K_T^{n-}(w)) + \int_0^T h_t^m(w)d(K_t^- - K_t^{n-}) \to 2\epsilon^1 K_T^-(w).
\end{aligned}$$

Thus, we can obtain

$$\lim_{n\to\infty}\left[\sup_{0\leq t\leq T}\int_0^T (V_t - Y_t)d(K_t^- - K_t^{n-})\right] \leq 2\epsilon^1 K_T^-(w). \tag{4.17}$$

Finally, by (4.16) and (4.17), we can get

$$\begin{aligned}
&\int_0^T (V_t - Y_t)dK_t^- \\
=&\int_0^T (V_t - Y_t)d(K_t^- - K_t^{n-}) + \int_0^T (V_t - Y_t^n)dK_t^{n-} \leq 3\epsilon^1 K_T^-(w) + \epsilon^2.
\end{aligned}$$

As ϵ^1 and ϵ^2 are arbitrary, and $Y \leq L$, then we have (*iii*) $\int_0^T (V_t - Y_t)dK_t^- = 0$. □

4.2.2.2 Existence Theorem for the RABSDE (4.1) in the general frame

We have proved the existence of the penalized RBSDE (4.2), by the uniqueness theorem (Theorem 4.2.1) in Section 4.2.1, we can use the fixed point method (Banach fixed-point theorem) to prove the following existence and uniqueness theorem of RABSDE with two obstacles (4.1) in the general frame.

Theorem 4.2.3. ***(Existence and uniqueness theorem for RABSDEs with two obstacles and default risk)*** *Suppose that the anticipated processes ξ, α and β, the generator f, the obstacles L and V satisfy the assumptions* **H 3.1**, **H 3.2** *and* **H 4.1**. *Then RABSDE (4.1) with the coefficient $(f, \xi, \alpha, \beta, \delta^1, \delta^2, \delta^3, L, V)$ has a unique solution $(Y, Z, U, K^+, K^-) \in \mathcal{S}_{\mathcal{G}}^2(0, T + T^{\delta}; \mathbb{R}) \times \mathcal{L}_{\mathcal{G}}^2(0, T + T^{\delta}; \mathbb{R}^d) \times \mathcal{L}_{\mathcal{G}}^{2,\tau}(0, T + T^{\delta}; \mathbb{R}^k) \times \mathcal{A}_{\mathcal{G}}^2(0, T; \mathbb{R}) \times \mathcal{A}_{\mathcal{G}}^2(0, T; \mathbb{R})$.*

Proof. Denote $\mathcal{D} := \mathcal{S}_{\mathcal{G}}^2(0, T + T^{\delta}; \mathbb{R}) \times \mathcal{L}_{\mathcal{G}}^2(0, T + T^{\delta}; \mathbb{R}^d) \times \mathcal{L}_{\mathcal{G}}^{2,\tau}(0, T + T^{\delta}; \mathbb{R}^k)$. Define the following mapping:

$$\begin{aligned}
&\Phi: \mathcal{D} \to \mathcal{D}; \\
&(y, z, u) \to \Phi(y, z, u) := (Y, Z, U).
\end{aligned}$$

For any $(y, z, u) \in \mathcal{D}$, there exist the increasing processes K^+ and $K^- \in \mathcal{A}_{\mathcal{G}}^2(0, T; \mathbb{R})$, such that (Y, Z, U, K^+, K^-) is the solution of RABSDE with the coefficient $(f(s, Y_s, Y_{s+\delta^1(s)},$

$Z_s, Z_{s+\delta^2(s)}, U_s, U_{s+\delta^3(s)}), \xi, L, V)$.

First, we prove that Φ is a contraction mapping of $\mathcal{D}$.

For any (y, z, u), $(y', z', u') \in \mathcal{D}$, denote:

$$\begin{aligned}
\hat{y} &= y - y', & \hat{z} &= z - z', & \hat{u} &= u - u'; \\
\hat{Y} &= Y - Y', & \hat{Z} &= Z - Z', & \hat{U} &= U - U', \\
\hat{K}^+ &= K^+ - K'^+, & \hat{K}^- &= K^- - K'^-.
\end{aligned}$$

Applying the Itô's formula for rcll semi-martingale (Theorem 1.3.4), we have

$$\begin{aligned}
&|\hat{Y}_t|^2 + \int_t^T e^{cs}\left(c|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau > s\}}\gamma_s\right) ds \\
=&2\int_0^T e^{cs}\hat{Y}_s \big| f\big(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\big) \\
&\qquad - f\big(s, y'_s, y'_{s+\delta^1(s)}, z'_s, z'_{s+\delta^2(s)}, u'_s, u'_{s+\delta^3(s)}\big)\big| ds \\
&+ 2\int_t^T e^{cs}|\hat{Y}_s|(d\hat{K}_s^+ - d\hat{K}_s^-) \\
&- 2\int_t^T e^{cs}\hat{Y}_s\hat{Z}_s dB_s - \int_t^T e^{cs}\left(2\hat{Y}_{s-}\hat{U}_s + |\hat{U}_s|^2\right) dM_s,
\end{aligned}$$

Since $\int_t^T e^{cs}\hat{Y}_s^{(n)}\hat{Z}_s^{(n)} dB_s$ and $\int_t^T e^{cs}\big(2\hat{Y}_{s-}^{(n)}\hat{U}_s^{(n)} + |\hat{U}_s^{(n)}|^2\big) dM_s$ are $\mathcal{G}$-martingales, $|\hat{Y}_t^{(n)}|^2 \geq 0$, $\int_t^T e^{cs}|\hat{Y}_s|(d\hat{K}_s^+ - d\hat{K}_s^-) \leq 0$, by the Fubini's Theorem, **H 2.2** and the Lipschitz condition for f, it follows

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T e^{cs}\left(\frac{c}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau > s\}}\gamma_s\right) ds\right] \\
\leq&\frac{2}{c}\mathbb{E}\Big[\int_t^T e^{cs}\hat{Y}_s \big| f\big(s, y_s, y_{s+\delta^1(s)}, z_s, z_{s+\delta^2(s)}, u_s, u_{s+\delta^3(s)}\big) \\
&\qquad - f\big(s, y'_s, y'_{s+\delta^1(s)}, z'_s, z'_{s+\delta^2(s)}, u'_s, u'_{s+\delta^3(s)}\big)\big|^2 ds\Big] \\
\leq&\frac{2L^2}{c}\mathbb{E}\Big[\int_0^T e^{cs}\Big(|\hat{y}_s| + \mathbb{E}^{\mathcal{G}_s}|\hat{y}_{s+\delta^1(s)}| + |\hat{z}_s| + \mathbb{E}^{\mathcal{G}_s}|\hat{z}_{s+\delta^2(s)}| \\
&\qquad + |\hat{u}_s| 1_{\{\tau > s\}}\sqrt{\gamma_s} + \mathbb{E}^{\mathcal{G}_s}|\hat{u}_{s+\delta^3(s)}| 1_{\{\tau > s\}}\sqrt{\gamma_s}\Big)^2 ds\Big] \\
\leq&\frac{12L^2}{c}\mathbb{E}\left[\int_t^T e^{cs}|\hat{Y}_s|\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau > s\}}\gamma_s\right) ds\right] \\
&+ \frac{12L^2L^\delta}{c}\mathbb{E}\left[\int_t^{T+T^\delta} e^{cs}\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau > s\}}\gamma_s\right) ds\right] \\
\leq&\frac{12L^2 + 12L^2L^\delta}{c}\mathbb{E}\left[\int_t^{T+T^\delta} e^{cs}\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau > s\}}\gamma_s\right) ds\right].
\end{aligned}$$

where $L \geq 0$, $L^\delta \leq 0$ are constants. Set $c = 12L^2 + 12L^2L^\delta + 2$, it follows

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T+T^\delta} e^{cs}\left(\frac{c}{2}|\hat{Y}_s|^2 + |\hat{Z}_s|^2 + |\hat{U}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right] \\
\leq &\frac{1}{2}\mathbb{E}\left[\int_0^{T+T^\delta} e^{cs}\left(|\hat{y}_s|^2 + |\hat{z}_s|^2 + |\hat{u}_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right].
\end{aligned}$$

Therefore, Φ is a contraction mapping on $\mathcal{D}$ equipped with the norm

$$\|(Y,Z,U)\|_c := \left\{\mathbb{E}\int_0^{T+T^\delta} e^{cs}\left(|Y_s|^2 + |Z_s|^2 + |U_s|^2 1_{\{\tau>s\}}\gamma_s\right) ds\right\}^{\frac{1}{2}}.$$

From the Banach fixed-point theorem, there exists a unique fixed point $(Y, Z, U) \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^\delta; \mathbb{R}^k)$, with K^+ and K^-, is the solution of RABSDE (4.1). Combining with assumption **H 2.3** and **H 2.2**, it follows that $f(t, Y_t, Y_{t+\delta^1(t)}, Z_t, Z_{t+\delta^2(t)}, U_t, U_{t+\delta^3(t)}) \in \mathcal{L}^2_{\mathcal{G}}(0, T+T^\delta; \mathbb{R})$. □

4.3 Linear RABSDEs with Two Obstacles and Stochastic Differential Delay Equations

Similarly to Section 2.4 and Section 3.4, we study the relation between linear RABSDEs with two obstacles and SDDEs. Consider the following reflected anticipated BSDE with two obstacles and default risk (δ is a given constant, t_0 is the initial time, B is a d-dimensional standard Brownian motion). $(Y, Z, U, K^+, K^-) \in \mathcal{S}^2_{\mathcal{G}}(0, T+\delta; \mathbb{R}) \times \mathcal{L}^2_{\mathcal{G}}(0, T+\delta; \mathbb{R}^d) \times \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+\delta; \mathbb{R}^k) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R}) \times \mathcal{A}^2_{\mathcal{G}}(0, T; \mathbb{R})$ is a solution of the RABSDE (4.18):

$$\begin{cases}
-dY_t = \Big(\sigma_t Y_t + \hat{\sigma}_t \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}] + Z_t\theta_t + \mathbb{E}^{\mathcal{G}_t}[Z_{t+\delta}]\hat{\theta}_t + U_t\mu_t 1_{\{\tau>t\}} \\
\qquad\qquad + \mathbb{E}^{\mathcal{G}_t}[U_{t+\delta}]\hat{\mu}_t 1_{\{\tau>t\}} + l_t\Big) dt + dK_t^+ - dK_t^- \\
\qquad\qquad - Z_t dB_t - U_t dM_t, \qquad t \in [t_0, T]; \\
V_t \geq Y_t \geq L_t, \qquad t \in [0, T+\delta]; \\
Y_t = \xi_t, \qquad t \in [T, T+\delta]; \\
Z_t = \alpha_t, \qquad t \in (T, T+\delta]; \\
U_t = \beta_t, \qquad t \in (T, T+\delta]; \\
\int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (V_t - Y_t) dK_t^- = 0,
\end{cases} \tag{4.18}$$

and the following stochastic differential delay equations with default risk (SDDE):

$$\begin{cases}
dX_s = \left(\sigma_s X_s + \hat{\sigma}_{s-\delta}^T X_{s-\delta}\right) ds + \left(X_s\theta_s^T + X_{s-\delta}\hat{\theta}_{s-\delta}^T\right) dB_s \\
\qquad\quad + \left(X_{s-}\mu_s^T + X_{s-\delta}\hat{\mu}^T_{(s-\delta)-}\right) dM_s, \qquad s \in [t, T+\delta]; \\
X_t = 1; \\
X_s = 0, \qquad s \in [t-\delta, t).
\end{cases} \tag{4.19}$$

where σ, $\hat{\sigma}$, θ, μ and $\hat{\mu}$ are uniformly bounded. σ, $\hat{\sigma} \in \mathcal{L}^2_{\mathcal{G}}(t_0-\delta, T+T^\delta;\mathbb{R})$, θ, $\hat{\theta} \in \mathcal{L}^2_{\mathcal{G}}(t_0-\delta, T+T^\delta;\mathbb{R}^d)$, μ, $\hat{\mu} \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(t_0-\delta, T+T^\delta;\mathbb{R}^k)$. $\xi \in \mathcal{S}^2_{\mathcal{G}}(T, T+T^\delta;\mathbb{R})$, $\alpha \in \mathcal{L}^2_{\mathcal{G}}(T, T+T^\delta;\mathbb{R}^d)$, $\beta \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0, T+T^\delta;\mathbb{R}^k)$.

The following Theorem 4.3.1 illustrates the main result of this section.

Theorem 4.3.1. *The solution Y of the above RABSDE with two obstacles (3.27) can be given by the following form:*

$$\begin{aligned}
Y_t = ess \inf_{v_1\in\mathcal{T}_t} ess \sup_{v_2\in\mathcal{T}_t} \mathbb{E}^{\mathcal{G}_t}\Bigg[&\int_t^{v^1\wedge v^2} l_s X_s ds + L_{v_2}X_{v_2}1_{\{v_2\le v_1<T\}} + V_{v_1}X_{v_1}1_{\{v_1<v_2\}}\\
&+1_{\{v_1=v_2=T\}}\int_T^{T+\delta}\left(\xi_s\hat{\sigma}_{s-\delta}+\alpha_s\hat{\theta}_{s-\delta}+\beta_s\hat{\mu}_{s-\delta}1_{\{\tau>s\}}\right)X_{s-\delta}ds\\
&+X_T\xi_T1_{\{v_1=v_2=T\}}\Bigg]\\
= ess \sup_{v_2\in\mathcal{T}_t} ess \inf_{v_1\in\mathcal{T}_t} \mathbb{E}^{\mathcal{G}_t}\Bigg[&\int_t^{v^1\wedge v^2} l_s X_s ds + L_{v_2}X_{v_2}1_{\{v_2\le v_1<T\}} + V_{v_1}X_{v_1}1_{\{v_1<v_2\}}\\
&+1_{\{v_1=v_2=T\}}\int_T^{T+\delta}\left(\xi_s\hat{\sigma}_{s-\delta}+\alpha_s\hat{\theta}_{s-\delta}+\beta_s\hat{\mu}_{s-\delta}1_{\{\tau>s\}}\right)X_{s-\delta}ds\\
&+X_T\xi_T1_{\{v_1=v_2=T\}}\Bigg],
\end{aligned} \tag{4.20}$$

where X is the solution of the above SDDE (4.19).

Proof. From Theorem 2.4.1, we know that the SDDE (3.28) has a unique solution. Applying the Itô formula to X_sY_s on $s\in[t,T]$, and taking conditional expectation under $\mathcal{G}_t$, we can obtain

$$\begin{aligned}
&X_TY_T - X_tY_t\\
&=\int_t^T\Big(Y_s\hat{\sigma}_{s-\delta}X_{s-\delta}-\mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}]\hat{\sigma}_sX_s+Z_s\hat{\theta}_{s-\delta}X_{s-\delta}-\mathbb{E}^{\mathcal{G}_s}[Z_{s+\delta}]\hat{\theta}_sX_s\\
&\qquad+U_s\hat{\mu}_{s-\delta}X_{s-\delta}1_{\{\tau>s\}}-\mathbb{E}^{\mathcal{G}_s}[U_{s+\delta}]\hat{\mu}_sX_s1_{\{\tau>s\}}-X_sl_s\Big)ds\\
&\quad-\int_t^T X_s dK_s^+ + \int_t^T X_s dK_s^-\\
&\quad+\int_t^T\left[Y_s\left(X_s\theta_s^T+X_{s-\delta}\hat{\theta}^T_{s-\delta}\right)+X_sZ_s\right]dB_s\\
&\quad+\int_t^T\left[Y_s\left(X_{s-}\mu_s^T+X_{s-\delta}\hat{\mu}^T_{(s-\delta)-}\right)+X_sU_s\right]dM_s
\end{aligned}$$

When $s \in [t-\delta, t)$, $X_s = 0$, and $X_t = 1$, it follows:

$$\begin{aligned}
X_tY_t =& X_TY_T + \int_t^T X_s l_s ds + \int_t^T X_s dK_s^+ - \int_t^T X_s dK_s^- - \int_t^T Y_s\hat{\sigma}_{s-\delta}X_{s-\delta}ds \\
&+ \int_{t+\delta}^{T+\delta} Y_s\hat{\sigma}_{s-\delta}X_{s-\delta}ds - \int_t^T Z_s\hat{\theta}_{s-\delta}X_{s-\delta}ds + \int_{t+\delta}^{T+\delta} Z_s\hat{\theta}_{s-\delta}X_{s-\delta}ds \\
&- \int_t^T U_s\hat{\mu}_{s-\delta}X_{s-\delta}1_{\{\tau>s\}}ds + \int_{t+\delta}^{T+\delta} U_s\hat{\mu}_{s-\delta}X_{s-\delta}1_{\{\tau>s\}}ds \\
&+ \int_t^T \left[Y_s\left(X_s\theta_s^T + X_{s-\delta}\hat{\theta}_{s-\delta}^T\right) + X_sZ_s\right]dB_s \\
&+ \int_t^T \left[Y_s\left(X_{s-}\mu_s^T + X_{s-\delta}\hat{\mu}_{(s-\delta)-}^T\right) + X_sU_s\right]dM_s \\
=& X_T\xi_T + \int_t^T X_s l_s ds + \int_T^{T+\delta}\left(\xi_s\hat{\sigma}_{s-\delta} + \alpha_s\hat{\theta}_{s-\delta} + \beta_s\hat{\mu}_{s-\delta}1_{\{\tau>s\}}\right)X_{s-\delta}ds \\
&+ \int_t^T X_s dK_s^+ - \int_t^T X_s dK_s^- + \int_t^T \left[Y_s\left(X_s\theta_s^T + X_{s-\delta}\hat{\theta}_{s-\delta}^T\right) + X_sZ_s\right]dB_s \\
&+ \int_t^T \left[Y_s\left(X_{s-}\mu_s^T + X_{s-\delta}\hat{\mu}_{(s-\delta)-}^T\right) + X_sU_s\right]dM_s.
\end{aligned} \tag{4.21}$$

Denote

$$\begin{aligned}
\hat{Y}_t &= X_tY_t; \\
\hat{Z}_t &= Y_t\left(X_t\theta_t^T + X_{t-\delta}\hat{\theta}_{t-\delta}^T\right) + X_tZ_t; \\
\hat{Y}_t &= Y_t\left(X_{t-}\mu_t^T + X_{t-\delta}\hat{\mu}_{(t-\delta)-}^T\right) + X_tU_t; \\
\hat{K}_t^+ &= \int_0^t X_s dK_s^+; \qquad \hat{K}_t^- = \int_0^t X_s dK_s^-.
\end{aligned}$$

Therefore, $(\hat{Y}, \hat{Z}, \hat{U}, \hat{K}^+, \hat{K}^-)$ is a solution of the RBSDE (4.21), with the lower obstacle L_tX_t, the generator l_tX_t, the terminal condition $X_T\xi_T + \int_T^{T+\delta}\left(\xi_s\hat{\sigma}_{s-\delta} + \alpha_s\hat{\theta}_{s-\delta} + \beta_s\hat{\mu}_{s-\delta}1_{\{\tau>s\}}\right)X_{s-\delta}ds$. By Theorem 3.3.1, $X_t = 1$, it follows (2.6.1). □

4.4 Relation with the Obstacle Problems for Non-linear Parabolic PDEs

Harraj et. al [95] (2005) studied the relation between RBSDEs with two obstacles and Poisson jump and parabolic PDEs. In this section, we study the relation between RABSDEs with two obstacles mentioned above and non-linear parabolic PDEs in the defaultable setting.

We first consider a state process X which has an influence on the risk measure and the position. For each initial time $t \in [0, T+T^\delta]$ and each initial condition $x \in \mathbb{R}$, let $X^{t,x}$ be the unique solution of the following SDE (for simplicity, we consider a defaultable model of a single random default time):

$$X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x})dr + \int_t^s \theta(r, X_r^{t,x})dB_r + \int_t^s \mu(r, X_{r-}^{t,x})dM_r, \tag{4.22}$$

with the following assumptions:

H 4.2. *$\sigma : [0, T+T^\delta] \times \mathbb{R} \to \mathbb{R}$, $\theta : [0, T+T^\delta] \times \mathbb{R} \to \mathbb{R}$, $\mu : [0, T+T^\delta] \times \mathbb{R} \to \mathbb{R}$, and satisfy:*

- ***(a)*** *$\sigma(t,x)$ and $\theta(t,x)$ are continuous mappings, $\mu(t,x)$ is progressively measurable and invertible;*
- ***(b)*** *$\sigma(t,x)$, $\theta(t,x)$ and $\mu(t,x)$ are uniformly with respect to t and Lipschitz with respect to x, i.e. $\forall (t,x)$, $(t,x') \in [0,T] \times \mathbb{R}$, there exists a constant C_1, C_2, $C_3 \geq 0$, such that*

$$
\begin{aligned}
&|\sigma(t,x)| + |\theta(t,x)| \leq C_1(1+|x|);\\
&|\sigma(t,x) - \sigma(t,x')| + |\theta(t,x) - \theta(t,x')| \leq C_2|x-x'|;\\
&|\mu(t,x) - \mu(t,x')| \leq C_3|x-x'|1_{\{\tau > t\}};
\end{aligned}
$$

We consider the RABSDE below with two obstacles and default risk:

$$
\begin{cases}
(i) \quad Y_s^{t,x} = g(H_T, X_T^{t,x}) + \int_s^T f\big(r, H_r, X_r^{t,x}, Y_r^{t,x}, Y_{r+\delta^1(r)}^{t,x}, Z_r^{t,x}, Z_{r+\delta^2(r)}^{t,x},\\
\qquad\qquad U_r^{t,x}, U_{r+\delta^3(r)}^{t,x}\big)ds + \left(K_T^{t,x,+} - K_s^{t,x,+}\right) - \left(K_T^{t,x,-} - K_s^{t,x,-}\right)\\
\qquad\qquad - \int_s^T Z_r^{t,x} dB_r - \int_s^T U_r^{t,x} dM_r, \qquad s \in [0,T];\\
(ii) \quad L(s, X_s^{t,x}) \leq Y_s^{t,x} \leq V(s, X_s^{t,x}), \qquad s \in [0, T+T^\delta];\\
(iii) \quad Y_s^{t,x} = g(1, X_s^{t,x}), \qquad s \in (T, T+T^\delta];\\
(iv) \quad Z_s^{t,x} = \alpha_s, \qquad s \in (T, T+T^\delta];\\
(v) \quad U_s^{t,x} = \beta_s, \qquad s \in (T, T+T^\delta];\\
(vi) \quad \int_0^T (Y_s - L(s, X_s^{t,x}))dK_s^{t,x,+} = \int_0^T (V(s, X_s^{t,x}) - Y_s)dK_s^{t,x,-} = 0;\\
(vii) \quad \mathbb{E}\int_0^T \left(|Y_s^{t,x}|^2 + |Z_s^{t,x}|^2 + |U_s^{t,x}|^2 1_{\{\tau > s\}}\right) ds < \infty,
\end{cases}
\tag{4.23}
$$

with the following assumptions:

H 4.3. *$g \in \mathcal{C}(\mathbb{R})$, $f : [0, T+T^\delta] \times \{0,1\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $L : [0, T+T^\delta] \times \mathbb{R} \to \mathbb{R}$, $V : [0, T+T^\delta] \times \mathbb{R} \to \mathbb{R}$ satisfy:*

- ***(a)*** *$g(h,x)$ has at most polynomial growth at infinity, i.e. there exists constants $C_4 > 0$ and $p \in \mathbb{R}^+$, such that*

$$|g(h,x)| \leq C_4(1+|x|^p).$$

- ***(b)*** *$f(t,h,x,y,\bar{y}_r,z,\bar{z}_r,u,\bar{u}_r)$ is globally Lipschitz in $(y,\bar{y}_r,z,\bar{z}_r,u,\bar{u}_r)$ uniformly with respect to (t,x). There exists constants $C_5 > 0$ and $p \in \mathbb{R}^+$, for any $t \in [0,T]$, $r \in [t, T+T^\delta]$, y, $y' \in \mathbb{R}$, z, $z' \in \mathbb{R}$, u, $u' \in \mathbb{R}$, $\bar{y}$, $\bar{y}' \in \mathcal{S}_\mathcal{G}^2(t, T+T^\delta; \mathbb{R})$, $\bar{z}$, $\bar{z}' \in \mathcal{L}_\mathcal{G}^2(t, T+T^\delta; \mathbb{R})$, $\bar{u}$, $\bar{u}' \in \mathcal{L}_\mathcal{G}^{2,\tau}(t, T+T^\delta; \mathbb{R})$, there exists a constant $L \geq 0$, such that*

$$
\begin{aligned}
&|f(t,h,x,0,0,0,0,0,0)| \leq C_5(1+|x|^p);\\
&|f(t,h,x,y,\bar{y}_r,z,\bar{z}_r,u,\bar{u}_r) - f(t,h,x,y',\bar{y}'_r,z',\bar{z}'_r,u',\bar{u}'_r)|\\
\leq &L\Big(|y-y'| + \mathbb{E}^{\mathcal{G}_t}|\bar{y}_r - \bar{y}'_r| + |z-z'| + \mathbb{E}^{\mathcal{G}_t}|\bar{z}_r - \bar{z}'_r|\\
&\quad + |u-u'|1_{\{\tau>t\}}\sqrt{\gamma_t} + \mathbb{E}^{\mathcal{G}_t}|\bar{u}_r - \bar{u}'_r|1_{\{\tau>t\}}\sqrt{\gamma_t}\Big);
\end{aligned}
$$

(*c*) *$L(t,x)$ and $V(t,x)$ are jointly continuous in t and x. There exist constant $C_6 > 0$ and $p \in \mathbb{R}$, for any $(t,x) \in [0,T] \times \mathbb{R}^d$:*

$$L(t,x) \leq C_6\left(1+|x|^p\right), \qquad V(t,x) \geq -C_6\left(1+|x|^p\right);$$
$$L(t,x) \leq V(t,x), \qquad L(T,x) \leq g(h,x) \leq V(T,x).$$

Remark 4.4.1. *For any $s \in [0,t]$, we extent $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}, K_s^{t,x,+}, K_s^{t,x,-})$ by*

$$Y_s^{t,x} = Y_t^{t,x}, \quad Z_s^{t,x} = U_s^{t,x} = K_s^{t,x,+} = K_s^{t,x,-} = 0, \qquad s \in [0,t],$$

*under the assumptions **H** 4.2 and **H** 4.3, from the existence and uniqueness theorem for RABSDEs with two obstacles and default risk (Theorem 4.2.3), it follows that for each $(t,x) \in [0, T+T^\delta] \times \mathbb{R}$, there exists a unique $\mathcal{G}_s^t$-progressively measurable solution $(Y^{t,x}, Z^{t,x}, U^{t,x}, K^{t,x,+}, K^{t,x,-})$ for the RABSDE (4.23). Here we define $\mathcal{G}_s^t = \sigma\{B_s - B_t; t \leq s \leq T+T^\delta\} \vee \sigma\{H_s - H_t; t \leq s \leq T+T^\delta\}$.*

4.4.1 Related Parabolic PDEs

We consider the following related obstacle problem for a parabolic PDE:

$$\begin{cases} \min\Big(u(t,h,x) - L(t,x), \\ \qquad -\frac{\partial u}{\partial_t}(t,h,x) - \mathcal{L}^{t,x}u(t,h,x) - f^{t,x}(t,h,x)\Big) = 0; \\ \max\Big(u(t,h,x) - V(t,x), \\ \qquad -\frac{\partial u}{\partial_t}(t,h,x) - \mathcal{L}^{t,x}u(t,h,x) - f^{t,x}(t,h,x)\Big) = 0; \\ u(T,h,x) = g(h,x), \end{cases} \tag{4.24}$$

where $h := \{0,1\}$, $u : [0, T+T^\delta] \times \{0,1\} \times \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} \mathcal{L}^{t,x}u(t,h,x) :=& \sigma(t,x)\frac{\partial u}{\partial_x}(t,h,x) + \frac{1}{2}\theta^2(t,x)\frac{\partial^2 u}{\partial_x}(t,h,x) \\ &+ \left(\Delta u - \mu(t,x)\frac{\partial u}{\partial_x}(t,h,x)\right)(1-h)\gamma_t; \\ f^{t,x}(t,h,x) :=& f\Big(t,h,x,u(t,h,x), u\left(t+\delta^1(t),h,x\right), \theta(t,x)\frac{\partial u}{\partial_x}(t,h,x), \\ &\theta(t,x)\frac{\partial u}{\partial_x}\left(t+\delta^2(t),h,x\right), \Delta u(t,x), \Delta u(t+\delta^3(t),x)\Big), \\ \Delta u(t,x) :=& u\left(t,1,x+\mu(t,x)\right) - u(t,0,x). \end{aligned}$$

Remark 4.4.2. *In a defaultable market, $X_\cdot^{t,x}$ in (4.22) is the dynamics of asset with default risk, $g(H_T, X_T^{t,x})$ is the contingent that we want to replicate. We denote by $u^0(t,x) := u(t,0,x)$ the pre-default pricing function, and $u^1(t,x) := u(t,1,x)$ the post-default pricing function (see more in Remark 2.6.1 in Section 2.6).*

Since the function u defined in (4.24) is not smooth, we introduce the following definition of a viscosity solution of the parabolic obstacle problem (4.24) in a weaker sense (see more in Soner [96] (1988), Barles et al. [27] (1997), etc).

