Mathematical Methods in Solid Mechanics

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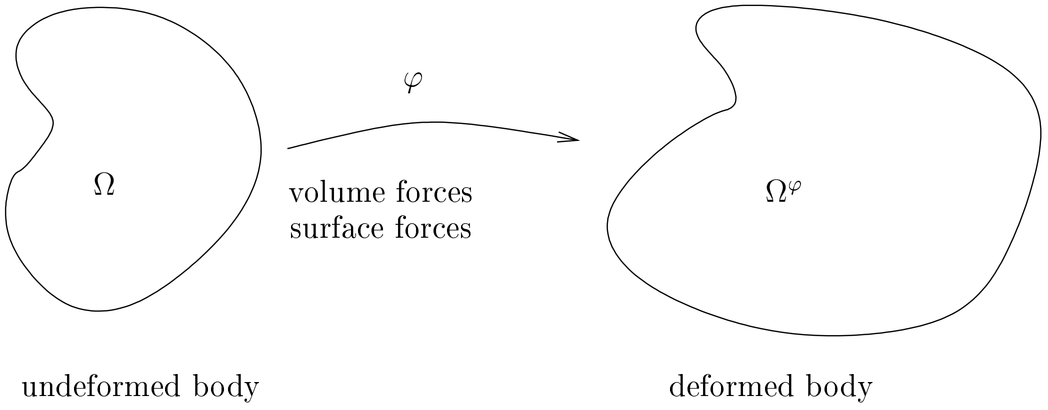
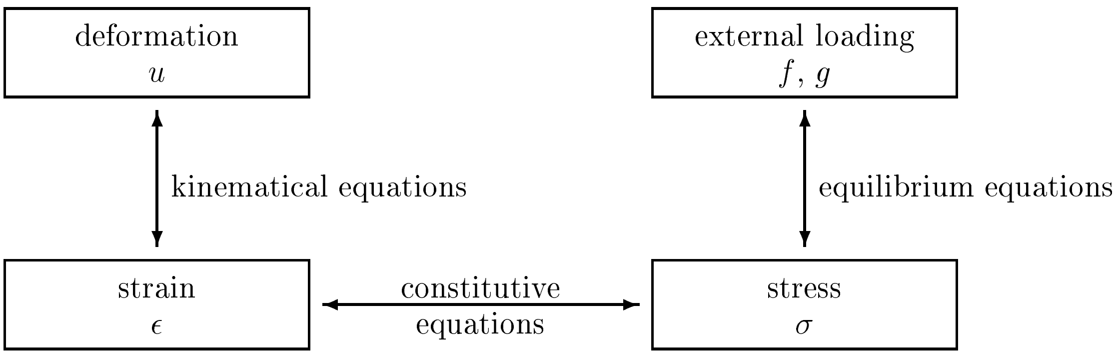
Contents

Introduction

Any material body Ω deforms when it is subjected to external forces.



The deformation is called **elastic**, if it is reversible and time independent, that means, if the deformation vanishes instantaneously when the forces are removed. A reversible, but time–dependent deformation is known as **viscoelastic**; in this case the deformation increases with time after application of load, and it decreases slowly after the load is removed. The deformation is called **plastic**, if it is irreversible or permanent.
The theories of elasticity an plasticity can be divided into two categories: One group is known as mathematical theories, the other as physical theories. Mathematical theories are formulated to represent experimental observations as well as physical principles in a general form (phenomenological theories). Physical theories on the other hand attempt to especially quantify plastic deformations at the microscopic level and explain why and how the deformations occur. The movements of atoms and the deformation of the crystals and grains are important considerations (micro–mechanics). Most applications, such as structural design or metal forming, are on the macroscopic scale. Here a mathematical theory is needed which allows to simulate the state of the body analytically and numerically and to compare the results with experimental data.
In this lecture we study the state of solid bodies under loading in the framework of continuum mechanics, that means, we assume that the states of the bodies can be described through functions (fields) that are defined on a continuum and depend on space, time and possibly on velocity variables.
In order to derive the mathematical models in form of partial differential equations or variational principles we follow the following scheme:



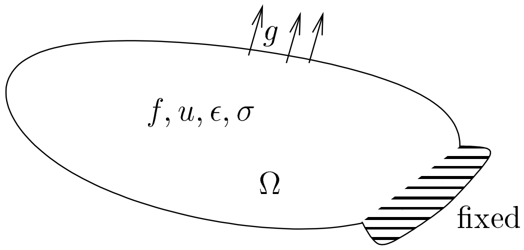


Fig. 1

Let us explain the strain–stress relations for metals under loading as a one–dimensional macroscopic model. From fundamental experimental observations one can gain insight into the basic behaviour of deformed materials.
We consider a metal wire under an axial load $P > 0$. The initial length of the specimen is l_0, the undeformed area of the cross section is A_0.

$$\begin{aligned} \hat{\sigma} &= \frac{P}{A_0} \qquad \text{is the nominal stress} \\ \hat{\epsilon} &= \frac{l - l_0}{l_0} \qquad \text{denotes the nominal strain.} \end{aligned}$$

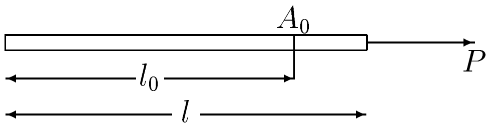


Fig. 2

The relation $\hat{\sigma} = \hat{\sigma}(\hat{\epsilon})$ depends on the material properties and the strength of the tensile force P. (For a compressive force $P < 0$, the stress $\hat{\sigma}$ is negative.)
If we slowly increase the loading, $\hat{\sigma}$ increases and the wire is expanded. Fig. 3 shows the typical behaviour of the $\hat{\sigma} - \hat{\epsilon}$ curve.

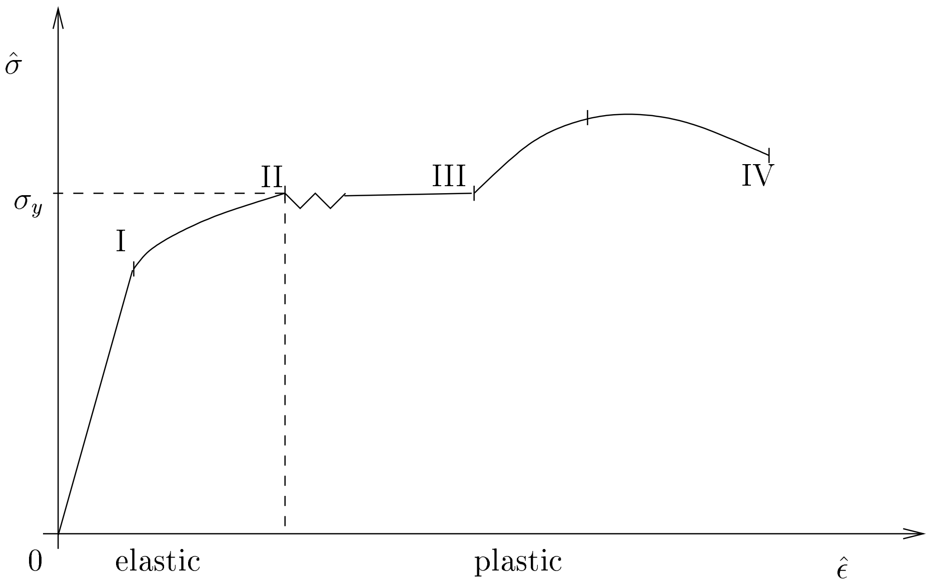


Fig. 3

0–I: Initially the relation between stress and strain is linear: $\hat{\sigma} = E\hat{\epsilon}$ is called Hooke's law, which forms the foundation of the theory of elasticity.
The constant E is called the elasticity module.

I–II: Beyond point I the increase in strain is not linear with the increase in stress, but the deformation is still elastic, that means the removal of stress will restore the material to its original shape. There exists a nonlinear relation

$$\hat{\sigma} = \sigma(\hat{\epsilon})$$

which is called nonlinear Hooke's law.

II–III: Point II is called the upper yield point. Here the wire begins to flow. The region II–III is significant for an increased rate of strain without essential increase of stress. There is a small local drop followed by several oscillations of the stress level. This forms a plateau until point III (flow region). Region II–III shows perfectly plastic behaviour. The lower bound of the stress oscillation determines the yield stress σ_y.

III–IV: If the stress keeps increasing at point III, the material hardens and the stress increases until the ultimate stress is reached at point IV. Region III–IV is known as material hardening.
After point IV the stress decreases with increase in strain representing the behaviour of instability (breaking).

In the entire region II–III–IV the material is plastically deforming.

Some metals, such as aluminum, copper and stainless steel, do not exhibit sharp yield points II and III. Instead, the yield of these materials is a gradual transition from a linear elastic to a nonlinear plastic behaviour. In these cases one usually defines an offset yield stress σ_y for which a definite plastic strain will be left after unloading.

Remarks:

1° Nominal stress $\hat{\sigma}$ and nominal strain $\hat{\epsilon}$ are defined based on the original dimensions. True stress σ and true strain ϵ are defined as

$$\begin{aligned} \sigma &= \frac{P}{A}, && A \text{ is the deformed cross area,} \\ \Delta\epsilon &= \frac{\Delta l}{l} && \Rightarrow \epsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln\frac{l}{l_0} \\ \epsilon &= \ln(1+\hat{\epsilon}). \end{aligned}$$

For $Al = A_0 l_0$ this equals to

$$\sigma = \hat{\sigma}(1+\hat{\epsilon}).$$

2° Real material behaviour in the plastic region is very complex. Idealization is necessary, especially in the hardening portion.

$$
\begin{aligned}
\sigma &= \sigma_y + H\epsilon^n && \text{(Ludwick 1909)}\\
\sigma &= H\epsilon^n && \text{(Holloman 1944)}\\
\sigma &= \sigma_y + (\sigma_s - \sigma_y)(1 - \epsilon^{-n\epsilon}) && \text{(Voce 1948)}\\
\sigma &= H(\epsilon_s + \epsilon)^n && \text{(Swift 1947)}\\
\sigma &= \sigma_y \tanh\left(\frac{E\epsilon}{\sigma_y}\right) && \text{(Prager 1938)}\\
\sigma &= \frac{\sigma}{E} + H\left(\frac{\sigma}{E}\right)^n && \text{(Ramberg and Osgood 1943)}
\end{aligned}
$$

$E, \sigma_y, \sigma_s, \epsilon_s, H$ and n are material constants and must be determined experimentally.

Historical remarks:

1638	Galilei	– resistance of solids to rupture
1660	Hooke	– Hooke's law
1705	J. Bernoulli	– bonded elastic bar, strain, 1D
1744	Euler	– differential equation of the elastica, 1D
1821	Navier	– general equations of equilibrium and vibration of elastic solids
1827	Cauchy	– stress, stress–strain relation for isotropic materials, 3D, linear case
1839	Green	– principle of minimal elastic potential energy, anisotropic materials
1864	Saint Venant, Tresca	– plastic deformations yield criterion
1872	Levy	– plasticity
1913	von Mises	– plasticity conditions
1924	Hencky, Prandtl, Taylor	– linear quasi–statical and quasi–dynamical behaviour
$1930 \sim 1950$	Signorini, Rivlin, Truesdell	– nonlinear elasticity
1958	Noll	– axiomatic of classical mechanics

Chapter 1

3D–Elastostatics

1.1 Kinematics

In continuum mechanics the term kinematics refers to the mathematical description of the deformation and motion of a body under loading. If the applied forces are time–independent, a new equilibrium position appears. The description of this state with the help of displacement fields, strain tensors and stress tensors is a central problem in elastostatics.

Deformation in $\mathbb{R}^3$

An elastic body occupies a reference configuration $\overline{\Omega}$ in $\mathbb{R}^3$. Mathematically spoken, $\overline{\Omega}$ is the closure of a domain Ω in $\mathbb{R}^3$, representing the volume of the body.

Definition 1: (domain)

$\Omega \subset \mathbb{R}^3$ is a domain, if Ω is an open, bounded and connected subset of $\mathbb{R}^3$. $\overline{\Omega}$ denotes its closure, $\partial\Omega$ its boundary.

Definition 2: (reference configuration, deformation, current configuration)

The initial undeformed domain Ω is called reference configuration. A mapping $\varphi\colon \overline{\Omega} \to \mathbb{R}^3$ is a C^1–deformation, if:

- *φ is differentiable in Ω, the deformation gradient*

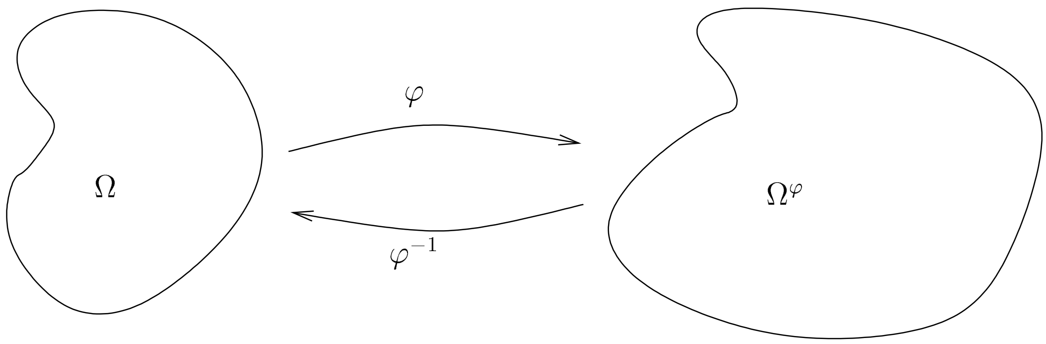
$$D\varphi = \nabla\varphi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \frac{\partial \varphi_1}{\partial x_3} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_3} \\ \frac{\partial \varphi_3}{\partial x_1} & \frac{\partial \varphi_3}{\partial x_2} & \frac{\partial \varphi_3}{\partial x_3} \end{pmatrix} \tag{1.1}$$

  is a continouos mapping. $D\varphi\colon \overline{\Omega} \to \mathbb{R}^{3,3}$.

- *φ is injective (one–to–one) on Ω, i.e. φ^{-1} exists.*

- φ *is orientation–preserving:* $\det(D\varphi(x)) > 0 \quad \forall x \in \Omega$.

$\varphi(\Omega) = \Omega^\varphi$ *is called the deformed configuration.*



reference configuration deformed configuration

The properties of the deformed configuration are prescribed as follows:

Proposition:
Let $\Omega \subset \mathbb{R}^3$ *be a domain with* $\Omega = \mathrm{int}(\overline{\Omega})$, $\partial\Omega = \partial\overline{\Omega}$. $\varphi\colon \overline{\Omega} \to \mathbb{R}^3$ *is a* C_1*–deformation and injective on* $\overline{\Omega}$. *Then* $\varphi(\Omega)$ *is a domain,* $\varphi(\overline{\Omega}) = \overline{\varphi(\Omega)}$, $\varphi(\Omega) = \mathrm{int}\varphi(\overline{\Omega})$, $\varphi(\partial\Omega) = \partial\varphi(\Omega) = \partial\varphi(\overline{\Omega})$.

Proof: [5, Th.1.2-8]
The mapping φ can be splitted into

$$\varphi = \mathrm{id} + u, \tag{1.2}$$

that means $\varphi(x) = x + u(x) \quad \forall x \in \overline{\Omega}$, id denotes the identity map.

Definition 3: (displacement)
The mapping $u\colon \overline{\Omega} \to \mathbb{R}^3$, *defined by the relation*

$$\varphi = \mathrm{id} + u$$

is the displacement.

The displacement gradient

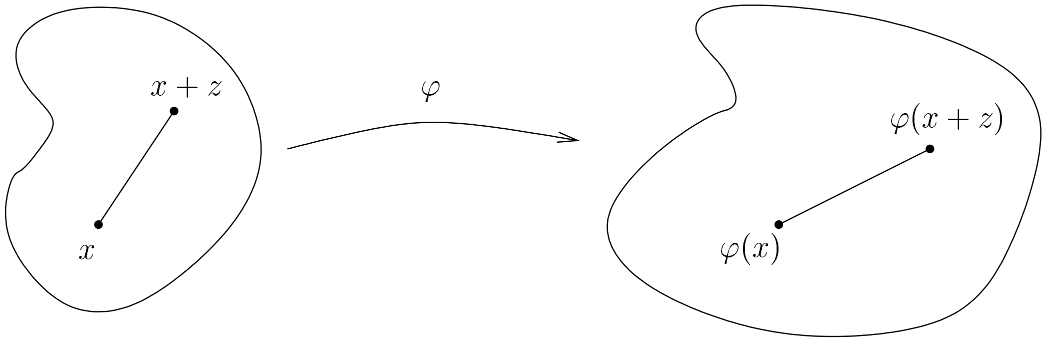
$$\nabla u := \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix},$$

$\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$ and the deformation gradient are related by the equation

$$\nabla\varphi = I + \nabla u. \tag{1.3}$$

Strain tensors

Let us consider two points x and $x+z$ from the reference configuration Ω.



The distance (with respect to the Euclidian norm) is:

$$\|x+z-x\|^2 = (z,z) = z^T z.$$

The image points $\varphi(x)$ and $\varphi(x+z)$ have the distance

$$\begin{aligned}
\|\varphi(x+z)-\varphi(x)\|^2 &= \|\nabla\varphi z + O(\|z\|)\|^2 \\
&= \|\nabla\varphi z\|^2 + O(\|z\|^2) \\
&= z^T(\nabla\varphi)^T\nabla\varphi z + O(\|z\|^2).
\end{aligned}$$

The matrix $\nabla\varphi^T\nabla\varphi$ is a local measure ($\|z\|$ has to be small) for the strain with respect to the deformation φ.

Definition 4: (strain tensor)
Let $\Omega \subset \mathbb{R}^3$ be a domain, $\varphi\colon \overline{\Omega} \to \mathbb{R}^3$ a C_1–deformation. The symmetric tensor

$$C = (\nabla\varphi)^T\nabla\varphi : \overline{\Omega} \to \mathbb{R}^{3,3} \tag{1.4}$$

is called the right Cauchy–Green strain tensor. The symmetric tensor

$$E = \frac{1}{2}(C-I) : \overline{\Omega} \to \mathbb{R}^{3,3} \tag{1.5}$$

is called the Green–St. Venant strain tensor.

Lemma:
For the displacement field u which corresponds to the deformation φ, it holds that:

$$C = I + (\nabla u)^T + \nabla u + \nabla u^T\nabla u \tag{1.6}$$

$$E = \frac{1}{2}(\nabla u^T + \nabla u + \nabla u^T\nabla u). \tag{1.7}$$

Proof: Insert (1.3) into (1.4). ■

Remark:

The left Cauchy Green strain tensor

$$B = \nabla\varphi\nabla\varphi^T$$

is also important. It plays an essential role explained later.

In view of showing that the tensor C is indeed a good measure of “strain“ in an intuitive sense of “change in form or size“, let us characterise a class of deformations which induce no “strain“.

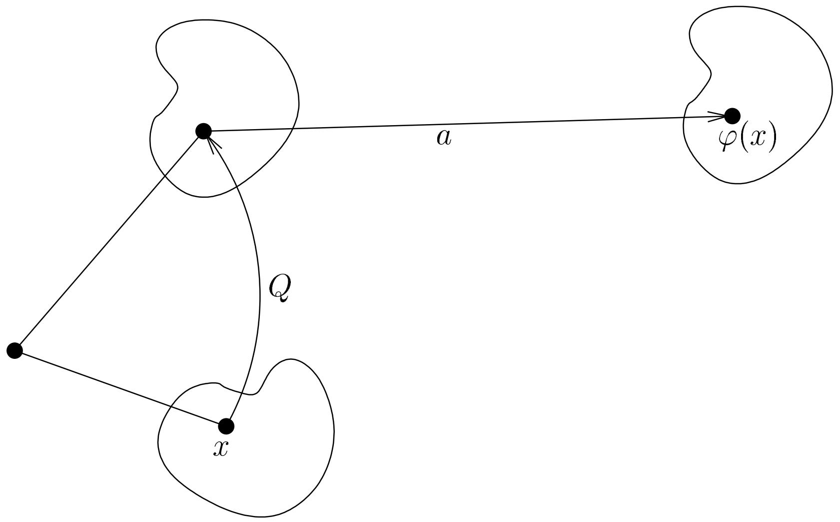
Definition 5: (rigid deformation)

A deformation φ is called a rigid deformation, if

$$\varphi(x) = Qx + a, \quad a \in \mathbb{R}^3, \quad Q \in \mathbb{R}^{3,3}$$

and Q is an orthogonal matrix with $\det Q = 1$.

In other words: The deformation is rigid, if the corresponding deformed configuration is obtained by rotating the reference configuration around the origin by the rotation Q and by translating it by the vector a.



For a rigid deformation it immediately follows that:

$$C = (\nabla\varphi)^T\nabla\varphi = Q^TQ = I, \quad E = 0, \quad \nabla u = -I + \nabla\varphi = -I + Q.$$

On the other hand: If $C = I$ and $\det\nabla\varphi(x) > 0$, then the corresponding deformation is necessarily rigid.

Theorem: [5, p.45]

Let Ω be an open and connected subset of $\mathbb{R}^n$ and φ a C^1–deformation. Then the following statements are equivalent:

(i) $C(x) = (\nabla\varphi)^T(x)\nabla\varphi(x) = I \qquad \forall x \in \Omega$

(ii) *for every $x \in \Omega$ there is a neighbourhood $U(x) \subset \Omega$, such that the Euclidian distances are unchanged under the map φ.*

$$\|\varphi(y) - \varphi(z)\| = \|y - z\| \qquad \forall y, z \in U(x).$$

I.e. locally the mapping φ is an isometry.

(iii) *There exists an orthogonal matrix $Q \in \mathbb{R}^{n,n}$ and a vector $a \in \mathbb{R}^n$ such that*

$$\varphi(x) = Qx + a \qquad \forall x \in \Omega.$$

Proof:

(iii) $\to$ (i): Evident

(i) $\to$ (ii): Since $\det\nabla\varphi(x) > 0 \quad \forall x \in \Omega$, it follows that φ is locally invertible in Ω. There exist open sets $U \subset \Omega$ and $V \subset \varphi(\Omega)$ containing the points x and $\varphi(x)$ respectively, and which are mapped bijectively in both directions:

$$U \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} V,$$

$\varphi\Big|_U$ and $\varphi^{-1}\Big|_V$ are continuously differentiable.
Applying the mean value theorem we get:

$$\begin{aligned}
\|\varphi(y) - \varphi(z)\| &\leq \|y - z\| \sup_{\xi\in(y,z)} \|\nabla\varphi(\xi)\| \\
&= \|y - z\| \sup_{\xi} \max_i (\lambda_i(\nabla\varphi^T(\xi)\nabla\varphi(\xi)))^{\frac{1}{2}} \\
&= \|y - z\|,
\end{aligned}$$

where λ_i are the eigenvalues of the matrix $\nabla\varphi^T(\xi)\nabla\varphi(\xi) = I$;
and

$$\begin{aligned}
\|y - z\| &= \|\varphi^{-1}\varphi(y) - \varphi^{-1}\varphi(z)\| \\
&\leq \|\varphi(y) - \varphi(z)\| \sup_{\eta\in(f(y),f(z))} \|\nabla_\eta\varphi^{-1}(\eta)\| \\
&= \|\varphi(y) - \varphi(z)\|.
\end{aligned}$$

In the last estimate we have used that

$$\begin{aligned}
\varphi^{-1}[\underbrace{\varphi(x)}_{\eta}] &= x \\
\nabla_x\left[\varphi^{-1}(\varphi(x))\right] &= \nabla_\eta\varphi^{-1}(\eta)\cdot\nabla_x\varphi(x) = I \\
\text{and} \quad \nabla_\eta\varphi^{-1}(\eta) &= (\nabla_x\varphi(x))^{-1} = \nabla\varphi(x)^T \text{ is an orthogonal matrix, too.}
\end{aligned}$$

We therefore conclude analogously to the first estimate.

(ii) → (iii): The relation $-\|y-z\|^2 + \|\varphi(y) - \varphi(z)\|^2 = 0$ reads

$$\sum_{i=1}^{n} [\varphi_i(y) - \varphi_i(z)]^2 - \sum_{i=1}^{n} (y-z)^2 = 0 \qquad \text{for } y, z \in U(x).$$

Differentiating with respect to y_i, we get

$$2\sum_{i=1}^{n} [\varphi_i(y) - \varphi_i(z)]\, \partial_j \varphi_i(y) - 2(y_j - z_j) = 0, \quad j = 1, ..., n.$$

Now we differentiate with respect to z_k:

$$-2\sum_{i=1}^{n} \partial_k \varphi_i(y) \partial_j \varphi_i(z) + 2\delta_{jk} = 0 \quad j, k = 1, ..., n$$

or

$$\begin{array}{rrcll} & \nabla\varphi^T(y)\nabla\varphi(z) & = & I & \text{for } y, z \in U \qquad (1.8) \\ \text{and} & \nabla\varphi^T(y)\nabla\varphi(y) & = & I & \text{for } y = z \in U \qquad (1.9) \\ \text{or} & \nabla\varphi(y)\nabla\varphi^T(y) & = & I & \text{for } y = z \in U. \end{array}$$

Multiplying (1.8) from the left by $\nabla\varphi(y)$, we get

$$\nabla\varphi(z) = \nabla\varphi(y),$$

which implies that $\nabla\varphi(y) = \text{const} = Q \quad \forall y \in U$, and moreover that $\forall y \in \Omega$, Q is an orthogonal matrix due to (1.9).
The map

$$\varphi(x) - Qx = g(x)$$

has the property that $\nabla g = 0$ in Ω.
Therefore $g(x) \equiv \text{const} = a$. ■

Polar factorization of a matrix: [5, p.94]
A complex number z can be written in polar coordinates

$$z = re^{i\varphi}, \qquad e^{i\varphi}e^{-i\varphi} = 1, \qquad r > 0.$$

If a matrix $A \in \mathbb{R}^{n,n}$ can be written as

$$A = QS, \tag{1.10}$$

where $Q \in \mathbb{R}^{n,n}$ is an orthogonal matrix, $S \in \mathbb{R}^{n,n}$ is a symmetric and positive definite, then relation (1.10) is called polar factorization.

Lemma:
A real invertible matrix $A \in \mathbb{R}^{n,n}$ can be polar factorized in a unique fashion as

$$A = QS \qquad \textit{or} \qquad A = \tilde{S}Q.$$

Furthermore

$$\det Q = \begin{cases} 1 & \textit{for } \det A > 0 \\ -1 & \textit{for } \det A < 0. \end{cases} \tag{1.11}$$

Proof:

A is invertible, hence A^TA is symmetric and positive definite ($A^TAx = \lambda x \Rightarrow (A^TAx, x) = \lambda(x,x) \Rightarrow (Ax, Ax) = \lambda(x,x) > 0 \quad \text{for } x \neq 0 \Rightarrow \lambda > 0$).
We set $S^2 = A^TA$, $S = \sqrt{A^TA}$. S is uniquely defined and positive definite. Now we define $Q = AS^{-1}$. Since $Q^TQ = S^{-1}A^TAS^{-1} = S^{-1}SSS^{-1} = I$, Q is an orthogonal matrix. From $\det A = \det Q \det S$ and $\det S > 0$, equation (1.11) follows.
Now we proof the uniqueness: Let $A = Q_1S_1 = Q_2S_2$. Then $A^TA = S_1^2 = S_2^2 \Rightarrow S_1 = S_2 \Rightarrow Q_1 = Q_2$. With $\tilde{S} = QSQ^T$, $\tilde{S}^T = QS^TQ^T = \tilde{S}$ is symmetric and positive definite.

$$A = QS = \tilde{S}Q.$$

Since $AA^T = \tilde{S}^2$, the uniqueness can be proved analogically. ■

Linear deformations:

Be $A \in \mathbb{R}^{3,3}$, $\det A > 0$, $\varphi(x) = Ax$.
Then $A = QS = QVDV^T$, where V is an orthogonal and D a diagonal matrix. The resulting strain tensors are

$$\begin{aligned} C &= (\nabla\varphi)^T\nabla\varphi = A^TA = VDV^TQ^TQVDV^T = VD^2V^T \\ E &= \frac{1}{2}(VD^2V^T - I). \end{aligned}$$

Linearised strain tensor:

The modelling of strains can be simplified for small ∇u.

Definition:

Let $\Omega \subset \mathbb{R}^3$ be a domain, $\varphi\colon \overline{\Omega} \to \mathbb{R}^3$ a C^1–deformation with displacement u.
The map

$$e = \epsilon = \frac{1}{2}(\nabla u^T + \nabla u)$$

$e\colon \vec{\Omega} \to \mathbb{R}^{3,3}$ *is a linearised strain tensor with the components*

$$e_{ij}(x) = \epsilon_{ij}(x) = \frac{1}{2}(\partial_i u_j(x) + \partial_j u_i(x)), \quad 1 \leq i,j \leq 3.$$

This linearisation leads to mathematical models for which the analysis and numerics is considerably simplier. Note that even for linear deformations $\varphi(x) = Ax = Ix + (A-I)x$ the deformation gradient $\nabla u = (A - I)$ can be large. For rigid deformations, we have

$$e = e(x) = \frac{1}{2}(Q^T + Q) - I,$$

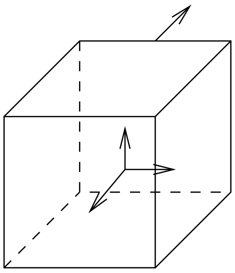
whereas $E = 0$.

Examples

We consider the domain $\Omega = (-1,1)^3$.

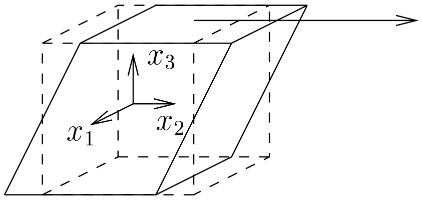
(i) $u(x) = (\alpha x_1, 0, 0)$, $\alpha \in \mathbb{R}$ describes stretching of the first coordinate.

$$e(x) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E(x) = \begin{pmatrix} \alpha + \frac{1}{2}\alpha^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$



(ii) $u(x) = (0, \alpha x_3, 0)$, $\alpha \in \mathbb{R}$, describes shearing in direction of the x_2–axis.

$$e(x) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix}, \qquad E(x) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & \alpha^2 \end{pmatrix}.$$



Volume, curve and surface integrals in the deformed configuration:

The variables $x^\varphi = \varphi(x)$ in the deformed configuration are called Euler variables, the variables x are called Lagrange variables. In general, the Euler variables are unknown and we have to transform the fields and quantities in the deformed configuration into corresponding fields and quantities in the reference configuration. Furthermore, equilibrium relations are formulated by volume and surface intrgrals in the deformed configuration and we have to express them in reference coordinates, too. We therefore express volume, curve and surface integrals in the deformed configuration through the corresponding quantities in the reference configuration.

