Multiscale Gravitational Field Recovery from GPS-Satellite-to-Satellite Tracking

by

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Abstract

The purpose of GPS-satellite-to-satellite tracking (GPS-SST) is to determine the gravitational potential at the earth's surface from measured ranges (geometrical distances) between a low-flying satellite and the high-flying satellites of the Global Positioning System (GPS). In this paper GPS-satellite-to-satellite tracking is reformulated as the problem of determining the gravitational potential of the earth from given gradients at satellite altitude. Uniqueness and stability of the solution are investigated. The essential tool is to split the gradient field into a normal part (i.e. the first order radial derivative) and a tangential part (i.e. the surface gradient). Uniqueness is proved for polar, circular orbits corresponding to both types of data (first radial derivative and/or surface gradient). In both cases gravity recovery based on satellite-to-satellite tracking turns out to be an exponentially ill-posed problem. As an appropriate solution method regularization in terms of spherical wavelets is proposed based on the knowledge of the singular system. Finally, the extension of this method is generalized to a non-spherical earth and a non-spherical orbital surface based on combined terrestrial and satellite data material.

Key words: GPS-satellite-to-satellite tracking, uniqueness, formulation as integral equation, regularization by wavelets.

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1 Introduction

Since the beginning of the space age, observed satellite orbit perturbations are exploited to recover the long-wavelength features of the figure of the earth. As a matter of fact, analyses of orbit perturbations are the only technique to determine precisely and homogeneously the long-wavelength gravitational field, whereas satellite altimetry over the oceans and terrestrial, ship- or airborne gravimetry are used in combination with the so-called satellite-only models of physical geodesy to increase the resolution up to the finer structures. Long-wavelength gravitational models are of great importance for the study of deepseated mass inhomogenities, especially within the earth's mantle. In oceanography, they provide the reference surface topography as basis to study the dynamics of ocean circulation.

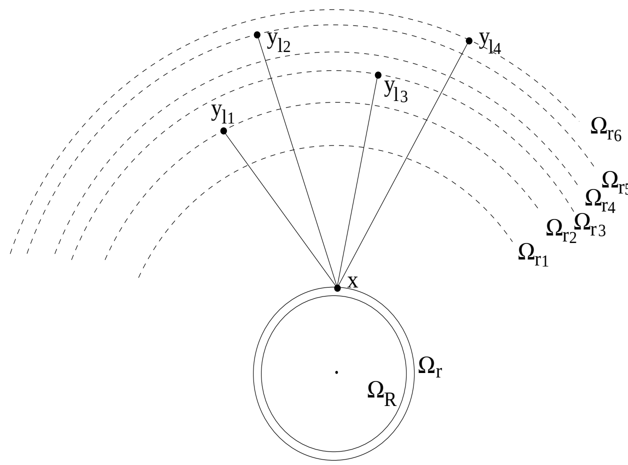
In order to achieve further progress in physical geodesy, oceanography and many areas of solid earth geophysics a dedicated satellite mission will be required, that has to provide a more accurate, higher resolved and especially a homogeneous gravity field model. All these improvements are expected to be realized by observing the relative motion of test particles (satellites) in near-earth orbits. Two measurement principles have been proposed so far: (1) satellite gravity gradiometry (SGG), where the test masses be located onboard a single satellite, and (2) satellite-to-satellite tracking (SST) between at least two separated space vehicles. For this latter technique two variants are under discussion: low-low- and high-low-intersatellite measurements. Due to the availability and the intriguing capabilities of the Global Positioning System the currently planned mission scenarios favour the high-low variant consisting of one low-earth-orbiter and the 24 high flying GPS-satellites. The measured force field includes non-gravitational accelerations due to e.g. air drag and solar radiation pressure, and satellite accelerations arising from the earth's gravitational potential. If all non-gravitational effects are measured and/or modelled correctly, the so reduced observations will help to recover the shortcomings in the underlying gravity field model and can be used for a model improvement. Because of its capability to provide a continuous data coverage GPS space based tracking for precise orbit determination is superior to any ground based system.

In the literature several proposals can be found for solving the satellite-to-satellite tracking problem mathematically (see, for example Ilk et al. (1978), Reigber (1989), Rummel (1975, 1979), Schwintzer et al. (1997)). A long list of publications concerning SST and SGG can be found in Tscherning et al. (1990). Generally two different approaches may be distinguished: the timewise and the spacewise approach (cf. Rummel et al. (1993)). The former one addresses gravitational field determination from the solution of the equations of motion for earth orbiting satellites, the latter solves the gravitational field under the assumption that the data are given in advance on a closed (orbital) surface. The investigations of this paper are part of the spacewise approach. They are led by the prerequisites of the German satellite CHAMP, which shall start with a Russian launch vehicle in 1999. The main characteristics of CHAMP are a (near-)polar, circular orbit, a low altitude (500 km ... 300 km), dense coverage of the orbit with GPS-satellite-to-satellite tracking data, and a direct measurement of the non-gravitational orbit perturbations by an onboard three-axes accelerometer (for more details the reader is referred to Reigber et al. (1996)).

This article motivates that satellite-to-satellite tracking is a wellsuited method for recovering globally and homogeneously the long to medium wavelengths of the earth's gravitational potential. In the language of constructive approximation it is characterized by a transition from multipole (spherical harmonic) modelling being responsible for the "low frequency contribution" to wavelet modelling being appropriate for the "higher frequency parts". Therefore, seen in the interdependence of decreasing space localization and increasing frequency localization of the uncertainty principle (cf. Freeden and Windheuser (1997)), satellite-to-satellite tracking may be understood as the interface of two important procedures, namely global expansion by ideally frequency localizing, but non space localizing polynomials (spherical harmonics) and multiscale expansion (multiresolution analysis) by frequency as well as space localizing wavelets. But in view of the amount of space/frequency localization it is also important to distinguish bandlimited wavelets from non-bandlimited ones. As a matter of fact, non-bandlimited wavelets show a much stronger space localization than their comparable bandlimited counterparts. We are led to the conclusion that for the long- to medium-wavelength feature bandlimited wavelets or moderately space localizing non-bandlimited wavelets should be used for approximation, which can be continued canonically by a multiscale analysis in terms of more and more space localizing non-bandlimited wavelets (for example, in the future satellite gravity gradiometry scenario).

1.1 GPS-Satellite-to-Satellite Tracking

In order to translate the GPS-satellite-to-satellite tracking problem into a mathematical language we introduce the following notations: Each point x of the three-dimensional Euclidean space $\mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$, $|x| \neq 0$ allows a unique representation of the form $x = r\xi$, $r = |x|$, $\xi = (\xi_1, \xi_2, \xi_3)^T$, where $\xi \in \mathbb{R}^3$, $|\xi| = 1$ is the uniquely determined directional unit vector of x. The sphere in $\mathbb{R}^3$ with radius r around the origin is denoted by Ω_r, i.e. $\Omega_r = \{x \in \mathbb{R}^3 \mid |x| = r\}$. Ω_r^{ext} is the exterior of Ω_r, while Ω_r^{int} is the interior of Ω_r. By definition, $\Omega(= \Omega_1)$ is the unit sphere in $\mathbb{R}^3$. A scalar (vectorial) function $F : \Omega_r \to \mathbb{R}$ ($f : \Omega_r \to \mathbb{R}^3$) which possesses k continuous derivatives on Ω_r is said to be of class $C^{(k)}(\Omega_r)$ ($c^{(k)}(\Omega_r)$), $0 \leq k \leq \infty$. As usual, ∇ and Δ denote the gradient and the Laplace operator in Euclidean space $\mathbb{R}^3$, respectively.



In this nomenclature the (idealized) formulation of satellite-to-satellite tracking may be given as follows: Let the "earth's sphere" Ω_R and the "satellite (orbital) sphere" Ω_r of the low satellite differ in the satellite height H, i.e. $r = R + H$, $H > 0$. The arrangement of the GPS satellites is such that at least four satellites are simultaneously visible above the horizon anywhere on the earth's surface Ω_R and Ω_r as well, all the time. Moreover, the GPS satellites may be supposed to be placed in circular orbits in six orbital spheres $\Omega_{r_1}, \ldots, \Omega_{r_6}$ with $r_i = R + H_i$ and $H \ll H_1, \ldots, H_6$; and n be the total number of GPS satellites. To every position $x \in \Omega_r$, therefore, there exist at least $m \geq 4$ visible GPS satellites $y_{l_1}, \ldots, y_{l_m}$, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$, such that the geometrical distances (ranges) $d_{l_i} = |x - y_{l_i}|$, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$, are measurable. Since the orbits of the GPS satellites are supposed to be known, the coordinates of the satellite located at x on the low orbital sphere Ω_r can be derived from simultaneous range measurements to the satellites. From this the relative positions $p_{l_i} = x - y_{l_i}$, $\quad l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$ of the satellites at x and y_{l_i} become available at time t. We do not enter into the details of gaining relative ranges d_{l_i} between the GPS-satellites and low flying satellites out of the original phase measurements, but only regard the geometrical configuration. For details of orbit determination we refer to Perosanz et al. (1997) and the references therein. The relative velocities v_{l_i} and accelerations a_{l_i} are obtainable by differentiating the relative positions with respect to t. We assume that the measurements are produced at a sufficiently dense rate so that (numerical) differentiation can be performed without any difficulty (note that the quantities v_{l_i} and a_{l_i}, however, are correlated). The interesting expressions now are the relative accelerations a_{l_i}, $i = 1, \ldots, m$, all of which are determined for inertial motion (in accordance with the Newton-Euler equation) by the gravitational field only and may be equated by the difference of the gradient field of the geopotential V, here evaluated at the locations of x and y_{l_i}, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$. To be more specific,

$$a_{l_i}(x) = (\nabla V)(x) - (\nabla V)(y_{l_i}), \ x \in \Omega_r \tag{1.1}$$

for $i = 1, \dots, m$. (Note that the gravitational force is considered to be independent of time t at a certain position. In other words, we assume that time-like variations of the field are so slow as to be neglected.) From (1.1) it follows that

$$(\nabla V)(x) = \sum_{i=1}^{m(x)} \alpha_i \left(a_{l_i}(x) + (\nabla V)(y_{l_i}) \right), \ x \in \Omega_r \tag{1.2}$$

for all selections $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ satisfying $\sum_{i=1}^m \alpha_i = 1$. As examples, we mention

(i) arithmetical summation

$$\alpha_i = \frac{1}{m}, \quad i = 1, \dots, m.$$

(The influence of the values $a_{l_i}(x) + (\nabla V)(y_{l_i})$ in arithmetical summation (i) to the representation of $(\nabla V)(x)$ is of equal weight.)