Definition 4.4.1. *(Viscosity solution) Define $u(t,h,x) \in \mathcal{C}([0,T+T^\delta]\times\mathbb{R})$ $(h=0,1)$, $u(T,h,x)=g(h,x)$.*

(a) *u is a viscosity subsolution of (4.24), if the following hold:*

- *$u(t,h,x) \leq V(t,x)$, for all $(t,x)\in[0,T+T^\delta]\times\mathbb{R}$, $(h=0,1)$;*
- *for any $q\in\mathcal{C}^2([0,T+T^\delta]\times\mathbb{R})$, $(t,x)\in[0,T+T^\delta)\times\mathbb{R}$ is a global maximum point of $u-q$, such that*

$$\begin{aligned}\min\Big(&u(t,h,x)-L(t,x),\\ &-\frac{\partial q}{\partial_t}(t,h,x)-\mathcal{L}^{t,x}q(t,h,x)-\bar{f}(t,h,x,u,q)\Big)\leq 0.\end{aligned} \tag{4.25}$$

(b) *u is a viscosity supersolution of (4.24), if the following hold:*

- *$u(t,h,x)\geq L(t,x)$, for all $(t,x)\in[0,T+T^\delta]\times\mathbb{R}$, $(h=0,1)$;*
- *for any $q\in\mathcal{C}^2([0,T+T^\delta]\times\mathbb{R})$, $(t,x)\in[0,T+T^\delta)\times\mathbb{R}$ is a global minimum point of $u-q$, such that*

$$\begin{aligned}\max\Big(&u(t,h,x)-V(t,x),\\ &-\frac{\partial q}{\partial_t}(t,h,x)-\mathcal{L}^{t,x}q(t,h,x)-\bar{f}(t,h,x,u,q)\Big)\geq 0,\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}\bar{f}(t,h,x,u,q):=&f\Big(t,h,x,u(t,h,x),u\left(t+\delta^1(t),h,x\right),\theta(t,x)\frac{\partial q}{\partial_x}(t,h,x),\\ &\theta(t,x)\frac{\partial q}{\partial_x}\left(t+\delta^2(t),h,x\right),\Delta q(t,x),\Delta q(t+\delta^3(t),x)\Big),\\ \Delta q(t,x):=&q\left(t,1,x+\mu(t,x)\right)-q(t,0,x).\end{aligned} \tag{4.27}$$

(c) *u is defined to be a viscosity solution of parabolic obstacle problem (4.24) if it is both a viscosity subsolution and supersolution.*

Similarly to the Definition 8.1 in El Karoui et. al [33], we introduce the definition of parabolic subjet and superjet below.

Definition 4.4.2. ***(Parabolic subjet and superjet)*** *Let $u(t,h,x)\in\mathcal{C}([0,T+T^\delta]\times\mathbb{R})$, $(h=0,1)$, for any $(t,x)\in[0,T]\times\mathbb{R}$.*

- *$\mathcal{P}^{t,h,x,+}u$ is the parabolic subjet of u with respect to (t,x), $(h=0,1)$, i.e. the set of the triple $(p,q,X)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, such that*

$$\begin{aligned}u(s,h,x')\geq&u(s,h,x)+p(s-t)+q(x'-x)+\frac{1}{2}\left\langle X(x'-x),x'-x\right\rangle\\ &+o(|s-t|+|x'-x|^2);\end{aligned}$$

- $\mathcal{P}^{t,h,x,-}u$ *is the parabolic superjet of* u *with respect to* (t,x), $(h=0,1)$, *i.e. the set of the triple* $(p,q,X) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, *such that*

$$\begin{aligned} u(s,h,x') \leq & u(s,h,x) + p(s-t) + q(x'-x) + \frac{1}{2}\left\langle X(x'-x), x'-x\right\rangle \\ & + o(|s-t| + |x'-x|^2). \end{aligned}$$

Lemma 4.4.1. $Y_t^{t,x}$ *defined in (4.23), then there exists a constant* $C \geq 0$, *for all* (t,x), $(t',x') \in [0, T+T^\delta] \times \mathbb{R}$, $p \in \mathbb{R}^+$, *such that*

$$\begin{aligned} & \mathbb{E}\left[\sup_{0\leq s\leq T}\left|Y_s^{t,x}\right|^2\right] \leq C\left(1+|x|^p\right); \\ & \mathbb{E}\left[\sup_{0\leq s\leq T}\left|Y_s^{t,x} - Y_s^{t',x'}\right|^2\right] \\ \leq & C\mathbb{E}\left(\left|g(H_T, X_T^{t,x}) - g(H_T, X_T^{t',x'})\right|^2\right) \\ & + \mathbb{E}\int_0^T \Big| 1_{[t,T]}(s)\Big(f\big(s, H_s, X_s^{t,x}, Y_s^{t,x}, Y_{s+\delta^1(s)}^{t,x}, Z_s^{t,x}, Z_{s+\delta^2(s)}^{t,x}, U_s^{t,x}, U_{s+\delta^3(s)}^{t,x}\big) \\ & \quad - f\big(s, H_s, X_s^{t',x'}, Y_s^{t',x'}, Y_{s+\delta^1(s)}^{t',x'}, Z_s^{t',x'}, Z_{s+\delta^2(s)}^{t',x'}, U_s^{t',x'}, U_{s+\delta^3(s)}^{t',x'}\big)\Big)\Big|^2 ds. \end{aligned} \tag{4.28}$$

Proof. Applying the Itô's formula for rcll semi-martingale (Theorem 1.3.4) to $\left|Y_s^{t,x}\right|^2$ on $[t,T]$, and taking expectation on both sides, we can obtain

$$\begin{aligned} & \mathbb{E}\left[\left|Y_s^{t,x}\right|^2 + \int_t^T \left|Z_s^{t,x}\right|^2 ds + \int_t^T \left|U_s^{t,x}\right|^2 1_{\{\tau>s\}}\gamma_s ds\right] \\ = & \mathbb{E}\left|g(H_T, X_T^{t,x})\right|^2 + 2\mathbb{E}\int_t^T Y_s^{t,x}\left(dK_s^{t,x,+} - dK_s^{t,x,-}\right) \\ & + 2\mathbb{E}\int_t^T Y_s^{t,x} f\big(s, H_s, X_s^{t,x}, Y_s^{t,x}, Y_{s+\delta^1(s)}^{t,x}, Z_s^{t,x}, Z_{s+\delta^2(s)}^{t,x}, U_s^{t,x}, U_{s+\delta^3(s)}^{t,x}\big) ds. \end{aligned}$$

Similarly,

$$\begin{aligned} & \mathbb{E}\left[\left|Y_s^{t,x} - Y_s^{t',x'}\right|^2 + \int_t^T \left|Z_s^{t,x} - Z_s^{t',x'}\right|^2 ds + \int_t^T \left|U_s^{t,x} - U_s^{t',x'}\right|^2 1_{\{\tau>s\}}\gamma_s ds\right] \\ = & \mathbb{E}\left|g(H_T, X_T^{t,x}) - g(H_T, X_T^{t',x'})\right|^2 \\ & + 2\mathbb{E}\int_t^T \left(Y_s^{t,x} - Y_s^{t',x'}\right)\left(dK_s^{t,x,+} - dK_s^{\prime t,x,+}\right) \\ & - 2\mathbb{E}\int_t^T \left(Y_s^{t,x} - Y_s^{t',x'}\right)\left(dK_s^{t,x,-} - dK_s^{\prime t,x,-}\right) \\ & + 2\mathbb{E}\int_t^T Y_s^{t,x}\Big(f\big(s, H_s, X_s^{t,x}, Y_s^{t,x}, Y_{s+\delta^1(s)}^{t,x}, Z_s^{t,x}, Z_{s+\delta^2(s)}^{t,x}, U_s^{t,x}, U_{s+\delta^3(s)}^{t,x}\big) \\ & \quad - f\big(s, H_s, X_s^{t',x'}, Y_s^{t',x'}, Y_{s+\delta^1(s)}^{t',x'}, Z_s^{t',x'}, Z_{s+\delta^2(s)}^{t',x'}, U_s^{t',x'}, U_{s+\delta^3(s)}^{t',x'}\big)\Big) ds. \end{aligned}$$

By the Lipschitz condition of the generator f, the assumptions **H 4.3 (a)** and **(b)**, and Gronwall's inequality, similarly to the proof of Lemma 4.2, we can obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le T}|Y_t^{n,k}|^2\Big]\\
\le &C\mathbb{E}\left(1+\left|g(H_T,X_T^{t,x})\right|^2+\int_0^T f\left(s,H_s,X_s^{t,x},0,0,0,0,0,0\right)ds\right)\\
\le &C\left(1+|x|^p\right).
\end{aligned}\tag{4.29}$$

where $C>0$ is a constant. It follows (4.28) consequently. □

4.4.2 Main Result

We now define

$$u(t,h,x):=Y_t^{t,x},\qquad (t,x)\in[0,T+T^\delta]\times\mathbb{R},\ \ h=\{0,1\},\tag{4.30}$$

since $Y_t^{t,x}$ is $\mathcal{G}_t^t$-measurable, hence u is a deterministic function.

Remark 4.4.3. *Set $Y_s^{t,x}=Y_t^{t,x}$ on $[0,t]$, let $(t_n,x_n)_{n\in\mathbb{N}}$ be a sequence of $[0,T+T^\delta]\times\mathbb{R}$ converging to (t,x). Applying Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|Y_{t_n}^{t_n,x_n}-Y_t^{t,x}|^2$ on $[0,T+T^\delta]$ and taking expectation on both sides, similarly to the proof of Lemma 4.2.1, we can obtain*

$$\begin{aligned}
&\left|u(t_n,h,x_n)-u(t,h,x)\right|^2=\left|Y_{t_n}^{t_n,x_n}-Y_t^{t,x}\right|^2\\
\le&\mathbb{E}\Big[\sup_{0\le s\le T+T^\delta}\left|Y_s^{t_n,x_n}-Y_s^{t,x}\right|^2\Big]\\
\le&C\mathbb{E}\left[\left|g(H_T,X_T^{t_n,x_n})-g(H_T,X_T^{t,x})\right|^2\right]\\
&+\mathbb{E}\int_0^T\Big[\Big|1_{[t,T]}(r)\Big(f\left(r,H_r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Y_{r+\delta^1(r)}^{t_n,x_n},Z_r^{t_n,x_n},Z_{r+\delta^2(r)}^{t_n,x_n},U_r^{t_n,x_n},U_{r+\delta^3(r)}^{t_n,x_n}\right)\\
&\quad-f\left(r,H_r,X_r^{t,x},Y_r^{t,x},Y_{r+\delta^1(r)}^{t,x},Z_r^{t,x},Z_{r+\delta^2(r)}^{t,x},U_r^{t,x},U_{r+\delta^3(r)}^{t,x}\right)\Big)\Big|^2\Big]dr.
\end{aligned}$$

From Lemma 4.4.1, it follows that $u(t_n,h,x_n)\to u(t,h,x)$ as $(t_n,h,x_n)\to(t,h,x)$, hence, $u\in\mathcal{C}([0,T+T^\delta]\times\mathbb{R})$.

We are going to use the approximation of the RABSDE (4.23) through the penalization method, which has been studied in Section 4.2.2.1. For each $(t,x)\in[0,T+T^\delta]\times\mathbb{R}$, $n\in\mathbb{N}$, let $({}^nY_s^{t,x},{}^nZ_s^{t,x},{}^nU_s^{t,x})_{s\le t\le T+T^\delta}$ be the solution of the penalized ABSDE below:

$$\begin{aligned}
{}^nY_s^{t,x}=&g(H_T,X_T^{t,x})+\int_s^T f\left(r,H_r,X_r^{t,x},{}^nY_r^{t,x},{}^nY_{r+\delta^1(r)}^{t,x},{}^nZ_r^{t,x},{}^nZ_{r+\delta^2(r)}^{t,x},{}^nU_s^{t,x},{}^nU_{r+\delta^3(r)}^{t,x}\right)ds\\
&+n\int_s^T\left({}^nY_s^{t,x}-L(s,X_s^{t,x})\right)^-ds-n\int_s^T\left({}^nY_s^{t,x}-V(s,X_s^{t,x})\right)^+ds\\
&-\int_s^T{}^nZ_r^{t,x}dB_r-\int_s^T{}^nU_r^{t,x}dM_r,\qquad s\in[t,T+T^\delta].
\end{aligned}\tag{4.31}$$

Lemma 4.4.2. *Consider the following penalized parabolic PDE:*

$$\begin{aligned} -\frac{\partial u_n}{\partial_t}(t,h,x) - \mathcal{L}^{t,x}u_n(t,h,x) - f^n(t,h,x) = 0, \quad (t,x) \in [0,T+T^\delta] \times \mathbb{R}, \;\; h = \{0,1\}; \\ u_n(T,h,x) = g(h,x), \qquad x \in \mathbb{R}, \;\; h = \{0,1\}. \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} f^n(t,h,x) :=& f\Big(t,h,x,u_n(t,h,x),u_n\left(t+\delta^1(t),h,x\right),\theta(t,x)\frac{\partial u_n}{\partial_x}(t,h,x), \\ & \theta(t,x)\frac{\partial u_n}{\partial_x}\left(t+\delta^2(t),h,x\right),\Delta u_n(t),\Delta u_n(t+\delta^3(t))\Big) \\ & - n\left(u_n(t,h,x)-L(t,x)\right)^- + n\left(u_n(t,h,x)-V(t,x)\right)^+; \\ \Delta u_n(t) :=& u_n\left(t,1,x+\mu(t,x)\right) - u_n(t,0,x). \end{aligned}$$

u_n defined in (4.32) has the following relation with the penalized ABSDE (4.31):

$${}^nY_t^{t,x} = u_n(t,H_t,x), \qquad (t,x) \in [0,T+T^\delta] \times \mathbb{R}, \;\; h = \{0,1\},$$

moreover,

$$\begin{aligned} {}^nZ_t^{t,x} =& \theta(t,X_t^{t,x})\frac{\partial u_n}{\partial_x}(t,H_t,X_t^{t,x}), \\ {}^nU_t^{t,x} =& 1_{\{t=\tau\}}\left(u_n(t,1,x+\mu(t,x)) - u_n(t,0,x)\right), \quad (t,x) \in [0,T+T^\delta] \times \mathbb{R}. \end{aligned}$$

Proof. Since $u \in \mathcal{C}([0,T+T^\delta] \times \mathbb{R})$, applying Itô formula for rcll semi-martingale (Theorem 1.3.4) to $u_n(s,H_s,X_s^{t,x})$, by (4.22), we can get

$$\begin{aligned} du_n(s,H_s,X_s^{t,x}) =& \frac{\partial u_n}{\partial_t}(s,H_{s-},X_{s-}^{t,x})ds + \frac{\partial u_n}{\partial_x}(s,H_{s-},X_{s-}^{t,x})dX_s^{t,x} \\ & + \frac{1}{2}\theta^2(s,X_{s-}^{t,x})\frac{\partial^2 u_n}{\partial_x}(s,H_{s-},X_{s-}^{t,x})ds \\ & + \Big(\Delta u_n(s) - \mu(s,X_{s-}^{t,x})\frac{\partial u}{\partial_x}(s,H_{s-},X_{s-}^{t,x})\Big)dH_s \\ =& \Big(\frac{\partial u_n}{\partial_t} + \mathcal{L}^{t,x}u_n\Big)(s,H_{s-},X_{s-}^{t,x})ds \\ & + \frac{\partial u_n}{\partial_x}(s,H_{s-},X_{s-}^{t,x})\theta(s,X_{s-}^{t,x})dB_s + \Delta u_n(s)dM_s; \\ u_n(T,H_T,X_T^{t,x}) =& g(H_T,X_T^{t,x}), \end{aligned} \tag{4.33}$$

where $\Delta u_n(s)$ is the jump at the default time τ:

$$\Delta u_n(s) := 1_{\{s=\tau\}}\left(u_n(s,1,X_s^{t,x}) - u_n(s,0,X_{s-}^{t,x})\right).$$

From (4.32) and Theorem 2.4.4 in Xu [97] (2010), it follows

$$
\begin{aligned}
&\Big(\frac{\partial u_n}{\partial_t}+\mathcal{L}^{t,x}u_n\Big)(s,H_{s-},X_{s-}^{t,x})\\
=&f^n(s,H_{s-},X_{s-}^{t,x})\\
=&f\big(s,H_{s-},X_{s-}^{t,x},u_n(s,H_{s-},X_{s-}^{t,x}),u_n\left(s+\delta^1(s),H_{s-},X_{s-}^{t,x}\right),\\
&\qquad\theta(s,X_{s-}^{t,x})\frac{\partial u_n}{\partial_x}(s,H_{s-},X_{s-}^{t,x}),\theta(s,X_{s-}^{t,x})\frac{\partial u_n}{\partial_x}\left(s+\delta^2(s),H_{s-},X_{s-}^{t,x}\right),\\
&\qquad\Delta u_n(s,),\Delta u_n(s+\delta^3(s))\big)-n\left(u_n(s,H_{s-},X_{s-}^{t,x})-L(s,X_s^{t,x})\right)^-\\
&\quad+n\left(u_n(s,H_{s-},X_{s-}^{t,x})-V(s,X_s^{t,x})\right)^+\\
=&f\big(s,H_s,X_s^{t,x},u_n(s,H_s,X_s^{t,x}),u_n\left(s+\delta^1(s),H_s,X_s^{t,x}\right),\\
&\qquad\theta(s,X_s^{t,x})\frac{\partial u_n}{\partial_x}(s,H_s,X_s^{t,x}),\theta(s,X_s^{t,x})\frac{\partial u_n}{\partial_x}\left(s+\delta^2(s),H_s,X_s^{t,x}\right),\\
&\qquad\Delta u_n(s),\Delta u_n(s+\delta^3(s))\big)-n\left(u_n(s,H_s,X_s^{t,x})-L(s,X_s^{t,x})\right)^-\\
&\quad+n\left(u_n(t,H_s,X_s^{t,x})-V(s,X_s^{t,x})\right)^+,
\end{aligned}
$$

and

$$\int_0^t\theta(s,X_{s-}^{t,x})\frac{\partial u_n}{\partial_x}(s,H_{s-},X_{s-}^{t,x})dB_s=\int_0^t\theta(s,X_s^{t,x})\frac{\partial u_n}{\partial_x}(s,H_s,X_s^{t,x})dB_s.$$

Set

$$
\begin{aligned}
{}^nY_s^{t,x}=&u_n(s,H_s,X_s^{t,x}),\quad {}^nZ_s^{t,x}=\theta(s,X_s^{t,x})\frac{\partial u_n}{\partial_x}(s,H_s,X_s^{t,x}),\\
{}^nU_s^{t,x}=&1_{\{s=\tau\}}\left(u_n(s,1,X_s^{t,x})-u_n(s,0,X_{s-}^{t,x})\right),
\end{aligned}
$$

consequently, $({}^nY_s^{t,x},{}^nZ_s^{t,x},{}^nU_s^{t,x})$ is the unique solution of ABSDE (4.31). □

Theorem 4.4.1. *Defined by (4.30), u is a viscosity solution of the parabolic obstacle problem (4.24).*

Proof. Lemma 4.4.2 implies that $L(t,x)\le u(t,h,x)\le V(t,x)$, for any $(t,x)\in[0,T+T^\delta]\times\mathbb{R}$, and $u(T,h,x)=g(h,x)$.

Step 1. We first prove that u is a viscosity subsolution.

Let $q\in\mathcal{C}^2([0,T+T^\delta]\times\mathbb{R})$, $(t^*,x^*)\in[0,T+T^\delta)\times\mathbb{R}$ be a global maximum point of $u-q$. If $u(t^*,h,x^*)=L(t^*,x^*)$, it follows (4.25).

Now we prove that if $u(t^*,h,x^*)>L(t^*,x^*)$, then

$$-\frac{\partial q}{\partial_t}(t,h,x)-\mathcal{L}^{t,x}q(t,h,x)-\bar{f}(t,h,x,u,q)\le 0,\tag{4.34}$$

where $\bar{f}$ is defined in (4.27). By Lemma 4.2.3, we can obtain that ${}^nY_t^{t,x}\to Y_t^{t,x}$, Since u_n and u are continuous, from Dini's theorem, it follows that $u_n(t,h,x)\to u(t,h,x)$

uniformly in compact set of $[0, T + T^\delta] \times \mathbb{R}$. Hence, there exists $N \in \mathbb{N}$, for all $n \geq N$, $u_n(t^*, h, x^*) \geq L(t^*, x^*)$, and (t^*, x^*) is a maximum point of $u_n - q$, such that

$$-\frac{\partial q}{\partial_t}(t^*, h, x^*) - \mathcal{L}^{t,x} q(t^*, h, x^*) - \bar{f}(t^*, h, x^*, u_n, q)$$
$$- n\left(u_n(t^*, h, x^*) - L(t^*, x^*)\right)^- + n\left(u_n(t^*, h, x^*) - V(t^*, x^*)\right)^+ \leq 0,$$

Let $n \to 0$, it follows that (4.25), therefore, u is a viscosity subsolution.

Step 2. Then we prove that u is a viscosity supersolution.

Let $q \in \mathcal{C}^2([0, T + T^\delta] \times \mathbb{R})$, $(t^*, x^*) \in [0, T + T^\delta) \times \mathbb{R}$ is a global minimum point of $u - q$. If $u(t^*, h, x^*) = V(t^*, x^*)$, it follows (4.26).

Now we prove that if $u(t^*, h, x^*) < V(t^*, x^*)$, then

$$-\frac{\partial q}{\partial_t}(t, h, x) - \mathcal{L}^{t,x} q(t, h, x) - \bar{f}(t, h, x, u, q) \geq 0, \tag{4.35}$$

where $\bar{f}$ is defined in (4.27). Since $u_n(t, h, x) \to u(t, h, x)$ uniformly in compact set of $[0, T + T^\delta] \times \mathbb{R}$, there exists $N \in \mathbb{N}$, for all $n \geq N$, $u_n(t^*, h, x^*) \leq V(t^*, x^*)$, and (t^*, x^*) is a minimum point of $u_n - q$, such that

$$-\frac{\partial q}{\partial_t}(t^*, h, x^*) - \mathcal{L}^{t,x} q(t^*, h, x^*) - \bar{f}(t^*, h, x^*, u_n, q)$$
$$- n\left(u_n(t^*, h, x^*) - L(t^*, x^*)\right)^- + n\left(u_n(t^*, h, x^*) - V(t^*, x^*)\right)^+ \geq 0,$$

Let $n \to 0$, it follows that (4.26), therefore, u is a viscosity supersolution. So we prove that u is a viscosity solution of the parabolic obstacle problem (4.24). □

Chapter 5

Numerical Algorithms for RABSDEs with Two Obstacles and Default Risk

In this chapter, we study numerical algorithms for RABSDEs with two obstacles driven by a Brownian motion and a mutually independent martingale in a defaultable setting. The generator of a RABSDE includes the present and future values of the solution. We introduce two main algorithms, a discrete penalization scheme and a discrete reflected scheme both based on a random walk approximation of the Brownian motion as well as a discrete approximation of the default martingale, and we study these two methods in both the implicit and explicit versions respectively. We give the convergence results of the algorithms, provide a numerical example and an application in American game options in order to illustrate the performance of the algorithms.

This chapter is organized as follows, we first introduce the discrete time framework in Section 5.1. We study the implicit and the explicit methods of two discrete schemes, i.e. the discrete penalization scheme in Section 5.2 and the discrete reflected scheme in Section 5.3. Section 5.4 completes the convergence results of the numerical algorithms which were provided in the previous sections. In Section 5.5 and Section 5.6, we illustrate the performance of the algorithms by a simulation example and an application in American game options in the defaulable setting.

Consider the RABSDE (5.1) below with two obstacles and default risk with coefficient (f,ξ,δ,L,V). $(Y,Z,U,K^+,K^-) := (Y_t,Z_t,U_t,K_t^+,K_t^-)_{0\le t\le T+\delta}$ is a solution for RABSDE with the generator f, the terminal value ξ_T, the anticipated processes ξ, the anticipated time δ ($\delta > 0$ is a constant), and the obstacles L and V, such that

$$\begin{cases} (i) & Y \in \mathcal{S}^2_{\mathcal{G}}(0,T+\delta;\mathbb{R}),\ \ Z \in \mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R}^d), \\ & U \in \mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R}^k),\ \ K^{\pm} \in \mathcal{A}^2_{\mathcal{G}}(0,T;\mathbb{R}); \\ (ii) & Y_t = \xi_T + \int_t^T f\big(s,Y_s,\mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}],Z_s,U_s\big)ds + (K_T^+ - K_t^+) \\ & \qquad -(K_T^- - K_t^-) - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, \qquad t\in[0,T]; \\ (iii) & V_t \ge Y_t \ge L_t, \qquad t\in[0,T]; \\ (iv) & Y_t = \xi_t, \qquad t\in(T,T+\delta]; \\ (v) & \int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (V_t - Y_t)dK_t^- = 0. \end{cases} \tag{5.1}$$

where K^+ and K^- are continuous increasing processes, therefore the jumps of the process Y are only from the default part. K^+ is to keep Y above the lower obstacle L, while K^- is to keep Y under the upper obstacle V.

Suppose that the terminal value ξ and the generator f satisfy the assumptions **H 3.1** and **H** 3.2, $f(t, y, \bar{y}_r, z, u)$ is increasing in $\bar{y}$, the obstacles L and V satisfy **H** 4.1. When $U \equiv +\infty$ ($L \equiv -\infty$), the corresponding RABSDE becomes a RABSDE with one lower obstacle L (upper obstacle U). In this chapter, we consider the following special case:

H 5.1. *Assume that the obstacles L and V are Itô processes with the following forms:*

$$L_t = L_0 + \int_0^t l_s^{(1)} ds + \int_0^t l_s^{(2)} dB_s + \int_0^t l_s^{(3)} dM_s;$$
$$V_t = V_0 + \int_0^t v_s^{(1)} ds + \int_0^t v_s^{(2)} dB_s + \int_0^t v_s^{(3)} dM_s,$$

where the processes $l^{(1)}$ and $v^{(1)}$ are rcll, the processes $l^{(2)}$, $l^{(3)}$, $v^{(2)}$ and $v^{(3)}$ are predictable, moreover, $\mathbb{E}\big[\int_0^T \big(|l_t^{(1)}|^2 + |l_t^{(2)}|^2 + |l_t^{(3)}|^2 + |v_t^{(1)}|^2 + |v_t^{(2)}|^2 + |v_t^{(3)}|^2\big)dt\big] < \infty$.

Remark 5.0.1. *If $V \geq L$, we can easily check that the assumption **H** 4.1 is satisfied. We can just set $X = L$ ($X = U$), $\sigma^{(1)} = l^{(2)}$ ($\sigma^{(1)} = v^{(2)}$), $\sigma^{(2)} = l^{(3)}$ ($\sigma^{(2)} = v^{(3)}$), $A^{\pm} = \int_0 l_s^{(1),\pm} ds$ ($A^{\pm} = \int_0 v_s^{(1),\pm} ds$), where $l^{(1),\pm}$ is the positive or negative part of $l^{(1)}$.*

5.1 Discrete Time Framework

The basic idea is to approximate the Brownian motion by a random walk approximation based on the binomial tree model, and a discrete approximation of the default martingale. In order to discretize $[0, T]$, for $n \in \mathbb{N}$, we introduce $\Delta^n := \frac{T}{n}$ and an equidistant time grid $(t_i)_{i=0,1,\ldots n,\ldots n^\delta}$ with step size Δ^n, where $t_i := i\Delta^n$, $n^\delta = n + \left[\frac{\delta}{\Delta^n}\right]$.