Volume integral and volume element in the deformed configuration:

It is well known that the changes of variables in multiple integrals over a domain $A^\varphi = \varphi(A) \subset \varphi(\Omega)$ are given by

$$\boxed{\int_{A^\varphi = \varphi(A)} u(x^\varphi)\, dx^\varphi = \int_A (u \circ \varphi)(x) |\mathrm{det} \nabla \varphi(x)|\, dx,}$$

assuming that $\varphi\colon A \to \varphi(A) = A^\varphi$ is an injective and continuously differentiable mapping with a continuous inverse $\varphi^{-1}\colon A^\varphi \to A$.
If φ is a C^1–deformation, then $|\det\nabla\varphi(x)| = \det\nabla\varphi(x) > 0$ and the volume element has the form

$$\boxed{dx^\varphi = \det\nabla\varphi\, dx.}$$

Curve integral and length element in the deformed configuration: [5, p.42]

Let $\gamma\colon [0,1) \to \Omega$ be a C^1–curve in the reference configuration Ω. Its length is

$$l(\gamma) = \int_0^1 \|\gamma'(t)\|\, dt = \int_0^1 \sqrt{\gamma'(t)^T\gamma'(t)}\, dt.$$

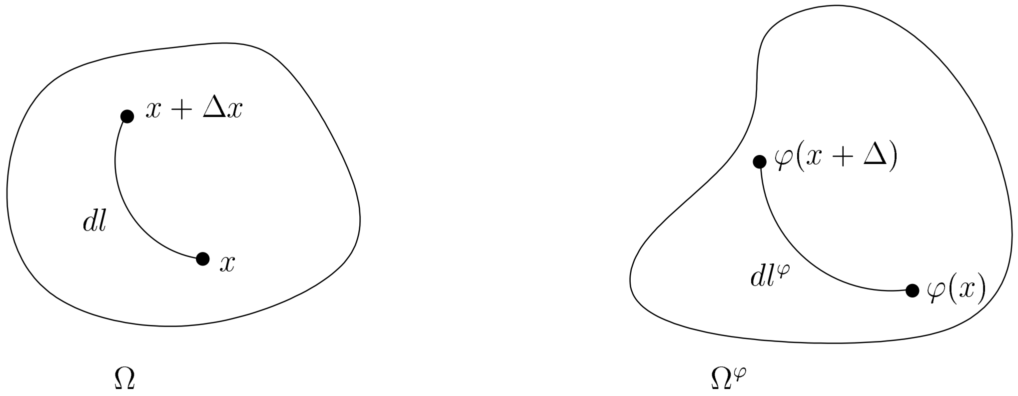
Let $\gamma^\varphi = \varphi(\gamma)$ be the deformed curve with a length of

$$\begin{aligned} l(\gamma^\varphi) &= \int_0^1 \|(\varphi\circ\gamma)'(t)\|\, dt \\ &= \int_0^1 \sqrt{\gamma'(t)^T\nabla\varphi(\gamma(t))^T\nabla\varphi(\gamma(t))\gamma'(t)}\, dt \\ &= \int_0^1 \sqrt{\gamma'(t)^T C(\gamma(t))\gamma'(t)}\, dt \end{aligned}$$

Symbolically the length elements are written as

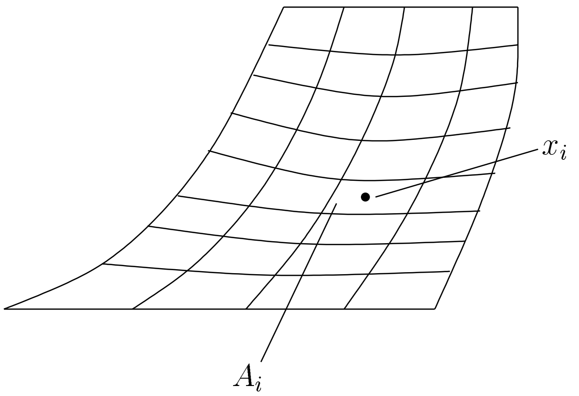
$$\boxed{\begin{aligned} dl &= (dx^T\cdot dx)^{\frac{1}{2}} \\ dl^\varphi &= (dx^T C\, dx)^{\frac{1}{2}} \end{aligned}}$$

with $x = \gamma(t)$, $l(\gamma) = \int\limits_\gamma dl$, $l^\varphi(\gamma_l) = \int\limits_{\gamma^\varphi} dl^\varphi$.



Surface integrals and area element in the deformed configuration:

Let A be a piece of a curved surface.



The surface integral of a function $u(x_1, x_2, x_3) = u(x)$ is defined as

$$\int_A u(x)\, da = \lim_{\substack{\Delta A_i \to 0 \\ n\to\infty}} \sum_{i=1}^{n} u(\vec{x}_i)\Delta A_i, \quad \Delta A_i = \text{measure of the area } A_i.$$

If $x_3 = x_3(x_1, x_2)$ is differentiable, then

$$\int_A u(x)\, da = \int_{A_{x_1 x_2}} u(x_1, x_2, x_3(x_1, x_2))\sqrt{1 + \left(\frac{\partial x_3}{\partial x_1}\right)^2 + \left(\frac{\partial x_3}{\partial x_2}\right)^2}\, dx_1\, dx_2.$$

If $x_i = x_i(u, v)$ are expressed in parameter form, then

$$\int_A f(x)\, da = \int_{A_{uv}}\int f(x(u, v))\sqrt{EG - F^2}\, du\, dv,$$

with $E = \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial u}\right)^2$, $F = \sum_{i=1}^{3}\frac{\partial x_i}{\partial u}\frac{\partial x_i}{\partial v}$, $G = \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial v}\right)^2$.

The surface integral occurs in Gauss' formula (also called fundamental Green's formula).

Given a domain Ω in $\mathbb{R}^3$ with normal vectors $n = (n_i)$ along $\partial\Omega$ and a smooth enough scalar function $u\colon \overline{\Omega} \to \mathbb{R}$, then

$$\boxed{\int_\Omega \partial_i u\, dx = \int_{\partial\Omega} u n_i\, da, \quad 1 \le i \le 3} \tag{1.12}$$

Inserting the product uv for u, we get the multi–dimensional formula for integration by parts:

$$\int_\Omega u\partial_i v\, dx = -\int_\Omega \partial_i u v\, dx + \int_{\partial\Omega} uvn_i\, da, \quad 1 \le i \le 3. \tag{1.13}$$

For a vector field $u\colon \overline{\Omega} \to \mathbb{R}^3$ with components $u_i\colon \overline{\Omega} \to \mathbb{R}$ Gauss' formula (1.17) yields

$$\boxed{\sum_i \int\limits_{\Omega} \partial_i u_i \, dx = \int\limits_{\Omega} \operatorname{div} u \, dx = \int\limits_{\partial\Omega} u \cdot n \, da} \tag{1.14}$$

with $\operatorname{div} u := \sum_{i=1}^{3} \partial_i u_i =: \partial_i u_i$. At this point Einstein's sum convention is used (sum up over the same indices).

Besides relation (1.19) for a vector field, we have to consider this relation for a tensor T.

Here, *tensor* means a second–order tensor

$$T = (T_{ij})_{i,j=1,2,3} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}.$$

For simplicity, we ignore the distinction between covariant and contravariant components and identify the set of all such tensors with the set $\mathbb{R}^{3,3}$ of all square matrices with an order of three.

For a smooth enough tensor field $T\colon \overline{\Omega} \to \mathbb{R}^{3,3}$ the divergence is defined as the following vector:

$$\operatorname{div} T := \begin{pmatrix} \partial_1 T_{11} + \partial_2 T_{12} + \partial_3 T_{13} \\ \partial_1 T_{21} + \partial_2 T_{22} + \partial_3 T_{23} \\ \partial_1 T_{31} + \partial_2 T_{32} + \partial_3 T_{33} \end{pmatrix} \tag{1.15}$$

Applying formula (1.14) to every row of T, we get

$$\boxed{\int\limits_{\Omega} \operatorname{div} T \, dx = \int\limits_{\partial\Omega} T n \, da.} \tag{1.16}$$

In the deformed domain Ω^{φ}, formula (1.16) reads:

$$\boxed{\int\limits_{\Omega^{\varphi}} \operatorname{div}^{\varphi} T^{\varphi} \, dx^{\varphi} = \int\limits_{\partial\Omega^{\varphi}} T^{\varphi} n^{\varphi} \, da^{\varphi}} \tag{1.17}$$

for a tensor field $T^{\varphi}\colon \overline{\Omega}^{\varphi} \to \mathbb{R}^{3,3}$.

The connection between (1.16) and (1.17) is given by the Piola transform:

Definition:

Let $\Omega \subset \mathbb{R}^3$ be a domain, φ a C^1–deformation and $T^{\varphi}:\ \varphi(\overline{\Omega}) \to \mathbb{R}^{3,3}$ a second order tensor.

The mapping $P\colon \mathbb{R}^{3,3} \to \mathbb{R}^{3,3}$ *with*

$$\begin{array}{rrcl} & PT^{\varphi} & = & T \\ \textit{and} & P^{-1}T & = & T^{\varphi} \\ \textit{defined by} & & & \\ & PT^{\varphi}(x^{\varphi}) & = & [\det \nabla\varphi(x)]T^{\varphi}(x^{\varphi})[\nabla\varphi(x)]^{-T} = T(x) \\ & P^{-1}T(x) & = & [\det \nabla\varphi(x)]^{-1}T(x)[\nabla\varphi(x)]^{T} = T^{\varphi}(x^{\varphi}) \end{array}$$

is called the Piola transform.

Remark:

$\mathrm{Cof}\, A := \det\, AA^{-T}$ is called the cofactor matrix of $A \in \mathbb{R}^{3,3}$.
With this notation $PT^{\varphi}(x^{\varphi}) = T^{\varphi}(x^{\varphi})\,\mathrm{Cof}(\nabla\varphi(x))$.

Theorem: (properties of the Piola transform)
Be $PT^{\varphi} = T$. *Then*

$$\textit{(i)} \qquad \operatorname{div} T(x) = \det(\nabla\varphi(x))\, \operatorname{div}^{\varphi} t^{\varphi}(x^{\varphi}) \quad \forall x^{\varphi} = \varphi(x), x \in \overline{\Omega} \tag{1.18}$$

$$\textit{(ii)} \qquad \int\limits_{\partial A} T(x)n\, da = \int\limits_{\partial\varphi(A)} T^{\varphi}(x^{\varphi})n^{\varphi}\, da^{\varphi} \quad \textit{for an arbitrary subdomain } A \subset \overline{\Omega} \tag{1.19}$$

$$\textit{(iii)} \qquad \det \nabla\varphi(x)|\nabla\varphi(x)^{-T}n|\, da = |\mathit{Cof}\nabla\varphi(x)n|\, da = da^{\varphi}. \tag{1.20}$$

Proof:

The key to this proof is the Piola identity:

$$\operatorname{div}\{\det(\nabla\varphi)\nabla\varphi^{-T}\} = \operatorname{div}\mathrm{Cof}\nabla\varphi = \vec{0}. \tag{1.21}$$

This identity can be proven by direct calculation, namely

$$\mathrm{Cof}\nabla\varphi = \det(\nabla\varphi)^{-T} = \frac{\det\nabla\varphi}{\det\nabla\varphi}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix},$$

where $A_{ij} = (-1)^{i+j}\det\, A'_{ij}$, A'_{ij} being the 2×2 matrix obtained by deleting the i^{th} row and the j^{th} column in the matrix $\nabla\varphi^{T}$ (keeping in mind that $(\nabla\varphi)^{-T} = (\nabla\varphi^{-1})^{-T} = (\nabla\varphi^{T})^{-1}$).

Since $(\nabla\varphi)^{T} = \begin{pmatrix} \partial_1\varphi_1 & \partial_1\varphi_2 & \partial_1\varphi_3 \\ \partial_2\varphi_1 & \partial_2\varphi_2 & \partial_2\varphi_3 \\ \partial_3\varphi_1 & \partial_3\varphi_2 & \partial_3\varphi_3 \end{pmatrix}$ we have

$$A_{ij} = \partial_{j+1}\varphi_{i+1}\partial_{j+2}\varphi_{i+2} - \partial_{j+2}\varphi_{i+1}\partial_{j+1}\varphi_{i+2} \quad 1 \le i,j \le 3,$$

where the indices are taken modulo 3.

The components of the vector div[Cof$\nabla\varphi$] are given by:

$$\begin{aligned}\sum_{j=1}^{3}\partial_j A_{ij} &= \partial_1[\partial_2\varphi_{i+1}\partial_3\varphi_{i+2}-\partial_3\varphi_{i+1}\partial_2\varphi_{i+2}]\\ &+ \partial_2[\partial_3\varphi_{i+1}\partial_1\varphi_{i+2}-\partial_1\varphi_{i+1}\partial_3\varphi_{i+2}]\\ &+ \partial_3[\partial_1\varphi_{i+1}\partial_2\varphi_{i+2}-\partial_2\varphi_{i+1}\partial_1\varphi_{i+2}]\\ &= 0 \qquad \text{for } i=1,2,3.\end{aligned}$$

We now prove (i):

The elements of the matrix $T = PT^\varphi$ are of the form

$$T_{ij}(x) = (\det\nabla\varphi(x))T^\varphi_{ik}(x^\varphi)((\nabla\varphi(x))^{-T})_{kj}.$$

Hence

$$\begin{aligned}\sum_j \partial_j T_{ij}(x) &= \sum_j \partial_j[T^\varphi_{ik}(x^\varphi)]\det\nabla\varphi(x)(\nabla\varphi(x))^{-T}_{kj}\\ &+ T^\varphi_{ik}(x^\varphi)\sum_j \partial_j[\det\nabla\varphi\nabla\varphi(x)^{-T}_{kj}].\end{aligned}$$

The second term vanishes as a consequence of the Piola identity. Writing $x^\varphi = \varphi(x)$ and applying the chain rule $\frac{\partial}{\partial x_j} = \sum_l \frac{\partial}{\partial\varphi_l}\frac{\partial\varphi_l}{\partial x_j}$, or shortly $\partial_j = \sum_l \partial^\varphi_l\partial_j\varphi_l$ to the first term, it becomes

$$\begin{aligned}\sum_j \partial_j T_{ij}(x) &= \sum_j\sum_l \partial^\varphi_l T^\varphi_{ik}(\varphi(x))\partial_j\varphi_l(x)(\det\nabla\varphi(x)\nabla\varphi(x)^{-T}_{kj})\\ &= \sum_l \partial^\varphi_l T^\varphi_{ik}(\varphi(x))\sum_j \underbrace{\partial_j\varphi_l(x)(\nabla\varphi(x))^{-T}_{kj}}_{\delta_{lk}}\det\nabla\varphi(x)\\ &= [\operatorname{div}^\varphi T^\varphi(x^\varphi)]_i\det\nabla\varphi(x).\end{aligned}$$

This relation means that

$$\operatorname{div}T(x) = \operatorname{div}^\varphi T^\varphi(\varphi(x))\det\nabla\varphi(x).$$

We now pass on to assertion (ii).
Let A be an arbitrary subdomain of $\overline{\Omega}$. Using the relation of the volume elements $dx^\varphi = \det\nabla\varphi(x)\,dx$, we get assertion (ii) from (1.16).

$$\begin{aligned}\int_{\partial A} T(x)n\,da &= \int_A \operatorname{div}T(x)\,dx = \int_A \operatorname{div}^\varphi T^\varphi(\varphi(x))\,\det\nabla\varphi(x)\,dx\\ &= \int_{\varphi(A)}\operatorname{div}^\varphi T^\varphi(x)^\varphi\,dx^\varphi = \int_{\partial\varphi(A)} T^\varphi(x^\varphi)n^\varphi\,da^\varphi.\end{aligned}$$

We come to assertion (iii).

We take $T(x) = \det \nabla\varphi(x)(\nabla\varphi(x))^{-T} = \mathrm{Cof}(\nabla\varphi)$. Then the inverse Piola transform yields

$$T^{\varphi}(x^{\varphi}) = P^{-1}T(x) = \frac{1}{\det \nabla\varphi(x)} \det \nabla\varphi(x)(\nabla\varphi(x))^{-T}(\nabla\varphi(x))^{T} = I.$$

Then relation (ii) reads

$$\int_{\partial A} \mathrm{Cof}(\nabla\varphi(x))n \, da = \int_{\partial\varphi(A)} n^{\varphi} \, da^{\varphi}.$$

Since $A \subset \overline{\Omega}$ is an arbitrary domain it follows that

$$\mathrm{Cof}(\nabla\varphi(x))n \, da = n^{\varphi} \, da^{\varphi}.$$

and

$$|\mathrm{Cof}(\nabla\varphi(x))n| \, da = da^{\varphi}.$$

1.2 Stresses and the equations of equilibrium

We will study the following situation:

A body occupying a deformed configuration $\overline{\Omega}^\varphi$ and subjected to applied body forces in its interior Ω^φ and to applied surface forces on a portion of its boundary is in static equilibrium when the fundamental stress principle of Euler and Cauchy is satisfied.

The forces are time–independent and can be:

- volume (body) forces.
  They are given through their space density $f^\varphi\colon \varphi(\Omega) \to \mathbb{R}^3$, that means the volume force F, which works in a subdomain $\varphi(A) \subset \varphi(\Omega)$ is given by
  $$F_{\varphi(A)} = \int_{\varphi(A)} f^\varphi(x^\varphi)\, dx^\varphi$$
- surface forces.
  They are given through their surface density $g^\varphi : \varphi(\Gamma_1) \to \mathbb{R}^3$, with $\Gamma_1 \subset \partial\Omega$, $\varphi(\Gamma_1) \subset \varphi(\partial\Omega)$. This means that the surface force $G_{\varphi(\gamma)}$, which works on a portion $\varphi(\gamma) \subset \varphi(\Gamma_1)$ is given by
  $$G_{\varphi(\gamma)} = \int_{\varphi(\gamma)} g^\varphi\, da^\varphi.$$

The applied forces describe the action of the outside world on the body. Body forces for example can be gravitational, electrostatic or thermal forces. Surface forces generally represent the action of another body along a portion Γ_1^φ.

The stress principle of Euler and Cauchy

Continuum mechanics for static problems is founded on the following stress principle, named after the fundamental contributions of Euler [1757, 1771] and Cauchy [1823, 1827].

Axiom (stress principle of Euler and Cauchy)

Consider a body occupying a deformed configuration $\overline{\Omega}^\varphi$, subjected to applied forces represented by densities $f^\varphi\colon \Omega^\varphi \to \mathbb{R}^3$, $g^\varphi\colon \Gamma_1^\varphi \to \mathbb{R}^3$.

Then there exists a vector field

$$t^\varphi\colon \overline{\Omega}^\varphi \times S^2 \to \mathbb{R}^3, \tag{1.22}$$

where S^2 is the unit sphere in $\mathbb{R}^3$ such that

(i) **Axiom of force balance**
For any subdomain $A^\varphi \subset \overline{\Omega}^\varphi$ and any point $x^\varphi \in \Gamma_1^\varphi \cap \partial A^\varphi$, where the unit outer normal vector n^φ to $\Gamma_1^\varphi \cap \partial A^\varphi$ exists,

$$t^\varphi(x^\varphi, n^\varphi) = g^\varphi(x^\varphi)$$

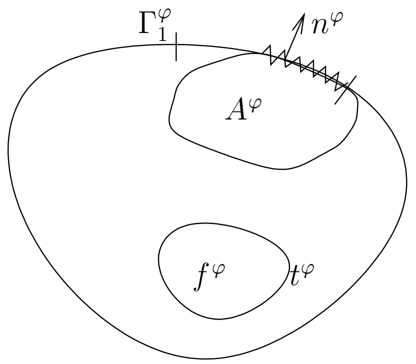
and

$$\int_{A^\varphi} f^\varphi(x^\varphi)\,dx^\varphi + \int_{\partial A^\varphi} t^\varphi(x^\varphi, n^\varphi)\,da^\varphi = 0.$$

(ii) **Axiom of moment balance**
For any subdomain $A^\varphi \subset \Omega^\varphi$

$$\int_{A^\varphi} x^\varphi \times f^\varphi(x^\varphi)\,dx^\varphi + \int_{\partial A^\varphi} x^\varphi \times t^\varphi(x^\varphi, n^\varphi)\,da^\varphi = 0.$$

The stress principle asserts the existence of a vector field that is defined on the boundaries of all subdomains $A^\varphi \subset \overline{\Omega}^\varphi$. Furthermore, it asserts that any subdomain A^φ of the deformed configuration $\overline{\Omega}^\varphi$, including $\overline{\Omega}^\varphi$ itself, is in static equilibrium. This means that its resulting vector vanishes (axiom of force balance) and that its resulting moment with respect to the origin vanishes (axiom of moment balance).



Definition: Cauchy stress vector

The vector $t^\varphi(x^\varphi, n^\varphi)$ is called the Cauchy stress vector across an oriented surface element with normal n^φ.

The Cauchy stress tensor:

We now derive important consequences from the stress principle. The first, which is due to Cauchy [1823, 1827], asserts that the dependence of the Cauchy stress vector $t^\varphi : \overline{\Omega}^\varphi \times S^2 \to \mathbb{R}^3$ is linear with respect to its second argument, i.e. at each point $x^\varphi \in \overline{\Omega}^\varphi$ there exists a tensor $T^\varphi(x^\varphi) \subset M^3$ such that $t^\varphi(x^\varphi, n) = T^\varphi(x^\varphi)n$ for all $n \in S^2$. The second consequence, again due to Cauchy [1827, 1828], is that the tensor field $T^\varphi\colon\ \Omega^\varphi \to M^3$ and the vector fields $f^\varphi\colon\ \Omega^\varphi \to \mathbb{R}^3$, $g^\varphi\colon\ \Gamma_1^\varphi \to \mathbb{R}^3$ are related by a partial differential equation in Ω^φ and by a boundary condition on Γ_1^φ respectively.

Theorem: (Cauchy's theorem)

Assume:

- The stress principle is valid.

- $\Omega \subset \mathbb{R}^3$ is a domain, $\varphi \colon \overline{\Omega} \to \mathbb{R}^3$ a C^1–deformation.
- $f^\varphi \colon \varphi(\overline{\Omega}) \to \mathbb{R}^3$ is continuous.
- $x^\varphi \to t^\varphi(x^\varphi, n)$ is continuously differentiable in $\overline{\Omega}^\varphi$ for each $n \in S^2$.
- $n \to t^\varphi(x, n)$ is continuous on S^2 for all $x^\varphi \in \overline{\Omega}^\varphi$.

Then there exists a continuously differentiable tensor field

$$T^\varphi \colon \varphi(\overline{\Omega}) \to \mathbb{R}^{3,3}$$

with

(a) $T^\varphi(x^\varphi)n = t^\varphi(x^\varphi, n) \quad$ for all $x^\varphi \in \overline{\Omega}^\varphi$and for all $n \in S^2$,

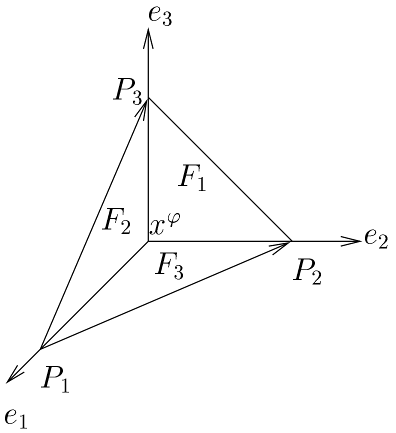
(b) $-\operatorname{div}^\varphi T^\varphi(x^\varphi) = f^\varphi(x^\varphi) \quad$ for all $x \in \Omega^\varphi$,

(c) $T^\varphi(x^\varphi) = T^\varphi(x^\varphi)^T \quad$ for all $X^\varphi \in \overline{\Omega}^\varphi$,

(d) $T^\varphi(x^\varphi)n^\varphi = g^\varphi(x^\varphi) \quad$ for all $x^\varphi \in \Gamma_1^\varphi$.

Proof:

(a) Let x^φ be a fixed point in Ω^φ. Since Ω^φ is open, there is a tetrahedron $H \in \Omega^\varphi$ with vertex x^φ and three faces $F_i, i = 1, 2, 3$, parallel to the coordinate planes and with a face F_0, whose normal vector is $n_0 = \sum_{i=1}^3 n_{i_0} e_i$, $n_{i_0} > 0$.



The normal vectors of F_i coincide with the vectors $-e_i$.

In the Lemma following this proof, we will show that

$$\operatorname{area} F_i = n_{i_0} \operatorname{area} F_0.$$

The axiom of force balance over the tetrahedron$H$ reads

$$\int\limits_H f^\varphi(y^\varphi)\, dy^\varphi + \int\limits_{\partial H} t^\varphi(y^\varphi, n^\varphi)\, da^\varphi = 0,$$

or, written in components

$$\begin{aligned}
&\int_H \begin{pmatrix} f_1^\varphi(y^\varphi) \\ f_2^\varphi(y^\varphi) \\ f_3^\varphi(y^\varphi) \end{pmatrix} dy^\varphi \\
&= -\left(\sum_{i=1}^{3} \int_{F_i} \begin{pmatrix} t_1^\varphi(y^\varphi, -e_i) \\ t_2^\varphi(y^\varphi, -e_i) \\ t_3^\varphi(y^\varphi, -e_i) \end{pmatrix} da^\varphi + \int_{F_0} \begin{pmatrix} t_1^\varphi(y^\varphi, n_0) \\ t_2^\varphi(y^\varphi, n_0) \\ t_3^\varphi(y^\varphi, n_0) \end{pmatrix} da^\varphi \right)
\end{aligned}$$

Because of the continuity of the integrants we can use the mean value theorem in every row: There are values $\eta_{ij}^\varphi \in F_i$, $i = 0, 1, 2, 3$, $j = 1, 2, 3$, such that

$$\begin{aligned}
\sup_{y^\varphi \in H} |f_j(y^\varphi)| \mathrm{vol}\, H &\geq |\sum_{i=1}^{3} t_j^\varphi(\eta_{ij}, -e_i) \mathrm{area}\, F_i + t_j^\varphi(\eta_{0j}, n_0) \mathrm{area}\, F_0| \\
&= \mathrm{area}\, F_0 |\sum_{i=1}^{3} t_j^\varphi(\eta_{ij}, -e_i) n_{i0} + t_j^\varphi(\eta_{0j}, n_0)|.
\end{aligned}$$

Keeping the vector n_0 fixed and the vertices v_i coalesce into the vertex x^φ, the continuity of t^φ implies that for $j = 1, 2, 3$

$$f_j(x^\varphi) \lim_{v_i \to x^\varphi} \frac{\mathrm{vol}\, H}{\mathrm{area}\, F_0} = 0 = |\sum_{i=1}^{3} t_j^\varphi(x^\varphi, -e_i) n_{i_0} + t_j^\varphi(x^\varphi, n_0)|.$$

(Also see the Lemma following this proof.)
Hence

$$-\sum_{i=1}^{3} t^\varphi(x^\varphi, -e_i) n_{i_0} = t^\varphi(x^\varphi, n_0).$$

Furthermore, the limit $n_0 \to e_j$, $n_{i_0} \to \delta_{ij}$ yields

$$-t^\varphi(x^\varphi, -e_j) = t^\varphi(x^\varphi, e_j) \quad \text{for } j = 1, 2, 3,$$

and therefore

$$-\sum_{i=1}^{3} t^\varphi(x^\varphi, -e_i) n_{i_0} = \boxed{\sum_{i=1}^{3} t^\varphi(x^\varphi, e_i) n_{i_0} = t^\varphi(x^\varphi, n_0).} \tag{1.23}$$

Passing on to a tetraheder mirrored on the axis, we get (1.23) for $n_{i-0} < 0$ as well, and therefore (1.24) folds for all $n \in S^2$.

$$\sum_{i=1}^{3} t^\varphi(x^\varphi, e_i) n_i = t^\varphi(x^\varphi, n). \tag{1.24}$$

We define T^φ by its components $(T^\varphi)_{ij}$.

$$(T^\varphi)_{ij}(x^\varphi) = t_i^\varphi(x^\varphi, e_j) \quad 1 \leq i, j \leq 3.$$

It follows from (1.24) that

$$t^\varphi(x^\varphi, n) = \sum_{j=1}^{3} t^\varphi(x^\varphi, e_j) n_j = T^\varphi(x^\varphi) n,$$

or, in components

$$t_i^\varphi(x^\varphi, n) = \sum_{j=1}^{3} T_{ij}^\varphi(x^\varphi) n_j.$$

(b) We apply Gauss' divergence theorem for tensor fields:
For an arbitrary subdomain $A^\varphi \subset \Omega^\varphi$, the axiom of force balance yields

$$\begin{aligned}
\int\limits_{A^\varphi} \operatorname{div} T^\varphi(x^\varphi)\, dx^\varphi &= \int\limits_{\partial A^\varphi} T^\varphi(x^\varphi) n^\varphi\, da^\varphi = \int\limits_{\partial A^\varphi} t^\varphi(x^\varphi, n^\varphi)\, da^\varphi \\
&= -\int\limits_{A^\varphi} f^\varphi(x^\varphi)\, dx^\varphi.
\end{aligned}$$

The assertion (b) follows immediately.

(c) We apply Gauss' formula to the surface integral in the axiom of moment balance:

$$\begin{aligned}
\int\limits_{\partial A^\varphi} x^\varphi \times t^\varphi(x^\varphi, n^\varphi)\, da^\varphi &= \int\limits_{\partial A^\varphi} \begin{vmatrix} i & j & k \\ x_1^\varphi & x_2^\varphi & x_3^\varphi \\ t_1^\varphi & t_2^\varphi & t_3^\varphi \end{vmatrix} da^\varphi \\
&= \int\limits_{\partial A^\varphi} \begin{pmatrix} x_2^\varphi t_3^\varphi - x_3^\varphi t_2^\varphi \\ x_3^\varphi t_1^\varphi - x_1^\varphi t_3^\varphi \\ x_1^\varphi t_2^\varphi - x_2^\varphi t_1^\varphi \end{pmatrix} da^\varphi = \int\limits_{\partial A^\varphi} \begin{pmatrix} x_2^\varphi \sum_j T_{3j}^\varphi n_j - x_3^\varphi \sum_j T_{2j}^\varphi n_j \\ x_3^\varphi \sum_j T_{1j}^\varphi n_j - x_1^\varphi \sum_j T_{3j}^\varphi n_j \\ x_1^\varphi \sum_j T_{2j}^\varphi n_j - x_2^\varphi \sum_j T_{1j}^\varphi n_j \end{pmatrix} da^\varphi \\
&= \int\limits_{A^\varphi} \begin{pmatrix} \sum_j \frac{\partial}{\partial x_j} \left[x_2^\varphi T_{3j}^\varphi - x_3^\varphi T_{2j}^\varphi\right] \\ \sum_j \frac{\partial}{\partial x_j} \left[x_3^\varphi T_{1j}^\varphi - x_1^\varphi T_{3j}^\varphi\right] \\ \sum_j \frac{\partial}{\partial x_j} \left[x_1^\varphi T_{2j}^\varphi - x_2^\varphi T_{1j}^\varphi\right] \end{pmatrix} dx^\varphi \\
&= \int\limits_{A^\varphi} \begin{pmatrix} \sum_j (\delta_{j2} T_{3j}^\varphi - \delta_{j3} T_{2j}^\varphi) + (x_2^\varphi \frac{\partial}{\partial x_j} T_{3j}^\varphi - x_3^\varphi \frac{\partial}{\partial x_j} T_{2j}^\varphi) \\ \sum_j (\delta_{j3} T_{1j}^\varphi - \delta_{j1} T_{3j}^\varphi) + (x_3^\varphi \frac{\partial}{\partial x_j} T_{1j}^\varphi - x_1^\varphi \frac{\partial}{\partial x_j} T_{3j}^\varphi) \\ \sum_j (\delta_{j1} T_{2j}^\varphi - \delta_{j2} T_{1j}^\varphi) + (x_1^\varphi \frac{\partial}{\partial x_j} T_{2j}^\varphi - x_2^\varphi \frac{\partial}{\partial x_j} T_{1j}^\varphi) \end{pmatrix} dx^\varphi \\
&= \int\limits_{A^\varphi} \begin{pmatrix} T_{32}^\varphi - T_{23}^\varphi \\ T_{13}^\varphi - T_{31}^\varphi \\ T_{21}^\varphi - T_{12}^\varphi \end{pmatrix} dx^\varphi - \int\limits_{A^\varphi} \begin{pmatrix} x_2^\varphi f_3^\varphi - x_3^\varphi f_2^\varphi \\ x_3^\varphi f_1^\varphi - x_1^\varphi f_3^\varphi \\ x_1^\varphi f_2^\varphi - x_2^\varphi f_1^\varphi \end{pmatrix} dx^\varphi.
\end{aligned}$$

It has been used that $\operatorname{div} T^\varphi(x^\varphi) = -f^\varphi(x^\varphi)$.
Since

$$\begin{aligned}
&\int\limits_{A^\varphi} x^\varphi \times f^\varphi(x^\varphi)\, dx^\varphi + \int\limits_{\partial A^\varphi} x^\varphi \times t^\varphi(x^\varphi, n^\varphi)\, da^\varphi = 0 \\
&\quad = \int\limits_{A^\varphi} x^\varphi \times f^\varphi(x^\varphi)\, dx^\varphi + \int\limits_{A^\varphi} \begin{pmatrix} T_{32}^\varphi - T_{23}^\varphi \\ T_{13}^\varphi - T_{31}^\varphi \\ T_{21}^\varphi - T_{12}^\varphi \end{pmatrix} dx^\varphi - \int\limits_{A^\varphi} x^\varphi \times f^\varphi(x^\varphi)\, dx^\varphi,
\end{aligned}$$

it follows that:

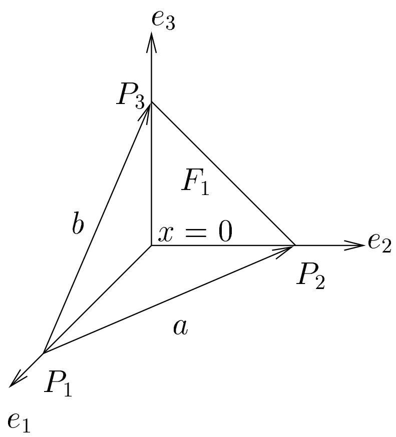
$$T^{\varphi}_{ik} = T^{\varphi}_{ki}.$$

(d) The boundary condition $T^{\varphi}(x^{\varphi})n^{\varphi} = g^{\varphi}(x^{\varphi})$ for all $x^{\varphi} \in \Gamma_1^{\varphi}$ is an immediate consequence of the definition of the Cauchy stress vector and of its relation to the tensor T^{φ}. ■

Lemma: (geometry of axis–parallel tetrahedron)
Let H be an axis–parallel tetrahedron with the four corner points

$$P_0 = x^{\varphi}, \quad P_i = x^{\varphi} + v_i, \quad v_i = c_i e_i, \quad c_i > 0,$$

and the faces F_i, in front of the points P_i, $1 \le i \le 3$.