(ii) binomial summation

$$\alpha_i = \frac{\binom{m-1}{i-1} z^i}{(1+z)^n}, \quad z > 0, \quad i = 1, \dots, m,$$

$$\alpha_i = \frac{\binom{m-1}{i-1} z^{m-i}}{(1+z)^n}, \quad z > 0, \quad i = 1, \dots, m.$$

(iii) Cesáro C_k-summation

$$\alpha_i = \frac{\binom{m-i+k-2}{m-i-1}}{\binom{m-1-k}{m-1}}, \quad i = 1, \dots, m, \quad k \geq 1.$$

(The influence of the values $a_{l_i}(x) + (\nabla V)(y_{l_i})$ in the summation types (ii) and (iii) to the representation of $(\nabla V)(x)$ is of weighted character. Note that the values a_{l_i} are correlated. Nevertheless, "pre-conditioning" by non-equally weighted summation (dependent on the data) seems to be a helpful variant not only for numerical purposes).

2 Spherical Approximation

Under the assumption that the geopotential V is sufficiently known at the altitude of GPS satellites a spherical approximation to (the V-representation of) GPS-satellite-to-satellite tracking reads as follows.

GPS-SST problem for the geopotential V (spherical approximation):

Assume that there is given a vector field $f_V \in c^{(\infty)}(\Omega_r)$ on the "satellite (orbital) sphere" Ω_r of the form (1.2). We look for a function $V : \overline{\Omega_R^{ext}} \to \mathbb{R}$ satisfying the following properties:

- V is continuous in $\overline{\Omega_R^{ext}}$ and twice continuously differentiable in Ω_R^{ext} (i.e. $V \in C^{(0)}(\overline{\Omega_R^{ext}}) \cap C^{(2)}(\Omega_R^{ext})$)
- V is harmonic in Ω_R^{ext} (i.e. $\Delta V = 0$ in Ω_R^{ext})
- V is regular at infinity (i.e. $|V(x)| = O(1/|x|), \ |x| \to \infty$)
- the gradient field of V coincides with the vector field f_V on the sphere Ω_r (i.e. $\nabla V|_{\Omega_r} = f_V$).

As it is well known, the geopotential V may be represented as sum $V = U + T$ with (known) reference potential U and the disturbing potential T. For the reference potential commonly the normal potential is used closely connected with the (Pizetti-Somigliana) reference ellipsoid, so that both level ellipsoid and the geoid have the same mass and the center of the ellipsoid and the earth's center of mass coincide. Expressed in terms of an $\mathcal{L}^2(\Omega_R)$-orthonormal system $\{Y^R_{n,k}\}$ of spherical harmonics this means that no zero and first degree term appear in the spherical harmonic expansion of T (note that $\{Y^R_{n,k}\}$ defined by $Y^R_{n,k}(x) = (1/R)Y_{n,k}(x/|x|)$, $x \in \Omega_R$ forms an $\mathcal{L}^2(\Omega_R)$-orthonormal basis in $\mathcal{L}^2(\Omega_R)$ provided that $\{Y_{n,k}\}$ constitutes an $\mathcal{L}^2(\Omega)$-orthonormal basis in $\mathcal{L}^2(\Omega)$). Because of the linearity of the operation of the gradient and the definition of the disturbing potential T this leads us to the expression

$$(\nabla T)(x) = \sum_{i=1}^{m(x)} \alpha_i \left(a_{l_i}(x) + (\nabla U)(y_{l_i}) + (\nabla T)(y_{l_i})\right) - (\nabla U)(x), \quad x \in \Omega_r, \tag{2.3}$$

for all choices $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ satisfying $\sum_{i=1}^m \alpha_i = 1$. For the high GPS satellites, the vectors $(\nabla T)(y_{l_i})$, $i = 1, \ldots, m$ may be considered to be known (note that the disturbing potential needed to model the orbit of so high satellites is known to extreme precision, hence, $(\nabla V)(y_{l_i})$ may be assumed to be known, too).

Summarizing our considerations we thus obtain the following mathematical formulation for the T-representation of GPS-satellite-to-satellite tracking:

GPS-SST problem for the disturbing potential T (spherical approximation):

Assume that there is given a vector field $f_T \in c^\infty(\Omega_r)$ on the "satellite (orbital) sphere" Ω_r of the form (2.3). We look for a function $T : \overline{\Omega_R^{ext}} \to \mathbb{R}$ satisfying the following properties:

- $T \in C^{(0)}(\overline{\Omega_R^{ext}}) \cap C^{(2)}(\Omega_R^{ext})$
- $\Delta T = 0$ in Ω_R^{ext}
- $|T(x)| = O(1/|x|^3), \ |x| \to \infty$
- $T^R_{n,k} = \int_{\Omega_R} T(x) Y^R_{n,k}(x)\, d\omega_R(x) = 0$ for $n = 0, 1$, $k = 1, \ldots, 2n+1$
- $\nabla T|_{\Omega_r} = f_T$.

Obviously we are confronted with two difficulties for the aforementioned problems, namely uniqueness (Chapter 2.1) and solution by (ill-posed) integral equations (Chapter 2.2).

Remark: The problem of determing a gravitational potential from a prescribed gradient field also plays an important role in gravity field modelling of a planet like Moon, Venus, etc. As a matter of fact, the approach developed below represents a new method in the approximation of Line-Of-Sight (LOS) problems occuring in planetary geodesy (cf. Moritz (1969), Sjogren et al. (1983), Barriot and Balmino (1992), Barriot (1994) and many others).

2.1 Uniqueness

The uniqueness can be adequately answered within the concept of vector spherical harmonics that will be recapitulated briefly.

2.1.1 Vector Spherical Harmonics

By $l^2(\Omega)$ we denote the space of (Lebesgue) square integrable vector fields on Ω. $l^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(f, g)_{l^2(\Omega)} = \int_\Omega f(\xi) \cdot g(\xi)\, d\omega(\xi).$$

For a given vector field $f : \Omega \to \mathbb{R}^3$, the field $f_{nor} : \xi \mapsto f_{nor}(\xi) = (\xi \cdot f(\xi))\xi,\ \xi \in \Omega$ is called the *normal part* of f, while $f_{tan} : \xi \mapsto f_{tan}(\xi) = f(\xi) - f_{nor}(\xi),\ \xi \in \Omega$ is called the *tangential part* of f. A vector field f is called *tangential* (resp. *normal*), if $f(\xi) = f_{tan}(\xi)$ (resp. $f(\xi) = f_{nor}(\xi)$) for all $\xi \in \Omega$.

The study of vector fields on the sphere can be greatly simplified in terms of the following operators: ∇^* (*surface gradient*), L^* (*surface curl gradient*), $\nabla^*\cdot$ (*surface divergence*), $L^*\cdot$ (*surface curl*), and Δ^* (*Beltrami operator*). Their representation in terms of polar coordinates and their role in integral formulas on the sphere are extensively discussed in Freeden et al. (1998). By use of these operators any vector field $f \in c^{(1)}(\Omega)$ admits the decomposition

$$f(\xi) = f_{nor}(\xi) + f_{tan}(\xi), \quad \xi \in \Omega,$$

where

$$\begin{aligned} f_{nor}(\xi) &= f^{(1)}(\xi), \quad \xi \in \Omega, \\ f_{tan}(\xi) &= f^{(2)}(\xi) + f^{(3)}(\xi), \quad \xi \in \Omega, \end{aligned}$$

and

$$\begin{aligned} f^{(1)}(\xi) \cdot f^{(i)}(\xi) &= 0, \quad i = 2, 3, \\ \nabla^*_\xi \cdot f^{(2)}(\xi) &= \nabla^*_\xi \cdot f_{tan}(\xi) \\ \nabla^*_\xi \cdot (f^{(2)}(\xi) \wedge \xi) &= 0 \\ \nabla^*_\xi \cdot f^{(3)}(\xi) &= 0 \\ \nabla^*_\xi \cdot (f^{(3)}(\xi) \wedge \xi) &= \nabla^*_\xi \cdot (f(\xi) \wedge \xi). \end{aligned}$$

Due to Backus (1967) (see also Meissl (1971)) there exist uniquely determined functions $F_i \in C^{(2)}(\Omega),\ i = 2, 3$ satisfying

$$\int_\Omega F_i(\xi)\, d\omega(\xi) = 0, \quad i = 2, 3,$$

such that

$$\begin{aligned} f^{(2)} &= \nabla^*_\xi F_2(\xi), \quad \xi \in \Omega, \\ f^{(3)} &= L^*_\xi F_3(\xi) = \xi \wedge \nabla^*_\xi F_3(\xi), \quad \xi \in \Omega. \end{aligned}$$

Freeden and Gervens (1993) give the explicit representation of $F_i,\ i = 2, 3$, in terms of Green's function with respect to the Beltrami operator Δ^*

$$G(\Delta^*; \xi, \eta) = \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln 2,$$

namely

$$\begin{aligned} F_2(\xi) &= -\int_\Omega G(\Delta^*; \xi, \eta) \nabla^*_\eta \cdot (f(\eta) - (\eta \cdot f(\eta))\eta)\, d\omega(\eta), \\ F_3(\xi) &= -\int_\Omega G(\Delta^*; \xi, \eta) \nabla^*_\eta \cdot (\eta \wedge f(\eta))\, d\omega(\eta). \end{aligned}$$

Using the operators $O^{(i)} : c^{(1)}(\Omega) \to C^{(0)}(\Omega),\ i = 1, 2, 3$, defined by

$$\begin{aligned} O^{(1)}_\xi f(\xi) &= \xi \cdot f(\xi), \\ O^{(2)}_\xi f(\xi) &= -\nabla^*_\xi \cdot (f(\xi) - (\xi \cdot f(\xi))\xi), \\ O^{(3)}_\xi f(\xi) &= \nabla^*_\xi \cdot (\xi \wedge f(\xi)), \end{aligned}$$

we are able to rewrite the decomposition formula as follows:

$$\begin{aligned} f^{(1)}(\xi) &= \left(O^{(1)}_\xi f(\xi)\right)\xi, \\ f^{(2)}(\xi) &= -\nabla^*_\xi \int\limits_\Omega G(\Delta^*;\xi,\eta) O^{(2)}_\eta f(\eta)\, d\omega(\eta), \\ f^{(3)}(\xi) &= -L^*_\xi \int\limits_\Omega G(\Delta^*;\xi,\eta) O^{(3)}_\eta f(\eta)\, d\omega(\eta). \end{aligned}$$