5.1.1 Random Walk Approximation of the Brownian Motion

We use a random walk to approximate the 1-dimensional standard Brownian motion:

$$\begin{aligned} B_0^n =& 0; \\ B_t^n :=& \sqrt{\Delta^n} \sum_{j=1}^{[t/\Delta^n]} \epsilon_j^n, \qquad t \in (0, T]; \\ \Delta B_i^n :=& B_i^n - B_{i-1}^n = \sqrt{\Delta^n} \epsilon_i^n, \qquad i \in [1, n^\delta], \end{aligned}$$

where $(\epsilon_i^n)_{i=1,\ldots n}$ is a $\{-1, 1\}$-value i.i.d. Bernoulli sequence with $\mathbb{P}(\epsilon_i^n = 1) = \mathbb{P}(\epsilon_i^n = -1) = \frac{1}{2}$. Denote $\mathcal{F}_i^n = \sigma\{\epsilon_1^n, \ldots \epsilon_i^n\}$, for any $i \in [1, n^\delta]$. By Donsker's invariance principle (Theorem A.2.1) and Skorokhod representation theorem (Theorem A.2.2), there exists a probability space, such that $\sup_{0 \leq t \leq T+\delta} |B_t^n - B_t| \to 0$, in $\mathcal{L}^2(\mathcal{G}_{T+\delta})$, as $n \to \infty$ (since $\epsilon_j \in \mathcal{L}^{2+\Delta^n}$), here $\mathcal{L}^{2+\Delta^n}$ is the space of the random variables satisfying $\mathbb{E}[\epsilon^{2+\Delta^n}] < \infty$.

5.1.2 Approximation of the Defaultable Model

We consider a defaultable model of a single uniformly distributed random default time $\tau \in (0, T]$. We define the discrete default process $h_i^n = h_{t_i}^n = 1_{\{\tau \leq t_i\}}$ ($i \in [1, n]$). Particularly, when $i \in [n+1, n^\delta]$, $h_i^n = 1$ (since default case already happened). We have the conditional expectations of h_i^n under $\mathcal{G}_{i-1}^n$:

$$\begin{aligned}\mathbb{E}\left[h_i^n = 1|h_{i-1}^n = 1\right] =& \mathbb{P}(\tau \le t_i|\tau \le t_{i-1}) = 1, && i \in [1,n];\\ \mathbb{E}\left[h_i^n = 1|h_{i-1}^n = 0\right] =& \mathbb{P}(\tau \le t_i|\tau > t_{i-1}) = \frac{\Delta^n}{T-t_{i-1}}, && i \in [1,n];\\ \mathbb{E}\left[h_i^n = 0|h_{i-1}^n = 0\right] =& \mathbb{P}(\tau > t_i|\tau > t_{i-1}) = \frac{T-t_i}{T-t_{i-1}}, && i \in [1,n].\end{aligned}$$

We have the following approximation for the discrete martingale M_t^n directly based on the definition of the martingale M (**H 1.1**):

$$\begin{aligned}M_0^n =& 0;\\ M_t^n :=& h_{[t/\Delta^n]}^n - \Delta^n \sum_{j=1}^{[t/\Delta^n]} (1-h_j^n)\gamma_j^n, && t \in (0,T];\\ \Delta M_i^n :=& h_i^n - h_{i-1}^n - \Delta^n(1-h_i^n)\gamma_i^n, && i \in [1,n],\end{aligned} \tag{5.2}$$

where the discrete intensity process $\gamma_i^n = \gamma_{t_i}^n \ge 0$ is an $\mathcal{F}_i^n$-adapted process. Denote $\mathbb{G} := \{\mathcal{G}_i^n;\ i \in [0,n^\delta]\}$, $\mathcal{G}_0^n = \{\Omega,\varnothing\}$, for any $i \in [1,n]$, $\mathcal{G}_i^n = \sigma\{\epsilon_1^n,...\epsilon_i^n,h_i^n\}$; for any $i \in [n+1,n^\delta]$, $\mathcal{G}_i^n = \sigma\{\epsilon_1^n,...\epsilon_i^n,h_n^n\}$, where h_i is independent from ϵ_1^n,...ϵ_i^n. From the martingale property of M_i, we can get

$$\mathbb{E}^{\mathcal{G}_{i-1}^n}\left[\Delta M_i^n\right] = \mathbb{E}^{\mathcal{G}_{i-1}^n}\left[h_i^n - h_{i-1}^n - \Delta^n(1-h_i^n)\gamma_i^n\right] = 0, \quad i \in [1,n],$$

therefore, the discrete intensity process has the following form (by the projection on $\mathcal{F}_{i-1}^n$):

$$\gamma_i^n = \frac{\mathbb{P}(t_{i-1} < \tau \le t_i|\mathcal{F}_{i-1}^n)}{\Delta^n\,\mathbb{P}(\tau > t_i|\mathcal{F}_{i-1}^n)} = \frac{1}{T-t_i}, \qquad i \in [1,\left[\frac{\tau}{\Delta^n}\right]].$$

Note that $\gamma_i^n = 0$, when $i = 0$ and $i \in [\left[\frac{\tau}{\Delta^n}\right]+1,n]$. If we set $\hat{\gamma}_t^n = \gamma_{[t/\Delta^n]}^n$ ($t \in [0,T]$), then as $n \to \infty$, from Remark 1.3.1 about the intensity process, it follows that $\hat{\gamma}_t^n$ converges to γ_t.

5.1.3 Approximations of the Anticipated Processes and the Generator

Consider the approximation ξ_n^n of the terminal value ξ, we have the following assumption:

H 5.2. $(\xi_i^n)_{i\in[n,n^\delta]}$ *is* $\mathcal{G}_i^n$*-measurable,* $\Psi : \{1,-1\}^n \to \mathbb{R}$ *is a real analytic function, such that*

$$\xi_i^n = \Psi\left(\epsilon_1^n,...\epsilon_i^n,h_i^n\right), \qquad i \in [n,n^\delta],$$

particularly, the terminal value $\xi_n^n = \Psi\left(\epsilon_1^n,...\epsilon_n^n,h_i^n\right)$ *is* $\mathcal{G}_n^n$*-measurable.*

For the approximation $(f^n(t_i,y,\bar{y},z,u))_{i\in[0,n]}$ of the generator f, we introduce the assumption below:

H 5.3. *for any* $i \in [0,n]$, $f^n(t_i,y,\bar{y},z,u)$ *is* $\mathcal{G}_i^n$*-adapted, and satisfies:*

(a) *there exists a constant $C > 0$, such that for all $n > 1 + 2L + 4L^2$,*

$$\mathbb{E}\left[\Delta^n \sum_{i=0}^{n-1} |f^n(\cdot, 0, 0, 0, 0)|^2\right] < C.$$

(b) *for any $i \in [0, n-1]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}$, $u, u' \in \mathbb{R}$, $\bar{y}, \bar{y}' \in \mathcal{S}^2_{\mathcal{G}}(t, T+\delta; \mathbb{R})$, there exists a constant $L \geq 0$, such that*

$$\begin{aligned}&|f^n(t_i, y, \bar{y}_i, z, u) - f^n(t_i, y', \bar{y}'_i, z', u')|\\ \leq & L\left(|y - y'| + \mathbb{E}^{\mathcal{G}_t}|\bar{y}_i - \bar{y}'_i| + |z - z'| + |u - u'| 1_{\{\tau > t_i\}} \sqrt{\gamma_i}\right),\end{aligned}$$

where $\bar{y}_i = \mathbb{E}^{\mathcal{G}^n_i}[y_{\bar{i}}]$, $\bar{i} = i + \left[\frac{\delta}{\Delta^n}\right]$.

As $n \to \infty$, $f^n(\left[\frac{t}{\Delta^n}\right], y, \bar{y}, z, u)$ converges to $f(t, y, \bar{y}, z, u)$ in $\mathcal{S}^2_{\mathcal{G}}(0, T+\delta; \mathbb{R})$.

5.1.4 Approximation of the Obstacles

$(L^n_i)_{i \in [0,n]}$ and $(V^n_i)_{i \in [0,n]}$ are the discrete versions of L and V, by Assumption 4.1, we can have the following approximations:

$$\begin{aligned} L^n_i &= L_0 + \Delta^n \sum_{j=0}^{i-1} l^{(1)}_j + \sum_{j=0}^{i-1} l^{(2)}_j \Delta B^n_{j+1} + \sum_{j=0}^{i-1} l^{(3)}_j \Delta M^n_{j+1};\\ V^n_i &= V_0 + \Delta^n \sum_{j=0}^{i-1} v^{(1)}_j + \sum_{j=0}^{i-1} v^{(2)}_j \Delta B^n_{j+1} + \sum_{j=0}^{i-1} v^{(3)}_j \Delta M^n_{j+1},\end{aligned}$$

where $l^{(k)}_j = l^{(k)}_{t_j}$, $v^{(k)}_j = v^{(k)}_{t_j}$ $(k = 1, 2, 3)$. By the Burkholder-Davis-Gundy inequality, it follows

$$V^n_i \geq L^n_i, \qquad \sup_n \mathbb{E}\left[\sup_i (L^n_i)^{+2} + \sup_i (V^n_i)^{-2}\right] < \infty.$$

We introduce the discrete version of assumption **H 4.1 (c)**:

H 5.4. *There exists a process X^n_i with the following form:*

$$X^n_i = X^n_0 - \sum_{j=0}^{i-1} \sigma^{(1)}_j \Delta B^n_{i+1} - \sum_{j=0}^{i-1} \sigma^{(2)}_j \Delta M^n_{i+1} + A^{+n}_i - A^{-n}_i,$$

where A^{+n}_i and A^{-n}_i are $\mathcal{G}^n_i$-adapted increasing processes, $\mathbb{E}\left[|A^{+n}_n|^2 + |A^{-n}_n|^2\right] < \infty$, such that

$$L^n_i \leq X^n_i \leq V^n_i, \qquad i \in [0, n].$$

We introduce two numerical algorithms below, discrete penalization scheme in Section 5.2 and discrete reflected scheme in Section 5.3. For each scheme, we study the implicit and explicit versions.

5.1.5 Computing the Conditional Expectations

When $i \in [1, n-1]$, we use the following formula to compute the conditional expectation for the function $f : \mathbb{R}^{i+2} \to \mathbb{R}$:

$$\begin{aligned}
&\mathbb{E}^{\mathcal{G}_i^n}\left[f\left(\epsilon_1^n, ..., \epsilon_{i+1}^n, h_{i+1}^n\right)\right] \\
=&\frac{1}{2}\Big(f\left(\epsilon_1^n, ..., \epsilon_i^n, 1, 1\right)|_{\epsilon_{i+1}^n=1, h_i^n=1} + f\left(\epsilon_1^n, ..., \epsilon_i^n, -1, 1\right)|_{\epsilon_{i+1}^n=-1, h_i^n=1}\Big) \\
&+ \frac{\Delta^n}{2(T-t_i)}\Big(f\left(\epsilon_1^n, ..., \epsilon_i^n, 1, 1\right)|_{\epsilon_{i+1}^n=1, h_i^n=0, h_{i+1}^n=1} \\
&\qquad + f\left(\epsilon_1^n, ..., \epsilon_i^n, -1, 1\right)|_{\epsilon_{i+1}^n=-1, h_i^n=0, h_{i+1}^n=1}\Big) \\
&+ \frac{T-t_{i+1}}{2(T-t_i)}\Big(f\left(\epsilon_1^n, ..., \epsilon_i^n, 1, 0\right)|_{\epsilon_{i+1}^n=1, h_i^n=0, h_{i+1}^n=0} \\
&\qquad + f\left(\epsilon_1^n, ..., \epsilon_i^n, -1, 0\right)|_{\epsilon_{i+1}^n=-1, h_i^n=0, h_{i+1}^n=0}\Big).
\end{aligned}$$

When $i \in [n, n^\delta]$, we have the following conditional expectation for the function $f : \mathbb{R}^{i+2} \to \mathbb{R}$:

$$\begin{aligned}
&\mathbb{E}^{\mathcal{G}_i^n}\left[f\left(\epsilon_1^n, ..., \epsilon_{i+1}^n, h_n^n\right)\right] \\
=&\frac{1}{2}f\left(\epsilon_1^n, ...\epsilon_i^n, 1, 1\right)|_{\epsilon_{i+1}^n=1} + \frac{1}{2}f\left(\epsilon_1^n, ...\epsilon_i^n, -1, 1\right)|_{\epsilon_{i+1}^n=-1}.
\end{aligned}$$

5.2 Discrete Penalization Scheme

We first use the methodology of penalization for the discrete scheme below. El Karoui et al. [33] (1997) proved the existence of RBSDE with one obstacle under a smooth square integrability assumption and Lipschitz condition through penalization method. Lepeltier and Martín [93] (2004) used the similar penalization method to prove the existence theorem of RBSDE with two obstacles and Poisson jump. Similarly to Lemma 4.2.1 in the Chapter 4, we consider the following special case of the penalized ABSDE for RABSDE (1.10):

$$\begin{aligned}
-dY_t^p =& f^n(t, Y_t^p, \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^p], Z_t^p, U_t^p)dt + dK_t^{+p} - dK_t^{-p} - Z_t^p dB_t - U_t^p dM_t, \quad t \in [0, T], \\
Y_t^p =& \xi_t, \qquad t \in [T, T+\delta],
\end{aligned} \tag{5.3}$$

where

$$K_t^{+p} = p\int_t^T (Y_s^p - L_s)^- ds, \qquad K_t^{-p} = p\int_t^T (Y_s^p - V_s)^+ ds.$$

By the existence and uniqueness theorem for ABSDEs with default risk (Theorem 3.2.3), there exists the unique solution for this penalized ABSDE (5.3). We will give the convergence of penalized ABSDE (5.3) to RABSDE (1.10) in Theorem 5.4.1 below.

5.2.1 Implicit Discrete Penalization Scheme

We first introduce the implicit discrete penalization scheme. In this scheme, p represents the penalization parameter. In practice, we can choose p which is independent of

n and much larger than n, this will be illustrated in the simulation Section 5.6.

$$\begin{cases} y_i^{p,n} = y_{i+1}^{p,n} + f^n(t_i, y_i^{p,n}, \bar{y}_i^{p,n}, z_i^{p,n}, u_i^{p,n})\Delta^n + k_i^{+p,n} - k_i^{-p,n} \\ \qquad -z_i^{p,n}\Delta B_{i+1}^n - u_i^{p,n}\Delta M_{i+1}^n, \qquad i \in [0, n-1]; \\ k_i^{+p,n} = p\Delta^n(y_i^{p,n} - L_i^n)^-, \qquad i \in [0, n-1]; \\ k_i^{-p,n} = p\Delta^n(y_i^{p,n} - V_i^n)^+, \qquad i \in [0, n-1]; \\ y_i^{p,n} = \xi_i^n, \qquad i \in [n, n^\delta], \end{cases} \tag{5.4}$$

where $\bar{y}_i^{p,n} = \mathbb{E}^{\mathcal{G}_i^n}[y_{\bar{i}}^{p,n}]$, $\bar{i} = i + [\frac{\delta}{\Delta^n}]$.

For the theoretical convergence results in Section 5.4, we first prove the convergence (Theorem 5.4.2) of implicit discrete penalization scheme (5.4) to the penalized ABSDE (5.3), then combining with Theorem A.2.3, we can get the convergence of the explicit discrete penalization scheme. By Theorem A.2.4 and Theorem 5.4.1, we can prove the convergence of the implicit discrete reflected scheme (5.10).

From Section 5.1.5, taking conditional expectation under $\mathcal{G}_i^n$, we can calculate $\bar{y}_i^{p,n}$ as follows:

$$\mathbb{E}^{\mathcal{G}_i^n}[y_{\bar{i}}^{p,n}] = \begin{cases} \frac{1}{2}\left(y_{\bar{i}}^{p,n}\Big|_{\epsilon_{\bar{i}}^n=1, h_i^n=1} + y_{\bar{i}}^{p,n}\Big|_{\epsilon_{\bar{i}}^n=-1, h_i^n=1}\right) \\ + \frac{\delta}{2(T-t_i)}\left(y_{\bar{i}}^{p,n}\Big|_{\epsilon_{\bar{i}}^n=1, h_i^n=0, h_{\bar{i}}^n=1} + y_{\bar{i}}^{p,n}\Big|_{\epsilon_{\bar{i}}^n=-1, h_i^n=0, h_{\bar{i}}^n=1}\right) \\ + \frac{T-t_i-\delta}{2(T-t_i)}\left(y_{\bar{i}}^{p,n}\Big|_{\epsilon_{\bar{i}}^n=1, h_i^n=0, h_{\bar{i}}^n=0} + y_{\bar{i}}^{p,n}\Big|_{\epsilon_{\bar{i}}^n=-1, h_i^n=0, h_{\bar{i}}^n=0}\right), & \bar{i} \in [i, n-1]; \\ \frac{1}{2}\left(\xi_{\bar{i}}^n\Big|_{\epsilon_{\bar{i}}^n=1} + \xi_{\bar{i}}^n\Big|_{\epsilon_{\bar{i}}^n=-1}\right), & \bar{i} \in [n, n^\delta]. \end{cases} \tag{5.5}$$

Similarly, $z_i^{p,n}$ and $u_i^{p,n}$ ($i \in [0, n-1]$) are given by

$$\begin{aligned} z_i^{p,n} =& \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^{p,n}\Delta B_{i+1}^n\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(\Delta B_{i+1}^n)^2]} = \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^{p,n}\sqrt{\Delta^n}\epsilon_{i+1}^n\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(\sqrt{\Delta^n}\epsilon_{i+1}^n)^2]} \\ =& \frac{1}{\sqrt{\Delta^n}}\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^{p,n}\epsilon_{i+1}^n\right] = \frac{y_{i+1}^{p,n}\Big|_{\epsilon_{i+1}=1} - y_{i+1}^{p,n}\Big|_{\epsilon_{i+1}=-1}}{2\sqrt{\Delta^n}}; \\ u_i^{p,n} =& \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^{p,n}\Delta M_{i+1}^n\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(\Delta M_{i+1}^n)^2]} \\ =& \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^{p,n}(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1})\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1})^2]} \\ =& \frac{y_{i+1}^{p,n}\Delta^n\Big|_{h_i=0, h_{i+1}=1} - y_{i+1}^{p,n}(T-t_{i+1})\Delta^n\gamma_{i+1}\Big|_{h_i=0, h_{i+1}=0}}{\Delta^n + (T-t_{i+1})(\Delta^n\gamma_{i+1})^2}. \end{aligned} \tag{5.6}$$

Note that $u_i^{p,n}$ only exists on $[0, [\frac{\tau}{\Delta^n}] - 1]$ (i.e. before the default event happens). By taking the conditional expectation of (5.4) in $\mathcal{G}_i^n$, it follows:

$$\begin{cases} y_i^{p,n} = (\Phi^{p,n})^{-1}\left(\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n]\right), & i \in [0, n-1]; \\ k_i^{+p,n} = p\Delta^n (y_i^{p,n} - L_i^n)^-, & i \in [0, n-1]; \\ k_i^{-p,n} = p\Delta^n (y_i^{p,n} - V_i^n)^+, & i \in [0, n-1]; \\ y_i^{p,n} = \xi_i^n, & i \in [n, n^\delta]; \\ z_i^{p,n} = \frac{1}{\sqrt{\Delta^n}}\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^{p,n}\epsilon_{i+1}^n\right], & i \in [0, n-1]; \\ u_i^{p,n} = \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^{p,n}\left(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1}\right)\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1})^2]}, & i \in [0, [\frac{\tau}{\Delta^n}] - 1]. \end{cases} \tag{5.7}$$

where $\Phi^{p,n}(y) = y - f^n(t_i, y, \bar{y}_i^{p,n}, z_i^{p,n}, u_i^{p,n})\Delta^n - p\Delta^n(y - L_i^n)^- + p\Delta^n(y - V_i^n)^+$. For the continuous time version $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n}, K_t^{+p,n}, K_t^{-p,n})_{0\le t\le T}$:

$$Y_t^{p,n} := y_{[t/\Delta^n]}^{p,n}, \qquad Z_t^{p,n} := z_{[t/\Delta^n]}^{p,n}, \qquad U_t^{p,n} := u_{[t/\Delta^n]}^{p,n},$$
$$K_t^{+p,n} := \sum_{i=0}^{[t/\Delta^n]} k_i^{+p,n}, \qquad K_t^{-p,n} := \sum_{i=0}^{[t/\Delta^n]} k_i^{-p,n}.$$

5.2.2 Explicit Discrete Penalization Scheme

In many cases, the inverse of mapping Φ is not easy to get directly, for example, if f is not a linear function with respect to y. So we introduce the following explicit discrete penalization scheme, we replace $y_i^{p,n}$ in f^n by $\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^{p,n}]$ in (5.4), it follows

$$\begin{cases} \tilde{y}_i^{p,n} = \tilde{y}_{i+1}^{p,n} + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}], \bar{\tilde{y}}_i^{p,n}, \tilde{z}_i^{p,n}, \tilde{u}_i^{p,n})\Delta^n + \tilde{k}_i^{+p,n} - \tilde{k}_i^{-p,n} \\ \qquad -\tilde{z}_i^{p,n}\Delta B_{i+1}^n - \tilde{u}_i^{p,n}\Delta M_{i+1}^n, & i \in [0, n-1]; \\ \tilde{k}_i^{+p,n} = p\Delta^n(\tilde{y}_i^{p,n} - L_i^n)^-, & i \in [0, n-1]; \\ \tilde{k}_i^{-p,n} = p\Delta^n(\tilde{y}_i^{p,n} - V_i^n)^+, & i \in [0, n-1]; \\ \tilde{y}_i^{p,n} = \xi_i^n, & i \in [n, n^\delta], \end{cases} \tag{5.8}$$

where $\bar{\tilde{y}}_i^{p,n} = \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{\bar{i}}^{p,n}]$, $\bar{i} = i + \left[\frac{\delta}{\Delta^n}\right]$. $\tilde{y}_i^{p,n}$, $\tilde{z}_i^{p,n}$ and $\tilde{u}_i^{p,n}$ can be calculated as (5.5) and (5.6). From Section 5.1.5, we can computer $\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^{p,n}]$ as follows:

$$\begin{aligned} \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}] =& \frac{1}{2}\left(\tilde{y}_{i+1}^{p,n}|_{\epsilon_{i+1}^n=1, h_i^n=1} + \tilde{y}_{i+1}^{p,n}|_{\epsilon_{i+1}^n=-1, h_i^n=1}\right) \\ &+ \frac{\Delta^n}{2(T-t_i)}\left(\tilde{y}_{i+1}^{p,n}|_{\epsilon_{i+1}^n=1, h_i^n=0, h_{i+1}^n=1} + \tilde{y}_{i+1}^{p,n}|_{\epsilon_{i+1}^n=-1, h_i^n=0, h_{i+1}^n=1}\right) \\ &+ \frac{T-t_{i+1}}{2(T-t_i)}\left(\tilde{y}_{i+1}^{p,n}|_{\epsilon_{i+1}^n=1, h_i^n=0, h_{i+1}^n=0} + \tilde{y}_{i+1}^{p,n}|_{\epsilon_{i+1}^n=-1, h_i^n=0, h_{i+1}^n=0}\right). \end{aligned}$$

By taking the conditional expectation of (5.8) under $\mathcal{G}_i^n$, we have the following explicit penalization scheme:

$$\begin{cases} \tilde{y}_i^{p,n} = \mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^{p,n}\right] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}], \bar{\tilde{y}}_i^{p,n}, \tilde{z}_i^{p,n}, \tilde{u}_i^{p,n})\Delta^n \\ \qquad + \tilde{k}_i^{+p,n} - \tilde{k}_i^{-p,n}, \qquad i \in [0, n-1]; \\ \tilde{k}_i^{+p,n} = \frac{p\Delta^n}{1+p\Delta^n}\Big(\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^{p,n}\right] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}], \bar{\tilde{y}}_i^{p,n}, \tilde{z}_i^{p,n}, \tilde{u}_i^{p,n})\Delta^n \\ \qquad - L_i^n\Big)^-, \qquad i \in [0, n-1]; \\ \tilde{k}_i^{-p,n} = \frac{p\Delta^n}{1+p\Delta^n}\Big(\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^{p,n}\right] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}], \bar{\tilde{y}}_i^{p,n}, \tilde{z}_i^{p,n}, \tilde{u}_i^{p,n})\Delta^n \\ \qquad - V_i^n\Big)^+, \qquad i \in [0, n-1]; \\ \tilde{y}_i^{p,n} = \xi_i^n, \qquad i \in [n, n^\delta]; \\ \tilde{z}_i^{p,n} = \frac{1}{2\sqrt{\Delta^n}}\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^{p,n}\epsilon_{i+1}^n\right], \qquad i \in [0, n-1]; \\ \tilde{u}_i^{p,n} = \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^{p,n}\left(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1}\right)\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1})^2]}, \qquad i \in [0, [\frac{\tau}{\Delta^n}] - 1]. \end{cases} \tag{5.9}$$

For the continuous time version $(\tilde{Y}_t^{p,n}, \tilde{Z}_t^{p,n}, \tilde{U}_t^{p,n}, \tilde{K}_t^{+p,n}, \tilde{K}_t^{-p,n})_{0\le t\le T}$:

$$\tilde{Y}_t^{p,n} := \tilde{y}_{[t/\Delta^n]}^{p,n}, \qquad \tilde{Z}_t^{p,n} := \tilde{z}_{[t/\Delta^n]}^{p,n}, \qquad \tilde{U}_t^{p,n} := \tilde{u}_{[t/\Delta^n]}^{p,n},$$

$$\tilde{K}_t^{+p,n} := \sum_{i=0}^{[t/\Delta^n]} \tilde{k}_i^{+p,n}, \qquad \tilde{K}_t^{-p,n} := \sum_{i=0}^{[t/\Delta^n]} \tilde{k}_i^{-p,n}.$$

Remark 5.2.1. *We give the following explanations of the derivation of $\tilde{k}_i^{+p,n}$ and $\tilde{k}_i^{-p,n}$:*

- *If $V_i^n > \tilde{y}_i^{p,n} > L_i^n$, we can get $\tilde{k}_i^{+p,n} = \tilde{k}_i^{-p,n} = 0$;*
- *if $\tilde{y}_i^{p,n} \le L_i^n$, we can get $\tilde{k}_i^{+p,n} = \frac{p\Delta^n}{1+p\Delta^n}\big(\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^{p,n}\right] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}], \bar{\tilde{y}}_i^{p,n}, \tilde{z}_i^{p,n}, \tilde{u}_i^{p,n})\Delta^n - L_i^n\big)^-$ and $\tilde{k}_i^{-p,n} = 0$. From (5.9), we know that p should be much larger than n to keep $\tilde{y}_i^{p,n}$ above the lower obstacle L_i^n;*
- *if $\tilde{y}_i^{p,n} \ge V_i^n$, we can get $\tilde{k}_i^{-p,n} = \frac{p\Delta^n}{1+p\Delta^n}\big(\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^{p,n}\right] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^{p,n}], \bar{\tilde{y}}_i^{p,n}, \tilde{z}_i^{p,n}, \tilde{u}_i^{p,n})\Delta^n - V_i^n\big)^+$ and $\tilde{k}_i^{+p,n} = 0$. From (5.9), we know that p should be much larger than n to keep $\tilde{y}_i^{p,n}$ under the upper obstacle V_i^n.*

5.3 Discrete Reflected Scheme

We can obtain the solution Y by reflecting between the two obstacles and get the increasing processes K^+ and K^- directly. So we can see clearly how the increasing processes work during the time interval $[0, T]$.