Let n_0 be the normal on F_0.
Then

$$n_{0i}\, area\, F_0 = area\, F_i$$

and

$$\lim_{h \to 0} \frac{vol\, H}{area\, F_0} = 0, \quad where\ c_1 = c_2 = c_3 = h.$$

Proof:

The face F_0 is spanned through the vectors

$$\begin{aligned} a &= \vec{P_1P_2} = c_2 e_2 - c_1 e_1 \\ b &= \vec{P_1P_3} = c_3 e_3 - c_1 e_1 \end{aligned}$$

The normal is given by $n_0 = \frac{a \times b}{|a \times b|}$ and its components by $n_{0i} = \frac{a \times b}{|a \times b|} e_i$. It follows that

$$\begin{aligned} n_{01}|a \times b| &= (a \times b)e_1 = \begin{vmatrix} i & j & k \\ -c_1 & c_2 & 0 \\ -c_1 & 0 & c_3 \end{vmatrix} e_1 = c_2 c_3, \\ n_{02}|a \times b| &= c_1 c_3, \\ n_{03}|a \times b| &= c_1 c_2. \end{aligned}$$

Since area $F_0 = \frac{|a \times b|}{2}$, it results that n_{0i} area $F_0 = \frac{1}{2} c_j c_k =$ area F_i.
Furthermore, vol $H = \frac{\text{area}\, F_3 \cdot h}{2} = \frac{h^3}{4}$ and area $F_0 = \frac{h^2\sqrt{3}}{2}$. This leads to

$$\lim_{h \to 0} \frac{\operatorname{vol} H}{\operatorname{area} F_0} = 0. \quad \blacksquare$$

Definition: (Cauchy stress tensor)

The symmetric tensor $T^\varphi(x^\varphi)$ is called the Cauchy stress tensor at $x^\varphi \in \overline{\Omega}^\varphi$. For $n \in S^2$, we have $t^\varphi(x^\varphi, n) = T^\varphi(x^\varphi)n$.

Examples:

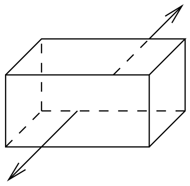
(i) Let

$$T^\varphi(x^\varphi) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \alpha e_1 \otimes e_1 = \alpha e_1 e_1^T.$$

For $\alpha > 0$ this is a pure tension in direction of the x_1–axis, for $\alpha < 0$ it is a pure compression.
The Cauchy stress vector is

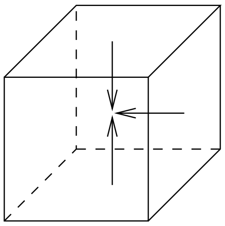
$$t^\varphi(x^\varphi, n) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} n_1 \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha(n \cdot e_1)e_1.$$

It vanishes on the faces, where $n_1 = 0$. (e.g. orthogonal to e_1).



(ii) Let

$$T^\varphi(x^\varphi) = -pI = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, p > 0.$$



The Cauchy stress tensor works uniformly from every direction. p is called the pressure.

(iii) Let

$$T^\varphi(x^\varphi) = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \alpha e_1 \otimes e_2 + \alpha e_2 \otimes e_1.$$

We then have $T^\varphi(x^\varphi)e_1 = \alpha e_2$, $T^\varphi(x^\varphi)e_2 = \alpha e_1$, $T^\varphi(x^\varphi)e_3 = 0$.
The Cauchy stress tensor is pure shear, with shear stress α relative to the directions e_1 and e_2.
The Cauchy stress vector is given by

$$t^\varphi(x^\varphi, n) = \alpha((e_1 \cdot n)e_2 + (e_2 \cdot n)e_1).$$

Moreover, we introduce the scalar normal stress and the shear–stress vector.

Definition:
Be $x^\varphi \in \varphi(\overline{\Omega})$, $T^\varphi(x^\varphi)$ denotes the Cauchy stress tensor in the point x^φ, $n \in S^2$. The real number

$$\boxed{T_N^\varphi = (T^\varphi n, n) = n^T T^\varphi n} \tag{1.25}$$

is the normal stress in the direction n, the vector

$$\boxed{T_S^\varphi = T^\varphi n - T_N^\varphi n} \tag{1.26}$$

is the shear stress or the tangential stress.

Remark:
The normal stress is the projection of the Cauchy stress tensor on n, whereas T_S^φ is the projection on the plane which is orthogonal to n.

If $x^\varphi \in \partial\Omega^\varphi$ and $n = n^\varphi$ is the external normal vector, then T_N^φ and T_S^φ are the normal and tangential components of the applied surface force. It holds

$$\|Tn\|_2^2 = T_N^2 + \|T_S\|_2^2.$$

The Cauchy stress tensor is symmetric. Therefore an orthonormal basis of eigenvectors exists in every point x^φ.

Definition: (principal stresses)
Let $T^\varphi(x^\varphi)$ be the Cauchy stress tensor at $x^\varphi \in \varphi(\overline{\Omega})$. The eigenvectors $\vec{n_1}$, $\vec{n_2}$, $\vec{n_3}$ of $T^\varphi(x^\varphi)$ are called principal stress directions, the corresponding eigenvalues $\vec{\tau_1}$, $\vec{\tau_2}$, $\vec{\tau_3}$ are called principal stresses.

The eigenvalues are the zeroes of $\det(T^\varphi(x^\varphi) - \tau I) = 0$; that means they are the zeroes of the characteristic polynomial

$$\tau^3 - I_{T^\varphi}\tau^2 + II_{T^\varphi}\tau - III_{T^\varphi} = 0,$$

where

$$\begin{aligned}
I_{T^\varphi} &= \operatorname{tr} I_\varphi = \tau_1 + \tau_2 + \tau_3, \\
II_{T^\varphi} &= \frac{1}{2}((\operatorname{tr} T_\varphi)^2 - \operatorname{tr} T_\varphi^2) = \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = \operatorname{tr}\operatorname{Cof} T^\varphi, \\
III_{T^\varphi} &= \det\, T^\varphi = \tau_1\tau_2\tau_3
\end{aligned}$$

are the principal invariants of the tensor T^φ. (They are invariant with respect to the choice of orthogonal coordinate systems.)

Remarks:

1° The Cauchy stress tensor T^φ can be splitted into the ball-tensor T_K^φ and the derivatoric tensor T_D^φ:

$$T^\varphi = T_K^\varphi + T_D^\varphi,$$

where

$$\begin{aligned}
T_K^\varphi &= \frac{1}{3}\operatorname{tr} T^\varphi I = \begin{pmatrix} \frac{1}{3}(\tau_1+\tau_2+\tau_3) & 0 & 0 \\ 0 & \frac{1}{3}(\tau_1+\tau_2+\tau_3) & 0 \\ 0 & 0 & \frac{1}{3}(\tau_1+\tau_2+\tau_3) \end{pmatrix} \\
\text{and} & \\
T_D^\varphi &= T^\varphi - T_K^\varphi.
\end{aligned}$$

2° The state of plane stress is typical for a membrane or a disc, and is characterised if one of the principal stress (say τ_3) equals zero. This means there is no stress in the direction $n = e_3$. Then

$$T^\varphi = \begin{pmatrix} T_{11}^\varphi & T_{12}^\varphi & 0 \\ T_{21}^\varphi & T_{22}^\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{21}^\varphi = T_{12}^\varphi.$$

The principal invariants are

$$\begin{aligned}
& T_{11}^\varphi + T_{22}^\varphi = \tau_1 + \tau_2 \\
& T_{11}^\varphi T_{22}^\varphi - (T_{12}^\varphi)^2 = \tau_1\tau_2 \\
\text{and} \quad & \tau_{1,2} = \frac{T_{11}^\varphi + T_{22}^\varphi}{2} \pm \sqrt{\frac{(T_{11}^\varphi - T_{22}^\varphi)^2}{4} + (T_{12}^\varphi)^2}.
\end{aligned}$$

3° Let $\tau_1 \geq \tau_2 \geq \tau_3$ be the principal stresses of T^φ. Then

$$\tau_1 = \max_{n\in S^2} n^T T_n^\varphi, \qquad \tau_3 = \min_{n\in S^2} n^T T_n^\varphi.$$

This means that the greatest and the smallest normal stresses are in the directions of the principal stresses.
$(T_{ni} = \tau_i n_i \Rightarrow n_i^T T n_i = \tau_i, i = 1, 2, 3)$

The equations of equilibrium and the principle of virtual work in the deformed configuration:

Let us first define the inner product of two matrices.

Definition:

Let $A, B \in \mathbb{R}^{n,n}$. The inner product of A and B is defined as

$$A : B = \sum_{i,j=1}^{n} a_{ij} b_{ij} = tr(A^T B) = tr(B^T A).$$

Lemma:

Let $\Omega \subset \mathbb{R}^n$ be a domain, $A\colon\ \Omega \to \mathbb{R}^{n,n}$, $b\colon\ \Omega \to \mathbb{R}^n$ continuously differentiable maps. Then

$$div[A(x)^T b(x)] = \langle div\, A(x), b(x)\rangle + A(x) : \nabla b(x) \quad \text{for all } x \in \Omega.$$

Proof:

$$\begin{aligned} \operatorname{div}[A(x)^T b(x)] &= \sum_{i,j} \partial_j a_{ij}(x) b_i(x) = \sum_{i,j} \{\partial_j a_{ij}(x) b_i(x) + a_{ij} \partial_j b_i(x)\} \\ &= \langle \operatorname{div} A(x), b(x)\rangle + A(x) : \nabla b(x). \ \blacksquare \end{aligned} \tag{1.27}$$

The Cauchy theorem says that the axiom of force leads to a boundary value problem in the deformed configuration:

$$-\operatorname{div} T^\varphi(x^\varphi) = f^\varphi(x^\varphi) \qquad \text{in } \Omega^\varphi \tag{1.28}$$

$$T^\varphi(x^\varphi) n^\varphi = g^\varphi(x^\varphi) \qquad \text{on } \Gamma_1^\varphi. \tag{1.29}$$

The partial differential equation (1.26) is given in "divergence form", which leads to a variational or weak formulation called **principle of virtual work** in mechanics.

Theorem:

The boundary value problem (1.28), (1.29)

$$\begin{aligned} -div^\varphi\, T^\varphi(x^\varphi) &= f^\varphi(x^\varphi) \qquad in\ \Omega^\varphi \\ T^\varphi(x^\varphi) n^\varphi &= g^\varphi(x^\varphi) \qquad on\ \Gamma_1^\varphi. \end{aligned}$$

is formally equivalent to the variational equations

$$\int_{\Omega^\varphi} T^\varphi : \nabla^\varphi v^\varphi \, dx^\varphi = \int_{\Omega^\varphi} f^\varphi \cdot v^\varphi \, dx^\varphi + \int_{\Gamma_1^\varphi} g^\varphi \cdot v^\varphi \, dx^\varphi \tag{1.30}$$

for all $v^\varphi\colon\ \Omega^\varphi \to \mathbb{R}^3$ from $V^\varphi = \{v \in C^\infty(\overline{\Omega}^\varphi) : v = 0 \text{ on } \partial\Omega^\varphi \backslash \overline{\Gamma_1}^\varphi\}$.

Proof:

(i) Assume T^φ to be a continuously differentiable solution of (1.28), (1.29). We multiply the partial differential equation (1.28) with an arbitrary element from V^φ (in the sense of the inner product in $L_2(\Omega^\varphi)$ and apply formula (1.18) for the integration by parts. It results that

$$\begin{aligned}
\int\limits_{\Omega^\varphi} -\mathrm{div}^\varphi T^\varphi \cdot v^\varphi \, dx^\varphi &= -\int\limits_{\Omega^\varphi} \sum_{i,j} \partial_j^\varphi T_{ij}^\varphi v_i^\varphi \, dx^\varphi \\
&= \int\limits_{\Omega^\varphi} \sum_{i,j} T_{ij}^\varphi \partial_j^\varphi v_i^\varphi \, dx^\varphi - \int\limits_{\partial\Omega^\varphi} \sum_{i,j} T_{ij}^\varphi v_i^\varphi n_j^\varphi \, da^\varphi \\
&= \int\limits_{\Omega^\varphi} T^\varphi : \nabla^\varphi v^\varphi \, dx^\varphi - \int\limits_{\Gamma_1^\varphi} (T^\varphi \cdot n^\varphi) v^\varphi \, da^\varphi \\
&= \int\limits_{\Omega^\varphi} f^\varphi v^\varphi \, dx.
\end{aligned}$$

The boundary condition (1.29) yields the variational equations (1.30):

$$\int\limits_{\Omega^\varphi} T^\varphi : \nabla^\varphi v^\varphi \, dx^\varphi = \int\limits_{\Omega^\varphi} f^\varphi v^\varphi \, dx^\varphi + \int\limits_{\Gamma_1^\varphi} g^\varphi v^\varphi \, da^\varphi .$$

(ii) Let T^φ be a solution of (1.30).
Then the above calculations imply that

$$\begin{aligned}
-\int\limits_{\Omega^\varphi} (\mathrm{div}^\varphi T^\varphi + f^\varphi) v^\varphi \, dx^\varphi &= 0 \qquad \forall v^\varphi \in V^\varphi \\
\text{and} \qquad \int\limits_{\Gamma_1^\varphi} (T^\varphi u^\varphi - g^\varphi) v^\varphi \, da^\varphi &= 0.
\end{aligned}$$

It follows from the variational lemma [12, p.90][13, p.72] that

$$\begin{aligned}
-\mathrm{div}^\varphi \, T^\varphi(x^\varphi) &= f^\varphi(x^\varphi) \qquad \text{in } \Omega^\varphi \\
T^\varphi(x^\varphi) n^\varphi &= g^\varphi(x^\varphi) \qquad \text{on } \Gamma_1^\varphi .
\end{aligned}$$

The equations of equilibrium and the principle of virtual work in the reference configuration:

Our goal is to formulate the Cauchy theorem in the reference configuration, that means to express the boundary value problem and its variational formulations in terms of the Lagrange variable x. In connection with that we remind of the Piola transform $P\colon \mathbb{R}^{3,3} \to \mathbb{R}^{3,3}$ defined by

$$\begin{aligned}
PT^\varphi(x^\varphi) &= T(x) = T^\varphi(x^\varphi)\mathrm{Cof}(\nabla\varphi) \\
&= T^\varphi(x^\varphi) \det(\nabla\varphi) [\nabla\varphi(x)]^{-T}
\end{aligned}$$

with the properties (1.18), (1.19), (1.20).

Definition:

$$T(x) = PT^\varphi(x^\varphi) = T^\varphi(x^\varphi)\,Cof(\nabla\varphi)$$

is called the first Piola–Kirchhoff stress tensor.

Theorem:

Assume that the stress principle is valid.
The first Piola–Kirchhoff stress tensor is a solution of the boundary value problem

$$\begin{aligned} -div\,T(x) &= f(x) = (\det\nabla\varphi(x))f^\varphi\circ\varphi(x) \qquad in\ \Omega && (1.31)\\ T(x)n(x) &= g(x) = (\det\nabla\varphi(x))\,|\nabla\varphi(x)^{-T}n|\,g^\varphi\circ\varphi(x) \quad on\ \Gamma_1 && (1.32)\\ and \\ \nabla\varphi(x)T(x)^T &= T(x)\nabla\varphi(x)^T. && (1.33)\end{aligned}$$

The boundary value problem (1.31), (1.32) is equivalent to the variational problem "Find a tensor $T\in M^3$ such that

$$\int_\Omega T(x) : \nabla v(x)\,dx = \int_\Omega f\cdot v\,dx + \int_{\Gamma_1} g\cdot v\,da \qquad \forall v\in V, \tag{1.34}$$

$V = \{u\in C^\infty(\Omega) : u = 0\ \ on\ \partial\Omega\backslash\Gamma_1\}$*."*

Proof:

Since

$$-\mathrm{div}\,T(x) = -\det\nabla\varphi\,\mathrm{div}^\varphi T^\varphi(x^\varphi) = \det\nabla\varphi f^\varphi(x^\varphi) = \det\nabla\varphi f^\varphi\circ\varphi(x)$$

(compare (1.18)), equation (1.31) is valid.
Furthermore, the relations (1.19) and (1.20) imply:

$$\begin{aligned} T(x)n(x)\,da &\overset{(1.19)}{=} T^\varphi(x^\varphi)\mathrm{Cof}\nabla\varphi n(x)\,da\\ &\overset{(1.19)}{=} T^\varphi(x^\varphi)n^\varphi(x^\varphi)\,da^\varphi\\ &\overset{(1.20)}{=} T^\varphi(x^\varphi)n^\varphi(x^\varphi)|\mathrm{Cof}\nabla\varphi(x)n(x)|\,da.\end{aligned}$$

Consequently

$$\begin{aligned} T(x)n(x) &= T^\varphi(x^\varphi)n^\varphi(x^\varphi)|\mathrm{Cof}\nabla\varphi(x)n(x)|\\ &= g^\varphi(x^\varphi)|\mathrm{Cof}\nabla\varphi(x)n(x)|\\ &= \det\nabla\varphi|\nabla\varphi(x)^{-T}n(x)|g^\varphi(\varphi(x)).\end{aligned}$$

The equivalence with the variational equations (1.34) is established as in the theorem for the deformed configuration (p.31).

Relation (1.33) says that the first Piola–Kirchhoff stress tensor

$$T(x)^T = (\nabla\varphi(x))^{-1} T(x) \nabla\varphi(x)^T$$

is non–symmetric in general. Indeed,

$$\begin{aligned} T(x)(\nabla\varphi(x))^T &= T^\varphi(x^\varphi)\det\nabla\varphi(x) \\ \nabla\varphi(x)T^T(x) &= (T^\varphi(x^\varphi))^T\det\nabla\varphi(x) = T^\varphi(x^\varphi)\det\nabla\varphi(x). \quad \blacksquare \end{aligned}$$

In order to avoid the non–symmetry of the first Piola–Kirchhoff stress tensor, one defines the second Piola–Kirchhoff stress tensor:

Definition:
The second Piola–Kirchhoff stress tensor is defined as

$$\Sigma(x) = \det\nabla\varphi\nabla\varphi(x)^{-1} T^\varphi(x^\varphi)\nabla\varphi(x)^{-T} = \nabla\varphi(x)^{-1} T(x).$$

The following theorem is evident:

Theorem:
Assume that the stress principle is valid. The second Piola–Kirchhoff stress tensor is symmetric and it is solution of the boundary value problem

$$\begin{aligned} -div\nabla\varphi(x)\Sigma(x) &= f(x) \qquad in\ \Omega && (1.35) \\ \nabla\varphi(x)\Sigma(x)n(x) &= g(x) \qquad on\ \Gamma_1, && (1.36) \end{aligned}$$

where the right hand sides f and g are defined by (1.31) and (1.32).

Remark:
In the equations (1.35) and (1.36), the unknowns Σ and φ appear. (Note that f and g also depend on φ.)
For simplification the notation *dead loads* is used. The density $f\colon \Omega \to \mathbb{R}^3$, or the density $g\colon \Gamma_1 \to \mathbb{R}^3$ is associated with a dead load, if it is considered independent of the particular deformation φ.
This is meaningful, if $\varphi(x)$ is close to the identity (small displacements).

1.3 Elastic materials and their constitutive equations

The three equations of equilibrium (1.35) together with the symmetry condition $\Sigma^T = \Sigma$ form an undetermined system for the nine unknowns (φ_1, φ_2, φ_3, Σ_{11}, Σ_{12}, Σ_{13}, Σ_{22}, Σ_{23}, Σ_{33}) in the reference configuration.
Because the equations of equilibrium are valid for each macroscopic continuum (gas, liquid, solid), the missing six equations should give informations about the constituting materials.

Elastic materials are characterised as follows:
At each point $x^\varphi = \varphi(x)$ of the deformed configuration the Cauchy stress tensor $T^\varphi(x^\varphi)$ is solely a function of $x \in \Omega$ and the deformation gradient $\nabla\varphi(x)$.
This relation is expressed with the help of the response function for the Cauchy stress.

Notation:

$$\begin{aligned}
\mathbb{R}^{3,3}_+ &= \{A : A \in \mathbb{R}^{3,3}, \det\ A > 0\} \\
S^{3,3} &= \{A : A \in \mathbb{R}^{3,3}, A \text{ is symmetric}\} \\
S^{3,3}_> &= \{A : A \in S^{3,3}, A \text{ is positive definite}\} \\
O^{3,3}_+ &= \{A : A \in \mathbb{R}^{3,3}, A \text{ is orthogonal}, \det A > 0\}.
\end{aligned}$$

Definition: (response function, constitutive equations)
A material occupying a domain $\Omega \subset \mathbb{R}^3$ is elastic, if there exists a mapping

$$R\colon\ \overline{\Omega} \times \mathbb{R}^{3,3}_+ \to S^{3,3},$$

such that in any deformed configuration (that means for every C^1–deformation $\varphi\colon\ \overline{\Omega} \to \mathbb{R}^3$) the Cauchy stress tensor can be expressed on the reference configuration:

$$T^\varphi(\varphi(x)) = R(x, \nabla\varphi(x)) \qquad \forall x \in \overline{\Omega}. \tag{1.37}$$

The mapping R is called response function *for the Cauchy stress tensor, the equation (1.37) is called the* constitutive equation *of the material.*

Remarks:

(i) Relation (1.37) means that elastic materials can only undergo a restricted class of deformations. Indeed, for any matrix $A \in \mathbb{R}^{3,3}_+$ exists a deformation φ such that $\nabla\varphi(x) = A$, (set $\varphi(x) = Ax$).

(ii) If the material is elastic, then it holds for the first and second Piola–Kirchhoff stress tensor

$$T(x) = \hat{T}(x, \nabla\varphi(x))$$

and

$$\begin{aligned}
\Sigma(x) &= (\nabla\varphi(x))^{-1} T(x) = (\nabla\varphi(x))^{-1}\hat{T}(x, \nabla\varphi(x)) \\
&= \hat{\Sigma}(x, \nabla\varphi(x)).
\end{aligned} \tag{1.38}$$

Indeed, it is

$$T^\varphi(x^\varphi) = R(x, \nabla(x)) = \frac{1}{\det \nabla\varphi} T(x)[\nabla\varphi(x)]^T$$

and therefore

$$T(x) = R(x, \nabla\varphi(x))\mathrm{Cof}\nabla\varphi(x) = \hat{T}(x, \nabla\varphi(x)). \qquad (1.39)$$

The relations

$$\begin{aligned} T(x) &= \hat{T}(x, \nabla\varphi(x)) \\ \Sigma(x) &= \hat{\Sigma}(x, \nabla\varphi(x)) \qquad \forall x \in \overline{\Omega} \end{aligned}$$

are also called constitutive equations (6 equations) for elastic materials; $\hat{T}$, $\hat{\Sigma}\colon \overline{\Omega} \to \mathbb{R}^{3,3}_+$ are the response functions for the first and second Piola–Kirchhoff stress tensor.

Definition:

A material in a reference configuration Ω is called homogeneous, if

$$T^\varphi(\varphi(x)) = R(\nabla\varphi(x)) \qquad \forall x \in \overline{\Omega};$$

otherwise the material is said to be nonhomogeneous.

It follows that for homogeneous materials

$$\begin{aligned} T(x) &= \hat{T}(\nabla\varphi(x)) \\ \Sigma(x) &= \hat{\Sigma}(\nabla\varphi(x)) \end{aligned}$$

The response function is essentially determined experimentally, realising some deformations $\nabla\varphi = A$.

Notice that the response function is *a priori* dependent on the particular orthogonal basis chosen and on the particular reference configuration considered.
These dependences must be studied carefully, using the axiom of material frame–indifference (also named axiom of objectivity) and the property of isotropy.

Material frame–indifference

Any observable quantity must be independent from the particular orthogonal basis in which it is computed. Our observable quantity here is the Cauchy stress vector. We keep the basis fixed and rotate the deformed configuration around the origin. $\psi = Q\varphi$ rotates the configuration $\overline{\Omega}^\varphi$ into the configuration $\overline{\Omega}^\psi$. Translations of the origin may be ignored since thay have no effect on the deformation gradient.
The axiom of objectivity says that the Cauchy stress vector is rotated by the same matrix Q as the configuration.

Axiom of objectivity for the stress vectors

Let $\Omega \subset \mathbb{R}^3$ be a domain, $\varphi, \psi\colon \overline{\Omega} \to \mathbb{R}^3$ C^1–deformations with $\psi = Q\varphi$, $Q \in O^{3,3}_+$. Then it holds for the Cauchy stress vectors that:

$$t^\psi(\psi(x), Qn) = Qt^\varphi(\varphi(x), n) \qquad \text{for } x \in \overline{\Omega}, n \in S^2. \qquad (1.40)$$

Corollary:

If the axiom of objectivity (frame–indifference) holds for the stress vectors, then the response function $R = R(x, A)$ for the stress tensor has the property

$$\begin{aligned} R(x, QA) &= QR(x, A)\,Q^T && (1.41) \\ \textit{and} & \\ \hat{T}(x, \nabla\psi(x)) &= \hat{T}(x, Q\nabla\varphi(x)) = Q\hat{T}(x, \nabla\varphi(x)) && (1.42) \\ \hat{\Sigma}(x, \nabla\psi(x)) &= \hat{\Sigma}(x, Q\nabla\varphi(x)) = \hat{\Sigma}(x, \nabla\varphi(x)). && (1.43) \end{aligned}$$

Indeed,

$$\begin{aligned} T^\psi(\psi(x))Qn &= t^\psi(\psi(x), Qn) = Qt^\varphi(\varphi(x), n) = QT^\varphi(\varphi(x))n \qquad \forall n \in S^2 \\ &\Leftrightarrow T^\psi(\psi(x))Q = QT^\varphi(\varphi(x)) \\ &\Leftrightarrow T^\psi(\psi(x)) = QT^\varphi(\varphi(x))Q^T \\ &\Leftrightarrow R(x, \nabla\psi(x)) = R(x, Q\nabla\varphi(x)) = QR(x, \nabla\varphi(x))Q^T. \qquad (1.44) \end{aligned}$$

Since for every $A \in \mathbb{R}^{3,3}_+$ a C^1–deformation φ with $\nabla\varphi = A$ exists (set $\varphi(x) = Ax$), it follows from (1.44) that

$$R(x, QA) = QR(x, A)Q^T.$$

Furthermore,

$$\begin{aligned} \hat{T}(x, \nabla\psi(x)) &= \hat{T}(x, Q\nabla\varphi(x)) = R(x, Q\nabla\varphi(x))\mathrm{Cof}(Q\nabla\varphi(x)) \\ &= QR(x, \nabla\varphi)Q^T \frac{1}{\det(Q\nabla\varphi)}(Q\nabla\varphi)^{-T} \\ &= QR(x, \nabla\varphi)\frac{1}{\det\nabla\varphi}Q^TQ^{-T}(\nabla\varphi)^{-T} \\ &= Q\hat{T}(x, \nabla\varphi) \\ \text{and} & \\ \hat{\Sigma}(x, \nabla\psi(x)) &= (Q\nabla\varphi(x))^{-1}\hat{T}(x, \nabla\psi(x)) = (\nabla\varphi)^{-1}\hat{T}(x, \nabla\varphi(x)) = \hat{\Sigma}(x, \nabla\varphi(x)). \end{aligned}$$

The above considerations imply: if (1.41) holds, then the frame–indifference axiom for the stress vectors is satisfied.

This leads to the following definition:

Definition: (objectivity fo the response function)

The response function R of an elastic material is objective, if

$$R(x, QA) = QR(x, A)Q^T \qquad \forall x \in \overline{\Omega}, \forall Q \in O^{3,3}_+, \forall A \in \mathbb{R}^{3,3}_+.$$

We summarise the results:

Theorem: (objectivity of an elastic material)
For an elastic material with the response functions R, $\hat{T}$ and $\hat{\Sigma}$ the following statements are equivalent:

(i) *R is objective.*

(ii) $\hat{T}(x, QA) = Q\hat{T}(x, A) \qquad \forall Q \in O_+^{3,3}, A \in \mathbb{R}_+^{3,3}, x \in \overline{\Omega}.$

(iii) $\hat{\Sigma}(x, QA) = \hat{\Sigma}(x, A) \qquad \forall Q \in O_+^{3,3}, A \in \mathbb{R}_+^{3,3}, x \in \overline{\Omega}$

(iv) *There exists a mapping* $\tilde{\Sigma} \colon \overline{\Omega} \times S_>^{3,3} \to S^{3,3}$ *with*

$$\hat{\Sigma}(x, A) = \tilde{\Sigma}(x, A^T A) \qquad \forall A \in \mathbb{R}_+^{3,3}, x \in \overline{\Omega}. \tag{1.45}$$

Proof: [5, Th.3.3–1, p.101]
We here consider (iii) $\to$ (iv).
Be $F, G \in \mathbb{R}_+^{3,3}$ with $F^T F = G^T G$. Then $Q = GF^{-1} \in O_+^{3,3}$ and $\hat{\Sigma}(x, F) = \hat{\Sigma}(x, QF) = \hat{\Sigma}(x, G)$ implies $\hat{\Sigma}(x, A) = \tilde{\Sigma}(x, A^T A)$.
Furthermore, (iv) $\to$ i:

$$\begin{aligned} R(x, QA) &= \hat{T}(x, QA)(\operatorname{Cof} QA)^{-1} = QA\hat{\Sigma}(x, QA)(\operatorname{Cof} QA)^{-1} \\ &= QA\tilde{\Sigma}(x, (QA)^T QA)\operatorname{Cof}(QA)^{-1} \end{aligned}$$

Since $\operatorname{Cof}(QA) = Q\operatorname{Cof} A$ (note that $\operatorname{Cof} Q = Q$, $\operatorname{Cof}(A \cdot B) = \operatorname{Cof} A \operatorname{Cof} B$), we get

$$\begin{aligned} R(x, QA) &= QA\tilde{\Sigma}(x, A^T A)(\operatorname{Cof} A)^{-1}Q^{-1} \\ &= QA\hat{\Sigma}(x, A)\operatorname{Cof} A^{-1}Q^{-1} \\ &= QAA^{-1}\hat{T}(x, A)\operatorname{Cof} A^{-1}Q^{-1} \\ &= QR(x, A)Q^{-1} \\ &= QR(x, A)Q^T. \end{aligned}$$

Assertion (iv) says, that the second Piola–Kirchhoff stress tensor

$$\Sigma(x) = \hat{\Sigma}(x, \nabla\varphi(x) = \tilde{\Sigma}(x, \nabla\varphi^T(x)\nabla\varphi(x))$$

depends on the right Cauchy–Green strain tensor $C = \nabla\varphi^T\nabla\varphi$ (strain–stress relation).

Isotropic elastic materials

We have studied how the axiom of the material frame–indifference restricts the form of the response function. Another property, the isotropy, yields further restrictions. The intuitive idea is, that at a given point, the behaviour of the material “is the same in all directions”. In contrast to this property, anisotropy means, that the response of the material depends on the direction.

We now introduce the mathematical definition of isotropy:

Definition: (isotropic elastic material)

An elastic material is isotropic at a point $x \in \Omega$, if its response function for the stress satisfies

$$R(x, FQ) = R(x, F) \qquad \forall Q \in O_+^{3,3}, F \in R_+^{3,3}. \tag{1.46}$$

The definition says: "The response function is independent of the rotation of the reference domain around point x."

Corollary:

The response functions of the first and the second Piola–Kirchhoff stress tensor for isotropic elastic materials have the properties

$$\begin{aligned} \hat{T}(x, FQ) &= \hat{T}(x, F)Q & (1.47) \\ \hat{\Sigma}(x, FQ) &= Q^T \hat{\Sigma}(x, F)Q \qquad \forall Q \in O_+^{3,3}, F \in \mathbb{R}_+^{3,3}. & (1.48) \end{aligned}$$

Indeed,

$$\begin{aligned} \hat{T}(x, \nabla\varphi Q) &\overset{(1.39)}{=} R(x, \nabla\varphi Q)\mathrm{Cof}\nabla\varphi Q \\ &\overset{(1.46)}{=} R(x, \nabla\varphi)\mathrm{Cof}\nabla\varphi Q \\ &\overset{(1.39)}{=} \hat{T}(x, \nabla\varphi)Q. \end{aligned}$$

Setting $\nabla\varphi = F$, we get (1.47).

$$\hat{\Sigma}(x, \nabla\varphi Q) = (\nabla\varphi Q)^{-1}\hat{T}(x, \nabla\varphi Q) = Q^{-1}\hat{\Sigma}(x, \nabla\varphi)Q = Q^T\hat{\Sigma}(x, \nabla\varphi)Q.$$

For $\nabla\varphi = F$, (1.48) follows.