Corresponding to the operators $O^{(i)}$ we introduce operators $o^{(i)} : C^{(1)}(\Omega) \to c^{(0)}(\Omega),\ i = 1,2,3,$ by setting

$$\begin{aligned} o^{(1)}_\xi F(\xi) &= F(\xi)\xi, \\ o^{(2)}_\xi F(\xi) &= \nabla^*_\xi F(\xi), \\ o^{(3)}_\xi F(\xi) &= L^*_\xi F(\xi) = \xi \wedge \nabla^*_\xi F(\xi). \end{aligned}$$

Then it is not difficult to show that for $F \in C^{(2)}(\Omega)$

$$O^{(j)}(o^{(i)} F(\xi)) = 0, \quad j \neq i.$$

Moreover, for $i = j$, we get

$$\begin{aligned} O^{(1)}(o^{(1)} F(\xi)) &= F(\xi), \\ O^{(2)}(o^{(2)} F(\xi)) &= -\Delta^*_\xi F(\xi), \\ O^{(3)}(o^{(3)} F(\xi)) &= -\Delta^*_\xi F(\xi). \end{aligned}$$

For convenience, we set

$$0_i = \begin{cases} 0 & , \quad i = 1 \\ 1 & , \quad i = 2,3. \end{cases}$$

Now, let Y_n be a scalar spherical harmonic of order n. Then it follows that

$$O^{(i)}(o^{(i)} Y_n(\xi)) = \mu^{(i)}_n Y_n(\xi)$$

for all $\xi \in \Omega$ and $i = 1,2,3$ and for $n = 0_i, 0_i + 1, \ldots$, where we have used the abbreviation

$$\mu^{(i)}_n = \begin{cases} 1 & , \quad i = 1 \\ n(n+1) & , \quad i = 2,3. \end{cases}$$

The vector fields

$$y^{(i)}_n(\xi) = o^{(i)}_\xi Y_n(\xi),\ \xi \in \Omega,\ n = 0_i, 0_i + 1, \ldots$$

are called *vector spherical harmonics of order n and type i*. $y^{(1)}_n$ describes a normal field, while $y^{(2)}_n$, $y^{(3)}_n$ are tangential fields of order n. Obviously, according to our construction (cf. Freeden and Gervens (1993)), we have

$$\begin{aligned} \xi \wedge y^{(1)}_n(\xi) &= 0, \\ \xi \cdot y^{(2)}_n(\xi) &= 0, & \nabla^*_\xi \cdot (\xi \wedge y^{(2)}_n(\xi)) &= 0, \\ \xi \cdot y^{(3)}_n(\xi) &= 0, & \nabla^*_\xi \cdot y^{(3)}_n(\xi) &= 0, \\ \xi \wedge y^{(2)}_n(\xi) &= y^{(3)}_n(\xi) \end{aligned}$$

and

$$y^{(i)}_n(\xi) \cdot y^{(j)}_n(\xi) = 0,\ i \neq j.$$

The vector fields $y^{(i)}_{n,k} = (\mu^{(i)}_n)^{-1/2} o^{(i)} Y_{n,k},\ n = 0_i, 0_i+1, \ldots,\ k = 1, \ldots, 2n+1$ form an $l^2(\Omega)$-orthonormal system with $\{Y_{n,k}\}$ being always assumed to be an $\mathcal{L}^2(\Omega)$-orthonormal system of scalar spherical harmonics; more explicitly,

$$\int\limits_{\Omega} y^{(i)}_{n,k}(\xi) \cdot y^{(j)}_{m,l}(\xi)\, d\omega(\xi) = \delta_{i,j}\delta_{n,m}\delta_{k,l}.$$

The kernels (null-spaces) of the operators $o^{(i)}$ are characterized as follows.

Lemma 2.1 *Assume that* $F \in C^{(2)}(\Omega)$. *Then*

$$o^{(i)}_{\xi} F(\xi) = 0,\ \xi \in \Omega$$

is true if

$$\begin{aligned} (i)\ F(\xi) &= 0, \quad i = 1 \\ (ii)\ F(\xi) &= c = const, \quad i = 2, 3. \end{aligned}$$

From Lemma 2.1 it follows immediately that the spaces

$$c^{(\infty)}_{(i)}(\Omega) = \{o^{(i)} F \mid F \in C^{(\infty)}(\Omega)\}, \quad i = 1, 2, 3$$

are orthogonal. Moreover, it is shown in Freeden et al. (1998) that the spaces $l^2_{(i)}(\Omega) = \overline{c^{(\infty)}_{(i)}(\Omega)}$ (where completion is understood in the sense of $\|\cdot\|_{l^2(\Omega)}$) satisfy

$$l^2(\Omega) = \oplus^3_{i=1} l^2_{(i)}(\Omega).$$

The projection operators with respect to this decomposition are denoted by

$$p_{(i)} : l^2(\Omega) \to l^2_{(i)}(\Omega),\ f \mapsto p_{(i)} f = f^{(i)},\ i = 1, 2, 3,$$

respectively. Note that the projection operators can be obtained analogously for the sphere Ω_r instead of Ω $(r > 0)$.

The vector spherical harmonics $\{y^{(i)}_{n,k}\}$, $i = 1, 2, 3$ with $n = 0_i, 0_i+1, \ldots$ and $k = 1, \ldots, 2n+1$ form an orthonormal basis of $l^2_{(i)}(\Omega)$. It should be mentioned that $\{y^{(i),R}_{n,k}\}$ defined by $y^{(i),R}_{n,k}(x) = (1/R) y^{(i)}_{n,k}(x/|x|)$, $x \in \Omega_R$, $i = 1, 2, 3$ forms an $l^2(\Omega_R)$-orthonormal basis in $l^2(\Omega_R)$ provided that $\{y^{(i)}_{n,k}\}$ constitutes an $l^2_{(i)}(\Omega)$-orthonormal basis in $l^2_{(i)}(\Omega)$, $i = 1, 2, 3$. Every field $f^{(i)} \in l^2_{(i)}(\Omega_R)$ can be expanded in terms of vector spherical harmonics as follows:

$$f^{(i)} = \sum_{n=0_i}^{\infty} \sum_{k=1}^{2n+1} f^{(i),R}_{n,j} y^{(i),R}_{n,k} \tag{2.4}$$

with

$$f^{(i),R}_{n,k} = \int\limits_{\Omega_R} f^{(i)}(x) \cdot y^{(i),R}_{n,k}(x)\, d\omega_R(x), \tag{2.5}$$

where the equality in (2.4) is understood in $\|\cdot\|_{l^2(\Omega_R)}$-sense.

2.1.2 Decomposition of the Gradient Field

After these results on spherical vector fields we are able to describe how the gradient field of a continuously differentiable function can be decomposed on the sphere into its $o^{(i)}$-components.

Suppose that $H : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}$ is a continuously differentiable function. Then it is well-known that

$$h(x) = \nabla H(x)|_{|x|=1}$$

allows the decomposition

$$h(x) = \left(\xi\frac{\partial}{\partial r} + \frac{1}{r}\nabla^*_\xi\right) H(r\xi)|_{r=1}.$$

Using the $o^{(i)}$-operators this is equivalent to

$$h(\xi) = o^{(1)}\left(\frac{\partial}{\partial r} H(r\xi)|_{r=1}\right) + \frac{1}{r}o^{(2)} H(r\xi)|_{r=1}.$$

In particular, in terms of "outer harmonics" $H_{-n-1,k} : x \mapsto H_{-n-1,k}(x)$, $H_{-n-1,k}(r\xi) = r^{-(n+1)}Y_{n,k}(\xi)$, $r > 0$, $\xi \in \Omega$ we obtain the decomposition

$$(\nabla H_{-n-1,k})(x)|_{|x|=1} = -(n+1)o^{(1)}Y_{n,k}(\xi) + o^{(2)}Y_{n,k}(\xi).$$

In terms of $l^2(\Omega)$-orthonormal vector harmonics this finally shows us that

$$(\nabla H_{-n-1,k})(x)|_{|x|=1} = -(n+1)y^{(1)}_{n,k}(\xi) + (n(n+1))^{1/2}y^{(2)}_{n,k}(\xi).$$

2.1.3 GPS-SST Problem for the Geopotential V (Spherical Approximation)

After these preparations we come to the GPS-satellite-to-satellite tracking problem for the gravitational potential V. Consider the space $Pot(\Omega_R^{ext})$ of all potentials $V : \overline{\Omega_R^{ext}} \to \mathbb{R}$ of the form

$$V(x) = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} V^R_{n,k}\left(\frac{R}{|x|}\right)^n \frac{1}{|x|}Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \overline{\Omega_R^{ext}} \tag{2.6}$$

with

$$V^R_{n,k} = \int_{\Omega_R} V(x)Y^R_{n,k}(x)\, d\omega_R(x), \tag{2.7}$$

where $V|_{\Omega_R}$ is assumed to be square-integrable on the sphere Ω_R (i.e. $V|_{\Omega_R} \in \mathcal{L}^2(\Omega_R)$). In other words, $Pot(\Omega_R^{ext})$ consists of all harmonic functions in Ω_R^{ext} corresponding to $\mathcal{L}^2(\Omega_R)$-Dirichlet data. In the $Pot(\Omega_R^{ext})$-context the uniqueness problem of satellite-to-satellite tracking reads as follows:

Let there be known from $V \in Pot(\Omega_R^{ext})$ the gradient field $f_V = \nabla V|_{\Omega_r}$. Is the potential uniquely determined by its prescribed values f_V on Ω_r?

The keypoint of our considerations is the decomposition formula of the gradient field discussed in Section 2.1.2. Based on this result we obtain

$$\begin{aligned} f_V(x) &= (\nabla V)(x) \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} V^R_{n,k}\left(\frac{R}{|x|}\right)^n \left(-\frac{n+1}{|x|^2}o^{(1)}Y_{n,k}\left(\frac{x}{|x|}\right) + \frac{1}{|x|^2}o^{(2)}Y_{n,k}\left(\frac{x}{|x|}\right)\right) \end{aligned} \tag{2.8}$$

for all $x \in \overline{\Omega_R^{ext}}$ provided that V is of the form (2.6).

The formula (2.8) leads us to the following theorem:

Theorem 2.2 *Let $V \in Pot(\Omega_R^{ext})$ be of the representation (2.6). Then the following statements are valid:*

(i) $p_{(3)}(\nabla V)\big|_{\Omega_r} = 0,$

(ii) $p_{(1)}(\nabla V)\big|_{\Omega_r} = 0$ *if and only if* $V = 0,$

(iii) $p_{(2)}(\nabla V)\big|_{\Omega_r} = 0$ *if and only if* $V(x) = c/|x|$ *for a constant* $c \in \mathbb{R}$.