5.3.1 Implicit Discrete Reflected Scheme

We have the following implicit discrete reflected scheme,

$$\begin{cases} y_i^n = y_{i+1}^n + f^n(t_i, y_i^n, \bar{y}_i^n, z_i^n, u_i^n)\Delta^n + k_i^{+n} - k_i^{-n} \\ \qquad -z_i^n \Delta B_{i+1}^n - u_i^n \Delta M_{i+1}^n, \qquad i \in [0, n-1]; \\ V_i^n \geq y_i^n \geq L_i^n, \qquad i \in [0, n-1]; \\ k_i^{+n} \geq 0, \quad k_i^{-n} \geq 0, \quad k_i^{+n} k_i^{-n} = 0, \qquad i \in [0, n-1]; \\ (y_i^n - L_i^n) k_i^{+n} = (y_i^n - V_i^n) k_i^{-n} = 0, \qquad i \in [0, n-1]; \\ y_i^n = \xi_i^n, \qquad i \in [n, n^\delta]. \end{cases} \tag{5.10}$$

where $\bar{y}_i^n = \mathbb{E}^{\mathcal{G}_i^n}[y_{\bar{i}}^n]$, $\bar{i} = i + \left[\frac{\delta}{\Delta^n}\right]$. $\bar{y}_i^n$, z_i^n and u_i^n can be calculated as (5.5) and (5.6). By taking conditional expectation of (5.10) under $\mathcal{G}_i^n$, it follows

$$\begin{cases} y_i^n = \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] + f^n(t_i, y_i^n, \bar{y}_i^n, z_i^n, u_i^n)\Delta^n + k_i^{+n} - k_i^{-n}, \quad i \in [0, n-1]; \\ V_i^n \geq y_i^n \geq L_i^n, \qquad i \in [0, n-1]; \\ k_i^{+n} \geq 0, \quad k_i^{-n} \geq 0, \quad k_i^{+n} k_i^{-n} = 0, \qquad i \in [0, n-1]; \\ (y_i^n - L_i^n) k_i^{+n} = (y_i^n - V_i^n) k_i^{-n} = 0, \qquad i \in [0, n-1]; \\ y_i^n = \xi_i^n, \qquad i \in [n, n^\delta]; \\ z_i^n = \frac{1}{\sqrt{\Delta^n}} \mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^n \epsilon_{i+1}^n\right], \qquad i \in [0, n-1]; \\ u_i^n = \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^n \left(h_{i+1}^n - h_i^n - \Delta^n (1 - h_{i+1}^n)\gamma_{i+1}\right)\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(h_{i+1}^n - h_i^n - \Delta^n (1 - h_{i+1}^n)\gamma_{i+1})^2]}, \qquad i \in [0, [\frac{\tau}{\Delta^n}] - 1]. \end{cases} \tag{5.11}$$

Lemma 5.3.1. *If Δ^n is small enough, (5.11) is equivalent to*

$$\begin{cases} y_i^n = \Phi^{-1}\left(\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] + k_i^{+n} - k_i^{-n}\right), \qquad i \in [0, n-1]; \\ k_i^{+n} = \left(\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] + f^n(t_i, L_i^n, \bar{L}_i^n, z_i^n, u_i^n)\Delta^n - L_i^n\right)^-, \quad i \in [0, n-1]; \\ k_i^{-n} = \left(\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] + f^n(t_i, V_i^n, \bar{V}_i^n, z_i^n, u_i^n)\Delta^n - V_i^n\right)^+, \quad i \in [0, n-1]; \\ y_i^n = \xi_i^n, \qquad i \in [n, n^\delta]; \\ z_i^n = \frac{1}{2\sqrt{\Delta^n}} \mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^n \epsilon_{i+1}^n\right], \qquad i \in [0, n-1]; \\ u_i^n = \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[y_{i+1}^n \left(h_{i+1}^n - h_i^n - \Delta^n (1 - h_{i+1}^n)\gamma_{i+1}\right)\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(h_{i+1}^n - h_i^n - \Delta^n (1 - h_{i+1}^n)\gamma_{i+1})^2]}, \qquad i \in [0, [\frac{\tau}{\Delta^n}] - 1]. \end{cases} \tag{5.12}$$

here $\Phi(y) = y - f^n(t_i, y, \bar{y}_i^n, z_i^n, u_i^n)\Delta^n$.

Proof. Similarly to Section 4.1 in Xu [63], for any $y \neq y'$, since f satisfies the Lipschitz condition, we can obtain

$$\left[\Phi(y) - \Phi(y')\right](y - y') \geq (1 - \Delta^n L)(y - y')^2 > 0.$$

When Δ^n is small enough, it follows that Φ is invertible and strictly increasing in y. Therefore,

$$y \geq L_i^n \Longleftrightarrow \Phi(y) \geq \Phi(L_i^n);$$
$$y \leq V_i^n \Longleftrightarrow \Phi(y) \leq \Phi(V_i^n).$$

Step 1. (5.11) $\Rightarrow$ (5.12).

If $V_i^n > L_i^n$, we can obtain that $\{y_i^n - L_i^n = 0\}$ and $\{y_i^n - V_i^n = 0\}$ are disjoint.

When $y_i^n > L_i^n$, it follows that $k_i^{+n} = 0$, from the monotonicity of Φ,

$$\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] - \Phi(L_i^n) = \Phi(y_i^n) - \Phi(L_i^n) + k_i^{-n} > 0,$$

When $y_i^n = L_i^n$, it follows that $k_i^{-n} = 0$. On the other hand,

$$\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] - \Phi(L_i^n) = \Phi(y_i^n) - \Phi(L_i^n) - k_i^{+n} \leq 0,$$

hence,

$$k_i^{+n} = \Phi(L_i^n) - \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] = \left(\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] - \Phi(L_i^n)\right)^- .$$

Moreover, we assume that $V_i^n = L_i^n$, therefore, $V_i^n = L_i^n = y_i^n$, it follows that $k_i^{+n} = 0$ or $k_i^{-n} = 0$.

If $k_i^{+n} = k_i^{-n} = 0$, we can get

$$\Phi(y_i^n) = \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] = \Phi(L_i^n) = \Phi(V_i^n).$$

If $k_i^{+n} > 0$ and $k_i^{-n} = 0$, it follows

$$k_i^{+n} = \Phi(y_i^n) - \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] = \Phi(L_i^n) - \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] = \left(\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] - \Phi(L_i^n)\right)^- .$$

Similarly,

$$k_i^{-n} = \left(\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] - \Phi(V_i^n)\right)^+ .$$

It follows (5.12).

Step 2: (5.12) $\Rightarrow$ (5.11).

If $k_i^{+n} > 0$, we can obtain that

$$\Phi(V_i^n) \geq \Phi(L_i^n) > \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n],$$

hence, $k_i^{-n} = 0$. It follows that $k_i^{+n} k_i^{-n} = 0$. So we can obtain

$$\Phi(y_i^n) = \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] + k_i^{+n} = \Phi(L_i^n).$$

Therefore, $y_i^n = L_i^n$ (Φ is a one to one map), then $(y_i^n - L_i^n)k_i^{+n} = 0$. Similarly, we can prove that $(y_i^n - V_i^n)k_i^{-n} = 0$.

Then we prove that $y_i^n \geq L_i^n$. We assume that $y_i^n < L_i^n$, when $k_i^n = 0$, it follows that $\mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] \geq \Phi(L_i^n)$. Then $\Phi(y_i^n) = \mathbb{E}^{\mathcal{G}_i^n}[y_{i+1}^n] \geq \Phi(L_i^n)$. But Φ is non decreasing, so it leads to absurdity. □

For the continuous time version $(Y_t^n, Z_t^n, U_t^n, K_t^{+n}, K_t^{-n})_{0\leq t\leq T}$:

$$Y_t^n := y_{[t/\Delta^n]}^n, \qquad Z_t^n := z_{[t/\Delta^n]}^n, \qquad U_t^n := u_{[t/\Delta^n]}^n,$$

$$K_t^{+n} := \sum_{i=0}^{[t/\Delta^n]} k_i^{+n}, \qquad K_t^{-n} := \sum_{i=0}^{[t/\Delta^n]} k_i^{-n}.$$

5.3.2 Explicit Discrete Reflected Scheme

We introduce the following explicit discrete reflected scheme by replacing y_i^n in the generator f^n by $\mathbb{E}[y_{i+1}^n|\mathcal{G}_i^n]$ in (5.10).

$$\begin{cases} \tilde{y}_i^n = \tilde{y}_{i+1}^n + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n], \bar{\tilde{y}}_i^n, \tilde{z}_i^n, \tilde{u}_i^n)\Delta^n + \tilde{k}_i^{+n} - \tilde{k}_i^{-n} \\ \qquad -\tilde{z}_i^n \Delta B_{i+1}^n - \tilde{u}_i^n \Delta M_{i+1}^n, \qquad i \in [0, n-1]; \\ V_i^n \geq \tilde{y}_i^n \geq L_i^n, \qquad i \in [0, n-1]; \\ \tilde{k}_i^{+n} \geq 0, \quad \tilde{k}_i^{-n} \geq 0, \quad \tilde{k}_i^{+n}\tilde{k}_i^{-n} = 0, \qquad i \in [0, n-1]; \\ (\tilde{y}_i^n - L_i^n)\tilde{k}_i^{+n} = (\tilde{y}_i^n - V_i^n)\tilde{k}_i^{-n} = 0, \qquad i \in [0, n-1]; \\ \tilde{y}_i^n = \xi_i^n, \qquad i \in [n, n^\delta]. \end{cases} \tag{5.13}$$

where $\bar{\tilde{y}}_i^n = \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{\bar{i}}^n]$, $\bar{i} = i + \left[\frac{\delta}{\Delta^n}\right]$. $\bar{\tilde{y}}_i^n$, $\tilde{z}_i^n$ and $\tilde{u}_i^n$ can be calculated as (5.5) and (5.6). By taking conditional expectation of (5.13) under $\mathcal{G}_i^n$:

$$\begin{cases} \tilde{y}_i^n = \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n], \bar{\tilde{y}}_i^n, \tilde{z}_i^n, \tilde{u}_i^n)\Delta^n \\ \qquad +\tilde{k}_i^{+n} - \tilde{k}_i^{-n}, \qquad i \in [0, n-1]; \\ V_i^n \geq \tilde{y}_i^n \geq L_i^n, \qquad i \in [0, n-1]; \\ \tilde{k}_i^{+n} \geq 0, \quad \tilde{k}_i^{-n} \geq 0, \quad \tilde{k}_i^{+n}k_i^{-n} = 0, \qquad i \in [0, n-1]; \\ (\tilde{y}_i^n - L_i^n)\tilde{k}_i^{+n} = (\tilde{y}_i^n - V_i^n)\tilde{k}_i^{-n} = 0, \qquad i \in [0, n-1]; \\ \tilde{y}_i^n = \xi_i^n, \qquad i \in [n, n^\delta]; \\ \tilde{z}_i^n = \frac{1}{2\sqrt{\Delta^n}}\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^n \epsilon_{i+1}^n\right], \qquad i \in [0, n-1]; \\ \tilde{u}_i^{p,n} = \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^n\left(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1}\right)\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1})^2]}, \qquad i \in [0, [\frac{\tau}{\Delta^n}] - 1]. \end{cases} \tag{5.14}$$

Similarly to the implicit reflected case (Lemma 5.3.1), we can obtain

$$\begin{cases} \tilde{y}_i^n = \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n], \bar{\tilde{y}}_i^n, \tilde{z}_i^n, \tilde{u}_i^n)\Delta^n \\ \qquad + \tilde{k}_i^{+n} - \tilde{k}_i^{-n}, \qquad i \in [0, n-1]; \\ \tilde{k}_i^{+n} = \Big(\mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n], \bar{\tilde{y}}_i^n, \tilde{z}_i^n, \tilde{u}_i^n)\Delta^n \\ \qquad - L_i^n\Big)^-, \qquad i \in [0, n-1]; \\ \tilde{k}_i^{-n} = \Big(\mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n] + f^n(t_i, \mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n], \bar{\tilde{y}}_i^n, \tilde{z}_i^n, \tilde{u}_i^n)\Delta^n \\ \qquad - V_i^n\Big)^+, \qquad i \in [0, n-1]; \\ \tilde{y}_i^n = \xi_i^n, \qquad i \in [n, n^\delta]; \\ \tilde{z}_i^n = \frac{1}{2\sqrt{\Delta^n}}\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^n \epsilon_{i+1}^n\right], \qquad i \in [0, n-1]; \\ \tilde{u}_i^{p,n} = \frac{\mathbb{E}^{\mathcal{G}_i^n}\left[\tilde{y}_{i+1}^n\left(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1}\right)\right]}{\mathbb{E}^{\mathcal{G}_i^n}[(h_{i+1}^n - h_i^n - \Delta^n(1-h_{i+1}^n)\gamma_{i+1})^2]}, \qquad i \in [0, [\frac{\tau}{\Delta^n}] - 1]. \end{cases} \tag{5.15}$$

For the continuous time version $(\tilde{Y}_t^n, \tilde{Z}_t^n, \tilde{U}_t^n, \tilde{K}_t^{+n}, \tilde{K}_t^{-n})_{0 \le t \le T}$:

$$\tilde{Y}_t^n := \tilde{y}_{[t/\Delta^n]}^n, \qquad \tilde{Z}_t^n := \tilde{z}_{[t/\Delta^n]}^n, \qquad \tilde{U}_t^n := \tilde{u}_{[t/\Delta^n]}^n,$$

$$\tilde{K}_t^{+n} := \sum_{i=0}^{[t/\Delta^n]} \tilde{k}_i^{+n}, \qquad \tilde{K}_t^{-n} := \sum_{i=0}^{[t/\Delta^n]} \tilde{k}_i^{-n}.$$

5.4 Convergence Results

We first state the convergence result from the Penalized ABSDE (5.16) to the RABSDE (1.10) in Theorem 5.4.1, which is the basis of the following convergence results of the discrete schemes we have studied above. We prove the convergence (Theorem 5.4.2) from the implicit discrete penalization scheme (5.4) to the penalized ABSDE (5.3) with the help of Lemma 5.4.1. Combining with Theorem A.2.3, we can get the convergence (Theorem 5.4.3) of the explicit discrete penalization scheme (5.8). By Theorem A.2.4, Lemma 5.4.1 and Theorem 5.4.1, we can prove the convergence of the implicit discrete reflected scheme (5.10). By Theorem A.2.3, Theorem 5.4.6 and Lemma A.2.4, the convergence (Theorem 5.4.5) of the explicit penalization discrete scheme (5.13) then follows. We first introduce the following discrete Gronwall's inequality (Lemma A.2.1). Theorem A.2.3, Theorem A.2.4, Lemma A.2.4 can be found in the Appendix A.2.

5.4.1 Convergence of the Penalized ABSDE to the RABSDE (1.10)

Theorem 5.4.1. *Suppose that the anticipated process ξ, the generator f satisfy **H 3.1** and **H 3.2**, $f(t, y, \bar{y}_r, z, u)$ is increasing in $\bar{y}$, the obstacles L and V satisfy **H 4.1** and **H 5.1**. We consider* *the following special case of the penalized ABSDE for RABSDE (5.1):*

$$\begin{cases} -dY_t^p = f^n\big(t, Y_t^p, \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^p], Z_t^p, U_t^p\big)dt + dK_t^{+p} - dK_t^{-p} \\ \qquad\qquad -Z_t^p dB_t - U_t^p dM_t, \qquad t \in [0,T]; \\ K_t^{+p} = \int_0^t p(Y_s^p - L_s)^- ds, \qquad t \in [0,T]; \\ K_t^{-p} = \int_0^t p(Y_s^p - V_s)^+ ds, \qquad t \in [0,T]; \\ Y_t^p = \xi_t, \qquad t \in [T, T+\delta]. \end{cases} \tag{5.16}$$

Then we have the limiting process (Y, Z, U, K^+, K^-) of $(Y^p, Z^p, U^p, K^{+p}, K^{-p})$, i.e., as $p \to \infty$, $Y_t^p \to Y_t$ in $\mathcal{S}_{\mathcal{G}}^2(0, T+\delta; \mathbb{R})$, $Z_t^p \to Z_t$ weakly in $\mathcal{L}_{\mathcal{G}}^2(0,T;\mathbb{R})$, $U_t^p \to U_t$ weakly in $\mathcal{L}_{\mathcal{G}}^{2,\tau}(0,T;\mathbb{R})$, K_t^{+p} $(K_t^{-p}) \to K_t^+$ (K_t^-) weakly in $\mathcal{A}_{\mathcal{G}}^2(0,T;\mathbb{R})$. Moreover, there exists a constant $C_{\xi,f,L,V}$ depending on ξ, $f(t,0,0,0,0)$, L and V, such that

$$\begin{aligned} \mathbb{E}\bigg[&\sup_{0\le t\le T} |Y_t^p - Y_t|^2 + \int_0^T |Z_t^p - Z_t|^2 dt + \int_0^T |U_t^p - U_t|^2 1_{\{\tau > t\}} \gamma_t dt \\ &+ \sup_{0\le t\le T} |(K_t^{+p} - K_t^+) - (K_t^{-p} - K_t^-)|^2 \bigg] \le \frac{1}{\sqrt{p}} C_{\xi,f,L,V}. \end{aligned}$$

Proof. First, we introduce the following ABSDE:

$$\begin{aligned} -dY_t^{p,q} =& f\big(t, Y_t^{p,q}, \mathbb{E}^{\mathcal{G}_t}[Y_{t+\delta}^{p,q}], Z_t^{p,q}, U_t^{p,q}\big)dt + q(Y_t^{p,q} - L_t)^- dt \\ & - p(Y_t^{p,q} - V_t)^+ dt - Z_t^{p,q} dB_t - U_t^{p,q} dM_t, \qquad t \in [0,T]. \end{aligned} \tag{5.17}$$

By the existence and uniqueness theorem for ABSDEs with default risk (Theorem 2.2.1), there exists the unique solution for ABSDE (5.17). Similarly to Lemma 4.2.1, it follows that as $q \to \infty$, $Y_t^{p,q} \uparrow \underline{Y}_t^p$ in $\mathcal{S}_{\mathcal{G}}^2(0, T+\delta; \mathbb{R})$, $Z_t^{p,q} \to \underline{Z}_t^p$ in $\mathcal{L}_{\mathcal{G}}^2(0,T;\mathbb{R})$, $U_t^{p,q} \to \underline{U}_t^p$ in $\mathcal{L}_{\mathcal{G}}^{2,\tau}(0,T;\mathbb{R})$, $\int_0^t q(Y_s^{p,q} - L_s)^- ds \to \underline{K}_t^p$ in $\mathcal{A}_{\mathcal{G}}^2(0,T;\mathbb{R})$. $(\underline{Y}^p, \underline{Z}^p, \underline{U}^p, \underline{K}^p)$ is a solution of the following RABSDE with one obstacle L:

$$\begin{aligned} -d\underline{Y}_t^p =& f\big(t, \underline{Y}_t^p, \mathbb{E}^{\mathcal{G}_t}[\underline{Y}_{t+\delta}^p], \underline{Z}_t^p, \underline{U}_t^p\big)dt + d\underline{K}_t^p \\ & - p(\underline{Y}_t^p - V_t)^+ dt - \underline{Z}_t^p dB_t - \underline{U}_t^p dM_t, \qquad t \in [0,T]. \end{aligned} \tag{5.18}$$

Let $p \to \infty$, it follows that $\underline{Y}_t^p \downarrow Y_t$ in $\mathcal{S}_{\mathcal{G}}^2(0, T+\delta; \mathbb{R})$, $\underline{Z}_t^p \to Z_t$ in $\mathcal{L}_{\mathcal{G}}^2(0,T;\mathbb{R})$, $\underline{U}_t^p \to U_t$ in $\mathcal{L}_{\mathcal{G}}^{2,\tau}(0,T;\mathbb{R})$. By the comparison theorem for ABSDEs with default risk (Theorem 2.3.1), we know that $\underline{K}_t^p$ is increasing, then $\underline{K}_T^p \uparrow K_T^{+p}$ and $\underline{K}_T^{p+1} - \underline{K}_T^p \ge \sup_{0\le t\le T}[\underline{K}_t^{p+1} - \underline{K}_t^p] \ge 0$, therefore, $\underline{K}_t^p \to K_t^{+p}$ in $\mathcal{A}_{\mathcal{G}}^2(0,T;\mathbb{R})$. By Lemma 4.2.2, there exists a constant C_1 depending on ξ, $f(t,0,0,0,0)$, δ, L and V, such that

$$\mathbb{E}\left[\sup_{0\le t\le T} \left|\underline{Y}_t^p - Y_t\right|^2 + \int_0^T \left|\underline{Z}_t^p - Z_t\right|^2 dt + \int_0^T \left|\underline{U}_t^p - U_t\right|^2 1_{\{\tau > t\}} \gamma_t dt \right] \le \frac{C_1}{\sqrt{p}}.$$

Similarly, let $p \to \infty$ in (5.17), it follows that $Y_t^{p,q} \downarrow \overline{Y}_t^q$ in $\mathcal{S}_{\mathcal{G}}^2(0, T+\delta; \mathbb{R})$, $Z_t^{p,q} \to \overline{Z}_t^p$

in $\mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R})$, $U^{p,q}_t \to \overline{U}^p_t$ in $\mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R})$, $\int_0^t p(\underline{Y}^p_s - V_s)^+ ds \to \overline{K}^q_t$ in $\mathcal{A}^2_{\mathcal{G}}(0,T;\mathbb{R})$. $(\overline{Y}^q, \overline{Z}^q, \overline{U}^q, \overline{K}^q)$ is a solution of the following RABSDE with one obstacle V:

$$\begin{aligned}-d\overline{Y}^q_t =& f\big(t, \overline{Y}^q_t, \mathbb{E}^{\mathcal{G}_t}[\overline{Y}^q_{t+\delta}], \overline{Z}^q_t, \overline{U}^q_t\big)dt + q(\overline{Y}^q_t - L_t)^- dt \\ & - d\overline{K}^q_t - \overline{Z}^q_t dB_t - \overline{U}^q_t dM_t, \qquad t \in [0,T].\end{aligned} \tag{5.19}$$

Let $q \to \infty$, it follows that $\overline{Y}^q_t \downarrow Y_t$ in $\mathcal{S}^2_{\mathcal{G}}(0,T+\delta;\mathbb{R})$, $\overline{Z}^q_t \to Z_t$ in $\mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R})$, $\overline{U}^q_t \to U_t$ in $\mathcal{L}^{2,\tau}_{\mathcal{G}}(0,T;\mathbb{R})$, $\overline{K}^p_t \to K^{-p}_t$ in $\mathcal{A}^2_{\mathcal{G}}(0,T;\mathbb{R})$. Moreover, there exists a constant C_2 depending on ξ, $f(t,0,0,0,0)$, δ, L and V, such that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\overline{Y}^q_t - Y_t\right|^2 + \int_0^T \left|\overline{Z}^q_t - Z_t\right|^2 dt + \int_0^T \left|\overline{U}^q_t - U_t\right|^2 1_{\{\tau>t\}}\gamma_t dt\right] \le \frac{C_2}{\sqrt{q}}.$$

By the comparison theorem for ABSDEs with default risk (Theorem 2.3.1), it follows that $\underline{Y}^p_t \le Y^p_t \le \overline{Y}^p_t$, for any $t \in [0,T]$. Therefore,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|Y^p_t - Y_t\right|^2\right] \le \frac{C_3}{\sqrt{p}},$$

where $C_3 \ge 0$ is a constant. Similarly to the proof of Lemma 4.2.3, applying Itô formula for rcll semi-martingale (Theorem 1.3.4), we can obtain

$$\mathbb{E}\left[\int_0^T |Z^p_t - Z_t|^2 dt + \int_0^T |U^p_t - U_t|^2 1_{\{\tau>t\}}\gamma_t dt\right] \le \frac{C_4}{\sqrt{p}},$$

where $C_4 \ge 0$ is a constant. Since

$$\begin{aligned}K^{+p}_t - K^{-p}_t =& Y^p_0 - Y^p_t - \int_0^t f\big(s, Y^p_s, \mathbb{E}^{\mathcal{G}_s}[Y^p_{s+\delta}], Z^p_s, U^p_s\big)ds \\ & - \int_0^t Z^p_s dB_s - \int_0^t U^p_s dM_s; \\ K^+_t - K^-_t =& Y_0 - Y_t - \int_0^t f\big(s, Y_s, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}], Z_s, U_s\big)ds \\ & - \int_0^t Z_s dB_s - \int_0^t U_s dM_s.\end{aligned}$$

By the convergence of Y^p, Z^p, U^p and the Lipschitz condition of f, it follows

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|(K^{+p}_t - K^+_t) - (K^{-p}_t - K^-_t)|^2\right] \\ \le& \lambda\mathbb{E}\left[\sup_{0\le t\le T}|Y^p_t - Y_t|^2 + \int_0^T |Z^p_t - Z_t|^2 dt + \int_0^T |U^p_t - U_t|^2 1_{\{\tau>t\}}\gamma_t dt\right] \le \frac{C_5}{\sqrt{p}},\end{aligned}$$

where λ, $C_5 \ge 0$ are constants. Since $\mathbb{E}[|K^{+p}_T|^2 + |K^{-p}_T|^2] < \infty$, there exist processes $\hat{K}^+$ and $\hat{K}^-$ in $\mathcal{A}^2_{\mathcal{G}}(0,T;\mathbb{R})$ are the weak limits of K^{+p} and K^{-p} respectively. Since for any

$t \in [0,T]$, $\underline{Y}_t^p \leq Y_t^p \leq \overline{Y}_t^p$, we can get

$$\begin{aligned} dK_t^{+p} &= p(Y_t^p - L_t)^- dt \leq p(\overline{Y}_t^p - L_t)^- dt = d\overline{K}_t^{+p}; \\ dK_t^{-p} &= p(Y_t^p - V_t)^+ dt \geq p(\underline{Y}_t^p - V_t)^+ dt = d\underline{K}_t^{-p}. \end{aligned}$$

Hence, $d\hat{K}_t^+ \leq dK_t^+$, $d\hat{K}_t^- \geq dK_t^-$, it follows that $d\hat{K}_t^+ - d\hat{K}_t^- \leq dK_t^+ - dK_t^-$. On the other hand, the limit of Y^p is Y, so $d\hat{K}_t^+ - d\hat{K}_t^- = dK_t^+ - dK_t^-$, it follows that $d\hat{K}_t^+ = dK_t^+$, $d\hat{K}_t^- = dK_t^-$, then $\hat{K}_t^+ = K_t^+$, $\hat{K}_t^- = K_t^-$. □

5.4.2 Convergence of the Implicit Discrete Penalization Scheme

We first introduce the following lemma to prove the convergence result from penalized ABSDE (5.16) to implicit penalization scheme.

Lemma 5.4.1. *Under **H** 5.2 and **H** 5.3, $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n})$ converges to (Y_t^p, Z_t^p, U_t^p) in the following sense:*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t^{p,n} - Y_t^p|^2 + \int_0^T |Z_t^{p,n} - Z_t^p|^2 dt + \int_0^T |U_t^{p,n} - U_t^p|^2 1_{\{\tau>t\}} \gamma_t dt\right] = 0, \tag{5.20}$$

for any $t \in [0,T]$, as $n \to \infty$, $K_t^{+p,n} - K_t^{-p,n} \to K_t^{+p} - K_t^{-p}$ in $\mathcal{L}_{\mathcal{G}}^2(0,T;\mathbb{R})$.

Proof. **Step 1**. First, we consider the continuous and discrete time equations by Picard's method.

In the continuous case, set $Y^{p,\infty,0} = Z^{p,\infty,0} = U^{p,\infty,0} = 0$, then let $(Y_t^{p,\infty,m+1}, Z_t^{p,\infty,m+1}, U_t^{p,\infty,m+1})$ be the solution of the following BSDE:

$$\begin{cases} Y_t^{p,\infty,m+1} &= \xi_T + \int_t^T f^n\big(s, Y_s^{p,\infty,m}, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}^{p,\infty,m}], Z_s^{p,\infty,m}, U_s^{p,\infty,m}\big) ds \\ &\quad + \int_t^T q(Y_s^{p,\infty,m} - L_s)^- ds - \int_t^T p(Y_s^{p,\infty,m} - V_s)^+ ds \\ &\quad - \int_t^T Z_s^{p,\infty,m+1} dB_s - \int_t^T U_s^{p,\infty,m+1} dM_s, \qquad t \in [0,T]; \\ Y_t^{p,\infty,m+1} &= \xi_t, \qquad t \in (T, T+\delta], \end{cases} \tag{5.21}$$

where $(Y_t^{p,\infty,m}, Z_t^{p,\infty,m}, U_t^{p,\infty,m})$ is the Picard approximation of (Y_t^p, Z_t^p, U_t^p).