Theorem: (Rivlin–Ericksen representation theorem, 1955)

A response function of an elastic material is objective and isotropic if and only if

$$R(x, F) = \tilde{R}(x, FF^T) = \sum_{i=0}^{2} \beta_i(x, I_S, II_S, III_S)S^i \tag{1.49}$$

for all $F \in R_+^{3,3}$.
Here, $S = FF^T$, $I_S = tr\, S$, $II_S = tr\, Cof S$, $III_S = \det S$.
In particular, $S = \nabla\varphi\nabla\varphi^T$ is the left Cauchy–Green strain tensor.

Proof: [5, p.109ff]

From the Rivlin–Ericksen representation theorem we get a stress–strain relation for an elastic isotropic material:

Theorem: (stress-strain relation)

Let $\Omega \subset \mathbb{R}^3$ be a domain occupied by an isotropic elastic material. Assume that the axiom of objectivity is valid. Then there is a relation between the second Piola–Kirchhoff stress tensor Σ and the right Cauchy–Green strain tensor $C = (\nabla\varphi)^T\nabla\varphi$.

$$\Sigma(x) = \sum_{i=0}^{2} \gamma_i(x, I_C, II_C, III_C)[C(x)]^i. \tag{1.50}$$

Here, I_C, II_C, III_C are the invariants of the matrix $C(x)$, $\gamma_i\colon\ \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$.

Proof:

$$\Sigma(x) = \hat{\Sigma}(x, \nabla\varphi) = \tilde{\Sigma}(x, \nabla\varphi^T\nabla\varphi) = \tilde{\Sigma}(x, C) = \hat{\Sigma}(x, \sqrt{C})$$

since $C \in S^{3,3}_{>}$, $\sqrt{C}$ is symmetric and positive definite (compare Lemma p.13).

$$\begin{aligned}
\hat{\Sigma}(x,\sqrt{C}) &= (\sqrt{C})^{-1}R(x,\sqrt{C})\mathrm{Cof}\sqrt{C} \\
&= \det\sqrt{C}(\sqrt{C})^{-1}\tilde{R}(x,C)(\sqrt{C})^{-1} \\
&= \sqrt{III_C}(\sqrt{C})^{-1}\left[\sum_{i=0}^{2}\beta_i(x,I_C,II_C,III_C)C^i\right](\sqrt{C})^{-1} \\
\text{It is } (\sqrt{C})^{-1}(\sqrt{C})^{-1} &= C^{-1} \\
(\sqrt{C})^{-1}C(\sqrt{C})^{-1} &= I \\
(\sqrt{C})^{-1}C^2(\sqrt{C})^{-1} &= (\sqrt{C})^{-1}\sqrt{C}\sqrt{C}\sqrt{C}\sqrt{C}(\sqrt{C})^{-1} = C. \\
\text{Hence } \Sigma(x) &= \sqrt{III_C}C^{-1}\left[\sum_{i=0}^{2}\beta_i(x,I_C,II_C,III_C)C^i\right] \\
&= \sqrt{III_C}(\beta_0 C^{-1} + \beta_1 I + \beta_2 C) \qquad (1.51)
\end{aligned}$$

Furthermore, the Hamilton–Cayley theorem yields

$$C^{-1} = \frac{1}{III_C}(II_C I - I_C C + C^2). \tag{1.52}$$

Inserting (1.52) into (1.51), we get the representation (1.50) with

$$\begin{aligned}
\gamma_0 &= (III_C)^{\frac{1}{2}}(\beta_0\frac{II_C}{III_C} + \beta_1), \\
\gamma_1 &= (III_C)^{\frac{1}{2}}(-\beta_0\frac{I_C}{III_C} + \beta_2), \\
\gamma_2 &= (III_C)^{\frac{1}{2}}\beta_o\frac{1}{III_C}. \quad \blacksquare
\end{aligned}$$

The constitutive equations near the reference configuration

If $\varphi = \mathrm{id}$ and $\nabla\varphi = I$, then we describe the stresses in the reference configuration

$$\begin{aligned}\Sigma(x) &= T^{\varphi}(x^{\varphi}) = T^{\varphi}(x) = T(x)\\ &= \sum_{i=0}^{2} \gamma_i(x,3,3,1)I.\end{aligned}$$

This means that the residual stress $\Sigma(x) = T_R(x)$ is a pressure. A reference configuration $\overline{\Omega}$ is called a natural state, if the residual stress tensor vanishes at all points $x \in \overline{\Omega}$ (unstressed state).

We now study the situation that we are close to the reference configuration. It is

$$\Sigma(x) = \tilde{\Sigma}(x, C) = \tilde{\Sigma}(x, I + 2E),$$

where $E = \frac{1}{2}(C - I)$ is the Green strain tensor, $C = \nabla\varphi^T\nabla\varphi$.
We consider the mapping

$$f(E) = \tilde{\Sigma}(x, I + 2E) = \sum_{i=0}^{2} \gamma_i(x, I_C, II_C, III_C)(I + 2E)^i \tag{1.53}$$

and take the Taylor expansion (Frechet derivative) for small E and fixed x.

Theorem:

Let an elastic material with a objective and isotropic response function at a point $x \in \overline{\Omega}$ be given. Assume that the coefficients γ_i of the representation (1.53) are continuously differentiable. Then functions $p, \lambda, \mu\colon \overline{\Omega} \to \mathbb{R}$ exist, such that

$$\Sigma(x) = \tilde{\Sigma}(x, C) = -p(x)I + \lambda(x)(tr\, E(x))I + 2\mu(x)E(x) + o(E, x). \tag{1.54}$$

Proof:

It is

$$\begin{aligned}f(E) &= \gamma_0(x, I_{I+2E(x)}, II_{I+2E(x)}, III_{I+2E(x)})I\\ &+ \gamma_1(x, I_{I+2E(x)}, II_{I+2E(x)}, III_{I+2E(x)})(I + 2E(x))\\ &+ \gamma_2(x, I_{I+2E(x)}, II_{I+2E(x)}, III_{I+2E(x)})(I + 2E(x))^2.\end{aligned}$$

Since the coefficients γ_i are continuously differentiable, we expand them at the element $E = 0$ to

$$\begin{aligned}\gamma_i(x, I_{I+2E(x)}, II_{I+2E(x)}, III_{I+2E(x)}) &= \gamma_i(x, I_I, II_I, III_I)\\ &+ \frac{\partial\gamma_i}{\partial I}(I_I, II_I, III_I)(I_{I+2E} - I_I)\\ &+ \frac{\partial\gamma_i}{\partial II}(I_I, II_I, III_I)(II_{I+2E} - II_I)\\ &+ \frac{\partial\gamma_i}{\partial III}(I_I, II_I, III_I)(III_{I+2E} - III_I)\\ &+ o(E).\end{aligned}$$

It is

$$\begin{aligned}
I_{I+2E} &= \mathrm{tr}(I+2E) = 3 + 2\mathrm{tr}\, E \\
II_{I+2E} &= \frac{1}{2}\left\{[\mathrm{tr}(I+2E)]^2 - \mathrm{tr}(I+2E)^2\right\} = 3 + 4\mathrm{tr}\, E + o(E) \\
III_{I+2E} &= \frac{1}{6}\left\{[\mathrm{tr}(I+2E)]^3 - 3\mathrm{tr}(I+2E)\mathrm{tr}(I+2E)^2 + 2\mathrm{tr}(I+2E)^3\right\} \\
&= 1 + 2\mathrm{tr}\, E + o(E).
\end{aligned}$$

Hence

$$\gamma_i(x, I_{I+2E}, II_{I+2E}, III_{I+2E}) = \gamma_i(x,3,3,1) + \beta_i(x)\mathrm{tr}\, E + o(E), \tag{1.55}$$

where $\beta_i(x) = \left(2\frac{\partial\gamma_i}{\partial I} + 4\frac{\partial\gamma_i}{\partial II} + 2\frac{\partial\gamma_i}{\partial III}\right)(x,3,3,1)$.
Inserting (1.55) into (1.53), we get

$$\begin{aligned}
\Sigma(x) = \tilde{\Sigma}(x, I+2E) &= \gamma_0 I + \beta_0 \mathrm{tr}\, EI \\
&+ \gamma_1 I + \beta_1 \mathrm{tr}\, EI + \gamma_1 2E + \beta_1 \mathrm{tr}\, E2E \\
&+ \gamma_2 I + \beta_2 \mathrm{tr}\, EI + \gamma_2 4E + \beta_2 \mathrm{tr}\, E4E + o(E) \\
&= (\gamma_0+\gamma_1+\gamma_2)(x,3,3,1)I + (\beta_0+\beta_1+\beta_2)(x,3,3,1)\mathrm{tr}\, EI \\
&+ (-\gamma_1 + 2\gamma_2)2E + o(E).
\end{aligned}$$

With

$$\begin{aligned}
(\gamma_0+\gamma_1+\gamma_2)(x,3,3,1) &= -p(x) \\
(\beta_0+\beta_1+\beta_2)(x,3,3,1) &= \lambda(x) \\
(\gamma_1+2\gamma_2)(x,3,3,1) &= \mu(x)
\end{aligned}$$

assertion (1.54) follows. ■

Corollary:
If the reference configuration is in natural state, then $-p(x) = 0$ and $\Sigma(x) = \tilde{\Sigma}(x, C) = \lambda(x) tr\, E(x) I + 2\mu(x)E(x) + o(E(x))$.
*If the material in addition is homogeneous, then $\lambda(x) = \lambda$ and $\mu(x) = \mu$ are constants. In this case they are called **Lamé constants**.*

Experimental information about the Lamé constants

The strain–stress relation

$$\Sigma(x) = \lambda \mathrm{tr}\, E(x) I + 2\mu E(x) + o(E(x)) \tag{1.56}$$

for a homogeneous isotropic elastic material is based on the knowledge of the Lamé constants λ and μ.
They have to be determined experimentally. Here, we simply impose restrictions on the admissible value of the Lamé constants of any "real" elastic homogeneous isotropic material, considering "ideal" experiments.

We will proceed as follows:

- The reference configuration $\overline{\Omega}$ has a simple geometric form (a ball, a rectangular block, a circular cylinder) and it is a natural state.
- The family of deformations has a particularly simple form, related to pressure, simple shear, pure axial tension.

Let us describe the admissible family of C^1–deformations and the resulting strain–stress relations in the undeformed and the deformed configurations.

Lemma:

Let Ω be a reference configuration in a natural state, occupied by a homogeneous, isotropic elastic material, φ^ϵ a family of C^1–deformations

$$\varphi^\epsilon(x) = x + \epsilon G x + r(\epsilon, x), \qquad G \in \mathbb{R}^{3,3}_+,$$

with $r(\epsilon,x) = o(\epsilon,x)$, $\nabla_x r(\epsilon,x) = o(\epsilon,x)$, $\nabla_x^2 r(\epsilon,x) = o(\epsilon,x)$.

$$\begin{aligned} \textit{Then}\ \ \Sigma^\epsilon(x) &= \epsilon[\lambda(tr\, G)I + \mu(G+G^T)] + o(\epsilon,x) \\ \textit{and}\ \ T^{\varphi_\epsilon}(\varphi^\epsilon(x)) &= \epsilon[\lambda(tr\, G)I + \mu(G+G^T)] + o(\epsilon,x). \end{aligned}$$

Proof:

It is

$$\begin{aligned} \nabla\varphi^\epsilon(x) &= I + \epsilon G + o(\epsilon,x) \\ \text{and}\ \ C^\epsilon(x) &= \nabla\varphi^{\epsilon T}\nabla\varphi^\epsilon \\ &= [I + \epsilon G^T + o(\epsilon,x)][I + \epsilon G + o(\epsilon,x)] \\ &= I + \epsilon(G^T + G) + o(\epsilon,x), \\ E^\epsilon(x) &= \frac{1}{2}(C^\epsilon - I) = \frac{\epsilon}{2}(G^T+G) + o(\epsilon,x) \\ \text{and finally}\ \ \Sigma^\epsilon(x) &= \lambda\epsilon \mathrm{tr}\, GI + 2\mu\frac{\epsilon}{2}(G^T+G) + o(\epsilon,x). \end{aligned}$$

Since $T^\varphi(\varphi(x)) = T(x)(\mathrm{Cof}\nabla\varphi)^{-1} = \nabla\varphi(x)\Sigma(x)(\mathrm{Cof}\nabla\varphi)^{-1}$, we get:

$$\begin{aligned} T^{\varphi_\epsilon}(\varphi_\epsilon(x) &= \nabla\varphi_\epsilon(x)\Sigma^\epsilon(x)\frac{1}{\det\nabla\varphi_\epsilon(x)}(\nabla\varphi_\epsilon(x))^T \\ &= [I + \epsilon G + o(\epsilon,x)][\epsilon(\lambda \mathrm{tr}\, GI + \mu(G^T+G)) + o(\epsilon,x)]\cdot \\ &\quad \left[\frac{1}{\det(I + \epsilon G + o(\epsilon,x))}\right][I + \epsilon G^T + o(\epsilon,x)] \\ &= \frac{1}{1+o(\epsilon,x)}[\epsilon(\lambda \mathrm{tr}\, GI + \mu(G^T+G)) + o(\epsilon,x)] \\ &= \epsilon[\lambda \mathrm{tr}\, GI + \mu(G^T+G)] + o(\epsilon,x). \ \ \blacksquare \end{aligned}$$

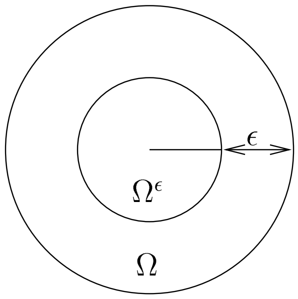
1. experiment:

Let $\Omega = B_1(0)$ be the unit ball. A uniform compression is described by

$$\varphi^\epsilon(x) = x - \epsilon x + o(\epsilon) \qquad \epsilon > 0.$$

Note the $r^\epsilon = (1-\epsilon)R + o(\epsilon)$.
Here, $G = -I$ and $T^{\varphi_\epsilon}(\varphi_\epsilon(x)) = \epsilon[-3\lambda - 2\mu]I + o(\epsilon) = -\epsilon\pi I + o(\epsilon)$, with $\pi = 3\lambda + 2\mu > 0$ and

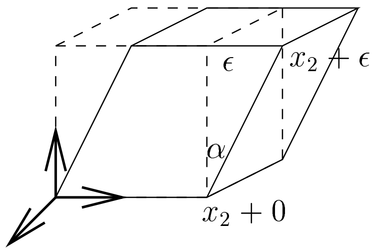
$$\boxed{\frac{\text{pressure}}{\epsilon} = 3\lambda + 2\mu}, \left[\frac{N}{m^2}\right].$$



$\kappa = \frac{1}{3}(3\lambda + 2\mu)$ is called the bulk module (or the modulus of compression).

2. experiment:

Let $\Omega = (0,-1) \times (0,1)^2$ be a cube. Then $\varphi_\epsilon(x) = x + \epsilon \begin{pmatrix} 0 \\ x_3 \\ 0 \end{pmatrix} + o(\epsilon, x)$ describes a simple shear.



It is $G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $T^{\varphi_\epsilon}(\varphi_\epsilon(x)) = \epsilon\mu \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + o(\epsilon, x)$.
It follows, that

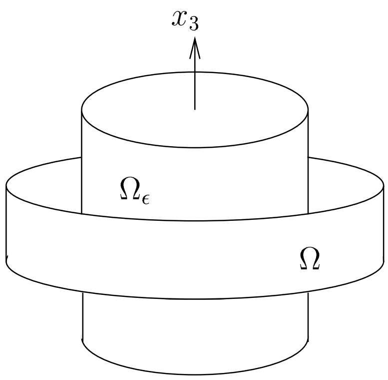
$$T_{23}^{\varphi_\epsilon} = T_{32}^{\varphi_\epsilon} = \mu\epsilon, \qquad \mu > 0 \quad \text{or } \mu = \frac{T_{23}^{\varphi_\epsilon}}{\epsilon}.$$

The number μ is called shear module

$$\mu = \frac{\text{shear stress}}{\tan\alpha}, \qquad \tan\alpha = \frac{\epsilon}{1} = \epsilon.$$

3. experiment:

Let Ω be a circular cylinder with the x_3–axis.



"uniform traction" of the cylinder

$\varphi_\epsilon(x) = x + \epsilon \begin{pmatrix} -\nu x_1 \\ -\nu x_2 \\ x_3 \end{pmatrix} + o(\epsilon, x)$, $\nu \in \mathbb{R}$,
where $\nu > 0$ has to be determined. We get

$$G = \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} T^{\varphi_\epsilon}(\varphi^\epsilon(x)) &= \epsilon \left[\lambda(-2\nu+1)I + \mu \begin{pmatrix} -2\nu & 0 & 0 \\ 0 & -2\nu & 0 \\ 0 & 0 & 2 \end{pmatrix} \right] + o(\epsilon, x) \\ &= \epsilon T + o(\epsilon, x), \end{aligned}$$

with $T = \lambda(1-2\nu)I + \mu \begin{pmatrix} -2\nu & 0 & 0 \\ 0 & -2\nu & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Assume $T_{11} = T_{22} = 0$, then $\lambda(1-2\nu) - 2\mu\nu = 0$ and $\nu = \frac{\lambda}{2(\lambda+\mu)}$.
Since $\mu > 0$ and $(3\lambda+2\mu) > 0$, and therefore $(\lambda+\mu) > 0$, the definition of ν is meaningful.
The natural assumption $\nu > 0$ leads to $\lambda > 0$.
The dimensionless number ν is called the Poisson ration and

$$\nu = \frac{\frac{d-d^\epsilon}{d}}{\frac{h^\epsilon - h}{h}},$$

where d and d^ϵ are the diameters, and h and h^ϵ are the length of Ω and Ω^ϵ, respectively.

For $\lambda > 0$, $\mu > 0$ it results that $0 < \nu < \frac{1}{2}$.

The component $T_{33} = \lambda(1-2\nu) + 2\mu = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = E$ is the Young modulus, $E = \frac{T_{33}^{\varphi_\epsilon}}{\epsilon}$, $\epsilon = \frac{h^\epsilon - h}{h}$.

The Lamé constants, the Poisson ration and the Young modulus are related to each other by the following equations:

$$\boxed{\begin{array}{ll} \nu = \dfrac{\lambda}{2(\lambda+\mu)}, & E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu} \\[2ex] \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}, & \mu = \dfrac{E}{2(1+\nu)} \end{array}}$$

Lamé constants of a homogeneous isotropic material

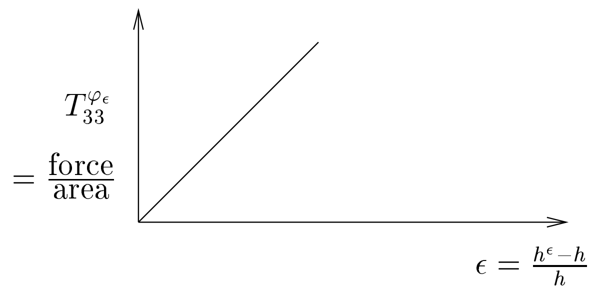
	E	ν	λ	μ	$\frac{1}{3}(3\lambda+2\mu)$
	(10^{10} N/m^2)		(10^{10} N/m^2)	(10^{10} N/m^2)	(10^{10} N/m^2)
Steel	21	0.28	10	8.2	16
Iron	20	0.28	9.9	7.8	15
Copper	11	0.34	8.7	4.1	11
Bronze	10	0.31	6.2	3.8	8.8
Aluminium	7.0	0.34	5.6	2.6	7.3
Glass	5.5	0.25	2.2	2.2	3.7
Nickel	2.2	0.30	1.3	0.85	1.8
Lead	1.8	0.44	4.6	0.63	5.0
Rubber	0.037	0.485	0.40	0.012	0.41

$1\text{GPa} = 10^9\text{N/m}^2$

Remarks:

- From the first experiment it follows, that the radius of the ball is not changing ($\epsilon \approx 0$) under any given pressure, if $3\lambda + 2\mu$ is very large ($\lambda \to \infty$). In this case the material is incompressible.
  The incompressibility can also be expressed with the help of the Poisson ratio:
  $$\lim_{\lambda\to\infty+} \frac{\lambda}{2(\lambda+\mu)} = \lim_{\lambda\to\infty+} \nu(\lambda,\mu) = \frac{1}{2}.$$
  Materials with $\lambda \gg \mu$ (e.g. rubber, see table above) are nearly incompressible.

- The third experiment leads to a uniaxial stress–strain curve
  $$T_{33}^{\varphi_\epsilon} = \epsilon E + o(\epsilon, x), \qquad \text{where } \epsilon = \frac{h^\epsilon - h}{h}.$$



  E is the ascent of the curve in the origin and it is also called elasticity modulus.

St. Venant–Kirchhoff materials and the linearised problem

If we neglect the higher–order terms in the expansion (1.56) of the second Piola–Kirchhoff stress tensor, we obtain a response function, suggested by St. Venant (1844) and Kirchhoff (1852).

Definition: (St. Venant–Kirchhoff materials)
An elastic material is a St. Venant–Kirchhoff material, if its response function is of the form

$$\Sigma(x) = \hat{\Sigma}(x, \nabla\varphi) = \tilde{\Sigma}(x, C) = \tilde{\Sigma}(x, 2E + I) = \lambda tr\, E(\lambda) I + 2\mu E(x), \tag{1.57}$$

where λ and μ are constants.

Clearly, such a material is homogeneous and the reference configuration is a natural state. E can be expressed in terms of the displacement fields u:

$$E = E(u) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u) \tag{1.58}$$

We now formulate boundary value problems for the unknowns Σ and φ, and for the unknown displacement field u, respectively, using the equilibrium equations (1.35), (1.36) and the stress–strain relation (1.57).

(Nonlinear) boundary value problems

(i) Let $\Omega \subset \mathbb{R}^3$ be a domain with the boundary $\partial\Omega$. $\partial\Omega$ is divided into two relatively open portions Γ_0 and Γ_1, with $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$.
The C^1–deformation $\varphi : \overline{\Omega} \to \mathbb{R}^3$ and the symmetric stress tensor $\Sigma : \overline{\Omega} \to \mathbb{R}^{3,3}$ have to be determined, such that

$$\begin{aligned} -\mathrm{div}(\nabla\varphi(x))\Sigma(x) &= f(x) \quad (= \hat{f}(x, \varphi(x))) \quad \text{in } \Omega \\ \varphi(x) &= \varphi_0(x) \quad \text{for } x \in \Gamma_0 \\ \nabla\varphi(x)\Sigma(x)n(x) &= g(x) \quad (= \hat{g}(x, \varphi(x))) \quad \text{for } x \in \Gamma_1 \end{aligned}$$

for given f, φ_0, g.

(ii) Let Ω, Γ_0, Γ_1 be defined as above, and let Ω be occupied by an St. Venant-Kirchhoff material.
A displacement field has to be found, such that

$$-\mathrm{div}\,\{(I + \nabla u(x))[\lambda \mathrm{tr}\, E(u)(x) + 2\mu E(u)(x)]\} = f(x) \tag{1.59}$$

$$u(x) = 0 \qquad \text{for } x \in \Gamma_0 \tag{1.60}$$

$$(I + \nabla u(x))[\lambda \mathrm{tr}\, E(u)(x) + 2\mu E(u)(x)]n(x) = g(x) \qquad \text{for } x \in \Gamma_1 \tag{1.61}$$

for given f and g, and $\varphi_0 = \mathrm{id}$.

We linearise the boundary value problem (1.59), (1.60) and (1.61), introducing

$$e(u) = \frac{1}{2}(\nabla u^T + \nabla u) \tag{1.62}$$

$$\sigma(e) = \lambda \mathrm{tr}\, eI + 2\mu e = \sigma(e(u)) \tag{1.63}$$

The tensors e and σ are not physical quantities, but approximations of the strain and the stress tensors.

Definition: (Hooke's law)
The relation

$$\sigma(e) = \lambda(tr\, e)I + 2\mu e$$

is called Hooke's law for isotropic, elastic, homogeneous materials.

The **linearised boundary value problem** (1.59), (1.60), (1.61) reads:

$$\begin{aligned} -\mathrm{div}(\lambda \mathrm{tr}\, e(u)I + 2\mu e(u)) &= f && \text{in } \Omega && (1.64)\\ u &= 0 && \text{on } \Gamma_0 && (1.65)\\ \sigma(e)n &= g && \text{on } \Gamma_1 && (1.66)\end{aligned}$$

or, written for the displacement field u:

$$\begin{aligned} -\left[\mu\Delta u + (\lambda+\mu)\mathrm{grad}(\mathrm{div}\, u)\right] &= f(x) && \text{in } \Omega && (1.67)\\ u\Big|_{\Gamma_0} &= 0 && && (1.68)\\ \sigma[u]n\Big|_{\Gamma_1} &= g && && (1.69)\end{aligned}$$

Indeed,

$$\begin{aligned} -\mathrm{div}\,\sigma(u) &= -\mathrm{div}\begin{pmatrix} \lambda \mathrm{div}\, u + 2\mu\partial_1 u_1 & \mu(\partial_1 u_2 + \partial_2 u_1) & \mu(\partial_1 u_3 + \partial_3 u_1)\\ \mu(\partial_1 u_2 + \partial_2 u_1) & \lambda \mathrm{div}\, u + 2\mu\partial_2 u_2 & \mu(\partial_2 u_3 + \partial_3 u_2)\\ \mu(\partial_1 u_3 + \partial_3 u_1) & \mu(\partial_2 u_3 + \partial_3 u_2) & \lambda \mathrm{div}\, u + 2\mu\partial_3 u_3 \end{pmatrix}\\ &= -\begin{pmatrix} \mu\Delta u_1 + (\lambda+\mu)\partial_1 \mathrm{div}\, u\\ \mu\Delta u_2 + (\lambda+\mu)\partial_2 \mathrm{div}\, u\\ \mu\Delta u_3 + (\lambda+\mu)\partial_3 \mathrm{div}\, u \end{pmatrix}. \end{aligned}$$

The differential equation (1.67) can be written with the help of the Lamé operator:

$$-Lu = -[\mu\Delta u + (\lambda+\mu)\mathrm{grad}(\mathrm{div}\, u)] = f \qquad (1.70)$$

or in matrix form:

$$-\begin{pmatrix} \mu\Delta + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 & (\lambda+\mu)\partial_1\partial_3\\ (\lambda+\mu)\partial_2\partial_1 & \mu\Delta + (\lambda+\mu)\partial_2^2 & (\lambda+\mu)\partial_2\partial_3\\ (\lambda+\mu)\partial_3\partial_1 & (\lambda+\mu)\partial_2\partial_3 & \mu\Delta + (\lambda+\mu)\partial_3^2 \end{pmatrix}\begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix} = \begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix}.$$

The differential equation system (1.70) is called Lamé or Lamé–Navier system for the displacement fields.

The matrix–vector form of Hooke's law

It is convenient to write the strain and stress tensors in vector form:

$$\vec{e} = \begin{pmatrix} e_{11}\\ e_{22}\\ e_{33}\\ 2e_{12}\\ 2e_{23}\\ 2e_{13}\end{pmatrix}, \qquad \vec{\sigma} = \begin{pmatrix} \sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{13}\end{pmatrix}.$$

Hooke's law then reads:

$$\vec{\sigma} = M\vec{e}, \qquad \text{where } M = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}. \tag{1.71}$$

M is called elasticity matrix.

Writing

$$\vec{e} = D\vec{u}, \qquad \text{with } D = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ \partial_2 & \partial_1 & 0 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \end{pmatrix}, \tag{1.72}$$

we get $L\vec{u} = D^T\vec{\sigma} = D^T M\vec{e} = D^T MD\vec{u}$ and the "stress vector" (traction)

$$\sigma n = N^T\vec{\sigma} = N^T MD\vec{u}$$

with

$$N = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \\ n_2 & n_1 & 0 \\ 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \end{pmatrix}. \tag{1.73}$$

Remark:

The elasticity matrix M can be more general when taking into account anisotropic materials.

1.4 Weak formulation of the linearised boundary value problem — The principle of virtual work

We have already formulated the principle of virtual work for the stress tensors $\Sigma(x)$ and $T^\varphi(x^\varphi)$ in the undeformed and the deformed domain. We now use the same ideas, starting from the boundary value problem (1.64), (1.65), (1.66) or from the equivalent displacement formulation

$$
\begin{aligned}
-L\vec{u} = -D^T M D\vec{u} &= \vec{f} \qquad \text{in } \Omega && (1.74)\\
\vec{u}\Big|_{\Gamma_0} &= 0 && (1.75)\\
N^T M D\vec{u}\Big|_{\Gamma_1} &= \vec{g}\Big|_{\Gamma_1}. && (1.76)
\end{aligned}
$$

Lemma:

If the solution $u \in [C^2(\overline{\Omega})]^3$ of the problems (1.64), (1.65), (1.66) or (1.74), (1.75), (1.76) exists for continuous right hand sides f and g, then

$$
\begin{aligned}
a(u,v) &= \int_\Omega \sigma(u) : e(v)\,dx \\
&= \int_\Omega \{\lambda tr\, e(u)\, tr\, e(v) + 2\mu e(u) : e(v)\}\,dx \\
&= \lambda \int_\Omega div\ u\ div\ v\,dx + 2\mu \int e(u) : e(v)\,dx \\
&= \int_\Omega M D\vec{u} D\vec{v}\,dx \\
&= \int_{\Gamma_1} gv\,da + \int_\Omega f v\,dx \qquad (1.77)
\end{aligned}
$$

for all $v \in V$.
V is the closure of the linear function space $\{u \in [C^\infty(\overline{\Omega})]^3,\ u\Big|_{\Gamma_0} = 0\}$ with respect to the norm

$$
\|u\|_V = \sum_{i=1}^{3} \left[\int_\Omega (|u_i|^2 + |\partial_1 u_i|^2 + |\partial_2 u_i|^2 + |\partial_3 u_i|^2)\,dx \right]^{\frac{1}{2}}. \qquad (1.78)
$$

Proof:

We multiply (1.64) with an element $v \in V$ and integrate on Ω.
Since

$$
-\int_\Omega \operatorname{div}\sigma(u)\cdot v\,dx = \int_\Omega \sigma(u) : \nabla v\,dx - \int_{\partial\Omega} \sigma(u)n\cdot v\,da
$$

$$
\begin{aligned}
&= \int\limits_{\Omega} f v \, dx \qquad \text{(see page 32)} \\
\text{and } \sigma(u) : \nabla v &= \sigma(u) : e(v),
\end{aligned}
$$

it becomes

$$
a(u,v) = \int\limits_{\Omega} \sigma(u) : e(u) \, dx = \int\limits_{\partial\Omega} \sigma(u) u \cdot v \, da + \int\limits_{\Omega} f v \, dx = \int\limits_{\Gamma_1} g v \, da + \int\limits_{\Omega} f v \, dx.
$$

Due to the relation $\sigma(u) = \lambda \mathrm{tr}\, e(u) + 2\mu e(u)$, the first part of the assertion follows. The last part of the assertion is evident, because $\vec{\sigma}(u) = M D \vec{u}$, $\vec{e}(v) = D\vec{v}$ and $\vec{\sigma}(u) \cdot \vec{e}(v) = \sigma(u) : e(v)$. ■

Definition: (weak formulation)

The weak formulation of the boundary value problem (1.64), (1.65), (1.66) or (1.74), (1.75), (1.76) reads:

Find an element $u \in V$ for given densities $f \in V'$, $g \in H^{\frac{1}{2}}(\partial\Omega)$, such that

$$
a(u,v) = \int\limits_{\Gamma_1} g v \, da + \int\limits_{\Omega} f v \, dx \qquad \forall v \in V. \tag{1.79}
$$

The element u is called the weak solution.

Remark:

Whether the weak solution u is a solution of the corresponding classical boundary value problem too is not easy to answer in general. It depends on the solutions regularity, which is influenced by the geometry of the domain and the regularity of the right hand sides.

We now discuss the solvability of the weak–formulated boundary value problem (1.79). The fundamental Lemma of Lax–Milgram gives the answer.

Lemma of Lax–Milgram: [14, Satz 17.9, p.264]

Let V be a Hilbertspace, $a(\cdot,\cdot) \colon V \times V \to \mathbb{R}$ a bilinear form, $F \in V'$. If there are constants $c_1, c_2 > 0$ such that

$$
\begin{aligned}
a(u,v) &\leq c_1 \|u\|_V \|v\|_v \qquad \forall u, v \in V && (1.80) \\
a(u,u) &\geq c_2 \|u\|_V^2 \qquad \forall u \in V, && (1.81)
\end{aligned}
$$

then there exists a uniquely defined solution $u \in V$ of

$$
a(u,v) = \langle F, v \rangle \qquad \forall v \in V
$$

and

$$
\|u\|_V \leq \frac{1}{c_2} \|F\|_{V'}. \tag{1.82}
$$

The property (1.81) is called V–ellipticity or positive definiteness of the bilinear form.