In conclusion, the gravitational potential V in $\overline{\Omega_R^{ext}}$ is uniquely determined by the prescribed $p_{(1)}$-component of its gradient field, i.e. the first radial derivative of V on Ω_r. Moreover, it can be recovered apart from an additive zero-order term of its series expansion from its $p_{(2)}$-component, i.e. the surface gradient field of V on Ω_r. Our results about the V-formulation of satellite-to-satellite tracking, therefore, have shown that the problem of developing the gravitational potential in $\overline{\Omega_R^{ext}}$ from given gravitational field information on Ω_r is overdetermined, it suffices to prescribe the first radial derivative or the surface gradient on Ω_r separately.

2.1.4 GPS-SST Problem for the Disturbing Potential T (Spherical Approximation)

According to our construction the moments $T^R_{0,1}$ and $T^R_{1,k}$, $k = 1, 2, 3$ of the disturbing potential vanish, i.e.

$$T(x) = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} T^R_{n,k} \left(\frac{R}{|x|}\right)^n \frac{1}{|x|} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \overline{\Omega_R^{ext}} \tag{2.9}$$

with

$$T^R_{n,k} = \int_{\Omega_R} T(x) Y^R_{n,k}(x)\, d\omega_R(x). \tag{2.10}$$

Therefore, from formula (2.8) and (2.9), we obtain

$$f_V(x) = f_T(x) + f_U(x),$$

where

$$\begin{aligned}
f_T(x) &= (\nabla T)(x) \\
&= \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} T^R_{n,k} \left(\frac{R}{|x|}\right)^n \left(-\frac{n+1}{|x|^2} o^{(1)} Y_{n,k}\left(\frac{x}{|x|}\right) + \frac{1}{|x|^2} o^{(2)} Y_{n,k}\left(\frac{x}{|x|}\right)\right), \\
f_U(x) &= (\nabla U)(x) \\
&= \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} U^R_{n,k} \left(\frac{R}{|x|}\right)^{n+1} \left(-\frac{n}{|x|^2} o^{(1)} Y_{n,k}\left(\frac{x}{|x|}\right) + \frac{1}{|x|^2} o^{(2)} Y_{n,k}\left(\frac{x}{|x|}\right)\right)
\end{aligned}$$

for all $x \in \overline{\Omega_R^{ext}}$ provided that V is of the form (2.6). But this shows us that the Fourier coefficients $(f_T)^{(1),R}_{0,1} = 0$ and $(f_T)^{(i),R}_{1,k} = 0$ for $i, k = 1, 2, 3$, i.e. the field of the disturbing potential does not contain any vector spherical harmonic contribution of order 0 and 1. This fact gives rise to the following theorem.

Theorem 2.3 *Let* $T \in Pot(\Omega_R^{ext})$ *be of the representation (2.9). Then the following statements are valid:*

(i) $p_{(3)}(\nabla T)\big|_{\Omega_r} = 0,$

(ii) $p_{(1)}(\nabla T)\big|_{\Omega_r} = 0$ *if and only if* $T = 0,$

(iii) $p_{(2)}(\nabla T)\big|_{\Omega_r} = 0$ *if and only if* $T = 0.$

In other words, the disturbing potential $T \in Pot(\Omega_R^{ext})$ is uniquely determined either by the first radial derivative of T or the surface gradient field of T on Ω_r.

2.2 Solution by Integral Equation

From our considerations presented above it follows that there are at least two different ways of determining the gravitational potential uniquely at (spherical earth's surface) Ω_R from GPS-satellite-to-satellite tracking measurements on the (spherical orbital) sphere Ω_r. When the disturbing potential is considered both approaches are unique.

2.2.1 Integral Equation for the Radial Derivative (Variant 1)

The first variant of satellite-to-satellite tracking simply reduces to the problem of determining the gravitational potential $V \in Pot(\Omega_R^{ext})$ from (negative) first order derivatives

$$-\frac{\partial}{\partial r} V(r\xi) = -\xi \cdot (\nabla V)(r\xi) = G(r\xi), \quad \xi \in \Omega.$$

As pointed out earlier it is also possible to consider the disturbing potential $T \in Pot(\Omega_R^{ext})$ with $T_{n,k}^R = 0$ for $n = 0, 1,\ k = 1, \ldots, 2n+1$ instead of the gravitational potential V itself. Since the satellite mission CHAMP, however, is intended to improve the coefficients $V_{0,1}^R$ and $V_{1,1}^R$, $V_{1,2}^R$, $V_{1,3}^R$, and no mathematical restrictions on the uniqueness are detectable, it is advisable to use the V-approach when radial derivatives are taken as input data (note that we base our considerations on the negative first order radial derivative to obtain positive singular values below). We are interested in expressing our problem in terms of an integral equation (cf. Freeden et al. (1997), Schneider (1997), and Thalhammer (1995)). To this end, we observe that

$$-\frac{\partial}{\partial r}\left(\frac{R}{r}\right)^{n+1} Y_{n,k}(\xi) = \left(\frac{R}{r}\right)^{n+1} \frac{n+1}{r} Y_{n,k}(\xi), \quad \xi \in \Omega.$$

Consequently the operator Λ of the (negative) radial derivative $-\partial/\partial r$ at height r is given by

$$(\Lambda V)(x) = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} \left(\frac{R}{r}\right)^{n} \frac{n+1}{r} V_{n,k}^R Y_{n,k}^r(x), \quad x = r\xi,\ x \in \Omega$$

and Λ has a positive symbol $\{\Lambda^\wedge(n)\}$, $\Lambda^\wedge(n) = \left(\frac{R}{r}\right)^n \frac{n+1}{r}$, $n = 0, 1, \ldots$. Therefore, the GPS-satellite-to-satellite tracking problem may be rewritten as an integral equation $\Lambda V = G$, where Λ is an injective pseudodifferential operator of order $-\infty$ (cf. Freeden et al. (1998)) given by

$$(\Lambda V)(x) = \int_{\Omega_R} K_\Lambda(x, y) V(y)\, d\omega_R(y) = G(x)$$

with

$$K_\Lambda(x, y) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \Lambda^\wedge(n) Y_{n,k}^R(y) Y_{n,k}^r(x) = -\frac{\partial}{\partial r} K(r\xi, R\eta)$$

and

$$K(r\xi, R\eta) = \frac{1}{4\pi R} \frac{r^2 - R^2}{(r^2 + R^2 - 2Rr(\xi \cdot \eta))^{3/2}}, \tag{2.11}$$

$(\xi, \eta) \in \Omega^2$. It is obvious that $K_\Lambda(\cdot, \cdot) \in C(\Omega_r) \times C(\Omega_R)$. Therefore, Λ is compact with infinite-dimensional range $\mathcal{R}(\Lambda)$ (cf. Louis (1989), Schneider (1997)). Hence, Λ^{-1} is not bounded on the space $\mathcal{L}^2(\Omega_R)$. Moreover, since $\overline{\mathcal{R}(\Lambda)}^{\|\cdot\|_{\mathcal{L}^2(\Omega_r)}} = \mathcal{L}^2(\Omega_r)$ it is clear from the Picard-criterion (see, e.g. Louis (1989)) that the integral equation

$$\Lambda V = G, \quad V \in \mathcal{L}^2(\Omega_R), \quad G \in \mathcal{L}^2(\Omega_r) \tag{2.12}$$

is solvable if and only if $G \in \mathcal{R}(\Lambda)$. An easy calculation shows that $\Lambda^* : \mathcal{L}^2(\Omega_r) \to \mathcal{L}^2(\Omega_R)$ given by

$$\Lambda^* G = \int\limits_{\Omega_r} K_\Lambda(x, \cdot) G(x) d\omega_r(x)$$

is the adjoint operator. Remembering Hadamard's introduction of a well-posed problem (existence, uniqueness, continuity of the inverse) we realize that the problem (2.12) is ill-posed as it violates the first and the third property. The singular system of (2.12) indicates the type of ill-posedness. Indeed, we have

$$\begin{aligned} \sigma_n &= \left(\frac{R}{r}\right)^n \frac{n+1}{r}, \quad n = 0, 1, \dots, \\ \Lambda Y^R_{n,k} &= \sigma_n Y^r_{n,k}, \quad n = 0, 1, \dots, \ k = 1, \dots, 2n+1, \\ \Lambda^* Y^r_{n,k} &= \sigma_n Y^R_{n,k}, \quad n = 0, 1, \dots, \ k = 1, \dots, 2n+1. \end{aligned}$$

$\{Y^R_{n,k}, Y^r_{n,k}, \sigma_n\}$ forms the singular system of problem (2.12). Since $(\Lambda^\wedge(n))^{-1}$ diverges exponentially for $n \to \infty$, the satellite-to-satellite tracking problem (2.12) of determining the gravitational potential V at earth's surface Ω_R from the (negative) first order radial derivative on the orbit Ω_r is an inverse problem that is exponentially ill-posed.

2.2.2 Integral Equation for the Surface Gradient (Variant 2)

The second variant of GPS-satellite-to-satellite tracking is the problem of determining $T \in Pot(\Omega_R^{ext})$ satisfying $T^R_{n,k} = 0$ for $n = 0, 1$, $k = 1, \dots, 2n+1$ from given surface gradients

$$\nabla^*_\xi T(r\xi) = g^{(2)}(r\xi), \quad \xi \in \Omega$$

with $g^{(2),r}_{n,k} = 0$ for $n = 0, 1$, $k = 1, \dots, 2n+1$. By virtue of Poisson's formula we obtain

$$\int\limits_{\Omega_R} k_\lambda(x, y) T(y) d\omega_R(y) = g^{(2)}(r\xi), \quad \xi \in \Omega,$$

where

$$k_\lambda(x, y) = \sum_{n=1}^{\infty} \sum_{k=1}^{2n+1} \left(\frac{R}{r}\right)^n Y^R_{n,k}(y) (\mu^{(2)}_n)^{1/2} y^{(2),r}_{n,k}(x)$$

for $x = r\xi$, $y = R\eta$, $(\xi, \eta) \in \Omega^2$. An easy calculation yields

$$k_\lambda(x, y) = \frac{1}{Rr} \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n P'_n(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi)$$