In the discrete case, set $y_i^{p,n,0} = z_i^{p,n,0} = u_i^{p,n,0} = 0$ (for any $i = 0, 2, ...n$), then let $(y_i^{p,n,m+1}, z_i^{p,n,m+1}, u_i^{p,n,m+1})$ be the solution of the discrete BSDE below:

$$\begin{cases} y_i^{p,n,m+1} &= y_{i+1}^{p,n,m+1} + f^n(t_i, y_i^{p,n,m}, \bar{y}_i^{p,n,m}, z_i^{p,n,m}, u_i^{p,n,m})\Delta^n \\ &\quad - z_i^{p,n,m+1}\Delta B_{i+1}^n - u_i^{p,n,m+1}\Delta M_{i+1}^n + p\Delta^n(y_i^{p,n} - L_i^n)^- \\ &\quad - p\Delta^n(y_i^{p,n} - V_i^n)^+, \quad i \in [0, n-1]; \\ y_i^{p,n,m+1} &= \xi_i^n, \qquad i \in [n, n^\delta]. \end{cases} \tag{5.22}$$

here $(Y_t^{p,n,m}, Z_t^{p,n,m}, U_t^{p,n,m})$ is the continuous time version of the discrete Picard approximation of $(y_i^{p,n,m}, z_i^{p,n,m}, u_i^{p,n,m})$.

Step 2. Then, we consider the following decomposition:

$$Y^{p,n} - Y^p = (Y^{p,n} - Y^{p,n,m}) + (Y^{p,n,m} - Y^{p,\infty,m}) + (Y^{p,\infty,m} - Y^p).$$

By Proposition 1 and Proposition 3 in Lejay et. al [98] and the definition of L_i^n and V_i^n, it follows (5.20). □

Theorem 5.4.2. *(**Convergence of the implicit discrete penalization scheme**) Under **H** 3.1 and **H** 5.3, $(Y_t^{p,n}, Z_t^{p,n}, U_t^{p,n})$ converges to (Y_t, Z_t, U_t) in the following sense:*

$$\lim_{p\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{p,n} - Y_t|^2 + \int_0^T |Z_t^{p,n} - Z_t|^2 dt + \int_0^T |U_t^{p,n} - U_t|^2 1_{\{\tau>t\}}\gamma_t dt\right] = 0, \tag{5.23}$$

for any $t \in [0,T]$, as $p\to\infty$, $n\to\infty$, $K_t^{+p,n} - K_t^{-p,n} \to K_t^+ - K_t^-$ in $\mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{R})$.

Proof. By Lemma 5.4.1 and Theorem 5.4.1, as $p\to\infty$, $n\to\infty$, it follows

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{p,n} - Y_t|^2 + \int_0^T |Z_t^{p,n} - Z_t|^2 dt + \int_0^T |U_t^{p,n} - U_t|^2 1_{\{\tau>t\}}\gamma_t dt\right]\\
\le& 2\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{p,n} - Y_t^p|^2 + \int_0^T |Z_t^{p,n} - Z_t^p|^2 dt + \int_0^T |U_t^{p,n} - U_t^p|^2 1_{\{\tau>t\}}\gamma_t dt\right]\\
&+ 2\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^{p} - Y_t|^2 + \int_0^T |Z_t^{p} - Z_t|^2 dt + \int_0^T |U_t^{p} - U_t|^2 1_{\{\tau>t\}}\gamma_t dt\right] \to 0.
\end{aligned}$$

For the increasing processes $K^{+p,n}$ and $K^{-p,n}$, by Theorem 5.4.1, we can obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(K_t^{+p,n} - K_t^{-p,n}\right) - (K_t^+ - K_t^-)\right]^2\\
\le& 2\mathbb{E}\left[\left(K_t^{+p,n} - K_t^{-p,n}\right) - \left(K_t^{+p} - K_t^{-p}\right)\right]^2\\
&+ 2\mathbb{E}\left[\left(K_t^{+p} - K_t^{-p}\right) - (K_t^+ - K_t^-)\right]^2\\
\le& 2\mathbb{E}\left[\left(K_t^{+p,n} - K_t^{-p,n}\right) - \left(K_t^{+p} - K_t^{-p}\right)\right]^2 + \frac{C}{\sqrt{p}},
\end{aligned}$$

where $C \ge 0$ is a constant depending on ξ, $f(t,0,0,0,0)$, δ, L and V. For each fixed p,

$$\begin{aligned}
K_t^{+p,n} - K_t^{-p,n} =& Y_0^{p,n} - Y_t^{p,n} - \int_0^t f^n(s, Y_s^{p,n}, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}^{p,n}], Z_s^{p,n}, U_s^{p,n})ds\\
&- \int_0^t Z_s^{p,n} dB_s^n - \int_0^t U_s^{p,n} dM_s^n;\\
K_t^{+p} - K_t^{-p} =& Y_0^p - Y_t^p - \int_0^t f(s, Y_s^p, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}^p], Z_s^p, U_s^p)ds\\
&- \int_0^t Z_s^p dB_s - \int_0^t U_s^p dM_s.
\end{aligned}$$

By Corollary 14 in Briand et. al [99], we know that as $n \to \infty$, $\int_0^\cdot Z_s^{p,n} dB_s^n \to \int_0^\cdot Z_s^p dB_s$ in $\mathcal{S}_\mathcal{G}^2(0,T;\mathbb{R})$, $\int_0^\cdot U_s^{p,n} dM_s^n \to \int_0^\cdot U_s^p dM_s$ in $\mathcal{S}_\mathcal{G}^2(0,T;\mathbb{R})$. By the Lipschitz condition of f and the convergence of $Y^{p,n}$, it follows that $K_t^{+p,n} - K_t^{-p,n} \to K_t^+ - K_t^-$ in $\mathcal{L}_\mathcal{G}^2(0,T;\mathbb{R})$. □

5.4.3 Convergence of the Explicit Discrete Penalization Scheme

By Theorem 5.4.2 and Theorem A.2.3, we can obtain the following convergence result of explicit penalization discrete scheme.

Theorem 5.4.3. *(Convergence of the explicit discrete penalization scheme) Under **H** 3.1 and **H** 5.3, $(\tilde{Y}_t^{p,n}, \tilde{Z}_t^{p,n}, \tilde{U}_t^{p,n})$ converges to (Y_t, Z_t, U_t) in the following sense:*

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{0\le t\le T} |\tilde{Y}_t^{p,n} - Y_t|^2 + \int_0^T |\tilde{Z}_t^{p,n} - Z_t|^2 dt + \int_0^T |\tilde{U}_t^{p,n} - U_t|^2 1_{\{\tau>t\}} \gamma_t dt\Big] = 0, \tag{5.24}$$

for any $t \in [0,T]$, as $n \to \infty$, $\tilde{K}_t^{+p,n} - \tilde{K}_t^{-p,n} \to K_t^+ - K_t^-$ in $\mathcal{L}_\mathcal{G}^2(0,T;\mathbb{R})$.

5.4.4 Convergence of the Implicit Discrete Reflected Scheme

By Theorem A.2.4, Lemma 5.4.1 and Theorem 5.4.1, we can prove the convergence of the implicit discrete reflected scheme (5.10).

Theorem 5.4.4. *(Convergence of the implicit discrete reflected scheme) Under **H** 5.3 and **H** 3.1, (Y_t^n, Z_t^n, U_t^n) converges to (Y_t, Z_t, U_t) in the following sense:*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} |Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2 dt + \int_0^T |U_t^n - U_t|^2 1_{\{\tau>t\}} \gamma_t dt\right] = 0, \tag{5.25}$$

and for any $t \in [0,T]$, as $n \to \infty$, $K_t^{+n} - K_t^{-n} \to K_t^+ - K_t^-$ in $\mathcal{L}_\mathcal{G}^2(0,T;\mathbb{R})$.

Proof. First, we prove (5.25).

From Theorem A.2.4, Lemma 5.4.1 and Theorem 5.4.1, we choose p much larger than n. For the fixed $p \in \mathbb{N}$, as $n \to \infty$, it follows

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} |Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2 dt + \int_0^T |U_t^n - U_t|^2 1_{\{\tau>t\}} \gamma_t dt\right] \\
\le& 3\mathbb{E}\left[\sup_{0\le t\le T} |Y_t^n - Y_t^{p,n}|^2 + \int_0^T |Z_t^n - Z_t^{p,n}|^2 dt + \int_0^T |U_t^n - U_t^{p,n}|^2 1_{\{\tau>t\}} \gamma_t dt\right] \\
&+ 3\mathbb{E}\left[\sup_{0\le t\le T} |Y_t^{p,n} - Y_t^p|^2 + \int_0^T |Z_t^{p,n} - Z_t^p|^2 dt + \int_0^T |U_t^{p,n} - U_t^p|^2 1_{\{\tau>t\}} \gamma_t dt\right] \\
&+ 3\mathbb{E}\left[\sup_{0\le t\le T} |Y_t^p - Y_t|^2 + \int_0^T |Z_t^p - Z_t|^2 dt + \int_0^T |U_t^p - U_t|^2 1_{\{\tau>t\}} \gamma_t dt\right]
\end{aligned}$$

$$\leq 3\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^{p,n}-Y_t^p|^2+\int_0^T|Z_t^{p,n}-Z_t^p|^2dt+\int_0^T|U_t^{p,n}-U_t^p|^2 1_{\{\tau>t\}}\gamma_t dt\right]$$
$$+\frac{3}{\sqrt{p}}C_{\xi,f,L,V}+\frac{3}{\sqrt{p}}\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n}\to 0.$$

For the increasing processes, for the fixed $p\in\mathbb{N}$, as $n\to\infty$,

$$\begin{aligned}
&\mathbb{E}\left[\left|(K_t^{+n}-K_t^{-n})-(K_t^+-K_t^-)\right|^2\right]\\
\leq&3\mathbb{E}\left[\left|(K_t^{+n}-K_t^{-n})-\left(K_t^{+p,n}-K_t^{-p,n}\right)\right|^2\right]\\
&+3\mathbb{E}\left[\left|\left(K_t^{+p,n}-K_t^{-p,n}\right)-\left(K_t^{+p}-K_t^{-p}\right)\right|^2\right]\\
&+3\mathbb{E}\left[\left|\left(K_t^{+p}-K_t^{-p}\right)-(K_t^+-K_t^-)\right|^2\right]\\
\leq&3\mathbb{E}\left[\left|\left(K_t^{+p,n}-K_t^{-p,n}\right)-\left(K_t^{+p}-K_t^{-p}\right)\right|^2\right]\\
&+\frac{3}{\sqrt{p}}C_{\xi,f,L,V}+\frac{3}{\sqrt{p}}\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n}\to 0.
\end{aligned}$$

□

5.4.5 Convergence of the Explicit Discrete Reflected Scheme

By Theorem A.2.3, Theorem 5.4.6 and Lemma A.2.4, we can get the convergence result of explicit penalization discrete scheme.

Theorem 5.4.5. *(**Convergence of the explicit discrete reflected scheme**) Under **H** 3.1 and **H** 5.3, $(\tilde{Y}_t^n,\tilde{Z}_t^n,\tilde{U}_t^n)$ converges to (Y_t,Z_t,U_t) in the following sense:*

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\leq t\leq T}|\tilde{Y}_t^n-Y_t|^2+\int_0^T|\tilde{Z}_t^n-Z_t|^2dt+\int_0^T|\tilde{U}_t^n-U_t|^2 1_{\{\tau>t\}}\gamma_t dt\right]=0, \quad (5.26)$$

for any $t\in[0,T]$, as $n\to\infty$, $\tilde{K}_t^{+n}-\tilde{K}_t^{-n}\to K_t^+-K_t^-$ in $\mathcal{L}_{\mathcal{G}}^2(0,T;\mathbb{R})$.

5.4.6 Distance between implicit discrete reflected and explicit discrete reflected schemes

Theorem 5.4.6. *(**Distance between implicit discrete reflected and explicit discrete reflected schemes**) Assumptions **H** 3.1 and **H** 5.3 hold, for any $p\in\mathbb{N}$:*

$$\begin{aligned}
\mathbb{E}\Bigg[&\sup_{0\leq t\leq T}\left|\tilde{Y}_t^n-Y_t^n\right|^2+\int_0^T\left|\tilde{Z}_t^n-Z_t^n\right|^2dt+\int_0^T\left|\tilde{U}_t^n-U_t^n\right|^2 1_{\{\tau>t\}}\gamma_t dt\\
&+\left|(\tilde{K}_t^{+n}-\tilde{K}_t^{-n})-(K_t^{+n}-K_t^{-n})\right|^2\Bigg]\leq\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n,p}(\Delta^n)^2.
\end{aligned} \quad (5.27)$$

where $\lambda_{L,T,\delta} \geq 0$ *is a constant depending on Lipschitz coefficient* L, T *and* δ, $C_{f^n,\xi^n,\Delta^n,L^n,V^n,p} \geq 0$ *is a constant depending on* $f^n(t_j,0,0,0,0)$, ξ^n, L^n, V^n *and* p.

Proof. From the definitions of the implicit discrete reflected scheme (5.4) and the explicit discrete reflected scheme (5.13), and the Lipschitz condition of f^n, we can obtain

$$\begin{aligned}
&\mathbb{E}\Big[|\tilde{y}_j^n - y_j^n|^2 + \Delta^n|\tilde{z}_j^n - z_j^n|^2 + \Delta^n|\tilde{u}_j^n - u_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}\Big] \\
=&\mathbb{E}\left|\tilde{y}_{j+1}^n - y_{j+1}^n\right|^2 + 2\Delta^n\mathbb{E}\Big[\big(f^n(t_j,\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n],\bar{\tilde{y}}_j^n,\tilde{z}_j^n,\tilde{u}_j^n) \\
&- f^n(t_j,y_j^n,\bar{y}_j^n,z_j^n,u_j^n)\big)\left(\tilde{y}_j^n - y_j^n\right)\Big] \\
&- (\Delta^n)^2\mathbb{E}\Big[f^n(t_j,y_j^n,\bar{y}_j^n,z_j^n,u_j^n) - f^n(t_j,\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n],\bar{\tilde{y}}_j^n,\tilde{z}_j^n,\tilde{u}_j^n) \\
&\qquad + (\tilde{k}_j^{+n} - k_j^{+n}) - (\tilde{k}_j^{-n} - k_j^{-n})\Big]^2 \\
&+ 2\mathbb{E}\left[\left(\tilde{y}_j^n - y_j^n\right)(\tilde{k}_j^{+n} - k_j^{+n}) - \left(\tilde{y}_j^n - y_j^n\right)(\tilde{k}_j^{-n} - k_j^{-n})\right] \\
\leq& 2\Delta^n L\mathbb{E}\Big[\Big(|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n] - y_j^n| + |\bar{\tilde{y}}_j^n - \bar{y}_j^n| + |\tilde{z}_j^n - z_j^n| \\
&\qquad + |\tilde{u}_j^n - u_j^n|(1-h_{j+1}^n)\sqrt{\gamma_{j+1}}\Big)\left(\tilde{y}_j^n - y_j^n\right)\Big],
\end{aligned} \tag{5.28}$$

since

$$\begin{aligned}
\left(\tilde{y}_j^n - y_j^n\right)(\tilde{k}_j^{+n} - k_j^{+n}) &= \left(\tilde{y}_j^n - L_j^n\right)\tilde{k}_j^{+n} + \left(y_j^n - L_j^n\right)k_j^{+n} \\
&\quad - \left(y_j^n - L_j^n\right)\tilde{k}_j^{+n} - \left(\tilde{y}_j^n - L_j^n\right)k_j^{+n} \leq 0; \\
\left(\tilde{y}_j^n - y_j^n\right)(\tilde{k}_j^{-n} - k_j^{-n}) &= \left(\tilde{y}_j^n - V_j^n\right)\tilde{k}_j^{-n} + \left(y_j^n - V_j^n\right)k_j^{-n} \\
&\quad - \left(y_j^n - V_j^n\right)\tilde{k}_j^{-n} - \left(\tilde{y}_j^n - V_j^n\right)k_j^{-n} \geq 0.
\end{aligned}$$

Taking sum from $j = i, ..., n-1$, it follows

$$\begin{aligned}
&\mathbb{E}\left[|\tilde{y}_i^n - y_i^n|^2 + \frac{\Delta^n}{2}\sum_{j=i}^{n-1}\left|\tilde{z}_j^n - z_j^n\right|^2 + \frac{\Delta^n}{2}\sum_{j=i}^{n-1}\left|\tilde{u}_j^n - u_j^n\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right] \\
\leq& 2\Delta^n L\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n] - y_j^n\right||\tilde{y}_j^{p,n} - y_j^{p,n}|\right] \\
&+ (2\Delta^n L + 4\Delta^n L^2)\mathbb{E}\left[\sum_{j=i}^{n-1}\left|\tilde{y}_j^n - y_j^n\right|^2\right].
\end{aligned}$$

By (5.14) and the Lipschitz condition of f^n, it follows

$$\begin{aligned}
&2\Delta^n L\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n] - y_j^n\right||\tilde{y}_j^n - y_j^n|\right] \\
\leq& 2\Delta^n L\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\left(\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n] - \tilde{y}_j^n\right) + \left(\tilde{y}_j^n - y_j^n\right)\right||\tilde{y}_j^n - y_j^n|\right]
\end{aligned}$$

$$\leq 2\Delta^n L \mathbb{E} \sum_{j=i}^{n-1} \Big[\Big(\Big| f^n(t_j, \mathbb{E}^{\mathcal{G}_j^n}[\bar{y}_{j+1}^n], \bar{y}_j^n, \tilde{z}_j^n, \tilde{u}_j^n)\Delta^n + \tilde{k}_j^{+n} - \tilde{k}_j^{-n} \Big| + \Big| \tilde{y}_j^n - y_j^n \Big| \Big) \Big| \tilde{y}_j^n - y_j^n \Big| \Big]$$

$$\leq (\Delta^n L)^2 \mathbb{E} \sum_{j=i}^{n-1} \Big[2|\tilde{y}_j^n|^2 + |\tilde{z}_j^n|^2 + |\tilde{u}_i^n|^2 (1 - h_{i+1}^n)\gamma_{i+1} \Big]$$

$$+ (\Delta^n)^2 \mathbb{E} \sum_{j=i}^{n-1} \Big[\big| f^n(t_j, 0, 0, 0, 0) \big|^2 \Big] + \mathbb{E} \sum_{j=i}^{n-1} \Big[|\tilde{k}_j^{+n}|^2 + |\tilde{k}_j^{-n}|^2 \Big]$$

$$+ \Big(8(\Delta^n L)^2 + 2\Delta^n L \Big) \mathbb{E} \left[\sum_{i=j}^{n-1} \Big| \tilde{y}_j^{p,n} - y_j^{p,n} \Big|^2 \right] + (\Delta^n L)^2 \mathbb{E} \left[\sum_{i=n}^{n^\delta - 1} |\xi_j^n|^2 \right].$$

From Lemma A.2.4, we can obtain

$$\mathbb{E} \left[|\tilde{y}_i^n - y_i^n|^2 + \frac{\Delta^n}{2} \sum_{j=i}^{n-1} \Big| \tilde{z}_j^n - z_j^n \Big|^2 + \frac{\Delta^n}{2} \sum_{j=i}^{n-1} \Big| \tilde{u}_j^n - u_j^n \Big|^2 (1 - h_{j+1}^n)\gamma_{j+1} \right]$$
$$\leq \Big(4\Delta^n L + 4\Delta^n L^2 + 8(\Delta^n L)^2 \Big) \mathbb{E} \left[\sum_{j=i}^{n-1} \Big| \tilde{y}_j^n - y_j^n \Big|^2 \right] + C_{f^n, \xi^n, L^n, V^n} \Delta^n. \tag{5.29}$$

where $C_{f^n, \xi^n, \Delta^n, L^n, V^n} \geq 0$ is a constant depending on $f^n(t_j, 0, 0, 0, 0)$, ξ^n, L^n and V^n. By the discrete Gronwall's inequality (Lemma A.2.1), when $\big(4\Delta^n L + 4\Delta^n L^2 + 8(\Delta^n L)^2\big) < 1$, we can get

$$\sup_i \mathbb{E} \, |\tilde{y}_i^n - y_i^n|^2 \leq C_{f^n, \xi^n, L^n, V^n} (\Delta^n)^2 e^{(4L + 4L^2 + 8\Delta^n L^2)T},$$

from (5.29), it follows

$$\mathbb{E} \left[\sum_{j=i}^{n-1} \Big| \tilde{z}_j^n - z_j^n \Big|^2 + \sum_{j=i}^{n-1} \Big| \tilde{u}_j^n - u_j^n \Big|^2 (1 - h_{j+1}^n)\gamma_{j+1} \right] \leq C_{f^n, \xi^n, L^n, V^n} (\Delta^n)^2.$$

Reconsidering (5.28), we take square, sup and sum over j, then take expectation, by Burkholder-Davis-Gundy inequality for the martingale parts, we can obtain

$$\mathbb{E} \left[\sup_i |\tilde{y}_i^n - y_i^n|^2 \right] \leq C\Delta^n \left[\sum_{i=0}^{n-1} \mathbb{E} \, |\tilde{y}_i^n - y_i^n|^2 \right] \leq CT \sup_i \mathbb{E} \, |\tilde{y}_i^n - y_i^n|^2.$$

It follows (A.13). □

5.5 One Numerical Simulation Example of RABSDE with Two Obstacles and Default Risk

For the convenience of computation, we consider the case when the terminal time $T = 1$, the calculation begins from $y_n^n = \xi^n$, and proceeds backward to solve $(y_i^n, z_i^n, u_i^n, k_i^{+n}, k_i^{-n})$

for $i = n-1, n-2, ..., 1, 0$. We use Matlab for the simulation. We consider a simple situation: the terminal value $\xi_T = \Phi(B_T, M_T)$ and anticipated process $\xi_t = \Phi(B_t)$ ($t \in (T, T+\delta]$); the obstacles $L_t = \Psi_1(t, B_t, M_t)$ and $V_t = \Psi_2(t, B_t, M_t)$, where Φ, Ψ_2 and Ψ_3 are real analytic functions defined on $\mathbb{R}$, $[0,T] \times \mathbb{R}$ and $[0,T] \times \mathbb{R}$ respectively. We take the following example ($n = 200$, anticipated time $\delta = 0.3$):

$$
\begin{aligned}
f(t, y, \bar{y}, z, u) &= \left| \frac{y}{2} + \frac{\bar{y}}{2} \right| + z + u, \qquad t \in [0, T]; \\
\Phi(B_t) &= |B_t| + M_T, \qquad t \in [T, T+\delta]; \\
\Psi_1(t, B_t, M_t) &= |B_t| + M_t + T - t, \qquad t \in [0, T]; \\
\Psi_2(t, B_t, M_t) &= |B_t| + M_t + 2\left(T - t\right), \qquad t \in [0, T];
\end{aligned}
$$

This example satisfies the assumptions **H 3.1**, **H 3.2** and **H 4.1** in the theoretical section. We choose the default time τ as a uniformly distributed random variable.

As the inverse for both implicit schemes in (5.7) and (5.12) is not easy to get directly, we only use explicit schemes below. We are going to illustrate the behaviors of the explicit reflected scheme by looking at the pathwise behavior for $n = 400$. Further, we will compare the explicit reflected scheme with the explicit penalization scheme for different values of the penalization parameter.

Figure 5.1 represents one path of the Brownian motion, Figure 5.2 and Figure 5.3 represent one path of the Brownian motion and one path of the default martingale when the default time $\tau = 0.7$ and 0.2 respectively.

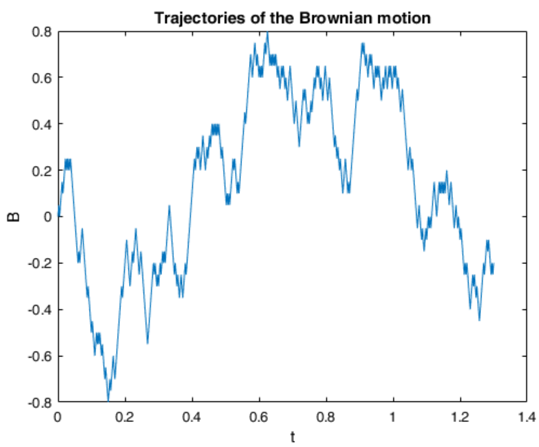

FIGURE 5.1: One path of the Brownian motion

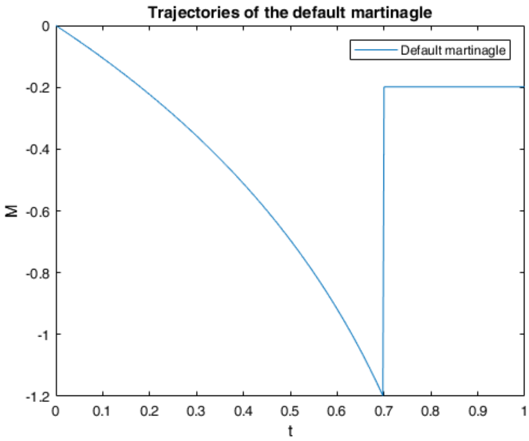

FIGURE 5.2: One path of the default martingale ($\tau = 0.7$)

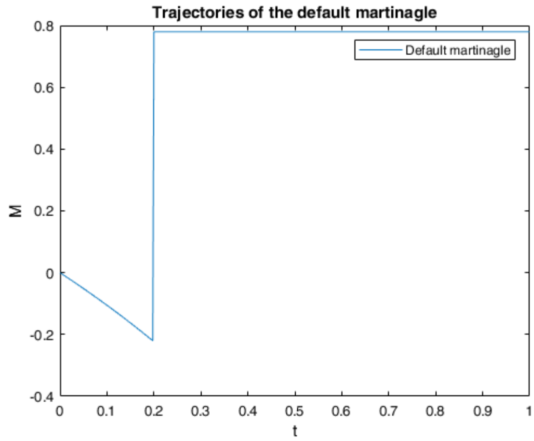

FIGURE 5.3: One path of the default martingale ($\tau = 0.2$)

Figure 5.4 and Figure 5.5 represent the paths of the solution $\tilde{y}^n$, increasing processes $\tilde{K}^{+n}$ and $\tilde{K}^{-n}$ in the explicit reflected scheme where the random default time $\tau = 0.7$. We can see that for all i, $\tilde{y}_i^n$ stays between the lower obstacle L_i^n and the upper obstacle V_i^n, the increasing process $\tilde{K}_i^{+n}$ (resp. $\tilde{K}_i^{-n}$) pushes $\tilde{y}_i^n$ upward (resp. downward), and they can not increase at the same time. In this example for $n = 400$, default time $\tau = 0.7$, we can get the reflected solution $\tilde{y}_0^n = 1.2563$ from the explicit reflected scheme.

Figure 5.4 and Figure 5.6 illustrate the influence of the jump on the solution $\tilde{y}^n$ at the different random default times, the reflected solution $\tilde{y}^n$ moves downwards after the default time (which can not be shown in Figure 5.7). From the approximation of the default martingale (5.2), M^n is larger with a larger default time.