Theorem:

The weak–formulated boundary value problem (1.79) has a uniquely defined solution $u \in V$, if $\text{meas}\,\Gamma_0 > 0$, and if the matrix M is positive definite. Here, $\langle F, v\rangle = \int\limits_\Omega f v\, dx + \int\limits_{\Gamma_1} g v\, da$.

Proof:

First, we show the bilinear form $a(\cdot,\cdot)$ to be bounded (continuous) (1.77):

$$\begin{aligned}
a(u,v) &= \int\limits_\Omega \sigma(u) : e(v)\, dx = \int\limits_\Omega M D\vec{u} \cdot D\vec{v}\, dx \\
&\le \sum_{i=1}^{6} \|(MD\vec{u})_i\|_{L_2(\Omega)} \|(D\vec{v})_i\|_{L_2(\Omega)} \\
&\le \max_{i,j} |m_{ij}| \Big[\|\partial_1 u_1\|\, \|\partial_1 v_1\| + \|\partial_2 u_2\|\, \|\partial_2 v_2\| + \|\partial_3 u_3\|\, \|\partial_3 v_3\| \\
&\quad + \|\partial_2 u_1 + \partial_1 u_2\|\, \|\partial_2 v_1 + \partial_1 v_2\| + \ldots + \|\partial_3 u_1 + \partial_1 u_3\|\, \|\partial_3 v_1 + \partial_1 v_3\| \Big] \\
&\le c \left[\sum_{i,j} \|\partial_i u_j\|_{L_2(\Omega)} \sum_{i,j} \|\partial_i v_j\|_{L_2(\Omega)} \right] \\
&\le c \left[\|u\|_V \cdot \|v\|_V \right].
\end{aligned}$$

Second, we verify estimate (1.81).
Due to M being positive definite, it holds

$$a(u,u) = \int\limits_\Omega M D\vec{u} \cdot D\vec{u}\, dx = \int\limits_\Omega \vec{e}(u)^T M \vec{e}(u)\, dx \ge c \int\limits_\Omega \vec{e}(u)^T \vec{e}(u)\, dx.$$

Korn's inequality says:
If $\text{meas}\,\Gamma_0 > 0$, then a constant $K > 0$ exists with

$$\int\limits_\Omega \vec{e}(u)^T \vec{e}(u)\, dx \ge K \|u\|_V^2 \qquad \forall u \in V.$$

This leads to (1.81). $\blacksquare$

Remarks:

(i) In the particular case when $\Gamma_1 = \emptyset$, Korn's inequality can be proved as follows:

$$\begin{aligned}
\int\limits_\Omega \vec{e}(u)^T \vec{e}(u)\, dx &= \int\limits_\Omega \sum_{i,j=1}^{3} e_{ij}^2(u)\, dx = \int\limits_\Omega \frac{1}{4} \sum_{i,j} (\partial_j u_i + \partial_i u_j)^2\, dx \\
&= \frac{1}{2} \sum_{i,j} \int\limits_\Omega (\partial_j u_i)^2\, dx + \frac{1}{2} \sum_{i,j} \int\limits_\Omega \partial_j u_i \partial_i u_j\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i,j}\int_\Omega (\partial_j u_i)^2\,dx + \frac{1}{2}\int_\Omega \underbrace{\left(\sum_i \partial_i u_i\right)\left(\sum_j \partial_j u_j\right)}_{\geq 0} dx \\
&\geq \frac{1}{2}\sum_{i,j}\int_\Omega (\partial_j u_i)^2\,dx.
\end{aligned}
$$

Here, we have used that $\int_\Omega \partial_j u_i\,\partial_i u_j\,dx = \int_\Omega \partial_i\partial_j\,u_i u_j\,dx = \int_\Omega \partial_i u_i\,\partial_j u_j\,dx$ for $u \in \overset{\circ}{H}{}^1(\Omega)$.
The Friedrichs–Poincaré equation then yields Korn's inequality.

(ii) The kernel of the linearised strain tensor e is:

$$
\ker e = \{u\colon\ u = a + b \times x\}_{a,b\in\mathbb{R}^3}, \quad \dim\ker e = 6.
$$

The elements of $\ker e$ are called rigid motions.
If $\operatorname{mes}\Gamma_0 > 0$, then the rigid motions cannot be solutions and this property leads to the V–ellipticity.

(iii) M is positive definite, if all eigenvalues of M are positive:

$$
\det(M - \alpha I) = (\mu - \alpha)^3(2\mu - \alpha)^2(3\lambda + 2\mu - \alpha) = 0.
$$

It follows that $\mu > 0$, $3\lambda + 2\mu > 0$ leads to positive eigenvalues.

(iv) The proof of the theorem is unchanged, if we replace the matrix M by a more general symmetric, positive definite matrix.

Furthermore, the estimate (1.82) describes the stability of the solution: For different force densities f_1, f_2 and g_1, g_2, the estimate is valid for the corresponding solutions u_1, u_2:

$$
\begin{aligned}
\|u_1 - u_2\|_V &\leq \frac{1}{c_2}\|F_1 - F_2\|_{V'} = \frac{1}{c_2}\sup_{\|v\|_V=1} |\langle F_1, v\rangle - \langle F_2, v\rangle| \\
&= \frac{1}{c_2}\sup_{\|v\|_V=1}\left|\int_\Omega (f_1 - f_2)v\,dx + \int_{\Gamma_1}(g_1 - g_2)v\,da\right| \\
&\leq \frac{1}{c_2}\sup_{\|v\|_V=1}\left\{\|f_1 - f_2\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + \|g_1 - g_2\|_{H^{-\frac{1}{2}}(\Gamma_1)}\|v\|_{H^{-\frac{1}{2}}(\Gamma_1)}\right\} \\
&\leq \frac{1}{c_2}\left[\|f_1 - f_2\|_{L_2(\Omega)} + \|g_1 - g_2\|_{H^{-\frac{1}{2}}(\Gamma_1)}\right],
\end{aligned}
$$

provided that $f_i \in L_2(\Omega)$, $g_i \in H^{\frac{1}{2}}(\Gamma_1)$, $i = 1, 2$.

The principle of the minimisation of the energy

Besides the weak formulated boundary value problem (1.76), one studies an equivalent minimisation problem of an energy functional.

Definition: (The quadratic functional $E\colon V \to \mathbb{R}$)
$E(u) = \frac{1}{2}a(u,u) = \frac{1}{2}\int\limits_\Omega \sigma(u) : e(u)\,dx$ is called elastic energy functional (for the linearised problem),
whereas

$$J(u) = E(u) - (\int\limits_\Omega f u\,dx + \int\limits_{\Gamma_1} gu\,da) = E(u) - \langle F, u\rangle \tag{1.83}$$

is called the total energy (for the linearised problem).

Definition: (convex functional)
A functional $J\colon K \subset \mathbb{R}$, defined on a convex subset K of a vector space V is convex on K, if for $u, v \in K$, $\alpha \in [0,1]$,

$$J(\alpha u + (1-\alpha)v) \leq \alpha J(u) + (1-\alpha)J(v).$$

J is strictly convex on K, if for $u, v \in K$, $u \neq v$, $\alpha \in (0,1)$

$$J(\alpha u + (1-\alpha)v) < \alpha J(u) + (1-\alpha)J(v).$$

Theorem: (convexity and derivatives) [15, p.247, 249ff]
Let $J\colon\ K \to \mathbb{R}$ be a functional defined and differentiable over a convex subset K of a normed vector space.

(a) J is convex on K, if and only if

$$J(v) \geq J(u) + J'(u)(v-u) \qquad \text{for all } u, v \in K.$$

(b) J is strictly convex on K, if and only if

$$J(v) > J(u) + J'(u)(v-u) \qquad \text{for all } u, v \in K,\ u \neq v.$$

Furthermore, a point u is a minimum of J on K, if and only if

$$J'(u)(v-u) \geq 0 \qquad \text{for all } v \in K.$$

If the set K is open, a point u is a minimum of J on K, if and only if $J'(u) = 0$.

Lemma:
Assume the bilinear form $a(\cdot,\cdot)$ defined by (1.79) to be V–elliptic (positive definite). The energy functional (1.83) is strictly convex, and for an uniquely defined element $u \in V$, it holds:

$$J(u) = \min_{v \in V} J(v) \Leftrightarrow a(u,v) = \langle F, v\rangle \qquad \forall v \in V. \tag{1.84}$$

Proof:

We use the above theorem and show (b):
First, we calculate $J(u)(v-u)$:

$$J(u+h) = J(u) + J'(u)h + \|h\|\epsilon(h),$$

where $\epsilon(h) \to 0$ as $h \to 0$.

$$\begin{aligned}
J(u+h) &= \frac{1}{2}a(u+h,u+h) - \langle F, u+h\rangle \\
&= \frac{1}{2}a(u,u) + a(u,h) + \frac{1}{2}a(h,h) - \langle F,u\rangle - \langle F,h\rangle \\
&= \frac{1}{2}a(u,u) - \langle F,u\rangle + a(u,h) - \langle F,h\rangle + \frac{1}{2}a(h,h) \qquad (1.85)\\
&= J(u) + J'(u)h + \|h\|\epsilon(h) \\
\text{where } J'(u)h &= a(u,h) - \langle F,h\rangle. \qquad (1.86)
\end{aligned}$$

Now, we verify condition (b):
Due to (1.86),

$$\begin{aligned}
J(u) + J'(u)(v-u) &= \frac{1}{2}a(u,u) - \langle F,u\rangle + a(u,v-u) - \langle F,v-u\rangle \\
&= -\frac{1}{2}a(u,u) + a(u,v) - \langle F,v\rangle \\
\text{and } J(u) &= \frac{1}{2}a(v,v) - \langle F,v\rangle > -\frac{1}{2}a(u,u) + a(u,v) - \langle F,v\rangle \\
&= J(u) + J'(u)(v-u), \\
\text{since} & \\
\frac{1}{2}a(v,v) + \frac{1}{2}a(u,u) - a(u,v) &= \frac{1}{2}a(u-v,u-v) > 0 \qquad \text{for all } u \neq v.
\end{aligned}$$

Relation (1.84) can be proved as follows:
Assume $a(u,v) = \langle F,v\rangle \quad \forall v \in V$.
Then

$$\begin{aligned}
J(u+tv) &= J(u) + t[a(u,v) - \langle F,v\rangle] + \frac{1}{2}t^2 a(v,v) \\
&= J(u) + \frac{1}{2}t^2 a(v,v) \geq J(u) \qquad \forall t \in \mathbb{R} \\
\text{and } J(u) &= \min_{v \in V} J(v).
\end{aligned}$$

Assume $J(u) = \min J(v)$. The minimum of a strictly convex functional is uniquely determined.
If $g(t) = J(u+v)$ is minimal for $t=0$, then $g'(0) = a(u,v) - \langle F,v\rangle = 0$, and $a(u,v) = \langle F,v\rangle \quad \forall v \in V$ follows. ■

If we consider a convex subset $K \subset V$ instead of the whole space V, we get an equivalent variational inequality:

Lemma: [2, p.53]

Assume that the bilinear form $a(\cdot, \cdot)$ defined by (1.79) is V–elliptic (positive definite). Let $K \subset V$ be a convex subset.
Then for $u \in K$

$$J(u) = \min_{v \in K} J(v) \Leftrightarrow a(u, v-u) \geq \langle F, v-u \rangle \qquad \forall v \in K.$$

1.5 Computational Mechanics — The Finite Element Method

In 1909, W. Ritz [10] published a method for the numerical solving of extremal problems of type (1.84), while B. Galerkin [6] developed a numerical method for solving variational problems of type (1.79). The basic idea of both methods is:
Do not solve the problems in spaces of infinite dimension, but in finite–dimensional subspaces. This leads to linear equation systems.

The Ritz method

Let $V_N \subset V$ be a finite–dimensional subspace of dimension N. We introduce a basis $\{b_1, ..., b_N\}$. Every element $w \in V_N$ has a uniquely determined representation

$$w = \sum_{i=1}^{N} w_i b_i.$$

Let $\vec{w} = (w_1, ..., w_N)^T$ be the coefficient vector and $P\vec{w} = w$. P is an isomorphism between $\mathbb{R}^N$ and V_N.

The extremal problem (1.84):

$$J(u) = \frac{1}{2}\, a(u,u) - \langle F, u\rangle = \min! \qquad \text{for } u \in V$$

reads in V_N: Find a coefficient vector $\vec{w}$, such that

$$\begin{aligned} J(w) = H(\vec{w}) &= \frac{1}{2}\, a\left(\sum_{i=1}^{N} w_i b_i, \sum_{j=1}^{N} w_j b_j\right) - \langle F, \sum_{i=1}^{N} w_i b_i\rangle \\ &= \min! \qquad \text{for } w \in V_N \quad (\text{or } \vec{w} \in \mathbb{R}^N). \end{aligned} \tag{1.87}$$

Lemma:

Assume that the bilinear form $a(\cdot,\cdot)$, defined by (1.79) is V–elliptic (positive definite). Then problem (1.87) has a uniquely defined solution $\vec{w} \in \mathbb{R}^N$, which can be computed as the solution of the linear equation system

$$\begin{pmatrix} a(b_1,b_1) & a(b_1,b_2) & \cdots & a(b_1,b_N) \\ a(b_2,b_1) & a(b_2,b_2) & \cdots & a(b_2,b_N) \\ \vdots & \vdots & \ddots & \vdots \\ a(b_N,b_1) & \cdots & \cdots & a(b_N,b_N) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} \langle F, b_1\rangle \\ \langle F, b_2\rangle \\ \vdots \\ \langle F, b_N\rangle \end{pmatrix}; \tag{1.88}$$

in short: $A\vec{w} = \vec{F}$.

Proof:

The necessary condition for an extremal value is:

$$\frac{\partial H}{\partial w_i} = 0 \qquad \text{for } i = 1, ..., N. \tag{1.89}$$

Since

$$\begin{aligned} H(w_1, w_2, ..., w_N) &= \frac{1}{2} w_1 \sum_{j=1}^{N} w_j \, a(b_1, b_j) - w_1 \langle F, b_1 \rangle \\ &+ \frac{1}{2} w_2 \sum_{j=1}^{N} w_j \, a(b_2, b_j) - w_2 \langle F, b_2 \rangle \\ &+ \vdots \\ &+ \frac{1}{2} w_N \sum_{j=1}^{N} w_j \, a(b_N, b_j) - w_N \langle F, b_N \rangle \end{aligned}$$

the relations (1.89) read:

$$\begin{aligned} \frac{\partial H}{\partial w_1} &= w_1 \, a(b_1, b_1) + \frac{1}{2} w_2 \, a(b_1, b_2) + \cdots + \frac{1}{2} w_N \, a(b_1, b_N) - \langle F, b_1 \rangle \\ & +\frac{1}{2} w_2 \, a(b_2, b_1) + \cdots + \frac{1}{2} w_N \, a(b_N, b_1) = 0 \\ &\vdots \\ \frac{\partial H}{\partial w_N} &= w_N \, a(b_N, b_N) + \frac{1}{2} w_1 \, a(b_N, b_1) + \cdots + \frac{1}{2} w_{N-1} \, a(b_N, b_{N-1}) - \langle F, b_N \rangle \\ & +\frac{1}{2} w_1 \, a(b_1, b_N) + \cdots + \frac{1}{2} w_{N-1} \, a(b_N, b_{N-1}) = 0. \end{aligned} \tag{1.90}$$

Due to the symmetry of the bilinear form $a(\cdot, \cdot)$, the equation system (1.88) results immediately.

The matrix

$$A = \begin{pmatrix} a(b_1, b_1) & \cdots & a(b_1, b_N) \\ \vdots & \ddots & \vdots \\ a(b_N, b_1) & \cdots & a(b_N, b_N) \end{pmatrix} \tag{1.91}$$

is symmetric and positive definite.

Indeed,

$$\begin{aligned} (A\vec{w}, \vec{w}) &= \sum_i w_i \left[\sum_j a(b_i, b_j) \, w_j \right] = \sum_i w_i \, a(b_i, w) \\ &= a(P\vec{w}, P\vec{w}) \geq c_2 \|P\vec{w}\|^2 > 0 \qquad \text{for } \vec{w} \neq 0. \end{aligned}$$

Therefore, the equation system (1.88) has a uniquely defined solution. Furthermore, this solution realises a minimum of (1.87) since the Hesse matrix

$$\begin{pmatrix} \frac{\partial^2 H}{\partial w_1^2} & \frac{\partial^2 H}{\partial w_2 \partial w_1} & \cdots & \frac{\partial^2 H}{\partial w_N \partial w_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial w_1 \partial w_N} & \cdots & \cdots & \frac{\partial^2 H}{\partial w_N^2} \end{pmatrix} = A$$

is positive definite. ■

The Galerkin method

We now consider the variational problem (1.79) in $V_N \times V_N$: Find an element $u_N \in V_N$, such that

$$a(u_N, v) = \langle F, v \rangle \qquad \forall v \in V_N. \tag{1.92}$$

Definition: (Galerkin solution)

The solution $u_N \in V_N$ of (1.92) is called Galerkin solution of the variational problem (1.79) $a(u,v) = \langle F, v \rangle$.

Taking the same basis $\{b_1, ..., b_N\}$ in V_N as before, it is easy to verify that:

$$u_N \in V_N \text{ is Galerkin solution } \Leftrightarrow a(u_N, b_i) = \langle F, b_i \rangle \qquad \text{for } i = 1, ..., N. \tag{1.93}$$

Lemma:

Be $\vec{F} = (\langle F, b_1 \rangle, \langle F, b_2 \rangle, ..., \langle F, b_N \rangle)^T$. The element $u_N = \sum_{i=1}^N c_i b_i$ is Galerkin solution, if and only if the coefficient vector $\vec{c} = (c_1, c_2, ..., c_N)^T$ is solution to the equation system (1.88):

$$A\vec{c} = \vec{F}.$$

Proof:

(a) Assume $u_N \in V_N$ is Galerkin solution. Inserting $u_N = \sum_{j=1}^N c_j b_j$ into (1.93) and using the bilinearity of $a(\cdot, \cdot)$, we get

$$a\left(\sum_j c_j b_j, b_i\right) = \sum_j a(b_j, b_i) c_j = \langle F, b_i \rangle, \quad i = 1, ..., N,$$

which immediately yields (1.88).

(b) Let $\vec{c}$ be solution of (1.88). The scalar multiplication of the system with an arbitrary vector $\vec{v} \in \mathbb{R}^N$ leads to

$$\begin{aligned} A\vec{c} \cdot \vec{v} &= \vec{F} \cdot \vec{v} = \sum_j \left[\sum_i a(b_i, b_j) c_i \right] v_j = a(P\vec{c}, P\vec{v}) \\ &= a(u_N, v) = \sum_i \langle F, b_i \rangle v_i = \langle F, v \rangle \quad \forall v \in V_N. \end{aligned}$$

Error estimate

We now estimate the error between the weak solution $u \in V$ and the Galerkin solution $u_N \in V_N$.

Lemma: (Céa '64) [3]

Let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$, which satisfies the conditions of the Lemma of Lax–Milgram (1.80) and (1.81):

$$\begin{aligned} a(u,v) &\le c_1 \|u\|_V \|v\|_V \qquad && \forall u, v \in V \\ a(u,u) &\ge c_2 \|u\|_V^2 && \forall u \in V. \end{aligned}$$

Then the error between the weak solution $u \in V$ of

$$a(u,v) = \langle F, v\rangle \qquad \forall v \in V, f \in V'$$

and the corresponding Galerkin solution $u_N \in V_N$ can be estimated:

$$\|u - u_N\|_V \le \frac{c_1}{c_2} \|u - v_N\|_V \qquad \forall v_N \in V_N, \tag{1.94}$$

or

$$\|u - u_N\|_V \le \frac{c_1}{c_2}\, dist\,(u, V_N) = \frac{c_1}{c_2} \inf_{v_N \in V_N} \|u - v_N\|_V. \tag{1.95}$$

Proof:

Since

$$\begin{aligned} a(u,v) &= \langle F, v\rangle \qquad && \forall v \in V_N \\ a(u_N,v) &= \langle F, v\rangle && \forall v \in V_N \end{aligned}$$

it follows that

$$a(u - u_N, v) = 0 \qquad \forall v \in V_N. \tag{1.96}$$

This means that $u - u_N$ is orthogonal on the space V_N with respect to the so–called energetic scalar product $[u,v] := a(u,v)$.

Hence,

$$\begin{aligned} \|u - u_N\|_V^2 &\overset{(1.81)}{\le} \frac{1}{c_2} |a(u - u_N, u - u_N)| \overset{(1.96)}{=} \frac{1}{c_2} |a(u - u_N, u - v_N)| \\ &\overset{(1.80)}{\le} \frac{c_1}{c_2} \|u - u_N\|_V \|u - v_N\|_V \qquad \text{for any } v_N \in V_N. \end{aligned}$$

Dividing this inequality by $\|u - u_N\| \ne 0$ (for $\|u - u_N\| = 0$ the estimates (1.94) and (1.95) are trivial) the estimates (1.94) and (1.95) follow. ∎

In order to guarantee, that the error $\|u - u_N\|_V$ is small if the dimension N is sufficiently large, we have to introduce an appropriate sequence of finite–dimensional subspaces, which converge to V.

We denote $V_i := V_{N_i}$ for $i \in \mathbb{N}$ and assume the approximation property

$$\lim_{i\to\infty} \operatorname{dist}(u, V_i) = 0 \qquad \forall u \in V. \tag{1.97}$$

Let us remark that (1.97) is satisfied, if

$$V_1 \subset V_2 \subset \cdots \subset V_i \subset \cdots \subset V, \qquad \bigcup_{i=1}^{\infty} V_i \text{ is dense in } V, \quad [7, p.154].$$

Corollary:

Assume, that the conditions of the Lax–Milgram Lemma (1.80) and (1.81) are satisfied for a given bilinear form and the property (1.97) holds. Then for the sequence of weak solutions $u_i \in V_i$ is holds that

$$\lim_{i\to\infty} \|u - u_i\|_V = 0.$$

Finite element methods

The computation of a Galerkin solution $u_N \in V_N$ (or of a minimiser of $J(u)$ in V_N) is reduced to solving the linear equation system (1.88)

$$A\vec{c} = \vec{F}.$$

In general, the matrix A is large and dense. We have to carry out N^2 integrations for getting the elements of A, N integrations for calculating the elements of $\vec{F}$ and N^3 operations for solving the resulting linear equation system. It therefore is very reasonable to choose a sequence of finite–dimensional subspaces $\{V_N \subset V\}$ with an appropriate basis $\{b_1, ..., b_N\}$, such that

- the integrations are elementar, and
- the matrix $A = (a(b_i, b_j))$ is sparse.

The best variant is that A is a diagonal matrix, or even $a(b_i, b_j) = \delta_{ij}$, which means to choose a basis that is orthonormal with respect to the energetic scalar product. Unfortunately, the orthonormality procedure is not efficient enough. The fundamental idea is to take elementary vector functions as basis in V_N, with small support, such that, in our case

$$a(b_i, b_j) = \int_\Omega \vec{e}(b_i)^T M \vec{e}(b_j)\, dx = \int_\Omega \sigma(b_i) : e(b_j)\, dx = \int_{\operatorname{supp} b_i \cap \operatorname{supp} b_j} \sigma(b_i) : e(b_j)\, dx$$

vanishes, if $\operatorname{supp} b_i \cap \operatorname{supp} b_j$ is a set of measure zero.
To construct such a basis, the domain Ω will be splitted into subdomains (triangles, bricks, polyhedrals, ...), which form the support for the basis functions, and then to take V_N–spaces of piecewise linear or piecewise polynomial vector fields on these small subdomains.

Piecewise linear elements

***n* = 1:**

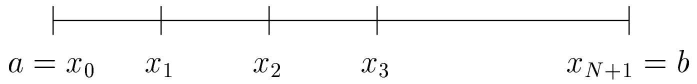
1° **The boundary value problem.**
We take $u_1 = u_2 = 0$, $u_3 = u = u(x)$, $x \in (a,b) = \Omega$ and consider the boundary value problem

$$\begin{aligned} -\mu\Delta u &= -\mu u''(x) = f(x) \quad \text{in } (a,b) \\ u(a) &= u(b) = 0. \end{aligned}$$

The weak formulation reads:
Find an element $u \in \overset{\circ}{H}{}^1(a,b)$ such that

$$a(u,v) = \int_a^b \mu u'(x) v'(x)\,dx = \int_a^b f(x)\,v(x)\,dx \qquad \forall v \in \overset{\circ}{H}{}^1(a,b).$$

2° **Partition of $\overline{\Omega}$.**
We choose the nodes $\{x_i : a = x_0 < x_1 < ... < x_{N+1} = b\}$. Then $\overline{\Omega} = [a,b] = \bigcup_{i=1}^{N+1} \overline{I}_i$, where $I_i = (x_{i-1}, x_i)$ are subintervalls.



3° **Construction of $V_N \subset V$.**

$$V_N = \{u \in C[a,b] : u\Big|_{I_i} = a_i x + b_i, 1 \le i \le N+1, u(a) = u(b) = 0\}.$$

Figure 4 shows graphs of some elements from V_N. Basis in V_N : $\{b_i(x) : b_i(x_j) = \delta_{ij}$, $b_i(x_0) = b_i(x_{N+1}) = 0, i,j = 1,...,N\}$.

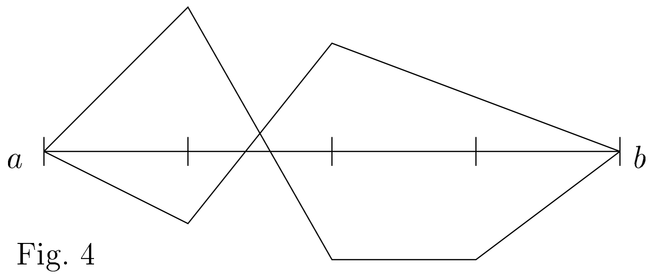
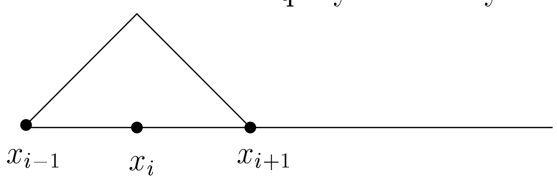


Fig. 4

Every basis element is uniquely defined by its values in the node points:



$b_i(x)$

The Galerkin solution $u_N(x) = \sum_{i=1}^N c_i b_i(x)$ has the property that $u_N(x_j) = c_j$.

4° **Setting up the equation system.**
We have

- $a(b_i, b_j) = 0$, if $|i-j| \geq 2$, i.e. A is a three–diagonal matrix.
- $a(b_i, b_{i-1}) = -\frac{1}{x_i - x_{i-1}}\mu$
- $a(b_i, b_i) = \left(\frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}\right)\mu$
- $a(b_i, b_{i+1}) = -\frac{1}{x_{i+1} - x_i}\mu$

Therefore

$$A = \mu \begin{pmatrix} \frac{1}{x_1-x_0} + \frac{1}{x_2-x_1} & -\frac{1}{x_2-x_1} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{x_2-x_1} & \frac{1}{x_2-x_1} + \frac{1}{x_3-x_2} & -\frac{1}{x_3-x_2} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{x_N-x_{N-1}} + \frac{1}{x_{N+1}-x_N} \end{pmatrix},$$

$$\vec{F} = \begin{pmatrix} \int\limits_{x_0}^{x_2} f\, b_1\, dx \\ \int\limits_{x_1}^{x_3} f\, b_2\, dx \\ \int\limits_{x_2}^{x_4} f\, b_3\, dx \\ \vdots \\ \int\limits_{x_{N-1}}^{x_{N+1}} f\, b_N\, dx \end{pmatrix}$$

and $\qquad A\vec{c} = \vec{F}.$

If $x_i = a + ih$, $h = \frac{b-a}{N-1}$, then

$$A = \frac{\mu}{h} \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & -1 & 2 \end{pmatrix}.$$

***n* = 2:**

1° **The boundary value problem.**
Let Ω be a polygonal domain, $u_3 = 0$, $x = (x_1, x_2) \in \Omega$, $\vec{u} = (u^{(1)}, u^{(2)})^T$. We consider the boundary value problem

$$\begin{aligned} -[\mu\Delta\vec{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\vec{u}] &= \vec{f} \qquad \text{in } \Omega \\ \vec{u} &= \vec{0} \qquad \text{on } \partial\Omega. \end{aligned}$$

The weak formulation (plane strain) reads:

Find an element $\vec{u} \in [\overset{\circ}{H^1}(\Omega)]^2 = V$ with

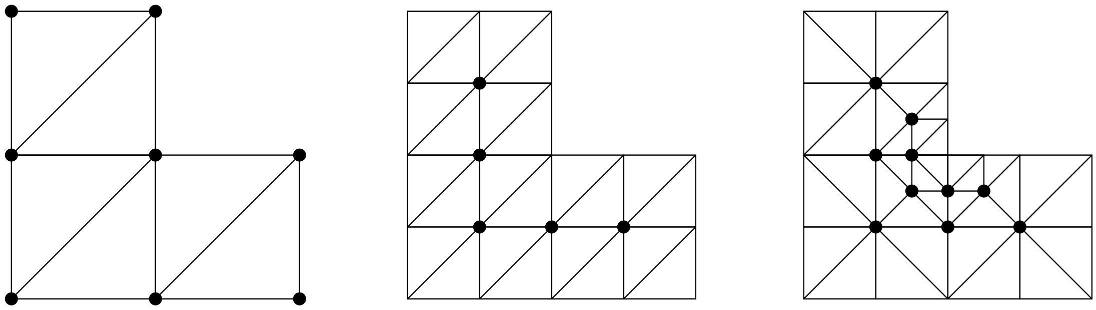
$$a(\vec{u},\vec{v}) = \int\limits_{\Omega} \vec{e}(\vec{u})^T M \vec{e}(\vec{v})\,dx = \int\limits_{\Omega} \vec{f}\cdot\vec{v}\,dx \qquad \forall \vec{v} \in [\overset{\circ}{H^1}(\Omega)].$$

Here

$$M = \begin{pmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \qquad \vec{e} = \begin{pmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{pmatrix}.$$

2° **Partition of $\overline{\Omega}$.**

We choose an admissible triangulation of Ω, $\overline{\Omega} = \bigcup_{i=1,\dots,t} \overline{T_i}$. (See Fig. hereafter.) N is the number of interior nodes, M denotes the number of all nodes (nodes are corner points of the triangles).



$t = 6$	$t = 24$	$t = 30$
$N = 0$	$N = 5$	$N = 10$
$M = 8$	$M = 21$	$M = 22$

3° **Construction of $V_{2N} \subset V$.**

We choose

$$V_{2N} = \left\{ \vec{u} = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}, u^{(k)} \in C(\overline{\Omega}) : u^{(k)}(\vec{x})\Big|_{T_m} = a^{(k)}_{m,1} + a^{(k)}_{m,2}x_1 + a^{(k)}_{m,3}x_2, \right.$$
$$\left. u^{(k)}\Big|_{\partial\Omega} = 0,\ k = 1,2,\ m = 1,\dots,t \right\}$$

with the basis

$$\{b_l(x)\}_{l=1,\dots,2N} = \left\{ \vec{b}_l(\vec{x}) : \vec{b}_{2i-1}(\vec{x}_j) = \begin{pmatrix} \delta_{ij} \\ 0 \end{pmatrix}, \vec{b}_{2i}(\vec{x}_j) = \begin{pmatrix} 0 \\ \delta_{ij} \end{pmatrix}, \right.$$
$$i,j = 1,\dots,N,\ l = 1,\dots,2N,\ \vec{x}_j \text{ are interior nodes,}$$
$$\left. \vec{b}_l(\vec{x}_J) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ at exterior nodes } \vec{x}_J,\ J = 1,\dots,M-N \right\}.$$

The basis elements $\vec{b}_l(x)$ are uniquely defined through their 3 values (for each component) in the corner points of a triangle.