(note that $\nabla^*_\xi F(\xi \cdot \eta) = F'(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi)$ for $F \in C^{(1)}[-1, 1]$). The operator $\lambda : \mathcal{L}^2_2(\Omega_R) \to l^2_{(2),2}(\Omega_r)$ of the surface gradient is given by

$$(\lambda T)(x) = \sum_{n=2}^{\infty} \sum_{k=1}^{2n+1} T^R_{n,k} \left(\frac{R}{r}\right)^n (\mu^{(2)}_n)^{1/2} y^{(2),r}_{n,k}(x),$$

where we have used the abbreviations

$$\mathcal{L}^2_2(\Omega_R) = \{F \in \mathcal{L}^2(\Omega_R) \mid F^R_{0,1} = F^R_{1,1} = F^R_{1,2} = F^R_{1,3} = 0\}$$

and

$$l^2_{(2),2}(\Omega_r) = \{f^{(2)} \in l^2_{(2)}(\Omega_r) \mid f^{(2),r}_{1,k} = 0, \ k = 1, 2, 3\}.$$

Again we are confronted with an ill-posed integral equation

$$\lambda T = g, \quad T \in \mathcal{L}_2^2(\Omega_R),\; g \in l^2_{(2),2}(\Omega_r), \tag{2.13}$$

where λ defines an injective operator from $\mathcal{L}_2^2(\Omega_R)$ onto $l^2_{(2),2}(\Omega_r)$. An easy calculation shows that $\lambda^* : l^2_{(2),2}(\Omega_r) \to \mathcal{L}_2^2(\Omega_R)$ given by

$$\lambda^* g = \int\limits_{\Omega_r} k_\lambda(x,\cdot)\cdot g(x)d\omega_r(x)$$

defines the adjoint operator of λ. It is not hard to verify that

$$\lambda^*\lambda Y^R_{n,k} = \left(\frac{R}{r}\right)^{2n} n(n+1) Y^R_{n,k}$$

(note that $\mu_n^{(2)} = n(n+1)$). We see that

$$\begin{aligned}
\sigma_n &= \left(\frac{R}{r}\right)^n \sqrt{n(n+1)}, \quad n = 2,3,\ldots,\\
\lambda Y^R_{n,k} &= \sigma_n y^{(2),r}_{n,k}, \quad n = 2,3,\ldots,\ k = 1,\ldots,2n+1\\
\lambda^* y^{(2),r}_{n,k} &= \sigma_n Y^R_{n,k}, \quad n = 2,3,\ldots,\ k = 1,\ldots,2n+1.
\end{aligned}$$

constitutes the singular system $\{Y^R_{n,k}, y^{(2),r}_{n,k}, \sigma_n\}$. Thus the problem (2.13) is exponentially ill-posed.

2.2.3 Functionalanalytic Synopsis

Next we want to formulate both satellite-to-satellite tracking variants for the disturbing potential in a unified functionalanalytic concept. For that purpose we introduce the following reference spaces:

	$(\mathcal{H},(\cdot,\cdot)_\mathcal{H})$	$(\mathcal{K},(\cdot,\cdot)_\mathcal{K})$
variant 1	$(\mathcal{L}^2(\Omega_R),(\cdot,\cdot)_{\mathcal{L}^2(\Omega_R)})$	$(\mathcal{L}^2(\Omega_r),(\cdot,\cdot)_{\mathcal{L}^2(\Omega_r)})$
variant 2	$(\mathcal{L}_2^2(\Omega_R),(\cdot,\cdot)_{\mathcal{L}^2(\Omega_R)})$	$\left(l^2_{(2),2}(\Omega_r),(\cdot,\cdot)_{l^2(\Omega_r)}\right)$

Integral equation and singular system are abbreviated as follows:

	A	$H_{n,k}$	$K_{n,k}$	σ_n
variant 1	Λ	$Y^R_{n,k}$	$Y^r_{n,k}$	$\left(\frac{R}{r}\right)^n \frac{n+1}{r}$
variant 2	λ	$Y^R_{n,k}$	$y^{(2),r}_{n,k}$	$\left(\frac{R}{r}\right)^n \sqrt{n(n+1)}$

Observe that $A : \mathcal{H} \to \mathcal{K}$ is a linear, compact operator satisfying $ker(A) = \{0\}$, $im(A) \subset \mathcal{K}$, $\overline{im(A)} = \mathcal{K}$ and $A^*A : \mathcal{H} \to \mathcal{H}$ is self-adjoint (where $ker(A)$ denotes the kernel, $im(A)$ the image, and A^* is the adjoint operator of A). Moreover, $\{H_{n,k}\}$ is a complete orthonormal system in $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ and $\{K_{n,k}\}$ is a complete orthonormal system in $(\mathcal{K}, (\cdot,\cdot)_{\mathcal{K}})$ such that the singular values $\{\sigma_n\}$ of A satisfy the relations

$$\begin{aligned} AH_{n,k} &= \sigma_n K_{n,k}, \\ A^* K_{n,k} &= \sigma_n H_{n,k}. \end{aligned}$$

2.2.4 Regularization

In what follows we restrict our considerations to the variant 1, i.e. the determination of the gravitational potential V from given radial derivatives. Similar methods apply to variant 2. Starting point is the integral equation

$$AV = G, \quad V \in \mathcal{H},\ G \in \mathcal{R}(A), \tag{2.14}$$

with an undisturbed right hand side G given by

$$G = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} G_{n,k} K_{n,k}$$

with $G_{n,k} = (G, K_{n,k})_{\mathcal{K}}$, $n = 0,1,\ldots$, $k = 1,\ldots,2n+1$. Then we obtain

$$A^{-1}G = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \sigma_n^{-1} G_{n,k} H_{n,k} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V_{n,k} H_{n,k} \tag{2.15}$$

with $V_{n,k} = (V, H_{n,k})_{\mathcal{H}}$, $n = 0,1,\ldots$, $k = 1,\ldots,2n+1$.

For arbitrary functions $G \in \mathcal{K}$ (note that in any practical application G is affected by observational errors) the right hand side of (2.15) is not necessarily convergent. To force convergence of (2.15) we have to replace the series (2.15) by a filtered singular value expansion

$$S_j G = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} F_{\gamma_j}(\sigma_n) G_{n,k} H_{n,k}.$$

The filter $\{F_{\gamma_j}(\sigma_n)\}$, $\gamma_j \in (0,\infty)$ with $\lim_{j\to\infty} \gamma_j = 0$ and $\lim_{j\to-\infty} \gamma_j = \infty$ (e.g. $\gamma_j = 2^{-j}$, $j \in \mathbb{Z}$) has to satisfy some properties known from the theory of inverse problems (cf. e.g. Tikhonov and Arsenin (1977)).

Theorem 2.4 *Let $\{S_j\}$, $\gamma_j \in (0,\infty)$, $j \in \mathbb{Z}$, such that $\lim_{j\to\infty} \gamma_j = 0$ and $\lim_{j\to-\infty} \gamma_j = \infty$, be a sequence of operators $S_j : \mathcal{K} \to \mathcal{H}$ with*

$$S_j K_{n,k} = F_{\gamma_j}(\sigma_n) H_{n,k} \tag{2.16}$$

for all $n = 0,1,\ldots$, $k = 1,\ldots,2n+1$, where $\{F_{\gamma_j}(\sigma_n)\}$ satisfies the three conditions:

(i) $\sup_n \left|F_{\gamma_j}(\sigma_n)\right| = c(\gamma_j) < \infty$

(ii) $\lim_{j\to\infty} \sigma_n F_{\gamma_j}(\sigma_n) = 1$ *for* $n = 0,1,\ldots$

(iii) $\sigma_n \left|F_{\gamma_j}(\sigma_n)\right| \leq c < \infty$ *uniformly with respect to* $\gamma_j > 0$ *and for* $n = 0,1,\ldots$.

Then $\{S_j\}$ is a regularization of A^{-1}, i.e. S_j is bounded on $\mathcal{K}$ for all $\gamma_j > 0$ and for all $G \in \mathcal{R}(A)$,

$$\lim_{j\to\infty} S_j G = A^{-1}G,$$

where the equality is understood in the $\|\cdot\|_{\mathcal{H}}$-sense.

For the proof the reader is directed to any textbook on inverse problems. $S_j G$ is called j-level regularization of the problem (2.14).

As examples we list three possible choices of the filter F_γ, namely the truncated singular value decomposition (TSVD), a smoothed version of TSVD and the Tikhonov filter (TF).

(i) TSVD

$$F_{\gamma_j}(\sigma_n) = \begin{cases} \sigma_n^{-1} & : \quad n \leq N(\gamma_j) \\ 0 & : \quad n > N(\gamma_j) \end{cases}$$

In TSVD we take the original coefficients of the series expansion up to a certain order $N(\gamma_j)$, where the error in the measured data may be strengthened. TSVD defines a bandlimited filter.

(ii) Smoothed TSVD

$$F_{\gamma_j}(\sigma_n) = \begin{cases} \sigma_n^{-1} & : \quad n \leq M(\gamma_j) \\ \sigma_n^{-1}\tau_{\gamma_j}(n) & : \quad n = M(\gamma_j)+1, \ldots, N(\gamma_j) \\ 0 & : \quad n > N(\gamma_j) \end{cases} ,$$

where τ_{γ_j} is assumed to be monotonically decreasing on $[M(\gamma_j), N(\gamma_j)]$.

(iii) TF

$$F_{\gamma_j}(\sigma_n) = \frac{\sigma_n}{\sigma_n^2 + \gamma_j^2}.$$

The influence of the Tikhonov filter can be characterized as follows. If the singular value σ_n is large compared to γ_j, the filter F_{γ_j} becomes nearly 1 and the regularization of the corresponding part of the solution is only weak. On the other hand, if σ_n is small, the filter damps the parts of the solution which might be strongly affected by errors in the mesurements. In contrast to TSVD, the high frequency contributions are not omitted completely, i.e. TF defines a non-bandlimited filter.

The idea discussed now in this paper is to represent the j-level regularization by means of wavelets thereby obtaining a $(j+1)$-level regularization by adding detail information to the j-level regularization. In doing so it becomes clear that any classical regularization method based on filtered approximation can be reformulated in terms of our wavelet method. Thus, any known parameter choice strategy depending on the filter technique under consideration is also applicable and, moreover, any of the corresponding error estimates are valid, too. This is the reason why we omit this discussion here.

From numerical tests with synthetic earth models it is known that the spherical harmonic methods introduced so long are appropriate tools to recover the low frequency part of the gravitational potential at the earth's surface from satellite-to-satellite tracking measurements along a satellite orbit. But when passing over to higher frequency contributions (more precisely, higher resolution in the space domain), the uncertainty principle tells us that more and more local structures have to come into play, which are not reflected by the spherical harmonic model with tolerable numerical effort. In fact, it is not suitable to model short wavelength phenomena by trial functions which show an ideal frequency localization, but no space localization as spherical harmonics do. Thus, multiscale techniques like wavelet methods based on frequency as well as space localizing trial functions have become more and more important in recent years (cf. Freeden and Windheuser (1997) and Freeden et al. (1998)). It is of importance to mention that combined spherical harmonic and wavelet expansions are developable (cf. Freeden and Windheuser (1997)), the spherical harmonic expansion of the gravitational potential being responsible for the low frequency part and the wavelet expansion being appropriate for the higher frequency parts.