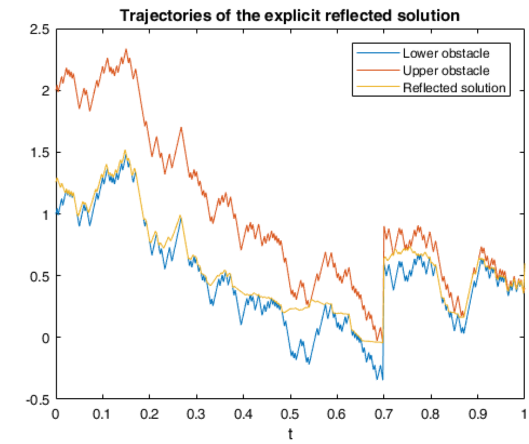

FIGURE 5.4: One path of $\tilde{y}^n$ in the explicit reflected scheme ($\tau = 0.7$)

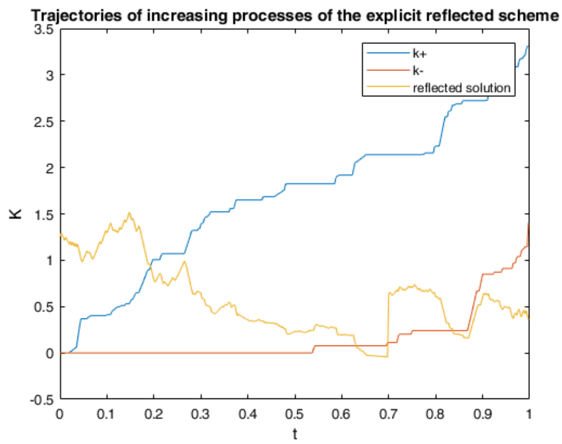

FIGURE 5.5: The paths of the increasing processes in the explicit reflected scheme ($\tau = 0.7$)

FIGURE 5.6: One path of $\tilde{y}^n$ in the explicit reflected scheme ($\tau = 0.2$)

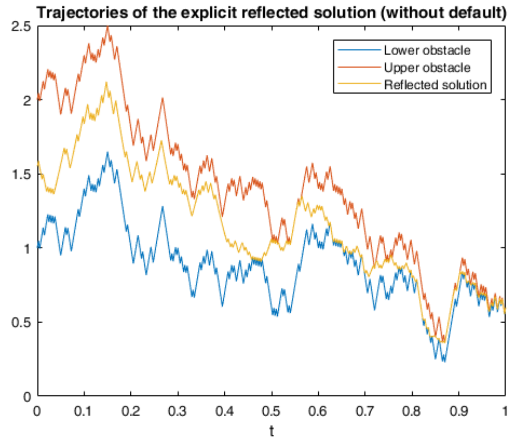

FIGURE 5.7: One path of $\tilde{y}^n$ in the explicit reflected scheme without default risk.

Table 5.1 and contains the comparison between the explicit reflected scheme and the explicit penalization scheme by the values of $\tilde{y}_0^n$ and $\tilde{y}_0^{p,n}$ with respect to the parameters n and p. As n increases, the reflected solution $\tilde{y}_0^n$ increases because of the choice of the coefficient. For fixed n, as the penalization parameter p increases, the penalization solution $\tilde{y}_0^{p,n}$ converges increasingly to the reflected solution $\tilde{y}_0^n$, which is obvious from the comparison theorem of BSDE with default risk. If p and n have a smaller difference (when $n = 10^3$, $p = 10^3$), the penalization solution $\tilde{y}_0^{p,n}$ is far from the reflected solution $\tilde{y}_0^n$. Hence,the penalization parameter p should be chosen as large as possible. Table 5.2 illustrates the comparison between the reflected solution $\tilde{y}_0^n$ and $\tilde{y}_0^{n*}$. Figure 5.7 represents the situation without the default risk, the reflected solution $\tilde{y}_0^{n*}$ has a larger value than in the situation when the default case happens (Figure 5.4).

TABLE 5.1: The values of the penalization solution $\tilde{y}_0^{p,n}$ ($\tau = 0.7$).

$\tilde{y}_0^{p,n}$	$p = 10^3$	$p = 10^4$	$p = 10^5$	$p = 10^6$
$n = 200$	1.2369	1.2394	1.2428	1.2452
$n = 400$	1.2458	1.2482	1.2496	1.2511
$n = 1000$	1.2343	1.2497	1.2527	1.2630

TABLE 5.2: The values of the reflected solution $\tilde{y}_0^n$ ($\tau = 0.7$) and $\tilde{y}_0^{n*}$.

	$\tilde{y}_0^n$	$\tilde{y}_0^{n*}$
$n = 200$	1.2469	1.5451
$n = 400$	1.2563	1.5507
$n = 1000$	1.2644	1.5614

5.6 Application in American Game Options in a Defaultable Setting

5.6.1 Model Description

American game options are a kind of a new derivative security, which enables both the broker and the trader to stop the contract at any time before the maturity. The trader can exercise the right to buy or sell a specified underlying security for a certain agreed price. The broker must pay a certain amount of penalty if the contract is terminated from his side. Hamadène [100] (2006) studied the relation between American game options and RBSDE with two obstacles driven by Brownian motion. See more in Kifer [101] (2000), Ma and Cvitanić [102] (2001), etc.

In this section, we consider the case with default risk. An American game option contract with maturity T involves a broker c_1 and a trader c_2:

- The broker c_1 has the right to cancel the contract at any time before the maturity T, while the trader c_2 has the right to early exercise the option;
- the trader c_2 pays an initial amount (the price of this option) which ensures an income L_{τ_1} from the broker c_1, where $\tau_1 \in [0, T]$ is an $\mathcal{G}$-stopping time;
- the broker has the right to cancel the contract before T and needs to pay V_{τ_2} to c_2. Here, the payment amount of the broker c_1 should be greater than his payment to the trader c_2 (if trader decides for early exercise), i.e. $V_{\tau_2} \geq L_{\tau_2}$, $V_{\tau_2} - L_{\tau_2}$ is the premium that the broker c_1 pays for his decision of early cancellation. $\tau_2 \in [0, T]$ is an $\mathcal{G}$-stopping time;
- if c_1 and c_2 both decide to stop the contract at the same time τ, then the trader c_2 gets an income equal to $Q_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}$.

5.6.2 The value of the American Game Option

Consider a financial market $\mathcal{M}$, we have a riskless asset $C_t \in \mathbb{R}$ with risk-free rate r:

$$\begin{cases} dC_t &= rC_t dt, \qquad t \in (0, T], \\ C_0 &= c_0, \qquad t = 0; \end{cases} \tag{5.30}$$

one risky asset $S_t \in \mathbb{R}$:

$$\begin{cases} dS_t &= S_t \left(\mu dt + \sigma dB_t + \chi dM_t\right), \qquad t \in (0, T], \\ S_0 &= s_0, \qquad t = 0, \end{cases} \tag{5.31}$$

where B_t is a 1-dimensional Brownian motion, μ is the expected return, σ is the volatility, χ is the parameter related to the default risk.

Consider a self-financing portfolio $\pi \in \mathbb{R}^2$ with strategy $\pi = \left(\beta_s^{(1)}, \beta_s^{(2)}\right)_{s\in[t,T]}$ trading on C and S respectively on the time interval $[t, T]$. $A^{\pi,\alpha}$ is the wealth process with the value α^A at time t, here is a non-negative $\mathcal{F}_t$-measurable random variable.

$$\begin{aligned} &A_s^{\pi,\alpha} = \beta_s^{(1)} C_s + \beta_s^{(2)} S_s, \qquad s \in [t,T]; \\ &A_s^{\pi,\alpha} = \alpha^A + \int_t^s \beta_u^{(1)} dC_u + \int_t^s \beta_u^{(2)} dS_u, \quad s \in [t,T]; \\ &\int_t^T \left[|\beta_u^{(1)}| + (\beta_u^{(2)} S_u)^2 \right] du < \infty; \end{aligned} \tag{5.32}$$

Let L be a positive local martingale with the following form:

$$\begin{cases} dL_t &= -L_{t-}\sigma^{-1}(\mu - r) dB_t, \qquad t \in (0,T], \\ L_0 &= 1, \qquad t = 0, \end{cases}$$

As for reasons of option pricing, we need a risk-neutral market setting, we will perform a change of measure. For simplicity, we only consider the case where the risk premium is removed by a Girsanov type transformation of the Brownian motion (Other transformations are possible, but we do not consider them here). By Girsanov's theorem, let $\mathbb{Q}$ be the corresponding equivalent measure of $\mathbb{P}$:

$$\frac{\mathbb{Q}}{\mathbb{P}}\Big|\mathcal{G}_T = L_T = \exp\left\{ -\sigma^{-1}(\mu - r) B_T - \frac{1}{2}\left(-\sigma^{-1}(\mu - r)\right)^2 T \right\},$$

here let $\mathbb{E}^{\mathbb{Q}}$ be the expectation, $B^{\mathbb{Q}}$ and $M^{\mathbb{Q}}$ be the Brownian motion and the default martingale under the measure $\mathbb{Q}$:

$$\begin{aligned} B_t^{\mathbb{Q}} &:= B_t + \sigma^{-1}(\mu - r)t; \\ M_t^{\mathbb{Q}} &:= M_t. \end{aligned}$$

Hence, the risky asset S_t defined in (5.31) can be converted into the following form under measure $\mathbb{Q}$:

$$\begin{cases} dS_t &= S_t \left(r dt + \sigma dB_t^{\mathbb{Q}} + \chi dM_t^{\mathbb{Q}} \right), \qquad t \in (0,T], \\ S_0 &= s_0, \qquad t = 0. \end{cases} \tag{5.33}$$

Denote by $R(s,\theta)$ the amount that the broker c_1 has to pay if the option is exercised by c_2 at s or canceled at the stopping time θ,

$$R(s,\theta) := V_\theta 1_{\{\theta < s\}} + L_s 1_{\{s < \theta\}} + Q_s 1_{\{\theta = s < T\}} + \xi 1_{\{\theta = s = T\}}, \quad s \in [t,T], \ \mathbb{P} - a.s.$$

Denote by (π, θ) a hedge for the broker against the American game option after time t, where π is defined in (5.32), $\theta \in [t,T]$ is a stopping time, satisfying

$$A_{s \wedge \theta}^{\pi,\alpha} \geq R(s,\theta), \qquad s \in [t,T], \ \mathbb{P} - a.s. \tag{5.34}$$

The hedging strategy (π, θ) for the broker has two components, a portfolio π and a stopping time θ. In an American game option contract, the broker is allowed to cancel the contract, so he only needs to hedge with a portfolio up to its cancellation time.

Similarly to El Karoui et. al [37] (1997), Karatzas and Shreve [103] (1998), we define the value of the option at time t by J_t, where $(J_t)_{0\le t\le T}$ is an rcll (right continuous with left limits) process, for any $t \in [0,T]$,

$$J_t := ess\inf\Big\{\alpha^A \ge 0;\ \mathcal{G}_t\text{-measurable such that there exists a hedge } (\pi,\theta) \text{ after } t, \pi \text{ is a self-financing portfolio after } t \text{ with corresponding wealth at time } t \text{ is } \alpha^A.\Big\} \tag{5.35}$$

Consider the following RBSDE with two obstacles and default risk, for any $t \in [0,T]$, there exist a stopping time θ_t, a process $\left(Z_s^{\pi,\alpha}\right)_{t\le s\le T}$ and the increasing processes $\left(K_s^{\pi,\alpha+}\right)_{t\le s\le T}$ and $\left(K_s^{\pi,\alpha-}\right)_{t\le s\le T}$, such that

$$\begin{cases} Y_t^{\pi,\alpha} = Y_{\theta_t}^{\pi,\alpha} + \left(K_{\theta_t}^{\pi,\alpha,+} - K_s^{\pi,\alpha,+}\right) - \left(K_{\theta_t}^{\pi,\alpha-} - K_s^{\pi,\alpha-}\right) - \int_s^{\theta_t} Z_u^{\pi,\alpha} dB_u^{\mathbb{Q}} \\ \qquad - \int_s^{\theta_t} U_u^{\pi,\alpha} dM_u^{\mathbb{Q}}, \qquad s \in [t,\theta_t]; \\ Y_T^{\pi,\alpha} = e^{-rT}\xi; \\ e^{-rs}L_s \le Y_s^{\pi,\alpha} \le e^{-rs}V_s, \qquad s\in[t,T]; \\ \int_t^{\theta_t}(Y_u^{\pi,\alpha} - e^{-ru}L_u)dK_u^{\pi,\alpha+} = \int_t^{\theta_t}(e^{-ru}V_u - Y_u^{\pi,\alpha})dK_u^{\pi,\alpha-} = 0; \end{cases} \tag{5.36}$$

For any $s \in [t,T]$, $e^{rt}Y_t^{\pi,\alpha}$ is the value of the option , i.e. $J_t = e^{rt}Y_t^{\pi,\alpha}$ (see Theorem 5.6.1). Similarly to Proposition 4.3 in Hamadène [100], we set

$$\begin{aligned} \theta_t^* &:= \inf\left\{s \ge t;\ Y_t^{\pi,\alpha} = e^{-rs}V_s\right\} \wedge T = \inf\left\{s\ge t;\ K_s^{\pi,\alpha-} > 0\right\}; \\ v_t^* &:= \inf\left\{s \ge t;\ Y_t^{\pi,\alpha} = e^{-rs}L_s\right\} \wedge T = \inf\left\{s\ge t;\ K_s^{\pi,\alpha+} > 0\right\}. \end{aligned} \tag{5.37}$$

The main result is represented in Theorem 5.6.1 below, for any $s \in [t,T]$, $e^{rt}Y_t^{\pi,\alpha}$ is the value of the game option, i.e. $J_t = e^{rt}Y_t^{\pi,\alpha}$.

Theorem 5.6.1. *For any $s \in [t,T]$, let the RBSDE (5.36) have a unique solution and assume that there exists a portfolio process π such that the infimum in (5.35) is attained. Then, $e^{rt}Y_t^{\pi,\alpha}$ is the value of the game option, i.e. $J_t = e^{rt}Y_t^{\pi,\alpha}$.*

Proof. **Step 1**. We first prove $J_t \ge e^{rt}Y_t^{\pi,\alpha}$.

Similarly to the proof method of Theorem 5.1 in Hamadène [100], for any fixed time $t \in [0,T]$, there exists a hedge (π,θ) after time t for the broker against the American game option. By (5.34) and (5.35), it follows that $\theta \ge t$, $\pi = \left(\beta_s^{(1)},\beta_s^{(2)}\right)_{s\in[t,T]}$ is a self-financing portfolio whose value at time t is A, satisfying $A_s^{\pi,\alpha} \ge R(s,\theta)$, here $s\in[t,T]$. By (5.32) and the Itô formula for rcll semi-martingale (Theorem 1.3.4), we can obtain

$$\begin{aligned} e^{-r(s\wedge\theta)}A_{s\wedge\theta}^{\pi,\alpha} =& e^{-rt}\alpha^A + \int_t^{s\wedge\theta}\beta_u^{(2)}e^{-ru}\sigma S_u dB_u^{\mathbb{Q}} + \int_t^{s\wedge\theta}\beta_u^{(2)}e^{-ru}\chi S_u dM_u^{\mathbb{Q}} \\ \ge& e^{-r(s\wedge\theta)}R(s,\theta). \end{aligned} \tag{5.38}$$

Let $v \in [t,T]$ be a $\mathcal{G}$-stopping time, set $s = v$ and take the conditional expectation in (5.38), it follows

$$e^{-rt}\alpha^A \geq \mathbb{E}^{\mathbb{Q}}\left[e^{-r(v\wedge\theta)}R(v,\theta)\Big|\mathcal{G}_t\right].$$

Hence, similarly to the result of Proposition 4.3 in Hamadène [100],

$$\begin{aligned}
e^{-rt}\alpha^A \geq & ess\sup_{v\geq t}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(v\wedge\theta)}R(v,\theta)\Big|\mathcal{G}_t\right]\\
\geq & ess\inf_{\theta\geq t} ess\sup_{v\geq t}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(v\wedge\theta)}R(v,\theta)\Big|\mathcal{G}_t\right]\\
= & Y_t^{\pi,\alpha}.
\end{aligned}$$

It follows $J_t \geq e^r t Y_t^{\pi,\alpha}$.

Step 2. Then prove $J_t \leq e^{rt}Y_t^{\pi,\alpha}$.

By the definition of θ_t^* in (5.37), it follows

$$Y_{s\wedge\theta_t^*}^{\pi,\alpha} = Y_t^{\pi,\alpha} - K_{s\wedge\theta_t^*}^{\pi,\alpha,+} + \int_t^{s\wedge\theta_t^*} Z_u^{\pi,\alpha}dB_u^{\mathbb{Q}} + \int_t^{s\wedge\theta_t^*} U_u^{\pi,\alpha}dM_u^{\mathbb{Q}}. \tag{5.39}$$

Since

$$Y_{\theta_t^*}^{\pi,\alpha}1_{\{\theta_t^*<T\}} = e^{-r\theta_t^*}U_{\theta_t^*}1_{\{\theta_t^*<T\}} \geq e^{-r\theta_t^*}Q_{\theta_t^*}1_{\{\theta_t^*<T\}},$$

therefore, by (5.39), we can obtain

$$\begin{aligned}
Y_{s\wedge\theta_t^*}^{\pi,\alpha} =& Y_s^{\pi,\alpha}1_{\{s<\theta_t^*\}} + Y_{\theta_t^*}^{\pi,\alpha}1_{\{\theta_t^*<s\}} + Y_{\theta_t^*}^{\pi,\alpha}1_{\{s=\theta_t^*<T\}} + \xi 1_{\{s=\theta_t^*<T\}}\\
\geq & e^{-rs}L_s1_{\{s<\theta_t^*\}} + e^{-r\theta_t^*}U_{\theta_t^*}1_{\{\theta_t^*<T\}} + e^{-r\theta_t^*}Q_{\theta_t^*}1_{\{s=\theta_t^*<T\}} + \xi 1_{\{s=\theta_t^*<T\}}\\
=& e^{-r(s\wedge\theta_t^*)}R(s,\theta_t^*).
\end{aligned}$$

It follows

$$Y_t^{\pi,\alpha} + \int_t^{s\wedge\theta_t^*} Z_u^{\pi,\alpha}dB_u^{\mathbb{Q}} + \int_t^{s\wedge\theta_t^*} U_u^{\pi,\alpha}dM_u^{\mathbb{Q}} \geq e^{-r(s\wedge\theta_t^*)}R(s,\theta_t^*), \qquad s\in[t,T].$$

Set

$$\begin{aligned}
\bar{A}_s =& e^{rs}\left(Y_t^{\pi,\alpha} + \int_t^{s\wedge\theta_t^*} Z_u^{\pi,\alpha}dB_u^{\mathbb{Q}} + \int_t^{s\wedge\theta_t^*} U_u^{\pi,\alpha}dM_u^{\mathbb{Q}}\right);\\
\beta_s^2 =& e^{rs}\left(Z_s^{\pi,\alpha}(\sigma S_s)^{-1} + U_s^{\pi,\alpha}(\chi S_s)^{-1}\right)1_{\{s\leq\theta_t^*\}};\\
\beta_s^1 =& \left(\bar{A}_s - \beta^{(2)}S_s\right)C_s^{-1}.
\end{aligned}$$

Obviously, $\bar{A}_s = \beta_s^{(1)} C_s + \beta_s^{(2)} S_s$. Applying the Itô formula for rcll semi-martingale (Theorem 1.3.4), we can obtain

$$\begin{aligned}\bar{A}_s =& e^{rt} Y_t^{\pi,\alpha} + \int_t^s r\bar{A}_u du + \int_t^s e^{ru} Z_u^{\pi,\alpha} 1_{\{s\leq\theta_t^*\}} dB_u^{\mathbb{Q}} + \int_t^s e^{ru} U_u^{\pi,\alpha} 1_{\{s\leq\theta_t^*\}} dM_u^{\mathbb{Q}} \\ =& e^{rs} Y_t^{\pi,\alpha} + \int_t^s e^{ru} \beta_u^{(1)} dC_u + \int_t^s e^{ru} \beta_u^{(2)} dS_u.\end{aligned}$$

So $\pi = \left(\beta_s^{(1)}, \beta_s^{(2)}\right)_{s\in[t,T]}$ is a self-financing portfolio with value $e^{rt} Y_t^{\pi,\alpha}$ at time t. Since for any $s \in [t, T]$, $\bar{A}_{s\wedge\theta_t^*} \geq R(s, \theta_t^*)$, then (π, θ_t^*) is a hedge strategy against this American game option, it follows that $J_t \leq e^{rt} Y_t^{\pi,\alpha}$. □

Therefore, from Theorem 5.6.1, we can obtain

$$\hat{R}_t(v, \theta_t^*) \leq Y_t^{\pi,\alpha} = \hat{R}_t(v_t^*, \theta_t^*) \leq \hat{R}_t(v_t^*, \theta),$$

where

$$\begin{aligned}\hat{R}_t(v, \theta) := \mathbb{E}^{\mathbb{Q}} \Big[& e^{-r\theta} V_\theta 1_{\{\theta<v\}} + e^{-rv} L_v 1_{\{v<\theta\}} + e^{-r\theta} Q_\theta 1_{\{\theta=v<T\}} \\ & + e^{-rT} \xi 1_{\{\theta=v=T\}} \big| \mathcal{G}_t \Big], \quad \mathbb{P} - a.s.\end{aligned}$$

5.6.3 Numerical Simulation

To calculate the value of the American game option, we use the same calculation method as in Section 5.5, starting from $Y_n^{\pi,\alpha} = \xi$, and proceeding backward to solve $(Y_i^{\pi,\alpha}, Z_i^{\pi,\alpha}, U_i^{\pi,\alpha}, K_i^{\pi,\alpha+}, K_i^{\pi,\alpha}$ for $i = n-1, ..., 1, 0$ with step size Δ^n. The forward SDEs (5.30) and (5.31) can be numerically approximated by the Euler scheme on the time grid $(t_i)_{i=0,1,...n}$:

$$\begin{aligned}C_{i+1} =& C_i + rC_i\Delta^n; \\ S_{i+1} =& S_i + S_i \left(\mu\Delta^n + \sigma\Delta B_i^n + \chi\Delta M_i^n\right).\end{aligned}$$

In this case, we consider parameters as below:

$$\begin{aligned}s_0 =& 1.5,\ \ T = 1,\ \ r = 1.1,\ \ \mu = 1.5,\ \ \sigma = 0.5,\ \ \chi = 0.2, \\ L_t =& (S_t - 1)^+,\ \ V_t = 2\,(S_t - 1)^+,\ \ \xi = 1.2\,(S_T - 1)^+,\end{aligned}$$

In the case $n = 400$, Figure 5.8 represents one path of the Brownian motion, Figure 5.9 and Figure 5.10 represent the paths of the solution $Y^{\pi,\alpha}$, increasing processes $K^{\pi,\alpha-}$ and $K^{\pi,\alpha-}$ in the explicit reflected scheme where the random default time $\tau = 0.2$. We can see that $Y_t^{\pi,\alpha}$ stays between the lower obstacle $e^{-rt}L_t$ and the upper obstacle $e^{-rt}V_t$. In this example for $n = 400$, default time $\tau = 0.2$, we can get the solution $Y_0^{\pi,\alpha} = 0.6857$ from the explicit reflected scheme, i.e. the value of the game option at $t = 0$ in the defaultable model. In the case without the default risk, $Y_0^{\pi,\alpha} = 0.7704$, which means the occurrence of the default event could reduce the value of $Y^{\pi,\alpha}$. Figure 5.11 represents the situation without the default risk, the solution $Y^{\pi,\alpha}$ has a larger value than in the situation when the default case happens (Figure 5.9).

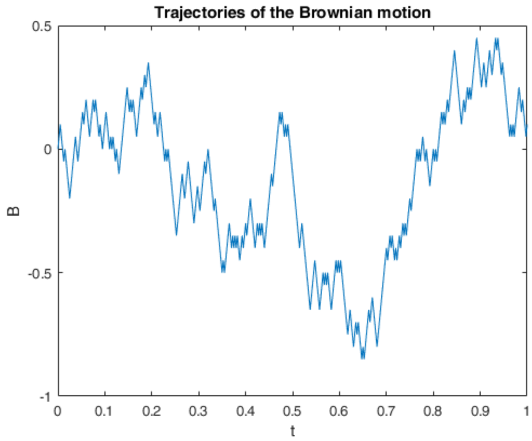

FIGURE 5.8: One path of the Brownian motion

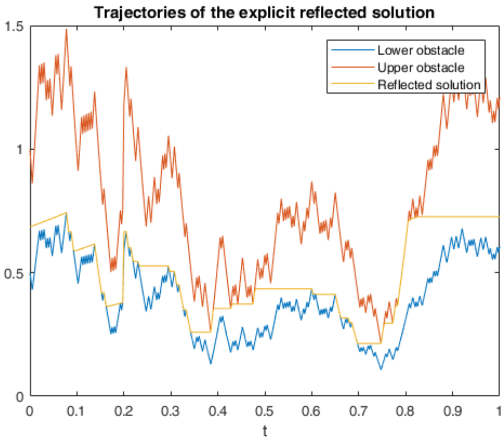

FIGURE 5.9: One path of $Y^{\pi,\alpha}$ in the explicit reflected scheme ($\tau = 0.2$)

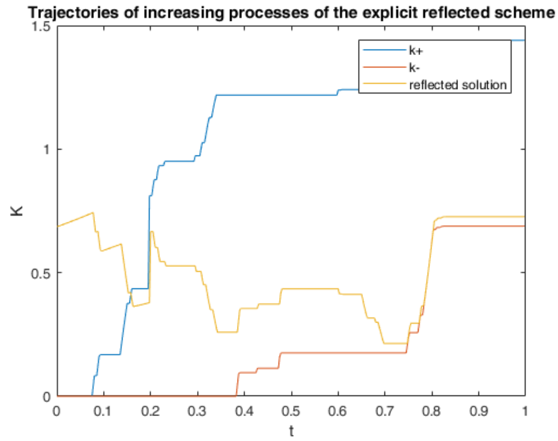

FIGURE 5.10: One path of the increasing processes in the explicit reflected scheme ($\tau = 0.2$)

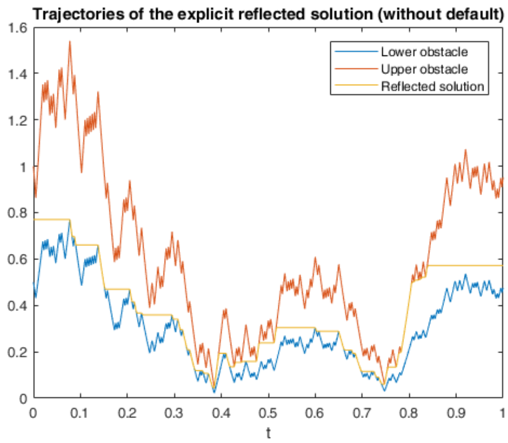

FIGURE 5.11: The paths of $Y^{\pi,\alpha}$ in the explicit reflected scheme without default risk

Appendix A

Appendix

A.1 Appendix for Chapter 3

Theorem A.1.1. ***(Burkholder-Davis-Gundy inequality)*** *(Meyer [88] (1966), p.304)*
For any $p \in [1, \infty)$, there exist the positive constants C_1^p and C_2^p, such that, for all the local martingales X with $X_0 = 0$ and stopping times v, the following inequality holds:

$$C_1^p \mathbb{E}\left[[X]_v^{\frac{p}{2}}\right] \le \mathbb{E}\left[\sup_{0\le s\le v} |X_s|^p\right] \le C_2^p \mathbb{E}\left[[X]_v^{\frac{p}{2}}\right],$$

where $[X]$ is the second variation of X.

Theorem A.1.2. ***(Section theorem)*** *(Meyer [88] (1966), p.220)*
Let X and Y be the stochastic processes, we can obtain the results below:

- ***Measurable selection**: If X and Y are jointly measurable and for each $\mathcal{F}$-measurable random time τ, $X_\tau = Y_\tau$ a.s., then it follows that $X = Y$.*
- ***Optional section**: If X and Y are optional and for each stopping time τ, $X_\tau = Y_\tau$ a.s., then it follows that $X = Y$.*
- ***Predictable section**: If X and Y are predictable and for each predictable stopping time τ, $X_\tau = Y_\tau$ a.s., then it follows that $X = Y$.*

Theorem A.1.3. ***(Dini's theorem)*** *(Meyer [88] (1966), p.202)*
Let S be a compact metric space, $f_n : S \to \mathbb{R}$ be a sequence of continuous functions. If $(f_n)_{n\in\mathbb{N}}$ is increasing, i.e. for all $x \in S$, $n \in \mathbb{N}$, $f_n(x) \le f_{n+1}(x)$, and $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a continuous function $f : S \to \mathbb{R}$, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Definition A.1.1. ***(Class $\mathcal{D}$)*** *Let $\theta := (\theta_t)_{0\le t\le T}$ be a $\mathcal{G}$-adapted rcll process. If $(\theta_v)_{v\in\mathcal{T}}$ is uniformly integrable, then the process θ is of class $\mathcal{D}[0, T]$. Here $\mathcal{T}$ is the set of all the stopping times on $[0, T]$.*

Definition A.1.2. ***(Snell envelope)*** *Suppose that θ is of class $\mathcal{D}[0, T]$, then its Snell envelope is defined as below:*

$$\mathcal{S}_t(\theta) = ess \sup_{\sigma\in\mathcal{T}_t} \mathbb{E}\left[\theta_\sigma|\mathcal{G}_t\right],$$

where $\mathcal{T}_t$ is the set of all the stopping times on $[t, T]$.