Furthermore, for an element $u_{2N} \in V_{2N}$ the following relation is valid:

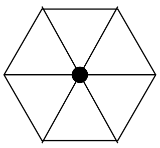
$$\begin{aligned}
\vec{u}_{2N}(\vec{x}) &= \sum_{l=1}^{2N} c_l \vec{b}_l(\vec{x}) \\
\vec{u}_{2N}(\vec{x}_j) &= \begin{cases} c_{2j-1}\binom{1}{0} + c_{2j}\binom{0}{1} = \binom{c_{2j-1}}{c_{2j}} & \text{for interior node points} \\ 0 & \text{for exterior node points.} \end{cases}
\end{aligned}$$

4° **Setting up the equation system.**
It is

$$a(\vec{b}_l, \vec{b}_{\tilde{l}}) = \int_\Omega \vec{e}(\vec{b}_l)^T M\, \vec{e}(\vec{b}_{\tilde{l}})\, dx = \sum_m \int_{T_m} \vec{e}(\vec{b}_l)^T M\, \vec{e}(\vec{b}_{\tilde{l}})\, dx,$$

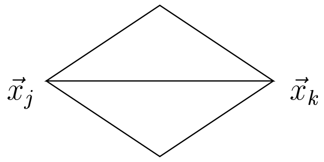
where we have to sum up over the following triangles

- $l = 2j-1, 2j$, $\tilde{l} = 2j-1, 2j$, $\vec{x}_j$ is a corner point of T_m



$\vec{x}_j$

- $l = 2j-1, 2j$, $\tilde{l} = 2k-1, 2k$, $j \neq k$, $\vec{x}_j$ and $\vec{x}_k$ are corner points of T_m.



This leads to a matrix A with a band structure.

***n* = 3:**

1° **The boundary value problem.**
Let Ω be a polyhedral domain. We consider the boundary value problem for the displacement field $\vec{u} = (u^{(1)}, u^{(2)}, u^{(3)})^T$,

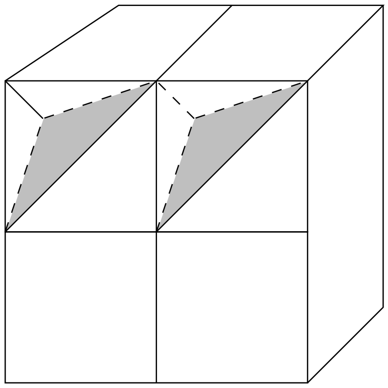
$$\begin{aligned}
-[\mu\Delta\vec{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\vec{u}] &= \vec{f} && \text{in } \Omega \\
\vec{u} &= \vec{0} && \text{on } \partial\Omega,
\end{aligned}$$

and its weak formulation:
Find a vector field $\vec{u} \in [\overset{\circ}{H}{}^1(\Omega)]^3 = V$, such that

$$a(\vec{u}, \vec{v}) = \int_\Omega \vec{e}(\vec{u})^T M \vec{e}(\vec{v})\, dx = \int_\Omega \vec{f} \cdot \vec{v}\, dx \qquad \forall \vec{v} \in [\overset{\circ}{H}{}^1(\Omega)]^3,$$

where M is the matrix (1.71).

2° **Partition of $\overline{\Omega}$.**
We divide $\overline{\Omega}$ into tetrahydrals taking an admissible partition [4]. $\overline{\Omega} = \bigcup_{i=1,\dots,t} \overline{T_i}$. We distinguish between the N interior nodes and the number M of all nodes (nodes are corner points of the tetrahydrals).



3° **Construction of $V_{3N} \subset V$.**
Let

$$V_{3N} = \left\{ \vec{u} = \begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{pmatrix}, u^{(k)} \in C(\overline{\Omega}) : u^{(k)}(\vec{x})\Big|_{T_m} = a^{(k)}_{m,1} + a^{(k)}_{m,2}x_1 + a^{(k)}_{m,3}x_2 + a^{(k)}_{m,4}x_3, \right.$$
$$\left. u^{(k)}\Big|_{\partial\Omega} = 0,\ k = 1,2,3,\ m = 1,\dots,t \right\}.$$

We introduce the basis

$$\{b_l(x)\}_{l=1,\dots,3N} = \left\{ \vec{b_l}(\vec{x}) : \vec{b}_{3i-2}(\vec{x}_j) = \begin{pmatrix} \delta_{ij} \\ 0 \\ 0 \end{pmatrix}, \vec{b}_{3i-1}(\vec{x}_j) = \begin{pmatrix} 0 \\ \delta_{ij} \\ 0 \end{pmatrix}, \vec{b}_{3i}(\vec{x}_j) = \begin{pmatrix} 0 \\ 0 \\ \delta_{ij} \end{pmatrix}, \right.$$
$$i,j = 1,\dots,N,\ l = 1,\dots,3N,\ \vec{x}_j \text{ are interior nodes,}$$
$$\left. \vec{b_l}(\vec{x}_J) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ at exterior nodes } \vec{x}_J,\ J = 1,\dots,M-N \right\}.$$

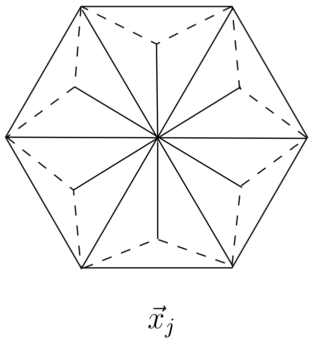
The basis elements are uniquely defined through their values at the 4 corner points of the tetrahydrons.

4° **Setting up the equation system.**
It is

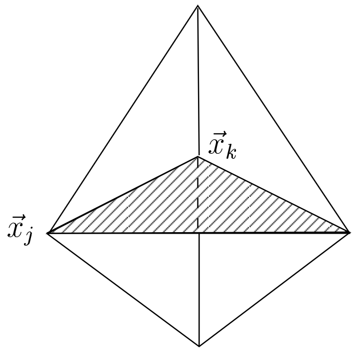
$$a(\vec{b_l}, \vec{b_{\tilde{l}}}) = \int_\Omega \vec{e}(\vec{b_l})^T M \, \vec{e}(\vec{b_{\tilde{l}}}) \, dx = \sum_m \int_{T_m} \vec{e}(\vec{b_l})^T M \, \vec{e}(\vec{b_{\tilde{l}}}) \, dx,$$

where we have to sum up over the following tetrahydrons:

- $l = 2j-2, 2j-1, 2j$, $\tilde{l} = 2j-2, 2j-1, 2j$, $\vec{x}_j$ is a corner point of T_m



- $l = 2j-2, 2j-1, 2j$, $\tilde{l} = 2k-2, 2k-1, 2k$, $j \neq k$, $\vec{x}_j$ and $\vec{x}_k$ are corner points of T_m



We get a matrix with a band structure.

Error estimates for piecewise linear elements

We cite a result for the two–dimensional case, which can also be applied to three–dimensional boundary value problems. We have shown that

$$\|u - u_{2N}\|_V \leq c \inf_{V \in V_{2N}} \|u - v\|_V.$$

The following theorem improves this estimate.

Theorem: [7]

Let V_{2N} be the space of continuous, piecewise linear vector fields with respect to an admissible triangulation of polygon Ω. Be $\alpha_0 > 0$ the smallest interior angle of the triangles and h the largest length of the sides of the triangles.
Then

$$\inf_{V \in V_{2N}} \|u - v\|_{[H^k(\Omega)]^2} \leq c(\alpha_0) h^{2-k} \|u\|_{[H^2(\Omega)]^2}.$$

for $k = 0, 1$ and all $u \in [H^2(\Omega)]^2 \cap V$.

Chapter 2

Plasticity

In the introduction we have discussed strain–stress relations, describing elastic and plastic deformations due to the applied loading [Fig. 3]. Because the strain–stress relations for elastic and plastic deformation are different, it is important to distinguish between elastic and plastic regions and to identify whether the deformation is elastic or plastic. One can use "yield surfaces" to identify different regions.

For the one–dimensional case (metal wire), we have introduced the yield stress σ_y as a value, that separates the elastic and plastic region of deformation. In general, constitutive models for plastic deformation are divided into two categories: one that assumes the existence of a yield point (yield surface) and one that does not. We are going to deal with the first category.

In the first part of this chapter, we discuss yield criteria, while in the second part, basic considerations as stress–strain relations (constitutive law of Prandtl–Reuß, Hencky) and hardening laws are dealt with.

We recommend the book:
Khan, A.S., S. Huang. *Continuum theory of Plasticity.* John Wiley, 1995
for further studies.

2.1 Yield criteria

In the three–dimensional case, the symmetric stress tensor is characterised by six independent stress components. Therefore, a stress state can be defined as a point in the six–dimensional space. All stress states that cause yielding can be thought to constitute a continuous surface, called yield surface, that divides the stress space into an elastic and a plastic domain. The yield surface is the boundary between these two domains.

The stress space and yield surfaces

In section 1.2, we have introduced the Cauchy stress vector t^φ and the Cauchy stress tensor T^φ in the deformed configuration. The Piola transform leads to the first and second Piola–Kirchhoff stress–tensors in the undeformed configuration:

$$\begin{aligned} T(x) &= T^\varphi(x^\varphi)\mathrm{Cof}\,(\nabla\varphi)\det(\nabla\varphi)[\nabla\varphi(x)]^{-T} \\ \Sigma(x) &= [\nabla\varphi(x)]^{-1}T(x). \end{aligned}$$

Assumption:
The differences between $T^\varphi(x^\varphi)$, $T(x)$ and $\Sigma(x)$ are negligible ($\nabla\varphi \sim I$).

Due to the symmetry of the Cauchy stress tensor, we define:

Definition: (stress space)

The stress space is the space $S^{3,3}$ of all symmetric matrices, which is isomorph to the 6–dimensional vector space $\mathbb{R}^6$. A point in $S^{3,3}$ describes a stress state.
All possible stress states corresponding to yielding, constitute a closed hypersurface in the stress space.

Definition: (yield surface)

Be $Z \subset S^{3,3}$ (or $\mathbb{R}^6$) a closed and convex subset, $0 \in \mathrm{int}\,Z$. The boundary ∂Z is called yield surface.

Definition: (yield function)

A continuous convex function $F\colon\ S^{3,3} \to \mathbb{R}$ is called yield function with respect to Z, if

$$\begin{aligned} Z &= \{\sigma \in S^{3,3} : F(\sigma) \leq 0\}, \\ \partial Z &= \{\sigma \in S^{3,3} : F(\sigma) = 0\}, \\ F(0) &< 0. \end{aligned}$$

Remark:

$F(\sigma) < 0$ describes the elastic deformation domain,
$F(\sigma) = 0$ describes the yield surface,
$F(\sigma) \geq 0$ describes the plastic deformation domain.

Example:

The ball $Z = B_R(0) = \{\sigma = (\sigma_1, ..., \sigma_6)^T, |\sigma| = \sqrt{\sum_{i=1}^{6} \sigma_i^2} \leq R\}$ is a closed convex subset.

$$F(\sigma) = |\sigma| - R$$

is the corresponding yield function.

We will now discuss more relevant examples. For this purpose, we remind of some quantities that have been defined in section 1.2:

The eigenvalues τ_1, τ_2, τ_3 of an element of $S^{3,3}$ are called the principal stresses, while the corresponding eigenvectors $\vec{n}^{(1)}$, $\vec{n}^{(2)}$, $\vec{n}^{(3)}$ are called the principal directions.
Since

$$\begin{pmatrix} \vec{n}_1^{(1)} & \vec{n}_1^{(2)} & \vec{n}_1^{(3)} \\ \vec{n}_2^{(1)} & \vec{n}_2^{(2)} & \vec{n}_2^{(3)} \\ \vec{n}_3^{(1)} & \vec{n}_3^{(2)} & \vec{n}_3^{(3)} \end{pmatrix}^T \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} \vec{n}_1^{(1)} & \vec{n}_1^{(2)} & \vec{n}_1^{(3)} \\ \vec{n}_2^{(1)} & \vec{n}_2^{(2)} & \vec{n}_2^{(3)} \\ \vec{n}_3^{(1)} & \vec{n}_3^{(2)} & \vec{n}_3^{(3)} \end{pmatrix} = \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}$$

the invariants of σ (invariant with respect to the choice of the orthogonal coordinate system) can be expressed with the help of the principal stresses:

$$\begin{aligned} I_1 &= \operatorname{tr}\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33} = \tau_1 + \tau_2 + \tau_3 \\ I_2 &= \tfrac{1}{2}((\operatorname{tr}\sigma)^2 - \operatorname{tr}\sigma^2) = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 \\ &= \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 \\ I_3 &= \det\sigma = \tau_1\tau_2\tau_3. \end{aligned}$$

In plasticity theory it is customary to decompose the stress tensor into two parts

$$\sigma_{ij} = p\delta_{ij} + s_{ij},$$

where $-p$ is the hydrostatic pressure:

$$p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}(\tau_1 + \tau_2 + \tau_3).$$

$-pI = -(p\delta_{ij})_{ij}$ is called the spherical or hydrostatic stress tensor.
The second part $(s_{ij})_{ij} = s = \sigma^d$ is called the deviatoric stress tensor. The principal deviatoric stresses s_i are related to the principal stresses τ_i by

$$s_i = \tau_i - p = \tau_i - \frac{1}{3}(\tau_1 + \tau_2 + \tau_3). \tag{2.1}$$

Remark:

The plastic deformations are very often independent of the hydrostatic pressure; that means:
If $\sigma, \tilde{\sigma}: [0,T] \to S^{3,3}$ and $\sigma - \tilde{\sigma} = pI$, $p \in \mathbb{R}$, then the corresponding deformations are equal.

Lemma: [2]

Be $\sigma \in S^{3,3}$. For $\sigma^d = \sigma - \frac{1}{3} tr\, \sigma I$ and $S^{3,3}_{dev} = \{\sigma^d : \sigma \in S^{3,3}\}$ it holds

$$S^{3,3} = S^{3,3}_{dev} \oplus P, \qquad \textit{where } P = \{pI, p \in \mathbb{R}\}.$$

Proof:

It is $\sigma - \tilde{\sigma} \in P$ and, due to (2.1), we have $\mathrm{tr}\, \sigma^d = 0$. Hence

$$(\sigma^d)^d = \sigma^d - \frac{1}{3}\mathrm{tr}\, \sigma^d I = \sigma^d. \tag{2.2}$$

Assume $\sigma = pI \in S^{3,3}_{dev}$, then $\sigma^d = \sigma - \frac{1}{3}\mathrm{tr}\, \sigma I = pI - pI = 0$.
Furthermore, $\sigma = \sigma^d = 0$, due to (2.2). Therefore, $\sigma \in S^{3,3}_{dev} \cap P$ yields $\sigma = 0$. ■

Remark:

For anisotropic material the orientation of the principal stresses is as important as their magnitude. When the material is isotropic, the material properties are the same in any direction and only the principal stresses (three-dimensional space!) describe the yield or failure behaviour.

von Mises and Tresca yield surfaces (isotropic case)

It was established that even at $25 \cdot 10^8 \frac{\mathrm{N}}{\mathrm{m}^2} = 2500\mathrm{MPa}$ there was no yielding in metals [8]. Therefore, the yielding in metals does not depend on the pressure; that means, it does not depend on the first invariant $I_1 = \mathrm{tr}\, \sigma$ of the stress tensor.

von Mises yield surface

The von Mises criterion (1913) assumes that the plastic yielding will only occur, when the second invariant of the deviatoric stress tensor σ^d reaches a critical value $-k^2$, that means

$$\begin{array}{lcll} -I_2(\sigma^d) - k^2 & = & 0 & \text{is the yielding surface} \\ -I_2(\sigma^d) < k^2 & & & \text{describes the elastic deformation.} \end{array}$$

Since

$$\begin{aligned}
I_2(\sigma^d) &= s_1s_2 + s_2s_3 + s_3s_1 \\
&= (\tau_1 - p)(\tau_2 - p) + (\tau_2 - p)(\tau_3 - p) + (\tau_3 - p)(\tau_1 - p) \\
&= \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 - p(2\tau_2 + 2\tau_2 + 2\tau_3) + 3p^2 \\
&= \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 - \frac{1}{3}(\tau_1 + \tau_2 + \tau_3)^2 \\
&= -\frac{1}{6}[(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2]
\end{aligned}$$

it follows, that the relation

$$\frac{1}{6}[(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2] - k^2 = 0 \tag{2.3}$$

describes the yielding surface.
Furthermore,

$$I_2(\sigma^d) = \frac{1}{2}(\mathrm{tr}\,\sigma^d)^2 - \frac{1}{2}\mathrm{tr}\,(\sigma^d)^d = -\frac{1}{2}[\sigma^d : \sigma^d]$$

which leads to the surface equation

$$\sigma^d : \sigma^d - 2k^2 = 0.$$

For ideal plastic materials the critical value is $2k^2 = \text{const}$. The constant $2k^2 > 0$ has to be determined in experiments. One possibility is to take a uniaxial tension and to measure the value k_0 such, that for $\sigma_{11} = k_0$ the plastic flow starts.
Then

$$\sigma = \begin{pmatrix} k_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \sigma^d = \sigma - \frac{1}{3}k_0 I = \begin{pmatrix} \frac{2}{3}k_0 & 0 & 0 \\ 0 & -\frac{1}{3}k_0 & 0 \\ 0 & 0 & -\frac{1}{3}k_0 \end{pmatrix}$$

and

$$F(\sigma) = \sigma^d : \sigma^d - 2k^2 = \left(\frac{2}{3}\right)^2 k_0^2 + \left(\frac{1}{3}\right)^2 k_0^2 + \left(\frac{1}{3}\right)^2 k_0^2 - 2k^2 = \frac{6}{9}k_0^2 - 2k^2 = 0,$$

if $k^2 = \frac{1}{3}k_0^2$. The von Mises surface equation reads for ideal plastic materials:

$$\boxed{\sigma^d : \sigma^d - \frac{2}{3}k_0^2 = 0}\,. \tag{2.4}$$

The form of the yield surface in the space of the principal stresses τ_1, τ_2, τ_3 is described by the equations (2.3). Consider the diagonal $\tau_1 = \tau_2 = \tau_3$ in the three–dimensional space. The plane through the origin $(0, 0, 0)$ and orthogonal to the diagonal is expressed by the relation

$$\tau_1 + \tau_2 + \tau_3 = 0. \tag{2.5}$$

The points which satisfy the relations (2.3) and (2.5) are situated in a circle at the plane (2.5) with radius $\sqrt{2}k = r$ and the midpoint $(0, 0, 0)$. (See Fig. 5)

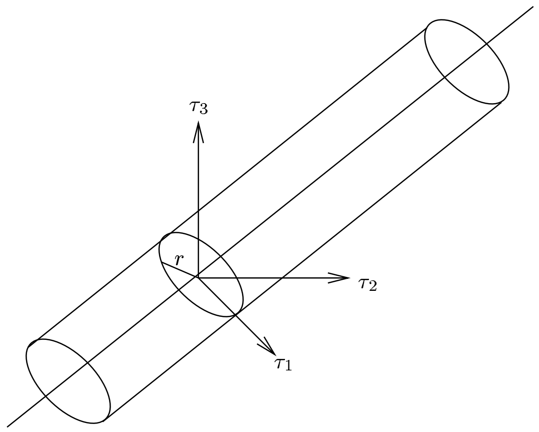


Fig. 5 von Mises yield surface

Inserting $\tau_1 = -\tau_2 - \tau_3$ into (2.3) then

$$\tau_2^2 + \tau_3^2 + \tau_2\tau_3 = k^2 = \frac{1}{2}(\tau_1^2 + \tau_2^2 + \tau_3^2)$$

and $\sqrt{2}k = r$.

Starting with the plane $\tau_1 + \tau_2 + \tau_3 = 3c$ and intersecting it with points of the yield surface (2.3), we again get a circle with radius $r = \sqrt{2}k$.

Summarising these results, we have:

Lemma:

The von Mises yield surface in the three–dimensional τ_1–τ_2–τ_3 space is a circle cylinder surface around the hydrostatic stress axis $\tau_1 = \tau_2 = \tau_3$ and with radius $r = \sqrt{2}k$.

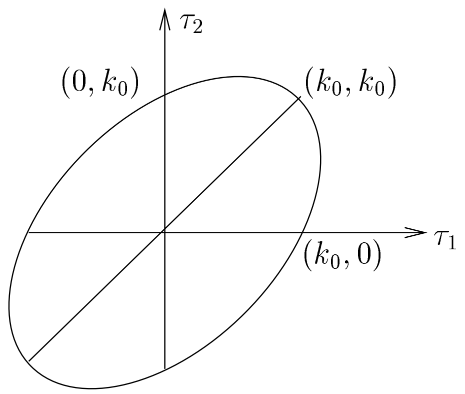
Corollary:

In the s_1–s_2–s_3 space, we have $s_1 = s_2 = s_3 = 0$ along the whole hydrostatic axis.

In case of plane stress ($\tau_3 = 0$), the equation (2.3) is reduced to

$$\tau_1^2 - \tau_1\tau_2 + \tau_2^2 = 3k^2 = k_0^2 \tag{2.6}$$

In the τ_1–τ_2 space, the equation (2.6) represents an ellipse. (Note that $\sqrt{3}k = k_0 = \sigma_y$.)



von Mises yield surface for plane stress conditions

Tresca yield surface

Tresca (1864) based on Coulomb's results on soil mechanics and his own experiments on metal extrusion, proposed a yield criterion for metallic solids, which is now well known as Tresca yield criterion. This criterion assumes that the plastic yielding will occur when the maximum shear stress reaches the critical value k_0 of the material.
Let us remind of the definition of the shear stress.
The stress vector $\sigma \cdot n$ is splitted into a vector parallel to n (normal direction) and a vector situated in a plane orthogonal to n (tangential direction).

$$\sigma n = \sigma_S + \sigma_N \cdot n \qquad \text{with } \sigma_N = (\sigma n, n) = n^T \sigma n.$$

Definition:

$\sigma_S = \sigma n - \sigma_N n$ is the shear stress.

Lemma:

Let $\tau_1 > \tau_2 > \tau_3$ be the principal stresses of the stress tensor $\sigma = diag(\tau_1, \tau_2, \tau_3)$. Then

$$\max_{n \in S^2} \|\sigma_S\|_2 = \frac{1}{2}(\tau_1 - \tau_3)$$

and the maximum is realised in the direction

$$n = \begin{pmatrix} \pm\frac{1}{2}\sqrt{2} \\ 0 \\ \pm\frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} \pm\cos 45^\circ \\ 0 \\ \pm\cos 45^\circ \end{pmatrix}.$$

Here, $\|\cdot\|$ denotes the Euclidian norm, S^2 is the unit sphere in $\mathbb{R}^3$.

Proof:

For $n \in S^2$ it holds

$$\|\sigma_S\|_2^2 = \|\sigma n\|_2^2 - \sigma_N^2 = \sum_{i=1}^{3} \tau_i^2 n_i^2 - \left(\sum_{i=1}^{3} \tau_i n_i^2\right)^2. \tag{2.7}$$

The maximum of $\|\sigma_S\|_2$ under the constraint $\sum_i n_i^2 = 1$ can be calculated by the Lagrange method:
Determine the factor α such that

$$0 = \frac{\partial}{\partial n_k}(\|\sigma_S\|_2^2 + \alpha\left(\sum n_i^2 - 1\right)), \qquad 1 \le k \le 3. \tag{2.8}$$

Inserting the expression (2.7) and (2.8), we get

$$0 = n_k(\alpha + \tau_k^2 - 2\tau_k \sigma_N), \qquad 1 \le k \le 3. \tag{2.9}$$

1.case: All components of n vanish, $n_1 = n_2 = n_3 = 0$.
Then $n \notin S^2$, that means this case is impossible.

2.case: Two components are zero, say $n_i = n_j = 0$ and $n_k \neq 0$.
Then $n = e_k$ and $\sigma_S = \tau_k e_k - \tau_k e_k = 0$.

3.case: Only one component is zero, $n_1 \neq 0$, $n_2 \neq 0$, $n_3 = 0$.
Then (2.9) implies:

$$\begin{array}{ll} & \alpha + \tau_1^2 - 2\tau_1\sigma_N = 0 \\ & \underline{\alpha + \tau_2^2 - 2\tau_2\sigma_N = 0} \\ & 0 + \tau_1^2 - \tau_2^2 - 2(\tau_1 - \tau_2)\sigma_N = 0 \\ \Rightarrow & \tau_1 + \tau_2 - 2\sigma_N = 0 \end{array}$$

and $\sigma_N = \frac{1}{2}(\tau_1 + \tau_2) = \tau_1 n_1^2 + \tau_2 n_2^2$, $n_1^2 + n_2^2 = 1$.
Since $\frac{1}{2}(\tau_1 + \tau_2) = (\tau_2 - \tau_1)n_2^2$, it follows that

$$n_1^2 = n_2^2 = \tfrac{1}{2}, \qquad n_1 = \pm\tfrac{1}{2}\sqrt{2}, \qquad n_2 = \pm\tfrac{1}{2}\sqrt{2}. \tag{2.10}$$

4.case: All components are non–zero.
Then, as in the third case,

$$\sigma_N = \tfrac{1}{2}(\tau_1 + \tau_2) = \tfrac{1}{2}(\tau_2 + \tau_3) = \tfrac{1}{2}(\tau_1 + \tau_3),$$

which leads to $\tau_1 = \tau_2 = \tau_3$ in contradiction to the assumption.

Since $\tau_1 > \tau_2 > \tau_3$, the third case with $n_1 \neq 0$, $n_2 = 0$, $n_3 \neq 0$ leads to the maximum:

$$\|\sigma_S\|_2 = \frac{1}{2}|\tau_1 - \tau_3|, \qquad n = \begin{pmatrix} \pm\sqrt{\frac{1}{2}} \\ 0 \\ \pm\sqrt{\frac{1}{2}} \end{pmatrix}. \quad \blacksquare$$

Definition:
The yield surface of Tresca is defined for every element $\sigma \in S^{3,3}$ by the yield function

$$F(\sigma) = \max_{n \in S^2} \|\sigma_S(\sigma, n)\|_2 - c_0 = \frac{1}{2}(\tau_{max} - \tau_{min}) - c_0 = 0, \tag{2.11}$$

where τ_{max}, τ_{min} are the largest and the smallest eigenvalue of σ respectively.

We have to show that the Tresca yield function F, defined by (2.11) is continuous and convex.
It is clear that $F(0) = -c_0 < 0$.
The shear stress $\sigma_S(\sigma, n) = \sigma n - (\sigma n, n)n$ is a linear mapping from $S^{3,3}$ to $\mathbb{R}^3$ for any fixed n, and it is continuous in both arguments. The norm is a continuous mapping, too. Therefore,

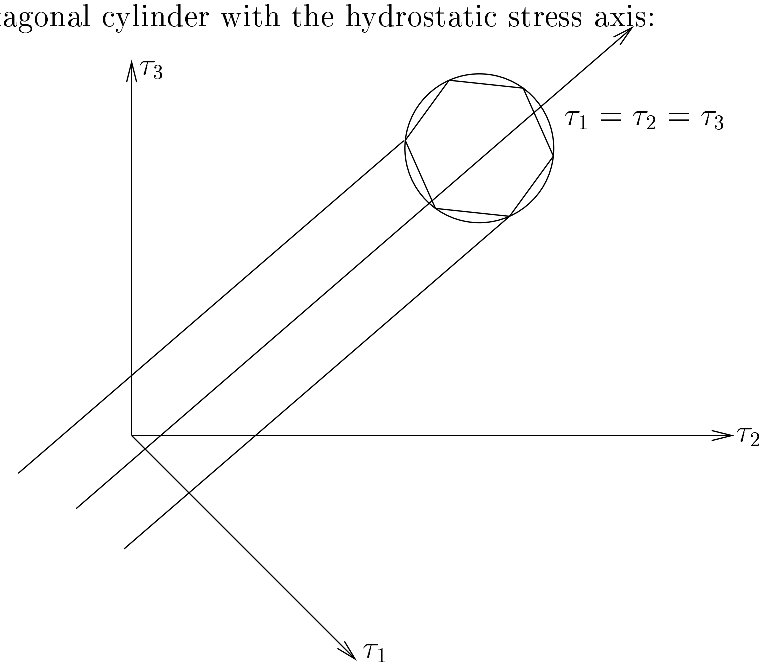
$$\tilde{F}(\sigma, n) = \|\sigma_S(\sigma, n)\|_2$$

is convex in σ and continuous in (σ, n).
The function $F(\sigma) = \max_{n \in S^2} \tilde{F}(\sigma, n) - c_0$ has the same properties. (Note that S^2 is a compact set.)

The Tresca yield function F only depends on the deviatoric stress tensor, due to

$$F(\sigma + \lambda I) = F(\sigma).$$

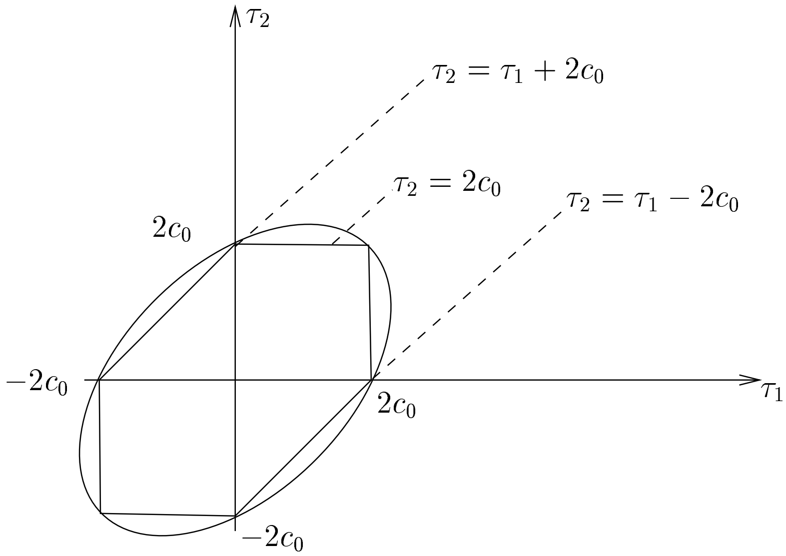
It can be shown ([8, p.86ff]) that in the principal stress space the Tresca yield surface is a hexagonal cylinder with the hydrostatic stress axis:



In case of plane stress ($\tau_3 = 0$), the criterion (2.11) implies

$$\begin{aligned} \tau_1 - \tau_2 &= \pm 2c_0 \\ \tau_1 &= \pm 2c_0 \\ \tau_2 &= \pm 2c_0. \end{aligned}$$

This leads to the Tresca hexagon in the τ_1–τ_2 plane.



The constant c_0 can be determined by different tests:

a) simple tension test

$$\tau_1 = \sigma_y, \quad \tau_2 = \tau_3 = 0.$$

Then $\frac{1}{2}\sigma_y = c_0$ and $2c_0 = k_0 = \sigma_y$ (von Mises).

b) pure shear test

$$\tau_1 = -\tau_3 = \tau_y \quad \tau_2 = 0.$$

Then $c_0 = \tau_y$.

Thus, for Tresca materials we get $\tau_y = \frac{\sigma_y}{2}$.

Yield criterion for anisotropic materials

For fully anisotropic materials the yield functions should be expressed in terms of six independent components of the stress tensor σ.

$$F(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}) = 0$$

In terms of the principal stresses τ_i and the associated principal directions $n^{(i)}$, the yield surface equation can be written as

$$\tilde{F}(\tau_1, \tau_2, \tau_3, n^{(1)}, n^{(2)}, n^{(3)}) = 0.$$

Both equations can be geometrically interpreted as a limiting envelope of the elastic domain in the six–dimensional stress space.

The search for yield criteria for anisotropic materials was started by Jackson, Smith and Lankford, Dorn and Hill in 1948/49 [8]. Basically, they tried to modify the Tresca or von Mises criterion.

Let us present Hill's yield criterion (1948) for orthotropic materials. (More exactly: for materials in which three mutually orthogonal planes of symmetry exist at each material point. The intersections of these planes are the principal axes of anisotropy.)

$$\begin{aligned} F(\sigma_{ij}) &= \frac{1}{2}[\alpha_{2233}(\sigma_{22} - \sigma_{33})^2 + \alpha_{1133}(\sigma_{33} - \sigma_{11})^2 + \alpha_{1122}(\sigma_{11} - \sigma_{22})^2 \\ &\quad + 2\alpha_{2323}\sigma_{23}^2 + 2\alpha_{1313}\sigma_{13}^2 + 2\alpha_{1212}\sigma_{12}^2 - 1] = 0 \end{aligned}$$

α_{ijkl} are material constants characterising the current state of anisotropic yield behaviour.

For $\alpha_{2323} = \alpha_{1313} = \alpha_{1212} = 3\alpha_{2233} = 3\alpha_{1133} = 3\alpha_{1122} = \frac{1}{2k^2}$ we get the von Mises criterion for isotropic materials.

Indeed,

$$\begin{aligned} \sigma^d : \sigma^d &= (\sigma_{11} - p)^2 + (\sigma_{22} - p)^2 + (\sigma_{33} - p)^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{13}^2 \\ &= \frac{2}{3}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) - \frac{2}{3}(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} + 2(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2) \\ &= \frac{1}{3}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{22})^2\right] + 2(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2) \\ \frac{2}{3}F(\sigma_{ij}) &= \frac{1}{2k^2}\sigma^d : \sigma^d - 1 = 0. \end{aligned}$$

More general, one can introduce a material tensor M of fourth–order and write the yield criterion for anisotropic material as

$$F(\sigma, M) \quad = \quad F(\sigma_{ij}, M_{ijkl}) = 0$$

when hydrostatic pressure independence is given.