Our aim now is to regularize the equation (2.14) by a multiresolution analysis in terms of spherical wavelets (cf. Schneider (1997)). As a matter of fact, the wavelet concept enables us to break up a complicated function into pieces at different scales and positions. The wavelet coefficients contain information of different frequencies, they mirror a space dependent regularization as the regularization parameter plays the role of the scale in the ordinary spherical wavelet theory. Altogether we are able to recover the gravitational potential V at the earth's surface in terms of expressions $\Psi_{j;y} = \Psi_j(y,\cdot)$ that are dilated and rotated versions of one function, viz. the mother wavelet Ψ.

The wavelet transform is based on the theory of singular integrals. To be specific, convolutions P_j (low-pass filter) and R_j (band-pass filter) are performed against the measured function G (given along the satellite orbit). Approximations of $A^{-1}G$ are obtainable by multiresolution analysis in accordance with the following scheme (cf. Freeden and Schneider (1998):

$$
\begin{array}{ccccccccccc}
P_0(G) & & P_1(G)\ldots & & P_j(G) & & P_{j+1}(G)\ldots & \stackrel{j\to\infty}{\to} & A^{-1}G \\
\Cap & & \Cap & & \Cap & & \Cap & & \Cap \\
\mathcal{V}_0 & \subset & \mathcal{V}_1 \ \ldots & \subset & \mathcal{V}_j & \subset & \mathcal{V}_{j+1} & \ldots = & \mathcal{H} \\
\mathcal{V}_0 + \mathcal{W}_0 & + & \ldots + \mathcal{W}_{j-1} & + & \mathcal{W}_j & + & \ldots & = & \mathcal{H} \\
\Cup \quad\ \Cup & & \Cup & & \Cup & & & & \Cup \\
P_0(G)+R_0(G) & + & \ldots + R_{j-1}(G) & + & R_j(G) & + & \ldots & = & A^{-1}G
\end{array}
$$

Consequently, V is represented by a two-parameter family reflecting the different scales of space localization and frequency localization. The decomposition and reconstruction scaling functions Φ_j and $\tilde{\Phi}_j$ are given by their series expansions

$$
\begin{aligned}
\Phi_j(y,\cdot) &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \Phi_j^{\wedge}(n) H_{n,k}(y) K_{n,k}, \quad y \in \Omega_R, \\
\tilde{\Phi}_j(y,\cdot) &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \Phi_j^{\wedge}(n) H_{n,k}(y) H_{n,k}, \quad y \in \Omega_R.
\end{aligned}
$$

The decomposition and reconstruction wavelets Ψ_j and $\tilde{\Psi}_j$ are defined in the same manner by

$$
\begin{aligned}
\Psi_j(y,\cdot) &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \Psi_j^{\wedge}(n) H_{n,k}(y) K_{n,k}, \quad y \in \Omega_R, \\
\tilde{\Psi}_j(y,\cdot) &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \Psi_j^{\wedge}(n) H_{n,k}(y) H_{n,k}, \quad y \in \Omega_R.
\end{aligned}
$$

Scaling functions and wavelets are related via the *scaling equation* in terms of their symbols

$$
\left(\Psi_j^{\wedge}(n)\right)^2 = \left(\Phi_{j+1}^{\wedge}(n)\right)^2 - \left(\Phi_j^{\wedge}(n)\right)^2 .
$$

The kernels Ψ_j and Φ_j are generated by dilation of the mother kernels Ψ and Φ, respectively. The scale index j serves as measure for decreasing frequency localization as well as a measure for increasing space localization. As mentioned above, the convolution operators

$$
P_j(G) = \tilde{\Phi}_j * \Phi_j * G = \int_{\Omega_R}\int_{\Omega_r} \Phi_j(y,x) G(x)\, d\omega_r(x) \tilde{\Phi}_j(y,\cdot)\, d\omega_R(y)
$$

and

$$
R_j(G) = \tilde{\Psi}_j * \Psi_j * G = \int_{\Omega_R}\int_{\Omega_r} \Psi_j(y,x) G(x)\, d\omega_r(x) \tilde{\Psi}_j(y,\cdot)\, d\omega_R(y)
$$

may be interpreted as *low-pass filter* and *band-pass filter*. The operator P_{J+1} can be decomposed in the following way

$$
P_{J+1} = P_J + R_J = P_0 + \sum_{j=0}^{J} R_j .
$$

The detail space $\mathcal{W}_J = \{R_J(G) \mid G \in \mathcal{K}\}$ contains the detail information needed to go from a regularized solution $P_J(G)$ to a regularized solution $P_{J+1}(G)$.

In terms of the above presented choices of the filter F_{γ_j} we are confronted with three possible representations of the coefficients $\Phi_j^\wedge(n)$ and $\Psi_j^\wedge(n)$:

(i) TSVD

$$\Phi_j^\wedge(n) = \begin{cases} \sigma_n^{-1/2} & , \quad n = 0, \dots, N_j \\ 0 & , \quad n \geq N_j + 1 \end{cases},$$

$$\Psi_j^\wedge(n) = \begin{cases} 0 & , \quad n = 0, \dots, N_j \\ \sigma_n^{-1/2} & , \quad n = N_j + 1, \dots, N_{j+1} \\ 0 & , \quad n \geq N_{j+1} + 1 \end{cases},$$

$$N_j = \begin{cases} 0 & for \quad j \in \mathbb{Z},\ j < 0 \\ 2^j - 1 & for \quad j \in \mathbb{Z},\ j \geq 0 \end{cases}.$$

(ii) Smoothed TSVD

$$\Phi_j^\wedge(n) = \begin{cases} \sigma_n^{-1/2} & , \quad n = 0, \dots, M_j \\ \sigma_n^{-1/2}(\tau_j(n))^{1/2} & , \quad n = M_j + 1, \dots, N_j \\ 0 & , \quad n \geq N_j + 1 \end{cases},$$

$$\Psi_j^\wedge(n) = \begin{cases} 0 & , \quad n = 0, \dots, M_j \\ \sigma_n^{-1/2}(1 - \tau_j(n))^{1/2} & , \quad n = M_j + 1, \dots, M_{j+1} \\ \sigma_n^{-1/2}(\tau_{j+1}(n) - \tau_j(n))^{1/2} & , \quad n = M_{j+1} + 1, \dots, N_j \\ \sigma_n^{-1/2}(\tau_{j+1}(n))^{1/2} & , \quad n = N_j + 1, \dots, N_{j+1} \\ 0 & , \quad n \geq N_{j+1} + 1 \end{cases},$$

$$N_j = \begin{cases} 0 & \text{for} \quad j \in \mathbb{Z}, j < 0 \\ 2^{j+1} - 1 & \text{for} \quad j \in \mathbb{Z}, j \geq 0 \end{cases}, M_j = \begin{cases} 0 & \text{for} \quad j \in \mathbb{Z}, j < 0 \\ 2^j - 1 & \text{for} \quad j \in \mathbb{Z}, j \geq 0 \end{cases}$$

and $\tau_j(t) = 2 - 2^{-j}(t+1)$, $t \in [2^j - 1, 2^{j+1} - 1]$, $j \in \mathbb{N}_0$.

(iii) TF

$$\Phi_j^\wedge(n) = \left(\frac{\sigma_n}{\sigma_n^2 + \gamma_j^2}\right)^{1/2}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{Z},$$

$$\Psi_j^\wedge(n) = \left(\frac{\sigma_n}{\sigma_n^2 + \gamma_{j+1}^2} - \frac{\sigma_n}{\sigma_n^2 + \gamma_j^2}\right)^{1/2}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{Z}$$

with $\{\gamma_j\}$, $j \in \mathbb{Z}$ being as always a sequence of real numbers satisfying $\lim_{j \to \infty} \gamma_j = 0$ and $\lim_{j \to -\infty} \gamma_j = \infty$.

2.2.5 Concluding Remarks

In the previous sections we have motivated that wavelets provide an appropriate framework to study the satellite-to-satellite tracking problem. Regularization methods are developed based on filtered singular integral decomposition as a multiresolution. The regularization parameter plays the role of the scale parameter in ordinary wavelet theory. For numerical purposes we propose two different ways of wavelet regularization, namely bandlimited truncated singular value decomposition and non-bandlimited Tikhonov regularization. In accordance with the uncertainty

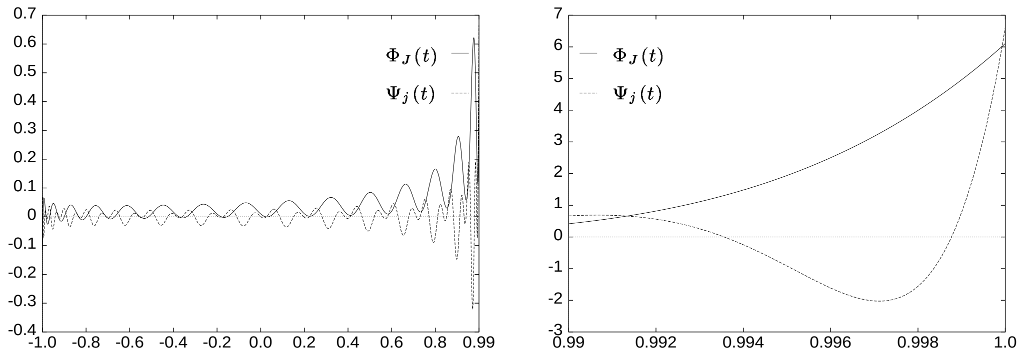


Figure 2.1: Spherical TSVD Regularization Wavelet Packet and Scaling Function

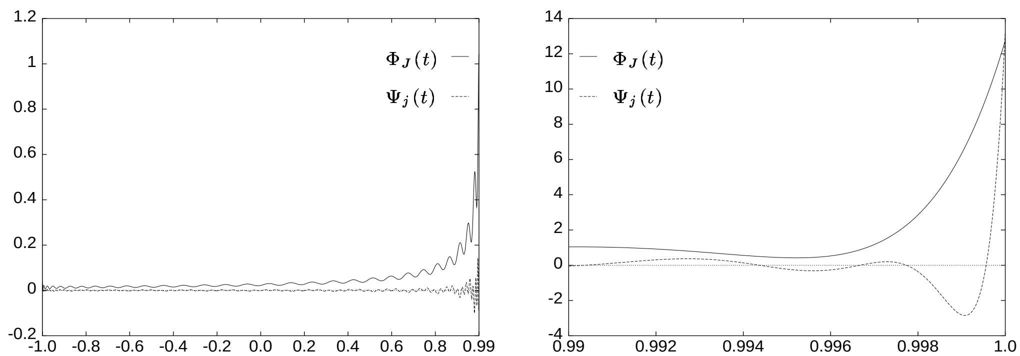


Figure 2.2: Spherical Smoothed TSVD Regularization Wavelet Packet and Scaling Function

principle (cf. Freeden and Windheuser (1997)) the constituting elements can be explained as follows:

Non-bandlimited regularization wavelets (cf. Figure 2.3) tend to be extremely space localizing, hence, huge data sets can be handled since only data in a small cap whose size is determined by the choice of the wavelet is needed for the purpose of numerical integration. On the other hand, a large number of wavelet coefficients depending on the choice of the wavelet for the regularization is needed, since the wavelet coefficients only give local information of a small neighbourhood.