Definition A.1.3. ***(Predictable projection)*** *Suppose that $\phi := (\phi_t)_{0\le t\le T}$ is of class $\mathcal{D}[0,T]$, then its predictable projection ${}^p\phi$ is a $\mathcal{G}$-predictable process and satisfies*

$$^p\phi_\sigma = \mathbb{E}\left[\phi_\sigma | \mathcal{G}_{\sigma-}\right],$$

where σ is a $\mathcal{G}$-predictable stopping time. Moreover, if ${}^p\phi_t = \phi_{t-}$, $\forall t \in [0,T]$, we call that ϕ is regular.

Proposition A.1.1. *$\mathcal{S}(\theta)$ is the smallest rcll super-martingale of class $\mathcal{D}[0,T]$ which dominates process θ, i.e. for any $t \in [0,T]$, $\mathcal{S}_t(\theta) \ge \theta_t$, $\mathbb{P}-a.s.$*

Theorem A.1.4. ***(Doob-Meyer decomposition theorem for Snell envelope)*** *There exists a unique decomposition of the Snell envelope:*

$$\mathcal{S}_t(\theta) = N_t - K_t^c - K_t^d, \qquad \forall t \in [0,T],$$

where N_t is a $\mathcal{G}_t$-martingale, $K = K^c + K^d$, K^c (resp. K^d) is the continuous (resp. discontinuous) part of K, and $K_0^c = K_0^d = 0$. Moreover, the following results hold:

- *if $\mathbb{E}\left[\sup_{0\le t\le T} |\mathcal{S}_t(\theta)|^2\right] < \infty$, then $\mathbb{E}|K_T|^2 < \infty$;*
- *$\{\Delta K^d > 0\} \subset \{\mathcal{S}_-(\theta) = \theta_-\}$, and $\Delta K^d = (\theta_- - \mathcal{S}_-(\theta))^+ 1_{\{\mathcal{S}_-(\theta)=\theta_-\}}$.*

Theorem A.1.5. *Let $\theta := (\theta_t)_{0\le t\le T}$ be a $\mathcal{G}$-adapted rcll process, and $\mathbb{E}\left[\sup_{0\le t\le T} |\theta_t|^2\right] < \infty$. For any $t \in [0,T]$, define the stopping time v_t as below:*

$$v_t := \inf\{s \ge t;\ K_s \ge K_t\} \wedge T,$$

if the Snell envelope $\mathcal{S}(\theta)$ is regular, then $K^d \equiv 0$, and v_t is optimal on $\mathcal{T}_t$, i.e. it satisfies:

- *$\mathbb{E}\left[\theta_{v_t}\right] = \sup_{v\ge t} \mathbb{E}\left[\theta_v\right]$;*
- *$\mathcal{S}_{v_t}(\theta) = \theta_{v_t}$, and $(\mathcal{S}_{v_t\wedge s}(\theta))_{s\ge t}$ is an $\mathcal{G}_s$-martingale.*

Lemma A.1.1. ***(Skorohod lemma)*** *Let x be a real-valued continuous function on $[0,\infty)$ with $x_0 \ge 0$. There exists a unique pair (y,k) of functions on $[0,\infty)$, such that*

(1) $y = x + k$;

(2) y is positive;

(3) $(k_t)_{t<\infty}$ is a continuous and increasing process with $k_0 = 0$ and $\int_0^\infty y_t dk_t = 0$.

Then the pair (y,k) is defined as the solution of the Skorohod problem. Moreover, k is given by

$$k_t = \sup_{s\le t} x_s^-.$$

A.2 Appendix for Chapter 5

The following Donsker's invariance principle is a functional extension of the central limit theorem.

Theorem A.2.1. ***(Donsker's invariance principle)*** *Let $(X_i)_{i=1,2,...}$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1. Denote $S_n := \sum_{i=1}^n X_i$, then the process $S := (S_n)_{n=1,2,...}$ is a random walk. Define the diffusively rescaled random walk as below*

$$\tilde{B}_t^n := \frac{S_{[nt]}}{\sqrt{n}}, \qquad t \in [0,1].$$

Since the random variables taking values in the Skorokhod space $\mathcal{D}[0,1]$, the random function $\tilde{B}^n$ converges in distribution to a standard Brownian motion $B := (B_t)_{0\le t\le 1}$, as $n \to \infty$.

Theorem A.2.2. ***(Skorokhod representation theorem)*** *Let $(\mu_n)_{n=1,2,...}$ be a sequence of probability measures on a metric space $\mathcal{D}$, such that μ_n converges weakly to a probability measure μ which is a distribution with separable support. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, as well as the random variables X and $(X_n)_{n=1,2,...}$ on this space, such that the laws of X and X_n are μ and μ^n respectively, moreover, for all $w \in \Omega$, $X_n(w)$ converges weakly to $X(w)$.*

Lemma A.2.1. ***(Discrete Gronwall's Inequality)*** *Suppose that a, b and c are positive constants, $b\Delta < 1$, $(\beta_i)_{i\in\mathbb{N}}$ is a sequence with positive values, such that*

$$\beta_i + c \le a + b\Delta \sum_{j=1}^{i} \beta_j, \quad i \in \mathbb{N},$$

then it follows

$$\sup_{i\le n} \beta_i + c \le aF_\Delta(b),$$

where $F_\Delta(b)$ is a convergent series with the following form:

$$F_\Delta(b) = 1 + \sum_{n=1}^{\infty} \frac{b^n}{n}(1+\Delta)...(1+(n-1)\Delta).$$

Lemma A.2.2. *(**Estimation result of implicit discrete penalization scheme**) Under **H** 5.2 and **H** 5.3 hold, for each $p \in \mathbb{N}$ and Δ^n, when $\left(\Delta^n + 3\Delta^n L + 4\Delta^n L^2 + (\Delta^n L)^2\right) < 1$, there exists a constant $\lambda_{L,T,\delta}$ depending on the Lipschitz coefficient L, T and δ, such that*

$$\begin{aligned}\mathbb{E}\Bigg[&\sup_i |y_i^{p,n}|^2 + \Delta^n \sum_{j=0}^{n-1} |z_j^{p,n}|^2 + \Delta^n \sum_{j=0}^{n-1} |u_j^{p,n}|^2 (1-h_{j+1}^n)\gamma_{j+1} \\ &+ \frac{1}{p\Delta^n} \sum_{j=0}^{n-1} \left(|k_j^{+p,n}|^2 + |k_j^{-p,n}|^2\right)\Bigg] \le \lambda_{L,T,\delta} C_{\xi^n, f^n, L^n, V^n},\end{aligned} \tag{A.1}$$

where $C_{\xi^n,f^n,L^n,V^n} \ge 0$ is a constant depending on ξ^n, $f^n(t_j,0,0,0,0)$, $(L^n)^+$ and $(V^n)^-$.

Proof. By the definition of the implicit penalization discrete scheme (5.4), applying the Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|y_j^{p,n}|^2$ on $j \in [i, n-1]$, it follows

$$\begin{aligned}
&\mathbb{E}\left[|y_i^{p,n}|^2 + \Delta^n \sum_{j=i}^{n-1} |z_j^{p,n}|^2 + \Delta^n \sum_{j=i}^{n-1} |u_j^{p,n}|^2 (1 - h_{j+1}^n)\gamma_{j+1}\right] \\
=&\mathbb{E}\,|\xi_n^n|^2 + 2\Delta^n \mathbb{E} \sum_{j=i}^{n-1} \left[y_j^{p,n} \left| f^n(t_j, y_j^{p,n}, \bar{y}_j^{p,n}, z_j^{p,n}, u_j^{p,n}) \right|\right] \\
&+ 2\mathbb{E} \sum_{j=i}^{n-1} \left[y_j^{p,n} k_j^{+p,n} - y_j^{p,n} k_j^{-p,n}\right].
\end{aligned} \tag{A.2}$$

Since

$$\begin{aligned}
y_j^{p,n} k_j^{+p,n} =& - p\Delta^n \left[\left(y_j^{p,n} - L_j^n\right)^-\right]^2 + p\Delta^n L_j^n \left(y_j^{p,n} - L_j^n\right)^- \\
=& - \frac{1}{p\Delta^n} \left(k_j^{+p,n}\right)^2 + L_j^n k_j^{+p,n}; \\
y_j^{p,n} k_j^{-p,n} =& p\Delta^n \left[\left(y_j^{p,n} - V_j^n\right)^+\right]^2 + p\Delta^n L_j^n \left(y_j^{p,n} - V_j^n\right)^+ \\
=& \frac{1}{p\Delta^n} \left(k_j^{-p,n}\right)^2 + V_j^n k_j^{-p,n}.
\end{aligned}$$

Moreover, by the Lipschitz condition of f^n, we can obtain

$$\begin{aligned}
&\mathbb{E}\Bigg[|y_i^{p,n}|^2 + \frac{\Delta^n}{2} \sum_{j=i}^{n-1} |z_j^{p,n}|^2 + \frac{\Delta^n}{2} \sum_{j=i}^{n-1} |u_j^{p,n}|^2 (1 - h_{j+1}^n)\gamma_{j+1} \\
&\quad + \frac{2}{p\Delta^n} \sum_{j=i}^{n-1} \left(|k_j^{+p,n}|^2 + |k_j^{-p,n}|^2\right)\Bigg] \\
\leq&\mathbb{E}\left[\,|\xi_n^n|^2 + \Delta^n L \sum_{j=n+1}^{n^\delta - 1} \left|\xi_j^n\right|^2\right] + \Delta^n \mathbb{E}\left[\sum_{j=i}^{n-1} |f^n(t_j, 0, 0, 0, 0)|^2\right] \\
&+ \left(\Delta^n + 3\Delta^n L + 4\Delta^n L^2 + (\Delta^n L)^2\right) \mathbb{E}\left[\sum_{j=i}^{n-1} |y_j^n|^2\right] \\
&+ \frac{1}{\lambda_1} \mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{+p,n}\right)^2 + \lambda_1 \mathbb{E} \sup_{i \leq j \leq n-1} ((L_j^n)^+)^2 \\
&+ \frac{1}{\lambda_1} \mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{-p,n}\right)^2 + \lambda_1 \mathbb{E} \sup_{i \leq j \leq n-1} ((V_j^n)^-)^2.
\end{aligned}$$

By the assumption **H 5.4**, similarly to the proof of Lemma 4.2.1, applying techniques of stopping times for the discrete case, it follows

$$\begin{aligned}&\mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{+p,n}\right)^2+\mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{-p,n}\right)^2\\ \leq & C_{\xi^n,f^n,X^n}\left[1+\Delta^n\mathbb{E}\sum_{j=i}^{n-1}\left(|z_j^{p,n}|^2+|u_j^{p,n}|^2(1-h_{j+1}^n)\gamma_{j+1}\right)\right].\end{aligned}$$

where $C_{\xi^n,f^n,X^n}\geq 0$ is a constant depending on ξ^n, $f^n(t_j,0,0,0,0)$, X^n. Since X^n can be dominated by L^n and V^n, we can replace it by L^n and V^n. By the discrete Gronwall's inequality (Lemma A.2.1), when $\left(\Delta^n+3\Delta^n L+4\Delta^n L^2+(\Delta^n L)^2\right)<1$, we can obtain

$$\begin{aligned}\sup_i\mathbb{E}\left[|y_i^{p,n}|^2\right]+\mathbb{E}\Bigg[\Delta^n\sum_{j=0}^{n}|z_j^{p,n}|^2+\Delta^n\sum_{j=0}^{n}|u_j^{p,n}|^2(1-h_{j+1}^n)\gamma_{j+1}\\ +\frac{1}{p\Delta^n}\sum_{j=0}^{n}\left(|k_j^{+p,n}|^2+|k_j^{-p,n}|^2\right)\Bigg]\leq\lambda_{L,T,\delta}C_{\xi^n,f^n,L^n,V^n},\end{aligned}$$

where $C_{\xi^n,f^n,L^n,V^n}\geq 0$ is a constant depending on ξ^n, $f^n(t_j,0,0,0,0)$, $(L^n)^+$ and $(V^n)^-$. Reconsidering (A.2), we take square, sup and sum over j, then take expectation, by Burkholder-Davis-Gundy inequality for the martingale parts, it follows

$$\begin{aligned}\mathbb{E}\left[\sup_i|y_i^{p,n}|^2\right]\leq & C_{\xi^n,f^n,L^n,V^n}+C\Delta^n\left[\sum_{i=0}^{n-1}\mathbb{E}\left|y_i^{p,n}\right|^2\right]\\ \leq & C_{\xi^n,f^n,L^n,V^n}+CT\sup_i\mathbb{E}\left|y_i^{p,n}\right|^2.\end{aligned}$$

It follows (A.1). □

Theorem A.2.3. ***(Distance between implicit discrete penalization and explicit discrete penalization schemes)*** *Under **H** 5.2 and **H** 5.3, for any $p\in\mathbb{N}$:*

$$\begin{aligned}\mathbb{E}\Bigg[\sup_{0\leq t\leq T}|\tilde{Y}_t^{p,n}-Y_t^{p,n}|^2+\int_0^T|\tilde{Z}_t^{p,n}-Z_t^{p,n}|^2dt+\int_0^T|\tilde{U}_t^{p,n}-U_t^{p,n}|^2 1_{\{\tau>t\}}\gamma_t dt\\ +\left|\left(\tilde{K}_t^{+p,n}-\tilde{K}_t^{-p,n}\right)-\left(K_t^{+p,n}-K_t^{-p,n}\right)\right|^2\Bigg]\leq\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n,p}(\Delta^n)^2,\end{aligned}\tag{A.3}$$

where $\lambda_{L,T,\delta}\geq 0$ is a constant depending on the Lipschitz coefficient L, the terminal T and δ, $C_{f^n,\xi^n,\Delta^n,L^n,V^n,p}\geq 0$ is a constant depending on $f^n(t_j,0,0,0,0)$, ξ^n, $(L^n)^+$, $(V^n)^-$ and p.

Proof. From the definitions of the implicit discrete penalization scheme (5.4) and the explicit discrete penalization scheme (5.8), as well as the Lipschitz condition of f^n, it follows

$$
\begin{aligned}
&\mathbb{E}\left[\left|\tilde{y}_j^{p,n}-y_j^{p,n}\right|^2+\Delta^n\left|\tilde{z}_j^{p,n}-z_j^{p,n}\right|^2+\Delta^n\left|\tilde{u}_j^{p,n}-u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right]\\
=&\mathbb{E}\left|\tilde{y}_{j+1}^{p,n}-y_{j+1}^{p,n}\right|^2+2\Delta^n\mathbb{E}\Big[\big(f^n(t_j,\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^{p,n}],\bar{\tilde{y}}_j^{p,n},\tilde{z}_j^{p,n},\tilde{u}_j^{p,n})\\
&-f^n(t_j,y_j^{p,n},\bar{y}_j^{p,n},z_j^{p,n},u_j^{p,n})\big)\left(\tilde{y}_j^{p,n}-y_j^{p,n}\right)\Big]\\
&-(\Delta^n)^2\mathbb{E}\Big[f^n(t_j,y_j^{p,n},\bar{y}_j^{p,n},z_j^{p,n},u_j^{p,n})-f^n(t_j,\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^{p,n}],\bar{\tilde{y}}_j^{p,n},\tilde{z}_j^{p,n},\tilde{u}_j^{p,n})\\
&\qquad+(\tilde{k}_j^{+p,n}-k_j^{+p,n})-(\tilde{k}_j^{-p,n}-k_j^{-p,n})\Big]^2\\
&+2\mathbb{E}\left[\left(\tilde{y}_j^{p,n}-y_j^{p,n}\right)\left((\tilde{k}_j^{+p,n}-k_j^{+p,n})-(\tilde{k}_j^{-p,n}-k_j^{-p,n})\right)\right]\\
\le&2\Delta^nL\mathbb{E}\Big[\Big(|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^{p,n}]-y_j^{p,n}|+|\bar{\tilde{y}}_j^{p,n}-\bar{y}_j^{p,n}|+|\tilde{z}_j^{p,n}-z_j^{p,n}|\\
&\qquad+|\tilde{u}_j^{p,n}-u_j^{p,n}|(1-h_{j+1}^n)\sqrt{\gamma_{j+1}}\Big)\left(\tilde{y}_j^{p,n}-y_j^{p,n}\right)\Big],
\end{aligned}
\tag{A.4}
$$

since the function $y\to(y-L_t^n)^--(y-V_t^n)^+$ is decreasing, then

$$
\begin{aligned}
&\mathbb{E}\left[\left(\tilde{y}_j^{p,n}-y_j^{p,n}\right)\left((\tilde{k}_j^{+p,n}-k_j^{+p,n})-(\tilde{k}_j^{-p,n}-k_j^{-p,n})\right)\right]\\
=&p\Delta^n\mathbb{E}\Big[\left(\tilde{y}_j^{p,n}-y_j^{p,n}\right)\Big(\left((\tilde{y}_j^{p,n}-L_j^n)^--(\tilde{y}_j^{p,n}-V_j^n)^+\right)\\
&\qquad-\left((y_j^{p,n}-L_j^n)^--(y_j^{p,n}-V_j^n)^+\right)\Big)\Big]\le 0.
\end{aligned}
$$

Taking sum from $j=i,...,n-1$, it follows

$$
\begin{aligned}
&\mathbb{E}\left[|\tilde{y}_i^{p,n}-y_i^{p,n}|^2+\frac{\Delta^n}{2}\sum_{j=i}^{n-1}\left|\tilde{z}_j^{p,n}-z_j^{p,n}\right|^2+\frac{\Delta^n}{2}\sum_{j=i}^{n-1}\left|\tilde{u}_j^{p,n}-u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right]\\
\le&2\Delta^nL\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^{p,n}]-y_j^{p,n}\right||\tilde{y}_j^{p,n}-y_j^{p,n}|\right]\\
&+\Delta^n(2L+4L^2)\mathbb{E}\left[\sum_{j=i}^{n-1}\left|\tilde{y}_j^{p,n}-y_j^{p,n}\right|^2\right].
\end{aligned}
$$

By (5.14), (5.8) and the Lipschitz condition of f^n, we can obtain

$$
\begin{aligned}
&2\Delta^nL\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^{p,n}]-y_j^{p,n}\right||\tilde{y}_j^{p,n}-y_j^{p,n}|\right]\\
\le&2\Delta^nL\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\left(\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^{p,n}]-\tilde{y}_j^{p,n}\right)+\left(\tilde{y}_j^{p,n}-y_j^{p,n}\right)\right||\tilde{y}_j^{p,n}-y_j^{p,n}|\right]\\
\le&2\Delta^nL\mathbb{E}\sum_{j=i}^{n-1}\Big[\Big(\Big|f^n(t_j,\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^{p,n}],\bar{\tilde{y}}_j^{p,n},\tilde{z}_j^{p,n},\tilde{u}_j^{p,n})\Delta^n+\tilde{k}_j^{+p,n}-\tilde{k}_j^{-p,n}\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\tilde{y}_j^{p,n}-y_j^{p,n}\right|\Big)\left|\tilde{y}_j^{p,n}-y_j^{p,n}\right|\Big] \\
\leq&(\Delta^n L)^2\mathbb{E}\sum_{j=i}^{n-1}\left[2|\tilde{y}_j^{p,n}|^2+|\tilde{z}_j^{p,n}|^2+|\tilde{u}_i^{p,n}|^2(1-h_{i+1}^n)\gamma_{i+1}\right] \\
&+(\Delta^n)^2\mathbb{E}\sum_{j=i}^{n-1}\left[\left|f^n(t_j,0,0,0,0)\right|^2\right] \\
&+(p\Delta^n)^2\mathbb{E}\sum_{j=i}^{n-1}\left[\left((\tilde{y}_i^{p,n}-L_i^n)^-\right)^2+\left((\tilde{y}_i^{p,n}-V_i^n)^+\right)^2\right] \\
&+\left(8(\Delta^n L)^2+2\Delta^n L\right)\mathbb{E}\left[\sum_{i=j}^{n-1}\left|\tilde{y}_j^{p,n}-y_j^{p,n}\right|^2\right]+(\Delta^n L)^2\mathbb{E}\left[\sum_{i=n}^{n^\delta-1}|\xi_j^n|^2\right],
\end{aligned}
$$

Therefore, there exists a constant $C_{f^n,\xi^n,L^n,V^n}\geq 0$ depending on $f^n(t_j,0,0,0,0)$, ξ^n, $(L^n)^+$ and $(V^n)^-$, such that

$$
\begin{aligned}
&\mathbb{E}\left[|\tilde{y}_i^{p,n}-y_i^{p,n}|^2+\frac{\Delta^n}{2}\sum_{j=i}^{n-1}\left|\tilde{z}_j^{p,n}-z_j^{p,n}\right|^2+\frac{\Delta^n}{2}\sum_{j=i}^{n-1}\left|\tilde{u}_j^{p,n}-u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right] \\
\leq&C_{f^n,\xi^n,L^n,V^n}(\Delta^n)^2+\left(8(\Delta^n L)^2+4\Delta^n L+4\Delta^n L^2\right)\mathbb{E}\left[\sum_{j=i}^{n-1}\left|\tilde{y}_j^{p,n}-y_j^{p,n}\right|^2\right].
\end{aligned}
$$

By the discrete Gronwall's inequality (Lemma A.2.1), when $\left(8(\Delta^n L)^2+4\Delta^n L+4\Delta^n L^2\right)<1$, we can get

$$
\sup_i \mathbb{E}\,|\tilde{y}_i^{p,n}-y_i^{p,n}|^2\leq C_{f^n,\xi^n,L^n,V^n}(\Delta^n)^2 e^{\left(8\Delta^n L^2+4L+4L^2\right)T},
$$

from (5.29), it follows

$$
\mathbb{E}\left[\sum_{j=i}^{n-1}\left|\tilde{z}_j^{p,n}-z_j^{p,n}\right|^2+\sum_{j=i}^{n-1}\left|\tilde{u}_j^{p,n}-u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right]\leq C_{f^n,\xi^n,L^n,V^n}(\Delta^n)^2.
$$

Reconsider (A.4), we take square, sup and sum over j, then take expectation, by Burkholder-Davis-Gundy inequality for the martingale parts, we can get

$$
\mathbb{E}\left[\sup_i|\tilde{y}_i^{p,n}-y_i^{p,n}|^2\right]\leq C\Delta^n\left[\sum_{i=0}^{n-1}\mathbb{E}\,|\tilde{y}_i^{p,n}-y_i^{p,n}|^2\right]\leq CT\sup_i\mathbb{E}\,|\tilde{y}_i^{p,n}-y_i^{p,n}|^2,
$$

it follows

$$
\begin{aligned}
&\mathbb{E}\left[\sup_i|\tilde{y}_i^{p,n}-y_i^{p,n}|^2+\Delta^n\sum_{j=0}^{n-1}\left|\tilde{z}_j^{p,n}-z_j^{p,n}\right|^2+\Delta^n\sum_{j=0}^{n-1}\left|\tilde{u}_j^{p,n}-u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right] \\
\leq&\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n,p}(\Delta^n)^2.
\end{aligned}
\tag{A.5}
$$

For the increasing processes, for each fixed p,

$$\begin{aligned}\tilde{K}_t^{+p,n}-\tilde{K}_t^{-p,n}=&\tilde{Y}_0^{p,n}-\tilde{Y}_t^{p,n}-\int_0^t f(s,\tilde{Y}_s^{p,n},\mathbb{E}^{\mathcal{G}_s}[\tilde{Y}_{s+\delta}^{p,n}],\tilde{Z}_s^{p,n},\tilde{U}_s^{p,n})ds\\&-\int_0^t \tilde{Z}_s^{p,n}dB_s^n-\int_0^t \tilde{U}_s^{p,n}dM_s^n;\\K_t^{+p,n}-K_t^{-p,n}=&Y_0^{p,n}-Y_t^{p,n}-\int_0^t f(s,Y_s^{p,n},\mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}^{p,n}],Z_s^{p,n},U_s^{p,n})ds\\&-\int_0^t Z_s^{p,n}dB_s^n-\int_0^t U_s^{p,n}dM_s^n.\end{aligned}$$

By the Lipschitz condition of f^n and (A.5), it follows

$$\mathbb{E}\left[\left|\left(\tilde{K}_t^{+p,n}-\tilde{K}_t^{-p,n}\right)-\left(K_t^{+p,n}-K_t^{-p,n}\right)\right|^2\right]\leq\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n,p}(\Delta^n)^2.$$

It follows (A.3). □

Theorem A.2.4. *(**Distance between implicit discrete penalization and implicit discrete reflected schemes**) Under **H** 5.2 and **H** 5.3, for any $p\in\mathbb{N}$:*

$$\begin{aligned}\mathbb{E}\Bigg[\sup_{0\leq t\leq T}|Y_t^n-Y_t^{p,n}|^2+\int_0^T|Z_t^n-Z_t^{p,n}|^2dt+\int_0^T|U_t^n-U_t^{p,n}|^2 1_{\{\tau>t\}}\gamma_t dt\\+\left|(K_t^{+n}-K_t^{-n})-\left(K_t^{+p,n}-K_t^{-p,n}\right)\right|^2\Bigg]\leq\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n}\frac{1}{\sqrt{p}},\end{aligned}\tag{A.6}$$

where $\lambda_{L,T,\delta}\geq 0$ is a constant depending on the Lipschitz coefficient L, the terminal time T and δ, $C_{f^n,\xi^n,L^n,V^n}\geq 0$ is a constant depending on $f^n(t_j,0,0,0,0)$, ξ^n, L^n and V^n.