Remark:

The yielding of porous materials is pressure sensitive. The yield criterion for these materials should include the influence of the hydrostatic pressure. Two classic yield criteria, the Coulomb–Mohr and the Drucker–Prager criteria describe the yielding of pressure dependent materials.

Coulomb–Mohr criterion: Improve the Tresca criterion assuming that the critical shear stress is not only related to the maximum shear stress, but also depends on the normal stress.

Drucker–Prager criterion: Add the hydrostatic stress term $\alpha_1 J_1(\sigma) = \alpha_1(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \alpha_1(\tau_1+\tau_2+\tau_3)$ to the von Mises criterion $F(J_1(\sigma), J_2(\sigma^d)) = \sqrt{-J_2(\sigma^d)} - \alpha_1 J_1(\sigma) - k = 0$.

Subsequent yield surface

So far, we only have discussed the initial yield surface at which the material yields for the first time on initial loading.
During the plastic deformation the subsequent yield surface will expand, translate and distort in the stress space. In order to describe this effect, we need more information about the plastic strain–stress relation. We here mention two classic models: the isotropic and the kinematic hardening models.

Isotropic hardening

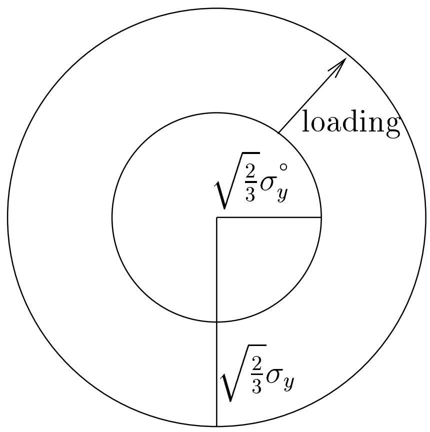
Isotropic hardening assumes that the subsequent yield surface is a uniform expansion of the initial yield surface, and that the material's isotropic response to yielding remains unchanged during plastic deformation.
If we ignore the effect of hydrostatic pressure on yield, we have

$$F(J_2(\sigma^d), J_3(\sigma^d), k) = f(J_2(\sigma^d), J_3(\sigma^d)) - k = 0.$$

k is a material constant characterising the isotropic hardening effect.
For the von Mises yield surface, we have

$$\sigma^d : \sigma^d - \frac{2}{3}k^2 = 0,$$

which describes a ball in the deviatoric space, with the radius $k = \sqrt{\frac{2}{3}}\sigma_y$. Initially, $\sigma_y = \sigma_y{}^{\circ}$.

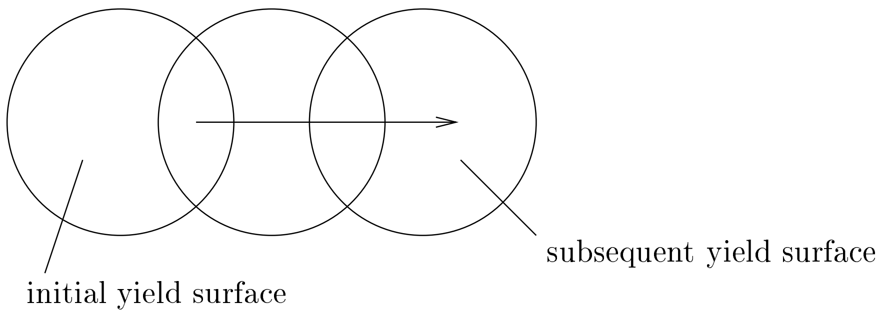


For perfectly plastic materials, k remains a constant during plastic deformation. In general, k depends on the accumulated plastic strain,

$$k = k(t), \qquad k(0) = \sigma_y \sqrt{\frac{2}{3}}.$$

Kinematic hardening

Prager (1956) suggested that the yield surface translates as a rigid body in the stress space during the plastic deformation, that means the shape of the yield surface remains unchanged.



This model can be written as

$$F(\sigma^d, \alpha) = f(\sigma^d - \alpha) - k_0 = 0,$$

where α is a second–order tensor, known as the back stress.

Generally, the yield surface, including the hardening effect, can be written as

$$F(\sigma, \alpha_i) = 0, \quad i = 1, ..., n,$$

where the α_i are hardening parameters.

2.2 Constitutive laws

It can be concluded from experiments that plastic deformation has the following features:

1. Plastic deformation is associated with the dissipation of energy and therefore irreversible. This can be seen in Fig. 6, which shows the $\sigma - \epsilon$ diagramm in an uniaxial tension test.

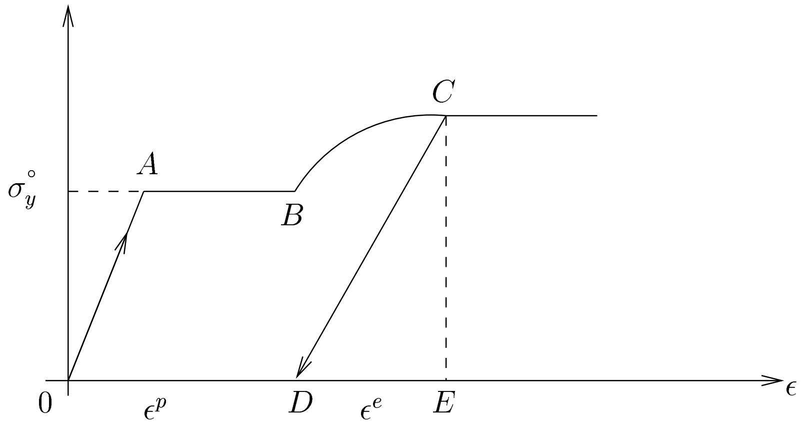


Fig. 6

When unloading occurs in point C, only a part of the strain $\epsilon^e(DE)$ can be recovered (elastic strain), while another part $\epsilon^p(0D)$ remains after the load is removed (plastic strain). This leads to an additive decomposition

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\right) = \epsilon_{ij}^e + \epsilon_{ij}^p. \tag{2.12}$$

This decomposition is correct for cases of infinitesimal strain only. The case of finite strains needs another model.

2. Due to its dissipation feature, the plastic deformation process is history or path dependent. Therefore, the constitutive equations for plastic deformation are formulated as differential equations or as incremental form (no one–to–one correspondence).

3. We assume, that the constitutive equations are invariant with respect to the time scale. The viscous effect is neglected.
For the (elastic) stress it holds that

$$\begin{aligned} \sigma : [t_0, t_1] \quad &= \quad I \to S^{3,3} \\ \sigma = C : \epsilon^e, \quad & \quad \epsilon^e = \tilde{C} : \sigma \\ \text{or} \qquad \vec{\sigma} = C\vec{\epsilon}^e, \quad & \quad \vec{\epsilon}^e = C^{-1}\vec{\sigma}, \quad C^{-1}\text{corresponds to } \tilde{C}. \\ \sigma(t) \in Z(t), \quad & \end{aligned} \tag{2.13}$$

$Z(t)$ will be determined by the evolution of the internal variables.
The plastic strain ϵ^p is considered as an internal variable, which incorporates the memory of the process.

We now pass on to the mathematical description.

Moving convex sets

Let Z be a closed convex subset of a Hilbert space H, equipped with the scalar product $\langle\cdot,\cdot\rangle$ and $0 \in \operatorname{int} Z$. For example: H is $S^{3,3}$ equipped with the scalar product $\langle A, B\rangle = A : B$, or: $H = \mathbb{R}^6$ (or $\mathbb{R}^3$) for anisotropic or isotropic materials equipped with the Euclidean scalar product.
A translational motion of Z from the initial position $Z(t_0) = Z$ to the final position $Z(t_f)$ is descibed by the relation

$$Z(t) = w(t) + Z, \qquad t \in [t_0, t_f], \tag{2.14}$$

$w\colon [t_0, t_f] \to H$. We consider a single moving point $v(t) \in Z(t)$ for $t \in [t_0, t_f]$.

$$v(t) = w(t) + z(t), \qquad z(t) \in Z \quad \forall t \in [t_0, t_f]. \tag{2.15}$$

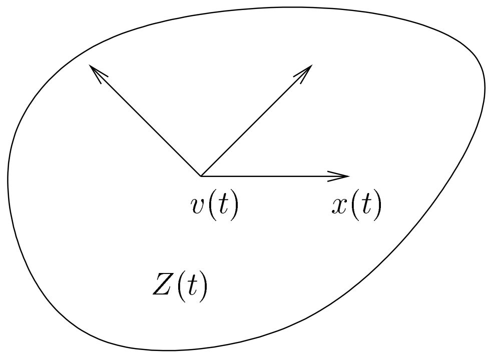
We want to specify the motions $w(t)$, satisfying (2.15), such that

$$\dot{w}(t) \in N_{Z(t)}(v(t)). \tag{2.16}$$

Here,

$$N_{Z(t)}(v(t)) = \{u \in H : \langle u, y(t) - v(t)\rangle \le 0 \quad \text{for all } \ y(t) \in Z(t)\} \tag{2.17}$$

is the normal cone to $Z(t)$ at $v(t)$.



Definition (2.17) means for $H = \mathbb{R}^{3(6)}$: $u \in N_{Z(t)}(v(t))$, if

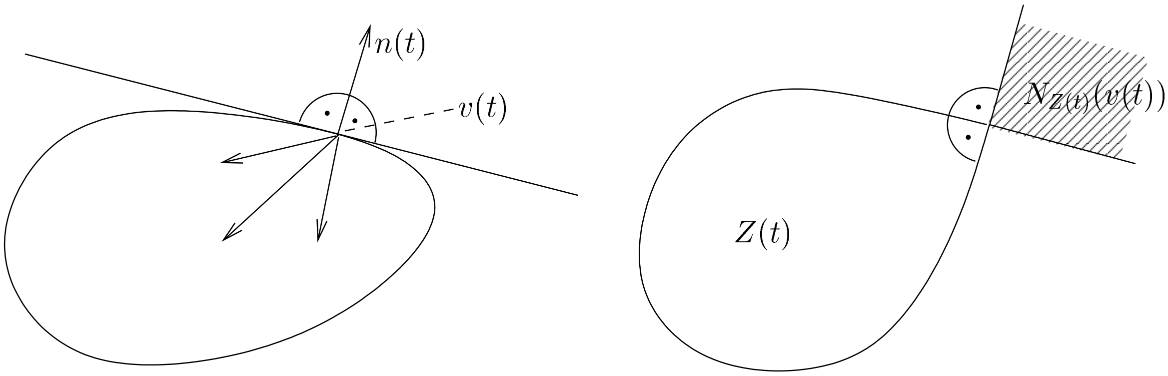
$$\cos(\measuredangle u, y(t) - v(t)) = \cos\alpha \le 0 \qquad \text{for all } y(t) \in Z(t),$$

or

$$\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \qquad \text{for all } y(t) \in Z(t). \tag{2.18}$$

If $v(t) \in \operatorname{int} Z(t)$, (Fig. above), then $N_{Z(t)}(v(t)) = \{0\}$. Set $y(t) = v(t) - \delta u(t)$ for a sufficiently small positive δ.
Hence, $\dot{w}(t) = 0$ and $w = \text{const}$. For $Z = Z(t_0)$, we get $w(t) \equiv 0$ and the convex set does not move.

If $v(t) \in \partial Z(t)$ has a normal, then condition (2.18) is illustrated by Fig (a), otherwise by Fig (b).



(a) (b)

For $H = \mathbb{R}^n$, $n = 3$ or 6, $N_{Z(t)}(v(t)) = \{\lambda(t)n(t)\}$, $\lambda(t) \geq 0$, n is the outer normal vector in point $v(t)$. If $v(t) \in \partial Z(t)$ does not have a normal (Fig.b), then $N_{Z(t)}(v(t))$ is a proper cone.

In case (a), we have a movement in the direction of the outer normal.

In order to calculate the translation $w(t)$ in case (a) for given $v = v(t)$, we transform (2.16) into a evolution variational inequality in $Z = Z(t_0)$.

(2.17) reads:

$$\langle \dot{w}(t), y(t) - v(t) \rangle \leq 0 \qquad \text{for all } y(t) \in Z(t).$$

Due to (2.14), we have $y(t) = w(t) + x$, $x \in Z$, and due to (2.15) $v(t) = w(t) + z(t)$, $z(t) \in Z$, and (2.17) becomes

$$\langle \dot{w}(t), x - z(t) \rangle \leq 0 \qquad \text{for all } x \in Z.$$

Thus, we get a system of relations for the calculation of w and z for given $v = v(t)$ and $z(t_0) = z^\circ$.

$$\boxed{\begin{array}{rcl} \langle \dot{w}(t), x - z(t) \rangle & \leq & 0 \qquad \forall x \in Z \\ v(t) & = & w(t) + z(t), \quad z(t) \in Z \\ z(t_0) & = & z^\circ \end{array}} \tag{2.19}$$

Eliminating $w(t) = v(t) - z(t)$ and setting $f = \dot{v}$, we arrive at the variational inequality

$$\boxed{\begin{array}{rcl} z(t) \in Z, \quad \langle \dot{z}(t), x - z(t) \rangle & \geq & \langle f(t), x - z(t) \rangle \\ z(t_0) & = & z^\circ \qquad\qquad \forall x \in Z \end{array}} \,. \tag{2.20}$$

Theorem:

If $v(t)$ is differentiable, then uniquely differentiable solutions of (2.19) and (2.20) exist.

According to this theorem,

$$w = P(v, z^\circ), \qquad z = S(v, z^\circ) \tag{2.21}$$

and it yields well–defined solution operators. P is called play operator, S is called stop operator [1].

Maximal dissipation principle (normality rule)

We start from decomposition (2.12).

$$\epsilon(t) = \epsilon^e(t) + \epsilon^p(t).$$

The consitutive law (2.13) for the elastic part is nothing else than Hooke's law.

A large class of constitutive laws for the plastic part is governed by the **normality rule**:

$$\boxed{\dot{\epsilon}^p(t) \in N_{Z(t)}(\sigma(t))}, \tag{2.22}$$

where $\operatorname{int} Z(t)$ describes the elastic region in the stress space, $\partial Z(t)$ the yield surface. $\sigma(t)$ is the elastic stress tensor, given by (2.13).

$$\dot{\vec{\epsilon^p}}(t) = \begin{cases} 0 & \text{if } \sigma(t) \in \operatorname{int} Z(t) \\ \lambda(t) n(t) & \text{if } \sigma(t) \in \partial Z(t), \quad \text{provided, } n(t) \text{ exists.} \end{cases}$$

An equivalent formulation of (2.22) is the so–called **maximual dissipation principle**

$$\dot{\epsilon}^p(t) : (\tau - \sigma(t)) \leq 0 \qquad \forall \tau \in Z(t). \tag{2.23}$$

The Prandtl–Reuss model

This model is often used in elastoplasticity. Here, $Z(t) = Z$ is time–independent, the yield surface ∂Z does not move during the evolution, $0 \in \operatorname{int} Z$.

Definition:

A pair $(\sigma, \epsilon) : [t_0, t_f] \to S^{3,3} \times S^{3,3}$ satisfies the constitutive law of Prandtl–Reuss with a given Hooke's law in an elastic region Z, if

$$\epsilon = \epsilon^e + \epsilon^p, \qquad \epsilon^e = \tilde{C}\sigma \quad (or\ \vec{\epsilon^e} = C^{-1}\vec{\sigma}),$$

$$\sigma(t) \in Z \qquad in\ [t_0, t_f]\ a.e.$$

$$\dot{\epsilon}^p(t) : (\tau - \sigma(t)) \leq 0 \qquad \forall \tau \in Z, \quad t \in [t_0, t_f]\ a.e.$$

Remark:

If (σ, ϵ) satisfies the constitutive law of Prandtl–Reuss and $\sigma(t) \in \operatorname{int} Z$, then $\dot{\epsilon}^p(t) = 0$. Plastic behaviour occurs only, if $\sigma(t) \in \partial Z$. In this case,

$$\dot{\epsilon}^p(t) = \lambda(t) n(t), \qquad \lambda(t) \geq 0,$$

where $n(t)$ is the outer normal in $\sigma(t) \in \partial Z$, provided, $n(t)$ exists.

Definition: Hencky material (time–independent)
A pair $(\sigma, \epsilon) \in S^{3,3} \times S^{3,3}$ *satisfies the consitutive law of Hencky with a given Hooke's law in an elastic region* Z, *if*

$$\epsilon = \epsilon^e + \epsilon^p, \qquad \epsilon^e = \tilde{C}\sigma,$$
$$\sigma \in Z,$$
$$\epsilon^p : (\tau - \sigma) \leq 0 \qquad \forall \tau \in Z.$$

Let us formulate the evolution variational inequalities of form (2.19) or (2.20) for the computation of ϵ^p.

We start with the decomposition

$$\epsilon = \epsilon^e + \epsilon^p$$

and apply the elastic tensor C (fourth order in the tensor form, second order in the vector form). Then the following relation holds in the stress space:

$$C\epsilon = C\epsilon^e + C\epsilon^p = \sigma + C\epsilon^p. \tag{2.24}$$

We consider $H = S^{3,3}$ endowed with the scalar product $\langle \sigma, \tau \rangle = \langle \tilde{C}\sigma : \tau \rangle$. Note, that the elastic matrix C is positive definite.

We rewrite the maximal dissipative principle

$$\tilde{C}C\dot{\epsilon}^p : (\tau - \sigma(t)) \leq 0 \qquad \tau \in Z$$

as

$$\langle \dot{w}(t), \tau - \sigma(t) \rangle = \langle \tilde{C}\dot{w}(t) : \tau - \sigma(t) \rangle \leq 0 \qquad \forall \tau \in Z, \tag{2.25}$$

with $\dot{w}(t) = C\dot{\epsilon}^p$. The relation (2.24) reads

$$w(t) + z(t) = v(t), \tag{2.26}$$

with $v(t) = C\epsilon$ and $z(t) = \sigma(t) \in Z$.

The initial value is

$$z(t_0) = \sigma(t_0) = C(\epsilon(t_0) - \epsilon^p(t_0)) = C\epsilon^e(t_0), \tag{2.27}$$

where $\epsilon^p(t_0)$ is a given value which represents the initial memory.

The relations (2.25), (2.26) and (2.27) have the form of (2.19), and (2.20) reads:
Find $\sigma(t) \in Z(t)$ with

$$\begin{array}{rcl} \langle \dot{\sigma}(t), x - \sigma(t) \rangle & \geq & \langle C\dot{\epsilon}(t), x - \sigma(t) \rangle \\ \text{and} \quad \sigma(t_0) & = & C\epsilon^e(t_0). \end{array}$$

There are solutions

$$\begin{array}{rcl} \sigma(t) & = & S(C\epsilon, \sigma(t_0)), \\ w(t) & = & P(C\epsilon, \sigma(t_0)). \end{array}$$

Constitutive laws of hardening type

As before, we assume

$$\epsilon(t) = \epsilon^e(t) + \epsilon^p(t) \qquad (2.12),$$
$$\dot{\epsilon}^p(t) \in N_{Z(t)}(\sigma(t)) \qquad (2.22).$$

Hence,

$$\dot{\vec{\epsilon^p}}(t) = \begin{cases} 0 & \text{if } \sigma(t) \in \operatorname{int} Z(t) \\ \lambda(t)n(t) & \text{if } \sigma(t) \in \partial Z(t), \quad \text{provided, } n(t) \text{ exists.} \end{cases} \tag{2.28}$$

Furthermore, we consider the case that the yield function which describes the yield surface $F(\vec{\sigma}, \vec{\alpha}) = 0$ depends on a vector (or tensor) of hardening parameters $\vec{\alpha} \in \mathbb{R}^e$.
Due to (2.28), we define an associate flow rule

$$\boxed{\dot{\vec{\epsilon^p}} = \beta \frac{\partial F(\vec{\sigma}, \vec{\alpha})}{\partial \vec{\sigma}}, \qquad \vec{\sigma} = \vec{\sigma}(t), \vec{\alpha} = \vec{\alpha}(t), \beta = \beta(t).} \tag{2.29}$$

The factor $\beta(t) \geq 0$ is unknown. We only know that $\beta(t) > 0$, when the plastic flow starts.

The yield function can have the following forms (compare section 2.1)

- ideal–plastic case
$$F(\vec{\sigma}, \vec{\alpha}) = f(\vec{\sigma})$$
- isotropic hardening
$$F(\vec{\sigma}, \alpha) = f(\vec{\sigma}) + h(\alpha) \qquad \text{with } \alpha \in \mathbb{R}$$
- kinematic hardening
$$F(\vec{\sigma}, \vec{\alpha}) = f(\vec{\sigma}, \vec{\alpha}) \qquad \text{with } \vec{\alpha} \in \mathbb{R}^{3(6)}.$$

The hardening parameters α_i, $\vec{\alpha} = (\alpha_1, ..., \alpha_l)$ have to be determined through experiments or calculated from certain models (W. Prager, Th. Lehmann, F. Odquist).

The hardening law describes the evolution of the yield surface and has a similar form as (2.29):

$$\boxed{\begin{array}{ll} \dot{\vec{\alpha}} = r(\vec{\sigma}, \vec{\alpha})\beta, & \beta = \beta(t) > 0 \\ \vec{\alpha}(0) = \vec{\alpha}_0 & r\colon\ \mathbb{R}^{3(6)} \times \mathbb{R}^l \to \mathbb{R}^l \end{array}}\ . \tag{2.30}$$

$\beta = \beta(t)$ is the same factor as in (2.29).

Examples for $r(\vec{\sigma}, \vec{\alpha})$:

- isotropic hardening, $\vec{\alpha} = \alpha \in \mathbb{R}^1$
$$r(\vec{\sigma}, \alpha) = \sqrt{\tfrac{2}{3} \tfrac{\partial F^T}{\partial \vec{\sigma}} \tfrac{\partial F}{\partial \vec{\sigma}}}, \quad \alpha_0 = 0 \qquad \text{(F. Odquist)}$$
$$r(\vec{\sigma}, \alpha) = \vec{\sigma}^T \tfrac{\partial F}{\partial \vec{\sigma}}, \qquad \alpha_0 = 0 \qquad \text{(G.I. Taylor)}$$
- kinematic hardening, $\vec{\alpha} \in \mathbb{R}^{3(6)}$
$$r(\vec{\sigma}, \vec{\alpha}) = \vec{\sigma} - \vec{\alpha}, \qquad \vec{\alpha_0} = 0 \qquad \text{(H. Ziegler)}$$
$$r(\vec{\sigma}, \vec{\alpha}) = c \tfrac{\partial F}{\partial \vec{\sigma}}(\vec{\sigma}, \vec{\alpha}), \quad \vec{\alpha_0} = 0, \quad c > 0 \text{ constant} \qquad \text{(W. Prager)}$$

Example

The von Mises yield surface is given by the equation in the principal stress space:

$$(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2 = 2\sigma_y^2, \tag{2.31}$$

where $\sigma_y = h(\alpha)$ is the uniaxial yield stress, depending on the scalar hardening parameter $\alpha = \alpha(t)$ (isotropic hardening).

We consider the yield function in the stress space

$$\begin{aligned} F(\vec{\sigma}, \alpha) &= \tilde{F}(\tau_1, \tau_2, \tau_3) = \frac{1}{\sqrt{2}}\sqrt{(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2} - h(\alpha) = 0 \\ &= \sqrt{\frac{3}{2}\sigma^d : \sigma^d} - h(\alpha) \\ &= \left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{11}\sigma_{22} - \sigma_{11}\sigma_{33} - \sigma_{22}\sigma_{33} + 3(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)\right]^{\frac{1}{2}} - h(\alpha) \\ &= f(\sigma) - h(\alpha). \end{aligned} \tag{2.32}$$

Writing $\vec{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13})^T$, we get

$$\begin{aligned} \frac{\partial F}{\partial \vec{\sigma}} &= \frac{1}{2f(\sigma)}(2\sigma_{11} - \sigma_{22} - \sigma_{33}, 2\sigma_{22} - \sigma_{11} - \sigma_{33}, 2\sigma_{33} - \sigma_{11} - \sigma_{22}, 6\sigma_{12}, 6\sigma_{23}, 6\sigma_{13})^T \\ &= \frac{1}{2}\frac{3\vec{s}}{f(\sigma)}. \end{aligned} \tag{2.33}$$

It follows that

$$\begin{aligned} \left|\frac{\partial F}{\partial \vec{\sigma}}\right| &= \frac{3}{2}\frac{|\vec{s}|}{f(\vec{\sigma})} = \frac{1}{2}\sqrt{6}\frac{\sqrt{\frac{3}{2}\sigma^d : \sigma^d + 3(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)}}{\sqrt{\frac{3}{2}\sigma^d : \sigma^d}} = \sqrt{\frac{3}{2}}\sqrt{1 + 2\frac{(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)}{\sigma^d : \sigma^d}} \\ &\leq \sqrt{\frac{3}{2}}\sqrt{2} = \sqrt{3} \end{aligned}$$

and

$$1 \leq r(\vec{\sigma}, \alpha) = \sqrt{\frac{2}{3}}\left|\frac{\partial F}{\partial \vec{\sigma}}\right| = \sqrt{1 + 2\frac{(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)}{\sigma^d : \sigma^d}} \leq \sqrt{2}.$$

Loading and unloading in the stress space

The yield surfaces identify elastic and plastic regions, the flow rule (2.29) describes the direction of the plastic flow, whereas loading and unloading criteria identify the characteristics of the deformation (which leads to the calculation of $\beta(t)$).

Let $F(\vec{\sigma}, \alpha)$ be the yield function for a work hardening material. For a fixed point $x \in \Omega$ and a fixed time $t \in [t_0, t_f]$ we have the actual stress state $\vec{\sigma} = \vec{\sigma}(x, t)$ and the vector of hardening parameters $\vec{\alpha} = \vec{\alpha}(x, t)$.

We distinguish between the following cases:

$$F(\vec{\sigma}, \vec{\alpha}) < 0 \qquad \text{elastic deformation}$$

$$F(\vec{\sigma},\vec{\alpha}) = 0 \begin{cases} \frac{\partial F^T}{\partial \vec{\sigma}}\dot{\vec{\sigma}} < 0 & \text{unloading} \\ \frac{\partial F^T}{\partial \vec{\sigma}}\dot{\vec{\sigma}} = 0 & \text{neutral loading} \\ \frac{\partial F^T}{\partial \vec{\sigma}}\dot{\vec{\sigma}} > 0 & \text{loading.} \end{cases}$$

Fig. 7 demonstrates the different situations for $n = \frac{\partial F}{\partial \vec{\sigma}}$ and $\dot{\vec{\sigma}}$:

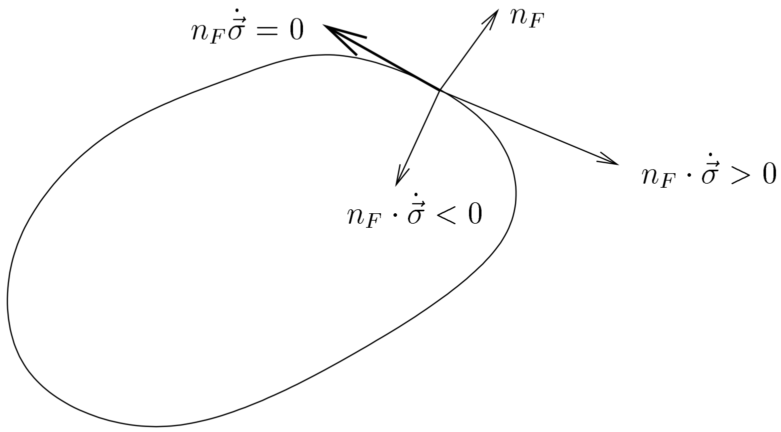


Fig. 7 loading and unloading

Let us discuss different loadings starting from an actual stress state at the yield surface.

a) unloading case

$$F(\vec{\sigma},\vec{\alpha}) = 0, \quad \frac{dF}{dt} = \frac{\partial F}{\partial \vec{\sigma}}\frac{\partial \vec{\sigma}}{dt} + \frac{\partial F}{\partial \vec{\alpha}}\frac{d\vec{\alpha}}{dt} < 0, \quad \frac{d\vec{\alpha}}{dt} = 0, \quad \dot{\epsilon}^p = 0.$$

Hence,

$$\frac{\partial F}{\partial \vec{\sigma}}\frac{\partial \vec{\sigma}}{dt} < 0.$$

b) neutral loading case

$$F(\vec{\sigma},\vec{\alpha}) = 0, \quad \frac{dF}{dt} = 0, \quad \frac{d\vec{\alpha}}{dt} = 0.$$

Hence,

$$\frac{\partial F}{\partial \vec{\sigma}}\dot{\vec{\sigma}} = 0.$$

c) loading case

$$F(\vec{\sigma},\vec{\alpha}) = 0, \quad \frac{dF}{dt} = 0, \quad \frac{d\vec{\alpha}}{dt} \neq 0,$$

$$\dot{\epsilon}^p \neq 0 \quad \text{and} \quad \dot{\epsilon}^p = \beta(t)\frac{\partial F}{\partial \vec{\sigma}} = \beta(t)n_F, \quad \beta(t) > 0.$$

It follows, that

$$\dot{\vec{\epsilon}}^p \cdot \dot{\vec{\sigma}} = \beta(t)\frac{\partial F}{\partial \vec{\sigma}} \cdot \frac{d\vec{\sigma}}{dt} > 0.$$

Calculation of $\beta(t)$

Plastic deformation ($\dot{\epsilon}^p(t) \neq 0$) occurs, if

(1) The actual stress state $\vec{\sigma} = \vec{\sigma}(x,t)$ is situated on the yield surface

$$F(\vec{\sigma}, \vec{\alpha}) = 0.$$

(2) There is loading

$$\frac{dF}{dt} = \frac{\partial F^T}{\partial \vec{\sigma}} \dot{\vec{\sigma}} + \frac{\partial F^T}{\partial \vec{\alpha}} \dot{\vec{\alpha}} = 0, \quad \frac{\partial F^T}{\partial \vec{\sigma}} \dot{\vec{\sigma}} > 0.$$

(3) The associate flow rule reads

$$\dot{\epsilon}^p = \beta \frac{\partial F}{\partial \vec{\sigma}}.$$

(4) The hardening law reads

$$\dot{\vec{\alpha}} = r(\vec{\sigma}, \vec{\alpha}) \beta(t), \quad \beta > 0, \quad \vec{\alpha}(0) = \vec{\alpha}_0.$$

(5)

$$\begin{aligned} \dot{\vec{\epsilon}} &= \dot{\vec{\epsilon}}^e + \dot{\vec{\epsilon}}^p \\ \dot{\vec{\sigma}} &= C \dot{\vec{\epsilon}}^e \qquad \text{(Hooke's law)}. \end{aligned}$$

In order to get a formula for the computation of $\beta = \beta(t)$, we proceed as follows: We multiply (5) with the elasticity matrix C and insert relation (3):

$$\dot{\vec{\sigma}} = C \dot{\vec{\epsilon}} - C \frac{\partial F}{\partial \vec{\sigma}} \, \beta(t), \tag{2.34}$$

(4) and (2) yield

$$\frac{\partial F^T}{\partial \vec{\sigma}} \dot{\vec{\sigma}} = -\beta(t) \frac{\partial F^T}{\partial \vec{\alpha}} \, r(\vec{\sigma}, \vec{\alpha}). \tag{2.35}$$

Inserting (2.34) into (2.35), we get

$$\frac{\partial F^T}{\partial \vec{\sigma}} \left[C \dot{\epsilon} - C \frac{\partial F}{\partial \vec{\sigma}} \, \beta(t) \right] + \beta(t) \frac{\partial F^T}{\partial \dot{\vec{\alpha}}} \, r(\vec{\sigma}, \vec{\alpha}) = 0.$$

and

$$\beta(t) = \frac{\frac{\partial F^T}{\partial \vec{\sigma}} \, C}{\frac{\partial F^T}{\partial \vec{\sigma}} \, C \, \frac{\partial F}{\partial \vec{\sigma}} - \frac{\partial F^T}{\partial \vec{\alpha}} \, r(\vec{\sigma}, \vec{\alpha})} \, \dot{\epsilon}(t). \tag{2.36}$$

$\beta(t)$ is well defined, since the denominator does not vanish (due to the positive definiteness of C and assumption (2)). Using relation (2.36), we can write (2.34) as constitutive equation

$$\boxed{\begin{aligned} \dot{\vec{\sigma}} &= \left(C - \frac{C \frac{\partial F}{\partial \vec{\sigma}} \frac{\partial F^T}{\partial \vec{\sigma}} \, C}{\frac{\partial F^T}{\partial \vec{\sigma}} \, C \, \frac{\partial F}{\partial \vec{\sigma}} - \frac{\partial F^T}{\partial \vec{\alpha}} \, r(\vec{\sigma}, \vec{\alpha})} \right) \dot{\vec{\epsilon}} \\ &= C_{ep}(\dot{\vec{\sigma}}, \vec{\alpha}) \dot{\vec{\epsilon}} \end{aligned}} \tag{2.37}$$

where C_{ep} denotes the elasto–plastic material matrix.