Bandlimited regularization wavelets (cf. Figures 2.1 and 2.2) show more moderate phenomena of space localization so that one can work only with smaller data sets in numerical evaluation, but the number of wavelet coefficients can be reduced, since they contain information of a more extended area. Moreover, a certain spectral band can be expressed exactly in terms of wavelets because of their bandlimited character. Fast computation by virtue of panel clustering and pyramid schemes is possible (cf. Freeden et al. (1998A)).

In conclusion, the proper choice of a spherical regularization wavelet for an inverse problem like satellite-to-satellite tracking depends on the practical requirements. Nevertheless, our investigations illustrate the following aspects in future modelling of the earth's gravitational potential: It is canonical to start with non-space localizing and ideally frequency localizing polynomials (i.e. spherical harmonics) to work out a trend approximation. For the intermediate case between long and short wavelength approximation, we should continue with moderately balanced space and frequency localizing trial functions. There are two choices, namely (smooth) bandlimited TSVD-wavelets or moderately space localizing non-bandlimited TF-wavelets. Finally, for the purpose of higher resolution (e.g. satellite gradiometry mission), more and more space localizing non-bandlimited TF-wavelets should come into play.

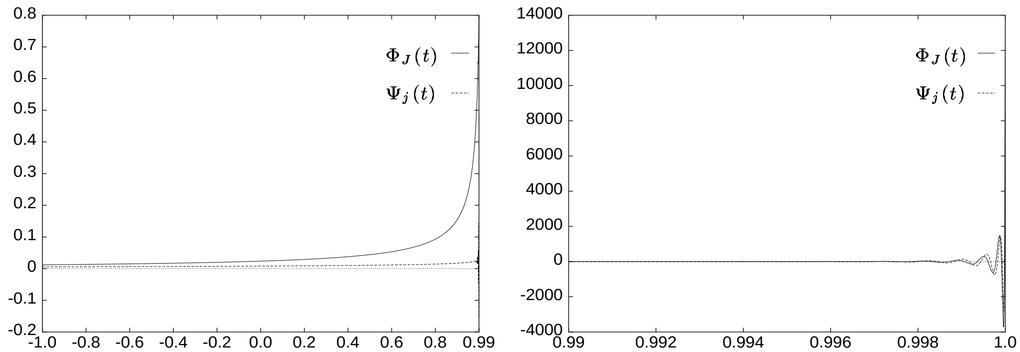


Figure 2.3: Spherical Tikhonov Regularization Wavelet Packet and Scaling Function

2.3 Locally Adapted Wavelet Regularization (Spherical Approximation)

The essential feature of regularization wavelets is the ability of realizing multiscale approximation for the regularized solution to the inverse problem by use of space localizing wavelets. This observation gives rise to establish a locally adapted regularization method for a local subdomain Ω_R^{loc} of Ω_R. According to the multiscale regularization scheme an approximation V_J to the potential V on the subdomain Ω_R^{loc} reads as follows:

$$V_J(x) = P_0(G)(x) + \sum_{j=0}^{J-1} R_j(G)(x), \quad x \in \Omega_R^{loc},$$

where $V_J = P_J(G)$ represents the regularized solution with respect to the regularization parameter γ_J. First P_0 and R_0 are determined on Ω_R^{loc} with the aid of panel clustering (cf. Freeden et al. (1998A), Glockner (1997)). In similarity to the ideas of thresholding a locally adapted regularization method can be developed as follows: Find a subdomain Ω_R^0 of Ω_R^{loc} such that $R_0(G)$ is negligible on $\Omega_R^{loc} \backslash \Omega_R^0$. In other words, we are ready to accept an error as long as the quality after compression is acceptable (e.g. of order smaller than 10^{-8}), hence, $R_0(G)$ is replaced by $R_0^{loc}(G)$ defined by

$$R_0^{loc}(G)(x) = \begin{cases} R_0(G)(x) & , \quad x \in \Omega_R^0 \\ 0 & , \quad x \in \Omega_R^{loc} \backslash \Omega_R^0 \end{cases} .$$

In doing so, less critical areas are extracted in potential determination. Numerical tests (cf. Bayer (1996), Glockner (1997), Bayer et al. (1998)), in fact, show that a considerable amount of data reduction does not change essentially the detail information of the potential. Continuing this process yields a sequence of subsets of Ω_R^{loc}

$$\Omega_R^{loc} \supsetneq \Omega_R^0 \supsetneq \Omega_R^1 \supsetneq \ldots \supsetneq \Omega_R^{J-1}.$$

such that

$$R_j^{loc}(G)(x) = \begin{cases} R_j(G)(x) & , \quad x \in \Omega_R^j \\ 0 & , \quad x \in \Omega_R^{loc} \backslash \Omega_R^j \end{cases} .$$

As result we therefore obtain

$$V_J^{loc} = P_0(G) + \sum_{j=0}^{J-1} R_j^{loc}(G)$$

for the approximation of V_J on Ω_R^{loc}. But this means that the detail information is added locally such that

$$\begin{aligned} V_J^{loc}\big|_{\Omega_R^{loc}\setminus\Omega_R^0} &= P_0(G), \\ V_J^{loc}\big|_{\Omega_R^j\setminus\Omega_R^{j+1}} &= P_{j+1}(G), \quad j = 0, \dots, J-2 \\ V_J^{loc}\big|_{\Omega_R^{loc}\setminus\Omega_R^{J-1}} &= P_J(G). \end{aligned}$$

Thus the computations of our method are based on locally adapted regularization parameters, and the compression is obtained by the idea of thresholding. The critical point of this procedure is the proper choice of the sequence $\{\Omega_R^j\}$ and the associated regularization parameters $\{\gamma_j\}$. In our tests performed so far, this is still a trial-and-error procedure. Moreover the determination of optimal regularization parameters is a challenge for future work. Nevertheless, our computations have shown (cf. Bayer (1996), Glockner (1997)), that treating inverse problems for the OSU91A model (as synthetic dataset) by locally adapted regularization wavelets shows, in fact, a considerable improvement (of about 40%) both in accuracy and computational efficiency.

3 Non-Spherical Approximation

Next we give a multiscale approximation for the disturbing potential on and outside the real earth based on the actual (orbital) data. For this purpose we assume that the surface of the earth Σ_{earth} is regular (in the sense that (i) Σ_{earth} divides the three-dimensional Euclidean space into a bounded region Σ_{earth}^{int} (inner space) and the unbounded region Σ_{earth}^{ext} (outer space) defined by $\Sigma_{earth}^{ext} = \mathbb{R}^3\setminus\overline{\Sigma_{earth}^{int}}$, $\overline{\Sigma_{earth}^{int}} = \Sigma_{earth}^{int} \cup \Sigma_{earth}$, (ii) Σ_{earth}^{int} contains the origin, (iii) Σ_{earth} is a closed and compact surface free of double points, (iv) Σ_{earth} is $C^{(2)}$-Hölder-smooth). Furthermore, the orbit Σ_{orb} is assumed to be a regular surface such that there exist two spheres Ω_a (Bjerhammar sphere), Ω_b satisfying (cf. Figure 3.1)

$$a < \inf_{x\in\Sigma_{earth}} |x| \leq \sup_{x\in\Sigma_{earth}} |x| < b < \inf_{x\in\Sigma_{orb}} |x|$$

and

$$\Sigma_{earth}^{ext} \subset \Omega_a^{ext}, \quad \Sigma_{orb}^{ext} \subset \Omega_b^{ext}.$$

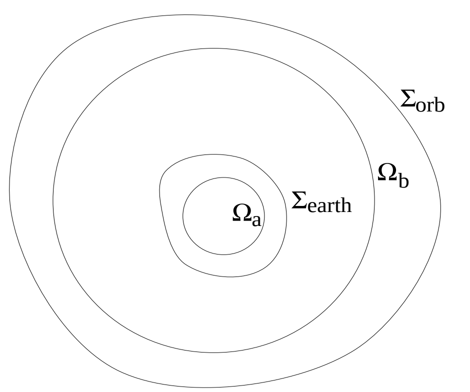


Figure 3.1:

The GPS-SST problem for the gravitational potential (non-spherical approximation):

Assume that there is given a vector field $f_V \in c^{(\infty)}(\Sigma_{orb})$ on the known "orbital surface" of the form (1.2). We look for a potential $V : \overline{\Sigma_{earth}^{ext}} \to \mathbb{R}$ satisfying the following properties:

- $V \in C^{(0)}(\overline{\Sigma_{earth}^{ext}}) \cap C^{(2)}(\Sigma_{earth}^{ext})$
- $\Delta V = 0$ in Σ_{earth}^{ext}

- $|V(x)| = O(1/|x|), \ |x| \to \infty$
- $-\frac{x}{|x|}\nabla_x V(x) = -\frac{x}{|x|} f_V(x), \ x \in \Sigma_{orb}$.

First we discuss the *uniqueness*. From our GPS-SST approach it is clear that the direction $-x/|x|$, at any point x on the orbital surface Σ_{orb} forms with the inner (unit) normal $\nu(x)$ on Σ_{orb} an angle satisfying

$$\inf_{x \in \Sigma_{orb}} \left(-\frac{x}{|x|} \cdot \nu(x) \right) > 0.$$

Under this condition, however, the oblique derivative problem of potential theory (cf. e.g. Miranda (1970), Freeden and Kersten (1981)) tells us that V is uniquely determined in Σ_{orb}^{ext}. By analytical continuation, therefore, V is uniquely determined in Σ_{earth}^{ext}.