Proof. By the definitions of the implicit discrete reflected scheme (5.10) and the implicit discrete penalization scheme (5.4), applying the Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|y_j^n-y_j^{p,n}|^2$ on $j\in[i,n-1]$, it follows

$$\begin{aligned}&\mathbb{E}\left[\left|y_i^n-y_i^{p,n}\right|^2+\Delta^n\sum_{j=i}^{n-1}\left|z_j^n-z_j^{p,n}\right|^2+\Delta^n\sum_{j=i}^{n-1}\left|u_j^n-u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right]\\=&2\Delta^n\mathbb{E}\sum_{j=i}^{n-1}\left[\left|f^n(t_j,y_j^n,\bar{y}_j^n,z_j^n,u_j^n)-f^n(t_j,y_j^{p,n},\bar{y}_j^{p,n},z_j^{p,n},u_j^{p,n})\right|\left(y_j^n-y_j^{p,n}\right)\right]\\&+2\mathbb{E}\sum_{j=i}^{n-1}\left[\left(y_j^n-y_j^{p,n}\right)\left(k_j^{+n}-k_j^{+p,n}\right)-\left(y_j^n-y_j^{p,n}\right)\left(k_j^{-n}-k_j^{-p,n}\right)\right].\end{aligned}\tag{A.7}$$

Since $(y_j^n-L_j^n)k_j^{+n}=(y_j^n-V_j^n)k_j^{-n}=0$ and $k_j^{+p,n}=p\Delta^n(y_j^{p,n}-L_j^n)^-$, $k_j^{-p,n}=p\Delta^n(y_j^{p,n}-V_j^n)^+$, it follows

$$\begin{aligned}
&\left(y_j^n - y_j^{p,n}\right)\left(k_j^{+n} - k_j^{+p,n}\right)\\
=&\left(y_j^n - L_j^n\right)k_j^{+n} - \left(y_j^{p,n} - L_j^n\right)k_j^{+n} - \left(y_j^n - L_j^n\right)k_j^{+p,n} + \left(y_j^{p,n} - L_j^n\right)k_j^{+p,n}\\
\leq&\left(y_j^{p,n} - L_j^n\right)^- k_j^{+n} - p\Delta^n\left((y_j^{p,n} - L_j^n)^-\right)^2\\
\leq&\left(y_j^{p,n} - L_j^n\right)^- k_j^{+n} = \frac{1}{p\Delta^n}k_j^{+p,n}k_j^{+n},
\end{aligned}$$

similarly, we can obtain

$$\left(y_j^n - y_j^{p,n}\right)\left(k_j^{-n} - k_j^{-p,n}\right) \geq -\left(y_j^{p,n} - V_j^n\right)^+ k_j^{-n} = \frac{1}{p\Delta^n}k_j^{-p,n}k_j^{-n}.$$

By (A.11), (A.12) and the Lipschitz condition of f^n, it follows

$$\begin{aligned}
&\mathbb{E}\left[\,|y_i^n - y_i^{p,n}|^2 + \Delta^n\sum_{j=i}^{n-1}\left|z_j^n - z_j^{p,n}\right|^2 + \Delta^n\sum_{j=i}^{n-1}\left|u_j^n - u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right]\\
\leq&\Delta^n(4L+4L^2)\mathbb{E}\left[\sum_{j=i}^{n-1}\left|y_j^n - y_j^{p,n}\right|^2\right]\\
&+2\mathbb{E}\sum_{j=i}^{n-1}\left[\left(y_j^{p,n} - L_j^n\right)^- k_j^{+n} + \left(y_j^{p,n} - V_j^n\right)^+ k_j^{-n}\right]\\
\leq&\Delta^n(4L+4L^2)\mathbb{E}\left[\sum_{j=i}^{n-1}\left|y_j^n - y_j^{p,n}\right|^2\right]\\
&+\frac{2}{\sqrt{p}}\left(\frac{1}{p\Delta^n}\mathbb{E}\sum_{j=i}^{n-1}\left(k_j^{+p,n}\right)^2\right)^{\frac{1}{2}}\left(\Delta^n\mathbb{E}\sum_{j=i}^{n-1}\left(k_j^{+n}\right)^2\right)^{\frac{1}{2}}\\
&+\frac{2}{\sqrt{p}}\left(\frac{1}{p\Delta^n}\mathbb{E}\sum_{j=i}^{n-1}\left(k_j^{-p,n}\right)^2\right)^{\frac{1}{2}}\left(\Delta^n\mathbb{E}\sum_{j=i}^{n-1}\left(k_j^{-n}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma A.2.2, Lemma A.2.3 and the discrete Gronwall's inequality (Lemma A.2.1), we can obtain

$$\begin{aligned}
&\sup_i \mathbb{E}\left[\,|y_i^n - y_i^{p,n}|^2\right] + \mathbb{E}\left[\Delta^n\sum_{j=0}^{n-1}\left|z_j^n - z_j^{p,n}\right|^2 + \Delta^n\sum_{j=0}^{n-1}\left|u_j^n - u_j^{p,n}\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right]\\
\leq&\lambda_{L,T,\delta}C_{f^n,\xi^n,L^n,V^n}\frac{1}{\sqrt{p}}.
\end{aligned}$$

where $C_{f^n,\xi^n,L^n,V^n} \geq 0$ is a constant depending on $f^n(t_j,0,0,0,0)$, ξ^n, L^n and V^n. Reconsider (A.7), we take square, sup and sum over j, then take expectation, by Burkholder-Davis-Gundy inequality for the martingale parts, it follows

$$\mathbb{E}\left[\sup_i |y_i^n - y_i^{p,n}|^2 + \Delta^n \sum_{j=0}^{n-1} \left|z_j^n - z_j^{p,n}\right|^2 + \Delta^n \sum_{j=0}^{n-1} \left|u_j^n - u_j^{p,n}\right|^2 (1-h_{j+1}^n)\gamma_{j+1}\right] \\ \leq \lambda_{L,T,\delta} C_{f^n,\xi^n,L^n,V^n} \frac{1}{\sqrt{p}}. \tag{A.8}$$

For the increasing processes, for each p,

$$\begin{aligned} K_t^{+n} - K_t^{-n} =& Y_0^n - Y_t^n - \int_0^t f\left(s, Y_s^n, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}^n], Z_s^n, U_s^n\right) ds \\ &- \int_0^t Z_s^n dB_s^n - \int_0^t U_s^n dM_s^n; \\ K_t^{+p,n} - K_t^{-p,n} =& Y_0^{p,n} - Y_t^{p,n} - \int_0^t f\left(s, Y_s^{p,n}, \mathbb{E}^{\mathcal{G}_s}[Y_{s+\delta}^{p,n}], Z_s^{p,n}, U_s^{p,n}\right) ds \\ &- \int_0^t Z_s^{p,n} dB_s^n - \int_0^t U_s^{p,n} dM_s^n. \end{aligned}$$

By the Lipschitz condition of f and (A.8), it follows

$$\mathbb{E}\left[\left|(K_t^{+n} - K_t^{-n}) - \left(K_t^{+p,n} - K_t^{-p,n}\right)\right|^2\right] \leq \lambda_{L,T,\delta} C_{f^n,\xi^n,L^n,V^n} \frac{1}{\sqrt{p}}.$$

It follows (A.6). □

Lemma A.2.3. *(**Estimation result of implicit discrete reflected scheme**) Under **H** 5.2 and **H** 5.3, for each $p \in \mathbb{N}$ and Δ^n, when $\Delta^n + 3\Delta^n L + 4\Delta^n L^2 + (\Delta^n)^2 L^2 < 1$, there exists a constant $\lambda_{L,T,\delta}$ depending on the Lipschitz coefficient L and the terminal time T, such that*

$$\mathbb{E}\left[\sup_i |y_i^n|^2 + \Delta^n \sum_{j=0}^{n-1} |z_j^n|^2 + \Delta^n \sum_{j=0}^{n-1} |u_j^n|^2 (1-h_{j+1}^n)\gamma_{j+1} \\ + \left|\sum_{j=0}^{n-1} k_j^{+n}\right|^2 + \left|\sum_{j=0}^{n-1} k_j^{-n}\right|^2\right] \leq \lambda_{L,T,\delta} C_{\xi^n,f^n,L^n,V^n}, \tag{A.9}$$

where $C_{\xi^n,f^n,L^n,V^n} \geq 0$ is a constant depending on ξ^n, $f^n(t_j,0,0,0,0)$, $(L^n)^+$ and $(V^n)^-$.

Proof. From the definition of implicit discrete reflected scheme (5.4), applying Itô formula for rcll semi-martingale (Theorem 1.3.4) to $|y_j^n|^2$ on $j \in [i, n-1]$, since $(y_i^n - L_i^n)k_i^{+n} = (y_i^n - V_i^n)k_i^{-n} = 0$ and the Lipschitz condition of f^n, it follows

$$\begin{aligned} &\mathbb{E}\left[|y_i^n|^2 + \Delta^n \sum_{j=i}^{n-1} \left|z_j^n\right|^2 + \Delta^n \sum_{j=i}^{n-1} \left|u_j^n\right|^2 (1-h_{j+1}^n)\gamma_{j+1}\right] \\ =& \mathbb{E}\,|\xi_n^n|^2 + 2\Delta^n \mathbb{E} \sum_{j=i}^{n-1} \left[y_j^n \left|f^n(t_j, y_j^n, \bar{y}_j^n, z_j^n, u_j^n)\right|\right] + 2\mathbb{E} \sum_{j=i}^{n-1} \left[y_j^n k_j^{+n} - y_j^n k_j^{-n}\right] \end{aligned}$$

$$
\begin{aligned}
\leq & \mathbb{E}\left[|\xi_n^n|^2 + \Delta^n \sum_{j=i}^{n-1} |f^n(t_j, 0,0,0,0)|^2\right] + \frac{\Delta^n}{2}\mathbb{E}\left[\sum_{j=i}^{n-1} |z_j^n|^2 + \left|u_j^n\right|^2 (1-h_{j+1}^n)\gamma_{j+1}\right] \\
& + \left(\Delta^n + 2\Delta^n L + 4\Delta^n L^2 + (\Delta^n L)^2\right)\mathbb{E}\left[\sum_{j=i}^{n-1} |y_j^n|^2\right] + \mathbb{E}\left[\sum_{j=i}^{n^\delta - 1} |y_j^n|^2\right] \\
& + \frac{1}{\lambda_1}\mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{+n}\right)^2 + \lambda_1 \mathbb{E}\left[\sup_{i\leq j\leq n-1} ((L_j^n)^+)^2\right] \\
& + \frac{1}{\lambda_1}\mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{-n}\right)^2 + \lambda_1 \mathbb{E}\left[\sup_{i\leq j\leq n-1} ((V_j^n)^-)^2\right].
\end{aligned}
\tag{A.10}
$$

where $\lambda_1 > 0$ is a constant. By the estimation of k_j^{+n} and k_j^{-n} in (5.12), it follows

$$
\begin{aligned}
k_j^{+n} &\leq \left(\mathbb{E}^{\mathcal{G}_j^n}[L_{j+1}^n] + f^n(t_j, L_j^n, \bar{L}_j^n, z_j^n, u_j^n)\Delta^n - L_j^n\right)^-; \\
k_j^{-n} &\leq \left(\mathbb{E}^{\mathcal{G}_j^n}[V_{j+1}^n] + f^n(t_j, V_j^n, \bar{V}_j^n, z_j^n, u_j^n)\Delta^n - V_j^n\right)^+.
\end{aligned}
\tag{A.11}
$$

where $L_i^n = V_i^n = \xi_i^n$, $(i \in [n, n^\delta])$. Therefore, by the Lipschitz condition of f^n,

$$
\begin{aligned}
\mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{+n}\right)^2 \leq & 6\mathbb{E}\sum_{j=i}^{n-1}\Big[\left|\mathbb{E}^{\mathcal{G}_j^n}[L_{j+1}^n] - L_j^n\right|^2 + (\Delta^n)^2 |f^n(t_j,0,0,0,0)|^2 \\
& + (\Delta^n L)^2 \Big(2\left|L_j^n\right|^2 + \left|z_j^n\right|^2 + \left|u_j^n\right|^2 (1-h_{j+1}^n)\gamma_{j+1}\Big)\Big] \\
& + 6(\Delta^n L)^2 \mathbb{E}\left[\sum_{j=n}^{n^\delta-1}\left|\xi_j^n\right|^2\right]; \\
\mathbb{E}\left(\sum_{j=i}^{n-1} k_j^{-n}\right)^2 \leq & 6\mathbb{E}\sum_{j=i}^{n-1}\Big[\left|\mathbb{E}^{\mathcal{G}_j^n}[V_{j+1}^n] - V_j^n\right|^2 + (\Delta^n)^2 |f^n(t_j,0,0,0,0)|^2 \\
& + (\Delta^n L)^2 \Big(2\left|V_j^n\right|^2 + \left|z_j^n\right|^2 + \left|u_j^n\right|^2 (1-h_{j+1}^n)\gamma_{j+1}\Big)\Big] \\
& + 6(\Delta^n L)^2 \mathbb{E}\left[\sum_{j=n}^{n^\delta-1}\left|\xi_j^n\right|^2\right].
\end{aligned}
\tag{A.12}
$$

Set $\lambda_1 = 48\Delta^n L^2$, it follows

$$
\begin{aligned}
& \mathbb{E}\left[|y_i^n|^2 + \frac{\Delta^n}{4}\sum_{j=i}^{n-1}\left|z_j^n\right|^2 + \frac{\Delta^n}{4}\sum_{j=i}^{n-1}\left|u_j^n\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right] \\
\leq & \mathbb{E}\left[|\xi_n^n|^2\right] + \left(\Delta^n L + \frac{\Delta^n}{4}\right)\mathbb{E}\left[\sum_{j=n}^{n^\delta-1}\left|\xi_j^n\right|^2\right]
\end{aligned}
$$

$$+\left(\Delta^n+3\Delta^n L+4\Delta^n L^2+(\Delta^n)^2L^2\right)\mathbb{E}\left[\sum_{j=i}^{n-1}|y_j^n|^2\right]$$
$$+48\Delta^n L^2\mathbb{E}\left[\sup_{i\le j\le n-1}((L_j^n)^+)^2+\sup_{i\le j\le n-1}((V_j^n)^-)^2\right]$$
$$+\frac{\Delta^n}{2}\mathbb{E}\sum_{j=i}^{n-1}\left[\left|L_j^n\right|^2+\left|V_j^n\right|^2\right]+\Delta^n\left(1+\frac{1}{4L^2}\right)\mathbb{E}\left[\sum_{j=i}^{n-1}|f^n(t_j,0,0,0,0)|^2\right]$$
$$+\frac{1}{4\Delta^n L^2}\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\mathbb{E}^{\mathcal{G}_j^n}[L_{j+1}^n]-L_j^n\right|^2+\left|\mathbb{E}^{\mathcal{G}_j^n}[V_{j+1}^n]-V_j^n\right|^2\right].$$

By the discrete Gronwall's inequality (Lemma A.2.1), when $\Delta^n+3\Delta^n L+4\Delta^n L^2+(\Delta^n)^2L^2<1$, we can obtain

$$\sup_i\mathbb{E}|y_i^n|^2+\mathbb{E}\left[\Delta^n\sum_{j=0}^{n-1}|z_j^n|^2+\Delta^n\sum_{j=0}^{n-1}|u_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}\right.$$
$$\left.+\sum_{j=0}^{n-1}\left(|k_j^{+n}|^2+|k_j^{-n}|^2\right)\right]\le\lambda_{L,T,\delta}C_{\xi^n,f^n,L^n,V^n},$$

where $C_{\xi^n,f^n,L^n,V^n}\ge 0$ is a constant depending on ξ^n, $f^n(t_j,0,0,0,0)$, $(L^n)^+$ and $(V^n)^-$. Reconsider (A.2), we take square, sup and sum over j, then take expectation, by Burkholder-Davis-Gundy inequality for the martingale parts, we can get

$$\begin{aligned}\mathbb{E}\left[\sup_i|y_i^n|^2\right]&\le C_{\xi^n,f^n,L^n,V^n}+C\Delta^n\left[\sum_{i=0}^{n-1}\mathbb{E}\,|y_i^n|^2\right]\\&\le C_{\xi^n,f^n,L^n,V^n}+CT\sup_i\mathbb{E}\,|y_i^n|^2.\end{aligned}$$

It follows (A.9). □

Lemma A.2.4. *(**Estimation result of explicit discrete reflected scheme**) Under **H** 5.2 and **H** 5.3, for each $p\in\mathbb{N}$ and Δ^n, when $\frac{7\Delta^n}{4}+2\Delta^n L+12\Delta^n L^2+10(\Delta^n L)^2<1$, there exists a constant $\lambda_{L,T,\delta}$ depending on the Lipschitz coefficient L, T and δ, such that*

$$\begin{aligned}\mathbb{E}\Bigg[&\sup_i|\tilde{y}_i^n|^2+\Delta^n\sum_{j=0}^{n-1}|\tilde{z}_j^n|^2+\Delta^n\sum_{j=0}^{n-1}|\tilde{u}_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}\\&+\left|\sum_{j=0}^{n-1}\tilde{k}_j^{+n}\right|^2+\left|\sum_{j=0}^{n-1}\tilde{k}_j^{-n}\right|^2\Bigg]\le\lambda_{L,T,\delta}C_{\xi^n,f^n,L^n,V^n},\end{aligned}\tag{A.13}$$

where $C_{\xi^n,f^n,L^n,V^n}\ge 0$ is a constant depending on ξ^n, $f^n(t_j,0,0,0,0)$, $(L^n)^+$ and $(V^n)^-$.

Proof. By the definition (5.13) of the explicit discrete reflected scheme, since $(\tilde{y}_j^n - L_j^n)\tilde{k}_j^{+n} = (\tilde{y}_j^n - V_j^n)\tilde{k}_j^{-n} = 0$ and the Lipschitz condition of f^n, it follows

$$
\begin{aligned}
&\mathbb{E}\Big[|\tilde{y}_j^n|^2 + \Delta^n|\tilde{z}_j^n|^2 + \Delta^n|\tilde{u}_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}\Big] \\
=&\mathbb{E}|\tilde{y}_{j+1}^n|^2 + 2\Delta^n\mathbb{E}\left[\tilde{y}_{j+1}^n f^n(t_j, \mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n], \bar{\tilde{y}}_j^n, \tilde{z}_j^n, \tilde{u}_j^n)\right] \\
&+ (\Delta^n)^2\mathbb{E}\left|f^n(t_j, \mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n], \bar{\tilde{y}}_j^n, \tilde{z}_j^n, \tilde{u}_j^n)\right|^2 \\
&- \mathbb{E}\left[|\tilde{k}_j^{+n}|^2 + |\tilde{k}_j^{-n}|^2\right] + 2\mathbb{E}\left[\tilde{y}_j^n(\tilde{k}_j^{+n} - \tilde{k}_j^{-n})\right] \\
\leq&\mathbb{E}|\tilde{y}_{j+1}^n|^2 + 2\Delta^n\mathbb{E}\Big[\tilde{y}_{j+1}^n \cdot f^n(t_j, \mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n], \bar{\tilde{y}}_j^n, \tilde{z}_j^n, \tilde{u}_j^n) \\
&+ \mathbb{E}\left|f^n(t_j,0,0,0,0)\right|\Big] + (\Delta^n)^2\mathbb{E}\left|f^n(t_j, \mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n], \bar{\tilde{y}}_j^n, \tilde{z}_j^n, \tilde{u}_j^n)\right|^2 \\
&- \mathbb{E}\left[|\tilde{k}_j^{+n}|^2 + |\tilde{k}_j^{-n}|^2\right] + 2\mathbb{E}\left[\tilde{y}_j^n(\tilde{k}_j^{+n} - \tilde{k}_j^{-n})\right] \\
\leq&\mathbb{E}|\tilde{y}_{j+1}^n|^2 + 2\Delta^n\mathbb{E}\Big[\tilde{y}_{j+1}^n\Big(L\left|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n]\right| + L|\bar{\tilde{y}}_j^n| + L|\tilde{z}_j^n| + L|\tilde{u}_j^n|(1-h_{j+1}^n)\sqrt{\gamma_{j+1}}\Big) \\
&+ |f^n(t_j,0,0,0,0)|\Big] + (\Delta^n)^2\mathbb{E}\Big[L\Big(\left|\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n]\right| + |\bar{\tilde{y}}_j^n| + |\tilde{z}_j^n| \\
&+ |\tilde{u}_j^n|(1-h_{j+1}^n)\sqrt{\gamma_{j+1}}\Big) + |f^n(t_j,0,0,0,0)|\Big]^2 + 2\mathbb{E}\left[\tilde{y}_j^n(\tilde{k}_j^{+n} - \tilde{k}_j^{-n})\right].
\end{aligned}
\tag{A.14}
$$

Since $\mathbb{E}^{\mathcal{G}_i^n}[\tilde{y}_{i+1}^n] = \frac{1}{2}\left(\tilde{y}_{i+1}^n|_{\epsilon_{i+1}^n=1} + \tilde{y}_{i+1}^n|_{\epsilon_{i+1}^n=-1}\right)$, we can obtain

$$
\begin{aligned}
&\mathbb{E}\Big[|\tilde{y}_j^n|^2 + \Delta^n|\tilde{z}_j^n|^2 + \Delta^n|\tilde{u}_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}\Big] \\
\leq&\mathbb{E}|\tilde{y}_{j+1}^n|^2 + \left(\Delta^n + 2\Delta^n L + 12\Delta^n L^2 + 5(\Delta^n L)^2\right)\mathbb{E}|\tilde{y}_{j+1}^n|^2 \\
&+ \left(\frac{\Delta^n}{4} + 5(\Delta^n L)^2\right)\mathbb{E}|\tilde{z}_j^n|^2 + \left(\frac{\Delta^n}{4} + 5(\Delta^n L)^2\right)\mathbb{E}|\tilde{u}_j^n|^2(1-h_{j+1}^n)\gamma_{j+1} \\
&+ \left(\frac{\Delta^n}{4} + 5(\Delta^n L)^2\right)\mathbb{E}|\bar{\tilde{y}}_j^n|^2 + \left(\Delta^n + 5(\Delta^n)^2\right)\mathbb{E}|f^n(t_j,0,0,0,0)|^2 \\
&+ 2\mathbb{E}\left[\tilde{y}_j^n(\tilde{k}_j^{+n} - \tilde{k}_j^{-n})\right].
\end{aligned}
$$

Taking sum from $j = i, ..., n-1$, it follows

$$
\begin{aligned}
&\mathbb{E}\left[|\tilde{y}_i^n|^2 + \sum_{j=i}^{n-1}|\tilde{z}_j^n|^2 + \sum_{j=i}^{n-1}|\tilde{u}_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}\right] \\
\leq&\mathbb{E}|\zeta_n^n|^2 + \left(\frac{\Delta^n}{4} + 5(\Delta^n L)^2\right)\mathbb{E}\left[\sum_{j=n}^{n^\delta-1}|\zeta_j^n|^2\right] \\
&+ \left(\Delta^n + 2\Delta^n L + 12\Delta^n L^2 + 5(\Delta^n L)^2\right)\mathbb{E}\left[\sum_{j=i}^{n-1}|\tilde{y}_{j+1}^n|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\Delta^n}{4}+5(\Delta^n L)^2\right)\mathbb{E}|\tilde{z}_j^n|^2+\left(\frac{\Delta^n}{4}+5(\Delta^n L)^2\right)\mathbb{E}|\tilde{u}_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}\\
&+\left(\frac{\Delta^n}{4}+5(\Delta^n L)^2\right)\mathbb{E}\left[\sum_{j=i}^{n-1}|\tilde{y}_j^n|^2\right]+\left(\Delta^n+5(\Delta^n)^2\right)\mathbb{E}\sum_{j=i}^{n-1}|f^n(t_j,0,0,0,0)|^2\\
&+\frac{1}{\lambda_1}\mathbb{E}\left(\sum_{j=i}^{n-1}\tilde{k}_j^{+n}\right)^2+\lambda_1\mathbb{E}\sup_{i\le j\le n-1}((L_j^n)^+)^2\\
&+\frac{1}{\lambda_1}\mathbb{E}\left(\sum_{j=i}^{n-1}\tilde{k}_j^{-n}\right)^2+\lambda_1\mathbb{E}\sup_{i\le j\le n-1}((V_j^n)^-)^2.
\end{aligned}
$$

where $\lambda_1 > 0$ is a constant. By the estimation of $\tilde{k}_j^{+n}$ and $\tilde{k}_j^{-n}$ in (5.15), it follows

$$
\begin{aligned}
\tilde{k}_j^{+n}&\le\left(\mathbb{E}^{\mathcal{G}_j^n}[L_{j+1}^n]+f^n(t_j,\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n],\bar{\tilde{y}}_j^n,\tilde{z}_j^n,\tilde{u}_j^n)\Delta^n-L_j^n\right)^-;\\
\tilde{k}_j^{-n}&\le\left(\mathbb{E}^{\mathcal{G}_j^n}[V_{j+1}^n]+f^n(t_j,\mathbb{E}^{\mathcal{G}_j^n}[\tilde{y}_{j+1}^n],\bar{\tilde{y}}_j^n,\tilde{z}_j^n,\tilde{u}_j^n)\Delta^n-V_j^n\right)^+.
\end{aligned}
\tag{A.15}
$$

Therefore, by the Lipschitz condition of f^n, we can obtain

$$
\begin{aligned}
\mathbb{E}\left(\sum_{j=i}^{n-1}\tilde{k}_j^{+n}\right)^2\le&6\mathbb{E}\sum_{j=i}^{n-1}\Big[\left|\mathbb{E}^{\mathcal{G}_j^n}[L_{j+1}^n]-L_j^n\right|^2+(\Delta^n)^2|f^n(t_j,0,0,0,0)|^2\\
&\qquad+(\Delta^n L)^2\Big(\left|\tilde{y}_{j+1}^n\right|^2+\left|\tilde{y}_j^n\right|^2+\left|\tilde{z}_j^n\right|^2+\left|\tilde{u}_j^n\right|^2(1-h_{j+1}^n)\gamma_{j+1}\Big)\Big]\\
&+6(\Delta^n L)^2\mathbb{E}\left[\sum_{j=n}^{n^\delta-1}\left|\xi_j^n\right|^2\right];\\
\mathbb{E}\left(\sum_{j=i}^{n-1}\tilde{k}_j^{-n}\right)^2\le&6\mathbb{E}\sum_{j=i}^{n-1}\Big[\left|\mathbb{E}^{\mathcal{G}_j^n}[V_{j+1}^n]-V_j^n\right|^2+(\Delta^n)^2|f^n(t_j,0,0,0,0)|^2\\
&\qquad+(\Delta^n L)^2\Big(\left|\tilde{y}_{j+1}^n\right|^2+\left|\tilde{y}_j^n\right|^2+\left|\tilde{z}_j^n\right|^2+\left|\tilde{u}_j^n\right|^2(1-h_{j+1}^n)\gamma_{j+1}\Big)\Big]\\
&+6(\Delta^n L)^2\mathbb{E}\left[\sum_{j=n}^{n^\delta-1}\left|\xi_j^n\right|^2\right].
\end{aligned}
\tag{A.16}
$$

Set $\lambda_1 = 48\Delta^n L^2$, when Δ^n is small enough, it follows

$$
\begin{aligned}
&\mathbb{E}\left[|\tilde{y}_i^n|^2+\frac{\Delta^n}{3}\sum_{j=i}^{n-1}\left|\tilde{z}_j^n\right|^2+\frac{\Delta^n}{3}\sum_{j=i}^{n-1}\left|\tilde{u}_j^n\right|^2(1-h_{j+1}^n)\gamma_{j+1}\right]\\
\le&\mathbb{E}|\xi_n^n|^2+\left(\frac{\Delta^n}{2}+5(\Delta^n L)^2\right)\mathbb{E}\left[\sum_{j=n}^{n^\delta-1}|\xi_j^n|^2\right]\\
&+\left(\frac{5\Delta^n}{4}+2\Delta^n L+12\Delta^n L^2+5(\Delta^n L)^2\right)\mathbb{E}\left[\sum_{j=i}^{n-1}|\tilde{y}_{j+1}^n|^2\right]
\end{aligned}
$$

$$+\left(\frac{\Delta^n}{2}+5(\Delta^n L)^2\right)\mathbb{E}\left[\sum_{j=i}^{n-1}|\tilde{y}_j^n|^2\right]$$
$$+\left(\frac{\Delta^n}{4L^2}+\Delta^n+5(\Delta^n)^2\right)\mathbb{E}\sum_{j=i}^{n-1}|f^n(t_j,0,0,0,0)|^2$$
$$+48\Delta^n L^2\mathbb{E}\left[\sup_{i\le j\le n-1}((L_j^n)^+)^2+\sup_{i\le j\le n-1}((V_j^n)^-)^2\right]$$
$$+\frac{1}{4\Delta^n L^2}\mathbb{E}\sum_{j=i}^{n-1}\left[\left|\mathbb{E}^{\mathcal{G}_j^n}[L_{j+1}^n]-L_j^n\right|^2+\left|\mathbb{E}^{\mathcal{G}_j^n}[V_{j+1}^n]-V_j^n\right|^2\right].$$

By the discrete Gronwall's inequality (Lemma A.2.1), when $\frac{7\Delta^n}{4}+2\Delta^n L+12\Delta^n L^2+10(\Delta^n L)^2<1$, we can obtain

$$\sup_i \mathbb{E}|\tilde{y}_i^n|^2+\mathbb{E}\left[\Delta^n\sum_{j=0}^{n-1}|\tilde{z}_j^n|^2+\Delta^n\sum_{j=0}^{n-1}|\tilde{u}_j^n|^2(1-h_{j+1}^n)\gamma_{j+1}+\sum_{j=0}^{n-1}\left(|\tilde{k}_j^{+n}|^2+|\tilde{k}_j^{-,n}|^2\right)\right]$$
$$\le\lambda_{L,T,\delta}C_{\xi^n,f^n,L^n,V^n},$$

where $C_{\xi^n,f^n,L^n,V^n}\ge 0$ is a constant depending on ξ^n, $f^n(t_j,0,0,0,0)$, $(L^n)^+$ and $(V^n)^-$. Reconsider (A.14), we take square, sup and sum over j, then take expectation, by Burkholder-Davis-Gundy inequality for the martingale parts, it follows

$$\mathbb{E}\left[\sup_i|\tilde{y}_i^n|^2\right]\le C\Delta^n\left[\sum_{i=0}^{n-1}\mathbb{E}\,|\tilde{y}_i^n|^2\right]\le CT\sup_i\mathbb{E}\,|\tilde{y}_i^n|^2.$$

It follows (A.13). □

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