(2.37) yields an explicit form of the constitutive law for elasto–plastic materials with hardening effect:

$$\dot{\vec{\sigma}} = \begin{cases} C\epsilon(\dot{u}) & \text{for } F(\vec{\sigma},\vec{\alpha}) < 0 \vee \frac{\partial F^T}{\partial \vec{\sigma}}\, \dot{\vec{\sigma}} \leq 0 \\ C_{ep}(\sigma,\alpha)\epsilon(\dot{u}) & \text{for } F(\vec{\sigma},\vec{\alpha}) = 0 \wedge \frac{\partial F^T}{\partial \vec{\sigma}}\, \dot{\vec{\sigma}} > 0. \end{cases} \tag{2.38}$$

$$\dot{\vec{\alpha}} = \begin{cases} 0 & \text{for } F(\vec{\sigma},\vec{\alpha}) < 0 \vee \frac{\partial F^T}{\partial \vec{\sigma}}\, \dot{\vec{\sigma}} \leq 0 \\ r(\sigma,\alpha)\, \frac{\frac{\partial F^T}{\partial \vec{\sigma}}\, C}{\frac{\partial F^T}{\partial \vec{\sigma}}\, C\, \frac{\partial F}{\partial \vec{\sigma}} - \frac{\partial F^T}{\partial \vec{\alpha}}\, r(\vec{\sigma},\vec{\alpha})}\, \dot{\epsilon}(\vec{u}) & \text{for } F(\vec{\sigma},\vec{\alpha}) = 0 \wedge \frac{\partial F^T}{\partial \vec{\sigma}}\, \dot{\vec{\sigma}} > 0. \end{cases} \tag{2.39}$$

where $\vec{\epsilon}(\dot{u}) = \dot{\vec{\epsilon}}$.

Example

We consider the von Mises yield surface model (2.32) for isotropic hardening again.

$$\begin{aligned} F(\vec{\sigma},\alpha) &= F(\tau_1,\tau_2,\tau_3,\alpha) = \frac{1}{\sqrt{2}}\sqrt{(\tau_1-\tau_2)^2+(\tau_2-\tau_3)^2+(\tau_3-\tau_1)^2} - h(\alpha) \\ &= \sqrt{\frac{3}{2}\sigma^d : \sigma^d} - h(\alpha). \end{aligned}$$

The yield function (2.31), (2.32) has the derivative (2.33). Choosing an isotropic material, where the elasticity matrix C coincides with

$$M = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix},$$

we have

$$C\frac{\partial F}{\partial \vec{\sigma}} = \frac{1}{2\sqrt{\frac{3}{2}\sigma^d : \sigma^d}} \begin{pmatrix} 2\mu(2\sigma_{11}-\sigma_{22}-\sigma_{33}) \\ 2\mu(2\sigma_{22}-\sigma_{11}-\sigma_{33}) \\ 2\mu(2\sigma_{33}-\sigma_{11}-\sigma_{22}) \\ 6\mu\sigma_{12} \\ 6\mu\sigma_{23} \\ 6\mu\sigma_{13} \end{pmatrix}$$

and

$$\frac{\partial F^T}{\partial \vec{\sigma}}\, C\, \frac{\partial F}{\partial \vec{\sigma}} = \frac{1}{4(\frac{3}{2}\sigma^d : \sigma^d)} \cdot 2\mu 6(\frac{3}{2}\sigma^d : \sigma^d) = 3\mu. \tag{2.40}$$

It results that

$$C_{ep}(\vec{\sigma},\alpha) = M\left[I - \frac{\frac{\partial F}{\partial \vec{\sigma}}\frac{\partial F^T}{\partial \vec{\sigma}}\, M}{3\mu + h(\alpha)\, r(\vec{\sigma},\vec{\alpha})}\right].$$

2.3 The initial boundary value problem for materials of hardening type

First, we remind of the weak formulation of the boundary value problem for linear elastic, homogeneous isotropic materials.

Let $V = H^1(\Omega, \Gamma_0)$ be the closure of $\{u \in [C^\infty(\overline{\Omega})]^3,\ u\Big|_{\Gamma_0} = 0\}$ with respect to the norm

$$\|u\|_V = \sum_{i=1}^{3} \left[\int_\Omega \left(|u_i|^2 + |\partial_1 u_i|^2 + |\partial_2 u_i|^2 + |\partial_3 u_i|^2 \right) dx \right]^{\frac{1}{2}}.$$

The weak formulation of the boundary value problem

$$\begin{aligned} -Lu &= f \text{ in } \Omega \\ u\Big|_{\Gamma_0} &= 0 \\ \sigma[u]n\Big|_{\Gamma_1} &= g \end{aligned}$$

reads: Find an displacement field $u \in V$, such that

$$a(u,v) = \int_\Omega \sigma(u) : \epsilon(v)\, dx = \int_\Omega \vec{\epsilon}(u)^T C \vec{\epsilon}(v)\, dx = \int_{\Gamma_1} gv\, da + \int_\Omega fv\, dx = \langle F, v\rangle \qquad \forall v \in V.$$

The finite element method yields an approximated solution in a finite dimensional subspace $u_h = \sum_{i=1}^{N(h)} c_i^{(h)} e_i^{(h)}$, where $\vec{c}_h = \left(c_i^{(h)}\right)_{i=1,\dots,N(h)}$ is solution to the algebraic equation system (1.88)

$$A^{(h)} \vec{c}_h = \vec{F}_h.$$

We consider the same boundary conditions for the displacement and stress fields for elasto–plastic materials of hardening type

$$u\Big|_{\Gamma_0} = 0, \qquad \sigma \cdot n\Big|_{\Gamma_1} = 0.$$

Fixing a time t, this condition leads to the space $V = H^1(\Omega, \Gamma_0)$ as admissible space for the displacement fields.

The admissible stress space is a subspace S of $[L_2(\Omega)]^6$, and the admissible space for the hardening parameters is a subspace H of $[L_2(\Omega)]^l$ (for a fixed time t).

Remark:

There are sign rules in the relations (2.38), (2.39) which can be expressed by strains.

Proposition:

For points on the yield surface, it holds that

$$sign\,\frac{\partial F^T}{\partial \vec{\sigma}}\dot{\vec{\sigma}} = sign\,\frac{\partial F^T}{\partial \vec{\sigma}}\;C\dot{\vec{\epsilon}}.$$

Proof:

In the elastic case we have $\dot{\vec{\sigma}} = C\dot{\vec{\epsilon}}$. We therefore concentrate of the case $\frac{\partial F^T}{\partial \sigma}\dot{\vec{\sigma}} > 0$.

We multiply the equation

$$\dot{\vec{\sigma}} = \left[C - \frac{C\,\dfrac{\partial F}{\partial \vec{\sigma}}\dfrac{\partial F^T}{\partial \vec{\sigma}}\,C}{\psi(\vec{\sigma},\alpha)} \right] \cdot \vec{\epsilon}(\dot{u}) \tag{2.41}$$

by $\frac{\partial F^T}{\partial \vec{\sigma}}$ from the left.

$$\frac{\partial F^T}{\partial \vec{\sigma}}\dot{\vec{\sigma}} = \left[\frac{\partial F^T}{\partial \vec{\sigma}}C - \frac{\dfrac{\partial F^T}{\partial \vec{\sigma}}\;C\;\dfrac{\partial F}{\partial \vec{\sigma}}\left(\dfrac{\partial F^T}{\partial \vec{\sigma}}\;C\right)}{\psi(\vec{\sigma},\alpha)} \right]\;\vec{\epsilon}(\dot{u}) = \left[1 - \frac{\dfrac{\partial F^T}{\partial \vec{\sigma}}\;C\;\dfrac{\partial F}{\partial \vec{\sigma}}}{\psi(\vec{\sigma},\vec{\alpha})} \right]\;\frac{\partial F^T}{\partial \vec{\sigma}}\;C\;\vec{\epsilon}(\dot{u}).$$

The factor is positive, since

$$0 \le \frac{\dfrac{\partial F^T}{\partial \vec{\sigma}}\;C\;\dfrac{\partial F}{\partial \vec{\sigma}}}{\dfrac{\partial F^T}{\partial \vec{\sigma}}\;C\;\dfrac{\partial F}{\partial \vec{\sigma}} - \underbrace{\dfrac{\partial F^T}{\partial \vec{\alpha}}\;r(\vec{\alpha},\beta)}_{<0}} < 1.$$

The scalar function h_0

$$0 \le \frac{\dfrac{\partial F}{\partial \vec{\sigma}}\;C\;\dfrac{\partial F}{\partial \vec{\sigma}}}{\psi(\vec{\sigma},\vec{\alpha})} = h_0(\vec{\sigma},\vec{\alpha}) < 1 \tag{2.42}$$

will play a special role in the following.

Formulation of the initial boundary value problem:

Find $u \in V$, $\vec{\sigma} \in S$, $\vec{\alpha} \in H$, such that $\forall t \in [t_0, t_f]$ and a given $f \in V'$,

$$\begin{aligned} a(\dot{u},\dot{v}) &= \int_\Omega \sigma(\dot{u}) : \epsilon(\dot{v})\,dx \\ &= \int_\Omega \vec{\sigma}(\dot{u})^T \vec{\epsilon}(\dot{v})\,dx = (f,\dot{v}) \qquad \forall \dot{v} \in V \end{aligned} \tag{2.43}$$

with

$$\vec{\sigma}(\dot{u}) = \begin{cases} C\vec{\epsilon}(\dot{u}) & \text{for } F(\vec{\sigma},\vec{\alpha}) < 0 \vee \frac{\partial F}{\partial \vec{\sigma}}^T C\vec{\epsilon}(\dot{u}) \leq 0 \\ C_{ep}\vec{\epsilon}(\dot{u}) & \text{for } F(\vec{\sigma},\vec{\alpha}) = 0 \wedge \frac{\partial F}{\partial \vec{\sigma}}^T C\vec{\epsilon}(\dot{u}) > 0. \end{cases}$$

$$\dot{\vec{\alpha}} = \begin{cases} 0 & \text{for } F(\vec{\sigma},\vec{\alpha}) < 0 \vee \frac{\partial F}{\partial \vec{\sigma}}^T C\vec{\epsilon}(\dot{u}) \leq 0 \\ r(\vec{\sigma},\vec{\alpha})\frac{\frac{\partial F}{\partial \vec{\sigma}}^T C}{\psi(\vec{\sigma},\vec{\alpha})}\,\epsilon(\dot{u}) & \text{for } F(\vec{\sigma},\vec{\alpha}) = 0 \wedge \frac{\partial F}{\partial \vec{\sigma}}^T C\vec{\epsilon}(\dot{u}) > 0. \end{cases}$$

$u(x,0) = 0,\ \sigma(x,0) = 0,\ \alpha(x,0) = 0,\ \forall x \in \Omega.$
Here,

$$\psi(\vec{\sigma},\vec{\alpha}) = \frac{\partial F}{\partial \vec{\sigma}}^T C \frac{\partial F}{\partial \vec{\sigma}} - \frac{\partial F}{\partial \vec{\alpha}}^T r(\vec{\sigma},\vec{\alpha}).$$

We rewrite the variational equality (2.43), introducing the vector

$$\vec{S}_0(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}) = \begin{cases} 0 & \text{in the elastic case} \\ \vec{s}_0(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u})) = \frac{\frac{\partial F}{\partial \vec{\sigma}}\frac{\partial F}{\partial \vec{\sigma}}^T C\vec{\epsilon}(\dot{u})}{\psi(\vec{\sigma},\vec{\alpha})} & \text{else.} \end{cases}$$

Then, (2.43) becomes:

$$a(\dot{u},\dot{v}) = \int_\Omega \vec{\sigma}(\dot{u})^T\vec{\epsilon}(\dot{v})\,dx = \int_\Omega \left[\vec{\epsilon}(\dot{u}) - \vec{S}_0(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}))\right]^T C\vec{\epsilon}(\dot{v})\,dx = \langle f,\dot{v}\rangle. \tag{2.44}$$

This is a nonlinear form for which we will discuss the solvability and uniqueness.

Monotone operators

In the linear elastic case, we have considered bilinear forms $a(\cdot,\cdot)$ on $V\times V$ with the properties (1.80) and (1.81).

$$\begin{aligned} a(u,v) &\leq c_1\|u\|\|v\| \qquad \forall u,v\in V \\ a(u,u) &\geq c_2\|u\|^2 \qquad \forall u\in V. \end{aligned}$$

The bilinear form can be written in operator form

$$\langle Au,v\rangle := a(u,v), \qquad A\colon\ V\to V'.$$

V is a Banach space, V' its dual space.

The properties (1.80) (1.81) guarantee that A^{-1} exists and is continuous. A Galerkin scheme works and the Lemma of Cea yields

$$\|u-u_h\|_V \leq \frac{c_1}{c_2}\inf_{v_h\in V_h}\|u-v_h\|_V.$$

We now consider nonlinear operators A with similar properties. Be $A:\ V\to V'$. We generalise condition (1.81) [15, p.500, II_B]

Definition:

a) *The operator A is monotone* $\iff \langle Au - Av, u - v\rangle \geq 0 \ \forall u, v \in V$.

b) *A is strictly monotone* $\iff \langle Au - Av, u - v\rangle > 0 \ \forall u, v \in V,\ u \neq v$.

c) *A is uniformly monotone* $\iff \langle Au - Av, u - v\rangle \geq b(\|u - v\|)\|u - v\| \ \forall u, v \in V$, *where $b\colon\ [0,\infty) \to \mathbb{R}$ is a strongly monotone increasing continuous function with $b(0) = 0$, $b(t) \to \infty$ for $t \to \infty$.*

d) *A is strongly monotone* $\iff \exists c_2 > 0$ *with* $\langle Au - Av, u - v\rangle \geq c_2\|u - v\|^2 \ \forall u, v \in V$.

e) *A is coercive* $\iff \frac{\langle Au,u\rangle}{\|u\|_V} \to \infty$ *for* $\|u\| \to \infty$.

Theorem: [16]

$$d) \Rightarrow c) \begin{array}{l} \Rightarrow b) \Rightarrow a) \\ \Rightarrow e) \end{array}$$

Theorem: [16]
Let V be a reflexive Banach space with countable basis, $A\colon\ V \to V'$ and

(i) *A is monotone.*

(ii) *A is coercive.*

(iii) *A is continuous.*

Then for every $f \in V'$ an element $u \in V$ exists with

$$Au = f$$

or

$$\langle Au, v\rangle = a(u, v) = \langle f, v\rangle \qquad \forall v \in V.$$

Definition:
The operator $A\colon\ V \to V'$ is Lipschitz continuous, if there exists a positive constant C_1, such that

$$|\langle Au - Av, w\rangle| \leq C_1\|u - v\|\|w\| \qquad \forall u, v, w \in V.$$

or

$$\|Au - Av\|_{V'} \leq C_1\|u - v\|_V \qquad \forall u, v \in V.$$

Theorem:

Let $A\colon V \to V'$ be strongly monotone and Lipschitz continuous, $f \in V'$. Assume $V_h \subset V$, $\dim V_h < \infty$. There then exists a uniquely determined element $u_h \in V_h$ with

$$\langle Au_h, v_h \rangle = \langle f, v_h \rangle \qquad \forall v_h \in V_h$$

and

$$\|u - u_h\| \leq \frac{C_1}{C_2} \inf_{v_h \in V_h} \|v_h - u\|.$$

Proof:

The coefficients of the Galerkin solution $u_h(x) = \sum_i c_i^{(h)} e_i^{(h)}(x)$ satisfies a nonlinear equation system. Since

$$\langle Au_h - Au, v_h \rangle = 0 \qquad \forall v_h \in V_h$$

it follows, that

$$\|u_h - u\|_V^2 \leq \frac{1}{C_2} \langle Au_h - Au, u_h - u \rangle = \frac{1}{C_2} \langle Au_h - Au, v_h - u \rangle \leq \frac{C_1}{C_2} \|u_h - u\| \|v_h - u\|$$

and finally

$$\|u_h - u\|_V \leq \frac{C_1}{C_2} \|v_h - u\| \qquad \forall v_h \in V_h.$$

We apply these results to the investigation of the nonlinear variational equation (2.44), writing

$$\langle A(\vec{\sigma}, \vec{\alpha}) \dot{u}, \dot{v} \rangle = a(\dot{u}, \dot{v}). \tag{2.45}$$

The following hardening assumptions (compare with (2.42)) guarantee that A is strongly monotone and Lipschitz continuous:

Hardening assumptions:

For all $\vec{\sigma} \in \mathbb{R}^6$, $\vec{\alpha} \in \mathbb{R}^l$, there is a constant $\nu_0 \in [0, 1)$ such that a uniform estimate holds

$$0 \leq \frac{\frac{\partial F}{\partial \vec{\sigma}}^T C \frac{\partial F}{\partial \vec{\sigma}}}{\psi(\vec{\sigma}, \vec{\alpha})} = h_0(\vec{\sigma}, \vec{\alpha}) \leq \nu_0 < 1. \tag{2.46}$$

Theorem: [9, 11]

Assume that the hardening assumption (2.46) is satisfied. The operator $A(\vec{\sigma}, \vec{\alpha}) : V \to V'$, defined through (2.45), is strongly monotone and Lipschitz continuous.

Proof:

It is

$$\langle A(\vec{\sigma}, \vec{\alpha}) \dot{u}, \dot{v} \rangle = \int\limits_{\Omega} \left[\vec{\epsilon}(\dot{u}) - \vec{S}_0(\vec{\sigma}, \vec{\alpha}, \vec{\epsilon}(\dot{u})) \right]^T C \vec{\epsilon}(\dot{v}) \, dx.$$

First step:

We estimate the integral $\int\limits_\Omega \left[\vec{S_0}(\vec\sigma,\vec\alpha,\vec\epsilon(\dot u))\right]^T C\vec\epsilon(\dot v)\,dx$ in the plastic region ($s_0\neq 0$).

The integrand reads

$$\vec\epsilon(\dot u)^T C\frac{\partial F}{\partial\vec\sigma}\frac{\partial F^T}{\partial\vec\sigma}C\vec\epsilon(\dot u)\frac{1}{\psi(\vec\sigma,\vec\alpha)}$$

and can be written as

$$\vec\epsilon^T C^{\frac12}C^{\frac12}\frac{\partial F}{\partial\vec\sigma}\frac{\partial F^T}{\partial\vec\sigma}C^{\frac12}C^{\frac12}\vec\epsilon\frac{1}{\psi(\vec\sigma,\vec\alpha)}.$$

Denoting $x=C^{\frac12}\vec\epsilon$, $y=C^{\frac12}\frac{\partial F}{\partial\vec\sigma}$, the integrand can be estimated in the following way in $\mathbb{R}^6$ (with the usual Euclidean length of a vector).

$$x^Tyy^Tx=(x,y)(x,y)\leq|x|^2|y|^2=x^Txy^Ty$$

or

$$\vec\epsilon(\dot u)C\frac{\partial F^T}{\partial\vec\sigma}C\vec\epsilon(\dot u)\leq(\vec\epsilon(\dot u))^TC\vec\epsilon(\dot u)\frac{\partial F^T}{\partial\vec\sigma}C\frac{\partial F}{\partial\vec\sigma}.$$

Hence,

$$\int\limits_\Omega\vec{S_0}[\vec\sigma,\vec\alpha,\vec\epsilon(\dot u)]^TC\vec\epsilon(\dot u)\leq\int\limits_\Omega\vec\epsilon(\dot u)^TC\vec\epsilon(\dot u)\nu_0\,dx. \tag{2.47}$$

Second step:

We show that A is strongly monotone

$$\begin{aligned}&\langle A(\vec\sigma,\vec\alpha)\dot u-A(\vec\sigma,\vec\alpha)\dot v,\dot u-\dot v\rangle\\&\quad=\int\limits_\Omega[\vec\epsilon(\dot u-\dot v)]^TC\vec\epsilon(\dot u-\dot v)-\left[\vec{S_0}(\vec\sigma,\vec\alpha,\vec\epsilon(\dot u)-\vec{S_0}(\vec\sigma,\vec\alpha,\vec\epsilon(\dot v)\right]^TC\vec\epsilon(\dot u-\dot v)\,dx\end{aligned} \tag{2.48}$$

It is necessary to distinguish whether $\vec\epsilon(\dot u)$ and $\vec\epsilon(\dot v)$ are pure elastic strains or have a plastic part.

1. *case* $\frac{\partial F^T}{\partial\vec\sigma}C\vec\epsilon(\dot u)\leq 0 \quad\wedge\quad \frac{\partial F^T}{\partial\vec\sigma}C\vec\epsilon(\dot v)\leq 0\quad\vee\quad F(\vec\sigma,\vec\alpha)<0.$

   Since the elastic region in the stress space is a convex domain, we have for all $\vartheta\in[0,1]$, that

   $$\vec\epsilon_\vartheta=\vec\epsilon(\dot u)+\vartheta[\vec\epsilon(\dot u)-\vec\epsilon(\dot v)]$$

   leads to the relation

   $$\vec{S_0}(\vec\sigma,\vec\alpha,\vec{\epsilon_\vartheta})=0.$$

2. *case* $\frac{\partial F^T}{\partial\vec\sigma}C\vec{\epsilon_\vartheta}>0\quad\forall\vartheta\in[0,1]\quad\wedge\quad F(\vec\sigma,\vec\alpha)=0.$

   Then

   $$\vec{S_0}(\vec\sigma,\vec\alpha,\vec{\epsilon_\vartheta})=\vec{s_0}(\vec\sigma,\vec\alpha,\vec{\epsilon_\vartheta})\quad\forall\vartheta\in[0,1].$$

3. case There is a $\vartheta^* \in [0,1]$ with

$$\frac{\partial F^T}{\partial \vec{\sigma}} C\vec{\epsilon}_\vartheta \leq 0 \quad \forall \vartheta \leq \vartheta^* \quad (\text{elastic } \vec{\epsilon}(\dot{u}))$$

and

$$\frac{\partial F^T}{\partial \vec{\sigma}} C\vec{\epsilon}_\vartheta > 0 \quad \forall \vartheta > \vartheta^* \quad (\text{plastic } \vec{\epsilon}(\dot{v}))$$

Then

$$\vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u})) - \vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{v})) = \int\limits_0^1 \frac{d}{d\vartheta} S_0(\vec{\sigma},\vec{\alpha},\vec{\epsilon}_\vartheta)\, d\vartheta = \int\limits_{\vartheta^*}^1 \frac{d}{d\vartheta} s_0(\vec{\sigma},\vec{\alpha},\vec{\epsilon}_\vartheta)\, d\vartheta.$$

Third step:

A is Lipschitz continuous.
We have to show that

$$\begin{aligned}
&|\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v}, \dot{w}\rangle| \\
&\quad = |\int\limits_\Omega \vec{\epsilon}(\dot{u}-\dot{v})^T C\vec{\epsilon}(\dot{w})\, dx + \left[\vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u})) - \vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{v})\right]^T C\vec{\epsilon}(\dot{w})\, dx| \qquad (2.49)\\
&\quad \leq C_1 \|\dot{u}-\dot{v}\|_V \|\dot{w}\|_V \qquad \forall \dot{u},\dot{v},\dot{w} \in V.
\end{aligned}$$

We introduce the scalar product in $[L_2(\Omega)]^6$:

$$(\vec{f},\vec{g})_E = \int\limits_\Omega \overline{f}(x)^T C\overline{g}(x)\, dx$$

and the corresponding norm

$$\|\vec{f}\|_E = (\vec{f},\vec{f})_E^{\frac{1}{2}}.$$

The Schwarz inequality reads

$$(\vec{f},\vec{g})_E \leq \|\vec{f}\|_E \|\vec{g}\|_E.$$

The relation (2.49) can be written as

$$\begin{aligned}
&|(\vec{\epsilon}(\dot{u}-\dot{v}),\vec{\epsilon}(\dot{w}))_E + (\vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}) - \vec{S_0}(\vec{\sigma},\vec{\alpha},\epsilon(\dot{u}),\vec{\epsilon}(\dot{w}))_E| \\
&\qquad \leq \|\vec{\epsilon}(\dot{u}-\dot{v})\|_E \vec{\epsilon}(\dot{w})\|_E + \|\vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u})) - \vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{v}))\|_E \|\vec{\epsilon}(\dot{w})\|_E.
\end{aligned}$$

We now estimate (2.48)

1. case

$$\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v}, \dot{u}-\dot{v}\rangle = \int\limits_\Omega \vec{\epsilon}(\dot{u}-\dot{v})^T C\vec{\epsilon}(\dot{u}-\dot{v})\, dx \geq C_2 \|\dot{u}-\dot{v}\|_V^2$$

due to Korn's inequality.

2. case

$$\begin{aligned}
&\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v}, \dot{u}-\dot{v}\rangle \\
&\quad = \int\limits_{\Omega} \vec{\epsilon}(\dot{u}-\dot{v})^T C \vec{\epsilon}(\dot{u}-\dot{v}) - \vec{s_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}-\dot{v})^T C \vec{\epsilon}(\dot{u}-\dot{v})\,dx \\
&\quad \overset{(2.47)}{\geq} (1-\nu_0)\int\limits_{\Omega} \vec{\epsilon}(\dot{u}-\dot{v})^T C \vec{\epsilon}(\dot{u}-\dot{v})\,dx \geq (1-\nu_0)C_2\|\dot{u}-\dot{v}\|_V^2
\end{aligned}$$

3. case

$$\begin{aligned}
&\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v}, \dot{u}-\dot{v}\rangle \\
&\quad = \int\limits_{\Omega} \vec{\epsilon}(\dot{u}-\dot{v})^T C \vec{\epsilon}(\dot{u}-\dot{v}) - \left[\int\limits_{\vartheta^*}^{1} \frac{d}{d\vartheta}\vec{s_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon_\vartheta})\right]^T C\vec{\epsilon}(\dot{u}-\dot{v})\,dx \\
&\quad = \int\limits_{\Omega} \vec{\epsilon}(\dot{u}-\dot{v})^T C \vec{\epsilon}(\dot{u}-\dot{v}) - (1-\vartheta^*)\left[\vec{s_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}-\dot{v})\right]^T C\vec{\epsilon}(\dot{u}-\dot{v}) \\
&\quad \overset{(2.47)}{\geq} (1-(1-\vartheta^*)\nu_0)C_2\|\dot{u}-\dot{v}\|_V^2.
\end{aligned}$$

In order to estimate

$$\|\vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}) - \vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{v})\|_E$$

we have to consider the three cases as before again.

1. case

$$|\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v}, \dot{w}\rangle| \leq \|\vec{\epsilon}(\dot{u}-\dot{v})\|_E\|\vec{\epsilon}(\dot{w})\|_E \leq C_1\|\dot{u}-\dot{v}\|_V\|\dot{w}\|_V.$$

2. case

$$|\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v}, \dot{w}\rangle| \leq \|\vec{\epsilon}(\dot{u}-\dot{v})\|_E\|\vec{\epsilon}(\dot{w})\|_E + \|s_0(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}-\dot{v})\|_E\|\vec{\epsilon}(\dot{w})\|_E \tag{2.50}$$

It is

$$\begin{aligned}
\|s_0(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}-\dot{v})\|_E^2 &= \int\limits_{\Omega} s_0^T C s_0\,dx \\
&= \int\limits_{\Omega} \vec{\epsilon}(\dot{u}-\dot{v})^T C \frac{\partial F^T}{\partial\vec{\sigma}}\left[\frac{\partial F^T}{\partial\vec{\sigma}} C \frac{\partial F}{\partial\vec{\sigma}}\right]\frac{\partial F^T}{\partial\vec{\sigma}} C\vec{\epsilon}(\dot{u}-\dot{v})\frac{1}{\psi^2}\,dx \\
&\overset{(2.46)}{\leq} \nu_0 \int\limits_{\Omega} \vec{\epsilon}(\dot{u}-\dot{v})^T C \frac{\partial F}{\partial\vec{\sigma}}\frac{\partial F^T}{\partial\vec{\sigma}} C\vec{\epsilon}(\dot{u}-\dot{v})\frac{1}{\psi}\,dx \\
&\overset{(2.47)}{\leq} \nu_0^2 \int\limits_{\Omega} \vec{\epsilon}(\dot{u}-\dot{v})^T C\vec{\epsilon}(\dot{u}-\dot{v})\,dx
\end{aligned}$$

Therefore, (2.50) can be estimated

$$|\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v}, \dot{w}\rangle| \leq (1+\nu_0)C_1\|\dot{u}-\dot{v}\|_V\|\dot{w}\|_V.$$

3. case

$$\vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u})) - \vec{S_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{v})) = \int\limits_{\vartheta^*}^{1} \frac{d}{d\vartheta}\vec{s_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon_\vartheta})\,dx = (1-\vartheta^*)\vec{s_0}(\vec{\sigma},\vec{\alpha},\vec{\epsilon}(\dot{u}-\dot{v})).$$

Repeating the calculations of the second step, we get

$$|\langle A(\vec{\sigma},\vec{\alpha})\dot{u} - A(\vec{\sigma},\vec{\alpha})\dot{v},\dot{w}\rangle| \le (1+(1-\vartheta^*)\nu_0)C_1\|\dot{u}-\dot{v}\|_V\|\dot{w}\|_V. \quad \blacksquare$$

2.4 Some remarks on materials of hardening type

We have used the von Mises yield function

$$F(\vec{\sigma}, \alpha) = f(\vec{\sigma}) - h(\alpha) = \sqrt{\frac{3}{2}\,\sigma^\alpha : \sigma^\alpha} - h(\alpha).$$

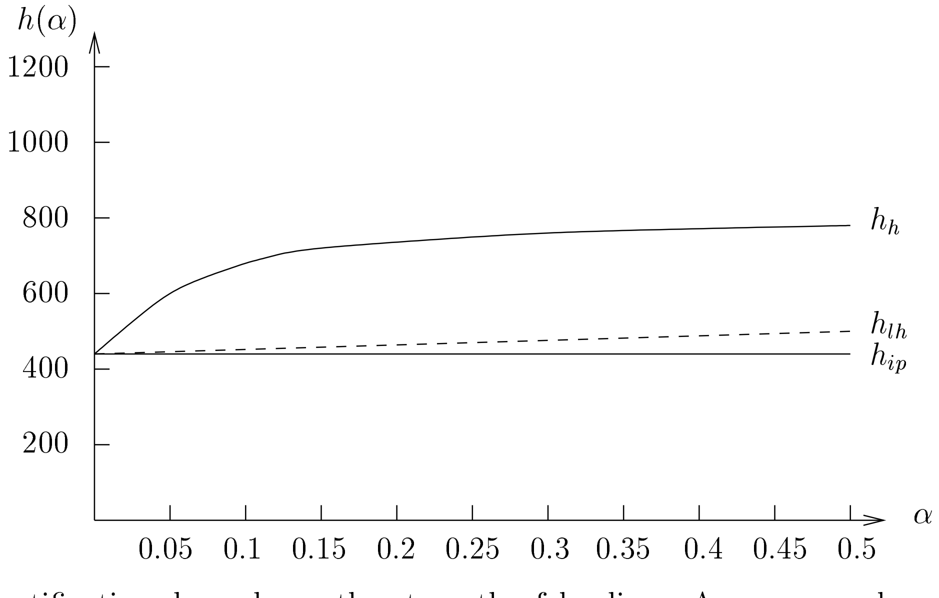
The derivative $\frac{\partial F}{\partial \alpha} = \frac{dh}{d\alpha}$ is needed for the computation of $\psi(\vec{\sigma}, \alpha)$.
Usually, the function $h(\cdot)$ is given as

$$h(\alpha) := \sigma_0 + k\alpha + (\sigma_\infty - \sigma_0)(1 - e^{-\eta\alpha}),$$

where σ_0 is the (uniaxial) yield stress, σ_∞ the saturated (uniaxial) yield stress, k a material dependent hardening factor, and η the hardening exponent.

Example: aluminium

ideal plastic	$h_{ip}(\alpha) = 450 \quad \left[\frac{N}{mm^2} = 10^6 \frac{N}{m^2}\right]$
linear hardening	$h_{lh}(\alpha) = 450 + 129.24\alpha \quad \left[\frac{N}{mm^2}\right]$
hardening	$h_h(\alpha) = 450 + 129.24\alpha + 265(1 - e^{-16.93\alpha}) \quad \left[\frac{N}{mm^2}\right]$



The plastification depends on the strength of loading. As an example, an aluminium rectangle with a circular hole, loaded in y–direction by $4.5\frac{kN}{mm^2}$ is considered in [11].

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