Theorem 3.1 *The GPS-SST problem for the gravitational potential in non-spherical approximation corresponding to the 'oblique data' on Σ_{orb} is uniquely determined.*

For our further considerations it remains to determine the remainder part V on and outside the real earth's surface from (discrete) data on the orbit Σ_{orb}. Freeden and Schneider (1998, 1998A) devised different strategies to solve this problem by constructive multiscale analysis. From these strategies we recapitulate below (without proof) two very promising concepts which are based on the Runge-Walsh assumption that says, for arbitrarily given value $\varepsilon > 0$, the potential V may be approximated by a bandlimited potential $V^{(m)} \in Pot(\Omega_a^{ext})$ (i.e. $\int_{\Omega_a} V^{(m)}(x) Y_{n,k}(x)\, d\omega_a(x) = 0$, $n = m+1, m+2, \ldots$ for some positive integer m) such that both $V^{(m)}$ approximates V in ε-accuracy on $(\overline{\Sigma_{earth}^{ext}})$:

$$\sup_{x \in \overline{\Sigma_{earth}^{ext}}} \left| V(x) - V^{(m)}(x) \right| < \varepsilon,$$

and V is consistent with $V^{(m)}$ on any given <u>finite</u> set of linear (observation) functionals $L_1^M, \ldots, L_M^M$ with

$$L_i^M V = L_i^M V^{(m)}, \quad i = 1, \ldots, M, \quad M = (m+1)^2.$$

As a matter of fact, the Runge-Walsh method of approximating the earth's gravitational potential from a 'wavelet potential' outside the Bjerhammar sphere is the key idea of our constructive approximation presented now.

Reconstructing the auxiliary bandlimited potential $V^{(m)}$ from the known finite set of given linear functionals L_i^M, $i = 1, \ldots, M$ then leads to a multiscale approximation of $V^{(m)}$ in terms of bandlimited (TSVD / smoothed TSVD) wavelets.

$$\begin{aligned}
\hat{\Phi}_j(x,y) &= \sum_{n=0}^{m_j} \sum_{k=1}^{2n+1} \hat{\Phi}_j^{\wedge}(n) \left(\frac{a}{|x|}\right)^{n+1} Y_{n,k}^a(x) \left(\frac{a}{|y|}\right)^{n+1} Y_{n,k}^a(y), \\
\tilde{\hat{\Phi}}_j(x,y) &= \sum_{n=0}^{m_j} \sum_{k=1}^{2n+1} \hat{\Phi}_j^{\wedge}(n) \left(\frac{a}{|x|}\right)^{n+1} Y_{n,k}^a(x) \left(\frac{a}{|y|}\right)^{n+1} Y_{n,k}^a(y), \\
\hat{\Psi}_j(x,y) &= \sum_{n=1}^{m_j} \sum_{k=1}^{2n+1} \hat{\Psi}_j^{\wedge}(n) \left(\frac{a}{|x|}\right)^{n+1} Y_{n,k}^a(x) \left(\frac{a}{|y|}\right)^{n+1} Y_{n,k}^a(y), \\
\tilde{\hat{\Psi}}_j(x,y) &= \sum_{n=1}^{m_j} \sum_{k=1}^{2n+1} \hat{\Psi}_j^{\wedge}(n) \left(\frac{a}{|x|}\right)^{n+1} Y_{n,k}^a(x) \left(\frac{a}{|y|}\right)^{n+1} Y_{n,k}^a(y),
\end{aligned}$$

$m_j = 2^{j+1} \geq 0$, $x, y \in \overline{\Omega_a^{ext}}$ with

$$\begin{aligned}
\hat{\Phi}_j^{\wedge}(n) &= \sqrt{\sigma_n} \Phi_j^{\wedge}(n), \\
\hat{\Psi}_j^{\wedge}(n) &= \sqrt{\sigma_n} \Psi_j^{\wedge}(n),
\end{aligned}$$

$n = 0, 1, \ldots$ (Note that, as usual, $\Psi_j^\wedge(0) = 0$, i.e. the 0th moment of $\widehat{\Psi}_j$ and $\tilde{\widehat{\Psi}}_j$ vanish for all j, which is characteristic in wavelet theory. It tells us that $\widehat{\Psi}_j$, $\tilde{\widehat{\Psi}}_j$ cannot just be a 'bump' function on A, but must oscillate. This justifies the name 'wavelet'.) More explicitely, we have the following result (cf. Freeden and Schneider (1998A)):

Theorem 3.2 *Let J be chosen such that $m_J > m$. Assume that, from the potential V, there are known the data $v_i = L_i V$, $i = 1, \ldots, M$. Furthermore, let $\{y_1^{M_j}, \ldots, y_{M_j}^{M_j}\}$, $M_j = (2m_j + 1)^2$, $j = 0, \ldots, J$ be systems of points on the sphere Ω_a. Then, under the assumptions stated above, the fully discrete J-level wavelet approximation of $V^{(m)}$ which serves as Runge-Walsh approximation of V (in uniform sense on $\overline{\Sigma_{earth}^{ext}}$) reads as follows:*

(a)

$$\left(V^{(m)}\right)_J = \sum_{j=0}^{J} \sum_{i=1}^{M_j} b_i^j \sum_{n=0}^{m} \sum_{k=1}^{2n+1} \sum_{s=1}^{M} a_s^{n,k} \widehat{\Psi}_j^\wedge(n) t_s H_{-n-1,k}(a; y_i^{M_j}) \tilde{\widehat{\Psi}}_j(y_i^{M_j}, \cdot),$$

where we have used the abbreviation

$$H_{-n-1,k}(a;x) = \left(\frac{a}{|x|}\right)^{n+1} Y_{n,k}^a(x).$$

The weights $a_1^{n,k}, \ldots, a_M^{n,k}$, $n = 1, \ldots, m$, $k = 1, \ldots, 2n+1$ satisfy the linear equations

$$\sum_{s=1}^{M} a_s^{n,k} L_s^M H_{-i-1,l}(a; \cdot) = \delta_{n,i}\delta_{k,l},$$

$i = 1, \ldots, m$, $l = 1, \ldots, 2n+1$, and $b_1^j, \ldots, b_{M_j}^j$, $j = 0, \ldots, J$ satisfy the linear equations

$$\sum_{i=1}^{M_j} b_i^j K_{Harm_{1,\ldots,2m_k}}(y_l^{M_j}, y_i^{M_j}) = \int_{\Omega_a} K_{Harm_{1,\ldots,2m_j}}(y_l^{M_j}, x)\, d\omega_a(x), \quad l = 1, \ldots, M_j. \tag{3.17}$$

For $x, y \in \overline{\Omega_a^{ext}}$, $K_{Harm_{1,\ldots,2m_j}}(\cdot,\cdot)$ is given by

$$K_{Harm_{1,\ldots,2m_j}}(x,y) = \sum_{n=1}^{2m_j} \frac{2n+1}{4\pi a^2} \left(\frac{a^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}, \frac{y}{|y|}\right).$$

(b)

$$\left(V^{(m)}\right)_J = \sum_{j=0}^{J} \sum_{i=1}^{M_j} b_i^j \sum_{s=1}^{M} a_s^{j,i} t_s \tilde{\widehat{\Psi}}_j(y_i^{M_j}, \cdot),$$

where the weights $a_1^{j,i}, \ldots, a_M^{j,i}$, $j = 0, \ldots, J$, $i = 1, \ldots, M_j$ are given by

$$\sum_{s=1}^{M} a_s^{j,i} L_l^M L_s^M K_{Harm_{q+1,\ldots,m}}(\cdot,\cdot) = L_l^M \widehat{\Psi}_j(y_i^{M_j}, \cdot)$$

$l = 1, \ldots, M$ and $b_1^j, \ldots, b_{M_j}^j$, $j = 0, \ldots, J$ satisfy (3.17).

It should be remarked that a large number of linear systems must be solved. But if we look carefully to the linear equations we realize that we are always confronted with the same coefficient matrix (that is assumed to be invertible). As a matter of fact, the coefficient matrix must be inverted once, all weights for numerical integration can be obtained by a matrix-vector multiplication and stored elsewhere (in an a-priori step for computation). In addition, it should be mentioned that the solution of the linear systems determining the weights of the reconstruction step (3.17) can be avoided completely if we place the knots for numerical integration of the wavelet coefficients for each detail step $j = 0, \dots, J$ on a special longitude-latitude grid on the sphere Ω_a. The corresponding set of integration weights for reconstruction purposes are explicitly available without solving any linear system (for more details concerning the determination of the weights the reader is referred to a paper due to Driscoll and Healy (1994)).

Until now the linear (observation) functionals have not been specified in more detail. In fact, the choice of the linear functionals enable us to develop two important variants of wavelet approximation in reality.

Satellite-only Multiscale Approximation

In this case $L_1^M, \dots, L_M^M$ are understood to be the SST-functionals

$$v_i = L_i^M V^{(m)} = -\frac{x}{|x|} \cdot \nabla_x V|_{x=x_i}, \quad i = 1, \dots, M.$$

As result we get a satellite-only multiscale approximation of V on $\overline{\Sigma_{earth}^{ext}}$ from discrete data on the real (orbital) surface.

In practice, however, we are confronted with the situation that terrestrial data as well as SST-satellite data are available for the potential V. This gives rise to a combined approach.

Terrestrial / Satellite Multiscale Approximation

The set $\{L_1^M, \dots, L_M^M\}$ consists of linear functionals of the following three types: $x \mapsto -\frac{x}{|x|} \cdot \nabla_x V(x)$, $x \in \Sigma_{orb}$ (SST-functional on real orbit), $x \mapsto \lambda(x) \cdot \nabla_x V(x)$, $x \in \Sigma_{earth}$ (Neumann (λ-oblique derivative) functional for points on continents), and $x \mapsto V(x)$, $x \in \Sigma_{earth}$ (Dirichlet functional for points on oceans), so that the linear equations stated above are uniquely solvable in terms of these functionals. Again we obtain a multiscale wavelet approximation, but now to a heterogeneous data set.

Considering our solution $(V^{(m)})_J$ in the context of satellite-to-satellite tracking we realize that actually a truncated singular value decomposition or some modified version of it (TSVD / smoothed TSVD) is performed, since the wavelets are bandlimited. Non-bandlimited (Tikhonov (TF) like) fully discretized wavelets require the solution of more sophisticated linear equations. Nevertheless, they should be investigated in more detail (e.g. for the gradiometry problem) because of their much stronger space localization properties. This aspect will be discussed in a forthcoming paper.

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